Fedosov's quantization of semisimple coadjoint orbits.

by

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Abstract

In this thesis, we consider Fedosov's quantization $(C^\infty(\mathcal{O})[[\hbar]], *)$ of the functions on a coadjoint orbit $\mathcal{O}$ of a Lie group G corresponding to the trivial cohomology class in $H^2(\mathcal{O}, \mathbb{R}[[\hbar]])$ and G-invariant torsion free symplectic connection on $\mathcal{O}$ (when it exists). We show that the map $\mathbf{g} \ni X \mapsto \frac{1}{\hbar}X \in \frac{1}{\hbar}C^\infty(\mathcal{O})[[\hbar]]$ is a homomorphism of Lie algebras, where we consider $\frac{1}{\hbar}C^\infty(\mathcal{O})[[\hbar]]$ as a Lie algebra with the bracket $[f,g] = f * g - g * f$, $f, g \in \frac{1}{\hbar}C^\infty(\mathcal{O})[[\hbar]]$. This map defines an algebra homomorphism μ from $\mathrm{U}(\mathbf{g})((\hbar))$ to the algebra $C^\infty(\mathcal{O})((\hbar))$. We construct a representation of $C^\infty(\mathcal{O})((\hbar))$. In case G is a simple Lie group and $\mathcal{O}$ is a semisimple coadjoint orbit, we show that μ maps $\mathrm{U}(\mathbf{g})((\hbar))$ onto the algebra of polynomial functions on $\mathcal{O}$ endowed with a star product. In particular, we compute the infinitesimal character of the corresponding representation of $\mathrm{U}(\mathbf{g})((\hbar))$. Also, we describe the corresponding version of the above results over the ring $\mathbb{R}[[\hbar]]$.

Thesis Supervisor: Bertram Kostant
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Contents

Chapter 1

Introduction

This thesis concerns with the equivariant Fedosov's quantization of the semisimple coadjoint orbits. In order to motivate this study we briefly state the history of the deformation quantization.

The problem of the existence of deformation quantization of an arbitrary Poisson manifold is an old one, it probably dates back to the beginning of this century. In the early eighties it was solved for the symplectic manifolds by De Wilde and Lecompte (see [4]). They proved that for any symplectic manifold there exists a star product such that $f * g = fg + \frac{\hbar}{2}\{f, g\} + \mathrm{O}(\hbar^2)$, where $\{f, g\}$ is a Poisson bracket of the functions f and g. The big disadvantage of the results of Lecompte and De Wilde (see [4],[5]) is that their proof of existence is not explicit. They showed that all the cohomological abstractions vanish although the cohomology groups in which they lie may be non-zero.

In the middle of eighties B. Fedosov gave completely different prove of the same theorem (see [7]). First of all, his proof is simpler and has a big advantage, since he gave an explicit procedure for constructinf a star product (see [6]). Starting with a torsion free symplectic connection ∇, he constructed a canonical connection ∇^F on the vector bundle $W = \prod_{n=0}^{\infty} \mathrm{S}^n(T^*M)[[\hbar]]$. In general, one needs a flat connection on W which satisfies two additional properties. Such connection is called Fedosov's connection. Although he gave an explicit construction, it is very seldom possible to write down explicit formulas of a star product using this construction. Even a Fedosov's connection can be written explicitly only in few cases (see [13]). The canonical Fedosov's connection is not known except in the trivial cases.

There are many other papers which are devoted to the problem of existence of deformation quantization (see [12], [8], [9], [2], ...). I would like to mention here that the general problem of existence of formal deformation quantization of an arbitrary Poisson manifold is still open.

In this thesis we are trying to understand Fedosov's construction in the case of semisimple coadjoint orbits. Although we can not explicitly compute the star product, we get some results concerning the algebra structure of the algebra of polynomial functions endowed with star product. Below we briefly state a couple of the results obtained here and then we give a description of the contents section by section.

Every semisimple coadjoint orbit admits a canonical G-invariant torsion free symplectic connection. We apply Fedosov's construction to this case and we get a G-invariant star product on functions. Moreover, the space of polynomial functions on a semisimple coadjoint orbit $\mathcal{O}$ is closed under a G-invariant star product. As $\mathbf{R}((\hbar))$ algebra, the algebra $\mathcal{A}$ of polynomial functions endowed with the star product is generated by the linear polynomials on $\mathcal{O}$. One of the main results of this thesis answers to the following question: "What are the relations between the generators of the algebra $\mathcal{A}$?" We give another description of the algebra $\mathcal{A}$ of polynomial functions on a semisimple orbit $\mathcal{O}$ as a quotient of the universal enveloping algebra of the Lie algebra $\mathbf{g}((\hbar))$ over the field $\mathbf{R}((\hbar))$ by some primitive ideal $\mathcal{I}_{\mathcal{O}}$. If we complexify the algebra $\mathcal{A}$ then we get that the primitive ideal $\mathcal{I}_{\mathcal{O}} \otimes_{\mathbf{R}((\hbar))} \mathbf{C}((\hbar))$ is the annihilator of the Verma module with the highest weight $\frac{\lambda}{\hbar} - \rho_{\mathbf{u}_+}$.

We briefly describe the contents of this thesis, section by section.

In section 2.1 we give a brief introduction to Fedosov's construction and we state all necessary results from Fedosov's papers.

In section 2.2 we analyze the Fedosov's construction in the equivariant setting. Let μ be a moment map. We prove that in many cases (see theorem 2.2.2) the map $\frac{\mu}{\hbar}$ is a homomorphism from the Lie algebra $\mathbf{g}$ to the Lie algebra of quantized function with the commutator $[f, g] = f * g - g * f$.

In section 3.1 we construct some representations of the quantized algebra of functions.

In section 3.2 we analyze the structure of a particular representation V. We construct a subrepresentation $\widetilde{V}$ of V and show that $\widetilde{V} \otimes_{\mathbf{C}[[v]]} \mathbf{C}((v))$ is a generalized Verma module for $\mathbf{g}_{\mathbf{C}}((v))$ (where v is a square root of $\hbar$). We calculate the highest weight of this generalized Verma module.

In section 3.3 we show that the map from $\mathrm{U}(\mathbf{g}) \otimes_{\mathbf{R}} \mathbf{C}((v))$ to the polynomial functions is surjective and calculate the kernel of this map. The theorem 3.3.1 contains the main results concerning this map.

In section 3.4 we analyze another representations of the quantized algebra of polynomial function and we show that they are irreducible quotients of Verma modules.

In section 4.1 we consider an example of the coadjoint orbits of $\mathrm{SU}(2)$ which shows that the canonical Fedosov's connection is non-trivial even in that case.

In section 4.2 we consider an example of a minimal nilpotent orbit of a real symplectic group $\mathrm{Sp}(2n, \mathbf{R})$.

Most of the results of this thesis will appear in [1].

Chapter 2

G-equivariant Fedosov's quantization.

2.1 Recollection of Fedosov's quantization.

2.1.1 Local picture.

Let V be a $2n$ dimensional real vector space and ω be a symplectic form on it. We can consider the corresponding Heisenberg algebra $\mathcal{H}$ - a one dimensional central extension of the trivial Lie algebra V given by the cocycle ω. As a vector space $\mathcal{H} = V \oplus \mathbf{R}\hbar$. Moreover we can define a grading on $\mathcal{H}$.

Definition 2.1.1 *The degree of any vector $v \in V$ is 1, and the degree of $\hbar$ is 2.*

It is easy to see that $\mathcal{H}$ becomes a graded Lie algebra. Let us consider the universal enveloping algebra $\mathrm{U}(\mathcal{H})$. As a vector space it is canonically isomorphic to $\bigoplus_{k=0}^{\infty} \mathrm{S}^k\mathcal{H}$. We denote the latter one by A. A is a graded vector space, since $\mathcal{H}$ is a graded vector space. Multiplication in $\mathrm{U}(\mathcal{H})$ induces a $*$ multiplication on A. Thus, A becomes a non-commutative algebra. It is obvious that the grading respects $*$ product. Let $\mathbf{m}$ be a maximal ideal generated by the elements of degree greater than 0. Denote by $\widehat{A}$

the completion of A in $\mathbf{m}$-adic topology.

$$\widehat{A} = \lim_{\leftarrow} A/\mathbf{m}^k A. \tag{2.1}$$

Algebra $\widehat{A}$ has a natural filtration $\widehat{A}_k = \mathbf{m}^k \widehat{A}$. $\widehat{A}$ is a module over the commutative ring $\mathbf{R}[[\hbar]]$, since $\hbar$ lies in the center of the Lie algebra $\mathcal{H}$. Let us notice that as a vector space $\widehat{A}$ is canonically isomorphic to $\prod_{k=0}^{\infty} \mathrm{S}^k(V)[[\hbar]]$.

Symplectic group $Sp(V)$ acts on the vector space $\widehat{A}$ and it is easy to see that the multiplication is invariant under this action. The space $\frac{1}{\hbar}\widehat{A}$ becomes a Lie algebra under the bracket $[f,g] = f * g - g * f$. The bracket is $\mathbf{R}[[\hbar]]$ linear. We have a map from $\frac{1}{\hbar}\widehat{A}$ to a Lie algebra of derivations $\mathrm{Der}(\widehat{A})$ of the algebra $\widehat{A}$. This map is just an adjoint action $\mathrm{ad} : \frac{1}{\hbar}\widehat{A} \to \mathrm{Der}(\widehat{A})$. Note that this map is surjective. Since the group $Sp(V)$ acts on $\widehat{A}$ we have a map σ from the Lie algebra $sp(V)$ to $\mathrm{Der}(\widehat{A})$. On the other hand we have a natural map ρ from $sp(V)$ to $\frac{1}{\hbar}\widehat{A}$ which is given by

$$\rho \, : sp(V) \to \{\frac{1}{\hbar} \cdot quadratic\ forms\} \subset \frac{1}{\hbar}\widehat{A}.$$

It is obvious that $\sigma = \mathrm{ad} \circ \rho$.

Finally, what we got is a Harish-Chandra pair $(Sp(V), \frac{1}{\hbar}\widehat{A}, \rho)$.

Remark 2.1.1 *Here is another description of the algebra $\widehat{A}$. Form ω induces a symplectic form $\widetilde{\omega}$ on V^*. Choose a basis p_i, q_i $(1 \le i \le n)$ in V such that $\widetilde{\omega} = \sum_{i=1}^{n} dp_i \wedge dq_i$. Then $\widehat{A}$ as a vector space is isomorphic to $\mathbf{R}[[(p_i), (q_i), \hbar]]$. The star product of two functions $f, g \in \mathbf{R}[[(p_i), (q_i), \hbar]]$ is given by the following formula:*

$$f * g = \exp\left(\frac{\hbar}{2}\sum_{i=1}^{n}(\partial_{p'_i}\partial_{q''_i} - \partial_{q'_i}\partial_{p''_i})\right)[f(p', q', \hbar)g(p'', q'', \hbar)]|_{\substack{p' = p'' \\ q' = q''}} \tag{2.2}$$

2.1.2 Global picture.

We are given a manifold M $(\dim M = 2n)$ with a symplectic form ω. We also have a torsion free symplectic connection $\widetilde{\nabla}$ on the tangent bundle TM. $\widetilde{\nabla}$ induces connection ∇ on the cotangent bundle T^*M. Denote by R the curvature tensor of the connection ∇. The form ω induces a symplectic structure on the fibers of T^*M. Each vector space T_p^*M inherits a symplectic form. Therefore, we obtain a principle $Sp(2n, \mathbf{R})$ bundle P over M together with a connection. In the section 2.1.1 we constructed an algebra $\widehat{A}$ which is a representation of $Sp(2n, \mathbf{R})$. With the principle $Sp(2n, \mathbf{R})$ bundle P and representation $\widehat{A}$ of $Sp(2n, \mathbf{R})$ we can associate a vector bundle $W = P \times_{Sp(2n,\mathbf{R})} \widehat{A}$. Since $\widehat{A}$ is an algebra and the product is $Sp(2n, \mathbf{R})$ invariant we obtain an algebra structure on W. We denote this product by $*$. Filtration $\widehat{A}_k$ on $\widehat{A}$ induces filtration W_k on W. Connection on P induces a connection on W. By abuse of notation we will denote it by ∇. Let us look closer at the vector bundle W. As a vector bundle it is canonically isomorphic to

$$\prod_{k=0}^{\infty} \mathrm{S}^k(T^*M)[[\hbar]] \cong W. \tag{2.3}$$

The connection ∇ is the natural connection on $\prod_{k=0}^{\infty} \mathrm{S}^k(T^*M)[[\hbar]]$ induced by the connection ∇ on T^*M.

Consider $W \otimes \Lambda^\bullet(T^*M)$. Let us introduce two operators on $W \otimes \Lambda^\bullet(T^*M)$. To define them we use the canonical identification of W with $\prod_{k=0}^{\infty} \mathrm{S}^k(T^*M)[[\hbar]]$. First, consider a canonical element in $\eta \in TM \otimes T^*M$ which corresponds to the identity morphism. We can write η (locally) as

$$\eta = \sum_i v_i \otimes \nu_i, \quad \text{where} \quad v_i \in \Gamma(TM) \text{ and } \nu_i \in \Gamma(T^*M). \tag{2.4}$$

Second, using the identification of W with $\prod_{k=0}^{\infty} \mathrm{S}^k(T^*M)[[\hbar]]$ we define an operator $\delta : W \otimes \Lambda^\bullet(T^*M) \to W \otimes \Lambda^\bullet(T^*M)$.

Definition 2.1.2 *If* $a \in W \otimes \Lambda^\bullet(T^*M)$, *equals* $a = \sum_i a_i \otimes \mu_i$, *where* $\mu_i \in \Lambda^\bullet(T^*M)$ *and* $a_i \in \prod_{k=0}^{\infty} S^k(T^*M)[[\hbar]]$ *then*

$$\delta(a) = \sum_{k,l} (\imath(v_k)a_l) \otimes (\nu_k \wedge \mu_l), \tag{2.5}$$

where $\imath(v_k)a_l$ *denotes contraction of* v_k *with* a_l.

We would like to define an operator δ^{-1}. Let us define it first on the space $\sum_{k=0}^{\infty} \mathrm{S}^k(T^*M)[[\hbar]] \otimes \Lambda^\bullet(T^*M)$ Then, we will see that this operator is continuous and it can be extended to the whole space $\prod_{k=0}^{\infty} \mathrm{S}^k(T^*M)[[\hbar]] \otimes \Lambda^\bullet(T^*M)$. The space $\sum_{k=0}^{\infty} \mathrm{S}^k(T^*M)[[\hbar]] \otimes \Lambda^\bullet(T^*M)$ has a natural grading by the sum of the degrees. Thus, any element $a \in \sum_{k=0}^{\infty} \mathrm{S}^k(T^*M)[[\hbar]] \otimes \Lambda^\bullet(T^*M)$ can be written as a sum of the homogeneous components

$$a = \sum_k a_k \otimes \mu_k, \tag{2.6}$$

where a_k is a homogeneous element in $\sum_{k=0}^{\infty} \mathrm{S}^k(T^*M)[[\hbar]]$ and μ_k is a homogeneous differential form.

