

Efficient Scheduling of Multi-Antenna Broadcast Systems

by

Krishna Prasanna Jagannathan

B. Tech., Electrical Engineering (2004)

Indian Institute of Technology, Madras, India

Submitted to the Department of Electrical Engineering and Computer Science

in partial fulfillment of the requirements for the degree of

Master of Science in Electrical Engineering and Computer Science

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

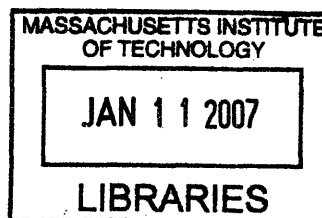
[September 2006]
August 2006

© Massachusetts Institute of Technology 2006. All rights reserved.

Author
Department of Electrical Engineering and Computer Science
August 16, 2006

Certified by
Eytan H. Modiano
Associate Professor
Thesis Supervisor

Accepted by
Arthur C. Smith
Chairman, Departmental Committee on Graduate Students



ARCHIVES

Efficient Scheduling of Multi-Antenna Broadcast Systems

by

Krishna Prasanna Jagannathan

B. Tech., Electrical Engineering (2004)

Indian Institute of Technology, Madras, India

Submitted to the Department of Electrical Engineering and Computer Science
on August 16, 2006, in partial fulfillment of the
requirements for the degree of
Master of Science in Electrical Engineering and Computer Science

Abstract

In this thesis, we study the problem of efficiently scheduling users in a multi-antenna Gaussian broadcast channel with M transmit antennas and K independent receivers each with a single receive antenna. We first focus on a scenario with two transmit antennas and statistically identical users, and analyze the gap between the full sum capacity and the rate that can be achieved by transmitting to a suitably selected pair of users. In particular, we consider a scheme that picks the user with the largest channel gain, and selects a second user from the next $L - 1$ largest ones to form the best pair, taking the orientation of channel vectors into account as well. We prove that the expected rate gap converges to $1/(L - 1)$ nats/symbol when the total number of users K tends to infinity. Allowing L to increase with K , it may be deduced that transmitting to a properly chosen pair of users is asymptotically optimal, while dramatically reducing the feedback overhead and operational complexity. Next, we tackle the problem of maximizing a *weighted* sum rate in a scenario with heterogeneous user characteristics. We establish a novel upper bound for the weighted sum capacity, which we then use to show that the maximum expected weighted sum rate can be asymptotically achieved by transmitting to a suitably selected subset of at most MC users, where C denotes the number of distinct user classes. Numerical experiments indicate that the asymptotic results are remarkably accurate and that the proposed schemes operate close to absolute performance bounds, even for a moderate number of users.

Thesis Supervisor: Eytan H. Modiano

Title: Associate Professor

Acknowledgments

Perhaps the most pleasant part of writing a dissertation is acknowledging the people who have played a role in making you what you are, both professionally and personally. Since professional debts are a little easier to acknowledge than personal ones, I shall begin with the former.

I express my deep gratitude to my advisor Prof. Eytan Modiano for his invaluable guidance, support and encouragement throughout the course of this work. His energetic and persevering attitude, combined with his clear, intuitive approach to tackling research problems have significantly influenced me. Working with him has been a pleasure on account of his friendly and informal demeanor as well.

Most of the work reported in this thesis was done in collaboration with Sem Borst and Phil Whiting, of the Mathematical Sciences Research Center at Bell Laboratories, Lucent Technologies, Murray Hill, NJ. I have immensely enjoyed and benefitted from working with Phil and Sem since the summer of 2005. I am grateful to them for their guidance and help throughout the past year. Sem's perfectionism towards research, and his serious, yet friendly attitude are exemplary. I am also greatly indebted to Phil for instilling in me the ability and willingness to look at things minutely, and the self-confidence to convert intuitively plausible propositions into precise mathematical statements. I also thank Hanan Weingarten (Technion) and Gerhard Kramer (Bell labs) for many useful suggestions and interesting discussions.

My profoundest expressions of gratitude go to my parents, K. Jagannathan and Geetha Jagannathan. I thank my father for what he is. While the people I mentioned above have taught me various things, my father has never taught me anything. He just *lives*, and that is lesson enough. I thank my mother for her constant affection, though it feels very inadequate to do so. I am grateful to Prabha for her companionship, and all the joy we share. I am also indebted to Dr. Dilip Veeraraghavan, professor of humanities and social sciences at IIT Madras, for triggering in me a great passion for Indian classical music.

I am thankful to Jay Kumar for his help and guidance on various topics, ranging from information theory to making delicious *Dosas*. I also thank all my group-mates for creating a great work-atmosphere. My thanks also go to all my friends and well-wishers I have not mentioned here, who have guided and helped me in various ways.

I thank God for His endless compassion.

Contents

1	Introduction	7
2	System Model and Known Results	12
2.1	Model description	12
2.2	Known information theoretic results	12
3	Scheduling in a Homogeneous System	15
3.1	Bounds for the Sum Capacity	15
3.2	Random channel vectors	17
3.3	Large- K asymptotics	18
3.3.1	Ratio asymptotics	19
3.3.2	Rate gap asymptotics	20
3.4	Large and small- P asymptotics	24
4	Scheduling in a Heterogeneous System	28
4.1	Bounds for the weighted sum rate	28
4.2	Random channel vectors	32
4.3	Large- K asymptotics	32
4.3.1	User selection schemes	33
4.3.2	Asymptotic optimality of the proposed schemes	35
5	Numerical Results	43
5.1	Homogeneous case	43
5.2	Heterogeneous case	45

6	Conclusions	51
A		52
A.1	Some useful determinant identities	52
A.2	Some results from extremal theory	54
A.3	Lemmas and proofs for the homogeneous case	56
A.3.1	Additional bounds for the sum capacity	56
A.3.2	Proof of Lemma 3.2.2	58
A.4	Lemmas and proofs for the heterogeneous case	59

Chapter 1

Introduction

The use of wireless communications for data as well as voice applications continues to experience tremendous growth. This continual growth creates an increasing pressure to squeeze the most out of the limited amount of wireless spectrum available. The use of antenna arrays offers a promising technique for improving spectrum efficiency so as to achieve higher data rates, larger capacity, better coverage, or a combination of these. The increase in data rates is of vital importance for enabling high-speed data applications in wireless environments. In dense urban areas where cell splitting and sectorization may have reached practical limitations, the capacity gain (supporting more users) is particularly relevant. The increase in coverage (installing fewer base stations) is especially attractive for service providers seeking to enter the market at an affordable capital investment.

The multi-antenna broadcast channel (BC) has been the subject of much research interest recently, owing primarily to the impressive capacity benefits that these systems can potentially offer. In order to achieve the full performance gain in any multi-antenna system, it is crucial that there be enough separation between the antenna elements in an array. Clearly, it is easier to realize this physical separation at a base station than on a mobile handset. Also, the incremental cost involved in setting up multiple antennas at a base station may be negligible compared to the total capital and operational costs. Thus, on the downlink from the base station to the mobile, it is easier to have multiple transmit than receive antennas, whereas the opposite is

true for the uplink.

In this thesis, we consider the downlink transmission from a single base station equipped with $M > 1$ transmit antennas to K independent data users each with a single receive antenna. In information-theoretic terms, this may be modeled as a multi-antenna broadcast channel (BC). Caire & Shamai [3] were the first to obtain the sum capacity expression for the Gaussian BC with two receivers, and to suggest the use of Dirty Paper Coding (DPC) [4] for transmitting over this channel. Viswanath & Tse [26] and Vishwanath *et al.* [25] extended the result for the sum capacity to an arbitrary number of users and receive antennas by exploiting a powerful duality relation with the multi-access channel which was further explored in Jindal *et al.* [12]. Recently, Weingarten *et al.* [30] showed that DPC in fact achieves the full capacity region of the multi-antenna Gaussian BC, thus providing a characterization of the entire capacity region.

Various researchers have investigated the sum capacity gains achievable in the above-described system by simultaneously transmitting to several users. In particular, Jindal & Goldsmith [10] show that the sum capacity gain over a TDMA strategy is approximately $\min\{M, K\}$, i.e., the minimum of the number of transmit antennas and the number of users. Jindal [8] demonstrates that the sum capacity grows with the SNR at rate $\min\{M, K\}$. In other words, multiple transmit antennas can potentially provide an M -fold gain in the sum capacity.

The above capacity results rely on the assumption that perfect channel state information is available at the transmitter, which usually involves feedback from the receivers. The amount of feedback overhead involved may be prohibitive, especially when the number of users is large, or just not be worth the actual gain in rate. In addition, DPC is quite a sophisticated technique and challenging to implement in an actual system.

Motivated by the above issues, extensive efforts have been made to devise practical transmission and coding schemes and find ways to reduce the amount of channel feedback information required. Hochwald *et al.* [6, 7] describe an algorithm based on

channel inversion and sphere encoding, and demonstrate that it closely approaches the sum capacity while being simpler to operate than DPC. Jindal [9] considers a multi-antenna BC with limited channel feedback information, and shows that the full sum capacity gain at high SNR values is achievable as long as the number of feedback bits grows linearly with the SNR (in dB).

As mentioned above, multiple transmit antennas can potentially yield an M -fold increase in the sum capacity. However, it is necessary that at least M users are served simultaneously in order to reap the full benefits. Transmitting to fewer than M users falls short of the maximum rate as it fails to fully exploit the available degrees of freedom. Transmitting to more than M users may be necessary to achieve the sum capacity in general, but the upper bound in [10] suggests that transmitting to a suitably selected subset of M near-orthogonal users is close to optimal. When the total number of users to choose from is sufficiently large, such a subset exists with high probability [21, 22].

Clearly, the above principle allows for a reduction of the amount of channel feedback and coding complexity. In particular, it suggests beam-forming (BF) schemes which construct M (random) orthogonal beams and serve the users with the largest channel gains on each of them with equal power. Transmission schemes along these lines are presented in Viswanath *et al.* [27], Sharif & Hassibi [17], and Vakili *et al.* [24]. Viswanathan & Kumaran [28] proposed fixed-beams and adaptive steerable-beams schemes grounded on that principle as well. Further related results may be found in Sharif & Hassibi [18, 19] who derive the asymptotic sum capacity for TDMA, DPC and beam-forming in the limit where the number of users grows large.

In this thesis, we propose scheduling schemes that transmit only to a small subset of users with favorable channel characteristics, and provide near-optimal performance when the total number of users to choose from is large. Extensive numerical experiments reveal that the scheduling schemes in fact operate remarkably close to absolute performance bounds, even when the number of users is fairly moderate. Since the proposed schemes only transmit to a small fraction of the users, they provide significant

scope for reducing the feedback overhead and operational complexity.

We first focus on a scenario with two-transmit antennas and statistically identical users, and analyze the gap between the full sum capacity and the rate that can be achieved by transmitting to a suitably selected pair of users. In particular, we consider a scheme that picks the user with the largest channel gain, and then selects a second user from the next $L - 1$ strongest ones to form the best possible pair with it, taking the orientation of channel vectors into account as well. We prove that the expected rate gap converges to $1/(L - 1)$ nats/symbol when the total number of users K tends to infinity. Allowing L to increase with K , we conclude that the gap asymptotically vanishes, and that the maximum expected sum rate is achievable by transmitting to a properly chosen pair of users. The fact that the rate gap decays as $1/(L - 1)$ also suggests that a modest value of L is adequate for most practical purposes. We remark that our scheme requires full channel feedback (i.e., both magnitude and phase information) only from the L strongest users. Finding the users with the largest channel gains can be accomplished using simple thresholding schemes wherein users with good channel gains feedback quantized versions of the magnitude of their channel vectors.

Next, we turn our attention to a more general system with M transmit antennas and heterogeneous user characteristics. For a heterogeneous system, the sum capacity is no longer an appropriate performance metric, because it does not reflect the potential fairness issues that arise. Hence, we will focus on maximizing a *weighted* sum rate, where the users with weaker channels would typically be assigned higher weights. Leaving fairness considerations aside, maximizing a weighted sum rate is also of critical importance in so-called queue-based scheduling strategies where the user weights are taken to be functions of the respective queue lengths. Queue-based scheduling strategies are particularly attractive because under mild assumptions they are known to achieve stability whenever feasible without explicit knowledge of the system parameters, see for instance [15, 20, 23].

Although the sum rate expression for the multi-antenna Gaussian BC and associ-

ated bounds have been thoroughly investigated, the problem of maximizing a general function over the capacity region has not attracted nearly as much attention. To the best of our knowledge, Viswanathan *et al.* [29] are among the few authors who consider the problem of attaining more general points on the boundary of the capacity region. In particular, they present an algorithm for finding the power allocation to achieve any weighted sum rate maximizing point. However, the optimization procedure is computationally demanding, especially for large numbers of users, and requires perfect channel state information. Lee & Jindal [13] study the problem of attaining the symmetric capacity, i.e., the maximum rate that can be provided to each of the users simultaneously.

In this work, we consider a M antenna broadcast system with a user population that consists of C distinct classes, where each class is assigned a non-negative weight. In this setting, we derive a generic upper bound for the weighted sum capacity, which includes as a special case the sum capacity bound in [10]. We then proceed to show that the upper bound is in fact attained for a particular ‘ideal’ configuration of MC channel vectors. Finally, we prove that a nearly ideal configuration of such channel vectors exists with high probability, and that the maximum expected weighted sum rate can thus be asymptotically achieved, when the total number of users grows large.

The remainder of this document is organized as follows. In Chapter 2, we present a detailed description of our system model, and review some known information theoretic results regarding the multi-antenna broadcast channel. In Chapter 3 we focus on the problem of maximizing the sum rate in a system with homogeneous users. Chapter 4 tackles the problem of maximizing the weighted sum rate in a system with heterogeneous user characteristics. In Chapter 5, we present the results of numerical simulations which indicate that the asymptotic results derived in the previous chapters are remarkably accurate even for moderate system sizes. Chapter 6 concludes the thesis.

Chapter 2

System Model and Known Results

2.1 Model description

We consider a broadcast channel (BC) with $M > 1$ transmit antennas and K receivers each with a single antenna, as schematically represented in Figure 2.1(a).

Let $\mathbf{x} \in \mathbb{C}^{M \times 1}$ be the transmitted vector signal and let $\mathbf{h}_k \in \mathbb{C}^{1 \times M}$ be the channel gain vector of the k -th receiver. Denote by $\mathbf{H} = [\mathbf{h}_1^\dagger \mathbf{h}_2^\dagger \cdots \mathbf{h}_K^\dagger]^\dagger$ the concatenated channel matrix of all K receivers. For now, the matrix \mathbf{H} is arbitrary but fixed. We assume that the transmitter has perfect channel state information, i.e., exact knowledge of the matrix \mathbf{H} . The circularly symmetric complex Gaussian noise at the k -th receiver is $n_k \in \mathbb{C}$ where n_k is distributed according to $\mathcal{CN}(0, 1)$. Thus the received signal at the k -th receiver is $y_k = \mathbf{h}_k \mathbf{x} + n_k$. The covariance matrix of the transmitted signal is $\Sigma_x = \mathbb{E}[\mathbf{x}\mathbf{x}^\dagger]$. The transmitter is subject to a power constraint P , which implies $\text{Tr}(\Sigma_x) \leq P$. (Here Tr denotes the trace operator, which is the sum of the diagonal elements of a square matrix.)

2.2 Known information theoretic results

Now we review some known results regarding the capacity region and the sum capacity of the multi-antenna Gaussian BC.

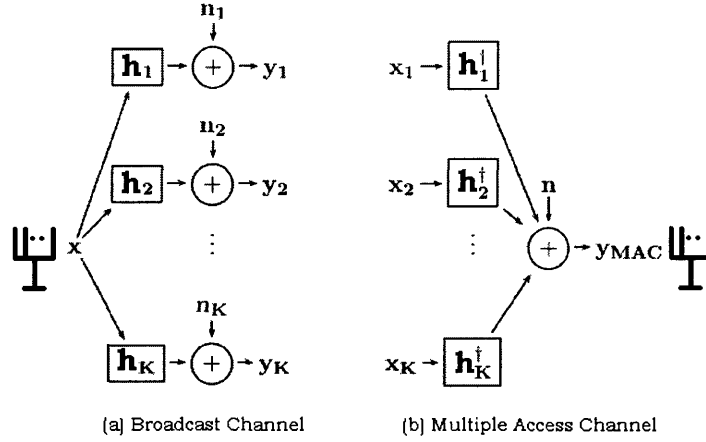


Figure 2-1: The multi-antenna BC (left) and the MAC (right) have the same capacity region.

