Invertible Sheaves on Generic Rational Surfaces and a Conjecture of Hirschowitz's

by

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Abstract

We study the cohomology of invertible sheaves $\mathcal{L}$ on surfaces X_r blowings-up of $\mathbf{P}_k^2$ at points $p_1, \ldots, p_r$ in general position (*generic rational surfaces*). The main theme is when such sheaves have the natural cohomology, i.e. at most one cohomology group is non zero.

Our approach is geometrical. On one hand we are lead to deform the configuration of points to a special position, notably surfaces with a reduced and irreducible anticanonical divisor, for which, thanks to the extensive work of Harbourne, the cohomology is known or computable. Semicontinuity theorems give then vanishing of the cohomology for line sheaves on generic surfaces satisfying a kind of positivity condition.

On the other hand we fiber our surfaces over the projective line, $\pi : X_r \to \mathbf{P}_k^1$, and reduce the problem to a cohomological estimate for locally free sheaves on the base. Indeed, under mild numerical assumptions on $\mathcal{L} = \mathcal{O}_{X_r}(dH - \Sigma_{i=1}^r m_i E_i)$ (where H denotes the divisor on X_r for which $\mathcal{O}_{X_r}(H) = \sigma^*(\mathcal{O}_{\mathbf{P}_k^2}(1))$, $\sigma : X_r \to \mathbf{P}_k^2$ being the blowing-up map and E_i the (-1)-curves $\sigma^{-1}(p_i)$) the $\pi_*\mathcal{L}$'s turn out to be locally free, hence, thanks to the Birkhoff-Grothendiek theorem, sums of invertible sheaves on $\mathbf{P}_k^1$.

We can thereby reduce a conjecture of André Hirschowitz —to the effect that the invertible sheaves $\mathcal{L}$ have the natural cohomology provided $c_1(\mathcal{L}).E \geq -1$ for any exceptional curve of the first kind E such that $2H.E \leq H.c_1(\mathcal{L})$— to the thesis that certain locally free sheaves are direct sums of $\mathcal{O}_{\mathbf{P}_k^1}(-1)$'s, or, equivalently, are semistable.

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CHAPTER 0

Introduction

La présence, en écartant de nous la seule réalité, celle qu'on pense, adoucit les souffrances, et l'absence les ranime.

Marcel Proust

The aim of the present work is to give new insights toward the study of invertible sheaves on *Generic Rational Surfaces*, that is the blowups of the projective plane at points in general position. The expectation is that such surfaces exhibit a *Bott-Weil* behavior (terminology due to Fedor Bogomolov), meaning for a given class of invertible sheaves their Hilbert polynomial grows asymptotically like the dimension of a single cohomology group,

$$\chi(\mathcal{L}^{\otimes n}) \sim h^{i(\mathcal{L})}(\mathcal{L}^{\otimes n}),$$

as in the case of Abelian Varieties or generalized flag varieties (quotients of reductive algebraic groups by parabolic subgroups —hence the terminology). In the case of generic rational surfaces we can even single out a class of invertible sheaves which *should* have at most one cohomology group nonzero (the so called *natural cohomology* or *non speciality*).

The Conjecture of André Hirschowitz we present in the first chapter best formulates the question we want to ask. We first overview the classical origins of the problem of determining the dimension of incomplete linear systems of planar curves of degree d subject to the condition of passing through r points with multiplicities and its algebraic counterpart. Already the Italian school had recognized the importance of a certain virtual dimension, called expressively *Postulation*, and the defect of coinciding with such dimension as an index of speciality. As in the case of linear system of curves, the modern viewpoint regards such speciality as the dimension of the first cohomology group of a coherent sheaf. Only here the sheaf in question is an ideal sheaf of a 0-dimenensional scheme, rather than an invertible sheaf. This (a feature common to higher dimensional geometry and only recently tackled in the case of surfaces by Reider's theory [**iR88**]) is the main source of difficulty. Blowing up the fixed points of our system we reduce the problem to the study of a complete linear system, and the postulation becomes the Euler-Poincaré characteristic of the pull-back sheaf. Here we dwell on the motivations and the geometrical setting of Hirschowitz's Conjecture giving a hopefully clear exposition of the beautifully terse article that inspired us [**aH89**].

In the second chapter we give shorter proofs of results gotten by Harbourne in the case of an irreducible reduced anticanonical divisor (*Harbourne Surfaces*) corresponding to the condition of the points lying on a cubic curve on $\mathbb{P}^2_k$. (Harbourne [**bH95**] has recently extended his description of numerically effective divisors to the case of arbitrary effective anticanonical divisors —*Anticanonical Surfaces.*)

The aim is to derive vanishings for the first cohomology of invertible sheaves on generic rational surfaces using the upper semicontinuity theorem. To wit, we consider the family of blowups (as constructed by [**bH82**] or [**sK81**]) of $\mathbb{P}^2_k$ within which the generic rational surfaces form an open subset. The Harbourne Surfaces will be at the boundary of this open variety. If we prove vanishing on them, we can then conclude the same for points in general position. This leads to the study of the intersection pairing on Harbourne's surfaces as well as generic ones. In particular the question arises whether there exist curves with self-intersection less than -1. On generic rational surfaces the absence of (-2)-curves is classically known [**iD83**]. In the case of rational curves [**MMM80**] prove that there aren't rational curves with self-intersection less than -1 as well as that, provided the anticanonical dimension [**fS83**] of our surface is zero, neither irrational curves with such self-intersection exist. Deformation theoretical considerations [**fB93, fB95**] suggest that (at least in characteristic zero) this should be the case for arbitrary curves. Following a remark of Hirschowitz's, we prove that his Conjecture implies the absence of such curves. It is very likely that the converse is also true, and speculate on the analogy with almost-complex deformations which characterizes such condition as a property of generality.

The third chapter expounds an original direction of study for generic rational surfaces. It is not hard to fiber them over a projective line. The fibration is flat and allows us, through an easy argument involving Leray spectral sequences, to prove that, under mild assumptions (weaker than the hypotheses of Hirschowitz's Conjecture), the direct images of our invertible sheaves are locally free. This enables us to reduce the computation of cohomology to a computation of degrees of line sheaves on $\mathbb{P}^1_k$.

This approach comes full circle in the concluding sections: we reformulate Hirschowitz's Conjecture in terms of properties of direct images. In particular we show how the presence of exceptional curves among the components of effective divisors lead to bounding the dimension of the first cohomology group of the corresponding line sheaf. Also we prove that the surjectivity in cohomology of the restriction to the generic fiber yields the vanishing of the first cohomolgy group. These results can be regarded as proofs of Hirschowitz's conjecture under some additional hypotheses. We expect to be able to relax such assumptions so as to get a full proof of the conjecture.

The study of such direct images, as well as those of vector sheaves, is interesting per se and we pursued it building on our knowledge of elementary cases. in particular we can compute the direct images for the linne sheaves with multiplicity equal to 1.

We hope such techniques will prove useful in dealing with the study of vector sheaves on generic rational surfaces, especially the problem of stability and exceptionality (the Rudakov's school). Even more generally one could consider this method as a new tool to compute cohomology of vector sheaves on varieties which can be nicely fibered over the projective line.

0.1. Terminology and notation. The language is that of modern Algebraic Geometry as in [**rH78**] or [**EGA**]. At the cost of several repetitions and redundancies, which I hope the reader will forgive, I have striven to make the mathematics accesible to anyone who has an uninhibited acquaintance with [**rH78**].

We have departed from the convention of considering "vector bundles" and "locally free sheaves" as synonyms to shorten the latter to "vector sheaves" (a term coined by Serge Lang). Correspondingly, invertible sheaves will be also called "line sheaves".

All the varieties considered are defined over an algebraically closed field k of arbitrary characteristic unless otherwise specified. In particular, when explicitly calculating direct images we will make use of the Kodaira and Ramanujam vanishing theorems. Hence we will have to assume $char(k) = 0$ (by the so called Lefschetz principle, cf., e.g., [**jH92**, p.187], we can always derive results over an arbitrary field of characteristic zero from the corresponding results over the complex numbers).

Although we have tried to be consistent in the notation and denoted sheaves with script upper case reserving Roman fonts for divisors, we have been somewhat sloppy in the terminology. For instance, we have spoken of intersection with a line sheaf $\mathcal{L}$ meaning with the corresponding divisor $c_1(\mathcal{L})$. By "curve" we will mean a reduced irreducible algebraic scheme of dimension 1. Generally we will deal with curves on surfaces, that is one-dimensional closed subvarieties of the ambient surface.

Given a real number a, we will denote by a^+ (resp. a^-) the positive part, viz. $(a + |a|)/2$ (resp. the negative part $-(a - |a|)/2$). Often, whenever there is no danger of ambiguity, we will shorten $\chi(\mathcal{F})^{\pm}$ to $\chi^{\pm}$ (where $\mathcal{F}$ is a coherent sheaf on a given variety).

0.2. Acknowledgments. First and foremost my heartfelt thanks go to Prof.s Fedor A. Bogomolov and Victor G. Kač who allowed me to keep working at MIT under the direction of Fedor. Fedor has been an inexhaustible source of inspiration for me. Many of the ideas presented here are due to him—of course, the form and validity of the arguments are only my responsibility.

It has been a pleasure and an honor to be able to work at this degree in the intellectually effervescent atmosphere of the MIT Math Department. Creative interaction with Prof.s Michael Artin, Sasha Beilinson, Steve L. Kleiman and Yuri I. Manin has enlightened my stay.

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With great pleasure I acknowledge my mathematical origins. They can be largely ascribed to the patience and confidence of my Master Thesis supervisor, Prof. Enrico Arbarello, who, back in Rome, initiated me to Algebraic Geometry as well as to the stern life of a mathematician. Obviously, the canonical provision that I am the sole responsible for such results earnestly applies.

CHAPTER 1

Hirschowitz's Conjecture

1. Classical Antecedents

It is a long standing problem in Algebraic Geometry to determine the dimension of linear systems of curves of degree $d \geq 1$ in $\mathbb{P}^2_k$ (k being an algebraically closed field) passing through r (closed) points $p_1, \ldots, p_r$ in general position with assigned multiplicities, resp. $m_1, \ldots, m_r \geq 1$.

Such a problem can be easily rephrased in a rather algebraic fashion. let us denote the maximal ideal of forms vanishing at the point p_i by I_i and by $\mathcal{I}_i$ the corresponding ideal sheaf and we let $R := k[T_0, T_1, T_2]$, the homogeneous coordinate ring of $\mathbb{P}^2_k$. Then set $I := \Pi_{i=1}^r I_i^{m_i}$, $A := R/I$ and let $I(d)$ and $A(d)$ denote the respective d-th graded component. One easily realizes that we are looking for the integers:

$$dim_k(I(d)) = h^0(\mathbb{P}^2_k, \mathcal{I}_Z(d)),$$

where $\mathcal{I}_Z$ is the structure ideal sheaf of Z, the (in general non-reduced– as soon as one of the m_i is greater than 1) punctual scheme $Proj(\mathcal{O}_{\mathbb{P}^2_k} / \bigcap_i \mathcal{I}_i^{m_i})$ which gives rise to the 0-cycle $\sum_i m_i p_i$ in $\mathbb{P}^2_k$.

On the algebraic side, the study of the Hilbert function,

$$H_Z(d) = dim_k(I_d) = h^0(\mathbb{P}^2_k, \mathcal{I}_Z(d)),$$

of such schemes, vividly called *fat points*, has been extensive (due to our ignorance, we only mention in the bibliography the articles we felt closer in spirit to our approach, such as [**GG91, aG89, aG93, aG89, sG89**]). Obviously the knowledge of such function would amount to the solution of the aforementioned problem. Unfortunately the state of the art knowledge gives only estimates on what the range of the Hilbert function can be, focussing, e.g., on the first (as well as successive) difference function

$$\Delta H_Z(d) = H_Z(d) - H_Z(d-1),$$

giving bounds for the minimal d for which H_Z can have the expected value (i.e. the *postulation*, cf. below) or trying to compute free resolution of the defining ideal $\mathcal{I}_Z$. (It ought to be mentioned that all such questions have their natural generalizations in higher dimension, as in, e.g., [**GM84, sG89, HS95**].) Indeed note that the structure sequence of Z twisted by $\mathcal{O}_{\mathbb{P}^2_k}(d)$ gives in cohomology:

$$0 \to H^0(\mathbb{P}^2_k, \mathcal{I}_Z(d)) \to H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(d)) \to H^0(Z, \mathcal{O}_Z(d)) \\ \to H^1(\mathbb{P}^2_k, \mathcal{I}_Z(d)). \tag{1.1}$$

One is inclined to expect, for generic points,

$$H_Z(d) = h^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(d)) - h^0(Z, \mathcal{O}_Z(d)),$$

i.e. the (affine) dimension of the "space of planar projective curves of degree d" minus the "number of conditions imposed by passing through the point p_i with multiplicity m_i, classically termed *Postulation*, that is the virtual dimension of the system. Hence the interest in the vanishing of $H^1(\mathbb{P}^2_k, \mathcal{I}_Z(d))$ and the resolutions of the structure ideal $\mathcal{I}_Z$.

Here the geometer would think in terms of spannedness at the points of Z, but as these points aren't reduced we are unable to obtain simple conditions on the very explicit sheaf $\mathcal{O}_{\mathbb{P}^2_k}(d)$ the way, for example, Reider's theory does. We can still formulate (on the blowing-up of $\mathbb{P}^2_k$ at the points p_i) some destabilizing argument, but because of the multiplicities we can't go far. Even the most sophisticated vanishing theorems for ideal sheaves [**aN90**] can't circumvent this obstacle.

On the other hand, we can easily eliminate the difficulty of having to deal with incomplete linear systems such as

$$|dH - \textstyle\sum_i m_i p_i|$$

the way classical algebraic geometers used to [**oZ70**] blowing up the support of Z.

Let $\sigma : X_r = X(p_1, \dots, p_r) \to \mathbb{P}^2_k$ be the blowing up of $\mathbb{P}^2_k$ at $p_1, \dots, p_r$, H a divisor on X_r such that $\sigma^*\mathcal{O}_{\mathbb{P}^2_k}(1) = \mathcal{O}_{X_r}(H)$, and $E_1, \dots, E_r$ the exceptional curves corresponding to the p_i's, then the above mentioned systems are isomorphic by pull back to the complete linear system [1]

$$|dH - \sum_{i=1}^{r} m_i E_i|. \tag{1.2}$$

We should emphasize that the assumption on the p_i's of being in general position means that the p_i's are in *d-general position* for any positive integer d: viz. any 3 of them don't lie on a line ($d = 1$, or linear general position), any 6 of them don't lie on a conic, and so on (any $(d+1)(d+2)/2$ do not lie on a curve of degree d). This can be formalized in the following definition [**jB79**] .

DEFINITION 1.1. A set of r (geometric) points $p_1, \dots, p_r \in \mathbb{P}^2_k$ are said to be *in general position* if:

(i) at most $\frac{(d+1)(d+2)}{2} - 1$ lie on a curve of degree d

(ii) for any p_i there is a curve C of degree

$$d := min\{d' \in \mathbb{Z}_+ : \frac{(d'+1)(d'+2)}{2} \geq r\}$$

such that $p_j \in C$ for any $j \neq i$ and $p_i \notin C$.

[1]More generally, for any locally free sheaf $\mathcal{F}$ on $\mathbb{P}^2_k$ $H^i(\mathbb{P}^2_k, \mathcal{F}) \cong H^i(X_r, \sigma^*\mathcal{F})$. Indeed the Leray spectral sequence associated to $\sigma^*\mathcal{F}$ degenerates at

$$E_2^{pq} = H^p(X, R^q\sigma_*(\sigma^*\mathcal{F})) \Rightarrow H^{p+q}(X_r, \sigma^*\mathcal{F}),$$

whence the isomorphism once one takes into account that (cf. [rH78, prop. V.3.4]) $R^q\sigma_*\mathcal{O}_{X_r} = 0$ for $q > 1$ so that

$$R^q\sigma_*\sigma^*\mathcal{F} = \mathcal{F} \otimes R^q\sigma_*\mathcal{O}_{X_r} = 0$$

(see the section on the Leray spectral sequence below for a similar argument).

