Efficiency Loss in a Class of Two-Sided Market Mechanisms

by

Sebastian James Neumayer

Submitted to the Department of Electrical Engineering and Computer Science
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Abstract

This thesis addresses the question of how to efficiently allocate resources among competing players in convex enviroments. We will analyze the efficiency loss of certain two-sided market mechanisms involving both consumers and suppliers that are natural extensions of Johari's thesis [5]. After gaining intuition about the mechanisms, we show that their worst case efficiency loss approaches 100%. We then introduce some supply-side market mechanisms in a network setting. In the market mechanisms we study, every player submits a bid which specifies a demand or supply function from a parameterized family. Then, the mechanism allocates resources so that supply meets demand.

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Chapter 1

Introduction

1.1 Objective

This thesis addresses the question of how to efficiently allocate resources among competing players in convex enviroments. We will analyze the efficiency loss of certain two-sided market mechanisms involving both consumers and suppliers. We will then introduce some supply-side market mechanisms in a network setting. In the market mechanisms to be studied, every player submits a bid which specifies a demand or supply function from a parameterized family. Then, the mechanism allocates resources so that supply meets demand.

1.2 Problem Motivation

The problem of how to effectively allocate resources in a competitive enviroment comes up frequently in all types of engineering and economic scenarios.

Consider the real life scenario where a pizza is to be split between two equally hungry people. How should the pizza be cut so that each person gets a fair share? We could have one person cut the pizza and the other person get first pick of their half. Since the person picking the pizza slice will take the biggest slice, this ensures the person slicing the pizza will cut it fairly evenly. However, what if the two people are not equally hungry; what sort of mechanism can be used to ensure efficient allocation?

We can imagine a mechanism where one player would pay the other player for the difference in pizza wanted. How do we ensure such a mechanism is fair?

Using a mechanism which efficiently allocates resources is important for other markets too. Two types of markets we are concerned with in this thesis are two-sided markets and supply-side markets in a network setting. As a supply side market, we could consider a consumer based electricity market where many suppliers (generators) compete to meet inelastic demand. Consumers do not have a way to react to prices in real time, so we model demand as fixed (inelastic) in this case. As a two-sided market we could have an industrial electricity market where suppliers (generators) and consumers (industrial plants) compete to buy and sell power. Industrial plants can conceivably change their production based on the price of power and that is why in this setup demand is modeled by consumers in a game. History and experience tell us that a poorly designed electricity market can skyrocket the price for power, producing socially undesirable outcomes. Knowing how particular market mechanisms perform in these types of scenarios can help market designers ensure efficient allocations.

1.3 Problem Statement

In this thesis we will analyze particular two-sided market mechanisms and introduce some supply-side network market mechanisms that are direct extensions of the mechanisms found in Johari's thesis [5]. In the market mechanisms we consider, each player submits a bid which parameterizes a family of supply or demand functions and then the mechanism distributes resources so that supply meets demand. We will consider worst case scenarios for the two-sided market mechanism and attempt to put an upper bound on its efficiency loss. We also assume a convex enviroment for our analysis. That is, we assume concave utilities, convex costs and convex constraints.

1.4 Previous Work

There has been a lot of activity in recent years in the analysis of worst case efficiency loss. Initially, economists who were studying market mechanisms were only interested in whether a Nash equilibrium results in an optimal allocation. Then in 2000, Roughgarden and Tardos considered the ratio of the optimal flow rate in a congestion network to the Nash flow rate. They called this quantity the 'price of anarchy' and showed that in a linear congestion network, this quantity is no more that $\frac{4}{3}$ [7]. A case which gives a $\frac{4}{3}$ price of anarchy is given by the well-known Braess's Paradox [7]. It should be noted that all results related to price of anarchy or efficiency loss (including this thesis) assume a convex enviroment.

The demand-side market mechanism we consider in this thesis is the same one Kelly presented in his work in 1997 [6]. He presented a mechanism where each consumer submitted a one-dimensional bid, which specified a demand function from a parameterized family. Then, each user received an allocation proportional to the ratio of their own bid to the sum of all players' bids. He showed that if consumers acted as price takers, the resulting allocation is socially optimal [6]. Hajek then showed by using modified cost functions that when players act as price anticipators this game has a unique Nash equilibrium [1].

Johari and Tsitsiklis took the analysis a step further and found that worst case efficiency in this one-sided game is $\frac{3}{4}$ [3]. They also showed that this result holds when the game is extended to the network setting. Later, they showed for a particular supply-side market mechanism with N suppliers (where suppliers' bids parameterize a family of supply curves), the worst case efficiency is $\frac{1}{1+\frac{1}{N-2}}$ [4]. There have been similar results on closely related market mechanisms by Hajek, Yang and others [2, 8].

1.5 Contributions

In this thesis, we show that a natural extension of Johari's one-sided mechanisms to a two-sided mechanism yields a worst case efficiency loss that approaches 100%.

Further, we show that other natural two-sided mechanisms where users submit supply functions parameterized by multidimensional bids and where bidding occurs in two-stages have efficiency loss that approaches 100% as well. In addition, we also define supply-side mechanisms that are extensions of Johari's and Kelly's supply side mechanisms to the network setting.

1.6 Organization Of Thesis

This thesis is divided into four additional chapters. Chapter 2 will be a brief introduction to basic game theory and other definitions useful in the analysis of efficiency loss. Chapter 3 will deal with some two-sided market mechanisms. A mechanism which naturally follows from Johari's work will be precisely defined and we will show the existence of a Nash equilibrium. We will then review numerical simulations and construct worst case scenarios to gain intuition for the mechanism. Using our intuition, we create a scenario where efficiency loss can come arbitrarily close to zero. Using the same setup, we show for other natural two-sided mechanisms, the efficiency loss approaches zero as well. Chapter 4 considers several supply-side market mechanisms in a network setting. We will introduce mechanisms where each supplier submits a multidimensional bid to the market mechanism (a bid for each consumer) as well as a mechanism where each supplier submits a one dimensional bid. Chapter 5 will summarize results and suggest areas of future research.

Chapter 2

Background Material

In this chapter we will present material which lays the groundwork for our analysis in the rest of the thesis. We will review elementary game theory and define some useful terminology.

2.1 Game Theory Basics

In the models we consider, each player in a competitive setting will simultaneously place a set of bids. Each player knows the other players' payoff functions, but does not know which action they will choose. The competitive setting in our thesis can be modeled as a strategic game. A strategic game is a triplet $< \mathbf{I}, (W_i)_{i \in \mathbf{I}}, (P_i)_{i \in \mathbf{I}} >$ where

- $\mathbf{I}$ is a finite set of players $\mathbf{I} = (1, ..., I)$.
- $(W_i)_{i \in \mathbf{I}}$ is a set of avaliable actions for player i. We denote $w_i \in W_i$ as an action for player i and $\vec{w}_{-i}$ as a vector of actions for all players except i. We also denote $W = \prod_i W_i$ as the set of all profiles of actions and also call $(w_i, \vec{w}_{-i}) \in W$ (or equivalently $\vec{w} \in W$) an outcome of the game.
- $(P_i)_{i \in \mathbf{I}}$ is a set of payoff functions where $P_i : W \rightarrow R$ is a function from the set of all outcomes to the real numbers.

This model of our market allows us to analyze what happens when players are rational. A rational player i will chose her bid, w_i, to maximize her payoff given $\vec{w}_{-i}$. If no one rational player would have decided to play differently given an outcome of a game, $\vec{w}$, then we call that outcome a Nash equilibrium of that game. More precisely, a pure strategy Nash equilibrium is an outcome $\vec{w}_{nash}$ such that $\forall\, i\, \in\, \mathbf{I}$ we have

$$P_i(w_{i,nash}, \vec{w}_{-i,nash}) \geq P_i(w_i, \vec{w}_{-i,nash}) \quad \forall\, w_i \in W_i$$

A Nash equilibrium is a stable outcome that no player wants to deviate from. We are interested in comparing a Nash equilibrium to the best possible outcome, called the socially optimal outcome. The socially optimal outcome of a game is the outcome which maximizes the net payoff. That is, $\vec{w}_{soc}$ is a socially optimal outcome if

$$\sum_{i \in \mathbf{I}} P_i(\vec{w}_{soc}) \geq \sum_{i \in \mathbf{I}} P_i(\vec{w}) \quad \forall\, \vec{w} \in W$$

In this thesis we deal with trying to allocate resources in a competitive setting. To do that, a market mechanism is necessary to allocate resources following a certain set of rules. Specifically, a market mechanism takes the set of all bids, $\vec{w}$, and dictates how much each player must supply or buy and at what price.

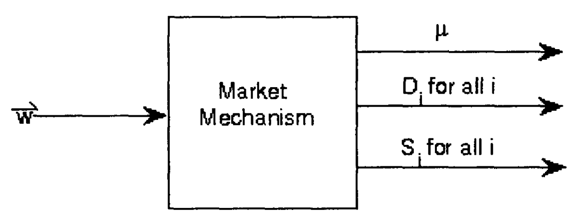

Figure 2-1: Market Mechanism Concept

2.2 Consumers, Suppliers, and Efficiency Loss

In this thesis, we deal with two types of players, consumers and suppliers. A consumer is characterized by her demand curve (the amount she wants at a given price). It is assumed that the higher the price, the lower the demand. A supplier is characterized by her supply curve (the amount she supplies at a given price). It is assumed that the higher the price, the greater the amount the supplier will supply. In the mechanisms we analyze, each supplier and consumer submit a bid to the market mechanism which parameterizes its supply and demand curves. Effectively, this bid selects a supply or demand curve from a parameterized family of curves thereby restricting the player from being able to submit arbitrary supply and demand curves. A sample family of supply and demand curves is given in figure 2-2.

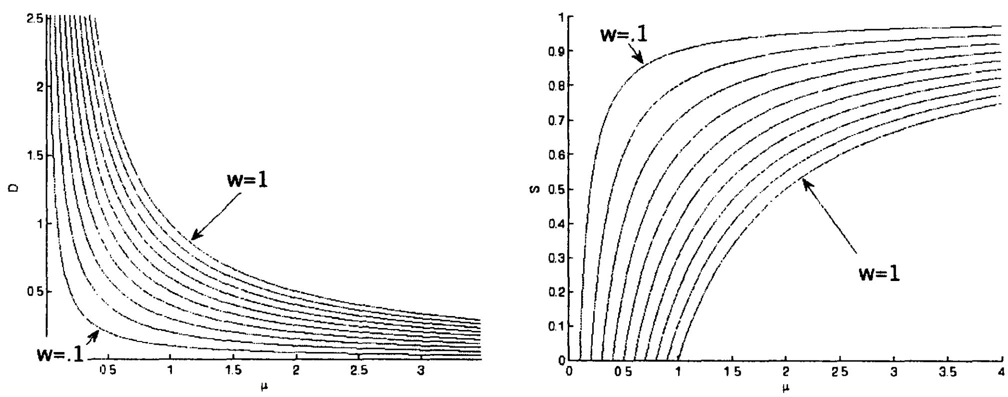

Figure 2-2: Example of Parameterized Curves. Supply on the left is given by $S = 1 - \frac{w}{\mu}$ and demand on the right is given by $D = \frac{w}{\mu}$

In our market, each supplier and consumer acts individually. We assume each player is rational and wishes to maximize her payoff. A price taker is a player who does not anticipate the effects of her bid on the price. She simply uses the bid which maximizes her payoff taking the price as a constant. On the other hand, a price anticipator foresees the effect her bid will have on the price and maximizes her payoff accordingly. In the two-sided mechanism we will study, it turns out that if players are price takers, the outcome is socially optimal.

Efficiency loss is the focus of this thesis. In crude terms, efficiency loss is a measure of the payoff lost when players act selfishly. Specifically, efficiency loss is one minus the ratio of the aggregate payoff in the worst case Nash equilibrium to the aggregate payoff in the socially optimal outcome. That is,

$$\text{Efficiency loss} = 1 - \frac{\min_{nash} \sum_i P_i(\vec{w}_{nash})}{\sum_i P_i(\vec{w}_{soc})}$$

We will study this efficiency loss in the context of two-sided markets and attempt to put upper bounds on it. These upper bounds will allow us to put some guarantees on the results of selfish behavior in the system with respect to the socially optimal solution.