Let $\pi(k)$, $k = 1, \dots, K$, be a permutation of $k = 1, \dots, K$. As shown in [25], the following rate vector is achievable using Dirty Paper Coding (DPC):

$$R_{\pi(k)} = \log \left(\frac{1 + \mathbf{h}_{\pi(k)} (\sum_{l \leq k} \Sigma_{\pi(l)}) \mathbf{h}_{\pi(k)}^\dagger}{1 + \mathbf{h}_{\pi(k)} (\sum_{l < k} \Sigma_{\pi(l)}) \mathbf{h}_{\pi(k)}^\dagger} \right), \quad k = 1, \dots, K.$$

The DPC region is defined as the convex hull of the union of all such rate vectors, over all positive semi-definite covariance matrices that satisfy the power constraint $\sum_{k=1}^K \text{Tr}(\Sigma_k) \leq P$, and over all possible permutations $\pi(k)$. As shown in [3, 30], DPC in fact achieves the entire capacity region denoted as \mathcal{C}_{BC} . The weighted sum capacity $\mathcal{C}_{BC}^{\mathbf{w}}(\mathbf{H}, P)$ for any weight vector $\mathbf{w} \in \mathbb{R}_+^K$ can therefore be written as

$$\mathcal{C}_{BC}^{\mathbf{w}}(\mathbf{H}, P) = \max_{\mathbf{R} \in \mathcal{C}_{BC}} \sum_{k=1}^K \mathbf{w}_k \mathbf{R}_k = \max_{\pi} \max_{\Sigma_k \geq 0, \sum_{k=1}^K \text{Tr}(\Sigma_k) \leq P} \sum_{k=1}^K w_{\pi(k)} \log \left(\frac{1 + \mathbf{h}_{\pi(k)} (\sum_{l \leq k} \Sigma_{\pi(l)}) \mathbf{h}_{\pi(k)}^\dagger}{1 + \mathbf{h}_{\pi(k)} (\sum_{l < k} \Sigma_{\pi(l)}) \mathbf{h}_{\pi(k)}^\dagger} \right). \quad (2.1)$$

Unfortunately, the maximization in (2.1) involves a non-concave function of the covariance matrices, which makes it hard to deal with analytically as well as numerically. However, in [25, 26], a duality is shown to exist between the BC and the Gaussian multiple-access channel (MAC) with a sum-power constraint P . That is, the dual MAC which is formed by reversing the roles of transmitters and receivers,

as represented in Figure 2.1(b), has the same capacity region as the BC.

Let $S_k := \sum_{l=1}^k R_l$ be the partial sum rate of the first k users. Note that the weighted capacity can be written in terms of the partial sum rates as $\mathcal{C}_{BC}^{\mathbf{w}}(\mathbf{H}, P) = \sum_{k=1}^K \Delta w_k S_k$, where $\Delta w_k := w_k - w_{k+1}$, with the convention that $w_{K+1} = 0$. Without loss of generality we may assume that $w_1 \geq w_2 \geq \dots \geq w_K$. Using the duality result, the weighted sum capacity (2.1) of the BC can thus be expressed in terms of the dual MAC weighted sum rate as

$$\mathcal{C}_{BC}^{\mathbf{w}}(\mathbf{H}, P) = \max_{\sum_{k=1}^K P_k \leq P} \sum_{k=1}^K \Delta w_k \log \det \left(I_M + \sum_{l=1}^k P_l \mathbf{h}_l^\dagger \mathbf{h}_l \right), \quad (2.2)$$

where $P_k \geq 0$ denotes the power allocated to the k -th receiver. As a special case of (2.2) with $w_k = 1$, $k = 1, \dots, K$, the sum capacity is obtained as

$$\mathcal{C}_{BC}^{\text{sum}}(\mathbf{H}, P) = \max_{\sum_{k=1}^K P_k \leq P} \log \det \left(I_M + \sum_{k=1}^K P_k \mathbf{h}_k^\dagger \mathbf{h}_k \right). \quad (2.3)$$

Since $\log \det(\cdot)$ is a concave function on the cone of positive-definite matrices, the problems in (2.2) and (2.3) only involve maximizing a concave objective function subject to convex constraints. Specialized algorithms have been developed to solve these problems [11, 29].

Chapter 3

Scheduling in a Homogeneous System

In this chapter, we study the problem of maximizing the sum rate in a system with $M = 2$ transmit antennas and statistically identical users. The sum capacity is a key metric of interest for the BC as it measures the maximum achievable total rate. Since it only considers the aggregate throughput, it does not reflect potential fairness issues that arise when users with widely disparate channel characteristics obtain vastly different throughput portions. In the present chapter, however, we focus on the case of statistically identical users, which by symmetry will obtain equal long-term throughput shares, so that fairness is not a major issue. In the next chapter, we will address the problem of maximizing a *weighted* sum rate in a system where the users may have different characteristics.

We will show that the sum capacity can be closely approached by transmitting to a suitably selected pair of users as the total number of users grows large. In preparation for that, we first present some useful lower and upper bounds for the sum capacity.

3.1 Bounds for the Sum Capacity

Denote by $\mathbf{h}_{(k)}$ the channel vector of the receiver with the k -th largest norm, i.e., $\|\mathbf{h}_{(1)}\|^2 \geq \|\mathbf{h}_{(2)}\|^2 \geq \dots \geq \|\mathbf{h}_{(K)}\|^2$. The next upper bound for the sum capacity is

established in [10]:

$$\mathcal{C}_{BC}^{\text{sum}}(\mathbf{H}, P) \leq M \log \left(1 + \frac{P}{M} \|\mathbf{h}_{(1)}\|^2 \right). \quad (3.1)$$

Observe that the above bound can be achieved when there are M receivers with orthogonal channel vectors tied for the maximum norm $\|\mathbf{h}_{(1)}\|^2$. From now on, we focus on the case of $M = 2$ transmit antennas, unless mentioned otherwise. The upper bound for the sum capacity in (3.1) then becomes

$$\mathcal{C}_{BC}^{\text{sum}}(\mathbf{H}, P) \leq 2 \log \left(1 + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2 \right). \quad (3.2)$$

Taking $P_i = P_j = P/2$ and $P_k = 0$ for all $k \neq i, j$ in Equation (2.3), we obtain a simple lower bound for the sum capacity

$$\mathcal{C}_{BC}^{\text{sum}}(\mathbf{H}, P) \geq C(\mathbf{h}_i, \mathbf{h}_j, P) := \log \det \left(I_2 + \frac{P}{2} (\mathbf{h}_i^\dagger \mathbf{h}_i + \mathbf{h}_j^\dagger \mathbf{h}_j) \right), \quad (3.3)$$

which corresponds to transmitting to users i and j at equal power.

For any two vectors $\mathbf{g}, \mathbf{h} \in \mathbb{C}^2$, let $U(\mathbf{g}, \mathbf{h}) := \frac{|\langle \mathbf{g}, \mathbf{h} \rangle|^2}{\|\mathbf{g}\|^2 \|\mathbf{h}\|^2}$ be the squared normalized inner product. Using Lemma A.1.2 in the Appendix, we obtain

$$C(\mathbf{h}_i, \mathbf{h}_j, P) = \log \left(1 + \frac{P}{2} (\|\mathbf{h}_i\|^2 + \|\mathbf{h}_j\|^2) + \frac{P^2}{4} \|\mathbf{h}_i\|^2 \|\mathbf{h}_j\|^2 V_{ij} \right), \quad (3.4)$$

with $V_{ij} = 1 - U(\mathbf{h}_i, \mathbf{h}_j)$.

The lower bound expression (3.4) reflects the fact that the sum rate for two users critically depends on the norms of the respective channel vectors and their degree of orthogonality. In particular, the sum rate is large when the channel vectors are nearly orthogonal and have large norms. Indeed, the lower bound coincides with the upper bound (3.2) when users i and j are orthogonal and tied for the maximum norm, i.e., $\|\mathbf{h}_i\|^2 = \|\mathbf{h}_j\|^2 = \|\mathbf{h}_{(1)}\|^2$ and $\langle \mathbf{h}_i, \mathbf{h}_j \rangle = 0$.

3.2 Random channel vectors

The lower and upper bounds for the sum capacity in the previous section hold for any arbitrary but fixed set of channel vectors. In order to derive meaningful asymptotic results, we will, in the remainder of this chapter, assume the channel vectors to be random and focus on the *expected* sum rate. We will adhere to the common assumption that the components of the channel vectors of the various users to be independent and distributed according to $\mathcal{CN}(0, 1)$, which corresponds to independent Rayleigh fading.

Remark 3.2.1 *The randomness in the channel vectors may be interpreted as variations resulting from fast fading due to multi-path propagation effects. The expected sum rate then represents the long-term system throughput. Implicitly, we make here the usual block fading assumption, where the frame length is short enough for the channel to remain (nearly) constant over the duration of a frame, yet sufficiently long to achieve a transmission rate close to the theoretical capacity.*

As mentioned earlier, the two-user sum rate critically depends on the norms of the channel vectors and their squared normalized inner product, and the statistical properties of these two quantities will therefore play a crucial role. The next lemma characterizes the distribution of the squared normalized inner product of two arbitrary channel vectors.

Lemma 3.2.1 *For any two users $i, j = 1, \dots, K, i \neq j$, the squared normalized inner product $U(\mathbf{h}_i, \mathbf{h}_j) := \frac{|\langle \mathbf{h}_i, \mathbf{h}_j \rangle|^2}{\|\mathbf{h}_i\|^2 \|\mathbf{h}_j\|^2}$ is independent of the norms of the respective channel vectors and distributed as the minimum of $(M - 1)$ i.i.d. uniform random variables in $[0, 1]$. In particular, when $M = 2$, the above quantity is uniform in $[0, 1]$.*

Proof

Since the normalized inner product is invariant under a unitary transformation of both vectors (i.e., a rotation of the co-ordinate axes), we can assume that one of the vectors, say \mathbf{h}_i , is oriented along the $[1 \ 0 \dots 0]$ direction. Thus $\mathbf{h}_i = [W_1 \ 0 \dots 0]$, where $W_1 = \|\mathbf{h}_i\|$. Also, since the distribution of circularly symmetric complex

Gaussian vectors is invariant under unitary transformations, we may assume that $\mathbf{h}_j = [X_1 + iY_1 \ X_2 + iY_2 \ \dots \ X_M + iY_M]$, where $X_1, X_2, \dots, X_M, Y_1, Y_2, \dots, Y_M$ are i.i.d. normal random variables, independent of W_1 . Thus, the quantity $U(\mathbf{h}_i, \mathbf{h}_j)$ is distributed as

$$\frac{X_1^2 + Y_1^2}{X_1^2 + Y_1^2 + X_2^2 + Y_2^2 + \dots + X_M^2 + Y_M^2} = \frac{Z_1}{Z_1 + \dots + Z_M},$$

where Z_1, \dots, Z_M are i.i.d. unit exponential random variables. The latter quantity may be interpreted as the ratio of the first and M -th event times in a Poisson process, which is known to be distributed as the minimum of $(M - 1)$ i.i.d. uniform random variables in $[0, 1]$, independent of $\|\mathbf{h}_j\| = \sqrt{Z_1 + \dots + Z_M}$ (as well as $\|\mathbf{h}_i\| = W_1$). See [16, p. 67].

□

We now turn the attention to the order statistics of the channel norms. The next lemma shows that the difference between the L -th largest and the maximum channel norm is asymptotically negligible in a certain sense, as long as L grows sufficiently slowly with K .

Lemma 3.2.2 *Let $L(K)$ be a sequence such that $L(K) = o(K^\delta)$ for any $0 < \delta < 1$ as $K \rightarrow \infty$ and $A, B, Q > 0$ positive constants. Then*

$$\lim_{K \rightarrow \infty} \mathbb{E} [\log (A + Q\|\mathbf{h}_{(1)}\|^2)] - \mathbb{E} [\log (B + Q\|\mathbf{h}_{(L(K))}\|^2)] = 0.$$

Proof

See Appendix. □

3.3 Large- K asymptotics

As mentioned earlier, the upper bound in (3.2) for the sum capacity can be achieved when there is a pair of orthogonal users tied for the maximum channel norm $\|\mathbf{h}_{(1)}\|^2$ by granting equal power to each of them. Intuitively, when the total number of users

is large, there exists with high probability a pair of users which are nearly orthogonal and have norms close to the maximum. This suggests that the sum capacity can be closely approached by transmitting to such a pair of users and allocating equal power to each of them.

We are now ready to formalize the above assertion. We will consider three heuristic selection schemes for scheduling a pair of users with equal power. Scheme I picks two arbitrary users among the L strongest ones. Scheme II selects an arbitrary user among the L strongest ones, and a second one from the same group to form the best pair, i.e., the pair that maximizes the sum rate. Scheme III picks the best pair among the L strongest users, i.e., the pair that maximizes the sum rate. Note that scheme II dominates scheme I and that scheme III in turn dominates scheme II, and that all three schemes coincide when $L = 2$.

3.3.1 Ratio asymptotics

We first establish ratio asymptotics for the above-described schemes. Specifically, the next theorem shows that the ratio of the rate achieved by scheme I to the upper bound in (3.2) converges to unity as the number of users grows large. Thus, scheme I is asymptotically optimal in a ratio sense, and hence so are the dominating schemes II and III.

Theorem 3.3.1 *For any fixed value of $L \geq 2$,*

$$\lim_{K \rightarrow \infty} \frac{\mathbb{E} [C(\mathbf{h}_{(i)}, \mathbf{h}_{(j)}, P)]}{\mathbb{E} [2 \log (1 + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2)]} = 1, \quad (3.5)$$

for all $i, j \leq L, i \neq j$.

Proof

It follows from Equations (3.2) and (3.3) the ratio is no larger than one for any fixed K and L . Thus, it suffices to show that the liminf of the ratio is no smaller

than one as $K \rightarrow \infty$. Lemma A.3.1 in the Appendix gives that

$$C(\mathbf{h}_{(i)}, \mathbf{h}_{(j)}, P) \geq 2 \log \left(1 + \frac{P}{2} \|\mathbf{h}_{(L)}\|^2 \right) + \log(V_{(i)(j)}),$$

with $V_{(i)(j)} := 1 - U(\mathbf{h}_{(i)}, \mathbf{h}_{(j)})$.

Lemma 3.2.1 implies

$$\mathbb{E} [\log(V_{(i)(j)})] = \int_{x=0}^1 \log(x) dx = [x(\log(x) - 1)]_{x=0}^1 = -1.$$

The proof is then completed using Lemma 3.2.2 with $A = B = 1$, $Q = P/2$, and noting that $\mathbb{E} [\log(1 + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2)] \rightarrow \infty$ as $K \rightarrow \infty$.

□

While the ratio asymptotics provide some initial understanding, they only offer limited practical insight. The fact that all three selection schemes are asymptotically optimal for any fixed value of L reflects the insensitivity of the results. In particular, the ratio asymptotics are too crude to capture the relative importance of the degree of orthogonality versus the magnitude of the channel vectors. Thus, they provide no indication of the relative performance of the various schemes and little guidance as to what a suitable choice of L might be for a given finite value of K . Also, the ratio asymptotics are too rough to discern any possible $o(\log \log K)$ gap between the sum rate achieved by any of these schemes and the capacity limit.

3.3.2 Rate gap asymptotics

In order to discriminate among the various selection schemes and gain a better sense of the performance impact of the parameter L , we now proceed to derive sharper asymptotics. In particular, we consider the *difference* between the expected sum rate and the upper bound in (3.2), which is not only more discerning than the ratio but also more physically meaningful.

Theorem 3.3.2 For any fixed value of $L \geq 2$, $l \leq L$, the difference

$$\mathbb{E} \left[2 \log \left(1 + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2 \right) \right] - \mathbb{E} \left[\max_{k=1, \dots, L, k \neq l} C(\mathbf{h}_{(l)}, \mathbf{h}_{(k)}, P) \right] \rightarrow \frac{1}{L-1}$$

as $K \rightarrow \infty$.

Proof

We first prove that the limsup of the difference is no larger than $1/(L-1)$.

Using Lemma A.3.1 in the Appendix, we obtain

$$\begin{aligned} & \max_{k=1, \dots, L, k \neq l} C(\mathbf{h}_{(l)}, \mathbf{h}_{(k)}, P) \\ & \geq \max_{k=1, \dots, L, k \neq l} 2 \log \left(1 + \frac{P}{2} \|\mathbf{h}_{(L)}\|^2 \right) + \log(V_{(l)(k)}) \\ & = 2 \log \left(1 + \frac{P}{2} \|\mathbf{h}_{(L)}\|^2 \right) + \max_{k=1, \dots, L, k \neq l} \log(V_{(l)(k)}), \end{aligned}$$

with $V_{(l)(k)} := 1 - U(\mathbf{h}_{(l)}, \mathbf{h}_{(k)})$.