It is not hard to prove (cf. [**jB79**]) that this geometric characterization is equivalent to more algebraic conditions such as those of the following proposition.

PROPOSITION 1.2. *A set of r points $Z = \{p_1, \ldots, p_r\} \subset \mathbb{P}^2_k$ is in general position if and only if for any subset $Y \subset Z$ of cardinality s one has*

$$h^0(\mathcal{I}_Y \otimes \mathcal{O}_{\mathbb{P}^2_k}(d)) = max\{(d+1)(d+2)/2 - s, 0\}.$$

From this generality assumption and the adjunction formula[2] we conclude all of the E_i's are irreducible smooth rational curves, or, equivalently, no $E_i - E_j$ for $(i \neq j)$ is effective[3]. Furthermore X_r has to be smooth. We will see that the generality of the p_i's has actually much stronger implications on the geometry of X_r.

Of course, one of the fundamental tools in computing the dimension of linear systems is the Riemann-Roch theorem—let's recall it now:

THEOREM A (Riemann-Roch Theorem for Surfaces). *If $\mathcal{L}$ is a line sheaf on a smooth algebraic surface Y,*

$$\chi(\mathcal{L}) = (c_1(\mathcal{L})^2 - c_1(\mathcal{L}).K_Y)/2 + \chi(\mathcal{O}_Y).$$

When Y is a rational surface, since $\chi(\mathcal{O}_Y)$ is a birational invariant,

$$\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_{\mathbb{P}^2_k}) = 1,$$

so that Theorem A can be rewritten as

$$\chi(\mathcal{L}) = (c_1(\mathcal{L})^2 - c_1(\mathcal{L}).K_Y)/2 + 1. \tag{1.3}$$

[2]Because the points p_i are different the exceptional divisors have to be irreducible. Then the adjunction formula gives the arithmetic genus

$$p_a(E_i) = (E_i^2 + K_{X_r}.E_i)/2 + 1 = 0,$$

whence it follows the E_i are smooth rational curves.

[3]This can happen if and only if some of the p_i are infinitely near (cf. [**mD80**]).

Since $\mathcal{K}_{X_r} = \mathcal{O}_{X_r}(-3H + \sum_{i=1}^r E_i)$, we get explicitly from (1.3):

$$\begin{aligned} &\chi(\mathcal{O}_{X_r}(dH - \textstyle\sum_{i=1}^r m_i E_i)) = \\ &= (d^2 - \textstyle\sum_{i=1}^r m_i^2 + 3d - \sum_{i=1}^r m_i)/2 + 1 = \\ &= (d+1)(d+2)/2 - \textstyle\sum_{r=1}^r m_i(m_i+1)/2. \end{aligned} \tag{1.4}$$

NOTATION . Henceforth we will denote the line sheaf $\mathcal{O}_{X_r}(dH - \sum_{i=1}^r m_i E_i)$ by $\mathcal{L}_{d,\mathbf{m}}$, where by $\mathbf{m}$ we understand the vector of multiplicities $(m_1, \dots, m_r)$.

REMARK 1.3. Notice the first summand in (1.4) is equal to $dim_k(R(d))$ $=$ *dim(space of curves of degree d in $\mathbb{P}^2_k$)*; moreover, the second term amounts to $deg(Z)$ or, in more classical terms, the number of conditions imposed by passing through the points p_i with multiplicities at least m_i.

It is classically known that

$$dim|dH - \Sigma_{i=1}^r m_i E_i| \geq (d+1)(d+2)/2 - \Sigma_{i=1}^r m_i(m_i+1)/2,$$

a bound that can also be gotten from (1.4) if $h^2(X_r, \mathcal{L}_{d,\mathbf{m}}) = 0$. It turns out it is not difficult to prove this vanishing whenever $d \geq 1$, as is natural to assume [4](the requirement being only $d \geq -2$):

LEMMA 1.4 (cf. [**mD80**]). *On a smooth rational surface Y for a Cartier divisor D, if $h^0(\mathcal{O}_Y(D)) \neq 0$, then $h^2(\mathcal{O}_Y(D))$ vanishes. Furthermore, if we take Y to be a generic rational surface $X_r = X(p_1, ..., p_r)$ as defined above we have:*

(i) *if $D.H < 0$, then $h^0(X_r, \mathcal{O}_{X_r}(D)) = 0$;*
(ii) *if $D.H \geq -2$, then $h^2(\mathcal{O}_{X_r}, \mathcal{O}_{X_r}(D)) = 0$;*
(iii) *under the hypotheses of* (iii), *if moreover $D^2 - D.K_{X_r} \geq 0$, then $h^0(\mathcal{O}_{X_r}(D))$ does not vanish.*

PROOF. For the first claim notice that by Serre's duality

$$H^2(\mathcal{O}_Y(D)) \cong H^0(\mathcal{O}_Y(K_Y - D)).$$

[4]We are considering linear systems of curves of some assigned degree d in $\mathbb{P}^2_k$ which correspond to divisors of the same H-degree on the blowing-up X_r. A non positive degree would not make sense.

Since, by assumption, D is effective, the cohomology group on the left injects into $H^0(\mathcal{O}_Y(K_Y))$ which is known to vanish for rational surfaces (birational invariance of the genus, and, more generally, of the plurigenera).

(i) If $h^0(X_r, \mathcal{O}_{X_r}(D)) \neq 0$, then D is effective, so that H, which is numerically effective, can't intersect it negatively.

(ii) Applying Serre's duality as at the beginning, the non vanishing of $h^2(X_r, \mathcal{O}_{X_r}(D))$ would imply that $K_X - D$ is effective. Then H would intersect it nonnegatively, while, by assumption,

$$K_{X_r}.H - D.H = -3 - D.H < 0.$$

(iii) From the previous item we know $H^2(X_r, \mathcal{O}_{X_r}(D))$ vanishes, then, by Riemann-Roch:

$$h^0(X_r, \mathcal{O}_{X_r}(D)) = (D^2 - D.K_{X_r})/2 + h^1(X_r, \mathcal{O}_{X_r}(D)) + 1 \geq 1.$$

□

2. The Conjecture

2.1. Some Heuristics. First of all assume

$$d \geq m_1 \geq m_2 \ldots \geq m_r \geq 1. \tag{2.1}$$

Notice that the ordering on the m_i's is not restrictive, nor the condition $d \geq 1$ is; on the other hand $d \geq m_i$ is natural to ask if we hope for a (*generically*) nonzero $h^0(\mathcal{L}_{d,\mathbf{m}})$.

REMARK 2.1. By Lemma 1.4,

$$c_1(\mathcal{L}_{d,\mathbf{m}}).H = d \geq -2 \Rightarrow h^2(X_r, \mathcal{L}_{d,\mathbf{m}}) = 0.$$

Hence we have the following cases:

1. $\chi(\mathcal{L}_{d,\mathbf{m}}) > 0 \Rightarrow h^0(X_r, \mathcal{L}_{d,\mathbf{m}}) \neq 0$,
2. $\chi(\mathcal{L}_{d,\mathbf{m}}) = 0 \Rightarrow h^0(X_r, \mathcal{L}_{d,\mathbf{m}}) = h^1(\mathcal{L}_{d,\mathbf{m}})$,
3. $\chi(\mathcal{L}_{d,\mathbf{m}}) < 0 \Rightarrow h^1(X_r, \mathcal{L}_{d,\mathbf{m}}) \neq 0$.

Since a great deal of the geometry of X_r depends on the E_i's, it is natural to look for bounds of $c_1(\mathcal{L}_{d,\mathbf{m}}).E_i$, or more generally for $c_1(\mathcal{L}_{d,\mathbf{m}}).E$ where E is an exceptional curve. Let's first consider the following basic case.

PROPOSITION 2.2. *Let L be the proper transform of a line through p_j and p_k with $j \neq k$ and define $\chi^- := -(\chi - |\chi|)/2$. Then*

$$h^1(\mathcal{L}_{d,\mathbf{m}}) = \chi^- \Rightarrow c_1(\mathcal{L}_{d,\mathbf{m}}).L \geq -1 - \chi^-$$

.

PROOF. Let's consider the structure sequence associated with L:

$$0 \to \mathcal{I}_L \cong \mathcal{O}_{X_r}(-H + E_j + E_k) \to \mathcal{O}_{X_r} \to \mathcal{O}_L \to 0.$$

Tensoring with $\mathcal{L}_{d,\mathbf{m}}$ we get

$$0 \to \mathcal{O}_{X_r}((d-1)H - \sum_{i=1}^{r} m_i E_i + E_j + E_k) \to \mathcal{L}_{d,\mathbf{m}} \to \mathcal{O}_L(d - m_j - m_k) \to 0.$$

Now the long exact cohomology sequence yields

$$H^1(\mathcal{L}_{d,\mathbf{m}}) \to H^1(\mathcal{O}_L(d - m_j - m_k)) \to$$
$$\to H^2(\mathcal{O}_{X_r}((d-1)H - \sum_{i=1}^{r} m_i E_i + E_j + E_k)) = 0,$$

where the vanishing of the last term follows from Lemma 1.4. Since $L \cong \mathbb{P}^1_k$, the Riemann-Roch Theoremon curves and the numerical assumption give

$$\chi^- \geq h^1(\mathcal{O}_L(d - m_j - m_k)) = h^0(\mathcal{O}_L(-2 - d + m_j + m_k)) \geq$$
$$\geq -1 - d + m_j + m_k = -1 - c_1(\mathcal{L}_{d,\mathbf{m}}).L. \tag{2.2}$$

□

It turns out it is possible to generalize such inequalities to any intersection with an exeptional curve.

2.2. First digression on exceptional curves. By *exceptional curve* (of the first kind) we mean an irreducible curve which can be smoothly contracted to a point. Castelnuovo's criterion gives a numerical characterization for such curves (which thereby are also called *(-1)-curves E*) on a smooth surface Y:

$$E^2 = -1 = K_Y.E. \tag{2.3}$$

The adjunction formula then yields

$$2h^1(\mathcal{O}_E) - 2 = E.(E + K_Y) = -2,$$

so that –taking into account the irreducibility– $E \cong \mathbb{P}^1_k$.

If Y is the blowing-up of a surface Y' at r points, $Pic(Y) = Pic(Y') \oplus \mathbb{Z}^r$. In particular, for our X_r we choose the obvious basis for $PicX_r$ $\{\epsilon_0, \epsilon_1, \dots, \epsilon_r\}$, where $\epsilon_0 := [H]$ and $\epsilon_i := [E_i]$; and we fix a coordinate system by giving the isomorphism

$$\begin{array}{ccc} PicX_r & \to & \mathbb{Z}^{r+1} \\ [n_0H + \sum_{i=1}^r n_iE_i] & \mapsto & (n_0, -n_1, \dots, -n_r). \end{array} \tag{2.4}$$

(2.3) then can be rewritten as

$$e_0^2 - \sum_{i=1}^r e_i^2 = -1 = -3e_0 + \sum_{i=1}^r e_i, \tag{2.5}$$

for an exeptional curve of class $[E] \mapsto (e_0, \dots, e_r)$.

We can now generalize Proposition 2.2 to any exceptional curve.

2.3. Necessary Conditions.

PROPOSITION 2.3. *Let E be an exceptional curve of degree $e_0 := E.H \leq d+2$ on X_r, then (using the notation of the previous subsection)*

$$h^1(\mathcal{L}_{d,\mathbf{m}}) = \chi^- \Rightarrow c_1(\mathcal{L}_{d,\mathbf{m}}).E \geq -1 - \chi^-.$$

PROOF. Here too we tensor the structure sequence of $E \sim e_0H + \sum_{i=1}^r e_iE_i$ by $\mathcal{L}_{d,\mathbf{m}}$ taking into account that $\mathcal{I}_E \cong \mathcal{O}_{X_r}(-e_0H+\sum_{i=1}^r e_iE_i) = \mathcal{L}_{e_0,\mathbf{e}}^{-1}$:

$$0 \to \mathcal{L}_{d-e_0,\mathbf{m}-\mathbf{e}} \to \mathcal{L}_{d,\mathbf{m}} \to \mathcal{O}_E(de_0 - \sum_{i=1}^r m_ie_i) \to 0.$$

Now the long exact cohomology sequence gives:

$$H^1(X_r, \mathcal{L}_{d,\mathbf{m}}) \to H^1(E, O_E(de_0 - \sum_{i=1}^r m_ie_i)) \to H^2(X_r, \mathcal{L}_{d-e_0,\mathbf{m}-\mathbf{e}}).$$

The last cohomology group vanishes because of the assumption on the degree of E and Lemma 1.4. Hence

$$\begin{aligned} \chi^- = h^1(\mathcal{L}_{d,\mathbf{m}}) &\geq h^1(\mathcal{O}_E(de_0 - \sum_{i=1}^r m_ie_i)) = \\ &= h^0(\mathcal{O}_E(-2 - de_0 + \sum_{i=1}^r m_ie_i)) \geq \\ &\geq -1 - de_0 + \sum_{i=1}^r m_ie_i = -1 - c_1(\mathcal{L}_{d,\mathbf{m}}).E. \end{aligned} \tag{2.6}$$

□

REMARK 2.4. Notice if $\chi^- = 0$ (i.e. $\chi(\mathcal{L}_{d,\mathbf{m}}) \geq 0$), Proposition 2.3 yields necessary conditions for the vanishing of $h^1(\mathcal{L}_{d,\mathbf{m}})$.

On the other hand, if $\chi^- > 0$ and $h^0(\mathcal{L}_{d,\mathbf{m}}) = 0$, obviously $h^1(\mathcal{L}_{d,\mathbf{m}}) = \chi^-$ and we obtain necessary conditions for the vanishing of $h^0(\mathcal{L}_{d,\mathbf{m}})$.

We can summarize this saying that the above are necessary conditions for $\mathcal{L}_{d,\mathbf{m}}$ to be non special after the following definition.

DEFINITION 2.5. A line sheaf $\mathcal{L}$ on an algebraic variety V is said to be *special* if at least two of the cohomology groups $H^i(V, \mathcal{L})$ don't vanish. Otherwise we will say that $\mathcal{L}$ is *non special* or that has the *natural cohomology*.

2.4. The Conjecture. Let us consider what happens if the intersection of $c_1(\mathcal{L}_{d,\mathbf{m}})$ with some exceptional curve E of large degree is not bounded below.