Definition 2.1.3 *We define δ^{-1} by*

$$\delta^{-1}(a) = \sum_{k,l} \frac{1}{\deg(a_l) + \deg(\mu_l)} \nu_k \cdot a_l \otimes \imath(v_k)\mu_l, \tag{2.7}$$

where $\cdot$ denotes the usual multiplication in the symmetric algebra $\sum_{k=0}^{\infty} S^k(T^*M)[[\hbar]]$.

Let us introduce one more notation. When $a \in W$, denote by a_0 the function

$$a_0 \stackrel{def}{=} a - \left(\delta \circ \delta^{-1}(a) + \delta^{-1} \circ \delta(a)\right) \in C^\infty(M)[[\hbar]]. \tag{2.8}$$

This gives us a map from the sections of W to the functions $C^\infty(M)[[\hbar]]$. We denote this map by ϕ.

$$\phi(a) = a_0, \qquad \text{where} \;\; a \in W. \tag{2.9}$$

2.1.3 Fedosov's connection.

It is well know that the quantizations of the Poisson algebra of functions $C^\infty(M)$ on a symplectic manifold M are parametrized by two dimensional cohomology group $H^2(M, \mathbf{R}[[\hbar]])$. Assume we have an element of $H^2(M, \mathbf{R}[[\hbar]])$ represented by the two form $\nu \in \Gamma(\Lambda^2(T^*M) \otimes \mathbf{R}[[\hbar]])$.

Definition 2.1.4 *Fedosov's connection is a connection, which satisfies two properties*

- *It can be written in the form*

$$\nabla^F = -\delta + \nabla + \left[\frac{1}{\hbar}\mathbf{r}, \cdot\right], \quad \textit{where} \;\; \mathbf{r} \in W_3 \otimes \Lambda^1(T^*M). \tag{2.10}$$

- *The curvature of* ∇^F *is equal to* $\frac{1}{\hbar}\omega + \nu$.

The curvature Ω of ∇^F is given by

$$\Omega = \frac{1}{\hbar}\omega + R + \frac{1}{\hbar}\left(-\delta\mathbf{r} + \nabla\mathbf{r} + \frac{1}{\hbar}\mathbf{r}^2\right). \tag{2.11}$$

Thus, one would like to solve the equation

$$R + \frac{1}{\hbar}\left(-\delta\mathbf{r} + \nabla\mathbf{r} + \frac{1}{\hbar}\mathbf{r}^2\right) = \nu. \tag{2.12}$$

The theorem of Fedosov states

Theorem 2.1.1 (Fedosov) *(see [6]) Equation (2.12) has a unique solution, satisfying the conditions*

$$\begin{gathered} \mathbf{r} \in W_3 \otimes \Lambda^1(T^*M), \\ \delta^{-1}\mathbf{r} = 0. \end{gathered} \tag{2.13}$$

Moreover, $\mathbf{r}$ satisfies the equation

$$\mathbf{r} = \delta^{-1}(R - \nu) + \delta^{-1}\left(\nabla\mathbf{r} + \frac{1}{\hbar}\mathbf{r}^2\right). \tag{2.14}$$

It is easy to see that one can solve equation (2.14) by the iteration method. The uniqueness of the solution is also obvious.

Theorem 2.1.2 (Fedosov) *(see [6]) The flat sections of* ∇^F *are in one to one correspondence with the smooth functions on* M*. It is given by taking the zero's component of the section*

$$W \ni a \mapsto a_0 \in C^\infty(M)[[\hbar]]. \tag{2.15}$$

Denote the map in the opposite direction by σ. We can define a star product of two functions $f, g \in C^\infty(M)[[\hbar]]$ by

$$f * g \stackrel{def}{=} (\sigma(f) * \sigma(g))_0. \tag{2.16}$$

We denote by $\mathcal{A}$ the algebra of functions $C^\infty(M)[[\hbar]]$ endowed with the star product. By definition, it is isomorphic to the algebra of flat sections of W.

2.2 Equivariant Fedosov quantization.

Suppose G is a Lie group acting on a symplectic manifold (M, ω). Assume that we have a G-invariant torsion free symplectic connection $\widetilde{\nabla}$ on TM. It induces G-invariant connection ∇ on T^*M. Let $\nu \in \Gamma(\Lambda^2(T^*M) \otimes \mathbf{R}[[\hbar]])$ be a G-invariant closed two-form. We construct Fedosov's connection ∇^F as in section 2.1.3.

Lemma 2.2.1 *Fedosov's connection ∇^F is G-invariant.*

Proof: This is obvious, since the curvature R and the operator δ^{-1} are G-invariant, and we obtain $\mathbf{r}$ by iterations of the equation (2.14). □

From now and for all we assume that ∇^F is a G-invariant Fedosov's connection such that $(\mathbf{r})_0 = 0$ (the latter can be easily assumed since the subtraction of $(\mathbf{r})_0 = 0$ from $\mathbf{r}$ does not change connection on W.

2.2.1 G action on $\mathcal{A}$.

G acts in a natural way on the vector bundle W. One can see that the map ϕ : $W \to C^\infty(M)[[\hbar]]$ is G-invariant. The star product on W is G-invariant, since the symplectic form ω is G-invariant. The action of G on W maps the sections which are flat with respect to ∇^F to the flat sections, because ∇^F is G-invariant. Thus, the star product on functions is G-invariant, since the star product on W is G-invariant and ϕ is a G-invariant map.

The Lie algebra $\mathbf{g}$ of G acts by derivations on the algebra $\mathcal{A}$. If $H^1(M, \mathbf{R}) = 0$ then all $\mathbf{R}[[\hbar]]$-linear derivations of the algebra $\mathcal{A}$ (we denote them by $\mathrm{Der}(\mathcal{A})$) are inner in the sense that the map

$$\mathrm{ad}: \frac{1}{\hbar}\mathcal{A} \to \mathrm{Der}(\mathcal{A}) \tag{2.17}$$

is surjective. The kernel of ad is $\frac{1}{\hbar}\mathbf{R}[[\hbar]]$. Thus, we get a map from the Lie algebra $\mathbf{g}$ to the Lie algebra $\frac{1}{\hbar}\left(\mathcal{A}/\mathbf{R}[[\hbar]]\right)$.

Let X be an element of the Lie algebra $\mathbf{g}$. By abuse of notation let us denote by the same letter X the corresponding vector field on M (G acts on M). By L_X we denote the Lie derivative along the vector field X on M. The action of the element $X \in \mathbf{g}$ on W is given by L_X. L_X is a derivation of the algebra W. Consider the following derivation of W

$$A_X \stackrel{def}{=} L_X - \nabla^F_X. \tag{2.18}$$

First, A_X is a tensor. Second, A_X commutes with ∇^F. Indeed, let Z be any vector field on M, then

$$\begin{aligned}\left[\nabla^F_Z, A_X\right] = \left[\nabla^F_Z, L_X - \nabla^F_X\right] = \left[\nabla^F_Z, L_X\right] - \left[\nabla^F_Z, \nabla^F_X\right] = \\ = -\nabla^F_{[X,Z]} - \nabla^F_{[Z,X]} = 0.\end{aligned} \tag{2.19}$$

Let X, Y be elements of $\mathbf{g}$. Easy calculation shows that in $\mathrm{Der}(W)$

$$[A_X, A_Y] = \left[L_X - \nabla^F_X, L_Y - \nabla^F_Y\right] = L_{[X,Y]} - \nabla^F_{[X,Y]} = A_{[X,Y]}. \tag{2.20}$$

Third, A_X can be considered as a section of $\frac{1}{\hbar}W$, since A_X is a tensor and all $C^\infty(M)[[\hbar]]$ derivations of W are inner. Certainly, as a section of $\frac{1}{\hbar}W$ A_X is not uniquely determined. We can always add an element of W which lies in the center of it, *i.e.*, any function $f \in \frac{1}{\hbar}C^\infty(M)[[\hbar]]$. A_X is uniquely determined as a section of $\frac{1}{\hbar}W_1$.

Lemma 2.2.2 *Assume that $H^1(M, \mathbf{R}) = 0$. Up to addition of a constant there is a* *unique flat section* $\widehat{A}_X \in \frac{1}{\hbar}W$ *such that*

$$\mathrm{ad}(\widehat{A}_X) = A_X \tag{2.21}$$

Proof: The uniqueness is obvious. Therefore, we must prove only the existence of it. Consider A_X as a section of $\frac{1}{\hbar}W_1$. We must find a function $f \in \frac{1}{\hbar}C^\infty(M)[[\hbar]]$ which is a solution to the equation

$$\nabla^F(f + A_X) = 0. \tag{2.22}$$

This equation is equivalent to the following one

$$\mathrm{d}(f) = -\nabla^F(A_X). \tag{2.23}$$

Since $H^1(M, \mathbf{R}) = 0$, it is enough to show that $\mathrm{d}(\nabla^F(A_X)) = 0$. $\mathrm{d}(\nabla^F(A_X)) = \nabla^F(\nabla^F(A_X))$ since $\nabla^F(A_X)$ lies in the center of $\frac{1}{\hbar}W$, *i.e.*, $\frac{1}{\hbar}C^\infty(M)[[\hbar]]$ (because A_X commutes with ∇^F). $\nabla^F(\nabla^F(A_X)) = 0$ since ∇^F is a flat connection. □

As an immediate corollary we get that up to addition of a constant there is a unique function $h_X \in \frac{1}{\hbar}\mathcal{A}$ which corresponds to the flat section $\widehat{A}_X \in \frac{1}{\hbar}W$. Let us summarize the results above.

Theorem 2.2.1 (B. Kostant) *(see [10])*
(a) The map $\mathbf{g} \to \mathrm{Der}(W)$ *given by* $\mathbf{g} \ni X \mapsto A_X \in \frac{1}{\hbar}W_1$ *is a homomorphism of Lie algebras.* A_X *commutes with Fedosov's connection* ∇^F.
(b) Assume that $H^1(M, \mathbf{R}) = 0$. *The map* $\mathbf{g} \ni X \mapsto h_X \in \frac{1}{\hbar}\mathcal{A}$ *defines a central extension of the Lie algebra* $\mathbf{g}$ *by* $\mathbf{R}[[\hbar]]$. *In particular, if* $H^2(\mathbf{g}, \mathbf{R}) = 0$, *there exists a Lie algebra homomorphism from* $\mathbf{g}$ *to* $\frac{1}{\hbar}\mathcal{A}$. *Moreover, if* $H^1(\mathbf{g}, \mathbf{R}) = 0$, *then it is unique.*

2.2.2 Relation with the moment map.

Assume that we have a moment map $\mu : M \to \mathbf{g}^*$. We denote by f_X the function which corresponds to the element $X \in \mathbf{g}$, *i.e.*, $f_X = \mu^*(X)$. The natural question is whether there exists a relation between functions h_X an f_X. We will see that if the form $\nu = 0$ then in many interesting cases we can choose $h_X = \frac{1}{\hbar} f_X$.

Proposition 2.2.1 *Let $\nu = 0$ and $X \in [\mathbf{g}, \mathbf{g}]$. If we consider A_X as a section of $\frac{1}{\hbar} W_1$, then*

$$A_X = \frac{1}{\hbar}\left(\sigma(f_X) - f_X\right). \tag{2.24}$$

It means that we can choose h_X to be equal to $\frac{1}{\hbar} f_X$.

Proof: It is enough to show that if $Y, Z \in \mathbf{g}$ then $A_{[Y,Z]} = \sigma(f_{[Y,Z]}) - f_{[Y,Z]}$. We know that $A_{[Y,Z]} = [A_Y, A_Z] - ([A_Y, A_Z])_0$. First, let us show that $[A_Y, A_Z]$ is an honest flat section of $\frac{1}{\hbar} W$. Indeed,

$$\nabla^F\left([A_Y, A_Z]\right) = \left[\nabla^F(A_Y), A_Z\right] + \left[A_Y, \nabla^F(A_Z)\right]. \tag{2.25}$$

We have seen that $\nabla^F(A_Y)$ and $\nabla^F(A_Z)$ lie in the center of the Lie algebra $\frac{1}{\hbar} W$. We obtain that $\nabla^F\left([A_Y, A_Z]\right) = 0$ which proves that $[A_Y, A_Z]$ is an honest flat section of $\frac{1}{\hbar} W$. Therefore, it is enough to show that $([A_Y, A_Z])_0 = \frac{1}{\hbar} f_{[Y,Z]}$. Let us calculate explicitly $([A_Y, A_Z])_0$.

$$\begin{gathered} A_Y = L_Y - \nabla^F_Y = L_Y - \left(-\delta_Y + \nabla_Y + \tfrac{1}{\hbar}\mathbf{r}(Y)\right) = \\ = (L_Y - \nabla_Y) + \left(\delta_Y - \tfrac{1}{\hbar}\mathbf{r}(Y)\right), \quad \text{and} \\ A_Z = (L_Z - \nabla_Z) + \left(\delta_Z - \tfrac{1}{\hbar}\mathbf{r}(Z)\right) \end{gathered} \tag{2.26}$$

$L_Y - \nabla_Y$ is a $C^\infty(M)[[\hbar]]$ linear derivation of W. Moreover, it is represented by some section $\mathbf{a}_Y \in \frac{1}{\hbar}\mathrm{S}^2(T^*M)[[\hbar]] \subset \frac{1}{\hbar} W$. Similarly, $L_Z - \nabla_Z$ is represented by some section $\mathbf{a}_Z \in \frac{1}{\hbar}\mathrm{S}^2(T^*M)[[\hbar]] \subset \frac{1}{\hbar} W$. Note, that if $\mathbf{a} \in \frac{1}{\hbar}\mathrm{S}^2(T^*M)[[\hbar]] \subset \frac{1}{\hbar} W$ and s is any section of $\frac{1}{\hbar} W$, then

$$([\mathbf{a}, s])_0 = 0. \tag{2.27}$$

Therefore, we obtain

$$
\begin{aligned}
&([A_Y, A_Z])_0 = \\
&= \left(\left[(L_Y - \nabla_Y) + \left(\delta_Y - \tfrac{1}{\hbar}\mathbf{r}(Y)\right), (L_Z - \nabla_Z) + \left(\delta_Z - \tfrac{1}{\hbar}\mathbf{r}(Z)\right)\right]\right)_0 = \\
&= \left(\left[L_Y - \nabla_Y, (L_Z - \nabla_Z) + \left(\delta_Z - \tfrac{1}{\hbar}\mathbf{r}(Z)\right)\right]\right)_0 - \\
&- \left(\left[L_Z - \nabla_Z, \delta_Y - \tfrac{1}{\hbar}\mathbf{r}(Y)\right]\right)_0 + \left(\left[\delta_Y - \tfrac{1}{\hbar}\mathbf{r}(Y), \delta_Z - \tfrac{1}{\hbar}\mathbf{r}(Z)\right]\right)_0 = \\
&= \left(\left[\delta_Y - \tfrac{1}{\hbar}\mathbf{r}(Y), \delta_Z - \tfrac{1}{\hbar}\mathbf{r}(Z)\right]\right)_0
\end{aligned} \tag{2.28}
$$

Easy, straightforward calculations show

$$
\begin{aligned}
&[\delta_Y, \delta_Z] = \tfrac{1}{\hbar}\omega(Y, Z), \\
&[\delta_Y, \mathbf{r}(Z)] + [\mathbf{r}(Y), \delta_Z] = (\delta\mathbf{r})(Y, Z), \quad \text{and} \\
&[\mathbf{r}(Y), \mathbf{r}(Z)] = \mathbf{r}^2(Y, Z).
\end{aligned} \tag{2.29}
$$

Thus, we get

$$
([A_Y, A_Z])_0 = \left(\frac{1}{\hbar}\omega(Y, Z) - \frac{1}{\hbar}(\delta\mathbf{r})(Y, Z) + \frac{1}{\hbar^2}\mathbf{r}^2(Y, Z)\right)_0 . \tag{2.30}
$$

It follows from the equation (2.12) that

$$
\begin{aligned}
&\left(-\tfrac{1}{\hbar}(\delta\mathbf{r})(Y, Z) + \tfrac{1}{\hbar^2}\mathbf{r}^2(Y, Z)\right)_0 = \\
&= \left(\nu(Y, Z) - R(Y, Z) - \tfrac{1}{\hbar}\left(\nabla\mathbf{r}\right)(Y, Z)\right)_0 = (\nu(Y, Z))_0 ,
\end{aligned} \tag{2.31}
$$

since $R_0 = 0$ and $(\nabla\mathbf{r})_0 = 0$ (this follows immediately from the assumption that $(\mathbf{r})_0 = 0$). Finally, we obtain

$$
([A_Y, A_Z])_0 = \frac{1}{\hbar}\omega(Y, Z) + \nu(Y, Z) = \frac{1}{\hbar}f_{[Y,Z]} + \nu(Y, Z). \text{ [1]} \tag{2.32}
$$

[1] see [10]

If $\nu = 0$ then $([A_Y, A_Z])_0 = \frac{1}{\hbar} f_{[Y,Z]}$. $\square$

Corollary 2.2.1 *If $H^1(\mathbf{g}, \mathbf{R}) = 0$ (i.e. $[\mathbf{g}, \mathbf{g}] = \mathbf{g}$) and $\nu = 0$ then the map $\frac{1}{\hbar}\sigma \circ \mu^* : \mathbf{g} \to \frac{1}{\hbar} W$ is a homomorphism of Lie algebras.*

Proof: Follows immediately from the proposition (2.2.1).