For compactness, denote

$$\Delta(L) := \log \left(1 + \frac{P}{2} \|\mathbf{h}_{(L)}\|^2 \right) - \log \left(1 + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2 \right).$$

Then,

$$\begin{aligned} & 2\mathbb{E} \left[\log \left(1 + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2 \right) \right] - \mathbb{E} \left[\max_{k=1, \dots, L, k \neq l} C(\mathbf{h}_{(l)}, \mathbf{h}_{(k)}, P) \right] \\ & \leq -2\mathbb{E} [\Delta(L)] - \mathbb{E} \left[\max_{k=1, \dots, L, k \neq l} \log(V_{(l)(k)}) \right]. \end{aligned}$$

Taking $A = B = 1$, $Q = P/2$ in Lemma 3.2.2, it follows that $\limsup_{K \rightarrow \infty} -\mathbb{E} [\Delta(L)] = 0$.

Using Lemma 3.2.1, a straightforward calculation yields

$$\mathbb{E} \left[\max_{k=1, \dots, L, k \neq l} \log(V_{(l)(k)}) \right] = \int_{x=0}^1 \log(x)(L-1)x^{L-2} dx = \left[x^{L-1} \left(\log(x) - \frac{1}{L-1} \right) \right]_{x=0}^1 = -\frac{1}{L-1}.$$

We now show that the liminf of the difference is no smaller than $1/(L-1)$.

Using Lemma A.3.4 in the Appendix, we obtain

$$\begin{aligned}
& \max_{k=1,\dots,L,k \neq l} C(\mathbf{h}_{(l)}, \mathbf{h}_{(k)}, P) \\
& \leq \max_{k=1,\dots,L,k \neq l} 2 \log \left(\frac{1}{\epsilon} + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2 \right) + \log(\max\{\epsilon, V_{(l)(k)}\}) \\
& \leq 2 \log \left(\frac{1}{\epsilon} + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2 \right) + \log(\max\{\epsilon, \max_{k=1,\dots,L,k \neq l} V_{(l)(k)}\}).
\end{aligned}$$

Thus,

$$\begin{aligned}
& 2\mathbb{E} \left[\log \left(1 + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2 \right) \right] - \mathbb{E} \left[\max_{k=1,\dots,L,k \neq l} C(\mathbf{h}_{(l)}, \mathbf{h}_{(k)}, P) \right] \geq \\
& 2\mathbb{E} \left[\log \left(1 + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2 \right) \right] - 2\mathbb{E} \left[\log \left(\frac{1}{\epsilon} + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2 \right) \right] - \mathbb{E} \left[\log \left(\max\{\epsilon, \max_{k=1,\dots,L,k \neq l} V_{(l)(k)}\} \right) \right].
\end{aligned}$$

Taking $A = 1$, $B = \frac{1}{\epsilon}$, $Q = P/2$ in Lemma 3.2.2, it follows that for any $\epsilon > 0$,

$$\limsup_{K \rightarrow \infty} \mathbb{E} \left[\log \left(1 + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2 \right) \right] - \mathbb{E} \left[\log \left(\frac{1}{\epsilon} + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2 \right) \right] = 0.$$

Using Lemma 3.2.1, a straightforward computation yields

$$\begin{aligned}
& \mathbb{E} \left[\log \left(\max\{\epsilon, \max_{k=1,\dots,L,k \neq l} V_{(l)(k)}\} \right) \right] \\
& = \int_{x=\epsilon}^1 \log(x)(L-1)x^{L-2} dx + \epsilon^{L-1} \log(\epsilon) \\
& = \left[x^{L-1} \left(\log(x) - \frac{1}{L-1} \right) \right]_{x=\epsilon}^1 + \epsilon^{L-1} \log(\epsilon) \\
& = \frac{\epsilon^{L-1} - 1}{L-1}.
\end{aligned}$$

Letting $\epsilon \downarrow 0$, the result follows. □

Theorem 3.3.3 For any fixed value l and sequence $L(K)$ with $\lim_{K \rightarrow \infty} L(K) \rightarrow \infty$,

$$2\mathbb{E} \left[\log \left(1 + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2 \right) \right] - \mathbb{E} \left[\max_{k=1, \dots, L(K), k \neq l} C(\mathbf{h}_{(l)}, \mathbf{h}_{(k)}, P) \right] \rightarrow 0$$

as $K \rightarrow \infty$.

Proof

Equations (3.2) and (3.3) imply that the above difference is non-negative for any fixed K . Thus, it suffices to show that the limsup of the difference is non-positive.

This follows by observing that

$$\limsup_{K \rightarrow \infty} \mathbb{E} \left[\max_{k=1, \dots, L(K), k \neq l} C(\mathbf{h}_{(l)}, \mathbf{h}_{(k)}, P) \right] - \mathbb{E} \left[\log \left(1 + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2 \right)^2 \right] \leq$$

$$\limsup_{K \rightarrow \infty} \mathbb{E} \left[\max_{k=1, \dots, L, k \neq l} C(\mathbf{h}_{(l)}, \mathbf{h}_{(k)}, P) \right] - \mathbb{E} \left[\log \left(1 + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2 \right)^2 \right]$$

for any fixed value of L , and then invoking Theorem 3.3.2 and letting $L \rightarrow \infty$.

□

The next corollaries follow as immediate consequences from Theorems 3.3.2 and 3.3.3.

Corollary 3.3.1 For any fixed value of L , $l \leq L$,

$$\mathbb{E} [C_{BC}^{\text{sum}}(\mathbf{H}, P)] - \mathbb{E} \left[\max_{k=1, \dots, L, k \neq l} C(\mathbf{h}_{(l)}, \mathbf{h}_{(k)}, P) \right] \rightarrow \frac{1}{L-1}$$

as $K \rightarrow \infty$.

The above corollary shows that the asymptotic performance gap of scheme II decays as $1/(L-1)$, which suggests that a relatively moderate value of L may be adequate for most practical purposes.

Corollary 3.3.2

$$\mathbb{E} [C_{BC}^{\text{sum}}(\mathbf{H}, P)] - \mathbb{E} [C(\mathbf{h}_{(1)}, \mathbf{h}_{(2)}, P)] \rightarrow 1$$

as $K \rightarrow \infty$.

The above corollary corresponds to a special case of scheme I with $L = 2$, and shows that simply selecting the two strongest users leaves a performance gap of 1 nats/symbol.

Corollary 3.3.3 *For any fixed value l and sequence $L(K)$ with $\lim_{K \rightarrow \infty} L(K) = \infty$,*

$$\mathbb{E} [C_{BC}^{\text{sum}}(\mathbf{H}, P)] - \mathbb{E} \left[\max_{k=1, \dots, L(K), k \neq l} C(\mathbf{h}_{(l)}, \mathbf{h}_{(k)}, P) \right] \rightarrow 0$$

as $K \rightarrow \infty$.

The above corollary shows that scheme II is asymptotically optimal when sufficiently many users are considered, and thus implies that the dominating scheme III is asymptotically optimal as well. As a by-product, we conclude that the upper bound (3.2) is asymptotically tight.

In conclusion, the above results show that scheme II is asymptotically optimal in the sense that the absolute gap to the sum capacity vanishes to zero provided $L(K) \rightarrow \infty$ as $K \rightarrow \infty$. Thus, transmitting to a suitably selected pair of users is asymptotically optimal, where one of them may in fact be arbitrarily chosen from a fixed short list. The gain from considering all pairs of users, as in scheme III, is asymptotically negligible. However, picking an arbitrary pair of users, as in scheme I, is not optimal even when the users are the two strongest ones.

3.4 Large and small- P asymptotics

We now take a brief look at small and large- P asymptotics for a fixed system size K , where the channel vectors are no longer assumed to be random.

Proposition 3.4.1 *For any i, j with $|\langle \mathbf{h}_i, \mathbf{h}_j \rangle|^2 < \|\mathbf{h}_i\|^2 \|\mathbf{h}_j\|^2$, $\lim_{P \rightarrow \infty} \frac{C(\mathbf{h}_i, \mathbf{h}_j, P)}{2 \log(1 + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2)} = 1$.*

Proof

It is clear that the ratio is smaller than one for any value of P . Thus, it suffices to show that the liminf of the ratio is not smaller than one as $P \rightarrow \infty$.

Using Lemma A.3.1, we obtain

$$C(\mathbf{h}_i, \mathbf{h}_j, P) \geq \log \left(1 + \frac{P}{2} \|\mathbf{h}_i\|^2 \right) + \log \left(1 + \frac{P}{2} \|\mathbf{h}_j\|^2 \right) + \log(V_{ij}),$$

with $V_{ij} := 1 - \frac{|\langle \mathbf{h}_i, \mathbf{h}_j \rangle|^2}{\|\mathbf{h}_i\|^2 \|\mathbf{h}_j\|^2} > 0$.

The result now follows readily. \square

Corollary 3.4.1 *For any i, j with $|\langle \mathbf{h}_i, \mathbf{h}_j \rangle|^2 < \|\mathbf{h}_i\|^2 \|\mathbf{h}_j\|^2$, $\lim_{P \rightarrow \infty} \frac{\mathcal{C}_{BC}(\mathbf{h}_i, \mathbf{h}_j, P)}{\mathcal{C}_{BC}(\mathbf{h}_1, \dots, \mathbf{h}_K, P)} = 1$.*

Proof

Follows from Proposition 3.4.1 and the fact that $\mathcal{C}_{BC}(\mathbf{h}_i, \mathbf{h}_j, P) \leq \mathcal{C}_{BC}(\mathbf{h}_1, \dots, \mathbf{h}_K, P) \leq 2 \log \left(1 + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2 \right)$. \square

Like in the large- K regime, we find that considering ratios is too crude to provide much practical insight in the large- P regime. As the above proposition indicates, transmitting to any pair of users which are not perfectly co-linear, is asymptotically optimal in that sense, which does not offer any meaningful guidance as to how to perform user selection in an actual system. In contrast, considering absolute differences *does* yield valuable insight, as the next lemma shows, and in fact suggests a specific criterion for user pair selection: the best pair of users in the large- P regime is the one that maximizes the expression $r(i, j) := \|\mathbf{h}_i\|^2 \|\mathbf{h}_j\|^2 V_{ij}$.

Proposition 3.4.2 *For any i, j ,*

$$\lim_{P \rightarrow \infty} \left[2 \log \left(1 + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2 \right) - C(\mathbf{h}_i, \mathbf{h}_j, P) \right] = \lim_{P \rightarrow \infty} \left[2 \log \left(1 + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2 \right) - \mathcal{C}_{BC}(\mathbf{h}_i, \mathbf{h}_j, P) \right] = \log \left(\frac{\|\mathbf{h}_{(1)}\|^4}{\|\mathbf{h}_i\|^2 \|\mathbf{h}_j\|^2 V_{ij}} \right).$$

Proof

Since $C(\mathbf{h}_i, \mathbf{h}_j, P) \leq \mathcal{C}_{BC}(\mathbf{h}_i, \mathbf{h}_j, P)$, it suffices to consider the limsup of the first term and the liminf of the second term.

We first prove that the limsup is no larger than the stated quantity.

Using Lemma A.3.1, we obtain

$$C(\mathbf{h}_i, \mathbf{h}_j, P) \geq 2 \log \left(\frac{P}{2} \right) + \log (\|\mathbf{h}_i\|^2) + \log (\|\mathbf{h}_j\|^2) + \log(V_{ij}). \quad (3.6)$$

Also,

$$2 \log \left(1 + \frac{P}{2} \|\mathbf{h}_{(1)}\|^2 \right) = 2 \log \left(\frac{P}{2} \right) + 2 \log \left(\frac{2}{P} + \|\mathbf{h}_{(1)}\|^2 \right). \quad (3.7)$$

Subtracting (3.6) from (3.7) and letting $P \rightarrow \infty$, the first part of the assertion follows.

Next, we deal with the liminf of the second expression.

$$\begin{aligned} & \max_{P_i + P_j \leq P} 1 + P_i \|\mathbf{h}_i\|^2 + P_j \|\mathbf{h}_j\|^2 + P_i P_j \|\mathbf{h}_i\|^2 \|\mathbf{h}_j\|^2 U_{ij} \\ & \leq 1 + P(\|\mathbf{h}_i\|^2 + \|\mathbf{h}_j\|^2) + \frac{P^2}{4} \|\mathbf{h}_i\|^2 \|\mathbf{h}_j\|^2 V_{ij}. \end{aligned}$$

Thus,

$$\mathcal{C}_{BC}(\mathbf{h}_i, \mathbf{h}_j, P) \leq 2 \log \left(\frac{P}{2} \right) + \log \left(\frac{4}{P^2} + \frac{4}{P} (\|\mathbf{h}_i\|^2 + \|\mathbf{h}_j\|^2) + \|\mathbf{h}_i\|^2 \|\mathbf{h}_j\|^2 V_{ij} \right) \quad (3.8)$$

Subtracting (3.8) from (3.7), and then letting $P \rightarrow \infty$, the result follows. \square

In the small- P regime, the gain obtained by using Dirty Paper Coding approaches unity [10]. That is, it is asymptotically optimal to transmit to the user with the largest channel gain.

Proposition 3.4.3

$$\lim_{P \downarrow 0} \mathcal{C}_{BC}(\mathbf{h}_1, \dots, \mathbf{h}_K, P)/P = \lim_{P \downarrow 0} \log(\det(1 + P\|\mathbf{h}_{(1)}\|^2))/P = \|\mathbf{h}_{(1)}\|^2.$$

Proof

Follows directly from Equation (3.2).

□

Chapter 4

Scheduling in a Heterogeneous System

In the previous chapter, we studied the problem of maximizing the sum rate in a two-antenna system with statistically identical users, and showed that transmitting to a suitably selected pair of users asymptotically achieves the maximum expected sum rate. We now turn our attention to a more general system with M transmit antennas and heterogeneous user characteristics (i.e., channels are not necessarily i.i.d). As mentioned earlier, the sum capacity is no longer an appropriate performance metric now, because it does not reflect the potential fairness issues that arise when users with different channel statistics obtain vastly different throughput portions. Hence, we will focus on the problem of maximizing a *weighted* sum rate, and we will demonstrate that transmitting to a properly selected group of users asymptotically achieves the maximum expected weighted sum rate, although scheduling just two users will no longer be sufficient in general.

4.1 Bounds for the weighted sum rate

We first establish a generic upper bound for the weighted sum rate for an arbitrary number of M transmit antennas. Let w_k be the weight associated with the k -th user. For notational convenience, define $\Delta w_k := w_k - w_{k+1}$ with the convention that

$w_{K+1} = 0$. Without loss of generality, we assume that the users are indexed such that $w_1 \geq w_2 \geq \dots \geq w_K$.

Theorem 4.1.1 *For any given set of channel vectors,*

$$\mathcal{C}_{BC}^w(\mathbf{H}, P) \leq \max_{\sum_{k=1}^K P_k \leq P} \Delta w_1 \log(1 + P_1 \|\mathbf{h}_1\|^2) + M \sum_{k=2}^K \Delta w_k \log \left(1 + \sum_{l=1}^k \frac{P_l}{M} \|\mathbf{h}_l\|^2 \right).$$

Proof

Equation (2.2) yields that $\mathcal{C}_{BC}^w(\mathbf{H}, P) = \sum_{k=1}^K \Delta w_k S_k$, with

$$S_k = \log \det \left(I_M + \sum_{l=1}^k P_l \mathbf{h}_l^\dagger \mathbf{h}_l \right).$$

Clearly,

$$S_1 = \log(1 + P_1 \|\mathbf{h}_1\|^2). \quad (4.1)$$

Using Hadamard's inequality for Hermitian positive semi-definite matrices [5, p. 502], and the concavity of the log function, we obtain

$$S_k \leq \sum_{m=1}^M \log \left(1 + \sum_{l=1}^k P_l h_{lm}^\dagger h_{lm} \right) \leq \log \left(1 + \sum_{l=1}^k \frac{P_l}{M} \|\mathbf{h}_l\|^2 \right) \quad (4.2)$$

for all $k = 2, \dots, K$.

Substituting inequalities (4.1) and (4.2), the statement of the theorem follows. \square

The next upper bound follows as a straightforward corollary of Theorem 4.1.1.

Corollary 4.1.1 *For any given set of channel vectors,*

$$\mathcal{C}_{BC}^w(\mathbf{H}, P) \leq M \max_{\sum_{k=1}^K P_k \leq P/M} \sum_{k=1}^K \Delta w_k \log \left(1 + \sum_{l=1}^k P_l \|\mathbf{h}_l\|^2 \right). \quad (4.3)$$

In order to develop a suitable asymptotic framework, we assume that there are C classes of users, with K_c the number of class- c users and $\sum_{c=1}^C K_c = K$. Let $\mathbf{h}_k^{(c)}$ be

the channel vector of the k -th class- c user. With minor abuse of notation, we let w_c be the weight associated with class c , and define $\Delta w_c := w_c - w_{c+1}$, with the convention that $w_{C+1} = 0$ as before. Let T_c be the total rate received by class c . Thus the weighted sum rate is $T := \sum_{c=1}^C w_c T_c$. Without loss of generality, we assume that the classes are indexed such that $w_1 \geq w_2 \geq \dots \geq w_C$. Let $\mathbf{h}_{(k)}^{(c)}$ be the channel vector of the class- c user with the k -th largest norm, i.e., $\|\mathbf{h}_{(1)}^{(c)}\|^2 \geq \|\mathbf{h}_{(2)}^{(c)}\|^2 \geq \dots \geq \|\mathbf{h}_{(K_c)}^{(c)}\|^2$.