PROPOSITION 2.6. *If for an exceptional curve E of class $(e_0, e_1, \dots, e_r)$ with H-degree $e_0 > d/2$ we have $c_1(\mathcal{L}_{d,\mathbf{m}}).E < -1$ then $h^0(\mathcal{L}_{d,\mathbf{m}}) = 0$.*

PROOF. Consider the structure sequence of the scheme $E^{(2)} := Spec(\mathcal{O}_{X_r}/\mathcal{O}_{X_r}(-2E))$ tensored by $\mathcal{L}_{d,\mathbf{m}}$:

$$0 \to \mathcal{L}_{d-2e_0,\mathbf{m}-2\mathbf{e}} \to \mathcal{L}_{d,\mathbf{m}} \to \mathcal{O}_{E^{(2)}} \otimes \mathcal{L}_{d,\mathbf{m}} \to 0.$$

The beginning of the corresponding long cohomology exact sequence is

$$0 \to H^0(X_r, \mathcal{L}_{d-2e_0,\mathbf{m}-2\mathbf{e}}) \to H^0(X_r, \mathcal{L}_{d,\mathbf{m}}) \to H^0(E^{(2)}, \mathcal{L}_{d,\mathbf{m}}|_{E^{(2)}}).$$

Since, by assumption, $d < 2e_0$ the first cohomology group in this sequence vanishes. To derive the vanishing of the last cohomology group we will make use of the exact sequence

$$0 \to \mathcal{I}_E/\mathcal{I}_E^2 \to \mathcal{O}_{E^{(2)}} \to \mathcal{O}_E \to 0. \tag{2.7}$$

Notice that $\mathcal{I}_E/\mathcal{I}_E^2 \cong \mathcal{N}_{E/X_r}^{-1} \cong \mathcal{O}_E(-E)$ (which follows from tensoring the structure sequence of E by $\mathcal{O}_{X_r}(-E)$). Now tensor (2.7) by $\mathcal{L}_{d,\mathbf{m}}$ and pass to cohomology:

$$0 \to H^0(E, \mathcal{O}_E((d-e_0)e_0 - \sum_{i=0}^r (m_i - e_i)e_i)) \to H^0(E^{(2)}, \mathcal{L}_{d,\mathbf{m}}|_{E^{(2)}})$$
$$\to H^0(E, \mathcal{O}_E(de_0 - \sum_{i=0}^r m_ie_i)).$$

Both side terms vanish as they are groups of global sections of line sheaves on $E \cong \mathbb{P}_k^1$ with negative degree:

$$de_0 - \sum_{i=1}^r m_ie_i < de_0 - \sum_{i=1}^r m_ie_i - e_0^2 + \sum_{i=1}^r e_i^2 \le -1,$$

and $H^0(E^{(2)}, \mathcal{L}_{d,\mathbf{m}}|_{E^{(2)}})$ vanishes. □

None the less examples of special line sheaves are ready at hand: for such is the line sheaf $\mathcal{L} = \mathcal{O}_{X_r}(2E)$, where E is an exceptional curve.

Indeed $h^2(\mathcal{L}) = 0$ by Lemma 1.4, and Riemann-Roch yields

$$h^0(\mathcal{L}) = ((2E)^2 - K_{X_r}.(2E))/2 + 1 + h^1(\mathcal{L}) = h^1(\mathcal{L}),$$

which, together with the non vanishing of $h^0(\mathcal{L})$, gives the speciality of $\mathcal{L}$.

Hirschowitz conjectures that the necessary conditions (cf. Remark 2.4) for the non speciality of line sheaves on X_r are sufficient as well.

In view of Proposition 2.6 we don't need to intersect the line sheaf $\mathcal{L}_{d,\mathbf{m}}$ with exceptional curves of degree $e_0 > d/2$. On the other hand, the above example, illustrates the necessity of intersecting with curves of degree up to $d/2$. This motivates the following:

CONJECTURE 2.7 (Hirschowitz [**aH89**]). *A line sheaf $\mathcal{L}_{d,\mathbf{m}}$ with $d \geq m_1 \geq m_2 \dots m_r \geq 1$ is non special, if for any exceptional curve E on X_r of class $(e_0, e_1, \dots, e_r)$ with $e_1 \geq e_2 \geq \dots \geq e_r$ one has*

$$c_1(\mathcal{L}_{d,\mathbf{m}}).E \geq -1.$$

REMARK 2.8. Conjecture 2.7 gives the dimension of the space of global sections of line sheaves on X_r. Indeed, if a line sheaf $\mathcal{L}_{d,\mathbf{m}}$ satisfies the hypotheses of the Conjecture, then

$$h^0(\mathcal{L}_{d,\mathbf{m}}) = \chi^+ := \chi(\mathcal{L}_{d,\mathbf{m}})^+;$$

otherwise, we can use the following Lemma.

LEMMA 2.9. *Let $\mathcal{L}_{d,\mathbf{m}}$ be a line sheaf on X_r for which there is an exceptional curve E of class $(e_0, \dots, e_r)$ with $c_1(\mathcal{L}_{d,\mathbf{m}}).E =: -s < -1$ then*

$$H^0(X_r, \mathcal{L}_{d,\mathbf{m}} \otimes \mathcal{O}_{X_r}(-sE)) \cong H^0(X_r, \mathcal{L}_{d,\mathbf{m}})$$

PROOF. Tensoring the structure sequence of the nonreduced scheme $E^{(s)} := Spec(\mathcal{O}_{X_r}/\mathcal{O}_{X_r}(-sE))$ by $\mathcal{L}_{d,\mathbf{m}}$ and passing to cohomology we obtain

$$\begin{aligned}0 \to H^0(X_r, \mathcal{O}_{X_r}((d - se_0)H - \textstyle\sum_{i=0}^{r}(m_i - se_i)E_i)) \to H^0(X_r, \mathcal{L}_{d,\mathbf{m}}) \\ \to H^0(E^{(s)}, \mathcal{L}_{d,\mathbf{m}}|_{E^{(s)}}),\end{aligned} \tag{2.8}$$

so that we have to prove the cohomology group on the right vanishes. This can be achieved by considering the exact sequence

$$0 \to \mathcal{I}_E^{s-1}/\mathcal{I}_E^s \to \mathcal{O}_{E^{(s)}} \to \mathcal{O}_{E^{(s-1)}} \to 0.$$

Tensoring by $\mathcal{L}_{d,\mathbf{m}}$ and taking into account that $\mathcal{I}_E^{s-1}/\mathcal{I}_E^s \cong \mathcal{O}_E(-(s-1)E) \cong \mathcal{O}_{\mathbb{P}^1_k}(s-1)$ and the very definition of s we obtain the cohomology exact sequence:

$$\begin{aligned}0 \to H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-1)) \to H^0(E^{(s)}, \mathcal{L}_{d,\mathbf{m}}|_{E^{(s)}}) \\ \to H^0(E^{(s-1)}, \mathcal{L}_{d,\mathbf{m}}|_{E^{(s-1)}}) \to H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-1)).\end{aligned}$$

The vanishing of the first and last group yields

$$H^0(E^{(s)}, \mathcal{L}_{d,\mathbf{m}}|_{E^{(s)}}) \cong H^0(E^{(s-1)}, \mathcal{L}_{d,\mathbf{m}}|_{E^{(s-1)}}).$$

which, by descending induction, gives in turn the sought for vanishing of the group on the right in (2.8) and concludes the proof. □

CHAPTER 2

Curves on Generic and Harbourne's Surfaces

1. Symmetries of Generic Rational Surfaces

Any treatment of generic rational surfaces would be incomplete without mention of the beautiful symmetry inherent in the different ways they can be gotten as blowups of $\mathbb{P}^2_k$ [**bH85a**]. Indeed the set of "basic" exceptional curves E_i, an *Exceptional Configuration* in Harbourne's terminology[1], can be carried into another such configuration resulting from a different blowing-up. It is better practice not to regard any exceptional configurations as priviledged and correspondingly to keep in mind that the choice of generators of the Picard group is non canonical.

The group operating on the exceptional configurations is the Weyl group associated to a generalized root system that can be defined in terms of an exceptional configuration. Its geometric significance is expressed by the representation to the group of Cremona transformations (birational transformations) of the geometric quotient (cf. footnote below) parametrizing ordered blowups. Such representation is faithful for generic rational surfaces X_r with $r \geq 9$ [**DO88**, p.102]. Moreover, if one extends the field of definition to the field of rational functions over the $(2r-8)$-dimensional quasi-projective variety parametrizing generic rational surfaces[2] $K = k(T_1, \dots T_{2r-8})$, Hirschowitz [**aH88**] constructs

[1]As Brian Harbourne kindly pointed out this terminology was actually coined by Looijenga in [**eL81**]. Harbourne's definition, which is what we adopt, slightly differs from the original one.

[2]To be more precise, Hirschowitz only requires that the r points to be blown up are in linear general position. This gives an open condition in $(\mathbb{P}^2_k)^r$. Then he quotients by the diagonal action of $PGL(3)$. K will be the field of rational function on this quotient, which is a quasi-projective variety, for, as $PGL(3)$ acts

a representation of the Weyl group into the automorphism group itself. As for generic rational surfaces the automorphism group over the ground field is trivial [**mK88**], the automorphisms coming from the Weyl group act as Galois morphisms in $Gal(K/k)$.

Ample treatment of these beautiful results is given in the original work of Dolgachev [**iD83, DO88**], Harbourne [**bH82**] and Hirshcowitz [**aH88**]. We would like at least to recall the basic notions of root systems and Weyl groups of a generic rational surface in order to motivate a reformulation of Hirschowitz's Conjecture which will be utilized in the sequel. (For a recent and far reaching perspective of the rôle of Weyl groups in geometry cf. [**kM95**].)

1.1. Reminder on root systems and Weyl groups. We refer to [**vK90**] as the standard reference for root systems and to [**DO88**] for geometrical variants.

By *generalized Cartan matrix* we mean a matrix $A = (a_{ij})$ with $a_{ij} \in \mathbb{Z}$, $a_{ii} = 2$, $a_{ij} \leq 0$ if $i \neq j$ and $a_{ij} = 0$ if $a_{ji} = 0$.

A *realization* of A is a triple $(\mathfrak{h}, \Pi, \check{\Pi})$ where $\mathfrak{h}$ is a $\mathbb{C}$-vector space, $\Pi := \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}$ and $\check{\Pi} := \{\check{\alpha}_1, \dots, \check{\alpha}_n\} \subset \check{\mathfrak{h}}$[3](the dual space) are subsets of linearly independent vectors such that $a_{ij} = \check{\alpha}_i(\alpha_j)$, and verifying the dimensional relation $n - rk(A) = dim(\mathfrak{h}) - n$.

transitively on the 4-tuple of points in $\mathbb{P}^2_k$, the fixing of the first four points will give a section of the quotient. This can therefore be considered as an open in $(\mathbb{P}^2_k)^{r-4}$. We could repeat the same costruction starting for the open of r-tuples of points in general position and get an open subset of the quotient considered above, hence the same function field. It is intuitive that a coarse moduli space for the surfaces X_r is given by this quotient. See [**DO88**] for a construction using Mumford's geometric invariant theory. One could argue also using Kodaira-Spencer deformation theory [**kK86**, p.220-226] that the number of moduli is $2r - 8 = h^1(X_r, \mathcal{T}_{X_r})$, and these are all effective moduli.

We should say that below we will make use of the universal family of ordered blowups and in particular of the open in such family that corresponds to generic rational surfaces.

[3]We depart from the original definition in [**vK90**] in that we define the roots are element of the space $\mathfrak{h}$ rather than its dual, and similarly for the coroots.

Π is called the *root basis*, $\check{\Pi}$ the *coroot basis* and the free abelian group $Q := \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i$ endowed with the intersection pairing coming from duality the *root lattice*.

DEFINITION 1.1. A *standard hyperbolic lattice* is a lattice $H_r := \bigoplus_{i=0}^{r} \mathbb{Z}\epsilon_i$ endowed with the unimodular intersection pairing defined by $\epsilon_0^2 = 1$, $\epsilon_i^2 = -1$ and $\epsilon_i.\epsilon_j = 0$.

A *geometric marking* of a surface X_r is an isomorphism $\phi : H_r \to Pic(X_r)$ defined by a choice of generators of the Picard group, that is $\phi(\epsilon_0) = [H]$ and $\phi(\epsilon_i) = [E_i]$ for $i > 0$, which is also a lattice morphism, that is, it respects the intersection forms. [4]

PROPOSITION 1.2. *Any blowing-up X_r with a geometric marking gives rise to a realization of the Cartan matrix $A = 2I$.*

PROOF. Define $\alpha_i := \epsilon_i - \epsilon_{i+1}$ if $i < r$ and $\alpha_r := \epsilon_0 - \epsilon_1 - \epsilon_2 - \epsilon_3$; let Q be the associated root lattice, $\mathfrak{h} := Q \otimes \mathbb{C}$ and take as pairing the negative of the intersection pairing in the hyperbolic lattice . Because of dimensional reasons we can define coroots as identical to the corresponding roots [5] . The rest follows by mere computation. □

REMARK 1.3. The Lie algebra associated to such realizations is generally an infinite dimensional Kač-Moody algebra.

One can also prove that the Root lattice $Q \subset Pic(X_r)$ is the orthogonal of the canonical class $-3\epsilon_0 + \epsilon_1 + \cdots + \epsilon_r$.

[4]Whenever there is no danger of confusion we will identify the Picard group and the standard hyperbolic lattice for some fixed geometric marking.

[5]It is remarkable that this theory can be generalized to punctual blowups of higher dimensional projective spaces even at infinitely near points (*generalized Del Pezzo varieties*; cf [**DO88**]). But in this more general context the root and coroot basis will be subsets of the the complexified Chow groups of one-dimensional and one-codimensional Chow groups respectively, and the coefficients of the last root and coroot will differ to reflect the higher dimension. It is also worth remarking the formal analogy with the intersection lattice of K3-surfaces which is hyperbolic as well. Harbourne [**bH87** and **bH91**] makes this analogy more precise for a certain class of special surfaces. Furthermore, the Cremona representation itself could be regarded as a weak Torelli Theorem for generic rational surfaces.

The Weyl group W associated to a root lattice Q is defined to be the subgroup of the group of isometries $O(Q)$ (intersection preserving isomorphisms) generated by the reflections with respect to the hyperplanes defined by the zeroes of the simple coroots, that is

$$s_i : \mathfrak{h} \longrightarrow \mathfrak{h}$$
$$x \mapsto x + \check{\alpha}_i(x)\alpha_i.$$

The Weyl group W_r associated to a geometric marking gives the appropriate symmetry group for the space parametrizing ordered blowups via an action on the set of geometric markings. Certainly we want the symmetric group Σ_r on r elements to give rise to such symmetries—and $\Sigma_r \subset W_r$. On the other hand the whole group of isometries $O(H_r)$ does not necessarily preserves geometric markings. Because it is canonically defined in terms of the root basis, the Weyl group will.

On a generic rational surface one can prove that W_r acts transitively on the set of exceptional curves, that there are no (-2)-curves and that there is a faithful representation $W_r \longrightarrow Bir((\mathbb{P}^2_k)^r/PGL(3))$.

1.2. An equivalent form of Hirschowitz's Conjecture. Let us start with a definition that a priori seems much more restricitive than the hypotheses of Hirschowitz's Conjecture.

DEFINITION 1.4. A line sheaf $\mathcal{L}_{d,\mathrm{m}}$ is said to be *adequate* if

$$m_i = c_1(\mathcal{L}_{d,\mathrm{m}}).E_i =$$
$$= min\{c_1(\mathcal{L}_{d,\mathrm{m}}).E : E \text{ exceptional curve}, E \neq E_{i+1}, \dots , E_r\}.$$

Any line sheaf $\mathcal{L}_{d,\mathrm{m}}$ intersecting exceptional curves in at least -1 can be rendered adequate tensoring with ideal sheaves of suitable exceptional curves (viz. those which intersect its divisor class in low intersection numbers). More elegantly, one could consider the orbit of the class of $\mathcal{L}_{d,\mathrm{m}}$ in the Picard group under the action of the Weyl group W_r. Then the very definition of W_r insures that such an orbit will intersect the set of adequate classes non emptily. Furthermore, considering the cohomology long exact sequence coming form the inclusion of the above mentioned sheaves one realizes that it is enough to prove the Conjecture for adequate sheaves.

Hirschowitz [**aH89**] proves that adequacy follows form a particularly simple condition.

LEMMA 1.5. *A line sheaf $\mathcal{L}_{d,\mathbf{m}}$ on a generic rational surface satisfying*

$$d \geq m_1 + m_2 + m_3$$

is adequate.

In virtue of the above considerations we can now reformulate Hirschowitz's Conjecture as follows.