Let us see what happens in the case when $H^1(\mathbf{g}, \mathbf{R}) \neq 0$. Unfortunately, we can not prove in general case that the map $\frac{1}{\hbar}\mu^*$ is a homomorphism of Lie algebras. We can prove that only under additional assumption that the Lie algebra $\mathbf{g}$ acts transitively on the manifold M.

Proposition 2.2.2 *Assume that $\nu = 0$ and the Lie algebra $\mathbf{g}$ acts transitively on the manifold M. Then for any $X \in \mathbf{g}$*

$$A_X = \frac{1}{\hbar}\left(\sigma(f_X) - f_X\right). \tag{2.33}$$

Proof: The statement is local, therefore we can assume that there exists a flat section $\widehat{A}_X$ of $\frac{1}{\hbar}W$ such that $A_X = \widehat{A}_X - \left(\widehat{A}_X\right)_0$ for any $X \in \mathbf{g}$. To prove the proposition it is enough to show that $\left(\widehat{A}_X\right)_0 - \frac{1}{\hbar} f_X$ is a constant function. Since the Lie algebra $\mathbf{g}$ acts transitively on M it is enough to show that $Y\left(\left(\widehat{A}_X\right)_0 - \frac{1}{\hbar} f_X\right) = 0$ for any element $Y \in \mathbf{g}$.

$$Y\left(\left(\widehat{A}_X\right)_0 - \frac{1}{\hbar} f_X\right) = \left(L_Y\left(\widehat{A}_X - \frac{1}{\hbar}\sigma(f_X)\right)\right)_0. \tag{2.34}$$

Therefore, it is sufficient to prove that $\left(L_Y\left(\widehat{A}_X - \frac{1}{\hbar}\sigma(f_X)\right)\right)_0 = 0$. Let us calculate it.

$$\begin{aligned} \left(L_Y \widehat{A}_X\right)_0 &= \left(\left(L_Y - \nabla^F_Y\right)\widehat{A}_X\right)_0 = \left(\left[A_Y, \widehat{A}_X\right]\right)_0 = \\ &= ([A_Y, A_X])_0 = \tfrac{1}{\hbar} f_{[Y,X]} \end{aligned} \tag{2.35}$$

as follows from the proof of the proposition (2.2.1). On the other hand $(L_Y\sigma(f_X))_0 = Y(f_X) = f_{[Y,X]}$. Thus, we get that $Y\left(\left(\widehat{A}_X\right)_0 - \frac{1}{\hbar} f_X\right) = 0$. The proposition is proved. $\square$

Let us summarize the results.

Theorem 2.2.2 *Assume that $\nu = 0$ and we have a moment map $\mu : M \to \mathbf{g}^*$.*
(a) If $H^1(\mathbf{g}, \mathbf{R}) = 0$, then the map $\frac{1}{\hbar}\sigma \circ \mu^* : \mathbf{g} \to \frac{1}{\hbar}W$, $X \mapsto \frac{1}{\hbar}\sigma(f_X)$, is a homomorphism of Lie algebras.
(b) If the Lie algebra $\mathbf{g}$ acts transitively on M, then the map $\frac{1}{\hbar}\sigma \circ \mu^* : \mathbf{g} \to \frac{1}{\hbar}W$, $X \mapsto \frac{1}{\hbar}\sigma(f_X)$, is a homomorphism of Lie algebras.

Corollary 2.2.2 *Let $\mathcal{O}$ be a coadjoint orbit of a Lie group G. Assume that there exists a G invariant connection on $\mathcal{O}$, then, we can choose a G invariant torsion free symplectic connection on $\mathcal{O}$. Consider Fedosov's quantization corresponding to the zero cohomology class in $H^2(M, \mathbf{R}[[\hbar]])$, $\nu = 0$. Then the map $\mathbf{g} \ni X \mapsto \frac{1}{\hbar}\sigma(f_X) \in \frac{1}{\hbar}W$ (we can consider X as a function on $\mathcal{O}$, we denote this function by f_X) is a homomorphism of Lie algebras.*

Chapter 3

On the Fedosov's quantization of semisimple coadjoint orbits.

3.1 Representations associated with Fedosov's quantization.

In this section we are going to describe some representations of the algebra $\mathcal{A}$ which we will use later. Let V be a representation of the algebra $\mathcal{A}$. Let us define a support of V. We denote it by $\operatorname{supp}(V)$. We will describe the complement of $\operatorname{supp}(V)$ in M.

Definition 3.1.1 $x \in (M - \operatorname{supp}(V))$ *if and only if there exist some open neighborhood* $U \ni x$ *such that for any function* $f \in \mathcal{A}$ *with support in* U *and for any element* $v \in V$

$$f(v) = 0.$$

Let us choose some point $m \in M$. It is easy to see that the representations of $\mathcal{A}$ with support m are in one to one correspondence with the representations of $\widehat{A}$.

Consider Heisenberg algebra $\mathcal{H} = T^*_m M \oplus \mathbf{R}c$ - one dimensional central extension of the trivial Lie algebra $T^*_m M$ given by the cocycle ω. One can (canonically) associate to an irreducible representation (π, V) of $\mathcal{H}$ (on which c acts by 1) an irreducible representation of $\mathcal{A}$. This construction is due to Bertram Kostant (see [11]).

Construction: Let us choose a square root v of $\hbar$, *i.e.*, $v^2 = \hbar$. Consider the fiber at point $m \in M$ of the vector bundle $W \otimes_{\mathbf{R}[[\hbar]]} \mathbf{R}((v))$. There is a canonical algebra homomorphism $\mathbf{U}(\mathcal{H}) \to \left(W \otimes_{\mathbf{R}[[\hbar]]} \mathbf{R}((v))\right)_m$ which is given by

$$\begin{aligned} X &\mapsto \tfrac{X}{v}, \quad \text{for } X \in T^*_m M \\ c &\mapsto 1. \end{aligned} \tag{3.1}$$

One can extend this homomorphism to a $\mathbf{R}((v))$ linear continuous homomorphism α from $\mathbf{U}(\mathcal{H})\widehat{\otimes}_{\mathbf{R}}\mathbf{R}((v)) = \mathbf{U}(\mathcal{H})((v))$ to $\left(W \otimes_{\mathbf{R}[[\hbar]]} \mathbf{R}((v))\right)_m$. It is easy to see that α is isomorphism.

Therefore, we get that $(\pi \circ \alpha^{-1}, V\widehat{\otimes}_{\mathbf{R}}\mathbf{R}((v)))$ is a representation of the algebra $\left(W \otimes_{\mathbf{R}[[\hbar]]} \mathbf{R}((v))\right)_m$. Moreover, it is easy to see that $(\pi \circ \alpha^{-1}, V\widehat{\otimes}_{\mathbf{R}}\mathbf{R}[[v]])$ is a representation of the algebra $\left(W \otimes_{\mathbf{R}[[\hbar]]} \mathbf{R}[[v]]\right)_m$.

There is an algebra homomorphism from $\mathcal{A} \otimes_{\mathbf{R}[[\hbar]]} \mathbf{R}[[v]]$ to $\left(W \otimes_{\mathbf{R}[[\hbar]]} \mathbf{R}[[v]]\right)_m$ given by restriction to the point $m \in M$ (we identified functions on M with the flat sections of the vector bundle W). Taking the composition we get a representation of the algebra $\mathcal{A} \otimes_{\mathbf{R}[[\hbar]]} \mathbf{R}[[v]]$.

Let us consider a particular example of the above construction which we will use in the future. Assume that we have two complementary lagrangian subspaces V_1, V_2 of $T^*_m M$. Algebra $\mathcal{A}$ will be acting on the space

$$V = \left(\bigoplus_{k=0}^{\infty} \mathrm{S}^k(V_1^*)\right)[[v]] = \left(\bigoplus_{k=0}^{\infty} \mathrm{S}^k(V_2)\right)[[v]]. \tag{3.2}$$

Let us describe the action of the algebra W_m (the fiber of the vector bundle W at point m) on V. Using the form ω we can identify V_2 with V_1^*. If $X \in V_2$ and $z \in V$ then we define action of X on z as product of functions

$$X(v) = vX\dot{z}.$$

If $Y \in V_1$ and $z \in V$ then we define action of Y on V as a derivative in Y direction multiplied by v, *i.e.*,

$$Y(v) = v\frac{\partial z}{\partial Y}.$$

As was stated above the algebra $\mathcal{A}$ acts on V via evaluation of the section at point $m \in M$. The module V has a distinguished element $\mathbf{1} \in \mathrm{S}^0(V_1^*) \subset \left(\bigoplus_{k=0}^{\infty} \mathrm{S}^k(V_1^*)\right)[[v]]$. It is obvious that the representation V is reducible. It becomes topologically irreducible if we tensor $\mathcal{A}$ and V by $\mathbf{R}((v))$ over $\mathbf{R}[[v]]$.

Representation V is a faithful representation of the algebra $W_m \otimes_{\mathbf{R}[[\hbar]]} \mathbf{R}[[v]]$. This can be easily seen using the coordinates. Let us choose a basis $\{x_1, ..., x_k\}$ of the vector space V_2 and a dual basis $\{y_1, ..y_k\}$ of the vector space V_1, *i.e.*, $\omega(y_i, x_j) = \delta_{i,j}$ Then $V = \mathbf{R}[[x_1, ..., x_k]][[v]] = \mathbf{R}[[x_1, ..., x_k, v]]$, $W_m \otimes_{\mathbf{R}[[\hbar]]} \mathbf{R}[[v]]$ as a vector space is isomorphic to $\mathbf{R}[[\bar{x}_1, ..., \bar{x}_k, \bar{y}_1, ...\bar{y}_k, v]]$. $\bar{x}_i$ acts as a multiplication by vx_i and $\bar{y}_i$ acts as a derivation $v\frac{\partial}{\partial x_i}$. Now, the fact that the representation V is faithful became obvious. Moreover $V\hat{\otimes}_{\mathbf{R}[[\hbar]]}\mathbf{R}((v))$ is a faithful representation of $W_m \otimes_{\mathbf{R}[[\hbar]]} \mathbf{R}((v))$.

3.2 Fedosov's quantization of semisimple coadjoint orbits.

In this section we assume that the Lie group G is simple. The manifold M will be a semisimple coadjoint orbit $\mathcal{O}$ of the group G. First, let us summarize the necessary results about Lie groups and semisimple coadjoint orbits.

3.2.1 Some facts about semisimple coadjoint orbits.

Let us choose a point $\lambda \in \mathcal{O} \subset \mathbf{g}^*$. The stabilizer of the point λ in the Lie algebra $\mathbf{g}$ is a Levi subalgebra $\mathbf{l}$. Let $\mathbf{g_C}$ be a complexification of the Lie algebra $\mathbf{g}$ and $\mathbf{l_C}$ be a complexification of the Lie algebra $\mathbf{l}$. Let us choose $\mathbf{h} \subset \mathbf{l_C}$ a Cartan subalgebra of the Lie algebra $\mathbf{g_C}$ and a set of positive roots $\Delta_+ \subset \Delta$ with the property

$$\lambda \in \mathbf{h}^* \subset (\mathbf{n}_-^* \oplus \mathbf{h}^* \oplus \mathbf{n}_+^*) = \mathbf{g}_{\mathbf{C}}^*, \tag{3.3}$$

where $\mathbf{g_C} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$ is triangular decomposition of the Lie algebra $\mathbf{g_C}$. We have two parabolic subalgebras $\mathbf{p}_+ = \mathbf{l_C} + \mathbf{n}_+$ and $\mathbf{p}_- = \mathbf{l_C} + \mathbf{n}_-$. We can write

$$\mathbf{p}_+ = \mathbf{l_C} \oplus \mathbf{u}_+, \quad \mathbf{p}_- = \mathbf{l_C} \oplus \mathbf{u}_- \text{ and } \mathbf{g_C} = \mathbf{u}_- \oplus \mathbf{l_C} \oplus \mathbf{u}_+. \tag{3.4}$$

Denote by Σ the set of roots of $\mathbf{u}_+$. The set of roots of $\mathbf{u}_-$ is $-\Sigma$. Let us choose a basis $\{e_\alpha\}$, $\alpha \in \Sigma$ of $\mathbf{u}_+$ which respects the root decomposition. Similar, we can choose a basis $\{f_\alpha\}$, $\alpha \in \Sigma$ of $\mathbf{u}_-$. We denote by $\rho_\mathbf{u}$ the half sum of the roots in Σ.