The next corollary specializes the upper bound in (4.3) to a class-based system.

Corollary 4.1.2 *For any given set of channel vectors,*

$$\sum_{c=1}^C w_c T_c \leq M \max_{\sum_{c=1}^C P_c \leq P/M} \sum_{c=1}^C \Delta w_c \log \left(1 + \sum_{d=1}^c P_d \|\mathbf{h}_{(1)}^{(d)}\|^2 \right). \quad (4.4)$$

Proof

Using Corollary 4.1.1, we obtain

$$\begin{aligned} \sum_{c=1}^C w_c T_c &\leq M \max_{\sum_{c=1}^C \sum_{k=1}^{K_c} P_k^{(c)} \leq P/M} \sum_{c=1}^C \Delta w_c \log \left(1 + \sum_{d=1}^c \sum_{k=1}^{K_d} P_k^{(d)} \|\mathbf{h}_k^{(d)}\|^2 \right) \\ &\leq M \max_{\sum_{c=1}^C P_c \leq P/M} \sum_{c=1}^C \Delta w_c \log \left(1 + \sum_{d=1}^c P_d \|\mathbf{h}_{(1)}^{(d)}\|^2 \right). \end{aligned}$$

□

Note that when all weights are taken equal to one, the upper bound in (4.3) reduces to that in Equation (3.1) for the sum rate. Recall that the upper bound in (3.1) is tight in the sense that it can actually be achieved when there are M users with orthogonal channel vectors tied for the maximum norm. Likewise, the upper bound in (4.4) can be attained for a particular configuration of channel vectors. Specifically, assume that there are M unit orthogonal vectors $\mathbf{u}_m \in \mathbb{C}^M$, $m = 1, \dots, M$, i.e., $\|\mathbf{u}_m\| = 1$ for all m , $\langle \mathbf{u}_m, \mathbf{u}_n \rangle = 0$, $m \neq n$, and MC users, M from each class, with channel vectors $\mathbf{h}_{u_m}^{(c)}$, $c = 1, \dots, C$, $m = 1, \dots, M$, that satisfy the following two properties:

(i) within each class, all M users are tied for the maximum norm, i.e., $\|\mathbf{h}_{u_m}^{(c)}\|^2 = \|\mathbf{h}_{(1)}^{(c)}\|^2$ for all $c = 1, \dots, C$, $m = 1, \dots, M$;

(ii) the channel vector of one of the users of each class is parallel to \mathbf{u}_m and thus

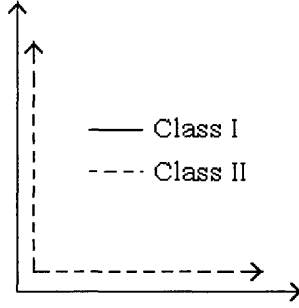


Figure 4-1: The optimal channel configuration for two user classes.

orthogonal to \mathbf{u}_n , $m \neq n$, i.e., $\langle \mathbf{u}_m, \mathbf{h}_{u_m}^{(c)} \rangle = \|\mathbf{h}_{u_m}^{(c)}\|$ and $\langle \mathbf{u}_n, \mathbf{h}_{u_m}^{(c)} \rangle = 0$ for all $c = 1, \dots, C$.

The second property implies that all the \mathbf{u}_m -users are orthogonal to all the \mathbf{u}_n -users, i.e., $\langle \mathbf{h}_{u_m}^{(c)}, \mathbf{h}_{u_n}^{(d)} \rangle = 0$ for all $c, d = 1, \dots, C$, $m \neq n$. For brevity, the above-described constellation of channel vectors will be referred to as the optimal configuration. Figure 4.1 provides a pictorial representation of the optimal configuration for the case of $C = 2$ user classes and $M = 2$ transmit antennas.

As mentioned above, the optimal configuration in fact achieves the upper bound in (4.4). In order to see this, let $P_1^*(K), \dots, P_C^*(K)$ be the optimizing power levels of the upper bound in (4.4) for given values of $\|\mathbf{h}_{(1)}^{(c)}\|^2$, $c = 1, \dots, C$, i.e.,

$$P^*(K) = (P_1^*(K), \dots, P_C^*(K)) := \arg \max_{\sum_{c=1}^C P_c \leq P/M} \sum_{c=1}^C \Delta w_c \log \left(1 + \sum_{d=1}^c P_d \|\mathbf{h}_{(1)}^{(d)}\|^2 \right).$$

Now suppose that we assign power $P_c^*(K)$ to all M class- c users in the optimal configuration, and arrange the users in order of increasing class index in the DPC sequence. Because of the orthogonality, the partial sum rate $S_c := \sum_{d=1}^c T_d$ of the first c classes will be

$$S_c = M \log \left(1 + \sum_{d=1}^c P_d^*(K) \|\mathbf{h}_{(1)}^{(d)}\|^2 \right).$$

Since the total weighted sum rate may be written as $\sum_{c=1}^C w_c T_c = \sum_{c=1}^C \Delta w_c S_c$, it follows that the optimal configuration indeed achieves the upper bound in (4.4).

For the sake of brevity, we introduce the following notation for the upper bound expression in (4.4):

$$U(w_c; \|\mathbf{h}_{(1)}^{(c)}\|^2; P) := M \max_{\sum_{c=1}^C P_c \leq P/M} \sum_{c=1}^C \Delta w_c \log \left(1 + \sum_{d=1}^c P_d \|\mathbf{h}_{(1)}^{(d)}\|^2 \right). \quad (4.5)$$

4.2 Random channel vectors

The bounds for the weighted sum rate in the previous section hold for any arbitrary but fixed set of channel vectors. In order to derive asymptotic results, we will as before, assume the channel vectors to be random and focus on the *expected* weighted sum rate. Within each class we assume the channel vectors to be independent and identically distributed, i.e., $\mathbf{h}_1^{(c)}, \mathbf{h}_2^{(c)}, \dots$ are i.i.d. copies of some random vector $\mathbf{h}^{(c)} \in \mathbb{C}^M$. Across the various classes, the channel vectors may however have different statistical characteristics. The numbers of users of the various classes are assumed to grow large in fixed proportions, i.e., $K_c = \alpha_c K$ for fixed coefficients $\alpha_1, \dots, \alpha_C$ with $\sum_{c=1}^C \alpha_c = 1$.

We assume that the channel vectors of all users in class c are Rayleigh faded with parameter $\beta_c, c = 1, \dots, C$. In other words, for each class c , $\mathbf{h}^{(c)} = \beta_c \mathbf{h}$, where the components of \mathbf{h} are independent and distributed according to $\mathcal{CN}(0, 1)$ as in the homogeneous case.

4.3 Large- K asymptotics

We now proceed to show that the upper bound in (4.4) is asymptotically achievable by transmitting to a judiciously chosen subset of MC users. In the case of homogeneous users, the key observation was that when the total number of users is large, there exists with high probability a pair of users which are nearly orthogonal and have norms close to the maximum. This intuitive insight was then formalized by

establishing that selecting such a pair of users and allocating equal power to each of them asymptotically achieves the maximum expected sum rate. Likewise, there exists with high probability a group of MC users with channel vectors close to the optimal configuration in the heterogeneous case when the total number of users is large. Thus, we will show that selecting such a group of MC users and allocating power P_c^* to the M class- c users, where

$$P^* = (P_1^*, \dots, P_C^*) := \arg \max_{\sum_{c=1}^C P_c \leq P/M} \sum_{c=1}^C \Delta w_c \log \left(\sum_{d=1}^c P_d \beta_d^2 \right)$$

asymptotically achieves the upper bound in (4.4). Define

$$V(w_c; \beta_c^2; P) := \max_{\sum_{c=1}^C P_c \leq P/2} \sum_{c=1}^C \Delta w_c \log \left(\sum_{d=1}^c P_d \beta_d^2 \right) = \sum_{c=1}^C \Delta w_c \log \left(\sum_{d=1}^c P_d^* \beta_d^2 \right). \quad (4.6)$$

Note that the power levels (P_1^*, \dots, P_C^*) are the limiting values of $(P_1^*(K), \dots, P_C^*(K))$ when the norms $\|\mathbf{h}_{(1)}^{(c)}\|^2$ grow large. It may in fact be shown that $(P_1^*(K), \dots, P_C^*(K))$ converge to (P_1^*, \dots, P_C^*) in probability, as $K \rightarrow \infty$. Finally, note also that the asymptotic powers (P_1^*, \dots, P_C^*) depend only on the values of the weights w_c and the relative channel qualities β_c of the various classes.

4.3.1 User selection schemes

We will now prove that transmitting to a carefully selected subset of MC users asymptotically achieves the upper bound (4.4) and thus maximizes the expected weighted sum rate. Motivated by the knowledge of the optimal channel configuration, we will consider the following two user selection schemes which will be referred to as the ‘list’ scheme and the ‘cone’ scheme, respectively.

List scheme

The ‘list’ scheme first identifies for each class the users with norms close to the maximum, and then selects a nearly orthogonal set of users among these. Specifically, the list scheme first selects the ML_c strongest users from class c . and divides them into M ‘groups’ of size L_c each, say in a round-robin fashion. Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_M \in \mathbb{C}^M$ be an arbitrary set of orthonormal vectors. From the M groups of L_c users formed above, the scheme picks, from group i , the user whose channel is most collinear to \mathbf{u}_i . That is, it selects the user in group i who maximizes the normalized inner product with $\mathbf{u}_i, i = 1, 2, \dots, M$. This procedure is repeated for every class $c = 1, 2, \dots, C$. This leads to a set of MC users, M from each class, which have a geometry close to the optimal configuration described earlier.

Cone scheme

Definition: Fix a $0 < \lambda < 1$. Then,

- (a) two vectors \mathbf{u} and \mathbf{v} are said to be λ -aligned if

$$U(\mathbf{u}, \mathbf{v}) := \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{\|\mathbf{u}\|^2 \|\mathbf{v}\|^2} \geq 1 - \lambda.$$

If this is true, we also say that \mathbf{u} lies in the λ -cone of \mathbf{v} and *vice-versa*.

- (b) two channel vectors \mathbf{u} and \mathbf{v} are said to be λ -orthogonal if

$$U(\mathbf{u}, \mathbf{v}) := \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{\|\mathbf{u}\|^2 \|\mathbf{v}\|^2} \leq \lambda.$$

The ‘cone’ scheme first identifies a group of users that are close to orthogonal, and then selects the ones with the largest norms among these. Specifically, let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_M \in \mathbb{C}^M$ be an arbitrary set of orthonormal vectors, and let $0 < \lambda < 1$ be a small tolerance margin. The cone scheme first considers the set of all channel vectors that are λ -aligned with \mathbf{u}_i , for $i = 1, 2, \dots, M$. From each of these M nearly orthogonal ‘cones’ of channel vectors, the scheme picks the strongest user from each class.

After selecting the users in the above-described fashion, both the list and the cone schemes allocate power P_c^* (defined in (4.6)) to all M class- c users.

4.3.2 Asymptotic optimality of the proposed schemes

Optimality of the List scheme

Define \widehat{T}_c as the rate received by class c under the list scheme, i.e., the sum rate of the M class- c users selected, and denote by $\widehat{T} := \sum_{c=1}^C w_c \widehat{T}_c$ the total weighted sum rate. The next theorem shows that the list scheme asymptotically maximizes the expected weighted sum rate, as long as the list size L_c grows with the system size as $L_c(K) = O(K^\delta)$ for some $\delta > 0$. Let $\mathbf{h}_{\mathbf{u}_i}^{(c)}$ denote the channel vector of the class c user who was chosen as being most collinear with $\mathbf{u}_i, i = 1, 2, \dots, M, c = 1, 2, \dots, C$.

Theorem 4.3.1 *In the list scheme described above, assume that the list size $L_c(K)$ for every class grows with the system size as $L_c(K) = O(K^\delta)$ for some $\delta > 0$. Then, the List scheme is asymptotically optimal in the sense that it closes the gap to the weighted sum capacity. Specifically, the gap between the upper bound in (4.5) and the weighted sum rate achieved using the List scheme converges to 0 as K becomes large:*

$$\lim_{K \rightarrow \infty} \mathbb{E} \left[U(w_c; \|\mathbf{h}_{(1)}^{(c)}\|^2; P) \right] - \mathbb{E} \left[\widehat{T} \right] = 0.$$

Proof

Since the above difference is always non-negative, it suffices to show that the limsup of the difference is non-positive.

We may write

$$U(w_c; \|\mathbf{h}_{(1)}^{(c)}\|^2; P) = M \sum_{c=1}^C \Delta w_c \log \left(1 + \sum_{d=1}^c P_d^*(K) \|\mathbf{h}_{(1)}^{(d)}\|^2 \right).$$

Using Lemma A.4.1 and the fact that $\Delta w_c \geq 0$ for all $c = 1, \dots, C$, we obtain

$$\limsup_{K \rightarrow \infty} \mathbb{E} \left[U(w_c; \|\mathbf{h}_{(1)}^{(c)}\|^2; P) \right] - M w_1 \log(B(K)) \leq M V(w_c; \beta_c^2; P). \quad (4.7)$$

Next, we lower bound the partial sum rate quantity $\widehat{S}_c = \sum_{d=1}^c \widehat{T}_d$ for each c . For any $c = 1, 2, \dots, C$, the partial sum rate \widehat{S}_c is given by the $Mc \times Mc$ determinant

$$\widehat{S}_c = \log \det(I_{Mc} + X_{Mc})$$

where

$$X_{Mc} = \begin{bmatrix} X_c^1 & Y_c^{1,2} & \dots & Y_c^{1,M} \\ & X_c^2 & \dots & Y_c^{2,M} \\ & & \ddots & \vdots \\ & & & X_c^M \end{bmatrix}. \quad (4.8)$$

In (4.8), the matrix X_{Mc} is Hermitian, so that the entries below the main diagonal can be obtained by taking the conjugates of the entries above the diagonal. The sub-matrices X_c 's and Y_c 's have the following interpretation:

- (i) For each $i = 1, \dots, M$, X_c^i is a $c \times c$ Hermitian matrix that involves only inner products between the c nearly collinear channel vectors $\mathbf{h}_{\mathbf{u}_i}^{(d)}$, $d = 1, \dots, c$.
- (ii) For each $i, j = 1, \dots, M, i \neq j$, $Y_c^{i,j}$ is a $c \times c$ matrix in which each entry is an inner product between one vector from the set $\mathbf{h}_{\mathbf{u}_i}^{(d)}$, $d = 1, \dots, c$ and the other from a nearly orthogonal set $\mathbf{h}_{\mathbf{u}_j}^{(d)}$, $d = 1, \dots, c$.

We now argue that the entries in each of the sub-matrices $Y_c^{i,j}$ become small with high probability as K becomes large. Let \mathcal{D}_c^ξ denote the event that for *every* $i, j = 1, \dots, M, i \neq j$, classes $d, f = 1, \dots, c$, and some $\xi > 0$, $U(\mathbf{h}_{\mathbf{u}_i}^{(d)}, \mathbf{h}_{\mathbf{u}_j}^{(f)}) < \frac{\xi}{K^{\frac{\delta}{2(M-1)}}}$.

That is

$$\mathcal{D}_c^\xi := \bigcap_{i \neq j} \bigcap_{d,f=1}^c \left\{ U(\mathbf{h}_{\mathbf{u}_i}^{(d)}, \mathbf{h}_{\mathbf{u}_j}^{(f)}) < \frac{\xi}{K^{\frac{\delta}{2(M-1)}}} \right\}. \quad (4.9)$$

It can be deduced from Lemma A.4.4 that there exists an $\eta_c > 0$ such that

$$\mathbb{P}\{\mathcal{D}_c^\xi\} > 1 - \frac{\eta_c}{K^\delta}. \quad (4.10)$$

Now, conditioned on the event \mathcal{D}_c^ξ , all the entries in each of the sub-matrices $Y_c^{i,j}$ tend to zero, since the magnitudes of the channel vectors increase as $O(\log K)$, while the normalized inner product terms decrease as $O(K^{-\delta/2(M-1)})$. Therefore, conditioned

on the event \mathcal{D}_c^ξ , we can asymptotically ignore all terms coming from the Y matrices, and expand the partial sum rate determinant as

$$\det(I_{M_c} + X_{M_c}) = \det(I_{M_c} + F_{M_c}) + \rho_K, \quad (4.11)$$

where $\rho_K \rightarrow 0$ and F_{M_c} is the block diagonal matrix

$$F_{M_c} = \text{diag}(X_c^1, X_c^2, \dots, X_c^M).$$

Notation: For a random variable X and an event A defined on the same probability space,

$$\mathbb{E}[X; A] := \mathbb{E}[X1_A],$$

where 1_A is the indicator function of the event A .