CONJECTURE 1.6. *Under the assumption that $d \geq m_1, \dots, \geq m_r \geq 1$ the line sheaf $\mathcal{L}_{d,\mathbf{m}}$is non special if*

$$d \geq m_1 + m_2 + m_3.$$

This formulation has actually an important interpretation in terms of root systems. Indeed one can express the hypotheses as

$$c_1(\mathcal{L}_{d,\mathbf{m}}).\phi(\alpha_i) \geq 0$$

for $i = 1, \dots, r$ and some geometric marking $\phi : H_r \rightarrow Pic(X_r)$. In this form the conjecture had already been proposed by Harbourne [**bH86**, p. 102] who calls classes of line sheaves satisfying the above condition $\mathcal{E}$-standard ($\mathcal{E}$ denoting a fixed geometric marking, in the terminology we adopted or an exceptional configuration in Harbourne's.). One of the main points of Hirschowitz's contribution [**aH89**] is to prove that such root system formulation is equivalent with the one involving intersections with exceptional curves which has a distinct Mori cone flavor.

2. Harbourne Surfaces

2.1. Using semicontinuity theorems. Thanks to the work of Brian Harbourne (cf. [**bH82, bH85**] and, for analogous results under much more general assumptions, the recent [**bH95**]), we have extensive knowledge of the cohomology of line sheaves on blowups of the projective plane at points lying on a cubic curve. One easily realizes that this is equivalent to having an irreducible and reduced divisor, a curve of arithmetic genus one, in the anticanonical linear system. This inspires the following definition:

DEFINITION 2.1. By *Harbourne Surface* we mean a blowup X_r of $\mathbb{P}^2_k$ such that there is a reduced irreducible divisor (a curve) $D \in |-K_{X_r}|$.

NOTATION . Below we will always denote this section by D.

We find it useful to consider a subclass of Harbourne surfaces enjoying a genericity property:

DEFINITION 2.2. A *Harbourne Surface* will be called *generic* if the " restriction homomorphism"

$$Pic(X_r) \xrightarrow{j^*} Pic(D)$$

induced by the inclusion $j : D \hookrightarrow X_r$ is injective.

One easily realizes that this means the trace of the exceptional curves E_i on D spans a $\mathbb{Z}$-linearly independent sublattice of $Pic(D)$. This can generically happen because $Pic(D)$ is much larger then $Pic(X_r)$. Indeed, if D is smooth, $Pic(D) \cong Pic^0(D) \oplus \mathbb{Z}$ and $Pic^0(D) \cong D$ as an algebraic variety. In the analytic category this condition of genericity corresponds to an open condition in D^r.

The reason we are considering Harbourne rather than generic rational surface is that we would like to derive vanishing theorems for the latter by semicontinuity.

More specifically, following a method of Kleiman (cf. [**sK81, DO88**] for a general approach and [**bH82**] for the projective plane and the functoriality), we can construct the family of ordered blowups of $\mathbb{P}^2_k$ by induction as follows. Let $\mathcal{X}_0 = \{p\} \subset \mathbb{P}^2_k$ be a point and $\mathcal{X}_1 = \mathbb{P}^2_k$. Supposing $\mathcal{X}_{r-1}$ constructed, we define $\mathcal{X}_r$ as the blowup of $\mathcal{X}_{r-1} \times_{\mathcal{X}_{r-2}} \mathcal{X}_{r-1}$ along the diagonal. (The blown up diagonals account for the different ways r-tuple of points can "converge".) Then one can prove [**bH82**] that the family $\mathcal{X}_{r+1} \to \mathcal{X}_r$ satisfies the functorial properties of the universal family of ordered blowups of the projective plane at r points. Moreover, there is a morphism $\phi : \mathcal{X}_r \to (\mathbb{P}^2_k)^r$ which is a composition of blowups and is an isomorphism outside the preimage of the diagonal.

In this family we will consider the open subfamily comprising generic rational surfaces. If one can prove vanishing theorems for cohomology of line sheaves at the boundary of such family, that is, for blown up points in special position (e.g., Harbourne surfaces), then the theorem of upper semicontinuity of the dimension of cohomology will yield the corrresponding generic vanishing. We are going to see that generic Harbourne surfaces are a "right" kind of special surfaces for this purpose.

2.2. ***nef*** **divisors on Harbourne surfaces.** Using more geometrical techniques we are able to give a simplified proof of a vanishing theorem of Harbourne under the assumption of genericity (on the other hand, our result is slightly stronger in that we don't assume the intersection with the anticanonical divisor is strictly positive).

LEMMA 2.3. *A nef line sheaf $\mathcal{M}$ with $c_1(\mathcal{M})$ on a generic Harbourne surface has vanishing first cohomology.*

PROOF. Harbourne recently proved [**bH95**] that a nef line sheaf on an anticanonical rational surface—so, in particular, on a Harbourne surface—is effective. We are therefore able to apply a well-known vanishing Theorem of C.P. Ramanujam [**cR72**] which grants $h^1(X_r, \mathcal{M}^{-1}) =$

0. Serre duality (as $\mathcal{K}_{X_r} \cong \mathcal{O}_{X_r}(-D)$) then gives the vanishing of $H^1(X_r, \mathcal{M}(-D))$.

But this last cohomology group fits in the long exact sequence coming form the structure sequence of D tensored by $\mathcal{M}$:

$$H^1(X_r, \mathcal{M}(-D)) \to H^1(X_r, \mathcal{M}) \to H^1(D, \mathcal{M}|_D).$$

Because of the assumption of injectivity of the restriction homomorphism of Picard groups the class of $\mathcal{M}|_D$ in $Pic(D)$ is non zero. D has arithmetic genus one, and the Riemann-Roch Theorem for embedded curves [**BPV84**, p. 51] gives the vanishing of the last cohomology group. □

REMARK 2.4. In the recent [**bH95**] Harbourne studies the vanishing of the first cohomology of nef line sheaves $\mathcal{L}$ on anticanonical rational surfaces X. It turns out in this more general case we don't always have vanishing. A necessary condition is that the intersection with the anticanonical divisor is zero. Moreover, if we don't have vanishing, $h^1(\mathcal{L}) + 1$ will give the number of connected components of a generic section of $\mathcal{L}(-K_X)$, a measure of the disconnectedness of $c_1(\mathcal{L})$. This also corroborates the intuition that, in ordered to get natural cohomology for nef line sheaves, one cannot relax the condition of genericity as reflected by the injectivity of the morphism of Picard groups induced by the restriction to the anticanonical divisor $Pic(X_r) \to Pic(D)$ [6] .

2.3. A vanishing theorem. Using Lemma 2.3 we are able to derive the vanishing of the first cohomology of line sheaves on generic

[6] As B. Harbourne remarked [**bH95c**], this injectivity condition, rather than the irreducibility of the anticanonical divisor, is the crucial assumption in studying vanishing of cohomology on anticanonical surfaces. Indeed [**bH95**, Theorem I.1] the non-vanishing of the first cohomology group of a line sheaf $\mathcal{L}$ on an anticanonical surface implies that $c_1(\mathcal{L})$ has components disjoint form those of the anticanonical divisor, hence non-trivial elements in the kernel of $Pic(X_r) \to Pic(D)$. It follows that a nef line sheaf has vanishing first cohomology on a *generic anticanonical surface* (obvious generalization of the above terminology).

rational surfaces which are not too negative in the sense of Theorem 2.6 below.

In the proof of the theorem we will have to resort to the Zariski decomposition for effective line sheaves [**oZ62**] [7]which we recall now.

LEMMA 2.5. *Given any effective line sheaf $\mathcal{L}$ on an algebraic surface, we can find two $\mathbb{Q}$-divisors M and N with M nef and N effective such that $c_1(\mathcal{L}) = M + N$ and :*

1. *$\mathcal{M}$ intersects every component of N trivially.*
2. *If $N \neq 0$ and N_i are the irreducible components of N, the quadratic form $\sum(N_i.N_j)x_i x_j$ (the x_i being abstract variables) is negative definite.*

We are now able to prove the following vanishing theorem.

THEOREM 2.6. *If an effective line sheaf $\mathcal{L}$ with $c_1(\mathcal{L}) > 0$ on a generic rational surface X_r intersects the anticanonical divisor and any exceptional curve non negatively, then $h^1(X_r, \mathcal{L}) = 0$.*

PROOF. Since the canonical divisor and exceptional curves as well as intersection pairings curves are stable under algebraic deformations [**kK63, wF84**], we can specialize to a generic Harbourne surface and there prove the vanishing by semicontinuity.

Suppose that X_r is generic Harbourne and take a Zariski decomposition $c_1(\mathcal{L}) = M + N$. As will be remarked in the next section, on a generic Harbourne surface the only curves with self-intersection negative are exceptional curves and the anticanonical curve D. Hence the components of N must be so. But, by assumption, $c_1(\mathcal{L}).N \geq 0$, while the very definition of Zariski decomposition would imply $c_1(\mathcal{L}).N < 0$. Therefore $N = 0$ and $\mathcal{L}$ is nef, so that Harbourne Lemma 2.3 gives the assertion. □

REMARK 2.7. Thanks to Harbourne's recent contribution [**bH95**] we can do without the assumption of positivity of the self-intersection

[7]In [**tF79**] Fujita generalizes this decomposition to pseudo-effective divisors.

of the line sheaf (specializing to anticanonical surfaces on which he proves the vanishing).

REMARK 2.8. The condition on the intersection with exceptional curves it is actually not restrictive from the point of view of Hirschowitz's Conjecture. Indeed, if the Conjecture is true for any effective line sheaf intersecting exceptional curves non negatively, then it will be true for any effective line sheaf intersecting them in no less than -1.

This claim can be proved as follows. Suppose that $\mathcal{L}$ is effective and satisfies the hypotheses of the Conjecture, then it will intersect at most a finite number of exceptional curves C_k in -1. Tensoring $\mathcal{L}$ by $\mathcal{O}_{X_r}(-\sum C_k)$ we obtain a line sheaf $\mathcal{L}' \subset \mathcal{L}$ that intersects any exceptional curve non negatively. Moreover, the C_k have to be disjoint, for if $C_k.C_l > 0$, the Riemann–Roch Theoremwould give $h^0(X_r, \mathcal{O}_{X_r}(C_k + C_l)) \geq 2$ so that the linear system $|C_k + C_l|$ would move and we would get the contradiction $c_1(\mathcal{L}).(C_k + C_l) \geq 0 \neq -2$. From the structure sequence of $\sum C_k$ tensored by $\mathcal{L}$, we get

$$H^1(X_r, \mathcal{L}') \to H^1(X_r, \mathcal{L}) \to H^1(\textstyle\sum C_k, \mathcal{L}|_{\sum C_k}).$$

$H^1(X_r, \mathcal{L}')$ vanishes because of the assumptions, while the second does because $c_1(\mathcal{L}).C_k \geq -1$.

REMARK 2.9. Thanks to this vanishing result we can restrict our investigation of Hirschowitz's Conjecture when the Euler-Poincaré characteristic is non negative to the case in which our line sheaves $\mathcal{L}_{d,\mathbf{m}}$ intersect the anticanonical sheaf negatively. If $h^1(X_r, \mathcal{L}_{d,\mathbf{m}}) \neq 0$ we have a non trivial extension of $\mathcal{O}_{X_r}$ by $\mathcal{L}_{d,\mathbf{m}}$

$$0 \to \mathcal{L}_{d,\mathbf{m}} \to \mathcal{E} \to \mathcal{O}_{X_r} \to 0$$

and we could conclude that $h^1(X_r, \mathcal{L}_{d,\mathbf{m}}) \leq 1$ provided $h^1(X_r, \mathcal{E}) = 0$. We remark that if $c_1(\mathcal{E})^2 = c_1(\mathcal{L}_{d,\mathbf{m}})^2 > 0$ (the non negativity of $\chi(\mathcal{L}_{d,\mathbf{m}})$ implies that this number is not smaller than $c_1(\mathcal{L}_{d,\mathbf{m}}).K_{X_r} - 2$, and we assumed the intersection with the canonical divisor to be positive), the sheaf $\mathcal{E}$ would be Bogomolov unstable (see, e.g., **[rL94,**

OSS80]). On the other hand, Brun [**jB79**] proves that we would have the vanishing of the first cohomology of $\mathcal{E}$ if $\mathcal{E}$ were stable. This give a further insight of how the "hard" cases of Hirschowitz's Conjecture are below the numerical threshold for which the vanishing or the bounding of first cohomology is known.

We conclude this section mentioning the strongest vanishing Theoremof which we are aware, due to Hirschowitz as well [**aH89**, p.212].

THEOREM 2.10. *If a line sheaf* $\mathcal{L}_{d,\mathbf{m}}$ *on a generic rational surface* X_r *is such that* $d \geq m_1 \geq \cdots \geq m_r \geq 0$ *and*

$$\sum_{i=1}^{r} m_i(m_i+1)/2 < \begin{cases} (d+2)(d+4)/4 & \text{if } d \text{ is even,} \\ (d+3)^2/4 & \text{if } d \text{ is odd,} \end{cases}$$

then $h^1(X_r, \mathcal{L}_{d,\mathbf{m}}) = 0$.

A clever case by case specialization gives the proof by semicontinuity.

v

3. Curves with Negative Self-intersection

From the discussion and motivation of Hirschowitz's Conjecture it was already apparent that, for the Conjecture to be true, our line sheaves can't contain (regarded as divisors) curves with very negative self-intersection.

In dealing with such problems on any smooth algebraic surface Y the main tool is the generalized adjunction formula for embedded curves which expresses [**BPV84**, p. 68] the arithmetic genus of a curve $C \subset Y$ as

$$p_a(C) = deg_C(\mathcal{K}_Y \otimes \mathcal{O}_C(C))/2 + 1.$$

Just assuming that $C^2 < 0$ we have the following three possibilities depending on the sign of the intersection with the canonical divisor K_Y:

1. If $K_Y.C < 0$, then $p_a(C) = 0$ and $C^2 = -1 = K_Y.C$, so that, by Castelnuovo's criterion, C is an exceptional curve.
2. If $K_Y.C = 0$, then $p_a(C) = 0$ as well. It follows that $C^2 = -2$ and C is an exceptional curve of the second kind. We recall that such curves cannot exist on a generic rational surface.
3. If $K_Y.C > 0$, then $p_a(C) > 0$ and nothing can be said in general.

On a Harbourne surface we are more interested in checking the sign of the intersection with the reduced and irreducible anticanonical divisor D. We can rephrase the above considerations saying that among the curves with negative self-intersection only exceptional curves (possibly also of second kind) can intersect D nonnegatively, while the third case can happen only if $C = D$ (in which case $C^2 = K_Y^2 = 9 - r$).

REMARK 3.1. On a generic Harbourne surface the only curve with negative self-intersection are the (-1)-curves and D. That the second case above is not possible follows from the injectivity of the homomorphism of Picard groups.

In dealing with a generic rational surface, the lack of information on the anticanonical divisor doesn't allow such strong limitations. Nevertheless, Miranda, Miyanishi and Murthy [**MMM80**] prove that there are no rational curves with negative self-intersection besides the (-1)-curves, so we can restrict our attention to irrational curves. If we assume the anticanonical dimension of our surface is zero (i.e $dim|-nK_{X_r}| \leq 0$ and there is at least an integer for which equality is attained —see, for example, [**fS83**]), then they also conclude that irrational curves with negative self-intersection can't exist.

A different insight, at least in characteristic zero, comes from the theory of pseudoconvex complex spaces. In [**fB93**] Bogomolov proves that almost complex deformations of Stein neighborhoods (the algebraic geometer can read "affine") of curves on a smooth complex surface with negative self-intersection destroy the complex structure on the curve, unless it is an exceptional curve. Since the notion of complex and almost complex structure for a curve coincide (for dimensional reasons), one can view the absence of such curve as a genericity condition for almost complex structures [**fB95**]. We believe this is the case also for complex (algebraic) deformations hence for generic rational surfaces:

THESIS 3.2. *On generic rational surfaces there are no curves with negative self-intersection except for exceptional curves of the first kind.*

REMARK 3.3. A positve answer to this question would virtually give the structure of the Mori cone $NE(X_r)$.