Chapter 3

A Two-Sided Market

3.1 Introduction

This chapter addresses the question of how to efficiently distribute resources amongst competitive players in a two-sided market. Such efficiency concerns can aid in the design of particular engineering systems such as power networks because every utility and generator acts only in their own self interest.

Much analysis has already been done by Johari on one-sided market mechanisms. In his thesis, he considers two market mechanisms: a supply-side market mechanism and a demand-side market mechanism, and puts certain bounds on the efficiency loss of these market mechanisms. In this chapter, we will consider a two-sided market mechanism which is a logical combination of the two one-sided market mechanisms found in Johari's thesis [1].

After defining and discussing the market mechanism, we prove that this mechanism is guaranteed to have a Nash equilibrium as long as there are at least two suppliers and one consumer in the market, under some reasonable assumptions on the cost and utility functions. Second, we identify utility and cost functions which maximize efficiency loss within a certain family of utility and cost functions. This allows us to get a better intuition about worst case scenarios for this market mechanism. Then, we will go through specific numerical examples to gain insight about the mechanism. Our intuition will help us to construct an example where the efficiency loss goes to one and discuss the example's characteristics. Last, we will analyze other two-sided mechanisms and reveal they also have a worst case efficiency loss which approaches one.

3.2 Assumptions On Players and Market Setup

In this section we define and discuss the two types of players, the market mechanism, supply and demand bidding functions, assumptions on the utility and cost functions and the payoff functions. The market consists of two types of players, consumers and suppliers. There are a total of Q consumers and R suppliers. Every consumer and supplier submits a nonnegative bid to the mechanism. The i^{th} consumer's bid is represented by w_i^c, and the i^{th} supplier's bid is represented by w_i^s. Define the vector of all players bids to be $\vec{w}$. Similarly, define the vector of all players bids excepts for the i^{th} consumer's bid to be $\vec{w}_{-i}^c$. Also define the vector of all players bids except the i^{th} supplier's bid to be $\vec{w}_{-i}^s$. These bids parameterize demand and supply functions, $D_i(\mu, w_i^c)$ and $S_i(\mu, w_i^s)$ which are given by:

$$D_i(\mu, w_i^c) = \frac{w_i^c}{\mu}$$

$$S_i(\mu, w_i^s) = 1 - \frac{w_i^s}{\mu}$$

These demand and supply functions are similar to the ones in the one-sided market mechanisms Johari used in his thesis. We assume that the i^{th} consumer's utility and the i^{th} supplier's cost can be quantified and are given by $U_i(D)$ and $C_i(S)$ on the range $D \geq 0$ and $S \leq 1$. We further assume that U_i is a non-negative strictly increasing twice differentiable continuous concave function and that the right derivative at 0, denoted by $U_i'(0)$, exists and is finite. This agrees with intuition that consumers prefer more to less, and that the marginal utility of a resource decreases with the amount received (Law of Decreasing Return). We also assume that C_i is a non-negative strictly increasing differentiable convex function where $C_i(S) = 0$ for $S \leq 0$. This agrees with intuition that suppliers have larger costs for supplying larger amounts of resources, and that marginal production cost always increases as more resources are being supplied because the production capacity is being pressed.

The market mechanism accepts all the consumer and supplier bids and then sets aggregate supply equal to aggregate demand to obtain the market clearing price, $\mu(\vec{w})$.

$$\begin{aligned}\sum_{j=1}^{Q} D_j(\mu(\vec{w}), w_j) &= \sum_{j=1}^{R} S_j(\mu(\vec{w}), w_j) \\ \sum_{j=1}^{Q} \frac{w_j^c}{\mu(\vec{w})} &= \sum_{j=1}^{R} (1 - \frac{w_j^s}{\mu(\vec{w})}) \\ \frac{\sum_{j=1}^{Q} w_j^c}{\mu(\vec{w})} &= R - \frac{\sum_{j=1}^{R} w_j^s}{\mu(\vec{w})}\end{aligned}$$

so

$$\mu(\vec{w}) = \frac{\sum_{j=1}^{Q} w_j^c + \sum_{j=1}^{R} w_j^s}{R}$$

Note that the market clearing price is a function of all the supplier and consumer bids. Also note that all demand and supply functions have price, μ, and a bid, w_i, as arguments. Since price is a function of all bids, we will sometimes abuse notation and use the shorthand $D_i(\vec{w})$ for $D_i(\mu(\vec{w}), w_i)$ and $S_i(\vec{w})$ for $S_i(\mu(\vec{w}), w_i)$. Given the market clearing price, $\mu(\vec{w})$, the market mechanism requires the i^{th} supplier to supply the amount given by the supply function, $S_i(\vec{w}) = 1 - \frac{w_i^s}{\mu(\vec{w})}$, and the i^{th} consumer to receive the amount given by the demand function, $D_i(\vec{w}) = \frac{w_i^c}{\mu(\vec{w})}$. If all bids are equal to zero, then $\mu(\vec{0}) = 0$ and both S and D are undefined. In this case, the only sensible way to have aggregate supply equal to aggregate demand is to set $S = 0$ and $D = 0$. The mechanism also requires that the i^{th} consumer 'pay' for their alotment in the amount of $\mu(\vec{w})D_i(\vec{w})$. It also requires that the i^{th} supplier be compensated in the amount of $\mu(\vec{w})S_i(\vec{w})$. This scheme is price indiscriminant; all consumers (suppliers) must pay (be compensated) at the same rate, $\mu(\vec{w})$.

Define the i^{th} consumer payoff to be

$$P_i^c(\vec{w}) = U_i(D_i(\vec{w})) - \mu(\vec{w})D_i(\vec{w})$$

Consumer payoff is the net gain of a consumer. The above definition is intuitive because the consumer gains from utility in the amount of $U_i(D_i(\vec{w}))$ but must pay for the amount consumed, $\mu(\vec{w})D_i(\vec{w})$.

Define the i^{th} supplier payoff to be

$$P_i^s(\vec{w}) = \mu(\vec{w})S_i(\vec{w}) - C_i(S_i(\vec{w}))$$

Supplier payoff is the net gain of a supplier. The above definition is intuitive because the supplier gains from compensation in the amount of $\mu(\vec{w})S_i(\vec{w})$ and has a cost given by $C_i(S_i(\vec{w}))$.

So far we have defined the players, payoffs, the parameterized supply and demand functions, the market mechanism and laid out the assumptions of the utility and cost functions.

3.3 Aggregate Payoff and Competitive Equilibrium

Aggregate payoff is defined as the sum of all the players payoffs. We will let J represent aggregate payoff. When the market is cleared, it can be alternatively represented by the aggregate utility minus the aggregate cost as shown below.

$$\begin{aligned}
J(\vec{w}) &= \sum_{j=1}^{Q} P_j^c(\vec{w}) + \sum_{j=1}^{R} P_j^s(\vec{w}) \\
&= \sum_{j=1}^{Q}(U_j(D_j(\vec{w})) - \mu(\vec{w})D_j(\vec{w})) + \sum_{j=1}^{R}(\mu(\vec{w})S_j(\vec{w}) - C_j(S_j(\vec{w}))) \\
&= \sum_{j=1}^{Q} U_j(D_j(\vec{w})) - \sum_{j=1}^{R} C_j(S_j(\vec{w})) + \mu(\vec{w})(\sum_{j=1}^{Q} S_j(\vec{w}) - \sum_{j=1}^{Q} D_j(\vec{w})) \\
&= \sum_{j=1}^{Q} U_j(D_j(\vec{w})) - \sum_{j=1}^{R} C_j(S_j(\vec{w}))
\end{aligned}$$

Aggregate payoff of a certain outcome, $J(\vec{w})$ can be seen as the total social welfare of the market.

A vector $\vec{w}_{comp}$ is said to be a competitive equilibrium if each player maximizes their payoff while treating the price as a constant. That is, each player acts as a price taker; they choose the best bid assuming their bid does not affect the market clearing price. So the player sees his payoff only as a function of her bid and the price, μ, not $\vec{w}$; that is, she sees payoff of the form $U_i(D_i(\mu, w_i^c)) - \mu D_i(\mu, w_i^c)$ and $\mu S_i(\mu, w_i^s) - C_i(S_i(\mu, w_i^s))$ and not like $U_i(D_i(\vec{w})) - \mu D_i(\vec{w})$ and $\mu S_i(\vec{w}) - C_i(S_i(\vec{w}))$. It has been shown by [6, 5] that in the one-sided mechanisms we are starting from, $\vec{w}_{comp}$ exists and is actually a socially optimal outcome. It is our belief that a similar result holds for our two-sided mechanism under consideration.

3.4 Existence of Nash Equilibrium

A vector $\vec{w}_{nash}$ is said to be a Nash equilibrium if it has the property that for each player i, w_i maximizes the i^{th} player's payoff given $\vec{w}_{-i}$. Intuitively, a Nash equilibrium is a set of bids such that no player can profitably deviate by changing his bid given that all the other bids are constant. In a Nash equilibrium, players act as price anticipators; they choose their bids knowing the effect they will have on the market clearing price. A Nash equilibrium is not guaranteed to be unique or to exist, so in this section we will prove that given at least two suppliers and one consumer, our two-sided market has a Nash equilibrium. We will first proceed by introducing a game where every player's bid is restricted to the interval $[\epsilon, M]$. Then we show sufficient conditions to apply Rosen's theorem to this $[\epsilon, M]$ game. (Rosen's theorem states a Nash equilibrium exists in every game in which the strategy space is compact and convex and the payoff of each player is continuous and quasi-concave). Then we use a limit argument by letting $\epsilon \to 0$ to show that there exists a Nash equilibrium when every player's bidding interval is extended to $[0, M]$. We then show that it is in no player's interest to bid above certain bounds in the $[0, M]$ game when M is large enough and there are at least two suppliers. By showing these strategic bids are bounded, we argue that a Nash equilibrium for the $[0, M]$ game when M is large enough is also a Nash equilibrium for the $[0, \infty)$ game when there are at least two suppliers.

3.4.1 Step 1: Applying Rosen's Theorem

Let us introduce a modified game where every player's bidding interval is restricted to $[\epsilon, M]$, where $0 < \epsilon < M < \infty$. We will show there exists a Nash equilibrium for this game as a result of Rosen's Theorem. It immediately follows that the strategy space of each player in this modified game is compact and convex because the bidding interval for every player is the closed interval $[\epsilon, M]$.

We now show the supplier and consumer payoff functions are concave by showing the second derivative is non-positive. The demand function is given by:

$$D_i(\vec{w}) = \frac{w_i^c}{\mu(\vec{w})} = R\frac{w_i^c}{\sum_{j=1}^{Q} w_j^c + \sum_{j=1}^{R} w_j^s}$$

Note that $D_i(\vec{w})$ is continuous when all the bids are restricted to $[\epsilon, M]$. Derivatives of the demand function are given by:

$$\frac{\partial D_i(\vec{w})}{\partial w_i^c} = R\frac{\sum_{j=1,i\neq j}^{Q} w_j^c + \sum_{j=1}^{R} w_j^s}{(\sum_{j=1}^{Q} w_j^c + \sum_{j=1}^{R} w_j^s)^2}$$

$$\frac{\partial^2 D_i(\vec{w})}{\partial w_i^{c2}} = -2R\frac{\sum_{j=1,i\neq j}^{Q} w_j^c + \sum_{j=1}^{R} w_j^s}{(\sum_{j=1}^{Q} w_j^c + \sum_{j=1}^{R} w_j^s)^3}$$

Since all bids are restricted to $[\epsilon, M]$, we have $\frac{\partial D_i(\vec{w})}{\partial w_i^c} > 0$ and $\frac{\partial^2 D_i(\vec{w})}{\partial w_i^{c2}} < 0$. The consumer payoff as a function of consumer's bids is given by:

$$\begin{aligned} P_i^c(\vec{w}) &= U_i(D_i(\vec{w})) - \mu(\vec{w})D_i(\vec{w}) \\ &= U_i(D_i(\vec{w})) - w_i^c \end{aligned}$$

Clearly, the payoff is continuous because U_i and D_i are continuous and the composition of continuous functions is continuous as well. Derivatives of the payoff are given by:

$$\frac{\partial P_i^c(\vec{w})}{\partial w_i^c} = U_i'(D_i(\vec{w}))\frac{\partial D_i(\vec{w})}{\partial w_i^c} - 1$$

$$\frac{\partial^2 P_i^c(\vec{w})}{\partial w_i^{c2}} = U_i''(D_i(\vec{w}))(\frac{\partial D_i(\vec{w})}{\partial w_i^c})^2 + U_i'(D_i(\vec{w}))\frac{\partial^2 D_i(\vec{w})}{\partial w_i^{c2}}$$

Now because we assumed U_i to be strictly increasing and concave and because $\frac{\partial D_i(\vec{w})}{\partial w_i^c} > 0$ and $\frac{\partial^2 D_i(\vec{w})}{\partial w_i^{c2}} < 0$, we know that $\frac{\partial^2 P_i^c(\vec{w})}{\partial w_i^{c2}} \leq 0$ and thus that the consumer payoff is concave with respect to her own bid. We will proceed in a similar fashion for the supplier's payoff.