We can now write the following series of lower bounds:

$$\begin{aligned} \mathbb{E}[\widehat{S}_c] - M \log(B(K)) &= \mathbb{E}[\log \det(I_{M_c} + X_{M_c})] - M \log(B(K)) \\ &\geq \mathbb{E}[\log \det(I_{M_c} + X_{M_c}); \mathcal{D}_c^\xi] - M \log(B(K)) \\ &\geq \mathbb{E}[\log(\det(I_{M_c} + F_{M_c}) + \rho_K)] \left(1 - \frac{\eta_c}{K^\delta}\right) - M \log(B(K)) \end{aligned}$$

Thus,

$$\begin{aligned} \liminf_{K \rightarrow \infty} \mathbb{E}[\widehat{S}_c] - M \log(B(K)) &\geq \liminf_{K \rightarrow \infty} \mathbb{E}[\log \det(I_{M_c} + F_{M_c})] - M \log(B(K)) \\ &= \liminf_{K \rightarrow \infty} \sum_{m=1}^M \mathbb{E}[\log \det(I_c + X_c^m)] - M \log(B(K)) \end{aligned} \quad (4.12)$$

Note that the expression $\log \det(I_c + X_c^m)$ in the above equation is equal to the sum rate of the c users which are nearly collinear to \mathbf{u}_m , in the absence of the other users. Now, consider a hypothetical scenario where these nearly collinear channel vectors in fact become *perfectly* collinear to \mathbf{u}_m , without any change in the channel norms. It is

clear that the partial sum rate corresponding to the original channel configuration is lower bounded by the partial sum rate of this hypothetical user configuration. This is because the users in the ‘perfectly parallel’ channel configuration suffer a higher level of interference than that in the original configuration, where the channel vectors were not perfectly parallel. We can therefore write

$$\log \det(I_c + X_c^m) \geq \log\left(1 + \sum_{d=1}^c P_d^* \|\mathbf{h}_{u_m}^{(d)}\|^2\right), m = 1, 2, \dots, M. \quad (4.13)$$

Now, continuing from (4.12),

$$\begin{aligned} \liminf_{K \rightarrow \infty} \mathbb{E} \left[\widehat{S}_c \right] - M \log(B(K)) &\geq \liminf_{K \rightarrow \infty} \sum_{m=1}^M \mathbb{E} \left[\log\left(1 + \sum_{d=1}^c P_d^* \|\mathbf{h}_{u_m}^{(d)}\|^2\right) \right] - M \log(B(K)) \\ &\geq \liminf_{K \rightarrow \infty} \sum_{m=1}^M \mathbb{E} \left[\log\left(1 + \sum_{d=1}^c P_d^* \|\mathbf{h}_{(L_d(K))}^{(d)}\|^2\right) \right] - M \log(B(K)) \\ &\geq M \left(\log \left(\sum_{d=1}^c P_d^* \beta_d^2 \right) + \log(1 - 2\delta) \right), \end{aligned}$$

where the final step follows from Lemma A.4.2. Since the above lower bound holds for every $c = 1, 2, \dots, C$, and since $\widehat{T} = \sum_{c=1}^C \Delta w_c \widehat{S}_c$, we can conclude that

$$\liminf_{K \rightarrow \infty} \mathbb{E} \left[\widehat{T} \right] - M w_1 \log(B(K)) \geq M \sum_{c=1}^C \Delta w_c \left(\log \left(\sum_{d=1}^c P_d^* \beta_d^2 \right) + \log(1 - 2\delta) \right). \quad (4.14)$$

Finally, subtracting (4.14) from (4.7), and noting that δ is arbitrary, we obtain

$$\limsup_{K \rightarrow \infty} \mathbb{E} \left[U(w_c; \|\mathbf{h}_{(1)}^{(c)}\|^2; P) \right] - \mathbb{E} \left[\widehat{T} \right] \leq 0.$$

□

The above theorem shows that scheduling a suitably selected group of MC users asymptotically achieves the upper bound (4.4) and thus maximizes the expected weighted sum rate. In fact, it shows that scheduling M users from each of the classes $c \in C^*$ is sufficient to asymptotically achieve the maximum expected weighted sum

rate, where $C^* := \{c : P_c^* > 0\}$.

In a similar fashion, it can be shown that the cone scheme described above also asymptotically achieves the maximum weighted sum rate, as long as the tolerance margin λ is scaled down at an appropriate rate. The proof of this statement is largely similar to the proof for the list scheme given above, and is sketched below.

Optimality of the Cone scheme

Let \tilde{T}_c denote the throughput obtained by class c under the cone scheme, and define $\tilde{T} := \sum_{c=1}^C w_c \tilde{T}_c$ to be the total weighted sum rate. Denote by $\mathbf{h}_{\mathbf{u}_i}^{(c)}$, $i = 1, \dots, M$, $c = 1, \dots, C$ the channel vector of the strongest class c user which lies in the λ -cone of \mathbf{u}_i . The following theorem shows that the cone scheme asymptotically closes the gap to weighted sum capacity, as long as the tolerance margin is scaled down with the system size as

$$\lambda \propto (\log K)^{-2}. \quad (4.15)$$

Theorem 4.3.2 *In the cone scheme described above, assume that the tolerance margin λ scales with the system size as $\lambda = \frac{\kappa}{(\log K)^2}$ for $\kappa > 0$. Then, the cone scheme is asymptotically optimal in the sense that it closes the gap to the weighted sum capacity. Specifically,*

$$\lim_{K \rightarrow \infty} \mathbb{E} \left[U(w_c; \|\mathbf{h}_{(1)}^{(c)}\|^2; P) \right] - \mathbb{E} \left[\tilde{T} \right] = 0.$$

Proof(Sketch)

The partial sum rate of the first c classes $\tilde{S}_c = \sum_{d=1}^c \tilde{T}_d$ is given by

$$\hat{S}_c = \log \det(I_{Mc} + X_{Mc})$$

where

$$X_{Mc} = \begin{bmatrix} X_c^1 & Y_c^{1,2} & \dots & Y_c^{1,M} \\ & X_c^2 & \dots & Y_c^{2,M} \\ & & \ddots & \vdots \\ & & & X_c^M \end{bmatrix}. \quad (4.16)$$

In the above equation, the matrix X_{Mc} has the same structure as the one in (4.8), except that the channel vectors are now chosen according to the cone scheme. Now, by Lemma A.4.3, we see that users in different cones are at least 3λ -orthogonal. Thus, if we scale down λ fast enough with K , the Y sub-matrices in X_{Mc} can be made arbitrarily small, and will not contribute to the determinant as K becomes large. More precisely, we can write

$$\det(I_{Mc} + X_{Mc}) = \det(I_{Mc} + F_{Mc}) + \rho_K, \quad (4.17)$$

where $\rho_K \rightarrow 0$ and F_{Mc} is a block diagonal matrix as in the previous proof. Following the arguments leading to equation (4.13), we can write

$$\liminf_{K \rightarrow \infty} \mathbb{E} \left[\tilde{S}_c \right] - M \log(B(K)) \geq \liminf_{K \rightarrow \infty} \sum_{m=1}^M \mathbb{E} \left[\log \left(1 + \sum_{d=1}^c P_d^* \|\mathbf{h}_{\mathbf{u}_m}^{(d)}\|^2 \right) \right] - M \log(B(K)) \quad (4.18)$$

Now, as shown in Lemma 3.2.1, the probability that an arbitrary channel vector lies in the λ -cone of \mathbf{u}_m is equal to

$$p_\lambda := 1 - (1 - \lambda)^{M-1} \quad (4.19)$$

Thus, the number of class d vectors in the λ -cone of \mathbf{u}_m (denoted by N_d^m) is a binomial random variable with mean $p_\lambda K_d$. Thus it can be seen that $\|\mathbf{h}_{\mathbf{u}_m}^{(d)}\|^2$ is distributed as the maximum among N_d^m i.i.d Erlang(M) random variables, where $N_d^m \sim \text{Binomial}(K_d, p_\lambda)$. Using Chebychev's inequality, and keeping the scaling rate of λ in mind (4.15), we get

$$\mathbb{P}\{N_d^M > p_\lambda K_d - \sqrt{K_d}\} > 1 - \frac{\kappa}{(\log K)^2}, d = 1, \dots, C,$$

from which it follows that

$$\mathbb{P}\{N_d^M > \lambda K_d / 2\} > 1 - \frac{\kappa}{(\log K)^2}, d = 1, \dots, C, \quad (4.20)$$

since $p_\lambda K_d - \sqrt{K_d} \geq \lambda K_d - \sqrt{K_d} > \lambda K_d/2$.

We can now write the following inequalities for each $m = 1, 2, \dots, M$:

$$\begin{aligned} & \mathbb{E} \left[\log \left(1 + \sum_{d=1}^c P_d^* \|\mathbf{h}_{u_m}^{(d)}\|^2 \right) \right] - \log(B(K)) \geq \\ & \mathbb{E} \left[\log \left(1 + \sum_{d=1}^c P_d^* \|\mathbf{h}_{u_m}^{(d)}\|^2 \right); N_d^m > \lambda K_d/2, d = 1, \dots, c \right] - \log(B(K)) \\ & \geq \mathbb{E} \left[\log \left(1 + \sum_{d=1}^c P_d^* \|\mathbf{h}_{\max, \lambda K_d/2}\|^2 \right) \right] \left(1 - \frac{\kappa}{(\log K)^2} \right)^c - \log B(K), \end{aligned} \quad (4.21)$$

where $\|\mathbf{h}_{\max, r}\|^2$ denotes the maximum among r i.i.d channel norms.

Similar to the result in Lemma A.2.1, we can show that for any $\epsilon > 0$, there exists a constant $C_\epsilon^{(c)}$ for each class c such that

$$\mathbb{P}\{\|\mathbf{h}_{\max, \lambda K_d/2}\|^2 > (1 - \epsilon)\beta_d^2 B(\lambda K_d/2)\} > 1 - \frac{C_\epsilon^{(c)}}{(\log \lambda K_d/2)^2}.$$

Thus, the expression in (4.21) is greater than or equal to

$$\log \left(1 + \sum_{d=1}^c P_d^* \beta_d^2 (1 - \epsilon) B(\lambda K_d/2) \right) \left(1 - \frac{\kappa}{(\log K)^2} \right)^c \prod_{d=1}^c \left(1 - \frac{C_\epsilon^{(c)}}{(\log \lambda K_d/2)^2} \right) - \log B(K). \quad (4.22)$$

Noting that the centering constant scales as $B(K) = O(\log K)$ (Section A.2 in the appendix), it is immediate that the liminf of the above expression is equal to

$$\log \left(\sum_{d=1}^c P_d^* \beta_d^2 \right).$$

Therefore, going back to (4.18),

$$\liminf_{K \rightarrow \infty} \mathbb{E} \left[\tilde{S}_c \right] - M \log(B(K)) \geq M \log \left(\sum_{d=1}^c P_d^* \beta_d^2 \right),$$

and finally, since $\tilde{T} := \sum_{c=1}^C \Delta w_c \tilde{S}_c$, we conclude that

$$\liminf_{K \rightarrow \infty} \mathbb{E} [\tilde{T}] - M w_1 \log(B(K)) \geq M \sum_{c=1}^C \Delta w_c \log \left(\sum_{d=1}^c P_d^* \beta_d^2 \right). \quad (4.23)$$

Now, subtracting (4.23) from (4.7), we get the desired result:

$$\limsup_{K \rightarrow \infty} \mathbb{E} \left[U(w_c; \|\mathbf{h}_{(1)}^{(c)}\|^2; P) \right] - \mathbb{E} [\tilde{T}] \leq 0.$$

□

Remark 4.3.1 *Inspection of the proof of theorem 4.3.1 reveals that $\mathbb{E} \left[U(w_c; \|\mathbf{h}_{(1)}^{(c)}\|^2; P) \right]$ asymptotically behaves as $M w_1 \log(B(K)) + M V(w_c; \beta_c^2; P)$ in the sense that the difference decays to zero asymptotically, and hence so does $\mathbb{E} [\hat{T}]$. In fact, it may be deduced that the total rate received by class 1 grows as $M[\log(B(K)) + \log(P_1^*) + 2 \log(\beta_1)]$, while the total rate received by class c , $c = 2, \dots, C$, asymptotically converges to $M [\log(\sum_{d=1}^c P_d^* \beta_d^2) - \log(\sum_{d=1}^{c-1} P_d^* \beta_d^2)]$. Thus, asymptotically, the lion's share of the aggregate throughput is accounted for by class 1.*

Chapter 5

Numerical Results

In this chapter, we present the results of some of the numerical experiments that we performed in order to evaluate the practical efficacy of the various scheduling schemes proposed in this thesis. We first present results for a homogeneous system with two antennas, followed by a heterogeneous system with two-antennas and two user classes. In both cases, the proposed schemes perform close to the capacity limit, even for relatively small system sizes considered here.

5.1 Homogeneous case

Here, we compare the sum rate obtained by the various user selection schemes with the TDMA rate. We also make a comparison with a beam-forming (BF) scheme along the lines described in [18] and [27].

We present numerical results for a system with two transmit antennas and $K = 25$ users in Figure 5.1. In Figure 5.1(a), we plot the ratio of the sum rate obtained by the various schemes to the TDMA sum rate, versus the SNR (in dB). The results shown here were an average over 100 channel realizations. The solid line corresponds to the optimal DPC scheme. The dotted line just underneath the solid line corresponds to scheme II with $L = 5$. It is clear that even for this moderate value of K , scheme II performs very well, in addition to being asymptotically optimal. The broken line corresponds to a special case of scheme I, where the two strongest users are scheduled

with equal power. It is clear that scheme II dominates scheme I quite significantly. It is also interesting to note that the upper bound in (3.2), although asymptotically tight, is quite loose for practical values of K and SNR. We finally observe that TDMA is optimal in the very low SNR regime. The absolute sum rate (in nats) for this system is graphed as a function of SNR in Figure 5.1(b).

The BF scheme proposed in [18] selects two users which have the best Signal-to-Interference-and-Noise Ratio (SINR) on each of the antennas. In particular, the transmitter forms random beams along the direction of two orthonormal vectors ϕ_1 and ϕ_2 , and selects two users $k_m^* := \arg \max_{k=1, \dots, K} \text{SINR}_{k,m}$, $m = 1, 2$, where

$$\text{SINR}_{k,m} := \frac{|\langle \mathbf{h}_k, \phi_m \rangle|^2}{2/P + |\langle \mathbf{h}_k, \phi_{3-m} \rangle|^2}.$$

The expected sum rate obtained (ignoring potential complications when $k_1^* = k_2^*$), is therefore

$$R_{BF} := \mathbb{E} \left[\log(1 + \text{SINR}_{k_1^*,1}) + \log(1 + \text{SINR}_{k_2^*,2}) \right].$$

The lower curves in Figure 5.1 plot the sum rate of this BF scheme compared with the other schemes. We observe that transmitting along two pre-determined beams without using actual phase information performs poorly, even though it is known to be asymptotically optimal in the limit of a large number of users. However, a plot of the quantity $C(\mathbf{h}_{k_1^*}, \mathbf{h}_{k_2^*})$ (*not* shown in the figure) revealed that this particular scheme actually does well in terms of *selecting* a pair of users.

Note that as $P \downarrow 0$, we have

$$R_{BF} \approx \frac{P}{2} \mathbb{E} \left[|\langle \mathbf{h}_{k_1^*}, \phi_1 \rangle|^2 + |\langle \mathbf{h}_{k_2^*}, \phi_2 \rangle|^2 \right] =$$

$$P \mathbb{E} \left[|\langle \mathbf{h}_{k_1^*}, \phi_1 \rangle|^2 \right] \leq P \mathbb{E} \left[\|\mathbf{h}_{(1)}\|^2 \right] \approx R_{TDMA}.$$

Denoting $g_{ij} := |\langle \mathbf{h}_{k_i^*}, \phi_j \rangle|^2$, we find that R_{BF} approaches

$$\mathbb{E} \left[\log \left(1 + \frac{g_{11}}{g_{12}} \right) + \log \left(1 + \frac{g_{22}}{g_{21}} \right) \right] = 2 \mathbb{E} \left[\log \left(1 + \frac{g_{11}}{g_{12}} \right) \right]$$

as $P \rightarrow \infty$. This shows that for any fixed number of users, the sum rate of the BF scheme saturates at a finite value as the transmit power becomes large, as is shown in Figure 5.1(b). In contrast, the TDMA sum rate R_{TDMA} grows without bound, albeit slowly.