In the fundamental paper [**aH89**, p.211], Hirschowitz shows how from his Conjecture follows the above thesis. We give a detailed proof below. We also remind that in [**bH94**] Harbourne shows that his conjecture, to the effect that on blowups of the projective plane at sufficiently general points nef line sheaves have natural cohomology and the only integral curves with negative self-intersection are the exceptional curves of the first kind, is equivalent to Hirschowitz's.

PROPOSITION 3.4. *If line sheaves $\mathcal{L}$ on generic rational surfaces which intersect exceptional curves E as*

$$c_1(\mathcal{L}).E \geq -1$$

are non special, then on such surfaces the only curves with negative self-intersection are the exceptional curves of the first kind.

PROOF. Suppose there is a curve C on X_r with $C^2 = -n \leq -2$. Take $\mathcal{L} = \mathcal{O}_{X_r}(C)$. As C itself cannot be exceptional and is reduced and irreducible, for any exceptional curve E we have $C.E \geq 0$. Therefore $\mathcal{L}$ satisfies the hypotheses of Hirschowitz's Conjecture 2.7.

Tensoring the structure sequence of C with $\mathcal{L}$ we get

$$0 \to \mathcal{O}_{X_r} \to \mathcal{L} \to \mathcal{L}|_C \to 0.$$

Passing to cohomology, we obtain

$$\begin{aligned} 0 \to H^0(X_r, \mathcal{O}_{X_r}) \to H^0(X_r, \mathcal{L}) \to H^0(C, \mathcal{L}|_C) \\ \to H^1(X_r, \mathcal{O}_{X_r}) \to H^1(X_r, \mathcal{L}) \to H^1(C, \mathcal{L}|_C) \\ \to H^2(X_r, \mathcal{O}_{X_r}). \end{aligned} \tag{3.1}$$

Now, since $deg\mathcal{L}|_C = -n \leq -2$, the first cohomology group of $\mathcal{L}|_C$ does not vanish[8] , while the zeroth does. Therefore both the zeroth and the first cohomology group of $\mathcal{L}$ don't vanish, which contradicts Hirschowitz's Conjecture.

□

[8]By virtue of the Serre duality Theorem on an embedded curve (see, for example, [**BPV84** p.55])

$$h^1(C, \mathcal{L}|_C) = h^0(C, \mathcal{L}|_C{}^{-1} \otimes \omega_C).$$

Then, the Riemann-Roch Theoremand adjunction formula for embedded curves [**BPV84** resp. p. 51 and p.68] yield

$$h^0(C, \mathcal{L}|_C{}^{-1} \otimes \omega_C) \geq -deg(\mathcal{L}|_C) + deg(\omega_C) + \chi(\mathcal{O}_C) = n + \chi(\omega_C) \geq 1$$

CHAPTER 3

The Rational Fibration Structure and Direct Images

1. The Geometric Idea

Instead of focussing on the blowing-up $X_r \xrightarrow{\sigma} \mathbb{P}^2_k$, we find it very useful to cosider a different strucure on X_r namely X_r can be regarded as a fibration over $\mathbb{P}^1_k$ whose generic fiber is $\mathbb{P}^1_k$. This is gotten as follows: start from the projection $\mathbb{P}^2_k \xrightarrow{\hat{\pi}_j} \mathbb{P}^1_k$ from p_j[1]: $\hat{\pi}_j$ is only a rational map, but if we blow up p_j we obtain a morphism. We keep blowing up the other points p_i $(i \neq j)$ so as to get a morphism $X_r \xrightarrow{\pi_j} \mathbb{P}^1_k$. It is clear that for any $q \in \mathbb{P}^1_k$ with $q \neq q_i := \hat{\pi}_j(p_i)$ if $i \neq j$), the fiber $\pi_j^{-1}(q) \cong \mathbb{P}^1_k$. On the other hand the degenerate fibers $\pi^{-1}(q_i)$ are isomorphic to two lines intersecting transversally and we have the linear equivalence $\pi_j^{-1}(q_i) \sim F + E_i$ for $i \neq j$ (where F denotes the generic fiber $F \sim H + E_j$).

REMARK 1.1. It is worth noting that the simple structure of such a fibration is a consequence of the points p_i being in linear general position. Otherwise we wouldn't be able to conclude that the fibers degenerate in at most two lines.

[1] $\hat{\pi}_j$ is the rational map defined by the subvector space of global sections of $\mathcal{O}_{\mathbb{P}^2_k}(1)$ that vanish at p_j. Up to a projective transformation we can assume p_j has homogeneous coordinates $[0,0,1]$, so that, if T_i are the standard homogeneous coordinates (corresponding to the polarization given by $\mathcal{O}_{\mathbb{P}^2_k}(1)$) the map is given by

$$[T_0, T_1, T_2] \mapsto [T_0, T_1].$$

We will study the direct image of $\mathcal{L}_{d,\mathbf{m}}$ on $\mathbb{P}^1_k$ and find that it has a remarkably simple nature. The problem of computing the cohomology of $\mathcal{L}_{d,\mathbf{m}}$ can be easily reduced to the corresponding one for the direct image by a standard spectral sequence argument which is described in the next section.

1.1. The Leray Spectral Sequence of $\pi_*\mathcal{F}$. Consider a coherent sheaf $\mathcal{F}$ on X_r, it is well known that the cohomology of $\mathcal{F}$ can be approximated by the cohomology of its direct images thanks to the particular version of the Grothendieck spectral sequence of composite functors commonly known as Leray spectral sequence [**aG57, rG58**][2].

In more precise terms, one says the Leray spectral sequences converges to $H^{\cdot}(X_r, \mathcal{F})$, in symbols:[3]

$$E_2^{p,q} = H^p(\mathbb{P}^1_k, R^q\pi_*\mathcal{F}) \Rightarrow H^{p+q}(X_r, \mathcal{F}).$$

By this, we mean there exist an ascending filtration

$$F^{\cdot}: \quad (0) = F^1 \subset F^2 \subset \dots F^{p+q} = H^{p+q}(X_r, \mathcal{F})$$

of the cohomology of $\mathcal{F}$whose associated graded module is isomorphic to a suitable stable value of $E_r^{p,q}$[4]

$$E_\infty^{p,q} \cong Gr^p(H^{p+q}(X_r, \mathcal{F})) = F^p/F^{p+1}.$$

[2]Here we consider the composition of the functor of global sections and its derived cohomology with the direct image and its derived functors. The terminology is due to the analogy with the Leray spectral sequence of a fibration in topology.

[3]Henceforth we will mostly omit the subscript j in π_j for the sake of brevity.

[4]That is the common value of the modules

$$E_r^{p,q} = E_{r+1}^{p,q} = \cdots = E_\infty^{p,q}.$$

Such collapsing is granted if, for example, the sequence is bounded, i.e. for each r and a fixed total degree d, there are only a finite number of possibly nozero terms $E_r^{p,q}$ on the "line" $p + q = d$. Being a first quadrant spectral sequence, the Leray spectral sequence is bounded.

If a spectral sequence has only the first two rows ($E_2^{p,q}$ with $q = 0$ or $q = 1$) possibly non zero, $E_\infty^{p,1} = Ker(d_2^{p,1})$ and $E_\infty^{p,0} = E_2^{p,0}/Im(d_2^{p-2,1})$. Therefore we have the exact sequence:

$$0 \to E_\infty^{p,1} \to E_2^{p,1} \xrightarrow{d_2^{p,1}} E_2^{p+2,0} \to E_\infty^{p+2,0} \to 0$$

The triviality of $E_\infty^{p,q}$ for $q \neq 0, 1$ gives a constraint on the filtration $F^\cdot$ so that

$$E_\infty^{p,1} = F^p/F^{p+1} = H^{p+1}(X_r, \mathcal{F})/F^{p+1} = H^{p+1}(X_r, \mathcal{F})/E_\infty^{p+1,0},$$

which yields

$$0 \to E_\infty^{p+1,0} \to H^{p+1}(X_r, \mathcal{F}) \to E_\infty^{p,1} \to 0.$$

Finally, combinng the last two short exact sequences, we obtain the *Gysin exact sequence*

$$\begin{aligned} \cdots \to H^{p-1}(X_r, \mathcal{F}) \to E_2^{p-2,1} \to E_2^{p,0} \\ \to H^p(X_r, \mathcal{F}) \to E_2^{p-1,1} \to E_2^{p+1,0} \to \\ \to H^{p+1}(X_r, \mathcal{F}) \to E_2^{p,1} \to E_2^{p+2,0} \to \cdots \end{aligned} \tag{1.1}$$

In our particular case $dim(\mathbb{P}_k^1) = 1$, so that if $p > 1$ then $E_2^{p,q} = 0$. Moreover, since for any $y \in \mathbb{P}_k^1$ we have $\dim(\pi^{-1}(y)) = 1$, $E_2^{p,q} = 0$ as soon as $q > 1$. Hence the Leray spectral sequence associated with $\pi_* \mathcal{F}$ can be nonzero at the level of E_2 only in the square $0 \leq p, q \leq 1$. From this follows the vanishing of the differential $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$ and $E_\infty = E_2$. Furthermore all the above considerations apply. Plugging in our data in the Gysin sequence we finally obtain the *fundamental exact sequence*

$$0 \to H^1(\mathbb{P}_k^1, \pi_* \mathcal{F}) \to H^1(X_r, \mathcal{F}) \to H^0(\mathbb{P}_k^1, R^1\pi_* \mathcal{F}) \to 0. \tag{1.2}$$

2. The $R^i\pi_*\mathcal{L}_{d,\mathbf{m}}$

It turns out it is not difficult to prove the vanishing of the first direct image of $\mathcal{L}_{d,\mathbf{m}}$, as well as the local freeness of its direct image. To this end, we will make use of the following basic result of Grauert (cf. [**rH78**, III.12.9] and, for the analytic category, [**GR84**, 10.5, p. 211])

LEMMA 2.1. *Let $V \xrightarrow{\pi} W$ be a projective morphism of noetherian schemes, $\mathcal{F}$ a coherent π-flat sheaf, W integral and*

$$h^i(w) := dim_{k(w)}H^i(V_w, \mathcal{F}|_{V_w})$$

constant on W (here $V_w = V \times_W Spec(k(w))$.

Then $R^i\pi_*\mathcal{F}$ is locally free and $\forall w \in W$

$$R^i\pi_*\mathcal{F} \otimes_{O_W} k(w) \cong H^i(V_w, \mathcal{F}|_{V_w}).$$

Recall that the morphism $\pi : X_r \to \mathbb{P}^1_k$ was gotten by resolving the indeterminacy of the rational projection from $p_j \in \mathbb{P}^2_k$ by means of the blowing-up at p_j. It is clear that π satisfies the hypotheses of Lemma 2.1 and the vanishing of $R^1\pi_*\mathcal{L}_{d,\mathbf{m}}$ easily follows under assumptions weaker than those of Hirschowitz's Conjecture:

PROPOSITION 2.2. *Suppose for $\mathcal{L}_{d,\mathbf{m}}$ we have for all i: $d \geq m_i + m_j, m_i \geq -1$, then*

$$R^1\pi_*\mathcal{L}_{d,\mathbf{m}} = 0$$

PROOF. The proof consists in applying Lemma 2.1 after showing

$$h^1(\pi^{-1}(q), \mathcal{L}_{d,\mathbf{m}}|_{\pi^{-1}(q)}) = 0 \ , \ \forall q \in \mathbb{P}^1_k.$$

Let $q_i := \hat{\pi}(p_i)$ with $i \neq j$ (the projections on $\mathbb{P}^1_k$ of the points of $\mathbb{P}^2_k$ before we blow them up), $U := \mathbb{P}^1_k - \bigcup_{i\neq j}\{q_i\}$ and F the generic fiber of π. Obviously π is smooth at any $q \in U$ and $\pi^{-1}(q) \cong F$. Furthermore we have the linear equivalence $F \sim H - E_j$ so that we can conclude:

$$h^1(F, \mathcal{L}_{d,\mathbf{m}}|_F) = h^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(d - m_j)) = 0.$$

Incidentally we notice that this already proves $R^1\pi_*\mathcal{L}_{d,\mathbf{m}}$ is a skyscraper sheaf with support on $\bigcup_{i\neq j}\{q_i\}$ (cf. [**EGA**, III (4.6.1)] and the Lemma 2.1 itself).

Let's denote by F_i the strict transform of the line in $\mathbb{P}^2_k$ passing through p_i and p_j. Clearly $F_i \sim H - E_i - E_j$ and for all degenerate fibers we have $\pi^{-1}(q_i) \sim F_i + E_i = F$. Tensoring the *decomposition sequence* of the reducible curve $F + E_i$ (cf. [**BPV84**, p. 48]) with $\mathcal{L}_{d,\mathbf{m}}$ we get

$$\begin{aligned} 0 \to \mathcal{L}_{d,\mathbf{m}} \otimes \mathcal{O}_{X_r}(-E_i)|_{F_i} \to \\ \mathcal{L}_{d,\mathbf{m}}|_{F_i+E_i} \to \mathcal{L}_{d,\mathbf{m}}|_{E_i} \to 0. \end{aligned} \tag{2.1}$$

The long cohomology exact sequence then gives

$$\begin{aligned} H^1(F_i, \mathcal{L}_{d,\mathbf{m}} \otimes \mathcal{O}_{X_r}(-E_i)|_{F_i}) \to \\ H^1(F_i + E_i, \mathcal{L}_{d,\mathbf{m}}|_{F_i+E_i}) \to H^1(E_i, \mathcal{L}_{d,\mathbf{m}}|_{E_i}). \end{aligned}$$

But

$$H^1(F_i, \mathcal{L}_{d,\mathbf{m}} \otimes \mathcal{O}_{X_r}(-E_i)|_{F_i}) \cong H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(d - m_j - m_i - 1)) = (0)$$

and

$$H^1(E_i, \mathcal{L}_{d,\mathbf{m}}|_{E_i}) \cong H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(m_i)) = (0)$$

by the hypotheses. □

It turns out that the zeroth direct image of $\mathcal{L}_{d,\mathbf{m}}$ is a vector sheaf.

THEOREM 2.3. *Under the same hypotheses as in Proposition 2.2, the direct image $\pi_*\mathcal{L}_{d,\mathbf{m}}$ is a vector sheaf of rank $d - m_j + 1$.*

PROOF. By Lemma 2.1 it is enough to show that

$$h^0(X_{r_q}, \mathcal{L}_{d,\mathbf{m}}|_{X_{r_q}}) = d - m_j + 1 \ , \ \forall q \in \mathbb{P}^1_k.$$

If $q \neq q_j$, we have

$$h^0(X_{r_q}, \mathcal{L}_{d,\mathbf{m}}|_{X_{r_q}} = h^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(d - m_j)) = d - m_j + 1$$

For degenerate fibers $X_{r_{q_i}} \sim F_i + E_i$, with $i \neq j$, we will make use of the decomposition sequence tensored with $\mathcal{L}_{d,\mathbf{m}}$ as in (2.1) once again. Its long exact cohomology sequence at the level of H^0 now gives the short exact sequence:

$$\begin{aligned} 0 \to H^0(F_i, \mathcal{L}_{d,\mathbf{m}} \otimes \mathcal{O}_{X_r}(-E_i)|_{F_i}) \to \\ H^0(F_i + E_i, \mathcal{L}_{d,\mathbf{m}}|_{F_i+E_i}) \to H^0(E_i, \mathcal{L}_{d,\mathbf{m}}|_{E_i}) \to 0, \end{aligned} \tag{2.2}$$

where the surjectivity on the right follows from the vanishing of the first cohomology group of $\mathcal{L}_{d,\mathbf{m}} \otimes \mathcal{O}_{X_r}(-E_i)|_{F_i}$ proved in the previous Proposition. Now the group on the left is isomorphic to $H^0(\mathbb{P}^1_k, O_{\mathbb{P}^1_k}(d - m_j - m_i - 1))$, and the one on the right to $H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(m_i)$ so that we can conclude

$$h^0(X_{r_{q_i}}, \mathcal{L}_{d,\mathbf{m}}|_{X_{r_{q_i}}}) = d - m_j - m_i + m_i + 1 = d - m_j + 1.$$

□

REMARK 2.4. In the above proof, after showing that the dimension of $\mathcal{L}_{d,\mathbf{m}}$ restricted to the fibers is generically constant, we could have argued that, since the direce image $\pi_* \mathcal{L}_{d,\mathbf{m}}$ is torsion free it has to be locally free because the codimension of the singularity set (the set on which a coherent sheaf fails to be locally free) has to be at least two (see [**OSS80**, Corollary p. 148]). We also note that the vanishing of the first direct image implies that [**dM74**, Corollary 3 p. 53]

$$\pi_* \mathcal{L}_{d,\mathbf{m}} \otimes_{\mathcal{O}_{\mathbb{P}^1_k}} k(q) \cong H^0(\pi^{-1}(q), \mathcal{L}_{d,\mathbf{m}}|_{\pi^{-1}(q)})$$

REMARK 2.5. We can consider the last two results as generalizations of standard facts for Hirzebruch surfaces (cf. [**rH78**, Lemmas V.2.4 and 2.1]). In fact, in the course of the proof of Proposition 3.1 below we will see that the projection π can be factored through the blowing-up of $r - 1$ exceptional curves and the standard fiber bundle map $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(1)) \to \mathbb{P}^1_k$.