Let us denote by K the coadjoint representation of the Lie algebra $\mathbf{g}$. By abuse of notation we also denote by K the coadjoint representation of $\mathbf{g_C}$. The tangent space to $\mathcal{O}$ at point λ can be canonically identified with $\mathbf{g}/\mathbf{l}$. For $X \in \mathbf{g}$ the corresponding tangent vector is $K(X)\lambda \in \mathbf{g}^*$. The cotangent space to $\mathcal{O}$ at point λ can be canonically identified with $\mathbf{g}/\mathbf{l}$ too. Any element of $\mathbf{g}$ is a 1-form on $\mathbf{g}^*$ and we can restrict it to the orbit $\mathcal{O}$. It is easy to see that if $Y \in \mathbf{l}$ and $X \in \mathbf{g}$ then $\langle Y, K(X)\lambda\rangle = -\langle X, K(Y)\lambda\rangle = 0$. The Kirillov-Kostant symplectic form on $\mathcal{O}$ is given by the formula $\omega|_\lambda(X,Y) = \langle [X,Y], \lambda\rangle$. The form ω induces a symplectic form $\widetilde{\omega}$ on the cotangent bundle, in particular on the vector space $T^*_\lambda\mathcal{O} = \mathbf{g}/\mathbf{l}$. It is given by

$$\widetilde{\omega}|_\lambda(X,Y) = \langle [X,Y], \lambda\rangle, \quad \text{where } X, Y \in \mathbf{g}. \tag{3.5}$$

Any semisimple orbit $\mathcal{O}$ admits two complementary G-invariant complex polarizations (non-unique). In order to describe them it's enough to to tell their restriction to point λ. Let us denote thses polarizations by E and F. We identify the complexification of $T_\lambda\mathcal{O}$ with $\mathbf{u}_+ \oplus \mathbf{u}_- = \mathbf{g_C}/\mathbf{l_C}$. We define $F|_\lambda$ as $\mathbf{u}_+$ and $E|_\lambda$ as $\mathbf{u}_-$ at point λ. The corresponding subalgebras of $\mathbf{g_C}$ are $\mathbf{p}_+$ and $\mathbf{p}_-$ respectively. Since the complexified tangent bundle is a direct sum of two bundles E and F the complexified cotangent bundle is a direct sum of two two bundles $\widetilde{E}$ and $\widetilde{F}$ (dual decomposition). We get $\widetilde{E}|_\lambda = \mathbf{u}_+$ and $\widetilde{F}|_\lambda = \mathbf{u}_-$.

3.2.2 An explicit formula for $\sigma(f_H)$, $H \in \mathbf{h}$

Consider $X \in \mathbf{l_C}$ as a function on $\mathcal{O}$. As before, we denote this function by f_X. We will calculate an explicit formula for the section $\sigma(f_X)|_\lambda$.

$$\sigma(f_X)|_\lambda = f_X(\lambda) + \hbar A_X|_\lambda. \tag{3.6}$$

We can calculate $A_X|_\lambda$ from the definition. As an operator $A_X = L_X - \nabla_X$. It is obvious that $\nabla_X|_\lambda = 0$ since $X_\lambda = 0$ $(K(X)\lambda = 0)$. First, let us compute the action of $L_X|_\lambda$ on $T^*_\lambda\mathcal{O} \otimes_{\mathbf{R}} \mathbf{C}$. The action is well defined since $X_\lambda = 0$. Let Y, Z be two elements of $\mathbf{g_C}$. We consider Y as a vector field and Z as 1-form. We have

$$\langle L_X(Z)|_\lambda, K(Y)\lambda\rangle = -\langle Z|_\lambda, K([X,Y])\lambda\rangle = \langle [X,Z], K(Y)\lambda\rangle. \tag{3.7}$$

Therefore, we obtain

$$L_X(Z)|_\lambda = [X,Z]|_\lambda. \tag{3.8}$$

The above formula shows that we have a homomorphism β from $\mathbf{l_C}$ to $\mathrm{sp}(T^*_\lambda\mathcal{O})$ and for any $Z \in \mathbf{g_C}$ 1-form (we consider $Z \in W$) $A_X(Z)|_\lambda = \beta(X)(Z)|_\lambda$. A_X is a derivation, therefore, it is true that for any section $s \in W$

$$A_X(s)|_\lambda = \beta(X)(s|_\lambda). \tag{3.9}$$

We have seen in section 2.1.1 that the action of $\mathrm{sp}(T^*_\lambda\mathcal{O} \otimes_{\mathbf{R}} \mathbf{C})$ on $W_\lambda \otimes_{\mathbf{R}} \mathbf{C}$ factorizes through the action of $\{\frac{1}{\hbar}\textit{quadratic forms}\} \subset \frac{1}{\hbar}W_\lambda \otimes_{\mathbf{R}} \mathbf{C}$. Therefore, we have

$$\sigma(f_X)_\lambda = f_X(\lambda) + \textit{some quadratic form} \tag{3.10}$$

and it is not hard to write down a formula for the corresponding quadratic form. We will do that for $H \in \mathbf{h}$.

As before, we identify $T^*_\lambda\mathcal{O} \otimes_{\mathbf{R}} \mathbf{C}$ with $\mathbf{g_C}/\mathbf{l_C} = \mathbf{u}_- \oplus \mathbf{u}_+$. Therefore, we have

$\beta(H)(e_\alpha) = \langle\alpha, H\rangle e_\alpha$ and $\beta(H)(f_\alpha) = -\langle\alpha, H\rangle f_\alpha$ for $\alpha \in \Sigma$. We get

$$A_X|_\lambda = \frac{1}{\hbar}\sum_{\alpha\in\Sigma}\frac{\langle\alpha, H\rangle e_\alpha f_\alpha}{\omega(f_\alpha, e_\alpha)}. \tag{3.11}$$

Finally, we obtain the formula for $\sigma(f_H)|_\lambda$

$$\sigma(f_H)|_\lambda = \langle\lambda, H\rangle + \sum_{\alpha\in\Sigma}\frac{\langle\alpha, H\rangle e_\alpha f_\alpha}{\omega(f_\alpha, e_\alpha)}. \tag{3.12}$$

Proposition 3.2.1 *For any $H \in \mathbf{h}$ we have the following formula for $\sigma(f_H)|_\lambda$*

$$\sigma(f_H)|_\lambda = \langle\lambda - \hbar\rho_{\mathbf{u}}, H\rangle + \sum_{\alpha\in\Sigma}\frac{\langle\alpha, H\rangle f_\alpha * e_\alpha}{\omega(f_\alpha, e_\alpha)}. \tag{3.13}$$

Proof: Follows immediately from the formula

$$f_\alpha * e_\alpha = f_\alpha e_\alpha + \frac{\hbar}{2}\omega(f_\alpha, e_\alpha) \tag{3.14}$$

and formula 3.12. □

3.2.3 Action of $\mathbf{h}$ on $W_\lambda \otimes_{\mathbf{R}} \mathbf{C}$.

From the corollary 2.2.2 follows that the map $\mathbf{g_C} \ni X \mapsto \frac{1}{\hbar}f_X \in W \otimes_{\mathbf{R}} \mathbf{C}$ is a homomorphism of Lie algebras. Therefore, the map $\mathbf{g_C} \ni X \mapsto \frac{1}{\hbar}f_X \in W_\lambda \otimes_{\mathbf{R}} \mathbf{C}$ - composition of the above map with the restriction is a homomorphism of Lie algebras. The main goal is to study this homomorphism. Here we will study the action of $\mathbf{h}$ on $W_\lambda \otimes_{\mathbf{R}} \mathbf{C}$.

Lemma 3.2.1 *We have a canonical isomorphism of vector spaces*

$$W_\lambda \otimes_{\mathbf{R}} \mathbf{C} = \mathbf{C}[[e_\alpha, f_\alpha, \hbar]], \quad \alpha \in \Sigma. \tag{3.15}$$

The star product is given by

$$a(e_\alpha, f_\alpha, \hbar) * b(e_\alpha, f_\alpha, \hbar) =$$

$$\exp\left(\tfrac{\hbar}{2}\textstyle\sum_{\alpha\in\Sigma}\omega(e_\alpha, f_\alpha)(\partial_{e'_\alpha}\partial_{f''_\alpha} - \partial_{f'_\alpha}\partial_{e''_\alpha})\right)[a(e'_\alpha, f'_\alpha, \hbar)b(e''_\alpha, f''_\alpha, \hbar)]|_{\substack{e'_\alpha = e''_\alpha \\ f'_\alpha = f''_\alpha}} \tag{3.16}$$

Proof: Obvious. □

The action of $\frac{1}{\hbar}\sigma(f_H)$ on $W_\lambda \otimes_{\mathbf{R}} \mathbf{C}$ coincides with the action of $A_H|_\lambda$. It was shown in section 3.2.2 that the action of $A_H|_\lambda$ on $W_\lambda \otimes_{\mathbf{R}} \mathbf{C}$ is given by $\beta(H)$, moreover, we have the following formulas

$$\beta(H)(e_\alpha) = \langle\alpha, H\rangle e_\alpha, \;\; \beta(H)(f_\alpha) = -\langle\alpha, H\rangle f_\alpha, \;\; \text{for } \alpha \in \Sigma. \tag{3.17}$$

The action of $A_H|_\lambda$ extends by derivation to the space $\mathbf{C}[[e_\alpha, f_\alpha, \hbar]]$.

Let us denote by L the root lattice of the Lie algebra $\mathbf{g}_{\mathbf{C}}$. Denote by L_+ a subsemigroup of L generated by the elements of Δ_+ and by L_- a subsemigroup of L generated by the elements of Δ_-. One have $L_- = -L_+$. Let α be an element of L. We denote by $W_\lambda^{(\alpha)}$ a subspace of $W_\lambda \otimes_{\mathbf{R}} \mathbf{C}$ which is an α eigenspace with respect to $\mathbf{h}$ action, *i.e.*, if $s \in W_\lambda^{(\alpha)}$ then $A_H|_\lambda(s) = \langle\alpha, H\rangle s$. The next proposition describes the structure of $W_\lambda \otimes_{\mathbf{R}} \mathbf{C}$.

Proposition 3.2.2 *We have an isomorphism*

$$W_\lambda \otimes_{\mathbf{R}} \mathbf{C} = \prod_{\alpha\in L} W_\lambda^{(\alpha)}. \tag{3.18}$$

Proof: Follows immediately from the isomorphism (3.15) and the fact that the space $\mathbf{C}[[e_\alpha, f_\alpha, \hbar]]$ is complete. □

Proposition 3.2.3 *For any $\alpha \in \Sigma$ we have*

$$\sigma(f_{e_\alpha}) \in W_\lambda^{(\alpha)}, \;\; \textit{and} \;\; \sigma(f_{f_\alpha}) \in W_\lambda^{(-\alpha)}. \tag{3.19}$$

Proof: It follows from the theorem 2.2.2 that for any $H \in \mathbf{h}$ we have $A_H(\frac{1}{\hbar}\sigma(f_{e_\alpha})) = \langle\alpha, H\rangle\frac{1}{\hbar}\sigma(f_{e_\alpha})$ and $A_H(\frac{1}{\hbar}\sigma(f_{f_\alpha})) = -\langle\alpha, H\rangle\frac{1}{\hbar}\sigma(f_{f_\alpha})$. This proves the proposition. □

3.2.4 Representation's structure.

In section 3.2.1 we described two complementary lagrangian subspaces of $T^*_\lambda\mathcal{O}$. We denote $\widetilde{E}_\lambda = \mathbf{u}_+$ by V_1 and $\widetilde{F}_\lambda = \mathbf{u}_-$ by V_2. Let us choose a square root of $\hbar$. We denote it by v. Applying the construction of section 3.1 we get a representation V of the algebra $W_\lambda \otimes_{\mathbf{R}[[\hbar]]} \mathbf{C}[[v]]$. As a vector space V is canonically isomorphic to $\left(\bigoplus_{k=0}^\infty \mathrm{S}^k(\mathbf{u}_-)\right)[[v]] = (\mathbf{C}[f_\alpha])[[v]]$, $\alpha \in \Sigma$ (see section 3.1).

Consider an action of $A_H|_\lambda$, $H \in \mathbf{h}$ on V. It is easy to see that

$$A_H|_\lambda(f_{\alpha_1}\cdots f_{\alpha_k}v^m) = -\langle\sum_{i=1}^k \alpha_i, H\rangle f_{\alpha_1}\cdots f_{\alpha_k}v^m. \tag{3.20}$$

Let us denote by $V^{(\alpha)}$ an eigenspace of the Lie algebra $\mathbf{h}$ ($H \in \mathbf{h}$ acts by $A_H|_\lambda$) on V with an eigenvalue $\alpha \in L_-$. It is obvious that $V^{(\alpha)}$ is a free module of finite rank over the ring $\mathbf{C}[[v]]$.

Lemma 3.2.2 *There exists a canonical isomorphism of vector spaces*

$$V = \prod_{\alpha \in L_-} V^{(\alpha)}. \tag{3.21}$$

Proof: Obvious. □

Lemma 3.2.3 *Let we have $x \in W_\lambda^{(\alpha)}$, $y \in W_\lambda^{(\beta)}$ and $z \in V^{(\gamma)}$. Then we have*

$$x * y \in W_\lambda^{(\alpha+\beta)} \quad \text{and} \quad x(z) \in V^{(\alpha+\gamma)}. \tag{3.22}$$

Proof: The statement immediately follows from the definitions and the fact that for any $H \in \mathbf{h}$

$$\langle\alpha, H\rangle x = A_H(x) = A_H|_\lambda * x - x * A_H|_\lambda. \tag{3.23}$$

□

Let us consider the following subspace $\widetilde{V}$ of V

$$\widetilde{V} \stackrel{def}{=} \sum_{\alpha \in L_-} V^{(\alpha)} \subset V. \tag{3.24}$$

Lemma 3.2.4 *We have*

$$\widetilde{V} = \left(\bigoplus_{k=0}^{\infty} S^k(\mathbf{u}_-) \right) \otimes_{\mathbf{C}} \mathbf{C}[[v]] = (\mathbf{C}[[v]])[f_\alpha], \;\; \alpha \in \Sigma. \tag{3.25}$$

Proof: Obvious. □

Proposition 3.2.4 *If $X \in \mathbf{g_C}$ then the action of $\sigma(f_X)$ on V preserves subspace $\widetilde{V}$, i.e.,*

$$\sigma(f_X)|_\lambda(\widetilde{V}) \subset \widetilde{V}, \;\; \textit{for any } X \in \mathbf{g_C}. \tag{3.26}$$

Proof: First, we have a decomposition

$$\widetilde{V} = \sum_{\alpha \in L_-} V^{(\alpha)}. \tag{3.27}$$

Therefore, it is enough to show that for any fixed $\alpha \in L_-$ and any $y \in V^{(\alpha)}$ $\sigma(f_X)|_\lambda(y) \in \widetilde{V}$.

Second, we can write X as a sum

$$X = X_\mathbf{l} + \sum_{\gamma \in \Sigma} (X_{-\gamma} + X_\gamma), \tag{3.28}$$

where X_γ is proportional to e_γ, $X_{-\gamma}$ is proportional to f_γ and $X_\mathbf{l} \in \mathbf{l}$. We have $\sigma(f_X) = \sum_{\gamma \in \Sigma} (\sigma(f_{X_{-\gamma}}) + \sigma(f_{X_\gamma})) + \sigma(f_{X_\mathbf{l}})$. Therefore, we can assume that X is either equal to e_γ, $\gamma \in \Sigma$ or f_γ, $\gamma \in \Sigma$ or $X \in \mathbf{l}$.

If $X \in \mathbf{l}$ then the fact that $\sigma(f_X)|_\lambda(y) \in \widetilde{V}$ immediately follows from the results in section 3.2.2.

Let us consider the other two cases. We assume now that X equals either e_γ or f_γ for some $\gamma \in \Sigma$. From the proposition 3.2.3 follows that $\sigma(f_X)|_\lambda \in W_\lambda^{(\delta)}$, where δ equals either γ or $-\gamma$.