5.2 Heterogeneous case

Background for the numerical results

The simulation results which are provided below are for a two-antenna, two-class system. The weights are taken to be $w_1 = 2$, $w_2 = 1$ (although we usually equivalently normalized these to sum to 1 over the users), and the coefficients $\beta_1 = 0.5$, $\beta_2 = 1.0$ determine the mean SNR's. The two populations of users are of equal size, $K_1 = K_2 = 10$. Under these circumstances, the asymptotically optimal power values are $P_1^* = 1/3$, $P_2^* = 1/6$, scaling out P , which is varied in most of the results below. We will state its value when necessary.

We now describe the schemes themselves. As far as the list and cone schemes are concerned, these are detailed in the text. Throughout, the asymptotically optimal power settings will be used, no power optimization is being employed. We will also consider TDMA, by which we mean the scheme that picks the user which has the maximum weighted rate when assigned full power, over all the users. Thus, it selects the k -th class- c user which maximizes

$$\max_{c=1,\dots,C} \max_{k=1,\dots,K_c} w_c \log(1 + P \|\mathbf{h}_k^{(c)}\|^2) = \max_{c=1,\dots,C} w_c \log(1 + P \|\mathbf{h}_{(1)}^{(c)}\|^2).$$

Finally, we consider two BF versions. The first version (referred to as BeamForm 2 in the figure) schedules one user in each beam, with the powers equally split and the user with the maximum weighted rate as determined by the SINR being the one selected for each beam. The second version (referred to as BeamForm 4) schedules one user from each class in each of the beams. In this case, each user is assigned its asymptotic power. Note that the latter scheme generalizes the BF technique proposed

in [18] to a class-based system. However, this scheme is not expected to perform well as the interference between users on the same beam cannot be resolved except by using DPC or some equivalent approach.

Graphs for basic schemes

Figure 5.2(a) shows results for all the main schemes as well as the upper bound and the average maximum weighted capacity limit. $L = 5$ was set for the list scheme and $\delta = 0.2$ for the cone scheme. (Further numerical experiments indicated that the performance of the list scheme is quite robust with respect to the list size L , so that the exact value is not that critical.) As expected, the upper bound (4.5) is loose and the list and cone schemes perform well at high SNR values. For low SNR values, TDMA outperforms these schemes. The BF schemes fall off at very high SNR as the figure shows.

As far as the list and cone schemes are concerned, good performance at high SNR is expected. However, at low SNR TDMA is close to optimal. (This latter conclusion follows from the linearity of the log.) Thus for low to moderate SNR's, one could make up for the loss of rate in the list scheme by optimizing the powers, instead of assigning asymptotically optimal powers to each user. Similarly, the cone scheme does well at high SNR but not at low SNR. This loss in performance can also be addressed by assigning the powers optimally. This is a concave optimization in three independent variables, and is therefore potentially a time-consuming calculation, since we have no explicit formula for determining the optimal powers.

Figure 5.2(b) shows the same results, but gives the ratio to TDMA. Note that unlike the homogeneous case, BF is not asymptotically optimal in terms of differences as the number of users is increased at fixed SNR. However, at low SNR's (below 0 dB) BeamForm 2 does better than cone or list. Figure 5.2(b) shows that BeamForm 2 performs consistently worse than TDMA, which was also observed in the homogeneous example which had a similar number of users. The results for BeamForm 4 are worse than those for BeamForm 2 as expected.

Additional compound schemes

We now look at simpler enhancements to avoid power optimization among the four selected users. One such enhancement to the list scheme is to identify the best possible pair among the already selected four users. Consider the two-user weighted sum rates obtained by scheduling all possible pairs of these users. The power is split equally while scheduling two users of the same class, but when scheduling one user from each class, we allocate them powers $2P_1^*$ and $2P_2^*$ respectively. The two-user scheme picks the pair that corresponds to the highest weighted sum rate among the six possible pairs.

We thus arrive at the following heuristic schemes. Compound scheme I selects the better among TDMA and the list schemes. Compound scheme II goes further and selects the best among TDMA, the two-user scheme above, and the original list scheme.

A three-user heuristic scheme was also considered, but since it did not provide any appreciable improvement, it has been omitted from the results.

In Figure 5.3, we compare the list scheme with the two heuristic schemes, Compound I and Compound II. These results are more clearly seen as a ratio to TDMA rather than the absolute rates which are difficult to distinguish. Since Compound I takes the best of TDMA and the list scheme, it cannot do worse than TDMA at any point and list at any point. Hence, it does well at low SNRs and at high SNRs. There is nevertheless a significant rate gap for this scheme for moderate SNR's, roughly in the range 0–5dB. Here TDMA falls off, but the list scheme is not yet in its most advantageous range. However, Compound II closes most of this gap as can be seen. The results in Figures 5.2 and 5.3 were averaged over 50 channel realizations.

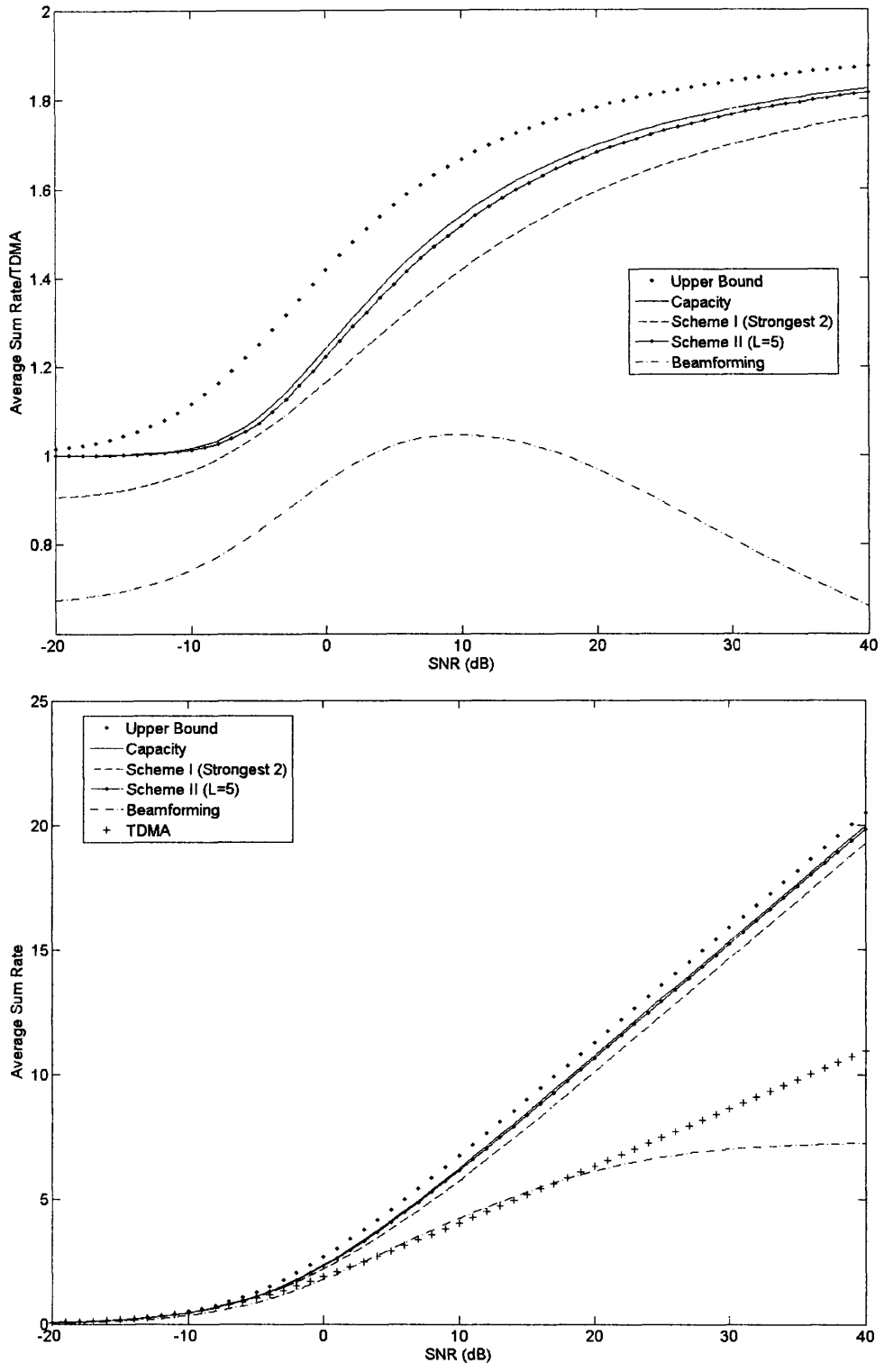


Figure 5-1: Homogeneous system: (a) Comparison of various user selection heuristics with TDMA, $K = 25$ users. (b) Absolute sum rate in nats vs SNR, $K = 25$ users.

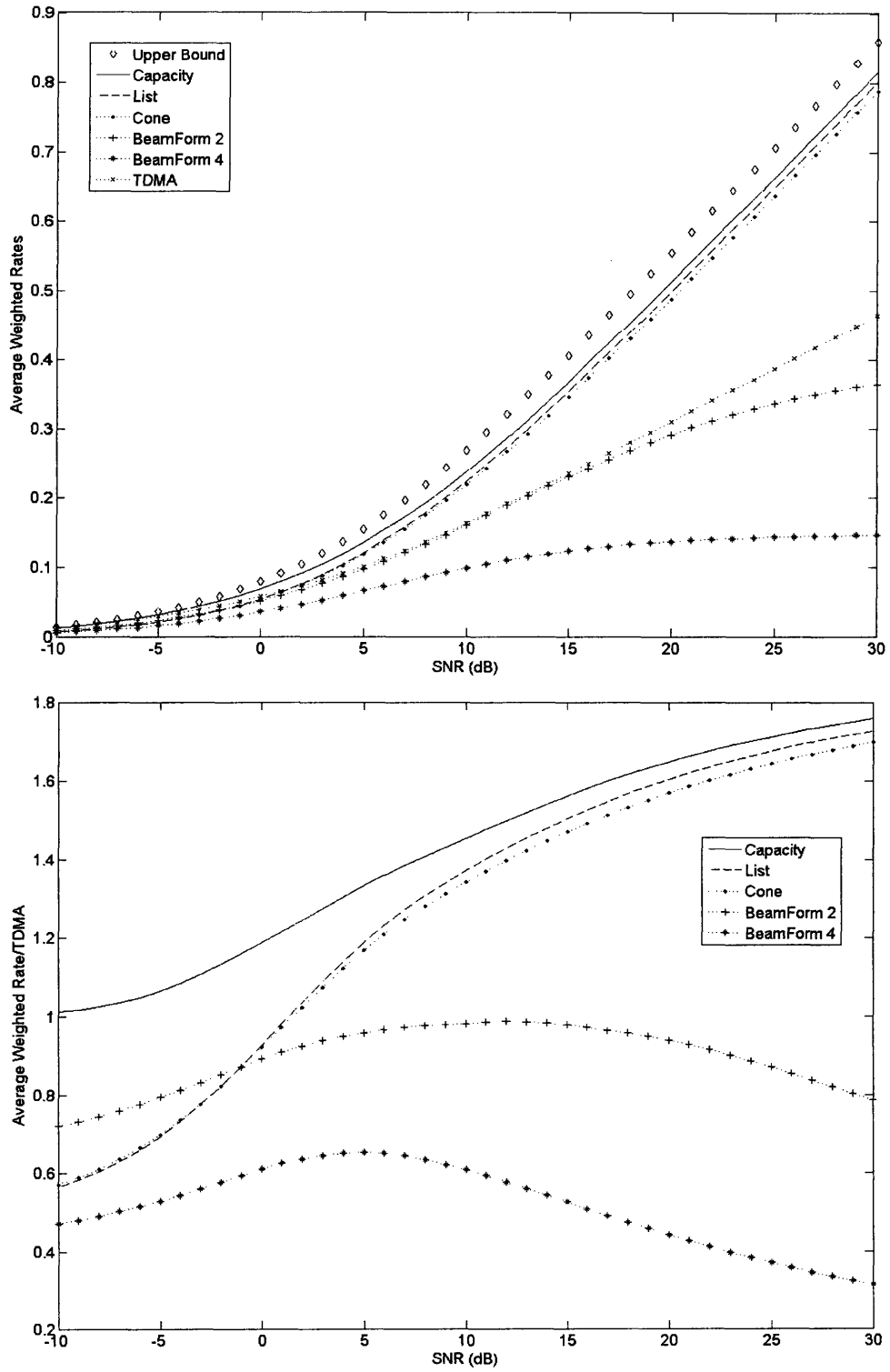


Figure 5-2: Heterogeneous system: (a) Absolute weighted rates for various schemes and upper bound (b) Ratio to TDMA for various schemes

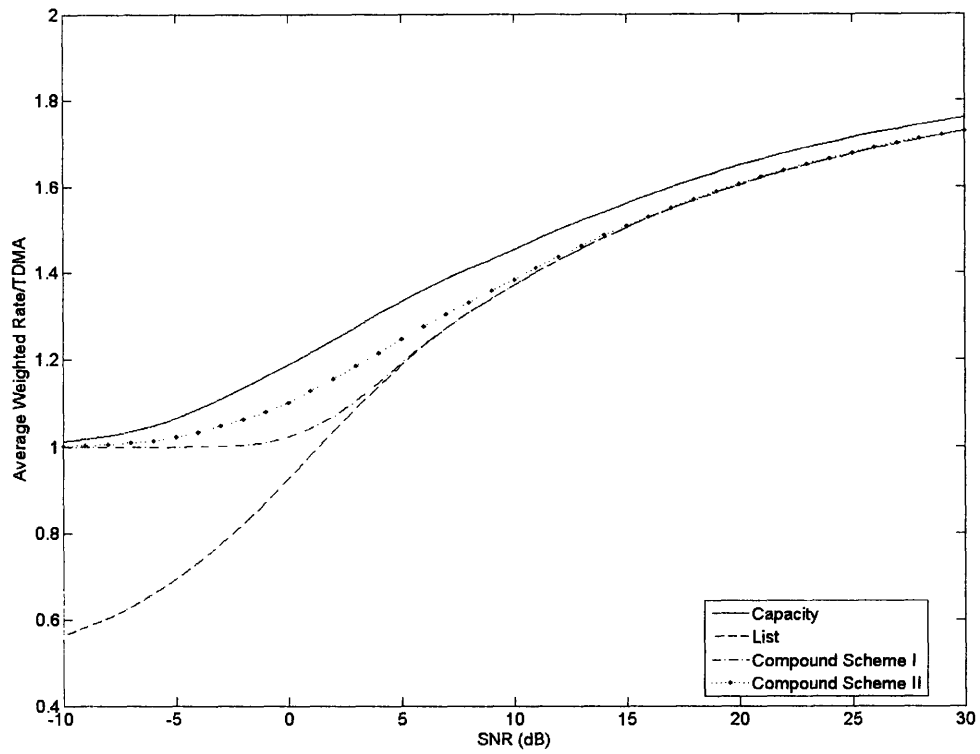


Figure 5-3: Relative weighted sum rates for compound schemes compared to TDMA.

Chapter 6

Conclusions

We studied the problem of maximizing the expected (weighted) sum rate in a Gaussian broadcast channel with multiple transmit antennas and K independent users each with a single receive antenna. We first focused on a scenario with two transmit antennas and statistically identical users, and analyzed the gap between the full sum capacity and the rate that can be achieved by transmitting to a suitably selected pair of users. In particular, we considered a scheme that picks the user with the largest channel gain, and selects a second user from the next $L - 1$ strongest ones to form the best pair, taking channel angles into account as well. We proved that the expected rate gap converges to $1/(L - 1)$ nats/symbol when the total number of users K tends to infinity. Allowing L to increase with K , it followed that transmitting to a properly chosen pair of users is asymptotically optimal, while dramatically reducing the feedback overhead and operational complexity. Next, we addressed the problem of maximizing a *weighted* sum rate in a scenario with M antennas and heterogeneous user characteristics. We established a novel upper bound for the weighted sum capacity, which we then used to show that the maximum expected weighted sum rate can be asymptotically achieved by transmitting to a properly chosen subset of at most MC users, where C denotes the number of distinct user classes. Numerical experiments showed that the asymptotic results are remarkably accurate and that the proposed schemes provide near-optimal performance, even for a moderate number of users.

Appendix A

A.1 Some useful determinant identities

In preparation for some of the subsequent proofs, we first establish a few useful determinant identities.

Lemma A.1.1 *For any K, M ,*

$$\det \left(I_M + \sum_{k=1}^K Q_k \mathbf{h}_k^\dagger \mathbf{h}_k \right) = \det(I_K + J),$$

with $J_{kl} := \sqrt{Q_k Q_l} \mathbf{h}_k \mathbf{h}_l^\dagger$, $k, l = 1, \dots, K$.