The supply function is given by:

$$S_i(\vec{w}) = 1 - \frac{w_i^s}{\mu(\vec{w})} = 1 - R\frac{w_i^s}{\sum_{j=1}^{Q} w_j^c + \sum_{j=1}^{R} w_j^s}$$

Note that $S_i(\vec{w})$ is continuous when all the bids are restricted to $[\epsilon, M]$. Its derivatives are given by:

$$\frac{\partial S_i(\vec{w})}{\partial w_i^s} = -R\frac{\sum_{j=1}^{Q} w_j^c + \sum_{j=1, i\neq j}^{R} w_j^s}{\left(\sum_{j=1}^{Q} w_j^c + \sum_{j=1}^{R} w_j^s\right)^2}$$

$$\frac{\partial^2 S_i(\vec{w})}{\partial w_i^{s2}} = 2R\frac{\sum_{j=1}^{Q} w_j^c + \sum_{j=1, i\neq j}^{R} w_j^s}{\left(\sum_{j=1}^{Q} w_j^c + \sum_{j=1}^{R} w_j^s\right)^3}$$

Since all bids are restricted to $[\epsilon, M]$, we have $\frac{\partial S_i(\vec{w})}{\partial w_i^s} < 0$ and $\frac{\partial^2 S_i(\vec{w})}{\partial w_i^{s2}} > 0$. The i^{th} supplier's payoff is given by:

$$\begin{aligned} P_i^s(\vec{w}) &= \mu(\vec{w})S_i(\vec{w}) - C_i(S_i(\vec{w})) \\ &= \frac{\sum_{j=1}^{Q} w_j^c + \sum_{j=1, i\neq j}^{R} w_j^s + (1-R)w_i^s}{R} - C_i(S_i(\vec{w})) \end{aligned}$$

The payoff is continuous because C_i and S_i are continuous, and the composition of continuous functions is continuous as well. Derivatives of the payoff are given by:

$$\frac{\partial P_i^s(\vec{w})}{\partial w_i^s} = \frac{1-R}{R} - C_i'(S_i(\vec{w}))\frac{\partial S_i(\vec{w})}{\partial w_i^s}$$

$$\frac{\partial^2 P_i^s(\vec{w})}{\partial w_i^{s2}} = -C_i''(S_i(\vec{w}))\left(\frac{\partial S_i(\vec{w})}{\partial w_i^s}\right)^2 + -C_i'(S_i(\vec{w}))\frac{\partial^2 S_i(\vec{w})}{\partial w_i^{s2}}$$

Since we assumed C_i to be strictly increasing and convex and because $\frac{\partial S_i(\vec{w})}{\partial w_i^s} < 0$ and $\frac{\partial^2 S_i(\vec{w})}{\partial w_i^{s2}} > 0$, we know that $\frac{\partial P_i^s(\vec{w})}{\partial w_i^{s2}} \leq 0$ and thus each supplier's payoff is concave with respect to his own bid.

In this $[\epsilon, M]$ game where each players bidding interval is restricted to $[\epsilon, M]$, we have shown the strategy space is compact and convex and that all players payoff functions are concave and continuous with respect to their own bid. This verifies the conditions in Rosen's theorem and therefore shows a Nash Equilibrium exists for any number of suppliers and consumers in this $[\epsilon, M]$ game.

3.4.2 Step 2: Extending the bidding interval to $[0, M]$

In this section we will show the existence of a Nash equilibrium in the $[0, M]$ game. We do this by creating a sequence of Nash equilibria from the $[\epsilon, M]$ game as $\epsilon \longrightarrow 0$. We show there exists a subsequence of this sequence which converges to some strategy, and that this strategy satisfies the optimality conditions for a Nash equilibrium in the $[0, M]$ game. We thus obtain a Nash equilibrium for the $[0, M]$ game.

In the previous section we showed that there exists a Nash equilibrium for the $[\epsilon, M]$ game. We denote such an equilibrium by $\vec{w}_{nash}(\epsilon)$. Let $\{\vec{w}_{nash}(\epsilon_k)\}$ be a sequence of these Nash equilibria based on the value of ϵ where $\epsilon_k = \frac{1}{k}$. Note that this sequence lies in a compact space because all bids lie in the interval $[0, M]$ (the finite Cartesian product of closed intervals is compact). Now, because $\{\vec{w}_{nash}(\epsilon_k)\}$ is a sequence in a compact space, we know that some subsequence of $\{\vec{w}_{nash}(\epsilon_k)\}$ converges to a vector $\vec{w}^*$ in our $[0, M]$ strategy space. Showing that $\vec{w}^*$ satisfies the optimality conditions for Nash equilibrium in the $[0, M]$ game will imply that $\vec{w}^*$ is a Nash equilibrium for the $[0, M]$ game.

In the $[\epsilon_k, M]$ game, $\{\vec{w}_{nash}(\epsilon_k)\}$ must satisfy the following optimality conditions:

$$\begin{aligned}
\frac{\partial P_i(\vec{w})}{\partial w_i}|_{\vec{w}=\vec{w}_{nash}(\epsilon_k)} &\leq 0 \quad \text{if } w_{i,nash}(\epsilon_k) = \epsilon_k \\
\frac{\partial P_i(\vec{w})}{\partial w_i}|_{\vec{w}=\vec{w}_{nash}(\epsilon_k)} &= 0 \quad \text{if } \epsilon_k < w_{i,nash}(\epsilon_k) < M \\
\frac{\partial P_i(\vec{w})}{\partial w_i}|_{\vec{w}=\vec{w}_{nash}(\epsilon_k)} &\geq 0 \quad \text{if } w_{i,nash}(\epsilon_k) = M
\end{aligned}$$

Plugging in for the payoff functions, we get:

if $w^c_{i,nash}(\epsilon_k) = \epsilon_k$:

$$U_i'\left(R\frac{\epsilon_k}{\epsilon_k+\sum_{j=1,j\neq i}^{Q}w_j^c(\epsilon_k)+\sum_{j=1}^{R}w_j^s(\epsilon_k)}\right)R\frac{\sum_{j=1,j\neq i}^{Q}w_j^c(\epsilon_k)+\sum_{j=1}^{R}w_j^s(\epsilon_k)}{(\epsilon_k+\sum_{j=1,j\neq i}^{Q}w_j^c(\epsilon_k)+\sum_{j=1}^{R}w_j^s(\epsilon_k))^2}-1\leq 0$$

if $\epsilon_k < w_{i,nash}^c(\epsilon_k) < M$:

$$U_i'\left(R\frac{w_i^c(\epsilon_k)}{\sum_{j=1}^{Q}w_j^c(\epsilon_k)+\sum_{j=1}^{R}w_j^s(\epsilon_k)}\right)R\frac{\sum_{j=1,j\neq i}^{Q}w_j^c(\epsilon_k)+\sum_{j=1}^{R}w_j^s(\epsilon_k)}{(\sum_{j=1}^{Q}w_j^c(\epsilon_k)+\sum_{j=1}^{R}w_j^s(\epsilon_k))^2}-1=0$$

if $w_{i,nash}^c(\epsilon_k) = M$:

$$U_i'\left(R\frac{M}{M+\sum_{j=1,j\neq i}^{Q}w_j^c(\epsilon_k)+\sum_{j=1}^{R}w_j^s(\epsilon_k)}\right)R\frac{\sum_{j=1,j\neq i}^{Q}w_j^c(\epsilon_k)+\sum_{j=1}^{R}w_j^s(\epsilon_k)}{(M+\sum_{j=1,j\neq i}^{Q}w_j^c(\epsilon_k)+\sum_{j=1}^{R}w_j^s(\epsilon_k))^2}-1\geq 0$$

if $w_{i,nash}^s(\epsilon_k) = \epsilon_k$:

$$C_i'\left(1-R\frac{\epsilon_k}{\epsilon_k+\sum_{j=1,j\neq i}^{R}w_j^s(\epsilon_k)+\sum_{j=1}^{Q}w_j^c(\epsilon_k)}\right)R\frac{\sum_{j=1,j\neq i}^{R}w_j^s(\epsilon_k)+\sum_{j=1}^{Q}w_j^c(\epsilon_k)}{(\epsilon_k+\sum_{j=1,j\neq i}^{R}w_j^s(\epsilon_k)+\sum_{j=1}^{Q}w_j^c(\epsilon_k))^2}+\frac{1-R}{R}\leq 0$$

if $\epsilon_k < w_{i,nash}^s(\epsilon_k) < M$:

$$C_i'\left(1-R\frac{w_i^s(\epsilon_k)}{\sum_{j=1}^{R}w_j^s(\epsilon_k)+\sum_{j=1}^{Q}w_j^c(\epsilon_k)}\right)R\frac{\sum_{j=1,j\neq i}^{R}w_j^s(\epsilon_k)+\sum_{j=1}^{Q}w_j^c(\epsilon_k)}{(\sum_{j=1}^{R}w_j^s(\epsilon_k)+\sum_{j=1}^{Q}w_j^c(\epsilon_k))^2}+\frac{1-R}{R}=0$$

if $w_{i,nash}^s(\epsilon_k) = M$:

$$C_i'\left(1-R\frac{M}{M+\sum_{j=1,j\neq i}^{R}w_j^s(\epsilon_k)+\sum_{j=1}^{Q}w_j^c(\epsilon_k)}\right)R\frac{\sum_{j=1,j\neq i}^{R}w_j^s(\epsilon_k)+\sum_{j=1}^{Q}w_j^c(\epsilon_k)}{(M+\sum_{j=1,j\neq i}^{R}w_j^s(\epsilon_k)+\sum_{j=1}^{Q}w_j^c(\epsilon_k))^2}+\frac{1-R}{R}\geq 0$$

Taking the limit as $\epsilon_k \to 0$ along a subsequence for which $\vec{w}_{i,nash}(\epsilon_k)$ converges to $\vec{w}^*$, we get the following conditions:

if $w_i^{*c} = 0$:

$$U_i'(0)\frac{R}{\sum_{j=1,j\neq i}^{Q}w_j^{*c}+\sum_{j=1}^{R}w_j^{*s}}-1\leq 0$$

if $0 < w_i^{*c} < M$:

$$U_i'\left(R\frac{w_i^{*c}}{\sum_{j=1}^{Q}w_j^{*c}+\sum_{j=1}^{R}w_j^{*s}}\right)R\frac{\sum_{j=1,j\neq i}^{Q}w_j^{*c}+\sum_{j=1}^{R}w_j^{*s}}{(\sum_{j=1}^{Q}w_j^{*c}+\sum_{j=1}^{R}w_j^{*s})^2}-1=0$$

if $w_i^{*c} = M$:

$$U_i'\left(R\frac{M}{M+\sum_{j=1,j\neq i}^{Q}w_j^{*c}+\sum_{j=1}^{R}w_j^{*s}}\right)R\frac{\sum_{j=1,j\neq i}^{Q}w_j^{*c}+\sum_{j=1}^{R}w_j^{*s}}{(M+\sum_{j=1,j\neq i}^{Q}w_j^{*c}+\sum_{j=1}^{R}w_j^{*s})^2}-1\geq 0$$

if $w_i^{*s} = 0$:

$$C_i'(1)\frac{R}{\sum_{j=1,j\neq i}^{R}w_j^{*s}+\sum_{j=1}^{Q}w_j^{*c}}+\frac{1-R}{R}\leq 0$$

if $0 < w_i^{*s} < M$:

$$C_i'\left(1-R\frac{w_i^{*s}}{\sum_{j=1}^{R}w_j^{*s}+\sum_{j=1}^{Q}w_j^{*c}}\right)R\frac{\sum_{j=1,j\neq i}^{R}w_j^{*s}+\sum_{j=1}^{Q}w_j^{*c}}{(\sum_{j=1}^{R}w_j^{*s}+\sum_{j=1}^{Q}w_j^{*c})^2}+\frac{1-R}{R}=0$$

if $w_i^{*s} = M$:

$$C_i'\Big(1 - R\frac{M}{M+\sum_{j=1,j\neq i}^{R} w_j^{*s}+\sum_{j=1}^{Q} w_j^{*c}}\Big)R\frac{\sum_{j=1,j\neq i}^{R} w_j^{*s}+\sum_{j=1}^{Q} w_j^{*c}}{(M+\sum_{j=1,j\neq i}^{R} w_j^{*s}+\sum_{j=1}^{Q} w_j^{*c})^2} + \frac{1-R}{R} \geq 0$$

