# Nonlinear Interaction of Long-Wave Disturbances with Short-Scale Oscillations in Stratified Flows 

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#### Abstract

A nonlinear mechanism that couples long-wave disturbances with short-scale oscillations in dispersive wave systems is investigated theoretically. This fully nonlinear interaction mechanism obtains when the phase speed of the short wave matches the speed of the long wave, and it causes solitary internal gravity waves to radiate short-scale oscillatory tails. In the context of density-stratified flows, the same resonance mechanism is also relevant to the generation of lee waves by flow over extended topography and to the formation of surface rips by long internal waves. In the weakly nonlinear-weakly dispersive regime, however, the short-wave amplitude is exponentially small (with respect to the long-wave amplitude) and lies beyond all orders of the usual expansion procedures. For this reason, in order to understand this long-short wave interaction theoretically, it becomes necessary to use techniques of exponential asymptotics.

In this thesis, a perturbation procedure in the wavenumber domain is presented for calculating the amplitude of short-scale oscillatory tails induced by steady long-wave disturbances. Through a series of model problems, it is demonstrated that the induced short-wave tail is sensitive to the details of the long-wave profile. Furthermore, in order to examine the physical importance of the resonance mechanism, the problems of lee waves and non-local gravity-capillary solitary waves are investigated. In the limit of weak dispersion and nonlinearity, the tail amplitude is calculated analytically following the wavenumberdomain approach. Numerical computations of large-amplitude wave disturbances are also carried out. Comparison with numerical results indicates that the asymptotic theory often remains reasonably accurate even when the effects of nonlinearity and dispersion are moderate, in which case the (formally exponentially small) tail amplitude is greatly enhanced by nonlinearity and can be quite substantial.

Finally, the techniques developed here are used to investigate the possibility of smallamplitude asymmetric solitary waves. In general, such waves exist and are related to envelope solitons with stationary crests near the minimum of the dispersion curve, where the phase velocity equals the group velocity. Both symmetric and asymmetric multi-modal solitary wavepackets are found, consistent with numerical computations.


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## CHAPTER 1

## GENERAL INTRODUCTION

### 1.1 Motivation and objective

Internal waves are gravity waves that propagate in density-stratified fluids and can transport energy and momentum. Internal-wave disturbances generated by tidal flow over bottom topography are common features in the ocean, due to existing temperature and salinity variations (see, for example, Farmer \& Smith 1980). Likewise, large-amplitude internal waves often form on the lee side of mountains (hence referred to as lee waves), and are a major factor in the development of storms (Lilly 1978).

On the basis of field observations, it is now recognized that internal-wave disturbances in the ocean often take the form of solitary waves (see, for example, Osborne \& Burch 1980). These are locally confined nonlinear waves of permanent form held together by the combined action of nonlinearity and dispersion. Recent findings, however, point to the fact that solitary internal waves may develop short-scale oscillations at the tails of the main disturbance (see Figure 1.1). In particular, Akylas \& Grimshaw (1992) studied solitary internal waves in a shallow stratified fluid layer bounded by rigid walls and pointed out that solitary waves of mode greater than one do not remain locally confined, in general. Rather, they develop oscillatory tails of infinite extent consisting of lower-mode small-amplitude short waves. These findings are supported by the field observations of Farmer \& Smith (1980) as well as the laboratory experiments of Davis \& Acrivos (1967).

Physically, the oscillatory tails are simply short waves whose phase speed matches the speed of the main disturbance. It is also useful to think of the solitary-wave core as a moving disturbance capable of exciting waves; accordingly, the tail amplitude is expected to be proportional to the spectral amplitude of the forcing at the tail wavenumber. For this reason, the calculation of the tail amplitude is a challenging task: since the spectral amplitude of a smooth, locally confined forcing is exponentially small in the large-wavenumber region of the
spectrum, the tails cannot be captured by the usual weakly nonlinear long-wave expansion. In fact. the effects of nonlinearity and dispersion of all orders are equally important in exciting short-wavelength oscillatory waves, and, to calculate the amplitude of these waves, one has to resort to the techniques of exponential asymptotics pioneered by Segur \& Kruskal (1987) (see also Kruskal \& Segur 1991; Segur, Tanveer \& Levine 1991 for a collection of recent articles on asymptotics 'beyond all orders'). Also, the dual demand of high accuracy and capability of resolving coexisting waves of very different lengthscales often renders numerical computations expensive, thus high-performance numerical methods are extremely desirable.

In addition to solitary internal waves, the fully nonlinear resonance mechanism described above is also relevant in a variety of other physical settings. The interaction of long internalwave disturbances over topography with relatively short lee waves is one example, while the appearance of short-scale surface waves-the so-called 'surface rips'-that often accompany internal-wave disturbances (Osborne \& Burch 1980) gives another. However, as the amplitude of the short-wave tails is formally exponentially small, an important issue is whether these tails would obtain appreciable amplitude under practical conditions, so that they are of physical significance. The overall objective of the present work is to address this issue by providing quantitative predictions of the tail amplitude. To this end, the problems of lee waves and non-local gravity-capillary solitary waves (as a model for surface rips) are studied in this thesis, using analytical and numerical techniques.

### 1.2 Background

Here we briefly discuss some background material, in order to put this work in the right perspective.

The study of solitary-wave dynamics has been based largely on weakly nonlinear longwave theories. The usual assumption is that the horizontal disturbance lengthscale is long compared with the thickness of the pycnocline-the region where appreciable density
stratification occurs. It is then possible to derive approximate evolution equations for the disturbance profile, using asymptotic expansions combining weak dispersive and nonlinear effects. This classical approach leads to the Korteweg-de Vries (KdV) equation for each internal-wave mode in shallow fluids (see, for example, Benney 1966), to the Benjamin-Davis-Ono (BDO) equation in deep fluids (Benjamin 1967; Davis \& Acrivos 1967; Ono 1975), and to the intermediate long-wave (ILW) equation in fluids of finite depth (Joseph 1977: Kubota, Ko \& Dobbs 1978). [The ILW equation may be viewed as encompassing the KdV and the BDO equations as limiting cases.] All these evolution equations admit solitary-wave solutions.

The first indication that solitary waves may cease to remain locally confined was given in the numerical study of Hunter \& Vanden-Broeck (1983). Using the full water-wave theory, they were unable to find localized gravity-capillary solitary waves of elevation for low surface tension. Instead, Hunter \& Vanden-Broeck (1983) computed symmetric periodic waves of permanent form with a single main hump, resembling a KdV solitary wave, which, however, was accompanied by short-wave oscillatory tails. Of course, it is now well known that this is due to the resonance mechanism described in the previous section: for low surface tension, there exist small-amplitude short-wavelength capillary waves that can travel with the same phase speed as a KdV solitary wave. On the other hand, when surface tension is neglected altogether or in the high-surface-tension regime, it is not possible for small-amplitude shortwavelength waves to move with a KdV solitary wave, and indeed no tails are found.