Proof

Define the $K \times M$ matrix \mathbf{H} by $H_{km} := \sqrt{Q_k} h_{km}$, $k = 1, \dots, K$, $m = 1, \dots, M$. The proof then follows easily from the identity relation $\det(I_M + \mathbf{H}^\dagger \mathbf{H}) = \det(I_K + \mathbf{H} \mathbf{H}^\dagger)$.

Indeed,

$$\begin{aligned}
& \det(I_M + \sum_{k=1}^K Q_k \mathbf{h}_k^\dagger \mathbf{h}_k) \\
&= \det \left(I_M + \begin{bmatrix} \sqrt{Q_1} \mathbf{h}_1 \\ \dots \\ \sqrt{Q_K} \mathbf{h}_K \end{bmatrix} \begin{bmatrix} \sqrt{Q_1} \mathbf{h}_1 \\ \dots \\ \sqrt{Q_K} \mathbf{h}_K \end{bmatrix}^\dagger \right) \\
&= \det(I_M + \mathbf{H}^\dagger \mathbf{H}) \\
&= \det(I_K + \mathbf{H} \mathbf{H}^\dagger) \\
&= \det \left(I_K + \begin{bmatrix} \sqrt{Q_1} \mathbf{h}_1 \\ \dots \\ \sqrt{Q_K} \mathbf{h}_K \end{bmatrix} \begin{bmatrix} \sqrt{Q_1} \mathbf{h}_1 \\ \dots \\ \sqrt{Q_K} \mathbf{h}_K \end{bmatrix}^\dagger \right) \\
&= \det(I_K + J).
\end{aligned}$$

□

Lemma A.1.2

$$\det(I_2 + \sum_{k=1}^K P_k \mathbf{h}_k^\dagger \mathbf{h}_k) = 1 + \sum_{k=1}^K P_k \|\mathbf{h}_k\|^2 + \sum_{k \neq l} P_k P_l \|\mathbf{h}_k\|^2 \|\mathbf{h}_l\|^2 V_{kl},$$

with $V_{kl} := 1 - U(\mathbf{h}_k, \mathbf{h}_l)$.

Proof

Expanding the determinant, we obtain

$$\begin{aligned}
& \det(I_2 + \sum_{k=1}^K P_k \mathbf{h}_k^\dagger \mathbf{h}_k) \\
&= \left(1 + \sum_{k=1}^K P_k h_{k1}^\dagger h_{k1}\right) \left(1 + \sum_{k=1}^K P_k h_{k2}^\dagger h_{k2}\right) - \left(\sum_{k=1}^K P_k h_{k2}^\dagger h_{k1}\right) \left(\sum_{k=1}^K P_k h_{k1}^\dagger h_{k2}\right) \\
&= 1 + \sum_{k=1}^K P_k \|\mathbf{h}_k\|^2 + \sum_{k=1}^K \sum_{l=1}^K P_k P_l [h_{k1}^\dagger h_{k1} h_{l2}^\dagger h_{l2} - h_{k2}^\dagger h_{k1} h_{l1}^\dagger h_{l2}] \\
&= 1 + \sum_{k=1}^K P_k \|\mathbf{h}_k\|^2 + \sum_{l \neq k} P_k P_l [h_{k1}^\dagger h_{k1} h_{l2}^\dagger h_{l2} + h_{l1}^\dagger h_{l1} h_{k2}^\dagger h_{k2} - h_{k2}^\dagger h_{k1} h_{l1}^\dagger h_{l2} - h_{l2}^\dagger h_{l1} h_{k1}^\dagger h_{k2}].
\end{aligned}$$

In order to complete the proof, it remains to be shown that

$$h_{k1}^\dagger h_{k1} h_{l2}^\dagger h_{l2} + h_{l1}^\dagger h_{l1} h_{k2}^\dagger h_{k2} - h_{k2}^\dagger h_{k1} h_{l1}^\dagger h_{l2} - h_{l2}^\dagger h_{l1} h_{k1}^\dagger h_{k2} = \|\mathbf{h}_k\|^2 \|\mathbf{h}_l\|^2 V_{kl},$$

which follows from

$$\begin{aligned}
& h_{k1}^\dagger h_{k1} h_{l2}^\dagger h_{l2} + h_{l1}^\dagger h_{l1} h_{k2}^\dagger h_{k2} - h_{k2}^\dagger h_{k1} h_{l1}^\dagger h_{l2} - h_{l2}^\dagger h_{l1} h_{k1}^\dagger h_{k2} \\
&= h_{k1}^\dagger h_{k1} h_{l1}^\dagger h_{l1} + h_{k1}^\dagger h_{k1} h_{l2}^\dagger h_{l2} + h_{k2}^\dagger h_{k2} h_{l1}^\dagger h_{l1} + h_{k2}^\dagger h_{k2} h_{l2}^\dagger h_{l2} \\
&- h_{k1}^\dagger h_{k1} h_{l1}^\dagger h_{l1} - h_{k2}^\dagger h_{k1} h_{l1}^\dagger h_{l2} - h_{l2}^\dagger h_{l1} h_{k1}^\dagger h_{k2} - h_{l2}^\dagger h_{l2} h_{k2}^\dagger h_{k2} \\
&= [h_{k1}^\dagger h_{k1} + h_{k2}^\dagger h_{k2}] [h_{l1}^\dagger h_{l1} + h_{l2}^\dagger h_{l2}] - [h_{k1}^\dagger h_{l1} + h_{k2}^\dagger h_{l2}] [h_{l1}^\dagger h_{k1} + h_{l2}^\dagger h_{k2}] \\
&= \|\mathbf{h}_k\|^2 \|\mathbf{h}_l\|^2 - |\langle \mathbf{h}_k, \mathbf{h}_l \rangle|^2 \\
&= \|\mathbf{h}_k\|^2 \|\mathbf{h}_l\|^2 V_{kl}.
\end{aligned}$$

□

A.2 Some results from extremal theory

Here, we quickly collect some useful results from the theory of extremal order statistics. The interested reader is referred to [1] for a detailed treatment.

Extremal theory deals with the behavior of the largest and smallest among K i.i.d.

random variables. In many cases, the centered versions of these variables have weak limits. For instance, consider K i.i.d. unit exponential random variables D_1, \dots, D_K , let $Y_K^{(1)} := \max(D_1, \dots, D_K)$, and define $Z_K^{(1)} := Y_K^{(1)} - \log K$. Then the distribution function of $Z_K^{(1)}$, $F_K^{(1)}(z)$ converges to $F_K^{(1)}(z) \rightarrow e^{-e^{-z}}$, [2].

A similar result holds for i.i.d Erlang(M) random variables which are a sum of M unit exponentials. In this case, the limiting distribution for the centered maximum remains the same as in the exponential case, but the centering constant $\log K$ is replaced by $B(K) = \log K + (M - 1) \log \log K - \log(M - 1)!$. Specifically let, E_1, E_2, \dots, E_K be i.i.d Erlang variables with M degrees of freedom. Let $Y_K^{(1)} := \max(E_1, \dots, E_K)$, and define $Z_K^{(1)} := Y_K^{(1)} - B(K)$. Then, $Z_K^{(1)} \Rightarrow e^{-e^{-z}}$, where \Rightarrow denotes convergence in distribution. This weak convergence result is straightforward to verify. Note that the sequence $B(K)$ is distinct from the sequence of the means $m_K := \mathbb{E}[Y_K]$. Similar results can also be derived for the centered second largest variable, third largest and so on.

A further useful fact that can be shown regarding the sequence of centered maxima $Z_K^{(1)}$ is that they are *uniformly integrable* [2]. It is a well known fact that a sequence of uniformly integrable random variables that have a weak limit also converge in the mean [2, p. 338]. Thus we can conclude that the sequence of means $\mathbb{E}[Z_K^{(1)}]$ converge:

$$\lim_{K \rightarrow \infty} \mathbb{E}[Z_K^{(1)}] = \lim_{K \rightarrow \infty} m_K - B(K) = \mathbb{E}[Z], \quad (\text{A.1})$$

where $\mathbb{E}[Z]$ is the mean of the limiting distribution function $e^{-e^{-z}}$. Interestingly, the mean of this distribution function turns out to be equal to the celebrated Euler-Mascheroni constant [14]:

$$\mathbb{E}[Z] = \gamma = 0.5772\dots$$

Finally, we state without proof another interesting and useful fact regarding the largest few realizations in a set of i.i.d Erlang random variables. Specifically, the following lemma states that for $0 < \delta < 1$, the largest K^δ random variables are unlikely to take values that are significantly lesser than the centering constant $B(K)$.

Lemma A.2.1 Given $\epsilon, \delta > 0$, $\delta < \epsilon/2$, define

$$p_{\delta, \epsilon}^K := \mathbb{P}\{Y_K^{(K^\delta)} \leq (1 - \epsilon)B(K)\},$$

where $Y_K^{(K^\delta)}$ is the K^δ order statistic among a set of K i.i.d unit Erlang(M) random variables. There exists a constant $C_{\delta, \epsilon}$ such that

$$p_{\delta, \epsilon}^K \leq \frac{C_{\delta, \epsilon}}{(\log(K))^2}.$$

Note that as a consequence of (A.1), the statement of the above lemma also holds when $B(K)$ is replaced by the sequence of means m_K in the definition of $p_{\delta, \epsilon}^K$.

A.3 Lemmas and proofs for the homogeneous case

A.3.1 Additional bounds for the sum capacity

Here we gather a few further bounds for the sum capacity that will be used in establishing Theorems 3.3.1 and 3.3.2. We first prove a simple lower bound.

Lemma A.3.1 For any i, j ,

$$C(\mathbf{h}_i, \mathbf{h}_j, P) \geq 2 \log \left(1 + \frac{P}{2} \|\mathbf{h}_{i \wedge j}\|^2 \right) + \log(V_{ij}),$$

with $\|\mathbf{h}_{i \wedge j}\|^2 := \min\{\|\mathbf{h}_i\|^2, \|\mathbf{h}_j\|^2\}$.

Proof

From Equation (3.4),

$$\begin{aligned} & C(\mathbf{h}_i, \mathbf{h}_j, P) \\ & \geq \log \left(\left(1 + \frac{P}{2} \|\mathbf{h}_i\|^2 \right) \left(1 + \frac{P}{2} \|\mathbf{h}_j\|^2 \right) V_{ij} \right) \\ & \geq 2 \log \left(1 + \frac{P}{2} \|\mathbf{h}_{i \wedge j}\|^2 \right) + \log(V_{ij}). \end{aligned}$$

□

We now state a few upper bounds. Define

$$F(\mathbf{h}_i, \mathbf{h}_j, P) := 1 + P\|\mathbf{h}_{i\vee j}\|^2 + \frac{P^2}{4}\|\mathbf{h}_{i\vee j}\|^4 V_{ij},$$

with $\|\mathbf{h}_{i\vee j}\|^2 := \max\{\|\mathbf{h}_i\|^2, \|\mathbf{h}_j\|^2\}$.

Lemma A.3.2 *For any i, j ,*

$$C_{BC}^{\text{sum}}(\mathbf{h}_i, \mathbf{h}_j, P) \leq \log(F(\mathbf{h}_i, \mathbf{h}_j, P)).$$

Proof

Using Equation (2.3) and Lemma A.1.1,

$$\begin{aligned} & C_{BC}^{\text{sum}}(\mathbf{h}_i, \mathbf{h}_j, P) \\ &= \max_{P_i+P_j \leq P} \log \det \left(I_2 + \frac{P}{2}(\mathbf{h}_i^\dagger \mathbf{h}_i + \mathbf{h}_j^\dagger \mathbf{h}_j) \right) \\ &= \max_{P_i+P_j \leq P} \log \left(1 + P_i\|\mathbf{h}_i\|^2 + P_j\|\mathbf{h}_j\|^2 + P_i P_j \|\mathbf{h}_i\|^2 \|\mathbf{h}_j\|^2 V_{ij} \right) \\ &\leq \max_{P_i+P_j \leq P} \log \left(1 + (P_i + P_j)\|\mathbf{h}_{i\vee j}\|^2 + P_i P_j \|\mathbf{h}_{i\vee j}\|^4 V_{ij} \right) \\ &= \log \left(1 + P\|\mathbf{h}_{i\vee j}\|^2 + \frac{P^2}{4}\|\mathbf{h}_{i\vee j}\|^4 V_{ij} \right). \end{aligned}$$

□

Lemma A.3.3 *For any $i, j, \epsilon \in (0, 1)$,*

$$F(\mathbf{h}_i, \mathbf{h}_j, P) \leq \left(\frac{1}{\epsilon} + \frac{P}{2}\|\mathbf{h}_{i\vee j}\|^2 \right)^2 \max\{\epsilon, V_{ij}\}.$$

Proof

By definition, for any $\epsilon \in (0, 1)$,

$$F(\mathbf{h}_i, \mathbf{h}_j, P) = 1 + P\|\mathbf{h}_{i\vee j}\|^2 + \frac{P^2}{4}\|\mathbf{h}_{i\vee j}\|^4 V_{ij} \leq \left(\frac{1}{\epsilon} + \frac{P}{2}\|\mathbf{h}_{i\vee j}\|^2 \right)^2 \max\{\epsilon, V_{ij}\}.$$

□

Lemma A.3.4 For any $i, j, \epsilon \in (0, 1)$,

$$C(\mathbf{h}_i, \mathbf{h}_j, P) \leq 2 \log \left(\frac{1}{\epsilon} + \frac{P}{2} \|\mathbf{h}_{i \vee j}\|^2 \right) + \log(\max\{\epsilon, V_{ij}\}).$$

Proof

Follows from Lemmas A.3.2 and A.3.3 and observing that $C(\mathbf{h}_i, \mathbf{h}_j, P) \leq C_{BC}^{\text{sum}}(\mathbf{h}_i, \mathbf{h}_j, P)$.

□

A.3.2 Proof of Lemma 3.2.2

Proof

First note that the difference is bounded from below by

$$\mathbb{E} \left[\log \left(1 + \frac{A - B}{B + Q \|\mathbf{h}_{(1)}\|^2} \right) \right],$$

so the liminf is non-negative since $\|\mathbf{h}_{(1)}\|^2 \rightarrow \infty$ almost surely as $K \rightarrow \infty$.

We now show that the limsup is non-positive.

Denoting $m_K := \mathbb{E} [\|\mathbf{h}_{(1)}\|^2]$ and applying Jensen's inequality, we obtain

$$\mathbb{E} [\log (A + Q \|\mathbf{h}_{(1)}\|^2)] \leq \log(A + Q m_K).$$

For any $\epsilon > 0$,

$$\begin{aligned} & \mathbb{E} [\log (B + Q \|\mathbf{h}_{(L)}\|^2)] \geq \\ & \log(B) + [\log (B + Q m_K (1 - \epsilon)) - \log(B)] \mathbb{P}\{\|\mathbf{h}_{(L)}\|^2 \geq m_K (1 - \epsilon)\}. \end{aligned}$$

Since $L(K) = o(K^\delta)$ for any $\delta > 0$, it follows that

$$\liminf_{K \rightarrow \infty} \mathbb{P}\{\|\mathbf{h}_{(L)}\|^2 \geq m_K (1 - \epsilon)\} \geq \liminf_{K \rightarrow \infty} \mathbb{P}\{\|\mathbf{h}_{(K^{\epsilon/4})}\|^2 \geq m_K (1 - \epsilon)\}.$$

We now use a fact that is derived using the theory of extremal order statistics. As stated in Lemma A.2.1, it can be shown that there exists a constant $C_{\epsilon/4,\epsilon}$ for which

$$\mathbb{P}\{\|\mathbf{h}_{(K^{\epsilon/4})}\|^2 \leq m_K(1 - \epsilon)\} \leq \frac{C_{\epsilon/4,\epsilon}}{(\log(K))^2}.$$

Combining the above inequalities and observing that $\log(1+Qm_k(1-\epsilon)) = o((\log(K))^2)$ as $K \rightarrow \infty$, we deduce that the limsup is bounded from above by

$$\limsup_{K \rightarrow \infty} \log(A + Qm_K) - \log(B + Qm_K(1 - \epsilon)).$$

The latter quantity is no larger than

$$\lim_{K \rightarrow \infty} \log\left(1 + \frac{A - B}{B + Qm_K}\right) - \log(1 - \epsilon) = -\log(1 - \epsilon),$$

because $m_K \rightarrow \infty$ as $K \rightarrow \infty$. Letting $\epsilon \downarrow 0$, the result follows. □

A.4 Lemmas and proofs for the heterogeneous case

The following lemmas are useful in proving Theorem 4.3.1.