These conditions hold as long as $\sum_{j=1}^{Q} w_j^{*c} + \sum_{j=1}^{R} w_j^{*s} \neq 0$. We will now show $\vec{w}^* \neq \vec{0}$ by showing $\sum_{j=1}^{Q} w_{j,nash}^c(\epsilon_k) + \sum_{j=1}^{R} w_{j,nash}^s(\epsilon_k) \geq c > 0$ for all k where c is some positive finite constant. If any player's bid is M in the Nash equilibrium for the $[\epsilon, M]$ game, then the sum of all players bids is also bounded by M in the Nash equilibrium. However, if no player bids M in the Nash equilibrium for the $[\epsilon, M]$ game, then from the Nash equilibrium conditions we must have $\frac{\partial P_i^c(\vec{w}(\epsilon_k))}{\partial w_i^c} \leq 0$ for every consumer i. This gives:

$$U_i'\Big(\frac{Rw_{i,nash}^c(\epsilon_k)}{\sum_{j=1}^{Q} w_{j,nash}^c(\epsilon_k) + \sum_{j=1,nash}^{R} w_j^c(\epsilon_k)}\Big)R\frac{\sum_{j=1,i\neq j}^{Q} w_{j,nash}^c(\epsilon_k) + \sum_{j=1}^{R} w_{j,nash}^c(\epsilon_k)}{(\sum_{j=1}^{Q} w_{j,nash}^c(\epsilon_k) + \sum_{j=1}^{R} w_{j,nash}^c(\epsilon_k))^2} - 1 \leq 0 \;\; \forall\, i$$

The market mechanism sets aggregate demand equal to aggregate supply, and since the maximum amount supplied by each supplier is one, demand is upper bounded by R. Also, since U is concave and strictly increasing, we have $0 < U'(R) < U'(A) < U'(0)$ where $0 < A < R$. Applying this to the inequality above yields:

$$\frac{\sum_{j=1,i\neq j}^{Q} w_{j,nash}^c(\epsilon_k) + \sum_{j=1}^{R} w_{j,nash}^c(\epsilon_k)}{(\sum_{j=1}^{Q} w_{j,nash}^c(\epsilon_k) + \sum_{j=1}^{R} w_{j,nash}^c(\epsilon_k))^2} \leq \frac{1}{RU_i'(R)} < \infty \quad \forall\, i$$

Adding all Q of these inequalities together (one for each consumer), we get:

$$\frac{(Q-1)\sum_{j=1}^{Q} w_{j,nash}^c(\epsilon_k) + Q\sum_{j=1}^{R} w_{j,nash}^c(\epsilon_k)}{(\sum_{j=1}^{Q} w_{j,nash}^c(\epsilon_k) + \sum_{j=1}^{R} w_{j,nash}^c(\epsilon_k))^2} \leq f$$

where f is some finite positive constant. Since all bids are non-negative, this implies:

$$\frac{(Q-1)(\sum_{j=1}^{Q} w_{j,nash}^c(\epsilon_k) + \sum_{j=1}^{R} w_{j,nash}^c(\epsilon_k))}{(\sum_{j=1}^{Q} w_{j,nash}^c(\epsilon_k) + \sum_{j=1}^{R} w_{j,nash}^c(\epsilon_k))^2} \leq f$$

So,

$$\frac{(Q-1)}{f} \leq \sum_{j=1}^{Q} w_{j,nash}^c(\epsilon_k) + \sum_{j=1}^{R} w_{j,nash}^c(\epsilon_k)$$

This means that for $Q \geq 2$, $\sum_{j=1}^{Q} w_{j,nash}^c(\epsilon_k) + \sum_{j=1}^{R} w_{j,nash}^c(\epsilon_k) \geq c > 0$. We leave as a conjecture that this is true for $Q = 1$. Since we have shown $\sum_{j=1}^{Q} w_{j,nash}^c(\epsilon_k) + \sum_{j=1}^{R} w_{j,nash}^c(\epsilon_k) \geq c > 0$ for all k. in taking the limit as $k \to \infty$, we know $\vec{w}^* \neq \vec{0}$. Since $\vec{w}^*$ is not equal to $\vec{0}$, we know $\vec{w}^*$ always satisfies the previous conditions. These previous conditions are also the same conditions for a Nash equilibrium in the $[0, M]$ game. We have thus shown that $\vec{w}^*$ is a strategy which satisfies the optimality conditions for Nash equilibrium in the $[0, M]$ game, which shows $\vec{w}^*$ is a Nash equilibrium for the $[0, M]$ game.

3.4.3 Step 3: Extending the bidding interval to $[0, \infty)$

Consider a $\vec{w}_{nash}$ which is a Nash equilibrium for the $[0, M]$ game in step 2; we want to show that when M is large enough, $\vec{w}_{nash}$ is a Nash equilibrium for the game where all players bids are restricted to $[0, \infty)$, assuming there exist at least two suppliers. Let a rational bid for player i be a bid which maximizes the players payoff, P_i, given everyone elses bid. We start by analyzing the payoff functions to get a bound on the rational bids for the $[0, M]$ game when M is large. By showing these rational bids are bounded, we argue that a Nash equilibrium for the $[0, M]$ game when M is large enough is also a Nash equilibrium for the $[0, \infty)$ game.

The i^{th} consumer payoff is given by:

$$P_i^c(\vec{w}) = U_i(D_i(\vec{w})) - \mu(\vec{w})D_i(\vec{w}) = U_i(D_i(\vec{w})) - w_i^c$$

If $w_i^c = 0$, then $P_i^c(\vec{w}) = U_i(D(0, \vec{w}_{-i}^c)) - 0 = U_i(0)$. Since aggregate supply must equal aggregate demand and because $S \leq 1$, we know aggregate demand cannot exceed the total number of suppliers, R. This implies that $U_i(D_i(\vec{w})) \leq U_i(R)$ for all possible $\vec{w}$. Next note that if $w_i^c > U_i(R) \geq U_i(D_i(\vec{w}))$, then $P_i^c(\vec{w}) < 0$. Since the i^{th} consumer's payoff is non-negative when w_i^c is zero (we assumed U to be non-negative), it is never in the i^{th} consumer's interest to bid more than $U_i(R)$, because the payoff would then be negative. So the rational i^{th} consumer will never choose to bid above $U_i(R)$, because choosing a bid of $w_i^c = 0$ will always result in a higher payoff. So a bid w_i^c which is greater than $\max_i U_i(R)$ is never a rational bid for any consumer in the $[0, M]$ game when M is large enough because letting $w_i^c = 0$ will always yield a higher payoff.

A similar argument holds for the supplier. We will first note for the $[0, \infty)$ game with only one supplier, the supplier payoff is given by

$$P_1^s(\vec{w}) = \sum_{j=1}^{Q} w_j^c - C_1\left(\frac{\sum_{j=1}^{Q} w_j^c}{w_1^s + \sum_{j=1}^{Q} w_j^c}\right)$$

Given $w_1^s = N$ and $w_i^c > 0$ for at least one consumer, the supplier is always better off choosing w_1^s above N because P_i^s is strictly increasing with w_1^s. This means that for a single supplier, there exists no Nash equilibrium because there exists no supplier bid that maximizes the supplier payoff function. Intuitively, a single supplier does not lead to a Nash equilibrium in the $[0, \infty)$ game because a single supplier has no competition so she has the ability to increase her bid without worry, so there is no bound on the 'best bid'.

Assuming $R \geq 2$, we will now show there exists a finite bound on w_i^s for which no rational supplier will bid above in the $[0, M]$ game when M is large. We first note that the supplier payoff is given by:

$$\begin{aligned} P_i^s(\vec{w}) &= \mu(\vec{w})S_i(\vec{w}) - C_i(S_i(\vec{w})) \\ &= \frac{\sum_{j=1}^{Q} w_j^c + \sum_{j=1, j\neq i}^{R} w_j^s}{R} + \left(\frac{1}{R} - 1\right)w_i^s - C_i\left(1 - \frac{Rw_i^s}{\sum_{j=1}^{Q} w_j^c + \sum_{j=1}^{R} w_j^s}\right) \end{aligned}$$

If $w_i^s = 0$, then $P_i^s(0, \vec{w}_{-i}^s) \geq \frac{\sum_{j=1}^{Q} w_j^c + \sum_{j=1, j\neq i}^{R} w_j^s}{R} - C_i(1)$ (there is inequality because if $\vec{w}_{-i}^s = \vec{0}$, then supply and thus cost become zero). Now note that C_i is non-negative, so $P_i^s(\vec{w}) \leq \frac{\sum_{j=1}^{Q} w_j^c + \sum_{j=1, j\neq i}^{R} w_j^s}{R} + (\frac{1}{R} - 1)w_i^s$ in general. This implies if $w_i^s > RC_i(1)$, then $P_i^s(\vec{w}) < \frac{\sum_{j=1}^{Q} w_j^c + \sum_{j=1, j\neq i}^{R} w_j^s}{R} + (R-1)C_i(1)$. Since the i^{th} supplier's payoff is greater when $w_i^s = 0$ than when $w_i^s > RC_i(1)$ and $R \geq 2$, it is never in the i^{th} suppliers interest to bid more than $RC_i(1)$ when there is more than one supplier because the i^{th} supplier can always increase her payoff by submitting $w_i^s = 0$ (Note this breaks down for $R = 1$). So, a bid w_i^s which is greater than $R \max_i C_i(R)$ is never a rational bid for any supplier in the $[0, M]$ game when $R \geq 2$ and M is large because letting $w_i^s = 0$ will always yield a higher payoff for the supplier.

In the previous section we have shown there exists a Nash equilibrium for the $[0, M]$ game. By showing that there exists a finite bound above which no consumer or supplier would rationally submit a bid in the $[0, M]$ game when $R \geq 2$ and M is large enough shows the Nash equilibrium for the $[0, M]$ game is also a Nash equilibrium for the $[0, \infty)$ game assuming there exist at least two suppliers.

3.5 Changing Utility and Cost Functions to Increase Efficiency Loss

We are interested in finding the worst-case efficiency loss of this mechanism because this will tell us in the worst case how much payoff is lost when every player acts selfishly relative to the maximum possible payoff. This helps to shed light on whether decentralization in a market of this type is a good or bad idea. In this section we analyze how to change utility and cost functions to increase efficiency loss while keeping the same Nash equilibrium. We will find that linear utility functions and piecewise linear cost functions will lead to greater efficiency loss than non-linear utility and cost functions with the same Nash equilibrium. This analysis will help bolster our intuition of the market mechanism. The argument for the utility functions is the same as in [5].