The numerical findings of Hunter \& Vanden-Broeck (1983) point to the fact that the KdV equation is not a valid model for gravity-capillary waves in the low-surface-tension regime because this model equation is derived on the assumption that only long waves are present, and hence does not take into account the short lengthscale of oscillatory tails. For this reason, Hunter \& Schuerle (1988) modified the KdV equation by adding a small fifth-order-derivative term, and proved the existence of traveling-wave solutions in the form of slightly perturbed KdV solitary waves with small-amplitude oscillations at the tails. Independently. Pomeau, Ramani \& Grammaticos (1988) also considered the fifth-order

KdV equation and calculated the (exponentially small) tail amplitude, using a nonlinear WKB technique devised earlier by Segur \& Kruskal (1987) (see also Kruskal \& Segur 1991). The technique focuses on the singularities of the KdV solitary-wave solution in the complex plane. Near the singularities, the long-wave expansion becomes disordered-all nonlinear and dispersive terms become equally important. Accordingly, inner solutions valid close to each singularity are constructed by 'summing' divergent asymptotic series through Borel summation (Bender \& Orszag 1978, §8.2) [or, equivalently, by expressing the inner solution as a Laplace transform (Byatt-Smith 1991; Grimshaw \& Jcshi 1995)]. The oscillatory tails are then determined on the real axis by matching these inner solutions with the long-wave expansion, imposing appropriate boundary conditions. This procedure was also followed by Akylas \& Grimshaw (1992) to discuss weakly non-local solitary internal waves in a shallow fluid layer. However, when applied to problems other than the KdV-type ones, the analysis in the complex plane can become prohibitive (see Chapter 3), and a more robust approach would be desirable. This prompts the development of the wavenumber-domain approach presented in Chapter 2.

Also, the fact that the tail amplitude lies beyond all orders of the long-wave expansion suggests that it is necessary to retain all nonlinear and dispersive effects in order to describe the tails accurately. Therefore, the predictions of model equations will agree only qualitatively with their corresponding full hydrodynamic equations. To examine the physical significance of the resonance mechanism in question, the full problems of lee waves and non-local gravity-capillary solitary waves are studied in this work. It is worth noting that, since the resonance mechanism in question is fully nonlinear, the mathematical problems posed by lee waves and non-local gravity-capillary waves are technically more difficult than that considered by Akylas \& Grimshaw (1992) because of the additional nonlinearities resulting from the presence of bottom topography and a free surface, respectively. In addition to calculating the short-wave amplitude asymptotically for smallamplitude wave disturbances, numerical computations of large-amplitude waves are also carried out, in order to confirm the predictions of the asymptotic theory and to explore the
behavior of short-wave tails for large-amplitude disturbances that are often observed in the field. From a theoretical point of view, the analytical and numerical techniques developed here are of interest in their own right; it is expected that these techniques will prove useful in other related problems dealing with nonlinear wave propagation in waveguides.

### 1.3 Outline of this work

A description of the organization and main results of this work follows.
In Chapter 2, a perturbation procedure in the wavenumber domain, devised by Akylas \& Yang (1995), is presented for calculating the amplitude of short-scale oscillatory tails induced by steady long-wave disturbances. In the limit of weak dispersion and nonlinearity, these tails have exponentially small amplitude that lies beyond all orders of the usual longwave expansion. The wavenumber-domain approach differs from other existing techniques of exponential asymptotics in that it is based on the picture that the appearance of oscillatory tails in the physical domain implies the presence of poles on the real axis (at the tail wavenumber) in the wavenumber domain. By working in the wavenumber domain, the residues of the poles can be determined by multiple-scaling techniques; once the residues are found, the amplitude and the phase of the tails are completely determined. Using the forced KdV (fKdV) equation as a simple model, the efficiency and robustness of the wavenumber-domain approach are demonstrated. More importantly, it is found that the induced short-wave tail is sensitive to the details of the long-wave profile.

In Chapter 3 we discuss the weakly non-local solitary-wave solutions of a model evolution equation in fluids of finite depth proposed by Yang \& Akylas (1995). In analogy with the fifth-order KdV equation, this model evolution equation adds to the ILW equation a small third-order-derivative term, to account for higher-order dispersive effects that give rise to oscillatory tails. The oscillatory tails accompanying finite-depth solitons are calculated following the wavenumber-domain approach presented in Chapter 2. However, unlike the 'sech ${ }^{2}$ ' profile of the KdV soliton that has double-pole singularities in the complex plane,
the profile of finite-depth solitons has simple poles. It turns out that this distinction leads to a significant phase modulation of the oscillatory tails of finite-depth solitons [in the case of the KdV soliton. the phase modulation vanishes to leading order]. As remarked earlier. by taking appropriate limit, the results of Chapter 3 also apply to non-local solitary waves in deep fluids.

To examine the physical significance of the resonance mechanism in question. we consider first the problem of lee waves in Chapter 4. This chapter is based on the work of Yang \& Akylas (1996a), which considers the flow of a continuously stratified fluid over a smooth bottom bump in a channel of finite depth. It is shown that, in the weakly nonlinear-weakly dispersive regime ( $\epsilon=a / h \ll 1, \mu=h / L \ll 1$, where $h$ is the channel depth and $a, L$ are the peak amplitude and the width of the obstacle respectively), the parameter $A=\epsilon / \mu^{p}$ (where $p>0$ depends on the obstacle shape) controls the effects of nonlinearity on the steady lee wavetrain that forms downstream of the obstacle for subcritical flow speeds.

For $A=O(1)$, when nonlinear and dispersive effects are equally important, the wavenumber-domain approach is extended to determine the lee-wave amplitude. The results suggest that nonlinearity may seriously affect the lee-wave pattern induced by a long smooth obstacle. However, the asymptotic theory is formally valid in the joint limit that $\epsilon, \mu \ll 1$ where lee waves have exponentially small amplitude and form only a small portion of the overall disturbance. A numerical procedure is then devised to examine the effects of nonlinearity on lee-wave patterns, and to provide independent confirmation of the asymptotic theory. Comparison with numerical results indicates that the asymptotic theory often remains reasonably accurate even for moderately small values of $\mu$ and $\epsilon$, in which case the (formally exponentially small) lee-wave amplitude is greatly enhanced by nonlinearity and can be quite substantial (see Figure 4.3). Moreover, these findings reveal that the range of validity of the classical linear lee-wave theory $(A \ll 1)$ is quite limited.