Therefore, the statement follows from the lemma 3.2.3. □

Consider a $\mathbf{C}[[v]]$ subalgebra of $\left(\sum_{\alpha\in L} W_\lambda^{(\alpha)}\right) \otimes_{\mathbf{C}[[\hbar]]} \mathbf{C}((v))$ generated by the elements $\frac{\sigma(f_{f_\alpha})}{v}$, $\alpha \in \Sigma$. We denote it by B. The algebra $B \subset \left(\sum_{\alpha\in L} W_\lambda^{(\alpha)}\right) \otimes_{\mathbf{C}[[\hbar]]} \mathbf{C}((v))$ acts on $\widetilde{V} \otimes_{\mathbf{C}[[v]]} \mathbf{C}((v))$. There is a distinguished element $\mathbf{1} \in \widetilde{V} \subset \widetilde{V} \otimes_{\mathbf{C}[[v]]} \mathbf{C}((v))$.

Proposition 3.2.5 $\mathbf{C}[[v]]$ *algebra B preserves $\widetilde{V}$. Moreover, we have*

$$B(\mathbf{1}) = \widetilde{V}. \tag{3.29}$$

Proof: To prove the first statement it is enough to show that $\frac{\sigma(f_{f_\alpha})}{v}$ preserves $\widetilde{V}$. We know that $\sigma(f_{f_\alpha})$ preserves $\widetilde{V}$. Therefore, it is sufficient to show that for any $y \in \widetilde{V}$ $\sigma(f_{f_\alpha})(y) = 0 \bmod(v)$. This follows from the fact that $\langle\lambda, f_\alpha\rangle = 0$ and the formulas in section 3.1.

To prove the second statement we need more information about $\sigma(f_{f_\alpha})$.

Lemma 3.2.5 *Let f be a function on a symplectic manifold M. Consider Fedosov's quantization of the functions on M. We have*

$$\sigma(f) = f + d(f) \mod(W_2). \tag{3.30}$$

Proof: The statement follows immediately from the Fedosov's construction. □

From the lemma 3.2.5 we get that $\sigma(f_{f_\alpha})|_\lambda = f_\alpha \bmod((W_\lambda)_2)$. From the results in section 3.1 follow that if $x \in (W_\lambda)_2$ then $x(\widetilde{V}) = 0 \mod(v^2)$. Therefore, we obtain that for any $y \in \widetilde{V}$

$$\frac{\sigma(f_{f_\alpha})}{v}(y) = f_\alpha y \mod(v). \tag{3.31}$$

As a corollary we get that

$$(B/vB)(\mathbf{1}) = \widetilde{V}/v\widetilde{V}. \tag{3.32}$$

We define $B^{(\alpha)}$, $\alpha \in L_-$ as an intersection $B \cap \left(W_\lambda^{(\alpha)} \otimes_{\mathbf{C}[[v]]} \mathbf{C}((v))\right)$. Let us consider the following decompositions

$$B = \sum_{\alpha\in L_-} B^{(\alpha)} \text{ and } \widetilde{V} = \sum_{\alpha\in L_-} V^{(\alpha)}. \tag{3.33}$$

It follows from the formulas (3.32), (3.33), (3.22) that $B^{(\alpha)}(\mathbf{1}) \subset V^{(\alpha)}$ and $B^{(\alpha)}(\mathbf{1}) = V^{(\alpha)} \mod(v)$. Since $B^{(\alpha)}$ and $V^{(\alpha)}$ are free modules of finite rank over $\mathbf{C}[[v]]$ we get

$$B^{(\alpha)}(\mathbf{1}) = V^{(\alpha)}. \tag{3.34}$$

□

Let us denote by $\widetilde{W}$ an algebra $\sum_{\alpha\in L} W_\lambda^{(\alpha)} \otimes_{\mathbf{C}[[\hbar]]} \mathbf{C}((v))$. The map $\mathbf{g_C}\otimes_{\mathbf{C}}\mathbf{C}((v)) \ni Xr(v) \mapsto \frac{r(v)}{v^2}\sigma(f_X)|_\lambda \in \widetilde{W}$ is a homomorphism of Lie algebras (see corollary 2.2.2). We can extend it to a homomorphism from the universal enveloping algebra of $\mathbf{g_C}\otimes_{\mathbf{C}} \mathbf{C}((v))$ (we denote it by $\mathbf{U}(\mathbf{g_C}\otimes_{\mathbf{C}}\mathbf{C}((v))) = \mathbf{U}(\mathbf{g})\otimes_{\mathbf{R}}\mathbf{C}((v)))$ to $\widetilde{W}$. Denote by $\mathbf{C}((v))_{\frac{\lambda}{v^2}-\rho_{\mathbf{u}}}$ a one dimensional (over $\mathbf{C}((v))$) representation of $\mathbf{p}_+((v))$ given by the character $\frac{\lambda}{v^2}-\rho_{\mathbf{u}}$, *i.e.*, $\mathbf{u}_+((v))$ and $[\mathbf{l},\mathbf{l}]((v))$ acts by zero and any element $H \in \mathbf{h}((v))$ acts as a multiplication by $\langle \frac{\lambda}{v^2} - \rho_{\mathbf{u}}, H\rangle$.

Theorem 3.2.1 $\mathbf{g_C}\otimes_{\mathbf{C}}\mathbf{C}((v)) = \mathbf{g_C}((v))$ *module* $\tilde{V}\otimes_{\mathbf{C}[[v]]}\mathbf{C}((v))$ *is a generalized Verma module, i.e.,*

$$\tilde{V}\otimes_{\mathbf{C}[[v]]}\mathbf{C}((v)) = \mathrm{Ind}_{\mathbf{p}_+((v))}^{\mathbf{g_C}((v))}\mathbf{C}((v))_{\frac{\lambda}{v^2}-\rho_{\mathbf{u}}} = M_{\frac{\lambda}{v^2}-\rho_{\mathbf{u}}}. \tag{3.35}$$

Proof: First, from the proposition 3.2.5 follows that

$$\mathbf{U}(\mathbf{g})\otimes_{\mathbf{R}}\mathbf{C}((v))(\mathbf{1}) = \tilde{V}\otimes_{\mathbf{C}[[v]]}\mathbf{C}((v)), \tag{3.36}$$

i.e., $\tilde{V}\otimes_{\mathbf{C}[[v]]}\mathbf{C}((v))$ is generated by $\mathbf{1}$.

Second, from lemma 3.2.3 follows that $\mathbf{u}_+(\mathbf{1}) = 0$. Really, $\mathbf{u}_+$ has the basis e_α, $\alpha\in\Sigma$ and $\sigma(f_{e_\alpha})(\mathbf{1}) \subset W_\lambda^{(\alpha)}(V^{(0)}) \subset V^{(\alpha)} = 0$ since, $\alpha \notin L_-$.

Combining these two statements we obtain that $\tilde{V}\otimes_{\mathbf{C}[[v]]}\mathbf{C}((v))$ is a quotient of $M_{\frac{\lambda}{v^2}-\rho_{\mathbf{u}}}$. The theorem follows from the easy calculation which shows that

$$\dim_{\mathbf{C}((v))}\left(V^{(\alpha)}\otimes_{\mathbf{C}[[v]]}\mathbf{C}((v))\right) = \dim_{\mathbf{C}((v))} M^{(\alpha)}_{\frac{\lambda}{v^2}-\rho_{\mathbf{u}}}. \tag{3.37}$$

□

3.2.5 One useful proposition.

Let us state few facts here which we will need later. First, $\tilde{V} \otimes_{\mathbf{C}[[v]]} \mathbf{C}((v))$ is dense in $V \otimes_{\mathbf{C}[[v]]} \mathbf{C}((v))$ and $\widetilde{W}$ is dense in $W_\lambda \otimes_{\mathbf{R}[[\hbar]]} \mathbf{C}((v))$. Second, $\tilde{V} \otimes_{\mathbf{C}[[v]]} \mathbf{C}((v))$ is a faithful representation of $\widetilde{W}$. That can be easily deduce from the previous statement and results of section 3.1.

Let us denote by $\tilde{\tau}$ the homomorphism from $\mathbf{U}(\mathbf{g}) \otimes_{\mathbf{R}} \mathbf{C}((v))$ to $\widetilde{W}$. Denote by $\mathcal{I}$ an ideal annihilating representation $M_{\frac{\lambda}{v^2} - \rho_{\mathbf{u}}}$.

Proposition 3.2.6 *The kernel of the homomorphism $\tilde{\tau}$ is the ideal $\mathcal{I}$. We have an embedding of $\mathbf{U}(\mathbf{g}) \otimes_{\mathbf{R}} \mathbf{C}((v))/\mathcal{I}$ into $\widetilde{W}$.*

Proof: Proposition follows immediately from the fact that $\tilde{V} \otimes_{\mathbf{C}[[v]]} \mathbf{C}((v))$ is a faithful representation of $\widetilde{W}$. □

3.3 Continuation.

3.3.1 General nonsense about polynomial functions.

Let us consider an algebra of polynomial functions *Pol* on a semisimple coadjoint orbit $\mathcal{O}$. The Lie group G acts on functions on $\mathcal{O}$. Consider all functions which transforms finitely under the action of group G. They form an algebra. It is well know that the latter algebra coincides with the algebra of all polynomial functions on the semisimple orbit. One can deduce it from the facts that the orbit $\mathcal{O}$ is closed and the action of the group G is algebraic. Moreover,it is easy to see that every finite-dimensional representation of G has a finite multiplicity in *Pol.* We denote the set of all finite dimensional representations of the Lie group G by R.

Consider a G-invariant star product on the algebra of smooth functions on $\mathcal{O}$. It follows from the above remarks that the algebra $Pol[[\hbar]]$ is closed under the star product. Moreover, the algebra $Pol \otimes_{\mathbf{R}} \mathbf{R}[[\hbar]]$ is closed under the star product. Let us denote $Pol \otimes_{\mathbf{R}} \mathbf{R}[[\hbar]]$ by $\mathcal{P}$ and it's complexification by $\mathcal{P}_{\mathbf{C}}$. We have a natural map μ^* (moment map) from $\mathbf{g}$ to $\mathcal{P}$ which is embedding.

Proposition 3.3.1 *The $\mathbf{R}[[\hbar]]$ algebra $\mathcal{P}$ is generated by the image of the map μ^*, i.e., by linear functions on $\mathcal{O}$.*

Proof: Let us denote by A a $\mathbf{R}[[\hbar]]$ subalgebra generated by the image of μ^*. Since the star product is just a usual multiplication modulo $\hbar$, we have $A/\hbar A = \mathcal{P}/\hbar\mathcal{P}$. We can decompose A and $\mathcal{P}$ into irreducible representations of the group G. We have

$$A = \bigoplus_{\pi \in R} A_\pi \quad \text{and} \quad \mathcal{P} = \bigoplus_{\pi \in R} \mathcal{P}_\pi. \tag{3.38}$$

It is easy to see that $A_\pi/\hbar A_\pi = \mathcal{P}_\pi/\hbar\mathcal{P}_\pi$. Since $A_\pi \subset \mathcal{P}_\pi$ and $\mathcal{P}_\pi$ is a free $\mathbf{R}[[\hbar]]$ module of finite rank we get that $A_\pi = \mathcal{P}_\pi$. □

From now on we are going to consider G-equivariant Fedosov's quantization. We have a map τ from $\mathbf{g}((\hbar))$ to $\mathcal{P} \otimes_{\mathbf{R}[[\hbar]]} \mathbf{R}((\hbar))$ given by

$$Xr(\hbar) \mapsto \frac{r(\hbar)}{\hbar}\sigma(f_X), \tag{3.39}$$

where $X \in \mathbf{g}$ and $r(\hbar) \in \mathbf{R}((\hbar))$. It is a homomorphism of Lie algebras (see corollary 2.2.2). We can extend it to an algebra homomorphism τ from $\mathbf{U}(\mathbf{g}) \otimes_{\mathbf{R}} \mathbf{R}((\hbar))$ to $\mathcal{P} \otimes_{\mathbf{R}[[\hbar]]} \mathbf{R}((\hbar))$. Let us denote by ${}^\hbar\mathbf{U}(\mathbf{g})$ a $\mathbf{R}[[\hbar]]$ subalgebra of $\mathbf{U}(\mathbf{g}) \otimes_{\mathbf{R}} \mathbf{R}((\hbar))$ generated by the elements $\hbar X$, $X \in \mathbf{g}$. It is easy to see that the restriction of the map τ to ${}^\hbar\mathbf{U}(\mathbf{g})$ maps ${}^\hbar\mathbf{U}(\mathbf{g})$ to $\mathcal{P}$. We denote it by τ_r.

Proposition 3.3.2 *The map τ_r from ${}^\hbar\mathbf{U}(\mathbf{g})$ to $\mathcal{P}$ is onto. In particular, the map τ from $\mathbf{U}(\mathbf{g}) \otimes_{\mathbf{R}} \mathbf{R}((\hbar))$ to $\mathcal{P} \otimes_{\mathbf{R}[[\hbar]]} \mathbf{R}((\hbar))$ is onto.*

Proof: First, let us observe that modulo $\hbar$ the map τ_r is surjective, *i.e.*,

$${}^\hbar\mathbf{U}(\mathbf{g})/\hbar{}^\hbar\mathbf{U}(\mathbf{g}) \xrightarrow{\tau_r} \mathcal{P}/\hbar\mathcal{P} = Pol \tag{3.40}$$

is onto. This follows from the fact that the star product modulo $\hbar$ is just a usual multiplication.

Second, let us decompose $\mathcal{P}$ into irreducible representations of the group G. We can write

$$\mathcal{P} = \bigoplus_{\pi \in R} \mathcal{P}_\pi = \bigoplus_{\pi \in R} (Pol)_\pi \, [[\hbar]]. \tag{3.41}$$

The homomorphism τ_r commutes with the action of the group G. Therefore, it's sufficient to show that the homomorphism from ${}^\hbar\mathbf{U}(\mathbf{g})_\pi$ to $\mathcal{P}_\pi$ is onto. But it is obvious, since $\mathcal{P}_\pi$ is a free module over $\mathbf{R}[[\hbar]]$ of finite rank and the homomorphism is surjective modulo $\hbar$ (follows from (3.40)).

The second statement is an immediate corollary of the first one. □

Let us denote by $\widetilde{\mu}$ a map from $\mathbf{U}(\mathbf{g_C}) \otimes_\mathbf{C} \mathbf{C}((\hbar))$ to $\mathcal{P}_\mathbf{C} \otimes_{\mathbf{C}[[\hbar]]} \mathbf{C}((\hbar))$, the complexification of the map τ. Let us denote ${}^\hbar\mathbf{U}(\mathbf{g}) \otimes_\mathbf{R} \mathbf{C}$ by ${}^\hbar\mathbf{U}(\mathbf{g_C})$ We denote the restriction of the map $\widetilde{\mu}$ to ${}^\hbar\mathbf{U}(\mathbf{g_C})$ by $\widetilde{\mu}_r$. It is obvious that $\widetilde{\mu}_r$ is a map from ${}^\hbar\mathbf{U}(\mathbf{g_C})$ to $\mathcal{P}_\mathbf{C}$.

We have a natural map i from $\mathcal{P} \otimes_{\mathbf{R}[[\hbar]]} \mathbf{C}((v))$ to $\widetilde{W}$.