We will start with the analysis of utility functions. We have assumed the i^{th} utility function is a non-negative strictly increasing continuous concave function defined on $D \geq 0$. Let $w_{i,nash}^c$ be the i^{th} consumer's bid in a Nash equilibrium and let $w_{i,soc}^c$ be the i^{th} consumer's bid which leads to a socially optimal outcome. We will represent the i^{th} consumer's demand at these outcomes by $D_{i,soc}$ and $D_{i,nash}$. Let us introduce a modified game where U_i is linearized about $D_{i,nash}$. Call this new utility function U_i^{mod}. It is obvious that both games contain a common Nash equilibrium because the derivative of the utility function is the same at $D_{i,nash}$ in both the initial and modified games. Since $U'^{mod}_i(D_{i,nash}) = U'_i(D_{i,nash})$, the same set of bids $\vec{w}_{nash}$ will

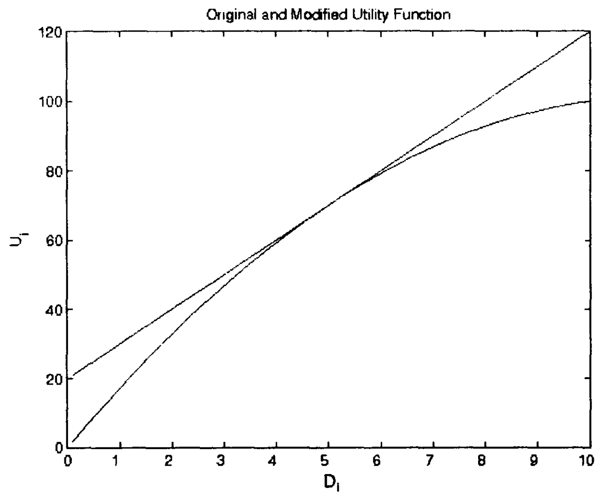

Figure 3-1: Linearization of the utility function. Note how $U_i^{mod} \geq U_i$

satisfy the optimality conditions for Nash equilibrium in both games. Now note that the modified utility function has increased on the entire range because linearizing a non-decreasing concave function results in a function that is larger or equal to the unmodified function on all points in the range. This implies that $U_i^{mod} \geq U_i$ with equality occuring at $D_{i,nash}$. Also note that changing the utility functions has no effect on the supplier payoff (P_s has no terms involving U), so rational supplier bids will be the same in both games. This shows that there is the same Nash equilibrium in the two games: one with unmodified cost and utility functions and one with a linearized i^{th} utility function about $D_{i,nash}$.

Since $U_i^{mod}(D_{i,nash}) = U_i(D_{i,nash})$ and the cost functions are unchanged, the aggregate surplus in a Nash equilibrium remains unchanged when the i^{th} utility function is linearized about the corresponding $D_{i,nash}$. This means that $J(\vec{w}_{nash}) = J^{mod}(\vec{w}_{nash})$ (remember J is aggregate payoff). However, $U_i^{mod} \geq U_i$, so we have $J^{mod}(\vec{w}) \geq J(\vec{w})$ for all feasible $\vec{w}$. Efficiency is defined as $\frac{J(\vec{w}_{nash})}{J(\vec{w}_{comp})}$ and since $J^{mod}(\vec{w}_{comp}) \geq J(\vec{w}_{comp})$ and $J^{mod}(\vec{w}_{nash}) = J(\vec{w}_{nash})$, linearizing the i^{th} utility function will only decrease efficiency.

Now that we have shown linearizing the i^{th} utility function to the form $U_i(D) = a_iD + b_i$ decreased efficiency, we will show that modifying the i^{th} utility function again to the form $U_i(D) = a_i D$ so that $U_i(0) = 0$ will further decrease efficiency. Since the modification is just a constant shift of the utility function, it does not affect

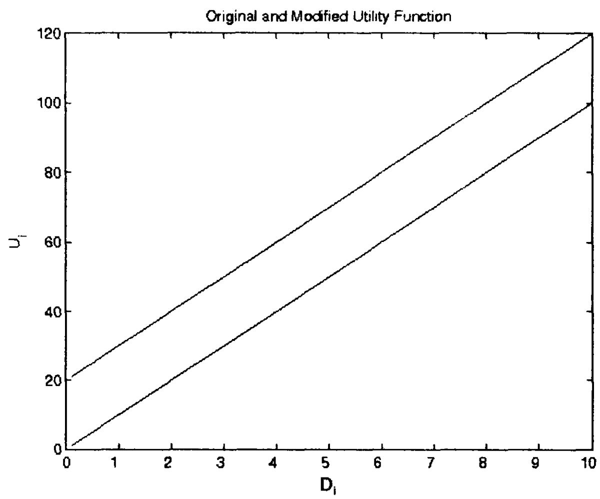

Figure 3-2: Affine shift of utility function where $U_i^{mod} + b_i = U_i$

the derivative of the utility function at any point and so the same $\vec{w}_{nash}$ satisfies the optimality conditions for Nash equilibrium in both games. It is also obvious that both games have the same set of bids which result in a socially optimal outcome. This means that although $\vec{w}_{nash}$ and $\vec{w}_{soc}$ remain unchanged, $J(\vec{w}_{nash})$ and $J(\vec{w}_{soc})$ both decrease by b_i. Ala $J^{mod}(\vec{w}_{nash}) = J(\vec{w}_{nash}) - b_i$ and $J^{mod}(\vec{w}_{soc}) = J(\vec{w}_{soc}) - b_i$. So the new efficiency is given by $\frac{J(\vec{w}_{nash})-b_i}{J(\vec{w}_{soc})-b_i}$ which is less than $\frac{J(\vec{w}_{nash})}{J(\vec{w}_{soc})}$ because $J(\vec{w}_{nash}) \leq J(\vec{w}_{soc})$. So linearizing the i^{th} utility function and then shifting it by a constant so that $U_i(0) = 0$ will decrease efficiency further. Any further shifts downward to decrease efficiency will break our assumption that U is non-negative. Given a game with utilities and costs specified, linearizing the utility function about a Nash eqilibium and shifting it so that $U(0) = 0$ yields the worst case efficiency of all games with the same Nash equlibrium and cost functions. Basically, this is the worst efficiency possible by changing our utility functions within the constraints of our assumptions and keeping the same Nash equilibrium. If we allow our Nash equilibrium to change, we can further decrease efficiency by playing with the slopes of our utility functions. This will be shown in a numerical excercise in section 6.3.

We will now analyze the cost functions in an analogous fashion. We have assumed that every cost function is a strictly increasing convex function where $C_i(S) = 0$ for $S \leq 0$. Let $w^s_{i,nash}$ be the i^{th} supplier's bid in a Nash equilibrium and let $w^s_{i,soc}$ be the i^{th} supplier's bid in a socially optimal outcome. Let us introduce a modified game where C_i is linearized about $S_i(\vec{w}_{nash})$ and then floored with a small linear function σS where σ is small enough to keep the convexity assumtions. It is obvious that both

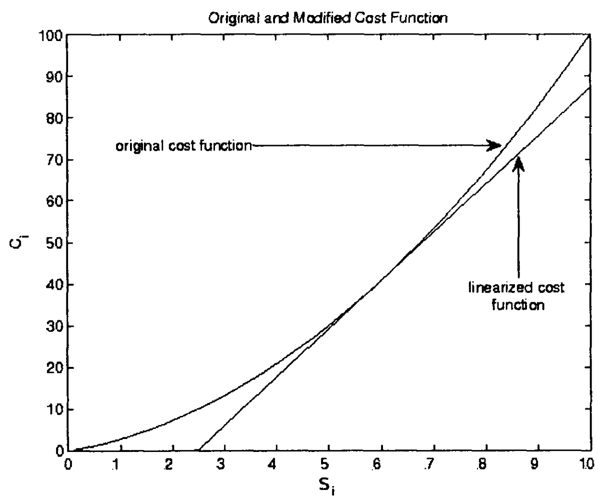

Figure 3-3: Piecewise linearization of the cost function

the initial and modified game share a Nash equilibrium because the derivatives of the cost functions remain the same at $S_i(\vec{w}_{nash})$. This means $\vec{w}_{nash}$ satisfies the optimality conditions for Nash equilibrium in both the modified and unmodified games. Now note that the modified cost function has decreased on the entire range because linearizing and flooring the cost function as above results in a function that is smaller than or equal to the unmodified function on all points in the range. This implies that $C_i^{mod} \leq C_i$ with equality occuring at $S_i(\vec{w}_{nash})$. Since $J(\vec{w}_{nash})$ did not change value and $J(\vec{w}_{soc})$ increased, this modification to the cost function will only decrease efficiency. Any further changes to decrease the cost function will either break convexity or other assumptions we have made. So given a game with utility and costs specified, linearizing, shifting, and flooring the utility and cost functions about a Nash equilibrium will yield a modified game which will have the worst case efficiency loss of all games with that Nash equilibrium. It should be noted though that further efficiency loss can be gained by changing the slopes of the utility or cost functions. This changes the Nash equilibrium and so is not covered by the above analysis, but we will go through some examples of this type of change in the next section.

3.6 Numerical Examples

In this section we will go through several numerical examples to gain insight into the market mechanism. We will first go through a simple example and explicitly show the steps to compute efficiency loss. We then proceed by looking at more complex examples and gain intuition about how efficiency loss changes with different market setups. Due to the complexity of these examples, we will not explictly show the steps; instead we will rely on a Matlab simulation to give our results.

3.6.1 Example: One Consumer and Two Suppliers

Our simple example will be a market consisting of two suppliers with identical quadratic costs and a single consumer with linear cost. Explicitly, the utility and costs functions are given by:

$$\begin{aligned} U_1(D) &= D \\ C_1(S) = C_2(S) &= S^2 \end{aligned}$$

The market clearing price is given by $\mu(\vec{w}) = \frac{w_1^c + w_1^s + w_2^s}{2}$. We will start our analysis with the competitive equilibrium, where each player acts as a price taker. That is, each player sees the market clearing price as a constant, μ. The payoffs for this setup are given by:

$$P_1^c(\mu, w_1^c) = U_1(D_1(\mu, w_1^c)) - \mu D_1(\mu, w_1^c) = \frac{w_1^c}{\mu} - w_1^c$$

$$P_1^s(\mu, w_1^s) = \mu S_1(\mu, w_1^s) - C_1(S_1(\mu, w_1^s)) = \mu - w_1^s - (1 - \frac{w_1^s}{\mu})^2$$

$$P_2^s(\mu, w_2^s) = \mu S_2(\mu, w_2^s) - C_2(S_2(\mu, w_2^s)) = \mu - w_2^s - (1 - \frac{w_2^s}{\mu})^2$$

Each player will choose a bid to optimize their payoff function with μ as constant. Taking the derivatives of the payoff functions, we get:

$$\begin{aligned} P'^c_1(\mu, w_1^c) &= \frac{1}{\mu} - 1 \\ P'^s_1(\mu, w_1^s) &= -1 + \frac{2}{\mu}(1 - \frac{w_1^s}{\mu}) \\ P'^s_2(\mu, w_2^s) &= -1 + \frac{2}{\mu}(1 - \frac{w_1^s}{\mu}) \end{aligned}$$

Setting these equal to zero and checking that our values don't lie outside the boundary, we get that $\mu_{comp} = 1$, $w_{1,comp}^c = 1$, $w_{1,comp}^s = w_{2,comp}^s = \frac{1}{2}$. This gives an aggregate surplus of $U_1(D(1,1)) - C_1(S(1, \frac{1}{2})) - C_2(S(1, \frac{1}{2})) = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$.

Now we will analyze the Nash equilibrium where each player acts as a price anticipator. That is, each player sees the market clearing price as a function of their own bids. The payoffs are given by:

$$P_1^c(\vec{w}) = U_1(D_1(\vec{w})) - \mu(\vec{w})D_1(\vec{w}) = \frac{w_1^c}{\frac{w_1^c + w_1^s + w_2^s}{2}} - w_1^c$$

$$P_1^s(\vec{w}) = \mu(\vec{w})S_1(\vec{w}) - C_1(S_1(\vec{w})) = \frac{w_1^c + w_1^s + w_2^s}{2} - w_1^s - (1 - \frac{w_1^s}{\frac{w_1^c + w_1^s + w_2^s}{2}})^2$$

$$P_2^s(\vec{w}) = \mu(\vec{w})S_2(\vec{w}) - C_2(S_2(\vec{w})) = \frac{w_1^c + w_1^s + w_2^s}{2} - w_2^s - (1 - \frac{w_2^s}{\frac{w_1^c + w_1^s + w_2^s}{2}})^2$$

Taking derivatives and setting them equal to zero gives rise to long expressions. After some algebra, the answer turns out to be $\mu_{nash} = .719, w_{1,nash}^c = .404, w_{1,nash}^s = w_{2,nash}^s = .517$. This gives an aggregate surplus of $U_1(D_1(\vec{w}_{nash})) - C_1(S_1(\vec{w}_{nash})) - C_2(S_2(\vec{w}_{nash})) = .404$. Taking the ratio of $J(\vec{w}_{nash})$ and $J(\vec{w}_{comp})$ gives an efficiency loss of 20%. Notice that $J(\vec{w}_{nash})$ is less than $J(\vec{w}_{comp})$. This is because the competitive equilibrium maximizes aggregate payoff and so $J(\vec{w}_{nash}) \leq J(\vec{w}_{comp})$. In the next example we will look at the outcomes with different numbers of suppliers and consumers and observe what happens to the efficiency loss.