The classical problem of gravity-capillary waves on a single liquid layer of uniform density is studied in Chapter 5. Recall that, in the low-surface-tension regime, gravitycapillary solitary waves may develop small-amplitude short-scale oscillatory tails. A
theoretically interesting question is whether the amplitude of the oscillatory tails can be evactly zero, under certain conditions. As remarked earlier, the effects of nonlinearity and dispersion of all orders are equally important in determining the tail amplitude; therefore. it would seem hopeless to answer this question by truncating the problem in some way, and then seeking certain bounds of the tail amplitude. A sensible and promising alternative is to use the techniques of exponential asymptotics. Once again, the wavenumber-domain approach turns out to be a success in this problem: the asymptotic analysis reveals that the amplitude never vanishes for low-surface-tension gravity-capillary waves. Chapter 5 is based on Yang \& Akylas (1996b).

Finally, in Chapter 6, the techniques developed in this thesis are used to investigate the possibility of small-amplitude asymmetric solitary waves based on the fifth-order KdV equation. In general, such waves exist and are related to envelope solitons with stationary crests near the minimum of the dispersion curve, where the phase velocity equals the group velocity. Previous numerical results suggest that asymmetric solitary waves bifurcate from finite-amplitude symmetric solitary waves. On the other hand, straightforward asymptotic analysis does not rule out the possibility of asymmetric solitary waves of infinitesimal amplitude. In turns out that the presence of exponentially small terms is the key to the resolution of this contradiction. The exponentially small terms are calculated following the wavenumber-domain approach presented in Chapter 2. It is found that, due to the radiation of oscillatory waves of exponentially small but growing (in space) amplitude, a single asymmetric wavepacket does not remain locally confined. However, multi-modal solitary waves that consist of several wavepackets can be constructed asymptotically. Both symmetric and asymmetric multi-modal solitary wavepackets are found. These multi-modal waves exist at finite amplitude, and, in a bifurcation diagram, a branch of asymmetric waves always intersects another branch of symmetric waves. It is therefore appropriate to attribute the existence of asymmetric solitary waves to a symmetrybreaking bifurcation from finite-amplitude symmetric solitary waves.

Future directions of research related to this work are discussed in Chapter 7.


Figlre 1.1 Image of internal-wave disturbance generated by stratified flow past a sill in the field experiment of Farmer \& Smith (1980). The streamlines indicate that the main disturbance is a mode-2 solitary-like wave and is followed by a train of smaller-amplitude mode-1 short waves. Adapted from Akylas \& Grimshaw (1992).

## CHAPTER 2

## WAVENUMBER-DOMAIN APPROACH

The motivation for working in the wavenumber domain comes from the physical picture that the tails are excited by the solitary-wave core: the appearance of short-scale oscillatory tails in the physical domain implies the presence of poles on the real axis in the wavenumber domain (at the tail wavenumber), and the amplitude of the tails is completely determined by the corresponding residues. Of course, as expected, the regular perturbation expansion in the wavenumber domain becomes disordered near the poles, but these non-uniformities can be readily handled and the residues can be calculated by multiple-scaling techniques.

In presenting the perturbation procedure in the wavenumber domain, we use, as a simple example, the forced Korteweg-de Vries (fKdV) equation and discuss the steady wavetrain induced in the far field by a prescribed, locally confined forcing disturbance. In the limit of weak dispersion and nonlinearity, the response obtains a form analogous to a weakly nonlocal solitary wave-an exponentially small, short-scale oscillatory tail appears downstream. For the purpose of illustrating the perturbation technique, however, this fKdV model is more instructive than the fifth-order KdV equation because it allows more flexibility in the choice of the forcing disturbance. It turns out that the precise form of the forcing plays an important part in determining the amplitude of the induced tail. Of course, as indicated at the end of this chapter ( $\S 2.5$ ), the wavenumber-domain approach is also readily applicable to the fifth-order KdV equation and to other non-local solitary-wave models (see Chapter 3).

### 2.1 Formulation

Suppose that $u(x)$ satisfies the steady fKdV equation,

$$
\begin{equation*}
\mu^{2} u_{x x}+u-\epsilon u^{2}=f(x) \quad(-\infty<x<\infty) \tag{2.1}
\end{equation*}
$$

subject to the boundary condition

$$
\begin{equation*}
u \rightarrow 0 \quad(x \rightarrow \infty) \tag{2.2}
\end{equation*}
$$

assuming that $f(x)$ is locally confined $(f(x) \rightarrow 0, x \rightarrow \pm \infty)$.
In the context of wave propagation, this boundary-value problem is a simple model for the steady-state wave pattern induced by a prescribed forcing disturbance $f(x)$ moving at constant (subcritical) speed in a nonlinear dispersive medium (Akylas 1988; Wu 1987); the parameters $\epsilon, \mu$ are measures of nonlinear and dispersive effects, respectively. The boundary condition (2.2) implies that waves may appear only downstream ( $x \rightarrow-\infty$ ); it derives from group-velocity arguments based on the unsteady fKdV equation. Our interest centers on the asymptotic behavior of the response when nonlinear and dispersive effects are equally important and weak, so the parameters $\epsilon, \mu$ in (2.1) are taken to be small $(0<\epsilon, \mu \ll 1)$. The appropriate relation between $\epsilon$ and $\mu$ for nonlinearity to balance dispersion depends on the form of the forcing $f(x)$, and specific examples will be given in the following sections.

The analysis commences by attempting a straightforward expansion of $u(x)$ in powers of $\epsilon, \mu^{2}$ :

$$
\begin{equation*}
u \sim f-\mu^{2} f_{x x}+\epsilon f^{2}+\cdots . \tag{2.3}
\end{equation*}
$$

According to (2.3), the response remains locally confined ( $u \rightarrow 0, x \rightarrow \pm \infty$ ) since the excitation is locally confined; however, the exact linear response apparently contradicts this conclusion. Specifically, taking Fourier transform,

$$
\hat{u}(k)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{-\mathrm{i} k x} u(x) \mathrm{d} x
$$

the linearized version of $(2.1)(\epsilon=0)$ yields
and

$$
\begin{gather*}
\hat{u}=\frac{\hat{f}(k ;}{1-\mu^{2} k^{2}},  \tag{2.4}\\
u=\int_{\mathcal{C}} e^{\mathrm{i} k x} \hat{u}(k) \mathrm{d} k, \tag{2.5}
\end{gather*}
$$

where the contour $\mathcal{C}$ is indented to pass above the poles of $\hat{u}$ on the real axis at $k= \pm 1 / \mu$ so that (2.2) is met. Hence, owing to the contribution of these poles, an oscillatory tail of
constant amplitude and wavenumber equal to $1 / \mu$ appears in $x<0$, and takes the form

$$
\begin{equation*}
u \sim-\frac{2 \pi}{\mu} \hat{f}(1 / \mu) \sin \frac{x}{\mu} \quad(x \rightarrow-\infty) \tag{2.6}
\end{equation*}
$$

assuming that $f(x)$ is even.
To reconcile the appearance of the tail (2.6) in the linear response with expansion (2.3), note that, in the limit $\mu \rightarrow 0$, the tail amplitude is exponentially small $[\hat{f}(1 / \mu)$ is exponentially small if $f(x)$ is smooth], and cannot be captured by an expansion in powers of $\mu^{2}$. Accordingly, in order to approximate the tail as $\epsilon, \mu \rightarrow 0$, expansion (2.3) has to be carried beyond all orders; one way to do this, following Segur \& Kruskal (1987) (see also Kruskal \& Segur 1991), is to use asymptotic matching with inner expansions valid near possible singularities of $u_{0}=f(x)$ in the complex plane.