3. Computation of Direct Images

3.1. Bounding Cohomology. First of all notice that Proposition 2.2, along with the fundamental exact sequence (1.2), yield

$$H^1(X_r, \mathcal{L}_{d,\mathbf{m}}) \cong H^1(\mathbb{P}^1_k, \pi_*\mathcal{L}_{d,\mathbf{m}}).$$

This along with the general fact (following from the Gysin exact sequence or the very definition of direct image) that

$$H^0(X_r, \mathcal{L}_{d,\mathbf{m}}) \cong H^0(\mathbb{P}^1_k, \pi_*\mathcal{L}_{d,\mathbf{m}}),$$

allows us to check the non speciality of $\mathcal{L}_{d,\mathbf{m}}$ via the cohomology of its direct images.

A famous Theoremof Grothendieck [**aG57a**] which, in different guise, goes back to Birkhoff, C. Segre and others [**OSS80**], asserts every vector sheaf on $\mathbb{P}^1_k$ is decomposable into a direct sum of line sheaves (a *dissocié* vector sheaf in the terminology of linkage theory), so that we can write

$$\pi_*\mathcal{L}_{d,\mathbf{m}} =: \bigoplus_{k=0}^{R} \mathcal{O}_{\mathbb{P}^1_k}(a_k),$$

where we assume

$$a_1 \leq a_2 \leq \ldots \leq a_R,$$

and $R := d - m_j + 1$. (Notice for each $\mathcal{L}_{d,\mathbf{m}}$ we have r direct images, each indexed by j, so that we should denote the degrees of the components of the direct image by a_{kj}; we will keep suppressing the index of the projection as long as that does not harm clarity.)

A more accurate study of the fibration π yields coarse upper bounds for the degrees a_k as follows

PROPOSITION 3.1. *Under the above hypotheses one has*

$$a_k \leq k + d - 1.$$

PROOF. The blowing-up $X_r \xrightarrow{\sigma} \mathbb{P}^2_k$ is an ordered one, i.e. it depends on the order in which we blow up the p_i's. One can decompose σ as

$$X_r = X_{i_r} \xrightarrow{\sigma_{i_r}} X_{i_{r-1}} \xrightarrow{\sigma_{i_{r-1}}} \cdots \xrightarrow{\sigma_{i_2}} X_{i_1} \xrightarrow{\sigma_{i_1}} \mathbb{P}^2_k,$$

where $X_{i_l} \xrightarrow{\sigma_{i_l}} X_{i_{l-1}}$ is the blowing-up of p_{i_l} (and, properly speaking, σ depends on the permutation $(i_1, i_2, \ldots, i_r)$).

It is classically known that X_{i_1}, the blowing-up of one point, is isomorphic to the first Hirzebruch (ruled) surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1_k} \bigoplus \mathcal{O}_{\mathbb{P}^1_k}(1))$; if one puts $i_1 = j$, obviuosly $\pi = \pi_j$ factors trough σ_{i_1}:

$$\pi : X_r \xrightarrow{\alpha_j} X_j \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(1)) \xrightarrow{\beta_j} \mathbb{P}^1_k,$$

where $\alpha_j := \sigma_{i_2} \circ \sigma_{i_3} \circ \cdots \circ \sigma_{i_r}$ and β_j is just the projection to $\mathbb{P}^1_k$. As for our line sheaves on X_r, one has[5]

$$\alpha_{j*}\mathcal{L}_{d,\mathbf{m}} = \mathcal{O}_{X_j}(dH - m_j E_j) \otimes \bigcap\nolimits_{i \neq j} \mathcal{I}^{m_i}_{p_i} \subset \mathcal{O}_{X_j}(dH - m_j E_j).$$

The direct images on the base scheme Y of tensor powers of the tautological line sheaves $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)$ for a vector sheaf $\mathcal{E}$ on Y are known to be the symmetric powers $S^n(\mathcal{E})$ (cf. [**EGA, III.2.1.15**]). Then using the projection formula [**rH78**, Ex. III.8.3] with $Y = \mathbb{P}^1_k$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_k} \bigoplus \mathcal{O}_{\mathbb{P}^1_k}(1)$, we get (here $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{O}_{X_{i_1}}(E_j)$[6] and $\mathcal{O}_{X_{i_1}}(H - E_j) = \beta_j^* \mathcal{O}_{\mathbb{P}^1_k}(1)$)

$$\beta_{j*}\mathcal{O}_{X_{i_1}}(dH - m_j E_j) = S^{d-m_j}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(1)) \otimes \mathcal{O}_{\mathbb{P}^1_k}(d) = \bigoplus\nolimits_{k=d}^{2d-m_j} \mathcal{O}_{\mathbb{P}^1_k}(k).$$

Note the rank of such direct image is exactly $R = d - m_j + 1$. Our direct image of $\mathcal{L}_{d,\mathbf{m}}$ is therefore a subvector sheaf (not a subbundle in

[5] Here we identify the points $p_i \in \mathbb{P}^2_k$, with p_i different from p_j, with the corresponding preimages through σ_j .

[6] It is known that [**rH78** , section V.2] for some section D of the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(1)) \xrightarrow{\beta_j} \mathbb{P}^1_k$ we have $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(D)$. Moreover D has to have self-intersection equal to -1. Therefore the divisor D has to be linearly equivalent to the unique exceptional curve E_j on the blowing-up of $\mathbb{P}^2_k$ at the point p_j.

the terminology of [**sS77**] and [**sL75**]!) of maximal rank of the above vector sheaf:

$$\pi_*\mathcal{L}_{d,\mathbf{m}} = \bigoplus_{k=1}^{R} \mathcal{O}_{\mathbb{P}^1_k}(a_k) \subset \beta_{j*}\mathcal{O}_{X_{i_1}}(dH - m_jE_j) = \bigoplus_{k=d}^{2d-m_j} \mathcal{O}_{\mathbb{P}^1_k}(k)$$

and this is only possible if the inequalities of the claim are satisfied.

□

REMARK 3.2. If $h^1(\mathcal{L}_{d,\mathbf{m}}) = 0$, then

$$\sum_{k=1}^{R} a_k = \chi(\mathcal{L}_{d,\mathbf{m}} \otimes \mathcal{O}_{X_r}(E_j - H))$$

(The proof is by brutal computation using $\mathcal{O}_{X_r}(H - E_j) = \pi^*\mathcal{O}_{\mathbb{P}^1_k}(1)$.) This can be rephrased to say that if the first cohomology group of $\mathcal{L}_{d,\mathbf{m}}$ (hence $\pi_*\mathcal{L}_{d,\mathbf{m}}$) vanishes then the degree of its direct image is determined by the twisting w.r.t. the generic fiber of the projection ($F_j \sim H - E_j$); and is better understood in connection with the more general result which follows.

THEOREM 3.3. *Consider a line sheaf $\mathcal{L}$ on a surface X_r such that:*

a) $R^1\pi_*\mathcal{L} = 0$;

b) let F be the generic fiber of π (recall $F \sim H + E_j$) and assume that the restricion map

$$H^0(X_r, \mathcal{L}) \xrightarrow{res_F} H^0(F, \mathcal{L}|_F)$$

is surjective.

Then

$$H^1(X_r, \mathcal{L}) = (0) = H^1(X_r, \mathcal{L} \otimes \mathcal{O}_{X_r}(-F))$$

and the degrees a_k in $\pi_*\mathcal{L} = \oplus_k \mathcal{O}_{\mathbb{P}^1_k}(a_k)$ are nonnegative.

PROOF. The short exact sequence

$$0 \to \mathcal{L} \otimes \mathcal{O}_{X_r}(-F) \to \mathcal{L} \to \mathcal{L}|_F \to 0$$

yields in cohomology

$$0 \to H^0(X_r, \mathcal{L} \otimes \mathcal{O}_{X_r}(-F)) \to H^0(X_r, \mathcal{L}) \xrightarrow{res_F} H^0(F, \mathcal{L}|_F)$$
$$\xrightarrow{\alpha} H^1(X_r, \mathcal{L} \otimes \mathcal{O}_{X_r}(-F)) \xrightarrow{\beta} H^1(X_r, \mathcal{L}) \to 0.$$

The vanishing of $H^1(\mathcal{L}|_F)$ following from that of the first direct image of $\mathcal{L}_{d,\mathbf{m}}$.

Since res_F is onto, $\alpha = 0$ and β is an isomorphism. Hence it suffices to prove that one of the cohomology groups in the assertion vanishes. To this effect, note that the vanishing of the first image of $\mathcal{L}$ implies, by virtue the fundamental exact sequence (1.2), that $H^1(X_r, \mathcal{L}) \cong H^1(\mathbb{P}^1_k, \pi_*\mathcal{L})$. Furthermore, the projection formula yields (taking into account that $\pi^*\mathcal{O}_{\mathbb{P}^1_k}(1) = \mathcal{O}_{X_r}(F)$)

$$R^1\pi_*(\mathcal{L} \otimes \mathcal{O}_{X_r}(-F)) \cong R^1\pi_*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^1_k}(-1) = 0,$$

so that the same isomorphism of first cohomology groups holds for $\mathcal{L} \otimes \mathcal{O}_{X_r}(-F)$ too.

We can then conclude that β descends to an isomorphism of cohomology groups on $\mathbb{P}^1_k$:

$$H^1(\mathbb{P}^1_k, \pi_*\mathcal{L}) \cong H^1(\mathbb{P}^1_k, \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^1_k}(-1))$$

which, as $\pi_*\mathcal{L}$ is sum of line sheaves, is possible only if both groups vanish.

The vanishing of the second of these cohomology groups yields at once the lower bound for the degrees a_k. □

REMARK 3.4. Even more generally, if a line sheaf $\mathcal{L}$ on X_r has positive Euler-Poincaré characteristic and intersects the generic fiber F positively, then from the cohomology of the structure sequence of F tensored by $\mathcal{L}$ it follows $\chi(\mathcal{L}(F)) > c_1(\mathcal{L}).F$ and $h^0(X_r, \mathcal{L}) > h^0(F, \mathcal{L}|_F)$.

REMARK 3.5. Notice that the line sheaf $\mathcal{L} = \mathcal{O}_{X_r}(H - E_i - E_j)$, with $i \neq j$, corresponding to the strict transform of the line passing through the points p_i and p_j satisfies the above theorem, but does not meet the requirements of the second formulation of Hirshowitz's Conjecture 1.6.

Indeed, it is easy to prove the vanishing of $H^i(X_r, \mathcal{L}(-F))$ for $i = 0, 1$, so that the whole cohomology of $\mathcal{L}$ is cocentrated on the generic fiber $F \sim H - E_j$. The same is still true if one takes $\mathcal{L} = \mathcal{O}_{X_r}(H - E - E_j)$, where E is any exceptional curve that intersect F trivially.

We would like to raise the following question:

PROBLEM 3.6. *Under what conditions on the line sheaves $\mathcal{L}_{d,\mathbf{m}}$ are their direct images ample vector bundles?*

An effective answer to such a question would give lower bounds for the degree of direct images as and ample vector on $\mathbb{P}^1_k$ has to be direct sum of ample line bundles (cf. [**rH66**]), hence their degreees have to be positive.

We hope that careful application of Viehweg's theory on the weakly positiveness of vector sheaves and their direct images will lead to some results in this direction at least in characteristic zero. On the other hand, we are already able to establish the ampleness of direct images, and actually give a full computation, in the case of few points blown up or of invertible sheaves whose sections are "not too singular", viz. the multiplicities m_i's are equal to 1.

We ought to mention that the problem of determining ampleness on generic rational surfaces is much harder than that of determining effectivity covered by Hirschowitz's Conjecture. It is not even clear what conditions one should impose in order to expect ampleness (cf. [**AH92**]). However the study of line sheaves with multiplicities $m_i = 1$ has lead to some results [**eB83**]. More recently Reider's theory [**oK94**] as well as a defomation theoretic argument [**gX95**] have yielded the following characterization of such sheaves:

PROPOSITION 3.7. *An invertible sheaf $\mathcal{O}_{X_r}(dH - E_1 - E_2 - \cdots - E_r)$ with $d > 3$ is ample if and only if it has positive self-intersection.*

REMARK 3.8. The necessity is well known to be required by Nakai's criterion, so the non trivial, or, better, original part consists in establishing the sufficiency of this condition. Here, as in the thesis 3.2, a lower bound for the self-intersection of curves plays a major role.

3.2. The case of two blown up points. Let us first remark that the problem of computing the cohomology of line sheaves on X_r for $r \leq 8$ was solved classically. Indeed in this case X_r admits an ample anticanonical sheaf and is, by definition, a Del Pezzo surface. It can easily be proved that the diophantine equations defining an exceptional class have only a finite number of solutions [**yM86**, Ch. 7 §25-26], so that we have only a finite number of exceptional curves $E_1, \dots, E_s$.

Mori's theory then insures [**sM82**, Theorems (1.2) p.135 and (2.1) p.141] that the cone of effective divisors $NE(X_r) \subset Num(X_r) \otimes_{\mathbb{Z}} \mathbb{R} =: N^1(X_r)$ (hence the ample cone $N_a(X_r) \subset N^1(X_r) = N_1(X_r)$ which, by Kleiman's criterion, is its dual) is a finitely generated polyhedral cone explicitly determined by the exceptional curves:

$$NE(X_r) = \sum_{k=1}^{s} \mathbb{R}_+[E_k].$$

Therefore given the "Picard coordinates" $(d, m_1, \dots, m_r)$ of a line bundle $\mathcal{L}_{d,\mathbf{m}}$, we are able to determine whether it admits non trivial global sections as well as well as the vanishing of its first cohomology group (e.g. checking if $[\mathcal{L}_{d,\mathbf{m}} \otimes \mathcal{K}_{X_r}^{-1}] \in N_a(X_r)$ and applying Kodaira's vanishing in characteristic zero–cf., e.g., [**EV92**, p. 4]).

When $r = 9$, X_r is a rational elliptic surface and $\mathcal{K}_{X_r} = \mathcal{O}_{X_r}$, but one can still obtain the Conjecture (cf. [**mN60a**], [**fS84**]).

Complications arise for $r \geq 10$: here the anticanonical sheaf is numerically indefinite and the infinite number of exceptional curves, hence ([**sM82**, Theorem(2.1) p. 141]) of extremal rays, makes the effective cone beyond reach of computation. Hirschowitz's Conjecture would give an alternative finite algorithm for the determination of effective divisors on the surfaces X_r.

We would like to illustrate our approach to the Conjecture by computing direct images in the case of X_2 defined over a field k with $char(k) = 0$. This will give a flavor of what we can expect in the general case.