3.6.2 Numerical Simulation

In the following sections we will go through examples that are very difficult to solve analytically, so we will rely on a computer program to solve these problems numerically. The program takes an initially chosen arbitrary strategy $\vec{w}$ and then for each player i, finds d_i which is the derivative of the i^{th} player's payoff function given the current $\vec{w}_{-i}$. We chose the next $\vec{w}$ to be the old $\vec{w}$ plus a small step in the $-\vec{d}$ direction. We repeat this process until the maximum change of any players bid is no more than δ.

While $\max_i |\epsilon \cdot d_i| > \delta$

For each player i

$d_i = \frac{\partial P_i(\vec{w})}{\partial w_i}$

End For

For each player i

$w_i = w_i - \epsilon \cdot d_i$

End For

End While

If this process converges, it stops at a strategy where for each player i, the i^{th} player has no incentive to change his bid to something outside the interval $[w_i - \delta, w_i + \delta]$. When δ is small, this approximates the conditions of a Nash equilibrium. So if $\vec{w}$ converges when δ is small, the vector $\vec{w}$ can be viewed as an approximate Nash equilibrium. We will apply this algorithim to the examples coming up in the next sections in order to aid us in finding the efficiency loss.

3.6.3 Example: Q Consumers and R Suppliers

We will now extend the previous example to an arbitrary number of suppliers and consumers where there are Q consumers with identical linear utilities, $U_i(D) = D$, and R suppliers with identical quadratic costs, $C_i(S) = S^2$. When we apply our algorithim and solve for the efficiency loss, we find some interesting but expected results. Below is a graph showing the efficiency loss for different Q's and R's (see graphs). Note that when $Q = 1$ and $R \to \infty$ the efficiency approaches 88%. When $R = 2$ and $Q \to \infty$ the efficiency approaches 93%. When $R \to \infty$ and $Q \to \infty$ the efficiency approaches 1. Note that efficiency monotonically increases with both R and Q. This makes intuitive sense because whenever another player enters the market, competition increases and so the market power of the players decreases and the efficiency increases. Also, when $R \to \infty$ and $Q \to \infty$, there are so many players that no one player can exert any type of market power because there is perfect competition. So as more players enter

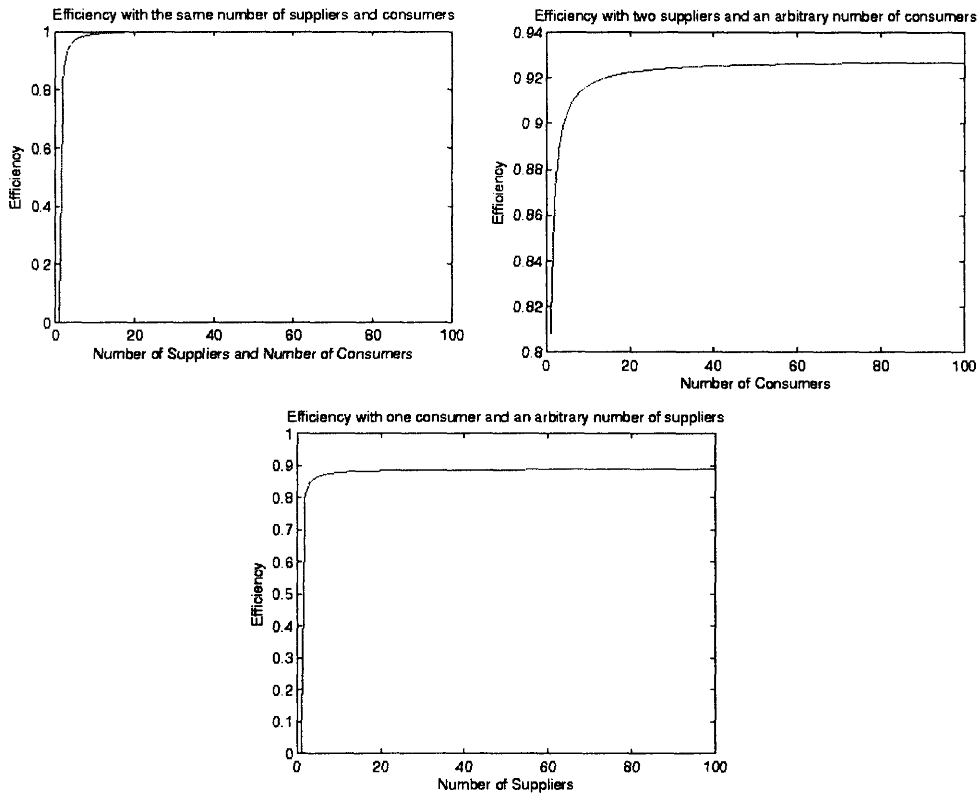

Figure 3-4: Efficiency with different number of suppliers and consumers

the market, the efficiency always increases, and in the limit the efficiency goes to 1.

3.6.4 Example: Consumers With Different Linear Utilities

In this section we analyze what happens when consumers have different linear utilities versus when they have identical linear utilities. We would expect that different utility functions among consumers will lead to lower efficiency than if all utilities were identical because the assymetry of utilities will cause more market power to be used. This is precisely what we find when we run our Matlab program. With five consumers with utilities given by $U_i(D) = D$ and five suppliers with costs given by $C_i(S) = S^2$ we find the efficiency to be 97.1%.

When we change the first utility function to be $U_1(D) = 2D$ we expect that this consumer will now exert more market power over the other players because his utility for every received amount is higher. This is precisely what we find; the efficiency becomes 72.1%. In the competitive equilibrium, all supply is given to this more eager utility.

Now we will change the first utility function to $U_1(D) = \frac{D}{10}$ and leave the other four at $U_i(D) = D$. We predict that the efficiency will again decrease because now the first player is effectively not in the game. This first player will receive no supply in the competitive equilibrium, and receives very little in the Nash equilibrium. Indeed, the efficiency for this setup is 96.6%, only slightly lower than our original case. In order to check our intuition, we run the game with four identical consumers and five identical suppliers. We would expect the efficiency to be nearly the same because we have taken out an inert player. Our intuition comes through; the efficiency loss in this case is 96.6%.

So when there are consumers with identical utility functions, we can decrease the efficiency in one of two ways. We can make a consumer's utility very small so she is effectively removed from the game. This will decrease efficiency because we have numerically shown efficiency is monotonically increasing with the number of consumers. We can also make a consumer's utility larger. This will effectively make her a stronger player and more likely to bid higher for the supply. This asymetry in turn reduces the market power of all other utilities and significantly decreases the efficiency.

3.7 A Zero Efficiency Case

In this example we carefully look at an example where efficiency goes to zero and try to best understand why this occurs. The setup for a simple zero efficiency game is a one consumer and two supplier game with the following utility and cost functions:

$$\begin{aligned} U_1(D) &= aD \\ C_1(S) = C_2(S) &= S \text{ if } S \leq .5 \\ &= 2S - .5 \text{ if } S \geq .5 \end{aligned}$$

We will assume $1 < a < \frac{3}{2}$. It should be noted that this example uses an example of the 'bad' cost and utility functions we found in section 5. Let u be the unit step function. Looking at the competitive equilibrium where players are price takers, we have the following payoff functions:

$$P_1^c(\mu, w_1^c) = U_1(D_1(\mu, w_1^c)) - w_1^c = \frac{aw_1^c}{\mu} - w_1^c$$

$$\begin{aligned} P_1^s(\mu, w_1^c) &= \mu S_1(\mu, w_1^s) - C_1(S_1(\mu, w_1^s)) \\ &= \mu - w_1^s - (1 - \frac{w_1^s}{\mu})u\Big(.5 - (1 - \frac{w_1^s}{\mu}) + 2(1 - \frac{w_1^s}{\mu}) - .5\Big)u\Big((1 - \frac{w_1^s}{\mu}) - .5\Big) \end{aligned}$$

$$\begin{aligned} P_2^s(\mu, w_2^c) &= \mu S_2(\mu, w_2^s) - C_1(S_2(\mu, w_2^s)) \\ &= \mu - w_2^s - (1 - \frac{w_2^s}{\mu})u\Big(.5 - (1 - \frac{w_2^s}{\mu}) + 2(1 - \frac{w_2^s}{\mu}) - .5\Big)u\Big((1 - \frac{w_2^s}{\mu}) - .5\Big) \end{aligned}$$

Optimizing, we get that $\mu_{comp} = a$, $w_{1,comp}^c = a$, $w_{1,comp}^s = w_{2,comp}^s = \frac{a}{2}$. This implies $S_1 = S_2 = \frac{1}{2}$ and $D_1 = 1$. Also, $P_{1,comp}^c = 0$ and $P_{1,comp}^s = P_{2,comp}^s = \frac{a}{2}$. This gives an aggregate surplus of $J(\vec{w}_{comp}) = (a - 1)$. It is interesting to note that when $\mu = a$, the consumer payoff always zero and independent of his bid w_i^c.

Now we will analyze the Nash equilibrium. We have:

$$P_1^c(\vec{w}) = \frac{2aw_1^c}{w_1^c + w_1^s + w_2^s} - w_1^c$$

$$P_1^s(\vec{w}) = \frac{w_1^c + w_1^s + w_2^s}{2} - w_1^s - (1 - \frac{w_1^s}{\frac{w_1^c+w_1^s+w_2^s}{2}})u\Big(.5 - (1 - \frac{w_1^s}{\frac{w_1^c+w_1^s+w_2^s}{2}})+$$
$$2(1 - \frac{w_1^s}{\frac{w_1^c+w_1^s+w_2^s}{2}}) - .5\Big)u\Big((1 - \frac{w_1^s}{\frac{w_1^c+w_1^s+w_2^s}{2}}) - .5\Big)$$

$$P_2^s(\vec{w}) = \frac{w_1^c + w_1^s + w_2^s}{2} - w_2^s - (1 - \frac{w_2^s}{\frac{w_1^c+w_1^s+w_2^s}{2}})u\Big(.5 - (1 - \frac{w_2^s}{\frac{w_1^c+w_1^s+w_2^s}{2}})+$$
$$2(1 - \frac{w_2^s}{\frac{w_1^c+w_1^s+w_2^s}{2}}) - .5\Big)u\Big((1 - \frac{w_2^s}{\frac{w_1^c+w_1^s+w_2^s}{2}}) - .5\Big)$$

We write down the Nash equilibrium conditions and solve them to find $w_{1,nash}^c = \frac{4a(a-1)}{(a+1)}$, $w_{1,nash}^s = w_{2,nash}^s = \frac{4a}{(a+1)^2}$, and $\mu_{nash} = \frac{2a(a^2+1)}{(a+1)^2}$ when a is close to 1. This gives an aggregate surplus of $J(\vec{w}_{nash}) = \frac{2(a^3-a^2-a+1)}{(a^2+1)^2}$. Demand and supply are given by $S_1 = S_2 = \frac{a^2-1}{a^2+1}$ and $D_1 = \frac{2(a^2-1)}{a^2+1}$.