Another approach is to work in the wavenumber domain; the goal is to construct a uniformly valid expression for $\hat{u}(k)$ so as to obtain an approximation to the tail as $\epsilon, \mu \rightarrow 0$ in terms of (2.5). Not unexpectedly, $\hat{u}(k)$, like its linear counterpart (2.4), has poles on the real axis at $k= \pm 1 / \mu$ and the corresponding residues, which determine the tail amplitude, depend on both nonlinear and dispersive terms.

In preparation for the ensuing analysis, the fKdV equation (2.1) is converted to an (integral) equation for $\hat{u}(k)$ :

$$
\begin{equation*}
\left(1-\mu^{2} k^{2}\right) \hat{u}-\epsilon \widehat{u^{2}}=\hat{f}(k) \tag{2.7}
\end{equation*}
$$

by taking Fourier transform formally. Expanding then $\hat{u}$ in powers of $\epsilon, \mu^{2}$ [or, equivalently, taking Fourier transform of (2.3) term by term] yields

$$
\begin{equation*}
\hat{u} \sim \hat{f}(k)\left(1+\mu^{2} k^{2}+\cdots\right)+\epsilon \int_{-\infty}^{\infty} \hat{f}(l) \hat{f}(k-l) \mathrm{d} l+\cdots \tag{2.8}
\end{equation*}
$$

As expected in view of the linear result (2.4), expansion (2.8) becomes disordered when $k=O(1 / \mu)$; but, in order to tackle this non-uniform behavior-particularly the convolution integral that derives from the nonlinear term in (2.7)-it is necessary to be more specific about the form of the forcing $f(x)$.

In the following sections, details are worked out for three choices of forcings that need to be handled somewhat differently.
2.2 The case $f(x)=\operatorname{sech}^{2} x$

As a first example, consider the forcing $f(x)=\operatorname{sech}^{2} x$ with

$$
\begin{equation*}
\hat{f}(k)=\frac{1}{2} k \operatorname{csch} \frac{\pi k}{2} . \tag{2.9}
\end{equation*}
$$

For this choice of $f(x)$, the analysis is relatively straightforward and closely parallels that of weakly non-local solitary-wave solutions of the fifth-order KdV equation (see $\S 2.5$ ).

In terms of (2.9), the convolution integral in (2.8) can be calculated explicitly,

$$
\widehat{f^{2}}=\frac{k}{12}\left(k^{2}+4\right) \operatorname{csch} \frac{\pi k}{2},
$$

and (2.8) becomes

$$
\begin{equation*}
\hat{u} \sim \frac{1}{2}\left(k+\mu^{2} k^{3}+\frac{1}{6} \epsilon k^{3}+\cdots\right) \operatorname{csch} \frac{\pi k}{2}+\cdots . \tag{2.10}
\end{equation*}
$$

Clearly, the choice $\epsilon=\mu^{2}$ ensures that nonlinear and dispersive effects are equally important. Accordingly, disordering occurs when $k=O(1 / \mu)=O\left(1 / \epsilon^{1 / 2}\right)$, and this suggests the twoscale expansion

$$
\begin{equation*}
\hat{u} \sim \frac{1}{\mu} U(\kappa) \operatorname{csch} \frac{\pi k}{2}+\cdots, \tag{2.11}
\end{equation*}
$$

in terms of the 'stretched' wavenumber variable $\kappa=\mu k$, with

$$
\begin{equation*}
U(\kappa) \sim \frac{1}{2} \kappa+\frac{7}{12} \kappa^{3}+\cdots \quad(\kappa \rightarrow 0) \tag{2.12}
\end{equation*}
$$

Substituting then (2.11) into (2.7), yields

$$
\begin{equation*}
\left(1-\kappa^{2}\right) U(\kappa)-\sinh \frac{\pi \kappa}{2 \mu} \int_{-\infty}^{\infty} \mathrm{d} \lambda \frac{U(\lambda) U(\kappa-\lambda)}{\sinh \frac{\pi \lambda}{2 \mu} \sinh \frac{\pi(\kappa-\lambda)}{2 \mu}}=\frac{1}{2} \kappa . \tag{2.13}
\end{equation*}
$$

However, in the limit $\mu \rightarrow 0$, the main contribution to the integral above comes from the range $0<\lambda<\kappa(\kappa>0), \kappa<\lambda<0(\kappa<0)$ and, to leading order, (2.13) reduces to a Volterra integral equation for $U(\kappa)$ :

$$
\begin{equation*}
\left(1-\kappa^{2}\right) U(\kappa)-2 \int_{0}^{\kappa} U(\lambda) U(\kappa-\lambda) \mathrm{d} \lambda=\frac{1}{2} \kappa \tag{2.14}
\end{equation*}
$$

The solution of (2.14) is posed as a power series:

$$
\begin{equation*}
U(\kappa)=\sum_{m=0}^{\infty} b_{m} \kappa^{2 m+1} \tag{2.15}
\end{equation*}
$$

where the coefficients $b_{m}$ satisfy the recurrence relation

$$
\begin{equation*}
b_{m}-b_{m-1}-2 \sum_{r=0}^{m-1} \frac{(2 m-2 r-1)!(2 r+1)!}{(2 m+1)!} b_{r} b_{m-r-1}=0 \quad(m \geq 1) \tag{2.16}
\end{equation*}
$$

with $b_{0}=\frac{1}{2}$. In particular, for $m=1,(2.16)$ gives $b_{1}=\frac{7}{12}$, consistent with (2.12).
In the limit $m \rightarrow \infty$, the convolution sum in (2.16) is subdominant, and

$$
b_{m} \sim C \quad(m \rightarrow \infty)
$$

where $C$ is a constant, the value of which can be found numerically from (2.16), $C=0.77$. Accordingly, the series (2.15) converges for $|\kappa|<1$, and $U(\kappa)$ has pole singularities at $\kappa= \pm 1:$

$$
U(\kappa) \sim-\frac{C}{2(\kappa \mp 1)} \quad(\kappa \rightarrow \pm 1)
$$

Hence, in view of (2.11), $\hat{u}(k)$ has pole singularities at $k= \pm 1 / \mu$ :

$$
\hat{u} \sim \mp \frac{C}{\mu^{2}} \frac{e^{-\pi / 2 \mu}}{k \mp 1 / \mu} \quad(k \rightarrow \pm 1 / \mu)
$$

and, returning to (2.5), it is seen that these poles contribute an exponentially small oscillatory tail for $x<0$ :

$$
\begin{equation*}
u \sim-4 \pi C \frac{e^{-\pi / 2 \mu}}{\mu^{2}} \sin \frac{x}{\mu} \quad(x \rightarrow-\infty) . \tag{2.17}
\end{equation*}
$$