In this case, if $\sigma^{-1}(p_i) = E_i$ $(i = 1, 2)$, the effective cone is

$$NE(X_2) = \mathbb{R}_+[E_1] + \mathbb{R}_+[E_2] + \mathbb{R}_+[H - E_1 - E_2],$$

while the ample one is

$$N_a(X_2) = \mathbb{R}_+[H - E_1] + \mathbb{R}_+[H - E_2] + \mathbb{R}_+[H].$$

A line sheaf $\mathcal{O}_{X_2}(dH - m_1E_1 - m_2E_2)$ will admit non trivial global sections iff

$$m_1 \geq 0,\ m_2 \geq 0,\ d \geq m_1 + m_2,$$

and will be ample iff

$$d > m_1,\ d > m_2,\ d > 0.$$

It is clear that a line sheaf satisfying the hypotheses of the Conjecture 1.6 (which in this case read $d > m_1+m_2$) imply $\mathcal{O}_{X_2}(dH - m_1E_1 - m_2E_2)$ as well as

$$\mathcal{O}_{X_2}(dH - m_1E_1 - m_2E_2)\otimes\mathcal{K}_{X_2}^{-1} = \mathcal{O}_{X_2}((d+3)H - (m_1+1)E_1 - (m_2+1)E_2)$$

are ample, hence, by the Kodaira vanishing theorem, $h^1(\mathcal{O}_{X_2}(dH - m_1E_1 - m_2E_2))$ vanishes. This means that Hirschowitz's Conjecture is true (similar consideration yield it for any X_r with $r \leq 8$). Insofar as direct images are concerned, $\pi_{j_*}\mathcal{O}_{X_2}(dH - m_1E_1 - m_2E_2)$ is a vector sheaf on $\mathbb{P}^1_k$ of rank $d - m_j + 1 =: R$, say

$$\pi_{j_*}\mathcal{O}_{X_2}(dH - m_1E_1 - m_2E_2) =: \bigoplus_{k=1}^{R} \mathcal{O}_{\mathbb{P}^1_k}(a_k).$$

with $a_1 \leq a_2 \leq \ldots \leq a_R$. Our objective is to find the degrees a_k's.

Note that, depending on the size of the multiplicities m_1, m_2 the direct images range between two extremal cases:

1. $d = m_1 + m_2$ which is the least required by Proposition 2.3 to yield a locally free sheaf.

2. $m_1 = 1 = m_2$.

(This, with obvious modifications, makes sense also in the the general case of r points blown-up.)

We are going to deal with such cases in the following

PROPOSITION 3.9. *Under the above assumptions we have*

(i) *If* $d = m_1 + m_2$ *and* $d > m_1 \geq m_2 > 0$, *then*

$$\pi_{j_*}\mathcal{O}_{X_2}(dH - m_1E_1 - m_2E_2) = \mathcal{O}_{\mathbb{P}^1_k}(m_j)^{\oplus d-m_j+1}.$$

(ii) *If* $m_1 = 1 = m_2$,

$$\pi_{j_*}\mathcal{O}_{X_2}(dH-E_1-E_2) = \mathcal{O}_{\mathbb{P}^1_k}(1)\oplus\mathcal{O}_{\mathbb{P}^1_k}(2)\oplus\cdots\oplus\mathcal{O}_{\mathbb{P}^1_k}(d-2)\oplus\mathcal{O}_{\mathbb{P}^1_k}(d-1)^{\oplus 2}.$$

PROOF. (i) Notice

$$\mathcal{O}_{X_2}(dH - m_1E_1 - m_2E_2)\otimes\mathcal{K}^{-1}_{X_2} = \mathcal{O}_{X_2}((d+3)H-(m_1+1)E_1-(m_2+1)E_2)$$

is ample (it lies in the ample cone), hence Kodaira vanishing grants the vanishing of $h^1(\mathcal{O}_{X_2}(dH - m_1E_1 - m_2E_2))$. A simple calculation then yields

$$h^0(\pi_{j_*}\mathcal{O}_{X_2}(dH - m_1E_1 - m_2E_2)) = h^0(\mathcal{O}_{X_2}(dH - m_1E_1 - m_2E_2)) =$$
$$\chi(\mathcal{O}_{X_2}(dH - m_1E_1 - m_2E_2)) = m_1m_2 + d + 1.$$

Whence we obtain the degree of the direct image vector sheaf

$$\sum_{k+1}^{R} a_k = m_j(d - m_j + 1) = m_j\ rk(\pi_{j_*}\mathcal{O}_{X_2}(dH - m_1E_1 - m_2E_2)).$$

Because of Proposition 3.1 and the fact that $\pi_{j_*}\mathcal{O}_{X_2}(dH - m_1E_1 - m_2E_2)$ has a vanishing first cohomology group we have v

$$-1 \leq a_k \leq d + k - 1.$$

On the other hand, as $\sum_{k=1}^{R} a_k = m_j rk(\pi_{j_*}\mathcal{O}_{X_2}(dH - m_1E_1 - m_2E_2))$, one has the upper bound

$$m_j \leq a_R.$$

Suppose we had strict inequality, then, taking $i \neq j$,

$$\begin{aligned} 1 &\leq h^0(\pi_{j_*}\mathcal{O}_{X_2}(dH - m_1E_1 - m_2E_2) \otimes \mathcal{O}_{\mathbb{P}^1_k}(-a_R)) = \\ &= h^0(\mathcal{O}_{X_2}((d-a_R)H + (a_R - m_j)E_j - (d-m_j)E_j) = 0, \end{aligned} \tag{3.1}$$

since $d - a_R < d - m_j$. We can therefore conclude the proof of the first item.

(ii) Here too the line sheaf

$$\mathcal{O}_{X_2}(dH - E_1 - E_2) \otimes \mathcal{K}_{X_2}^{-1} = \mathcal{O}_{X_2}((d+3)H - 2E_1 - 2E_2)$$

is ample, so that the Kodaira vanishing Theoremimplies that $h^1(\mathcal{O}_{X_2}(dH - E_1 - E_2)) = 0$. We can then obtain the dimension of the space global sections of $\mathcal{O}_{X_2}(dH - m_1E_1 - m_2E_2)$ as in the above case:

$$\begin{aligned} h^0(\pi_{j_*}\mathcal{O}_{X_2}(dH - E_1 - E_2)) &= \chi(\mathcal{O}_{X_2}(dH - E_1 - E_2)) \\ &= (d+1)(d+2)/2 - 2 = d(d+3) - 1. \end{aligned}$$

Proposition 3.1 along with the vanishing of the first cohomology group gives the following bounds for the degrees of the direct image of $\mathcal{O}_{X_2}(dH - E_1 - E_2)$

$$-1 \leq a_k \leq d + k - 1.$$

Notice that we cannot have (here $R = d$) $a_d = d$ since this would imply (again taking $i \neq j$)

$$1 \leq h^0(\pi_{j_*}\mathcal{O}_{X_2}(dH-E_1-E_2)\otimes\mathcal{O}_{\mathbb{P}^1_k}(-d)) = h^0(\mathcal{O}_{X_2}((d-1)E_j - E_i)) = 0.$$

On the other hand note that since $h^0(\pi_{j_*}\mathcal{O}_{X_2}(dH - E_1 - E_2)) = d(d+3)/2 - 1$ we have

$$\sum\nolimits_k a_k = 1 + 2 + 3 + \dots d - 1.$$

We are going to prove by induction on the degree d that the string $(a_1, \dots, a_d)$ can be filled only in the way we claimed.

For $d = 2$ one can directly compute (using cohomological dimension count as in the previous argument) that $\pi_{j_*}\mathcal{O}_{X_2}(2H - E_1 - E_2) = \mathcal{O}_{\mathbb{P}^1_k}(1)^{\oplus 2}$ (similarly $\pi_{j_*}\mathcal{O}_{X_2}(3H - E_1 - E_2) = \mathcal{O}_{\mathbb{P}^1_k}(1) \oplus \mathcal{O}_{\mathbb{P}^1_k}(2)^{\oplus 2}$).

Suppose the claim true for $d - 1$. Then

$$\mathcal{O}_{\mathbb{P}^1_k}(1) \oplus \mathcal{O}_{\mathbb{P}^1_k}(2) \dots \mathcal{O}_{\mathbb{P}^1_k}(d-2)^{\oplus 2} = \pi_{j_*}\mathcal{O}_{X_2}((d-1)H - E_1 - E_2) \subset \mathcal{O}_{X_2}(dH - E_1 - E_2)$$

Therefore for any $1 \le k \le d - 1$ we have the lower bound $k \le a_k$. It is now clear that the only possibility for the a_k to satisfy the above cohomological count is to be as claimed.

□

Based on these and other considerations, we venture the following educated guess:

THESIS 3.10. *The larger the multiplicities m_i's, the smaller the* maximal gap $max|a_k - a_l|$.

In particular we would expect that for borderline multiplicities (case 1) the direct image tend to be direct sums of the same line sheaf, while for multiplicities $m_i = 1$, the a_k tend to grow in an increasing sequence without gaps. This latter behavior resembles the notion of *character of a set of points* in $\mathbb{P}^2_k$ that Gruson and Peskine introduced in the study of space curves [**GP78**]. The "spread" of the a_k's also appears in results on semistable vector bundles à la Grauert-Mülich [**hS79**] [**mM81**].

REMARK 3.11. In the previous Proposition a few technical facts appear:

Sub-direct images tend to fit at the beginning of the larger direct image: i.e. if $\mathcal{L} \subset \mathcal{M}$, $\pi_{j_*}\mathcal{L} = \bigoplus_{k=1}^{R} \mathcal{O}_{\mathbb{P}^1_k}(a_k)$ and $\pi_{j_*}\mathcal{M} = \bigoplus_{k=1}^{S} \mathcal{O}_{\mathbb{P}^1_k}(b_k)$, then

$$\pi_{j_*}\mathcal{M}/\pi_{j_*}\mathcal{L} \cong \mathcal{N} \oplus \bigoplus_{k=R+1}^{S} \mathcal{O}_{\mathbb{P}^1_k}(b_k),$$

where $\mathcal{N}$ is a torsion sheaf on $\mathbb{P}^1_k$.

REMARK 3.12. As, in general, using vanishing Theoremsuch as Kodaira's, we can much more easily obtain the dimension of the space of global sections of a certain direct image and form this its degree as a vector sheaf on $\mathbb{P}^1_k$, the problem of the determination of the direct image itself is reduced to a question about combinatorial partition of the degrees a_k. We wonder if techniques from combinatorics, which have already succesfully been applied to the study of Chern classes, could also be employed in this connection.

3.3. The general case. In principle, because the cone of effective divisors is finite polyhedral, we have a finite algorithm to compute the global sections of any line sheaf on a Del Pezzo surface. Therefore we can mimic the computations in the case of the toy model of two points blown up to yield explicit direct images for line sheaves on X_r for $r \leq 8$.

In the general case, the knowledge of the direct images would yield Hirschowitz's Conjecture. Conversely, the vanishing of one of the cohomology groups determines the degree of the direct image as a vector sheaf on $\mathbb{P}^1_k$. [7] This enables us to recover the entire direct image in some particular case.

THEOREM 3.13. *Consider a line sheaf $\mathcal{L} = \mathcal{O}_{X_r}(dH - E_1 - E_2 - \cdots - E_r)$ on a surface X_r such that $h^1(\mathcal{L}) = 0$ and $r \leq d$. Then*

$$\pi_* \mathcal{L} = \bigoplus_{k=1}^{d-r-1} \mathcal{O}_{\mathbb{P}^1_k}(k) \oplus \bigoplus_{k=d-r}^{d} \mathcal{O}_{\mathbb{P}^1_k}(k-1)$$

PROOF. We will argue by induction on the number of points blown up r. For $r = 2$ the statement has already been proved in Proposition 3.9[8]. Suppose for $r - 1$ and $\mathcal{L}' = \mathcal{O}_{X_r}(dH - E_1 - E_2 - \cdots - E_{r-1})$

[7] Perhaps, the Hirzebruch-Riemann-Roch Theoremgives a more elegant way to relate this two quantities. It will be argued below that the case of Euler-Poincaé characteristic zero plays a crucial role exactly because in this case the vanishing of both cohomology groups would determine the direct image uniquely and viceversa.

[8] Then we supposed the characteristic of the ground field was zero in order to apply the Kodaira vanishing theorem and obtain the sought for direct image.

we have the thesis. Since $\mathcal{L} \subset \mathcal{L}'$, from the left exactness of the direct image functor it follows

$$\pi_* \mathcal{L} := \bigoplus_{k=1}^{d} \mathcal{O}_{\mathbb{P}^1_k}(a_k) \subset \pi_* \mathcal{L}'. \tag{3.2}$$

From the vanishing of the first cohomology group of $\mathcal{L}$ we can obtain the dimension of the space of global section of $\pi_* \mathcal{L}$ as the Euler-Poincaré characteristic of $\mathcal{L}$:

$$h^0(\mathbb{P}^1_k, \pi_* \mathcal{L}) = \frac{(d+1)(d+2)}{2} - r.$$

Translating this in a condition for the degrees a_k as in Proposition 3.9 we have

$$\sum a_k = 1 + 2 + \cdots + d - r.$$

Using again the vanishing of the first cohomology group of $\mathcal{L}$ and the containment in (3.2) we derive upper and lower bounds

$$\begin{array}{ll} -1 \leq a_k \leq k & if \quad k \leq d - r, \\ -1 \leq a_k \leq k - 1 & if \quad k \geq d - r + 1. \end{array} \tag{3.3}$$

It is now obvious that the only way we can "fill" the string of degrees a_k for $\pi_* \mathcal{L}$ is the one claimed. □

REMARK 3.14. If $char(k) = 0$, the vanishing of the first cohomology group would follow from the Ramanujam vanishing theorem. Indeed, letting $\mathcal{L} = \mathcal{O}_{X_r}(dH - E_1 - \cdots E_r)$,

$$h^1(X_r, \mathcal{L}) = h^1(X_r, \mathcal{L}^{-1} \otimes \mathcal{K}_{X_r}) = h^1(X_r, \mathcal{O}_{X_r}((-d-3)H)).$$

REMARK 3.15. A similar result holds for line sheaves $\mathcal{O}_{X_r}(dH - E_1 - \cdots - E_r)$ without the assumption $r \leq d$. This can still be proved

Obviously, if we suppose the vanishing of the first cohomology group, the same proof is still valid.

by induction using the above theorem. The notation for the direct image is awkward though.

PROPOSITION 3.16. *For an exceptional curve E on X_r we have*

$$\pi_{j_*}\mathcal{O}_{X_r}(E) = \mathcal{O}_{\mathbb{P}^1_k}(-1)^R \oplus \mathcal{O}_{\mathbb{P}^1_k}$$

where $R = E.(H - E_j)$; and either

$$\pi_{j_*}\mathcal{O}_{X_r}(-E) = \mathcal{O}_{\mathbb{P}^1_k}(-1),$$

or

$$\pi_{j_*}\mathcal{O}_{X_r}(-E) = 0$$

depending whether $R = 0$ or $R > 0$ respectively.

PROOF. By Lemma 2.1, if $\pi_{j_*}\mathcal{O}_{X_r}(E)$ is locally free it has to have rank (denoting again the generic fiber with F and recalling that $F \sim H - E_j$)

$$h^0(F, \mathcal{O}_{X_r}(E)|_F) \cong h^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(E.(H - E_j)).$$

Now the fact that $\pi_{j_*}\mathcal{O}_{X_r}(E)$ is a vector sheaf can be derived from the vanishing of the first direct image [**EGA**, III (9.)]. This, in turns follows form the fact that the intersection of E with every fiber is bounded below by -1, hence $h^1(\pi^{-1}(q), \mathcal{O}_{X_r}(E)|_{\pi^{-1}(q)}) = 0$ for any $q \in \mathbb{P}^1_k$. Using $1 = h^0(X_r, \mathcal{O}_{X_r}(E)) = h^0(\mathbb{P}^1_k, \pi_*\mathcal{O}_{X_r}(E))$ and (cf. (1.2)) $0 = h^1(X_r, \mathcal{O}_{X_r}(E)) = h^1(\mathbb{P}^1_k, \pi_*\mathcal{O}_{X_r}(E))$, we obtain the degrees of the direct image.