Taking $\frac{J(\vec{w}_{nash})}{J(\vec{w}_{comp})}$ yields an efficiency of $\frac{2(a^3-a^2-a+1)}{(a^2+1)^2(a-1)}$. Note that efficiency goes to 0 as $a \downarrow 1$. See graph below. Using our Matlab program, we also find that this efficiency

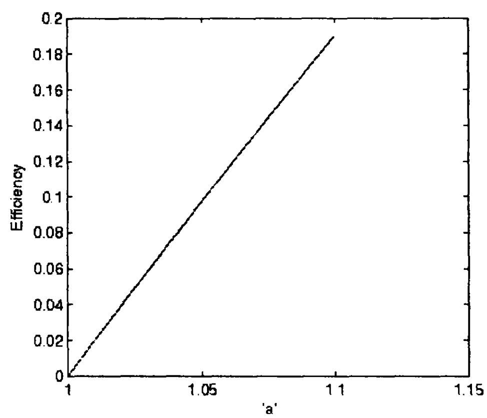

Figure 3-5: Efficiency as a function of a in the zero efficiency case

goes to 0 as $a \downarrow 1$ for any number of suppliers and consumers ($R \geq 2$).

Notable characteristics of this zero-efficiency case is that as $a \downarrow 1$, then both $J(\vec{w}_{comp}) \to 0$ and $J(\vec{w}_{nash}) \to 0$. The reason why efficiency goes to zero is simply because $J(\vec{w}_{nash})$ converges faster. Below are the reaction curves for the game with identical suppliers costs and consumer utilities. These reactions curves show that as

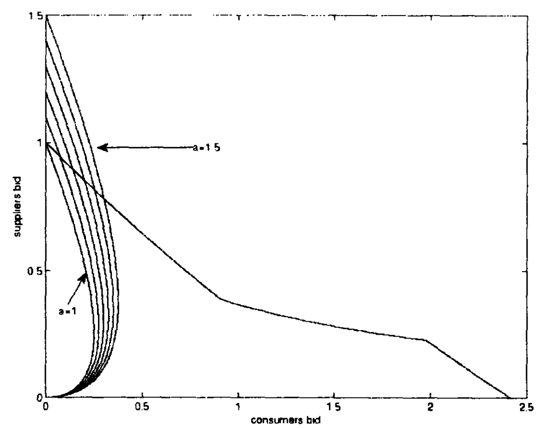

Figure 3-6: Reaction curves of supplier and consumers for values of 'a' from 1 to 1.5

$a \downarrow 1$, then the consumers bid tends towards zero and the suppliers bids tend toward one. This makes sense; as a tends toward one, there is less incentive for players to be in the game and these bids begin to approach a zero aggregate surplus scenario where nothing is traded. You can see the progression of these choices as a decreases by looking at the sequence of intersections on the reaction curve graph. Beyond this, we have not dug any deeper as to the reasons behind this zero efficiency case; this will make an interesting extension to this analysis.

3.8 Alternate Two-sided Mechanisms and Their Efficiency Loss

In this section, we will look at alternate two-sided mechanisms which mantain the flavor of Johari's one-sided mechanisms and analyze their worst case efficiency. The first mechanism we look at has each supplier submit a multidimensional bid instead of a single dimensional bid. We then look at a two-stage mechanism where the suppliers submit their bids and then consumers submit their bids and vice versa. It turns out that these new mechanisms have a worst case efficiency which approaches zero as well.

3.8.1 Multidimensional Bidding Mechanism

We introduce a new mechanism where each consumer i still submits a single bid, w_i^c, to the mechanism, but each supplier i now submits two bids, w_i^s and α_i to the market mechanism where both w_i and α_i are non-negative. Demand and supply are now determined by:

$$D_i(\vec{w},\vec{\alpha}) = \frac{w_i^c}{\mu(\vec{w},\vec{\alpha})}$$

$$S_i(\vec{w},\vec{\alpha}) = \alpha_i - \frac{w_i^s}{\mu(\vec{w},\vec{\alpha})}$$

The market mechanism sets aggregate demand equal to aggregate supply to determine price. So,

$$\mu(\vec{w},\vec{\alpha}) = \frac{\sum_{j=1}^{Q} w_j^c + \sum_{j=1}^{R} w_j^s}{\sum_{j=1}^{R} \alpha_j}$$

If $\sum_{j=1}^{R} \alpha_j = 0$, or $\sum_{j=1}^{R} w_j^s + \sum_{j=1}^{Q} w_j^c = 0$, then the market mechanism sets $\mu(\vec{w},\vec{\alpha}) = 0$, $D_i(\vec{w},\vec{\alpha}) = 0$, $S_i(\vec{w},\vec{\alpha}) = 0$ for all i. We keep the same assumptions on the cost and utility functions. The idea behind this market mechanism is to allow the supplier an extra degree of freedom by letting them choose the maximum amount supplied, α.

Nash Equilibrium with One Supplier and One Consumer

We will show this game has a Nash equilibrium with one supplier and one consumer. Then, using our intuition from the previous market mechanism, we will create a setup where efficiency approaches 0. To show there exists a Nash equilibrium we first look at the payoff functions with one consumer and one supplier.

$$P_1^c(\vec{w},\alpha_1) = U_1(D_1(\vec{w},\alpha_1)) - D_1(\vec{w},\alpha_1)\mu(\vec{w},\alpha_1) = U_1(\frac{w_1^c\alpha_1}{w_1^c + w_1^s}) - w_1^c$$

$$P_1^s(\vec{w},\alpha_1) = S_1(\vec{w},\alpha_1)\mu(\vec{w},\alpha_1) - C_1(S_1(\vec{w},\alpha_1)) = w_1^c - C_1(\frac{w_1^c\alpha_1}{w_1^c + w_1^s})$$

Notice that the supplier payoff function is strictly monotonically increasing in w_1^s and strictly monotonically decreasing in α_1 (remember C_1 is strictly monotonically increasing). The payment to the supplier from the consumer, $S\mu = w_1^c$, is not a function of the supplier's bid in this case. This means to maximize her payoff function, the supplier must minimize her cost. She can do this in one of two ways, either by letting $w_1^s \to \infty$ or setting $\alpha_1 = 0$. Minimizing $C_1(\frac{\alpha_1 w_1^c}{w_1^s + w_1^c})$ over $w_1^s \in [0, \infty)$ and $\alpha_1 \in [0, \infty)$ gives that cost can go as low as zero when $\alpha_1 = 0$ (w_1^s can be anything). Given $\alpha_1 = 0$, $P_1^c = U_1(0) - w_1^c$, which is maximized when $w_1^c = 0$. So $w_1^c = 0$, $\alpha_1 = 0$, $w_1^s \in (0, \infty)$ is a pure strategy Nash equilibrium. As a final note, notice that this scenario breaks down when there is more than one supplier because the 'payment' to the supplier, $S\mu$, becomes a function of supplier bids, and maximizing the payoff functions is not as trivial.

A Zero Efficiency Case

We will now give an example where efficiency goes to 0. We will assume the game has one supplier and one consumer and the costs are given by:

$$U_1(D) = aD$$

$$C_1(S) = S$$

The payoffs then become:

$$P_1^c(\vec{w}, \vec{\alpha}) = \frac{a w_1^c \alpha_1}{w_1^c + w_1^s} - w_1^c$$

$$P_1^s(\vec{w}, \vec{\alpha}) = w_1^c - \frac{w_1^c \alpha_1}{w_1^c + w_1^s}$$

As show previously, $w_1^c = 0$, $\alpha_1 = 0$, and $w_1^s \in (0, \infty)$ is a Nash equilibrium which results in a net payoff of zero. However, the expression for aggregate payoff is given by:

$$J(\vec{w}, \vec{\alpha}) = \frac{(a+1) w_1^c \alpha_1}{w_1^c + w_1^s}$$

This expression can be made arbitrarily large with a suitable choice of α_1. This shows that the efficiency in this case is equal to 0. However, this case has only one supplier and one consumer. It turns out if we let there be Q consumers and R suppliers and they bid symmetrically, aggregate payoff is given by $J(\vec{w}, \vec{\alpha}) = R\alpha(\frac{aQw^c+Rw^s}{Qw^c+Rw^s}) - R$. This can be arbitrarily large with the correct choice of α. It thus turns out that this mechanism has a worst case efficiency loss of 1 with R suppliers and Q consumers.

3.8.2 Two-Stage Bidding Mechanism

We will now consider some two-stage mechanisms where the consumer and supplier groups bid sequentially and all assumptions on rationality and on utility and cost functions still hold. At first, we will consider a mechanism where all consumers submit their bid in the first time slot and then all suppliers submit their bid in the second time slot. This can be modeled as a sequential game with two stages. The difference with this game is that the suppliers submit their bids knowing what every consumer bid and the consumers know the suppliers will react rationally to their actions. It is interesting to note that the aggregate payoff in the socially optimal outcome does not change from our initally proposed mechanism because our strategy space and payoff functions remain unchanged.

A Zero Efficiency Example

We will now run through an example which helps us understand this mechanism and shows a case where efficiency approaches zero. Consider a market with Q consumers and R suppliers and where $U_i(D) = aD$ for all i and $C_i(S) = S$ for all i. We first analyze what the supplier reactions will be given the consumers bids. We do this by maximizing P_i^s over w_i^s.

$$P_i^s(\vec{w}) = \mu(\vec{w})(1 - \frac{w_i^s}{\mu(\vec{w})}) - C_i(1 - \frac{w_i^s}{\mu(\vec{w})})$$

Taking the derivative and assuming symetry gives:

$$\frac{\partial P_i^s(\vec{w})}{\partial w_i^s} = \frac{1}{R} - 1 + \frac{(Rw^s + Qw^c)R - Rw^s}{(Rw^s + Qw^c)^2}$$

Setting the derivative to zero and solving for w^s yields,

$$w^s = \frac{(1-R)(2Qw^c - R) + \sqrt{((R-1)^2(2Qw^c - R)^2 - 4R(R-1)[(Qw^c)^2(1-\frac{1}{R}) - QRw^c]}}{2R(R-1)}$$

So this will be every supplier's rational reaction given the consumer's bids. The consumers know this as well and will choose their bids to maximize their payoff given this knowledge.

$$P_i^c(\vec{w}) = U_i(\frac{w_i^c}{\mu(\vec{w})}) - \mu(\vec{w})\frac{w_i^c}{\mu(\vec{w})}$$

$$P_i^c(\vec{w}) = \frac{aRw_i^c}{\sum_{j=1}^{Q} w_j^c + \sum_{j=1}^{R} w_j^s(\vec{w})} - w_i^c$$

Now we take the derivative remembering that supplier bids are now a function of consumer bids.

$$\frac{\partial P_i^c(\vec{w})}{\partial w_i^c} = \frac{(\sum_{j=1}^{Q} w_j^c + \sum_{j=1}^{R} w_j^s(\vec{w}))aR - aRw_i^c(\frac{1+\partial \sum_{j=1}^{R} w_j^s(\vec{w})}{\partial w_i^c})}{(\sum_{j=1}^{Q} w_j^c + \sum_{j=1}^{R} w_j^s(\vec{w}))^2} - 1$$

Setting this equal to zero gives an expression with many complicated terms. By creating a simulation simular to before in Matlab (all we have to do is modify our expression for the derivative of the consumer's payoff) and letting a approach 1, we find this example has an efficiency loss which approaches one, similar to the previous examples. It also turns out that if we consider the other two-stage mechanism where suppliers bid first and consumers bid second, we get the same efficiency loss result.

3.9 Conclusions and Extensions

We initially proposed a two-sided market mechanism based on the one-sided market mechanisms found in Johari's thesis [5]. After describing the mechanism, we showed there exists a Nash equilibrium, as long as there are at least two suppliers in the market. Then we described different pathological utilities and costs which increase efficiency loss. After going through a few numerical examples to boost our intuition, we lay out a case which gives zero efficiency. We then looked at alternate two-sided mechanisms like two-stage and multidimensional bidding mechanisms and showed those mechanisms also have a worst case efficiency that approaches zero. The results are negative for our two-sided mechanisms under consideration. Future research can explore if any general statements can be about efficiency loss in two-sided markets.

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Chapter 4

Some Supply-Side Market Mechanisms in a Network Setting

In this chapter we introduce two mechanisms to distribute resources in supply-side markets in a competitive network setting. The mechanisms we create are similar to the ones presented by Kelly and Johari [6, 5]. We are motivated to create these supply-side mechanisms because they can be used to describe situations where demand is inelastic or non-responsive like in a power network. In a power network, generators compete to meet a fixed demand at several locations. Demand is fixed because there is no way for consumers to react to the spot price of power. In the rest of the chapter we will create two mechanisms which model this type of scenario.