The asymptotic expression (2.17) is in good agreement with estimates of the tail amplitude obtained from numerical solutions of the boundary-value problem (2.1),(2.2) as shown in Figure 2.1. We also remark that the linear result (2.6), combined with (2.9), predicts the correct order of magnitude of the tail amplitude. However, the value of $C$, and hence the precise value of the amplitude, is significantly underestimated $\left[C=\frac{1}{2}\right.$ according to (2.6),(2.9)] if the nonlinear term in (2.1) is ignored: as indicated by the 'naive' expansion (2.10), with the choice $\epsilon=\mu^{2}$, the nonlinear and dispersive terms become equally important near the poles of $\hat{u}$ at $k= \pm 1 / \mu$.

In the following section, a slightly different forcing $f(x)$ is chosen for which nonlinearity alters the linear estimate for the induced wave tail more significantly.
2.3 The case $f(x)=\operatorname{sech} x$

Consider now the forcing $f(x)=\operatorname{sech} x$ for which

$$
\begin{equation*}
\hat{f}(k)=\frac{1}{2} \operatorname{sech} \frac{\pi k}{2} . \tag{2.18}
\end{equation*}
$$

Even though this forcing is qualitatively similar to the one considered in the previous section, here the analysis is more involved and the induced wave tail suffers a phase modulation. As will be seen, this complication derives from the fact that sech $x$ has simple-pole singularities in the complex plane (unlike the double poles of $\operatorname{sech}^{2} x$ ).

Using (2.18), the straightforward expansion (2.8) becomes

$$
\hat{u} \sim \frac{1}{2}\left(1+\mu^{2} k^{2}+\cdots\right) \operatorname{sech} \frac{\pi k}{2}+\frac{1}{2}(\epsilon k+\cdots) \operatorname{csch} \frac{\pi k}{2} .
$$

In this case, the balance $\epsilon=\mu$ brings nonlinear and dispersive effects at the same level, and the corresponding uniformly valid expansion takes the form

$$
\begin{equation*}
\hat{u} \sim \frac{1}{2} U(\kappa) \text { sech } \frac{\pi k}{2}+\cdots \tag{2.19}
\end{equation*}
$$

where $\kappa=\mu k$ and

$$
\begin{equation*}
U(\kappa) \sim 1+|\kappa|+\cdots \quad(\kappa \rightarrow 0) \tag{2.20}
\end{equation*}
$$

Inserting (2.19) into (2.7) then yields

$$
\begin{equation*}
\left(1-\kappa^{2}\right) U(\kappa)-\frac{1}{2} \cosh \frac{\pi \kappa}{2 \mu} \int_{-\infty}^{\infty} \mathrm{d} \lambda \frac{U(\lambda) U(\kappa-\lambda)}{\cosh \frac{\pi \lambda}{2 \mu} \cosh \frac{\pi(\kappa-\lambda)}{2 \mu}}=1 \tag{2.21}
\end{equation*}
$$

It is important to note that, unlike the example in $\S 2.2$, here, in view of $(2.20), U(\kappa) \sim 1$ as $\kappa \rightarrow 0$, a consequence of the simple-pole singularities of the sech $x$-forcing. As a result, it is permissible to approximate (2.21) by a simplified integral equation for $U(\kappa)$, analogous to (2.14),

$$
\begin{equation*}
\left(1-\kappa^{2}\right) U(\kappa)-\operatorname{sgn} \kappa \int_{0}^{\kappa} U(\lambda) U(\kappa-\lambda) \mathrm{d} \lambda=1 \tag{2.22}
\end{equation*}
$$

only if $\kappa$ is not too close to $\pm 1(1-|\kappa| \gg O(\mu))$, where $U(\kappa)$ is expected to be singular. Near these singularities, (2.22) breaks down and, later in this section, we shall present a separate local analysis to obtain the behavior of $U(i)$ as $\kappa \rightarrow \pm 1$ and the corresponding residues of $\hat{u}(k)$ at $k= \pm 1 / \mu$.

Attention is now focused on (2.22). By dominant balance, it can be readily shown that, on the basis of (2.22), $U(\kappa)$ has double-pole singularities at $\kappa= \pm 1$ :

$$
\begin{equation*}
U(\kappa) \sim \frac{C}{(\kappa \mp 1)^{2}} \quad(\kappa \rightarrow \pm 1) . \tag{2.23}
\end{equation*}
$$

The constant $C$ can be determined from a power-series solution of (2.22),

$$
U(\kappa)=\sum_{m=1}^{\infty} b_{m}|\kappa|^{m-1}
$$

the coefficients $b_{m}$ satisfy the recurrence relation

$$
b_{m}-b_{m-2}-\sum_{r=1}^{m-1} \frac{(r-1)!(m-r-1)!}{(m-1)!} b_{r} b_{m-r}=0 \quad(m \geq 3)
$$

with $b_{1}=b_{2}=1$, and it is easy to verify numerically, consistent with (2.23), that

$$
b_{m} \sim C m \quad(m \rightarrow \infty)
$$

with $C=0.94$.
However, as noted above, the simplified integral equation (2.22) is not a valid approximation to (2.21) close to the singularities of $U(\kappa)$ at $\kappa= \pm 1$. Accordingly, to obtain the structure of $\hat{u}(k)$ in the neighborhood of $k= \pm 1 / \mu$, we return to (2.21) and use a more accurate approximation to the integral term. Specifically, near $k=1 / \mu(\kappa=1)$, the asymptotic behavior (2.23) suggests the following rescaling of $U(\kappa)$

$$
\begin{equation*}
U(\kappa)=\frac{\Psi(\rho)}{\mu^{2}} \tag{2.24}
\end{equation*}
$$

in terms of the 'inner' wavenumber variable $\rho=(1-\kappa) / \mu$. Substituting then (2.24) into (2.21), it is found that, to leading order, $\Psi(\rho)$ satisfies the linear integral equation

$$
\begin{equation*}
\rho \Psi(\rho)-\frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d} \sigma e^{-\pi \sigma / 2} \operatorname{sech} \frac{\pi \sigma}{2} \Psi(\rho-\sigma)=0 . \tag{2.25}
\end{equation*}
$$