As for the second claim, the local freeness of $\pi_*\mathcal{O}_{X_r}(-E)$ is gotten in exacly the same way. Because $-E.F \leq 0$ (E is a curve!), $h^0(F, \mathcal{O}_{X_r}(-E)|_F) \leq 1$, so $\pi_*\mathcal{O}_{X_r}(-E)$ has at most rank one. If such direct image is non zero, then it has to be $\mathcal{O}_{\mathbb{P}^1_k}(-1)$ because both its cohomology groups vanish. □

As a consequence of this direct image computation we have a new strategy to prove Hirschowitz's Conjecture (see corollary below) at our disposal, in the case of effective line sheaves $\mathcal{L}_{d,\mathbf{m}}$, or, equivalently, of non negative Euler-Poincaré characteristic. In such case the Conjecture would follow from the presence of an exceptional curve among the components of a section of $\mathcal{L}_{d,\mathbf{m}}$ which intesect the genaric fiber of π_j as $\mathcal{L}_{d,\mathbf{m}}$ itself. Note that here we can take advantage of the fact that we have r fibration structures and, correspondingly, r possibilities for the intersection number.

COROLLARY 3.17. *If a line sheaf $\mathcal{L}_{d,\mathbf{m}}$ contains an exceptional curve E, with $E.(H-E_j) = d-m_j$, then its first cohomology group vanishes.*

PROOF. Because of the previous Proposition the direct image of $\mathcal{O}_{X_r}(E)$ is a vector sheaf on $\mathbb{P}^1_k$ with degrees bounded below by -1. The left exactness of the direct image functor and Theorem2.3 now prove that $\pi_*\mathcal{O}_{X_r}(E)$ is a subbundle of maximal rank of $pist\mathcal{L}_{d,\mathbf{m}}$. Therefore the same lower bound for the degrees of the latter holds, and from this we can conclude the vanishing of $h^1(\mathbb{P}^1_k, \pi_*\mathcal{L}_{d,\mathbf{m}}) = h^1(X_r, \mathcal{L}_{d,\mathbf{m}})$. □

REMARK 3.18. This condition on the intersection of E with the generic fiber $F \sim H - E_j$, is compatible withHirschowitz's Conjecture. Indeed, if $E \sim e_0H - e_1E_1 - \cdots - e_rE_r$, and $E.F = d - m_j = c_1(\mathcal{L}_{d,\mathbf{m}}).F$, then $d - m_j = e_0 - e_j$. The second formulation of the Conjecture 1.6 assumes that

$$d \geq m_1 + m_2 + m_3,$$

while Noether inequality [**DO88**, p. 77] for exceptional curves implies

$$e_0 < e_1 + e_2 + e_3.$$

If one assumes, as we do, the non restricitive ordering $e_0 \geq e_1 \geq \cdots \geq e_r$ and notices that the containment $\mathcal{O}_{X_r}(E) \subset \mathcal{L}_{d,\mathbf{m}}$ gives bounds

$m_i \leq e_i$, then one realizes that the two above inequalities can be satisfied simultaneously.

Furthermore, even if $E.F < d - m_j$, we would still have, taking direct images, a suvector sheaf of $\pi_* \mathcal{L}_{d,\mathbf{m}}$ with lower bounds for its degrees. This would give corresponding lower bounds on some of the degrees of $\pi_* \mathcal{L}_{d,\mathbf{m}}$, hence a limitation, albeit not a vanishing, on the dimension of its first cohomology group.

We hope that using the action of the Weyl group on exceptional configurations, we can arrange exceptional curves so as to bound all the degree of $\pi_* \mathcal{L}_{d,\mathbf{m}}$.

It is also worth noticing how the bound provided by this direct image approach to the Conjecture are essentially stronger than the previous approach.

Indeed, if we just assumed the containment of an exceptional E curve in $\mathcal{L}_{d,\mathbf{m}}$, the cohomology exact sequence and the bound $c_1(\mathcal{L}_{d,\mathbf{m}}).E \geq -1$ would only give

$$h^1(X_r, \mathcal{L}_{d,\mathbf{m}}) \leq h^1(X_r, \mathcal{L}_{d,\mathbf{m}}(-E)),$$

no matter how large the intersection $E.F$.

It is worth noticing that the universal family of blowups we considered in the previous chapter carries the structure of rational fibration we have been using in the present chapter. Then we cpould consider family of direct images of line shaves on surfaces X_r.

To see how direct images can change under specialization consider the example of the anticanonical sheaf $\mathcal{O}_{X_r}(-K_{X_r})$. One checks immediately that $-K_{X_r}.F = 2$ and that $-K_{X_r}$ intersects degenerate fibers positively as well. Therefore the first direct image vanishes and $\pi_* \mathcal{O}_{X_r}(-K_{X_r})$ is a vector shef of rank $h^0(F, \mathcal{O}_F(2)) = 3$. Let

$$\pi_* \mathcal{O}_{X_r}(-K_{X_r}) =: \mathcal{O}_{\mathbb{P}^1_k}(a) \oplus \mathcal{O}_{\mathbb{P}^1_k}(b) \oplus \mathcal{O}_{\mathbb{P}^1_k}(c),$$

with $a \leq b \leq c$ as usual. As the cohomology is preserved and (for $r > 9$) $h^0(X_r, \mathcal{O}_{X_r}(-K_{X_r})) = 0$, we obtain for the Riemann-Roch theorem

$$h^1(\mathbb{P}^1_k, \pi_*\mathcal{O}_{X_r}(-K_{X_r})) = -(K^2_{X_r} + 1) = r - 10.$$

Serre duality on $\mathbb{P}^1_k$ along with the vanishing of the zeroth cohomology give now conditions on the degrees

$$a + b + c = 7 - r$$

and

$$a, b, c < -1.$$

On the other hand, if we specialize our surface to an anticanonical surface, the anticanonical divisor will admit a section, so at least one degree will have to become non negative. If we specialize further to a Harbourne surface, $\mathcal{O}_{\mathbb{P}^1_k}$ will be among the direct summand of $\pi_*\mathcal{O}_{X_r}(-K_{X_r}))$. Indeed, in this case $h^0(X_r, \mathcal{O}_{X_r}(-K_{X_r})) = 1$ because the anticanonical section D doesn't move in its linear system.

We would like to conclude this section with an observation on the structure of the set $T_1(X_r) \subset N^1(X_r)$ of line sheaves with vanishing first cohomology.

Invertible sheaves on generic rational surfaces enjoy a property of "homogeneity" with respect to the vanishing of the first cohomology: tensoring a positive enough line sheaf with vanishing first cohomology with a non negative one, the resulting line sheaf has zero first cohomology as well. One could say the set $P(X_r)$ of "positive" (with the properties of $\mathcal{E}$ in the proposition below) line sheaves, or more generally vector sheaves, is a cone in $T_1(X_r)$ and that $P_1(X_r) + T_1(X_r) \subset T_1(X_r)$ [9].

[9] Although in our case the Picard scheme $Pic^0(X_r)$ is trivial, it would be interesting to explore possible connections with the Green-Lazarsfeld theory of generic vanishing (cf. [**GL87**, **GL91** and **AGL91**]).

This can be formulated using direct images and easily generalized to vector sheaves. (Incidentally, this will be the only place where we will consider direct images of higher rank vector sheaves.) The conditions of positivity we alluded to above can be precisely formulated using the degrees of the direct images. In the case of line sheaves we are claiming that the locus in $Num(X_r)$ of sheaves with vanishing first cohomology is not convex, but we can find a subset which, after being translated by a suitably positive line sheaf, will be convex and homogeneous with respect to the tensor product.

PROPOSITION 3.19. *Consider two vector sheaves $\mathcal{E}$ and $\mathcal{F}$ on X_r such that the "restricion homomorphism" in cohomology induced by the embedding of the generic fiber $j : F \hookrightarrow X_r$ is surjective , $h^1(X_r, \mathcal{E}) = 0 = h^1(X_r, \mathcal{F})$ and such that their direct images $\pi_*\mathcal{E} = \bigoplus \mathcal{O}_{\mathbb{P}^1_k}(a_k)$ and $\pi_*\mathcal{F} = \bigoplus \mathcal{O}_{\mathbb{P}^1_k}(b_l)$ have degrees bounded below as $0 \leq a_k$ and $-1 \leq b_l$.*

Then, the morphism induced by the cup product on the 0-th cohomology

$$\pi_*\mathcal{E} \otimes \pi_*\mathcal{F} \longrightarrow \pi_*(\mathcal{E} \otimes \mathcal{F})$$

is generically an isomorphism on $\mathbb{P}^1_k$ and $h^1(X_r, \mathcal{E} \otimes \mathcal{F}) = 0$

PROOF. Grauert's Theorem2.1 gives the isomorphism $\pi_*\mathcal{E} \otimes k(q) \cong H^0(F, \mathcal{E}|_F)$ and the analogous for $\mathcal{F}$ and $\mathcal{E} \otimes \mathcal{F}$ for generic $q \in \mathbb{P}^1_k$. Then the claim of the isomorphism follows from the assertion that the cup product is indeed surjective for vector sheaves over a $F \cong \mathbb{P}^1_k$.

Now, the condition on the degrees of the direct images of $\mathcal{E}$ and $\mathcal{F}$ implies that the degrees of $\pi_*\mathcal{E} \otimes \mathcal{F}$ are no less than -1. □

4. A Reformulation of Hirschowitz's Conjecture

4.1. Reduction to the case $\chi(\mathcal{L}_{d,\mathbf{m}}) = 0$. It turns ou that in the statement of Hirschowitz's Conjecture 1.6 we can assume $\chi(\mathcal{L}_{d,\mathbf{m}}) = 0$. This is not hard to prove reduce to this specific case:

PROOF. 1. $\chi(\mathcal{L}_{d,\mathbf{m}}) < 0$.

By decreasing multiplicities we can find another line sheaf $\mathcal{L} := \mathcal{O}_{X_r}(dH - \Sigma_{i=1}^r n_i E_i)$ with $n_i \leq m_i$ such that $\chi(\mathcal{L}) = 0$. From the exact sequence

$$0 \to \mathcal{L}_{d,\mathbf{m}} \to \mathcal{L} \to \mathcal{Q} \to 0,$$

we obtain

$$0 \to H^0(X_r, \mathcal{L}_{d,\mathbf{m}}) \to H^0(X_r, \mathcal{L}).$$

Now, if the Conjecture is true for line bundles of Euler-Poincaré characteristic zero, since $\mathcal{L}$ satisfies the hypotheses of the Conjecture (its degree are bounded by those of $\mathcal{L}_{d,\mathbf{m}}$ which does), it will have both cohomology groups zero, hence $h^0(\mathcal{L}_{d,\mathbf{m}}) = 0$, that is $\mathcal{L}_{d,\mathbf{m}}$ verifies the Conjecture as well.

2. $\chi(\mathcal{L}_{d,\mathbf{m}}) > 0$.

Consider additional points $p_{r+1}, \ldots, p_s \in \mathbb{P}_k^2$ such that $p_1, \ldots, p_s$ are in general position (this is possible for any positive integer $s \geq r$, see, e.g., [**jB79**]).

Choose s such that $\mathcal{L} := \mathcal{O}_{X_s}(dH - \sum_{i=1}^r m_i E_i - \sum_{i=r+1}^s E_i)$ satisfies $\chi(\mathcal{L}) = 0$.

We have the exact sequence

$$0 \to \mathcal{L} \to \mathcal{L}_{d,\mathbf{m}} \to \mathcal{O}_{\sum_{i=r+1}^s E_i} \to 0$$

which, passing to cohomology, yields

$$0 \to H^0(X_s, \mathcal{L}) \to H^0(X_s, \mathcal{L}_{d,\mathbf{m}}) \to H^0(X_s, \mathcal{O}_{\sum_{i=r+1}^s E_i}$$
$$\to H^1(X_s, \mathcal{L}) \to H^1(X_s, \mathcal{L}_{d,\mathbf{m}}) \to H^1(X_s, \mathcal{O}_{\sum_{i=r+1}^s E_i}) = 0$$

Now if $\mathcal{L}_{d,\mathbf{m}}$ satisfies the hypotheses of the Conjecture, so does $\mathcal{L}$, and if the Conjecture holds for $\mathcal{L}$, the latter will have first cohomology group zero which implies the same vanishing for $\mathcal{L}_{d,\mathbf{m}}$. (All the way we are identifying cohomology groups of $\mathcal{L}_{d,\mathbf{m}}$ on X_s and X_r.)

We have also obtained the dimension of the space of global section of $\mathcal{L}_{d,\mathbf{m}}$ as:

$$h^0(X_r, \mathcal{L}_{d,\mathbf{m}}) = \chi(\mathcal{L}_{d,\mathbf{m}}) = h^0(X_s, \mathcal{O}_{\Sigma_{i=r+1}^{s} E_i}).$$

□

4.2. Relations with Semistability of Vector Bundles. The isomorphism of the cohomology groups of $\mathcal{L}_{d,\mathbf{m}}$ and those of its direct image gives in particular the equality

$$\chi(\mathcal{L}_{d,\mathbf{m}}) = \chi(\pi_* \mathcal{L}_{d,\mathbf{m}})$$

Then the Theoremof Riemann-roch for vector bundles on a curve (viz. $\mathbb{P}_k^1$) or the general Grothendieck-Riemann-Roch Theoremgives

$$\begin{aligned}\chi(\mathcal{L}_{d,\mathbf{m}}) &= deg(\pi_* \mathcal{L}_{d,\mathbf{m}}) + rk(\pi_* \mathcal{L}_{d,\mathbf{m}}) \\ &= \sum_{k=1}^{R} a_k + d - m_j + 1\end{aligned}$$

In the key case $\chi(\mathcal{L}_{d,\mathbf{m}}) = 0$ we can therefore conclude

$$\sum_{k=1}^{R} a_k = -(d - m_j - 1) \tag{4.1}$$

In particular this last equality would be satisfied if

$$\forall k : a_k = -1 \tag{4.2}$$

This is what we have to prove to establish the Conjecture (in order for $\mathcal{L}_{d,\mathbf{m}}$ to have both cohomology group zero).

REMARK 4.1. In particular we would derive the Conjecture if we could prove that

$$h^1(\mathbb{P}_k^1, End(\pi_* \mathcal{L}_{d,\mathbf{m}})) = 0.$$

In general this expresses a condition of rigidity on the vector sheaf: the dimension of the first cohomology group of the endomorphism sheaf is the number of moduli of the deformation of a vector sheaf , or, in other words, the dimension of the tangent space to the local deformations. But, since moduli of vector sheaves on $\mathbb{P}^1_k$ are discreet (a consequence, for example, of Grothendieck's theorem), we can't use this vanishing to express geometrical constraints on our direct image.

From (4.1) follows that the slope of $\pi_*\mathcal{L}_{d,\mathbf{m}}$ is

$$\mu(\pi_*(\mathcal{L}_{d,\mathbf{m}})) = -1$$

As a dissocié vector sheaf is semistable if and only if it is direct sum of semistable subvector sheaves with the same slope (cf. e. g. [**sL75**]) we can reformulate Hirschowit's Conjecture as follows:

CONJECTURE 4.2. *An invertible sheaf $\mathcal{L}_{d,\mathbf{m}}$ on X_r such that $d \geq m_1 + m_2 + m_3$ has the natural cohomology if the invertible sheaf $\mathcal{L}$ with $\chi(\mathcal{L}) = 0$ derived from $\mathcal{L}_{d,\mathbf{m}}$ as in the previous section has semistable direct image $\pi_*\mathcal{L}$.*

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