Lemma 3.3.1 *The natural map i is embedding.*

Proof: Let us denote by $\mathcal{K}$ the kernel of the map i. Then it is obvious that the space $\mathcal{K}$ is invariant under the action of $\mathbf{g_C}((v))$ (since it is a free $\mathbf{C}((v))$ module and is invariant under the action of $\mathbf{g_C}$ (since the latter is given by the bracket with the sections of $\widetilde{W}$)). Therefore, the group G preserves $\mathcal{K}$. If $f \in \mathcal{K}$ and $f \neq 0$ then $\sigma(f)|_{g\lambda} = g(\sigma(g^{-1}(f))|_\lambda) = 0$ since $g^{-1}(f) \in \mathcal{K}$. Therefore, we get that $\sigma(f) = 0$ which is a contradiction. This proves that $\mathcal{K}$ is zero. □

3.3.2 Main results.

Let us remind some notation. We denoted by $\mathbf{C}((\hbar))_{\frac{\lambda}{\hbar}-\rho_\mathbf{u}}$ a one dimensional (over $\mathbf{C}((\hbar))$) representation of the Lie algebra $\mathbf{p}_+((\hbar))$ given by the character $\frac{\lambda}{\hbar} - \rho_\mathbf{u}$. We denoted by $M_{\frac{\lambda}{\hbar}-\rho_\mathbf{u}}$ a generalized Verma module induced from the representation $\mathbf{C}((\hbar))_{\frac{\lambda}{\hbar}-\rho_\mathbf{u}}$ of Lie algebra $\mathbf{p}_+((\hbar))$

$$M_{\frac{\lambda}{\hbar}-\rho_\mathbf{u}} \overset{def}{=} \mathrm{Ind}_{\mathbf{p}_+((\hbar))}^{\mathbf{g_C}((\hbar))} \mathbf{C}((\hbar))_{\frac{\lambda}{\hbar}-\rho_\mathbf{u}}. \tag{3.42}$$

Denote by $\mathcal{J}$ an ideal in $\mathbf{U}(\mathbf{g_C}) \otimes_\mathbf{C} \mathbf{C}((\hbar))$ annihilating representation $M_{\frac{\lambda}{\hbar}-\rho_\mathbf{u}}$.

Let us notice that the algebra ${}^{\hbar}\mathbf{U}(\mathbf{g_C})$ is a universal enveloping algebra of a Lie algebra ${}^{h}\mathbf{g_C}$ (defined over $\mathbf{C}[[\hbar]]$), where ${}^{h}\mathbf{g_C}$ as a $\mathbf{C}[[\hbar]]$ module is isomorphic to $\mathbf{g_C} \otimes_{\mathbf{C}} \mathbf{C}[[\hbar]] = \mathbf{g_C}[[\hbar]]$ and the bracket is given by the formula

$$[Xa(\hbar), Yb(\hbar)] \stackrel{def}{=} [X,Y]\hbar a(\hbar)b(\hbar), \quad X, Y \in \mathbf{g_C}. \tag{3.43}$$

Let us define subalgebras ${}^{h}\mathbf{l_C}$, ${}^{h}\mathbf{u}_+$, ${}^{h}\mathbf{u}_-$, ${}^{h}\mathbf{p}_+$, ${}^{h}\mathbf{p}_-$ and ${}^{h}\mathbf{h}$ of the Lie algebra ${}^{h}\mathbf{g_C}$.

Definition 3.3.1

$$\begin{aligned} &{}^{h}\mathbf{l_C} \stackrel{def}{=} \mathbf{l_C}[[\hbar]] \subset {}^{h}\mathbf{g_C}, \quad {}^{h}\mathbf{u}_+ \stackrel{def}{=} \mathbf{u}_+[[\hbar]] \subset {}^{h}\mathbf{g_C}, \quad {}^{h}\mathbf{u}_- \stackrel{def}{=} \mathbf{u}_-[[\hbar]] \subset {}^{h}\mathbf{g_C}, \\ &{}^{h}\mathbf{p}_+ \stackrel{def}{=} \mathbf{p}_+[[\hbar]] \subset {}^{h}\mathbf{g_C}, \quad {}^{h}\mathbf{p}_- \stackrel{def}{=} \mathbf{p}_-[[\hbar]] \subset {}^{h}\mathbf{g_C}, \quad {}^{h}\mathbf{h} \stackrel{def}{=} \mathbf{h}[[\hbar]] \subset {}^{h}\mathbf{g_C}. \end{aligned} \tag{3.44}$$

We denote by $\mathbf{C}[[\hbar]]_{\lambda-\hbar\rho_{\mathbf{u}}}$ a representation of the Lie algebra ${}^{\hbar}\mathbf{p}_+$ given by the character $\lambda - \hbar\rho_{\mathbf{u}}$. We denote by ${}^{\hbar}M_{\lambda-\hbar\rho_{\mathbf{u}}}$ a generalized Verma module induced from the representation $\mathbf{C}[[\hbar]]_{\lambda-\hbar\rho_{\mathbf{u}}}$ of the Lie algebra ${}^{h}\mathbf{p}_+$. Let us denote by $\mathcal{J}_r$ an ideal in ${}^{\hbar}\mathbf{U}(\mathbf{g_C})$ annihilating representation ${}^{\hbar}M_{\lambda-\hbar\rho_{\mathbf{u}}}$.

It is easy to see that $\mathcal{J} = \mathcal{J}_r \otimes_{\mathbf{C}[[\hbar]]} \mathbf{C}((\hbar))$ and $\mathcal{J}_r = {}^{\hbar}\mathbf{U}(\mathbf{g_C}) \cap \mathcal{J}$.

Theorem 3.3.1 *The kernel of the map $\widetilde{\mu}$ ($\widetilde{\mu}_r$) is the ideal $\mathcal{J}$ ($\mathcal{J}_r$). The map $\widetilde{\mu}$ ($\widetilde{\mu}_r$) induces a map from $\mathbf{U}(\mathbf{g_C}) \otimes_{\mathbf{C}} \mathbf{C}((\hbar))/\mathcal{J}$ $\quad ({}^{\hbar}\mathbf{U}(\mathbf{g_C})/\mathcal{J}_r)$ to $\mathcal{P}_{\mathbf{C}} \otimes_{\mathbf{C}[[\hbar]]} \mathbf{C}((\hbar))$ ($\mathcal{P}_{\mathbf{C}}$) which is an isomorphism of algebras, i.e.,*

$$\begin{gathered} \mathbf{U}(\mathbf{g_C}) \otimes_{\mathbf{C}} \mathbf{C}((\hbar))/\mathcal{J} \stackrel{\widetilde{\mu}}{\to} \mathcal{P}_{\mathbf{C}} \otimes_{\mathbf{C}[[\hbar]]} \mathbf{C}((\hbar)), \\ {}^{\hbar}\mathbf{U}(\mathbf{g_C})/\mathcal{J}_r \stackrel{\widetilde{\mu_r}}{\to} \mathcal{P}_{\mathbf{C}}. \end{gathered} \tag{3.45}$$

Proof: It is easy to see that $M_{\frac{\lambda}{v^2}-\rho_{\mathbf{u}}} = M_{\frac{\lambda}{\hbar}-\rho_{\mathbf{u}}} \otimes_{\mathbf{C}((\hbar))} \mathbf{C}((v))$. Therefore, the theorem follows from the propositions 3.2.6 and 3.3.2 and lemma 3.3.1. □

The completion of $\mathbf{U}(\mathbf{g_C}) \otimes_{\mathbf{C}} \mathbf{C}((\hbar))$ in $\hbar$-adic topology is $\mathbf{U}(\mathbf{g_C})((\hbar))$. We denote the completion of ${}^{\hbar}\mathbf{U}(\mathbf{g_C})$ by ${}^{\hbar}\widehat{U}(\mathbf{g_C})$. Let us denote by $\widehat{\mathcal{J}}$ and $\widehat{\mathcal{J}_r}$ the completions of $\mathcal{J}$ and $\mathcal{J}_r$ in $\hbar$-adic topology. We denote by $\widehat{\mathcal{P}}_{\mathbf{C}}$ an algebra $Pol[[\hbar]]$ endowed with the star product. We denote by $\widehat{\mathcal{P}}_{\mathbf{C},(\hbar)}$ an algebra $Pol((\hbar))$ endowed with the star product. Denote by $\widehat{\mu}$ and $\widehat{\mu}_r$ continuous extensions of the maps $\widetilde{\mu}$ and $\widetilde{\mu}_r$.

Corollary 3.3.1 *We have the following isomorphisms of algebras*

$$\begin{gathered} \mathbf{U}(\mathbf{g_C})((\hbar))/\widehat{\mathcal{J}} \overset{\widehat{\mu}}{\rightarrow} \widehat{\mathcal{P}}_{\mathbf{C},(\hbar)}, \\ {}^{\hbar}\widehat{U}(\mathbf{g_C})/\mathcal{J}_r \overset{\widehat{\mu}_r}{\rightarrow} \widehat{\mathcal{P}}_{\mathbf{C}}. \end{gathered} \tag{3.46}$$

3.3.3 Remarks.

Proposition 3.3.3 $M_{\frac{\lambda}{\hbar}-\rho_{\mathbf{u}}}$ *is an irreducible representation of* $\mathbf{g_C}((\hbar))$.

Sketch of the proof: Let us assume the opposite. Then, there exist a singular vector w in the module $M_{\frac{\lambda}{\hbar}-\rho_{\mathbf{u}}}$ which generates $\mathbf{g_C}((\hbar))$ submodule $M' \subset M_{\frac{\lambda}{\hbar}-\rho_{\mathbf{u}}}$. Choose v a square root of $\hbar$ as in section 3.1. Let us change the base field from $\mathbf{C}((\hbar))$ to $\mathbf{C}((v))$ We denote the module $M' \otimes_{\mathbf{C}((\hbar))} \mathbf{C}((v))$ by M and we have seen that $M_{\frac{\lambda}{\hbar}-\rho_{\mathbf{u}}} \otimes \mathbf{C}((\hbar))\mathbf{C}((v)) = M_{\frac{\lambda}{v^2}-\rho_{\mathbf{u}}}$. Denote the completions of M and $M_{\frac{\lambda}{\hbar}-\rho_{\mathbf{u}}}$ in v-adic topology by $\widehat{M}$ and $\widehat{M}_{\frac{\lambda}{v^2}-\rho_{\mathbf{u}}}$ respectively. It is easy to see that $\widehat{M}$ is a proper submodule of $\widehat{M}_{\frac{\lambda}{v^2}-\rho_{\mathbf{u}}}$. It is obvious that $\widehat{M}_{\frac{\lambda}{v^2}-\rho_{\mathbf{u}}} = V \otimes_{\mathbf{C}[[v]]} \mathbf{C}((v))$, where V was defined in section 3.2.4. Therefore, $\widehat{M}_{\frac{\lambda}{v^2}-\rho_{\mathbf{u}}}$ is a topologically irreducible representation of the algebra $\mathcal{A} \otimes_{\mathbf{C}[[\hbar]]} \mathbf{C}((v))$, moreover $\mathrm{Supp}(\widehat{M}_{\frac{\lambda}{v^2}-\rho_{\mathbf{u}}}) = \{\lambda\} \in \mathcal{O}$. Submodule $\widehat{M} \subset \widehat{M}_{\frac{\lambda}{v^2}-\rho_{\mathbf{u}}}$ is a closed submodule. $\widehat{M}$ is stable under the action of the subalgebra $Pol((v)) \subset \mathcal{A} \otimes_{\mathbf{C}[[\hbar]]} \mathbf{C}((v))$. Let us fix some small compact neighborhood U of the point $\lambda \in \mathcal{O}$. The proof of this proposition is based upon the fact that we can approximate any smooth function and its derivatives by the polynomial functions on U (really we need to approximate smooth function and its defivative only at point λ). The action of $\mathcal{A} \otimes_{\mathbf{C}[[\hbar]]} \mathbf{C}((v))$ on $\widehat{M}_{\frac{\lambda}{v^2}-\rho_{\mathbf{u}}}$ is given by differential operators. Therefore, the subalgebra $Pol((v))$ is dense in $\mathcal{A} \otimes_{\mathbf{C}[[\hbar]]} \mathbf{C}((v))$. The representation $\widehat{M}_{\frac{\lambda}{v^2}-\rho_{\mathbf{u}}}$ of $\mathcal{A} \otimes_{\mathbf{C}[[\hbar]]} \mathbf{C}((v))$ is continuous in this new topology which is stronger then just v-adic topology. Therefore, submodule $\widehat{M}$ (it is certainly closed submodule in the stronger topology) is stable under the action of $\mathcal{A} \otimes_{\mathbf{C}[[\hbar]]} \mathbf{C}((v))$, which is a contradiction. This ends the sketch of the proof of this proposition. □

Let us consider a more general situation then a semisimple coadjoint orbit. We have a connected symplectic manifold M. Let us assume that the manifold M satisfies all conditions of the theorem 2.2.2 b). Then we have an algebra homomorphism $\tilde{\mu}$ from $\mathbf{U}(\mathbf{g_C}) \otimes_{\mathbf{C}} \mathbf{C}((v))$ to a quantized functions $\mathcal{A} \otimes_{\mathbf{C}[[v]]} \mathbf{C}((v))$. Let us denote by Z the center of $\mathbf{U}(\mathbf{g_C})$. $Z \otimes_{\mathbf{C}} \mathbf{C}((v))$ is the center of $\mathbf{U}(\mathbf{g_C}) \otimes_{\mathbf{C}} \mathbf{C}((v))$.

Proposition 3.3.4 *If an element u lies in the center of $\mathbf{U}(\mathbf{g_C}) \otimes_{\mathbf{C}} \mathbf{C}((v))$, i.e., $u \in Z \otimes_{\mathbf{C}} \mathbf{C}((v))$ then $\tilde{\mu}(u)$ is a constant function on M, i.e., $\tilde{\mu}(u) \in \mathbf{C}((v)) \subset \mathcal{A} \otimes_{\mathbf{C}[[v]]} \mathbf{C}((v))$.*

Proof: The map $\tilde{\mu}$ is a G-invariant map. Therefore, it maps the center of $\mathbf{U}(\mathbf{g_C}) \otimes_{\mathbf{C}} \mathbf{C}((v))$ to the G-invariant functions on M, i.e., $\left(\mathcal{A} \otimes_{\mathbf{C}[[v]]} \mathbf{C}((v))\right)^G = \mathcal{A}^G \otimes_{\mathbf{C}[[v]]} \mathbf{C}((v))$. Since the Lie algebra $\mathbf{g}$ acts transitively on M the only G-invariant functions on M are constant functions. □

Let $\mathcal{O}$ be a coadjoint orbit of a Lie group G. Let us assume that $\mathcal{O}$ admits a G-invariant connection. Then we can find a G-invariant symplectic connection on $\mathcal{O}$ and construct a star product using Fedosov's method.