Furthermore, to be consistent with (2.23), the matching condition

$$
\begin{equation*}
\Psi(\rho) \sim \frac{C}{\rho^{2}} \quad(\rho \rightarrow \infty) \tag{2.26}
\end{equation*}
$$

is imposed.
The integral in equation (2.25) is of the convolution type, and this suggests using an integral-transform approach to solve for $\Psi(\rho)$ :

$$
\begin{equation*}
\Psi(\rho)=\int_{\mathcal{L}} e^{-\rho s} \Phi(s) \mathrm{d} s \tag{2.27}
\end{equation*}
$$

where the cuatour $\mathcal{L}$ extends from $s=0$ to $\infty$ with $\operatorname{Re} \rho s>0$ and $\operatorname{Im} s>0$. This choice of $\mathcal{L}$ implies that $\Psi(\rho)$ is analytic in $\operatorname{Im} \rho<0$ and, hence, does not contribute to the singularities of $\hat{u}(k)$ in $\operatorname{Im} k>0$, consistent with the radiation condition (2.2) that the induced wave tail vanishes in $x>0$. Proceeding then formally, (2.25) is transformed into a differential equation for $\Phi(s)$ :

$$
\frac{\mathrm{d} \Phi}{\mathrm{~d} s}-\frac{\Phi}{\sin s}=0
$$

where, in view of the matching condition (2.26), $\Phi(s) \sim C s(s \rightarrow 0)$.
The solution of this initial-value problem is

$$
\Phi(s)=2 C \tan \frac{s}{2}
$$

and, returning to (2.27), it is concluded that $\Psi(\rho)$ has a simple-pole singularity at $\rho=0$ :

$$
\begin{equation*}
\Psi(\rho) \sim 2 \mathrm{i} \frac{C}{\rho} \quad(\rho \rightarrow 0) \tag{2.28}
\end{equation*}
$$

This brings up the question of interpreting the singular integral in (2.25); the integration path has to be deformed in the complex plane to avoid the singularity at $\sigma=\rho$, and the way this is done is related to the radiation condition (2.2). However, as noted above, $\Psi(\rho)$ in fact obeys this radiation condition owing to the choice of the integration contour $\mathcal{L}$ in (2.27).

Finally, combining (2.28) with (2.19) and (2.24), it is now clear that $\hat{u}(k)$ has a simplepole singularity at $k=1 / \mu$, and (by a very similar reasoning) the same is true at $k=-1 / \mu$ :

$$
\hat{u} \sim-2 \mathrm{i} \frac{C}{\mu^{2}} \frac{e^{-\pi / 2 \mu}}{k \mp 1 / \mu} \quad(k \rightarrow \pm 1 / \mu)
$$

Therefore, returning to (2.5), it is seen that these poles contribute an exponentially small oscillatory tail in $x<0$ :

$$
\begin{equation*}
u \sim-8 \pi C \frac{e^{-\pi / 2 \mu}}{\mu^{2}} \cos \frac{x}{\mu} \quad(x \rightarrow-\infty) \tag{2.29}
\end{equation*}
$$

The above asymptotic result is in good agreement with estimates of the induced tail obtained by solving the problem (2.1),(2.2) numerically for various values of $\epsilon=\mu$ (see Figure 2.2).

It is interesting to compare expression (2.29) with the linear estimate (2.6) [taking $\hat{f}(1 / \mu)=\exp (-\pi / 2 \mu)$ according to (2.18)]:

$$
\begin{equation*}
u \sim-2 \pi \frac{e^{-\pi / 2 \mu}}{\mu} \sin \frac{x}{\mu} \tag{2.30}
\end{equation*}
$$

Note that the nonlinear term in (2.1) affects the order of magnitude of the amplitude and the phase of the tail. Both these effects arise from the fact that the forcing $f(x)=\operatorname{sech} x$ has simple-pole singularities at $x= \pm \mathrm{i} \pi / 2$ so $U(\kappa) \sim 1$ as $\kappa \rightarrow 0$; this, in turn, makes necessary the rescaling (2.24) near the singularities of $U(\kappa)$ at $\kappa=1^{\circ} \therefore$ ass the corresponding residues to become imaginary (see (2.28)). Similar features are expected to occur when short-wave tails are induced by long-wave profiles with simple-pole singularities.

The reason for the phase shift in the nonlinear response (2.29) can be further clarified by constructing a WKB solution of (2.1) representing a short-scale oscillatory disturbance superposed on the leading-order long wave $f(x)$ :

$$
\begin{equation*}
u=f(x)+\left\{v(x) e^{i x / \mu}+\text { c.c. }\right\} . \tag{2.31}
\end{equation*}
$$

Linearizing then (2.1) (with $\epsilon=\mu$ ) about $f(x)=\operatorname{sech} x$, one obtains to leading order

$$
v_{x}+\mathrm{i} f v=0
$$

and hence

$$
\begin{equation*}
v=A e^{-\mathrm{i} \phi(x)} \tag{2.32}
\end{equation*}
$$

where $\phi(x)=\tan ^{-1}(\sinh x)$ and the constant $A$ is related to the amplitude of the induced tail. Owing to the presence of the long wave, the short wave experiences a phase modulation which, according to (2.29), results to a $\pi / 2$ phase shift as $x \rightarrow-\infty$ in the nonlinear response relative to the linear result (2.30). On the other hand, it is worth noting that in the example
of $\S 2.2$, where $f(x)=\operatorname{sech}^{2} x$ and $\epsilon=\mu^{2}$, the analogous WKB solution of (2.1) in the form (2.31), yields (2.32) with $\phi(x)=\mu \tanh x$, and the corresponding phase modulation can be neglected to leading order, consistent with (2.17).

### 2.4 Gaussian forcing

Consider finally the Gaussian forcing, $f(x)=e^{-x^{2}}$, for which

$$
\begin{equation*}
\hat{f}(k)=\frac{1}{2 \sqrt{\pi}} e^{-k^{2} / 4} \tag{2.33}
\end{equation*}
$$

Note that, unlike the previous examples, this forcing has no singularities in the finite complex plane; it turns out that this difference has an important bearing on the analysis of the induced wave tail.