Lemma A.4.1 *For all $c = 1, \dots, C$,*

$$\limsup_{K \rightarrow \infty} \mathbb{E} \left[\log \left(1 + \sum_{d=1}^c P_d^*(K) \|\mathbf{h}_{(1)}^{(d)}\|^2 \right) \right] - \log(B(K)) \leq \log \left(\sum_{d=1}^c P_d^* \beta_d^2 \right).$$

Proof

Since $\|\mathbf{h}_{(1)}^{(d)}\|^2$ is a scaled version of a unit Erlang(M) random variable, we can write

$$\|\mathbf{h}_{(1)}^{(d)}\|^2 = \beta_d^2 [B(K) + Z^{(d)}(K)].$$

This is in view of Section A.2 above.

Then,

$$\begin{aligned}
& \mathbb{E} \left[\log \left(1 + \sum_{d=1}^c P_d^*(K) \|\mathbf{h}_{(1)}^{(d)}\|^2 \right) \right] - \log(B(K)) \\
&= \mathbb{E} \left[\log \left(1 + \sum_{d=1}^c P_d^*(K) \beta_d^2 (B(K) + Z^{(d)}(K)) \right) \right] - \log(B(K)) \\
&= \mathbb{E} \left[\log \left(\frac{1}{B(K)} + \sum_{d=1}^c P_d^*(K) \beta_d^2 \left(1 + \frac{Z^{(d)}(K)}{B(K)} \right) \right) \right] \\
&\leq \mathbb{E} \left[\log \left(\frac{1}{B(K)} + \sum_{d=1}^c \beta_d^2 \left(P_d^*(K) + \frac{P(Z^{(d)}(K))}{2B(K)} \right) \right) \right] \\
&\leq \log \left(\frac{1}{B(K)} + \sum_{d=1}^c \beta_d^2 \left(\mathbb{E}[P_d^*(K)] + \frac{P \mathbb{E}[(Z^{(d)}(K))]}{2B(K)} \right) \right),
\end{aligned}$$

where the last step follows from Jensen's inequality. We see from (A.1) that

$$\frac{\mathbb{E}[(Z^{(d)}(K))]}{B(K)} \rightarrow 0$$

as $K \rightarrow \infty$. Hence,

$$\limsup_{K \rightarrow \infty} \mathbb{E} \left[\log \left(1 + \sum_{d=1}^c P_d^*(K) \|\mathbf{h}_{(1)}^{(d)}\|^2 \right) \right] - \log(B(K)) \leq \limsup_{K \rightarrow \infty} \log \left(\sum_{d=1}^c \mathbb{E}[P_d^*(K)] \beta_d^2 \right).$$

As stated earlier, the powers $(P_1^*(K), \dots, P_C^*(K))$ converge to (P_1^*, \dots, P_C^*) in probability, as $K \rightarrow \infty$. Also, since the powers are bounded above by P/M , $P_c^*(K)$ is a sequence of uniformly integrable random variables for each class c . Using these two facts, one may deduce that $\lim_{K \rightarrow \infty} \mathbb{E}[P_d^*(K)] = P_d^*$ for $d = 1, 2, \dots, C$, see [2]. The result easily follows now since

$$\limsup_{K \rightarrow \infty} \log \left(\sum_{d=1}^c \mathbb{E}[P_d^*(K)] \beta_d^2 \right) = \log \left(\sum_{d=1}^c P_d^* \beta_d^2 \right).$$

□

Lemma A.4.2 For all $c = 1, \dots, C$, if $L_d(K) = O(K^\delta)$, $\delta > 0$, for all $d = 1, \dots, c$,

then

$$\liminf_{K \rightarrow \infty} \mathbb{E} \left[\log \left(1 + \sum_{d=1}^c P_d^* \|\mathbf{h}_{(L_d(K))}^{(d)}\|^2 \right) \right] - \log(B(K)) \geq \log \left(\sum_{d=1}^c P_d^* \beta_d^2 \right) + \log(1 - 2\delta).$$

Proof

Take $\epsilon \geq 2\delta$.

Define the event

$$L_{c,\epsilon}^K := \cup_{d=1}^c \left\{ \|\mathbf{h}_{(L_d(K))}^{(d)}\|^2 \leq (1 - \epsilon) \beta_d^2 B(K) \right\},$$

and its complement

$$\bar{L}_{c,\epsilon}^K := \cap_{d=1}^c \left\{ \|\mathbf{h}_{(L_d(K))}^{(d)}\|^2 \geq (1 - \epsilon) \beta_d^2 B(K) \right\}.$$

In view of Lemma A.2.1, it is easy to deduce that

$$\mathbb{P}\{L_{c,\epsilon}^K\} = o((\log B(K))^{-1}).$$

Next,

$$\begin{aligned} & \mathbb{E} \left[\log \left(1 + \sum_{d=1}^c P_d^* \|\mathbf{h}_{(L_d(K))}^{(d)}\|^2 \right) \right] - \log(B(K)) \\ & \geq \mathbb{E} \left[\log \left(1 + \sum_{d=1}^c P_d^* \|\mathbf{h}_{(L_d(K))}^{(d)}\|^2 \right); \bar{L}_{c,\epsilon}^K \right] - \log(B(K)) \\ & \geq \log \left(1 + (1 - \epsilon) \sum_{d=1}^c P_d^* \beta_d^2 B(K) \right) (1 - \mathbb{P}\{L_{c,\epsilon}^K\}) - \log(B(K)) \\ & \geq \left[\log(1 - \epsilon) + \log(B(K)) + \log \left(\sum_{d=1}^c P_d^* \beta_d^2 \right) \right] [1 - \mathbb{P}\{L_{c,\epsilon}^K\}] - \log(B(K)) \\ & \geq \log(1 - \epsilon) + \log \left(\sum_{d=1}^c P_d^* \beta_d^2 \right) - \left[\log(B(K)) + \log \left(\sum_{d=1}^c P_d^* \beta_d^2 \right) \right] \mathbb{P}\{L_{c,\epsilon}^K\}. \end{aligned}$$

Since $\mathbb{P}\{L_{c,\epsilon}^K\} = o((\log(B(K)))^{-1})$, the statement of the lemma follows. \square

Lemma A.4.3 *Let $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{C}^M$ be two orthonormal vectors. Let \mathbf{g} be λ -aligned with \mathbf{u}_1 and \mathbf{h} be λ -aligned with \mathbf{u}_2 , for some $\lambda < \frac{1}{3}$. Then, \mathbf{g} and \mathbf{h} are 3λ -orthogonal.*

Proof

Let $\mathbf{g} = [g_1, g_2, \dots, g_M]$ and $\mathbf{h} = [h_1, h_2, \dots, h_M]$. Without loss of generality, we may assume $\mathbf{u}_1 = [1, 0, \dots, 0]$ and $\mathbf{u}_2 = [0, 1, \dots, 0]$. By hypothesis, we have,

$$\frac{|g_1|^2}{\sum_{j=1}^M |g_j|^2} > 1 - \lambda$$

and

$$\frac{|h_2|^2}{\sum_{j=1}^M |h_j|^2} > 1 - \lambda.$$

Now,

$$\begin{aligned} U(\mathbf{h}, \mathbf{g}) &= \frac{|\sum_{j=1}^M g_j h_j|^2}{\sum_{j=1}^M |g_j|^2 \sum_{j=1}^M |h_j|^2} \\ &\leq \frac{\sum_{j=1}^M |g_j|^2 |h_j|^2}{\sum_{j=1}^M |g_j|^2 \sum_{j=1}^M |h_j|^2} \\ &< \frac{|g_1|^2(|h_1|^2 + \sum_{j=3}^M |h_j|^2) + |g_2|^2 |h_2|^2}{|g_1|^2 |h_2|^2} \\ &< \frac{|h_1|^2 + \sum_{j=3}^M |h_j|^2}{|h_2|^2} + \frac{|g_2|^2}{|g_1|^2} \\ &< \frac{2\lambda}{1 - \lambda} < 3\lambda. \end{aligned}$$

In the above, step 2 follows from triangle inequality and step 5 from the hypothesis.

□

Lemma A.4.4 *Consider the MC users chosen by the List scheme described earlier. For any $c, d = 1, 2, \dots, C$, and any $i, j = 1, 2, \dots, M, i \neq j$ there exist constants $\xi_{c,d}$ and $\eta_{c,d}$ such that*

$$\mathbb{P}\{U(\mathbf{h}_{\mathbf{u}_i^{(c)}}, \mathbf{h}_{\mathbf{u}_j^{(d)}}) < \frac{\xi_{c,d}}{K^{\frac{\delta}{2(M-1)}}}\} > 1 - \frac{\eta_{c,d}}{K^\delta}.$$

Proof

As shown in Lemma 3.2.1, the normalized inner product between any arbitrary channel vector and the unit vector \mathbf{u}_i has the distribution function

$$F_U(u) = 1 - (1 - u)^{M-1}.$$

For the special case of $M = 2$, this corresponds to a uniform distribution. Now, recall that $\mathbf{h}_{\mathbf{u}_i}^{(c)}$ is chosen as the vector which is most collinear with \mathbf{u}_i from among a group of $L_c = K_c^\delta$ class c users. In other words, the normalized inner product $U(\mathbf{h}_{\mathbf{u}_i}^{(c)}, \mathbf{u}_i)$ is chosen as the maximum among $L_c = K_c^\delta$ i.i.d variables, distributed according to $F_U(u)$ above. The distribution function of $U(\mathbf{h}_{\mathbf{u}_i}^{(c)}, \mathbf{u}_i)$ can therefore be written as

$$\mathbb{P}\{U(\mathbf{h}_{\mathbf{u}_i}^{(c)}, \mathbf{u}_i) < u\} = (1 - (1 - u)^{M-1})^{L_c}$$

from which it is easy to deduce that

$$\mathbb{P}\{U(\mathbf{h}_{\mathbf{u}_i}^{(c)}, \mathbf{u}_i) < 1 - K_c^{-\frac{\delta}{2(M-1)}}\} = (1 - \frac{1}{\sqrt{L_c}})^{L_c}.$$

We now use the fact that the function $f(x) = (1 - 1/x)^x, x > 1$ is bounded above by $1/e$, to upper bound the above probability as

$$\mathbb{P}\{U(\mathbf{h}_{\mathbf{u}_i}^{(c)}, \mathbf{u}_i) < 1 - K_c^{-\frac{\delta}{2(M-1)}}\} < e^{-\sqrt{L_c}}.$$

Even though this is a sharp bound on the above probability, the following loose bound will suffice here:

$$\mathbb{P}\{U(\mathbf{h}_{\mathbf{u}_i}^{(c)}, \mathbf{u}_i) > 1 - K_c^{-\frac{\delta}{2(M-1)}}\} > 1 - 1/L_c = 1 - K_c^{-\delta}.$$

Similarly, we can write, for $j \neq i$,

$$\mathbb{P}\{U(\mathbf{h}_{\mathbf{u}_j}^{(d)}, \mathbf{u}_j) > 1 - K_d^{-\frac{\delta}{2(M-1)}}\} > 1 - 1/L_d = 1 - K_d^{-\delta}.$$

Using the above pair of inequalities, and invoking Lemma A.4.3, we get

$$\mathbb{P}\{U(\mathbf{h}_{u_j}^{(d)}, \mathbf{h}_{u_i}^{(c)}) < \frac{3}{\min(K_c, K_d)^{\frac{\delta}{2(M-1)}}}\} > 1 - K_c^{-\delta} - K_d^{-\delta}.$$

Finally, noting that each $K_c = \alpha_c K$, with α_c constant, the result follows. \square

Bibliography

- [1] P. Billingsley (1968). *Convergence of Probability Measures*. Wiley & Sons.
- [2] P. Billingsley (1995). *Probability and Measure*. Third Edition, Wiley & Sons.
- [3] G. Caire, S. Shamai (2003). On the achievable throughput of a multiantenna Gaussian broadcast channel. *IEEE Trans. Inf. Theory* **49** (7), 1691–1706.
- [4] M. Costa (1983). Writing on dirty paper. *IEEE Trans. Inf. Theory* **29** (3), 439–441.
- [5] T. Cover, J. Thomas (1991). *Elements of Information Theory*. Wiley & Sons.
- [6] B.M. Hochwald, C.B. Peel, A.L. Swindlehurst (2005). A vector-perturbation technique for near-capacity multiantenna multiuser communication – part I: channel inversion and regularization. *IEEE Trans. Commun.* **53** (1), 195–202.
- [7] B.M. Hochwald, C.B. Peel, A.L. Swindlehurst (2005). A vector-perturbation technique for near-capacity multiantenna multiuser communication – part II: perturbation. *IEEE Trans. Commun.* **53** (3), 537–544.
- [8] N. Jindal (2005). A high SNR analysis of MIMO broadcast channels. Preprint.
- [9] N. Jindal (2006). Finite rate feedback MIMO broadcast channels. In: *Proc. Workshop on Information Theory and its Applications, UC San Diego*.
- [10] N. Jindal, A. Goldsmith (2005). Dirty-paper coding versus TDMA for MIMO broadcast channels. *IEEE Trans. Inf. Theory* **51** (5), 1783–1794.

- [11] N. Jindal, W. Rhee, S. Vishwanath, S.A. Jafar, A. Goldsmith (2005). Sum power iterative water-filling for multi-antenna Gaussian broadcast channels. *IEEE Trans. Inf. Theory* **51** (4), 1570–1580.
- [12] N. Jindal, S. Vishwanath, A. Goldsmith (2004). On the duality of Gaussian multiple-access and broadcast channels. *IEEE Trans. Inf. Theory* **50** (5), 768–783.
- [13] J. Lee, N. Jindal (2005). Symmetric capacity of MIMO downlink channels. Preprint.
- [14] Mathworld: <http://mathworld.wolfram.com/Euler-MascheroniConstant.html>
- [15] M.J. Neely, E. Modiano, C.E. Rohrs (2003). Dynamic power allocation and routing for time-varying wireless networks. In: *Proc. Infocom 2003*.
- [16] S. Ross (1996). *Stochastic Processes*, Second Edition, Wiley & Sons.
- [17] M. Sharif, B. Hassibi (2005). On the capacity of MIMO broadcast channels with partial side information. *IEEE Trans. Inf. Theory* **51** (2), 506–522.
- [18] M. Sharif, B. Hassibi (2005). Scaling laws of sum rate using time sharing, DPC and beamforming for MIMO broadcast channels. In: *Proc. ISIT 2005 Conf.*
- [19] M. Sharif, B. Hassibi (2006). Scaling laws of sum rate of time sharing, DPC and beamforming for MIMO BC with many users. Submitted for publication.
- [20] A.L. Stolyar (2004). MaxWeight scheduling in a generalized switch: state space collapse and workload minimization in heavy traffic. *Ann. Appl. Prob.* **14**, 1–53.
- [21] C. Swannack, E. Uysal-Biyikoglu, G. Wornell (2004). Low-complexity multi-user scheduling for maximizing throughput in the MIMO broadcast channel. In: *Proc. 42nd Annual Allerton Conf. Commun. Control, Comput.*
- [22] C. Swannack, E. Uysal-Biyikoglu, G. Wornell (2004). Finding NEMO: near-orthogonal sets for multiplexing and opportunistic scheduling in MIMO broadcast. Preprint.

- [23] L. Tassiulas, A. Ephremides (1992). Stability properties of constrained queuing systems and scheduling policies for maximum throughput in multihop radio networks. *IEEE Trans. Aut. Contr.* **37**, 1936–1948.
- [24] A. Vakili, A. Dana, M. Sharif, B. Hassibi (2005). Differentiated rate scheduling for MIMO broadcast channels. In: *Proc. 43rd Annual Allerton Conf. Commun. Control, Comput.*
- [25] S. Vishwanath, N. Jindal, A. Goldsmith (2003). Duality, achievable rates and sum-rate capacity of MIMO broadcast channels. *IEEE Trans. Inf. Theory* **49** (10), 2658–2668.
- [26] P. Viswanath, D.N.C. Tse (2003). Sum capacity of the vector Gaussian broadcast channel and uplink-downlink duality. *IEEE Trans. Inf. Theory* **49** (8), 1912–1921.
- [27] P. Viswanath, D.N.C. Tse, R. Laroia (2002). Opportunistic beamforming using dumb antennas. *IEEE Trans. Inf. Theory* **48** (6), 1277–1294.
- [28] H. Viswanathan, K. Kumaran (2005). Rate scheduling in multiple-antenna downlink wireless systems. *IEEE Trans. Commun.* **53** (4), 645–655.
- [29] H. Viswanathan, S. Venkatesan, H. Huang (2005). Downlink capacity evaluation of cellular networks with known interference cancellation. *IEEE J. Sel. Areas Commun.* **21** (5), 802–811.
- [30] H. Weingarten, Y. Steinberg, S. Shamai (2005). The capacity region of the Gaussian MIMO broadcast channel. *IEEE Trans. Inf. Theory*, to appear.