We will first create a mechanism based on the market mechanism found in Johari's thesis [5]. In this mechanism, each supplier submits a bid to each demand source. At each demand source there is a mechanism which sets aggregate supply equal to the fixed amount of demand at the source. The cost incurred by each supplier is a function of the amount supplied to each demand source. The second mechanism under consideration is a Kelly-like mechanism where every supplier submits a one dimensional bid to the market mechanism which parameterizes her own cost function. The market mechanism then solves an optimization problem which minimizes aggregate cost while ensuring supply meets demand at every location.

4.1 Supply-Side Mechanism with Multidimensional Bids

In this section we will describe a mechanism where each supplier submits a multidimensional bid to the market mechanism and then ensures aggregate supply matches demand at each demand source. This mechanism is inspired by a type of situation where a factory (supplier) can produce multiple types of goods and sells each type of goods to different demand sources. It could also represent a simple model of a powergrid where different generators sell power to different cities. The production costs of these goods are coupled and are described by the supplier's cost function. Let R be the number of suppliers (assumed to be greater than one) and Q be the

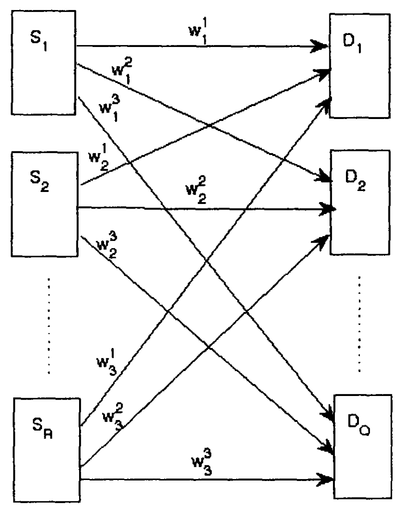

Figure 4-1: Graphical Representation of Market Setup

number of demand sources. Every supplier i submits a non-negative bid representing their supply function for the market at demand source j, denoted by w_i^j. So every supplier i is in fact submitting a multidimensional bid $\{w_i^1, w_i^2, \ldots, w_i^Q\}$ to the market mechanism. We call the set of bids directed at demand source j by $\vec{w^j}$. That is, $\vec{w^j} = \{w_1^j, w_2^j, \ldots, w_R^j\}$. The amount demanded at each demand source j is given by D_j. The market mechanism determines the amount each supplier i supplies to each demand source j by:

$$S_i^j(\vec{w^j}) = D_j - \frac{w_i^j}{\mu_j(\vec{w^j})}$$

When $\mu_j(\vec{w^j}) = 0$, we define $S_i^j(\vec{w^j}) = 0$ for all i. Notice the maximum amount avaliable for supply by a single supplier is D_j. This is to ensure a market clearing outcome is feasible. The market mechanism clears the market by setting aggregate supply equal to demand at each demand source (i.e., $\sum_{i=1}^{R} S_i^j(\vec{w^j}) = D_j$). This gives a market clearing price at demand source j of:

$$\mu_j(\vec{w^j}) = \frac{\sum_{i=1}^{R} w_i^j}{(R-1)D_j}$$

So there will be a total of Q market clearing prices, one for each demand source. It is as if there are Q separate markets, but coupled together by supplier costs. The cost incurred by supplier i is given by $C_i(S_i^1, S_i^2, \ldots, S_i^Q)$ and is a function of the amount supplied to each demand source. We assume that C_i is a non-negative, convex function that is strictly increasing with respect to all its arguments. The payoff function for each supplier is nearly identical to as before. Supplier i is paid for the amount supplied at every demand source in the aggregate amount of $\sum_{j=1}^{Q} \mu_j(\vec{w^j}) S_i^j(\vec{w^j})$ while incurring a cost of $C_i(S_i^1, S_i^2, \ldots, S_i^Q)$. This gives a total payoff for supplier i of:

$$P_i(\vec{w}) = \sum_{j=1}^{Q} \mu_j(\vec{w^j}) S_i^j(\vec{w^j}) - C_i(S_i^1, S_i^2, \ldots, S_i^Q)$$

4.1.1 An Example

To illustrate this mechanism we will go through a simple example. Let $R = 3$ and $Q = 2$ and also let the cost equal the sum of the supply. That is let $C_i(S_i^1, S_i^2) = S_i^1 + S_i^2$ for all i. Also, let $D_1 = D_2 = 1$.

The market clearing prices are given by:

$$\mu_j(\vec{w^j}) = \frac{w_1^j + w_2^j + w_3^j}{2} \quad \text{for j} = 1 \text{ and } 2$$

Costs are given by:

$$C_i(S_i^1(\vec{w^1}), S_i^2(\vec{w^2})) = 1 - \frac{w_i^1}{\mu_1(\vec{w^1})} + 1 - \frac{w_i^2}{\mu_2(\vec{w^2})} \text{ for i } = 1,\ 2,\text{ and } 3$$

Payoffs are the given by:

$$P_i(\vec{w}) = (\mu_1(\vec{w^1}) - 1)S_i^1(\vec{w^1}) + (\mu_1(\vec{w^2}) - 1)S_i^2(\vec{w^2}) \text{ for i } = 1,\ 2,\text{ and } 3$$

As before, the set of bids which maximizes aggregate payoff is the socially optimal solution. A Nash equilibrium (if it exists) can be similarly found by finding the outcome so that no one supplier can change her multidimensional bid to increase her payoff. We only present the mechanism here with no particular results. Although, we know from Johari if $Q = 1$, the worst case efficiency loss is given by $\frac{1}{1+\frac{1}{N-2}}$ [5]. It is an interesting future research direction to find significant results about the efficiency loss of this more general mechanism. Another interesting research direction is to consider a mechanism where the consumers and suppliers are not fully connected. That is, we could consider when some suppliers do not have the ability to supply to a particular demand source.

4.2 Supply Side Mechanism With Single Dimensional Bids

In a network with many consumers, submitting a bid for every demand source may be unrealistic and too complicated, so we create a simpler market mechanism where a supplier submits only a one-dimensional bid.

Let R be the number of suppliers and Q be the number of demand sources. Every supplier i submits a non-negative bid to the market mechanism, denoted by w_i. The amount demanded at each demand source j is given by D_j. Let A be a $Q \times R$ matrix which determines how supply is distributed to the demand sources. The vector of amount supplied at each demand source is given by $A\vec{S}$ where $\vec{S}$ is the supply vector.

The market mechanism determines the price, μ, and the amount each supplier i supplies by solving the following optimization problem:

$$\min_{\mu} \sum_{i=1}^{R} C_i(S_i(\vec{w}))$$

$$\text{s.t. } A\vec{S} \geq \vec{D}$$

$$\text{where } S_i(\vec{w}) = \sum_{j=1}^{Q} D_j - \frac{w_i}{\mu(\vec{w})}$$

In other words, the market mechanism minimizes the aggregate cost required to have supply meet demand. There is only one market clearing price in the market and it is obtained by solving the optimization above. This is unlike the previous mechanism which had a different price at each demand source.

In previous mechanisms in this thesis, the suppliers bid has parameterized a family of supply functions which the market mechanism uses to set aggregate demand to aggregate cost. In this mechanism takes another viewpoint and actually sees the cost functions as being parameterized. Marginal cost at some amount supplied is the same as price at some amount supplied. Given this relationship, we have:

$$\frac{dC_i(S_i(\vec{w}))}{dS(\vec{w})} = \mu(\vec{w}) = \frac{w_i^s}{\sum_{j=1}^{Q} D_j - S_i(\vec{w})}$$

which yeilds

$$C_i(S_i(\vec{w})) = -w_i^s \ln(\sum_{j=1}^{Q} D_j - S_i(\vec{w})) = -w_i^s \ln(\frac{w_i^s}{\mu(\vec{w})})$$

The above gives a parameterized family of cost functions which the market mechanism uses in its optimization.

Payoffs for each supplier i are given by:

$$P_i(\vec{w}) = \sum_{j=1}^{Q} \mu(\vec{w}) A_{ji} S_i(\vec{w}) - C_i(S_i(\vec{w}))$$

This structure is intuitive; the supplier i is paid for the amount supplied to each demand source j in the amount of $\mu(\vec{w})A_{ji}S_i(\vec{w})$ and at an incurred cost of $C_i(S_i(\vec{w}))$.

4.2.1 An Example

To illustrate this mechanism we go through a simple example. Let $D_1 = 1$, $D_2 = 1$, and $A = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix}$. The market mechanism will now solve the following problem:

$$\min_{\mu} \sum_{i=1}^{R} C_i(S_i(\vec{w})) = -w_1^s \ln(\frac{w_1^s}{\mu(\vec{w})}) - w_2^s \ln(\frac{w_2^s}{\mu(\vec{w})}) - w_3^s \ln(\frac{w_3^s}{\mu(\vec{w})})$$

$$\text{s.t.} \quad \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 - \frac{w_1^s}{\mu(\vec{w})} \\ 2 - \frac{w_2^s}{\mu(\vec{w})} \\ 2 - \frac{w_3^s}{\mu(\vec{w})} \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Given $\vec{w}$, the market mechanism will solve the above problem by finding the optimal $\mu(\vec{w})$. Each supplier must then supply the amount given by $S_i(\vec{w}) = \sum_{j=1}^{Q} D_j - \frac{w_i^s}{\mu(\vec{w})}$. The Nash equilibrium and socially optimal solution concept are the same as before, except now evaluating the outcome of a game is more numerically intensive.

The two mechanisms presented in this chapter are used to distribute resources in supply-side markets in a competitive network setting. The first mechanism is similar to the one found in Johari [5], and takes a multidimesional bid and sets aggregate demand to aggregate supply at every demand source. The second mechanism is a Kelly-like mechanism [6] which takes a single bid from each supplier and minimizes aggregate cost while ensuring supply meets demand. No analysis is presented, but it is an interesting future research direction as it would shed light on if these were potential appropriate markets for power networks.

Chapter 5

Conclusions

This thesis attempts to address the problem of how to efficiently allocate resources in a competitive enviroment. In this respect, we presented and analyzed certain mechanisms which may model different types of engineering and economic scenarios. The most relevant model seems to be the one for power networks; by understanding our mechanisms we may be able to gain future insight in how to increase the efficiency of deregulated power markets.

5.1 Research Summary

In this thesis we introduced some market mechanisms and tried to analyze their worst case efficiency loss. To do this, we first introduced some basic game theory concepts. We then introduced a two-sided mechanism which naturally followed from Johari's previous work. After showing the existence of a Nash equilibrium for the mechanism, we performed numerical simulations and constructed worst case scenarios to help us understand the mechanism. We then presented an example where efficiency comes arbitrarily close to zero. Next, we considered other mechanisms where suppliers submit multidimensional bids and where the players bid sequentially. We then showed these mechanisms have a worst case efficiency loss which approaches 100% as well. We moved on from two-sided mechanisms by constructing several supply-side mechanisms in a network setting. We did not analyze the efficiency loss of these mechanisms, but this is one of several areas of potential future research.

5.2 Areas for Future Research

There are several interesting paths we can take to continue to analyze efficiency loss. The most obvious direction is to try to find some more general statements about our two-sided market mechanism. Does there exist a parameterized family of demand and supply functions such that a market clearing mechanism is guaranteed to have an efficiency loss bounded away from 100%? Can we alter the two-sided market mechanism so that an efficient outcome is guaranteed (perhaps by imposing stricter assumptions or by increasing/decreasing the strategy space)?

As a next step, we can analyze the worst case efficiency loss of our supply-side mechanisms. The analysis may be difficult, but solving this problem will likely lend great insight into networked markets with fixed demand, like power markets.

As a final note, we should mention that the Nash equilibrium concept is inherently static and tells us nothing about how players might dynamically interact. We did not consider how players will interact to approach a Nash equilibrium or if their behavior will even converge to a Nash equlibrium. An interesting future direction is to model the response of market participants and see how they change their strategy over time as price, aggregate supply, and aggregate demand change and analyze the evolving efficiency loss. Modeling this is difficult as some players may react faster than others and may use historical information in different ways. The models neccessary to analyze this behavior must be quite robust and this is definitely an interesting open direction.

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