Corollary 3.3.2 *The restriction of the homomorphism $\tilde{\mu}$ to the center $Z \otimes_{\mathbf{C}} \mathbf{C}((v))$ of $\mathbf{U}(\mathbf{g_C}) \otimes_{\mathbf{C}} \mathbf{C}((v))$ is a character of the algebra $Z \otimes_{\mathbf{C}} \mathbf{C}((v))$. Therefore, we can associate a character $\chi : Z \otimes_{\mathbf{C}} \mathbf{C}((v)) \mapsto \mathbf{C}((v))$ to any coadjoint orbit $\mathcal{O}$ which admits a G-invariant connection.*

Proof: The statement follows immediately from proposition 3.3.4. □

3.4 Some representations of $\mathcal{P}_{\mathbf{C}} \otimes_{\mathbf{C}[[\hbar]]} \mathbf{C}((v))$.

In section 3.2.4 we constructed a representation $\tilde{V} \otimes_{\mathbf{C}[[v]]} \mathbf{C}((v))$ of the algebra $\mathcal{P}_{\mathbf{C}} \otimes_{\mathbf{C}[[\hbar]]} \mathbf{C}((v))$. It follows from the theorem 2.2.2 (or theorem 3.3.1) that $\tilde{V} \otimes_{\mathbf{C}[[v]]} \mathbf{C}((v))$ is a representation of the Lie algebra $\mathbf{g_C}((v))$. The theorem 3.2.1 states that this module is a generalized Verma module with the highest weight $\frac{\lambda}{v^2} - \rho_{\mathbf{u}}$. Let us reformulate this result. We have a triangular decomposition $\mathbf{g_C} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$. We can state the results of the theorems 3.2.1 and 3.3.4 in the following way.

Lemma 3.4.1 *The module* $\widetilde{V}\otimes_{\mathbf{C}[[v]]}\mathbf{C}((v))$ *is an irreducible quotient (quotient by the maximal submodule) of the Verma module with the highest weight* $\frac{\lambda}{v^2}-\rho_{\mathbf{u}}$.

Let us generalize this construction and the result of the lemma 3.4.1. In section 3.2.1 we chose the set of positive roots of the Lie algebra $\mathbf{g_C}$ (the choice is not unique). Let us fix it. With any choice of the set of positive roots (it is equivalent to a choice of an element of the Weyl group) we associate an irreducible representation of $\mathbf{g_C}((v))$.

Remark 3.4.1 *It can happen that with to different choices of the set of positive roots the associated representations are isomorphic (actually equal).*

Let us chose a set of positive roots Δ'_+. We denote by Δ'_- the corresponding set of negative roots. Denote by $\mathbf{n}'_+$ and $\mathbf{n}'_-$ the corresponding nilpotent subalgebras of $\mathbf{g_C}$. We have a triangular decomposition $\mathbf{g_C}=\mathbf{n}'_-\oplus\mathbf{h}\oplus\mathbf{n}'_+$. Let us denote by V_1 a subspace $(\mathbf{u}_+\oplus\mathbf{u}_-)\cap\mathbf{n}'_+$ of $\mathbf{u}_+\oplus\mathbf{u}_-=T^*_\lambda\mathcal{O}$ and by V_2 a subspace $(\mathbf{u}_+\oplus\mathbf{u}_-)\cap\mathbf{n}'_-$. It is easy to see that both V_1 and V_2 are lagrangian subspaces of $\mathbf{u}_+\oplus\mathbf{u}_-=T^*_\lambda\mathcal{O}$. If $\Delta'_+=\Delta_+$ then the choice of V_1 and V_2 coincides with the choice made in section 3.2.4. Let us denote by L'_+ a subsemigroup of L generated by the elements of Δ'_+ and by L'_- a subsemigroup of L generated by the elements of Δ'_-. One have $L'_-=-L'_+$. Let us denote the subset of roots $(\Sigma\cup-\Sigma)\cap\Delta'_+$ by Γ. Denote by Γ_+ the set $\Sigma\cap\Delta'_+$ and by Γ_- the set $(-\Sigma)\cap\Delta'_+$. We have $\Gamma=\Gamma_-\cup\Gamma_+$. We denote by ρ_Γ the half sum of the roots of Γ.

We can apply the construction of section 3.1 now. We get a representation V' of the algebra $W_\lambda\otimes_{\mathbf{R}[[\hbar]]}\mathbf{C}((v))$. As a vector space V' is canonically isomorphic to $\left(\oplus_{k=0}^{\infty}\mathrm{S}^k(\mathbf{u}_-)\right)[[v]]=(\mathbf{C}[f_\alpha,e_\beta])[[v]]$, $\alpha\in\Gamma_+$ and $\beta\in\Gamma_-$ (see section 3.1). It is easy to see that the action of $A_H|_\lambda$, $H\in\mathbf{h}$ on V' can be described by a formula similar to (3.20).

Let us denote by $V'^{(\alpha)}$ an eigenspace of the Lie algebra $\mathbf{h}$ ($H\in\mathbf{h}$ acts by $A_H|_\lambda$) on V' with an eigenvalue $\alpha\in L'_-$. It is obvious that $V'^{(\alpha)}$ is a free module of finite rank over the ring $\mathbf{C}[[v]]$.

Lemma 3.4.2 *There exists a canonical isomorphism of vector spaces*

$$V' = \prod_{\alpha \in L'_-} V'^{(\alpha)}. \tag{3.47}$$

Proof: Obvious. □

Let us consider a subspace $\widetilde{V'}$ of V'

$$\widetilde{V'} \stackrel{def}{=} \sum_{\alpha \in L'_-} V'^{(\alpha)} \subset V'. \tag{3.48}$$

Lemma 3.4.3 *We have*

$$\widetilde{V'} = \left(\bigoplus_{k=0}^{\infty} S^k(V_2)\right) \otimes_{\mathbf{C}} \mathbf{C}[[v]] = (\mathbf{C}[[v]])[f_\alpha, e_\beta], \ \ \alpha \in \Gamma_+, \ \beta \in \Gamma_-. \tag{3.49}$$

Proof: Obvious. □

Proposition 3.4.1 *$\widetilde{V'}$ is a representation of $\mathcal{P}_{\mathbf{C}}$, i.e., elements of $\mathcal{P}_{\mathbf{C}}$ preserve the subspace $\widetilde{V'} \subset V'$.*

Proof: Similar to the proof of the proposition 3.2.4. □

Let us denote by V a module $\widetilde{V'} \otimes_{\mathbf{C}[[v]]} \mathbf{C}((v))$. It becomes a representation of $\mathbf{U}(\mathbf{g_C}) \otimes_{\mathbf{C}} \mathbf{C}((v))$ via the homomorphism $\widetilde{\mu}$ (from $\mathbf{U}(\mathbf{g_C}) \otimes_{\mathbf{C}} \mathbf{C}((\hbar))/\mathcal{J}$ to $\mathcal{P}_{\mathbf{C}} \otimes_{\mathbf{C}[[v]]} \mathbf{C}((v))$). The module V contains a distinguished element $\mathbf{1} \in \widetilde{V'} \subset V$. Reasonings similar to the reasonings in section 3.2.4 show that the following proposition holds.

Proposition 3.4.2 *The $\mathbf{U}(\mathbf{g_C}) \otimes_{\mathbf{C}} \mathbf{C}((v))$ module V is generated by the element $\mathbf{1} \in V$, i.e.,*

$$\Big(\mathbf{U}(\mathbf{g_C}) \otimes_{\mathbf{C}} \mathbf{C}((v))\Big)(\mathbf{1}) = V. \tag{3.50}$$

Let us denote by $\mathbf{C}((v))_{\frac{\lambda}{v^2}-\rho_\Gamma}$ a one dimensional (over $\mathbf{C}((v))$) representation of $\mathbf{p}'_+((v))$ given by the character $\frac{\lambda}{v^2} - \rho_\Gamma$, where $\mathbf{p}'_+((v)) \stackrel{def}{=} \mathbf{h} \oplus \mathbf{n}'_+$. We denote by $M'_{\frac{\lambda}{v^2}-\rho_\Gamma}$ a Verma module with the highest weight $\frac{\lambda}{v^2} - \rho_\Gamma$, *i.e.*,

$$M'_{\frac{\lambda}{v^2}-\rho_\Gamma} \stackrel{def}{=} \mathrm{Ind}_{\mathbf{p}'_+((v))}^{\mathbf{g_C}((v))} \mathbf{C}((v))_{\frac{\lambda}{v^2}-\rho_\Gamma}. \tag{3.51}$$

Proposition 3.4.3 *Representation V of the Lie algebra $\mathbf{g}_{\mathbf{C}}((v))$ is a quotient of the Verma module $M'_{\frac{\lambda}{v^2}-\rho_\Gamma}$.*

Proof: The proof is similar to the proof of the theorem 3.2.1. □

Proposition 3.4.4 *V is an irreducible representation of the Lie algebra $\mathbf{g}_{\mathbf{C}}((v))$.*

Proof: Similar to the proof of the proposition 3.3.4. □

Combining the last two results we get

Theorem 3.4.1 *The module V as a representation of the Lie algebra $\mathbf{g}_{\mathbf{C}}((v))$ is isomorphic to the irreducible quotient of the Verma module $M'_{\frac{\lambda}{v^2}-\rho_\Gamma}$.*

$$V = \left(\bigoplus_{k=0}^{\infty} S^k(V_2)\right) \otimes_{\mathbf{C}} \mathbf{C}((v)) = (\mathbf{C}((v)))[f_\alpha, e_\beta], \;\; \alpha \in \Gamma_+, \;\; \beta \in \Gamma_-. \qquad (3.52)$$

Proof: Obvious. □

We can state the theorem 3.4.1 in the following way

Theorem 3.4.2 *To each choice of the set of positive roots Δ'_+ the associated representation of $\mathbf{g}_{\mathbf{C}}((v))$ is an irreducible quotient of the Verma module (induced from $\mathbf{h} \oplus \mathbf{n}'_+$) with the highest weight $\frac{\lambda}{v^2} - \rho_\Gamma$.*

Proof: The statement is equivalent to the theorem 3.4.1. □

Chapter 4

Examples.

In this chapter we will consider two examples. The first one is the case of the coadjoint orbits of the group SU(2). This is the most elementary case. The second example is the case of the minimal nilpotent orbits of the symplectic group $\mathrm{Sp}(2n, \mathbf{R})$. In the first example although the group is the simplest possible one, the answer is not trivial. In the second case although we are talking about nilpotent orbits all the calculations are elementary since the orbit is a quotient of $\mathbf{R}^{2n} - 0$ by the multiplication by ± 1. The reason why we give this example here is that it shows that the results similar to the case of semisimple orbits may be true in case of nilpotent orbits when they admit G-invariant connection.

4.1 SU(2) example.

Let us look at the example of the group $G = \mathrm{SU}(2)$ closer. The coadjoint orbits are spheres except the trivial one. We can identify the dual space $\mathbf{g}^*$ to the Lie algebra $\mathbf{g}$ with the Lie algebra $\mathbf{g}$ itself using the trace form. Therefore any element of the space $\mathbf{g}^*$ can be represented by a matrix

$$\begin{pmatrix} i\frac{\lambda}{2} & X + iY \\ -X + iY & -i\frac{\lambda}{2} \end{pmatrix}, \quad \text{where} \quad X, Y, \lambda \in \mathbf{R}. \tag{4.1}$$

Every coadjoint orbit contains the point $\widetilde{\lambda}$

$$\widetilde{\lambda} = \begin{pmatrix} i\frac{\lambda}{2} & 0 \\ 0 & -i\frac{\lambda}{2} \end{pmatrix}, \quad \text{for some } \lambda \in \mathbf{R}. \tag{4.2}$$

If the coadjoint orbit contains the point $\widetilde{\lambda}$ then it contains the point $-\widetilde{\lambda}$

$$-\widetilde{\lambda} = \begin{pmatrix} -i\frac{\lambda}{2} & 0 \\ 0 & i\frac{\lambda}{2} \end{pmatrix}. \tag{4.3}$$

In this way all the coadjoint orbits are in one to one correspondence with the non-negative real numbers. From now on we fix some positive number λ.

We are going to try to understand the representation which we constructed in section 3.2.4.

Let us denote by E, H, F the following basis in $\mathrm{sl}(2, \mathbf{C})$

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.4}$$

One can consider E, H, F as functions on the coadjoint orbits. We denote by e and f the sections of $T^*\mathcal{O}$ at point $\widetilde{\lambda}$ corresponding to $\mathrm{d}(E)$ and $\mathrm{d}(F)$. The symplectic form $\widetilde{\omega}$ at point $\widetilde{\lambda}$ is equal to

$$\widetilde{\omega}|_{\widetilde{\lambda}}(e, f) = i\lambda, \tag{4.5}$$

see 3.5.

From the corollary 2.2.2 follows that $\mathrm{sl}(2, \mathbf{C}) \ni X \mapsto \frac{1}{\hbar}\sigma(X) \in W$ is a homomorphism of Lie algebra. Moreover, we have that

$$\frac{1}{\hbar}\sigma(H)|_{\widetilde{\lambda}} = \frac{i\lambda}{\hbar} - 1 + \frac{2i}{\lambda\hbar} f * e, \tag{4.6}$$

see proposition 3.2.1.

We denote by v the square root of $\hbar$. Let us identify the algebra $W|_{\widetilde{\lambda}} \otimes_{\mathbf{R}[[\hbar]]} \mathbf{C}[[v]]$

with the completion of the algebra of differential operators - $\mathbf{C}[[vz, v\frac{d}{dz}]][[v]]$, by

$$f \mapsto i\sqrt{\lambda}vz \quad \text{and} \quad e \mapsto \sqrt{\lambda}v\frac{d}{dz}. \tag{4.7}$$

We denote by $\widetilde{V}$ the vector space of polynomials in z over the field $\mathbf{C}((v))$, *i.e.*, $\mathbf{C}((v))[z]$. It follows from the proposition 3.2.4 that the space $\widetilde{V}$ is a representation of $\mathbf{g}((v)) = \text{sl}(2, \mathbf{C})((v))$. The theorem 3.2.1 states that $\widetilde{V}$ is a Verma module with the highest weight $\frac{\widetilde{\lambda}}{v^2} - \rho$. Let us try to write down explicitly the elements $E, F, H \in \text{sl}(2, \mathbf{C})((v))$ as a differential operators acting on $\widetilde{V}$.

Since $\sigma(E)|_{\widetilde{\lambda}} \in W^{(\alpha)}_{\widetilde{\lambda}}$ and $\sigma(F)|_{\widetilde{\lambda}} \in W^{(-\alpha)}_{\widetilde{\lambda}}$ it follows that we can write

$$\begin{aligned} &\tfrac{1}{\hbar}\sigma(E)|_{\widetilde{\lambda}} = \tfrac{1}{\hbar}e\phi(ef), \quad \text{where } \phi(x) \text{ is some function,} \\ &\tfrac{1}{\hbar}\sigma(F)|_{\widetilde{\lambda}} = \tfrac{1}{\hbar}f\psi(ef), \quad \text{where } \psi(x) \text{ is some function.} \end{aligned} \tag{4.8}$$

Now let us determine the relation between ϕ and ψ. Since our construction was real, the sections of the complex conjugate function should be complex conjugate. It is obvious that the functions E and F are complex conjugate on the coadjoint orbits and also the sections e and f of the cotangent bundle $T^*\mathcal{O}$ at point $\widetilde{\lambda}$ are complex conjugate. Therefore, we see that function ψ is complex conjugate to ϕ, *i.e.*, $\psi(x) = \bar{\phi}(x)$ (since ef is real). (If $\phi(x) = \sum_i \phi_i x^i$, where $\phi_i \in \mathbf{C}[[v]]$ then $\bar{\phi}(x) = \sum_i \bar{\phi}_i x^i$ $(\bar{v} = v)$.)