Again, it is helpful to begin by expanding $\hat{u}(k)$ in powers of $\epsilon, \mu^{2}$; inserting (2.33) into (2.8) with $\epsilon=\mu^{2}$, one obtains correct to $O\left(\mu^{4}\right)$ :

$$
\begin{align*}
\hat{u} \sim & \frac{1}{2 \sqrt{\pi}}\left(1+\mu^{2} k^{2}+\mu^{4} k^{4}+\cdots\right) e^{-k^{2} / 4}+\frac{\mu^{2}}{2 \sqrt{2 \pi}}\left(1+\frac{3}{2} \mu^{2} k^{2}+\cdots\right) e^{-k^{2} / 8} \\
& +\frac{\mu^{4}}{\sqrt{3 \pi}}(1+\cdots) e^{-k^{2} / 12}+\cdots . \tag{2.34}
\end{align*}
$$

From the way the various terms have been grouped, it is clear that there are two sources of non-uniformities (as $k \rightarrow \infty$ ) in this expansion: first, there are secular terms, similar to those in (2.10), multiplying each of the exponentials. There is a second, more severe, non-uniformity in (2.34), however, owing to the exponentials themselves, each approaching zero less rapidly than the previous one.

The first non-uniformity in (2.34) can be handled in the same way as before, by introducing the two-scale expansion

$$
\begin{equation*}
\hat{u} \sim \sum_{m=0}^{\infty} \mu^{2 m} U_{m}(\kappa) \exp \left[-k^{2} / 4(m+1)\right]+\cdots, \tag{2.35}
\end{equation*}
$$

in terms of the scaled wavenumber $\kappa=\mu k$. Comparing (2.35) with (2.11), (2.19), note that here we have to deal with an infinite sequence of unknowns, $U_{m}(\kappa)(m=0,1,2, \cdots)$, rather than one; this complication $E$, related to the second non-uniformity noted above.

Upon substituting (2.35) into (2.7) and collecting terms with equal powers of $\mu$, one has

$$
\begin{aligned}
& \left(1-\kappa^{2}\right) U_{m}(\kappa)-\frac{1}{\mu} \sum_{p=0}^{m-1} \int_{-\infty}^{\infty} \mathrm{d} \lambda U_{p}(\lambda) U_{m-p-1}(\kappa-\lambda) \\
& \quad \times \exp \left[-\frac{1}{4 \mu^{2}} \frac{m+1}{(m-p)(p+1)}\left(\lambda-\frac{p+1}{m+1} \kappa\right)^{2}\right]=0 \quad(m \geq 1),
\end{aligned}
$$

where, in view of (2.34),

$$
\begin{equation*}
U_{0}(\kappa)=\frac{1}{2 \sqrt{\pi}\left(1-\kappa^{2}\right)} \tag{2.36}
\end{equation*}
$$

In the limit $\mu \rightarrow 0$, the above sequence of integral equations can be simplified considerably, however, by evaluating the integrals asymptotically using Laplace's method:

$$
\begin{align*}
& \left(1-\kappa^{2}\right) U_{m}(\kappa)-\frac{2 \sqrt{\pi}}{\sqrt{m+1}} \sum_{p=0}^{m-1} \sqrt{(p+1)(m-p)}  \tag{2.37}\\
& \quad \times U_{p}\left(\frac{p+1}{m+1} \kappa\right) U_{m-p-1}\left(\frac{m-p}{m+1} \kappa\right)=0 \quad(m \geq 1)
\end{align*}
$$

This is now a recurrence relation that allows one to determine $U_{m}(\kappa)(m \geq 1)$ explicitly in terms of $U_{0}(\kappa)$ in (2.36). Specifically, for $m=1,2$,

$$
U_{1}(\kappa)=2 \sqrt{2} \pi U_{0}(\kappa)\left[U_{0}(\kappa / 2)\right]^{2}, U_{2}(\kappa)=\frac{32 \pi^{2}}{\sqrt{3}} U_{0}(\kappa) U_{0}(2 \kappa / 3)\left[U_{0}(\kappa / 3)\right]^{3}
$$

and it is easy to verify that these expressions are consistent with expansion (2.34) in the limit $\kappa \rightarrow 0$. Thus, the first non-uniformity appearing in the straightforward expansion (2.34) has been removed, by effectively summing the corresponding series of secular terms.

It is clear from (2.36), (2.37) that $U_{m}(\kappa)(m=0,1,2, \ldots)$ have simple-pole singularities on the real $\kappa$-axis (for $|\kappa| \geq 1$ ) which, according to (2.35), translate into singularities of $\hat{u}(k)$ on the real $k$-axis. In particular, the poles at $\kappa= \pm 1$ are common to all $U_{m}(\kappa)$ and, as it turns out, make the dominant contribution to the tail amplitude. For this reason, attention is focused on these singularities, and we write

$$
\begin{equation*}
U_{m}(\kappa)=\frac{Q_{m}(\kappa)}{\sqrt{\prime}^{-}\left(1-\kappa^{2}\right)\left[1-\kappa^{2} /(m+1)^{2}\right]^{m+1}} \quad(m \geq 2) \tag{2.38}
\end{equation*}
$$

where, from (2.37), $Q_{m}(\kappa)$ satisfy the recurrence relation

$$
\begin{align*}
& Q_{m}(\kappa)=\frac{2 \sqrt{m}}{\sqrt{m+1}} \frac{Q_{m-1}(m \kappa /(m+1))}{\left[1-m^{2} \kappa^{2} /(m+1)^{2}\right]} \\
& \quad+\frac{2}{\sqrt{m+1}} \sum_{p=1}^{m-2} \sqrt{(p+1)(m-p)}  \tag{2.39}\\
& \quad \times \frac{Q_{p}\left(\frac{p+1}{m+1} \kappa\right) Q_{m-p-1}\left(\frac{m-p}{m+1} \kappa\right)}{\left[1-\left(\frac{r+1}{m+1} \kappa\right)^{2}\right]\left[1-\left(\frac{m-p}{m+1} \kappa\right)^{2}\right]} \quad(m \geq 2)
\end{align*}
$$

with

$$
Q_{1}=1 / 2 \sqrt{2}
$$

Returning now to the integral representation (2.5) for $u(x)$, taking into account (2.35). (2.36) and (2.38), it is seen that the contribution of the poles of $\hat{u}(k)$ at $k= \pm 1 / \mu$ results to the appearance of an oscillatory tail for $x<0$ :

$$
u \sim A \sin \frac{x}{\mu} \quad(x \rightarrow-\infty)
$$

with amplitude

$$
\begin{equation*}
A=-\frac{\sqrt{\pi}}{\mu} \exp \left(-1 / 4 \mu^{2}\right)-\frac{2 \sqrt{\pi}}{\mu^{3}} \sum_{m=2}^{\infty} \frac{Q_{m-1}(1)}{\left(1-1 / m^{2}\right)^{m}} \mu^{2 m} \exp \left(-1 / 4 m \mu^{2}\right) \tag{2.40}
\end{equation*}
$$

The first term in the above expression is the linear result (2.6). It is worth noting, however, that, unlike the examples discussed earlier, here linear theory does not provide even the correct exponential order of magnitude of the tail amplitude; to obtain the correct magnitude of $A$, it is necessary to sum the series in (2.40).