In the Fedosov's construction of the connection there is an initial condition on the tensor $\mathbf{r}$, *i.e.*, $\delta^{-1}(\mathbf{r}) = 0$. If we translate this condition into a condition on the functions ϕ and ψ we will see that these functions should be equal *i.e.*, $\phi = \psi$. In general, we do not require $\delta^{-1}(r) = 0$ from the Fedosov's connection, $\delta^{-1}(\mathbf{r})$ can be almost any G-invariant section of W (one can describe explicitly the class of the admissible initial conditions). If we fix an initial condition (*i.e.*, $\delta^{-1}(\mathbf{r})$) then we get some conditions on the functions ϕ and ψ.

We will write $\frac{1}{\hbar}\sigma(E)|_{\widetilde{\lambda}}$ and $\frac{1}{\hbar}\sigma(F)|_{\widetilde{\lambda}}$ as sections of $\frac{1}{v^2}W \otimes_{\mathbf{R}[[\hbar]]} \mathbf{C}[[v]]$ using non-commutative star product in $W \otimes_{\mathbf{R}[[\hbar]]} \mathbf{C}((v))$. We get

$$\frac{1}{\hbar}\sigma(E)|_{\widetilde{\lambda}} = \frac{1}{v^2} e * g(ef) \text{ and } \frac{1}{\hbar}\sigma(F)|_{\widetilde{\lambda}} = \frac{1}{v^2} f * \bar{g}(ef), \tag{4.9}$$

where $g(x)$ is some function (since $*$ product is real). Here under $g(ef)$ we understand the following section

$$g(ef) = \sum_{n=0}^{\infty} g_n \, (ef) * \cdots * (ef), \quad \text{where } \; g(x) = \sum_{n=0}^{\infty} g_n x^n, \;\; g_n \in \mathbf{C}[[v]]. \tag{4.10}$$

The initial condition $\delta^{-1}(\mathbf{r}) = 0$ gives a condition

$$\bar{g}(x + i\lambda v^2) = g(x) \tag{4.11}$$

on the function $g(x)$.

One can easily see that on $\widetilde{V}$ the operator $ef = \frac{e*f+f*e}{2}$ acts as $\frac{i\lambda v^2}{2}(z\frac{d}{dz} + \frac{d}{dz}z)$. Let us denote the operator $\frac{i\lambda}{2}(z\frac{d}{dz} + \frac{d}{dz}z)$ by Δ. We have

$$\frac{1}{\hbar}\sigma(H)|_{\widetilde{\lambda}} = \frac{i\lambda}{v^2} + \frac{2i}{\lambda}\Delta. \tag{4.12}$$

We know that

$$\frac{1}{\hbar}\sigma(E)|_{\widetilde{\lambda}} = \frac{\sqrt{\lambda}}{v}\frac{d}{dz}\, g(v^2\Delta) \text{ and } \frac{1}{\hbar}\sigma(F)|_{\widetilde{\lambda}} = \frac{i\sqrt{\lambda}}{v} z\, \bar{g}(v^2\Delta), \tag{4.13}$$

for some function $g(x)$.

We would like to find this function $g(x)$. In order to do that we will use the fact that $[\frac{1}{\hbar}\sigma(E)|_{\widetilde{\lambda}}, \frac{1}{\hbar}\sigma(F)|_{\widetilde{\lambda}}] = \frac{1}{\hbar}\sigma(H)|_{\widetilde{\lambda}}$. This give us some equation on the function $g(x)$. We will show that this equation together with the equation (4.11) (coming from the initial condition on $\mathbf{r}$) has a unique solution. We will not write down this solution explicitly. One can try to do that but the formulas that we got are very long and are not explicit enough.

We must compute the following commutator

$$[\frac{\sqrt{\lambda}}{v}\frac{d}{dz}\,g(v^2\Delta), \frac{i\sqrt{\lambda}}{v}z\,\bar{g}(v^2\Delta)] = \frac{i\lambda}{v^2}[\frac{d}{dz}\,g(v^2\Delta), z\,\bar{g}(v^2\Delta)]. \tag{4.14}$$

It is easy to check that $\Delta z = z(\Delta + i\lambda)$ and $\Delta\frac{d}{dz} = \frac{d}{dz}(\Delta - i\lambda)$. Therefore, we can write $[\frac{d}{dz}\,g(v^2\Delta), z\,\bar{g}(v^2\Delta)]$ as

$$\begin{aligned}
&[\tfrac{d}{dz}\,g(v^2\Delta), z\,\bar{g}(v^2\Delta)] = \tfrac{d}{dz}\,g(v^2\Delta)z\,\bar{g}(v^2\Delta) - z\,\bar{g}(v^2\Delta)\tfrac{d}{dz}\,g(v^2\Delta) = \\
&\tfrac{d}{dz}(g(v^2\Delta)z)\bar{g}(v^2\Delta) - z(\bar{g}(v^2\Delta)\tfrac{d}{dz})g(v^2\Delta) = \\
&\tfrac{d}{dz}(z\,g(v^2(\Delta + i\lambda)))\bar{g}(v^2\Delta) - z(\tfrac{d}{dz}\,\bar{g}(v^2(\Delta - i\lambda)))g(v^2\Delta) = \\
&(\tfrac{1}{2} + \tfrac{\Delta}{i\lambda})g(v^2(\Delta + i\lambda))\bar{g}(v^2\Delta) - (-\tfrac{1}{2} + \tfrac{\Delta}{i\lambda})g(v^2\Delta)\bar{g}(v^2(\Delta - i\lambda)).
\end{aligned} \tag{4.15}$$

Let us denote the product $(-\frac{1}{2} + \frac{\Delta}{i\lambda})g(v^2\Delta)\bar{g}(v^2(\Delta - i\lambda))$ by $G(\Delta)$. Then we can write the equation (4.15) as an equation on the function $G(\Delta)$ as follows

$$i\lambda(G(\Delta + i\lambda) - G(\Delta)) = i\lambda + \frac{2i}{\lambda}v^2\Delta, \tag{4.16}$$

or equivalently

$$G(\Delta + i\lambda) - G(\Delta) = 1 + \frac{2}{\lambda^2}v^2\Delta. \tag{4.17}$$

It is easy to see (direct calculation) that

$$G_0(\Delta) = (\frac{\Delta}{i\lambda} - \frac{1}{2})(\frac{v^2\Delta}{\lambda^2} - \frac{iv^2}{2\lambda} + 1) \tag{4.18}$$

is a particular solution of the equation (4.17). One can get all other solutions by adding functions which do not depend on Δ. The general solution has form

$$G(\Delta) = G_0(\Delta) + C(v), \text{ where } C(v) \in \mathbf{C}[[v]]. \tag{4.19}$$

Let us denote by $F(v^2\Delta)$ the product $g(v^2\Delta)\bar{g}(v^2(\Delta - i\lambda))$. We see that $G(\Delta) = (\frac{\Delta}{i\lambda} - \frac{1}{2})F(v^2\Delta)$. Let us put $F_0(v^2\Delta) = \frac{v^2\Delta}{\lambda^2} - \frac{iv^2}{2\lambda} + 1$. It is obvious that $G_0(\Delta) =$

$(\frac{\Delta}{i\lambda}-\frac{1}{2})F_0(v^2\Delta)$. Therefore, we get the following equation

$$C(v)=(\frac{\Delta}{i\lambda}-\frac{1}{2})(F(v^2\Delta)-F_0(v^2\Delta)). \tag{4.20}$$

It is easy to see that this equation has no solutions since $F(v^2\Delta)-F_0(v^2\Delta)\in \mathbf{C}[[v^2\Delta,v]]$.

As a consequence we get that the only possible solution is $F(\Delta)=F_0(\Delta)$. Therefore, we obtain that the function $g(x)$ satisfies the following equation

$$g(x)\bar{g}(x-i\lambda v^2)=1-\frac{iv^2}{2\lambda}+\frac{x}{\lambda^2}. \tag{4.21}$$

Let us see that there exist a unique $g(x)$ which satisfies both equations (4.21) and (4.11). Indeed, from (4.11) we get $\bar{g}(x-i\lambda v^2)=g(x-2i\lambda v^2)$. Plugging that in the equation (4.21) we obtain

$$g(x)g(x-2i\lambda v^2)=1-\frac{iv^2}{2\lambda}+\frac{x}{\lambda^2}. \tag{4.22}$$

It is easy to see (by induction) that the equation (4.22) has a unique solution. Unfortunately, I do not know any nice formula for this solution.

Let us summarize all the above.

Proposition 4.1.1 *Applying Fedosov's construction to the coadjoint orbits of* SU(2) *we obtain the following representation of* sl(2, **C**)

$$\begin{aligned} E &\mapsto \tfrac{\sqrt{\lambda}}{v}\tfrac{d}{dz}\, g\left(\tfrac{i\lambda v^2}{2}\left(z\tfrac{d}{dz}+\tfrac{d}{dz}z\right),v^2\right), \\ H &\mapsto \tfrac{i\lambda}{v^2}-\left(z\tfrac{d}{dz}+\tfrac{d}{dz}z\right), \\ F &\mapsto \tfrac{i\sqrt{\lambda}}{v}z\, g\left(\tfrac{i\lambda v^2}{2}\left(z\tfrac{d}{dz}+\tfrac{d}{dz}z-2\right),v^2\right), \end{aligned} \tag{4.23}$$

where the function $g(x,y)\in\mathbf{C}[[x,y]]$ *satisfies the following equation*

$$g(x,y)g(x-2i\lambda y,y)=1-\frac{iy}{2\lambda}+\frac{x}{\lambda^2}. \tag{4.24}$$

4.2 Minimal nilpotent orbit of $\mathrm{Sp}(2n, \mathbf{R})$.

This example is a trivial one and we give it here only as an example of the nilpotent orbit which admits a G-invariant symplectic connection. Let us choose v a square root of $\hbar$, *i.e.*, $v^2 = \hbar$. Let us consider a $2n$ dimensional vector space $\mathbf{R}^{2n}$ with the coordinates $x_1, \cdots, x_n, y_1, \cdots, y_n$ and a symplectic form $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$. The minimal nilpotent orbit $\mathcal{O}$ of $\mathrm{Sp}(2n, \mathbf{R})$ is $(\mathbf{R}^{2n} - 0)/(\pm 1)$. The trivial flat connection d is $\mathrm{Sp}(2n, \mathbf{R})$-invariant. Therefore, Fedosov's connection on $\mathbf{R}^{2n}$ (or $\mathcal{O}$) is just $\nabla^F = d - \delta$. The moment map tells us that $\mathrm{sp}(2n, \mathbf{R})$ is isomorphic to the vector space of quadratic polynomials on $\mathbf{R}^{2n}$. The Poisson bracket of two functions f and g is

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right). \tag{4.25}$$

Let us denote by p_i the section dx_i and by q_i the section dy_i of the cotangent bundle. Let us write down the flat sections of W which corresponds to the quadratic functions.

$$\begin{aligned} \sigma(x_i x_j) &= x_i x_j + x_i p_j + x_j p_i + p_i p_j \in W \\ \sigma(y_i y_j) &= y_i y_j + y_i q_j + y_j q_i + q_i q_j \in W \qquad \text{for } i, j = 1 \cdots n. \\ \sigma(x_i y_j) &= x_i y_j + x_i q_j + y_j p_i + p_i q_j \in W \end{aligned} \tag{4.26}$$

One can see that these formulas descend to the coadjoint orbit $\mathcal{O}$. It is obvious that if we fix some point $m \in \mathbf{R}^{2n}$ ($x_i = x_i^{(0)}$, $y_i = y_i^{(0)}$) then

$$p_i \mapsto v \frac{\partial}{\partial z_i} \quad \text{and} \quad q_i \mapsto v z_i \tag{4.27}$$

gives us a representation of W_m. Moreover, if we restrict ourselves to the Lie subalgebra $\mathrm{sp}(2n, \mathbf{R})$ of $\frac{1}{v^2} W_m$ spanned by $\frac{1}{v^2}\sigma(x_i x_j)$, $\frac{1}{v^2}\sigma(y_i y_j)$ and $\frac{1}{v^2}\sigma(x_i y_j)$ we get its representation V in the space of polynomials in the variables $z_1, \cdots, z_n$ over the field $\mathbf{C}((v))$, *i.e.*, $V = \mathbf{C}((v))[z_1, \cdots, z_n]$. One can see that if $m \in \mathbf{R}^{2n}$ is not 0 then this representation is irreducible. If $m = 0$ then V is a sum of two irreducible representations $V = V^{(even)} \oplus V^{(odd)}$.

Let us choose the point $m \in \mathbf{R}^{2n}$ with the coordinates $x_i = 0$, $y_j = 0$ for $i = 1, \cdots, n$ and $j = 2, \cdots, n$ and $y_1 = 1$. Then we get the formulas

$$\begin{aligned}
&\tfrac{1}{v^2} x_i x_j \mapsto \tfrac{\partial}{\partial z_i}\tfrac{\partial}{\partial z_j} \quad \text{for } i, j = 1, \cdots, n \\
&\tfrac{1}{v^2} y_i y_j \mapsto z_i z_j \quad \text{for } i, j = 2, \cdots, n \\
&\tfrac{1}{v^2} x_i y_j \mapsto \tfrac{1}{2}\left(z_j \tfrac{\partial}{\partial z_i} + \tfrac{\partial}{\partial z_i} z_j\right) \quad \text{for } i = 1, \cdots, n \text{ and } j = 2, \cdots, n \\
&\tfrac{1}{v^2} y_1 x_i \mapsto \tfrac{\partial}{\partial z_i} + \tfrac{1}{2}\left(z_1 \tfrac{\partial}{\partial z_i} + \tfrac{\partial}{\partial z_i} z_1\right) \quad \text{for } i = 1, \cdots, n \\
&\tfrac{1}{v^2} y_1 y_i \mapsto z_i + z_1 z_i \quad \text{for } i = 2, \cdots, n \\
&\tfrac{1}{v^2} y_1^2 \mapsto 1 + 2z_1 + z_1^2
\end{aligned} \tag{4.28}$$

It is easy to see that this representation is a maximal irreducible quotient of Verma module with the highest weight λ, where $\langle \lambda, \frac{1}{v^2} x_i y_i \rangle = 1$. It is well known that the ideal in $\mathrm{U}(\mathrm{sp}(2n, \mathbf{C}))$ annihilating this representation is the Joseph ideal (we denote it by $\mathcal{J}$).

It is obvious that this picture descends to the minimal nilpotent coadjoint orbit $\mathcal{O}$. It is obvious that if we restrict ourselves to the polynomial functions on $\mathbf{R}^{2n}$ then this representation is faithful. The constructed map maps $\mathrm{U}(\mathrm{sp}(2n, \mathbf{C})) \otimes_{\mathbf{C}} \mathbf{C}((v))$ in the polynomial functions $\mathbf{C}((v))[x_i, y_j]$ on $\mathbf{R}^{2n}$. Therefore, the kernel of this map coincides with the ideal in $\mathrm{U}(\mathrm{sp}(2n, \mathbf{C})) \otimes_{\mathbf{C}} \mathbf{C}((v))$ annihilating the representation above. This ideal is exactly $\mathcal{J} \otimes_{\mathbf{C}} \mathbf{C}((v))$.

In this example we see that in our formulas we can put v to be a number, *i.e.*, there is no any problems with convergence. This tells that the star product on the coadjoint orbit $\mathcal{O}$ endows the polynomial functions on $\mathcal{O}$ with the algebra structure which is isomorphic to $\mathrm{U}(\mathrm{sp}(2n, \mathbf{C}))/\mathcal{J}$.

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