The infinite sum in (2.40) can be evaluated asymptotically for $\mu \ll 1$ using a technique analogous to Laplace's method for integrals (Bender \& Orszag 1978): as $\mu \rightarrow 0$, the terms of the sum peak sharply around $m=M$, where $M$ is the integer closest to

$$
\frac{1}{2 \mu \sqrt{-2 \ln \mu}}
$$

so the main contribution to the sum comes from this neighborhood. Accordingly, near $m=M$,

$$
\mu^{2 m} \exp \left(-1 / 4 m \mu^{2}\right) \approx \exp \left(-1 / 2 \mu^{2} M\right) \exp \left(-q^{2} / 4 \mu^{2} M^{3}\right) \quad(|q| \ll M)
$$

and $A$ may be approximated as

$$
\begin{equation*}
A \sim-\frac{2 \sqrt{\pi}}{\mu^{3}} Q_{M-1}(1) \exp (-\sqrt{-2 \ln \mu} / \mu) \sum_{q=-\infty}^{\infty} \exp \left(-q^{2} / 4 \mu^{2} \cdot M^{3}\right) \tag{2.41}
\end{equation*}
$$

However.

$$
\sum_{q=-\infty}^{\infty} \exp \left(-q^{2} / 4 \mu^{2} M^{3}\right) \sim 2 \sqrt{\pi} \mu M^{3 / 2} \quad(\mu \rightarrow 0)
$$

and. finally, (2.41) yields

$$
\begin{equation*}
A \sim-\frac{\pi}{2^{1 / 4} \mu^{7 / 2}} \frac{Q_{M-1}(1)}{(-\ln \mu)^{3 / 4}} \exp (-\sqrt{-2 \ln \mu / \mu}) \tag{2.42}
\end{equation*}
$$

where $Q_{M-1}(1)$ has to be computed numerically as a function of $\mu$ from the recurrence relation (2.39).

Figure 2.3 shows a comparison, for various values of $\mu^{2}=\epsilon$, of the asymptotic expression (2.42) for the tail amplitude with results obtained by direct numerical integration of the boundary-value problem (2.1),(2.2); the agreement is quite reasonable. On the other hand. as already remarked, the linear estimate (2.6) [with $\hat{f}(1 / \mu)$ obtained from (2.33)] grossly underpredicts the amplitude of the tail.

### 2.5 Discussion

As illustrated by the examples above, the amplitude of the induced short-wave tail is quite sensitive to the details of the long-wave profile. Successive nonlinear and dispersive corrections to the long wave, obtained from the long-wave expansion, tend to be steeper in the physical domain, so they become important in the wavenumber domain-particularly in the large-wavenumber region of the spectrum. Hence, they affect the residues of the poles that occur on the real axis and in turn the amplitude of the oscillatory tail. Specifically, in the case of the $\operatorname{sech}^{2} x$-forcing, the contribution of the nonlinear corrections to the pole residues is as important as that of the leading-order (linear) response, so linear theory gives the right order of magnitude, but not the precise value, of the tail amplitude. On the other hand, in the case of the sech $x$-forcing, the nonlinear interaction of the short-wave tail with
the long wave alters the order of magnitude of the tail amplitude and causes a phase shift in the tail oscillations. Finally, in the case of the Gaussian forcing, the large-wavenumber region of the spectrum is dominated by the nonlinear terms. and the prediction of linear theory for the tail amplitude fails in an even more substantial way.

The wavenumber-domain approach is also suitable for determining the amplitude of the tails of weakly non-local solitary waves. In particular, the analysis for the steady fifth-order KdV equation,

$$
\begin{equation*}
-c u+3 u^{2}+u_{x x}+\mu^{2} u_{x x x x}=0, \tag{2.43}
\end{equation*}
$$

follows along the lines of the example discussed in $\S 2.2$.
Briefly, the straightforward expansion in the physical domain, analogous to (2.3), takes the form

$$
u \sim u_{0}+\mu^{2}\left(\frac{15}{2} u_{0}^{2}-10 \gamma^{2} u_{0}\right)+\cdots
$$

where $u_{0}=2 \gamma^{2} \operatorname{sech}^{2} \gamma x$ is the KdV solitary wave with speed $c=4 \gamma^{2}+O\left(\mu^{2}\right), \gamma>0$. In the wavenumber domain one then has

$$
\hat{u} \sim\left(k+\frac{5}{2} \mu^{2} k^{3}+\cdots\right) \operatorname{csch} \frac{\pi k}{2 \gamma},
$$

which suggests the uniformly valid two-scale expression

$$
\begin{equation*}
\hat{u} \sim \frac{1}{\mu} U(\kappa) \operatorname{csch} \frac{\pi k}{2 \gamma}, \tag{2.44}
\end{equation*}
$$

in terms of $\kappa=\mu k$.
Substituting (2.44) into (2.43) after taking Fourier transform, it is found that, to leading order, $U(\kappa)$ satisfies an integral equation similar to (2.14):

$$
\kappa^{2}\left(\kappa^{2}-1\right) U(\kappa)+6 \int_{0}^{\kappa} U(\lambda) U(\kappa-\lambda) \mathrm{d} \lambda=0
$$

Again, the solution can be expressed as a power series

$$
U(\kappa)=\sum_{m=0}^{\infty} b_{m} \kappa^{2 m+1}
$$

where

$$
b_{m-2}-b_{m-1}+6 \sum_{r=0}^{m-1} \frac{(2 m-2 r-1)!(2 r+1)!}{(2 m+1)!} b_{r} b_{m-r-1}=0 \quad(m \geq 2)
$$

with $b_{0}=1, b_{1}=\frac{5}{2}$. In fact. this recurrence relation is identical [if $b_{m}$ is multiplied with $\left.2(-1)^{m+1}\right]$ to the one encountered previously by Pomeau et al. (1988) and Grimshaw \& Joshi (1995) in analyzing the inner solution near the singularity of $u_{0}(x)$ at $x=\mathrm{i} \pi / 2 \gamma$. Accordingly,

$$
b_{m} \sim C \quad(m \rightarrow \infty),
$$

where $C=-\frac{1}{2} K(K=-19.97)$ in their notation, and $U(\kappa)$ has pole singularities at $\kappa= \pm 1$ :

$$
U(\kappa) \sim \frac{C \kappa}{1-\kappa^{2}} \quad(\kappa \rightarrow \pm 1) .
$$

Hence, in view of (2.44), $\hat{u}$ has poles at $k= \pm 1 / \mu$,

$$
\hat{u}(k) \sim \mp \frac{C}{\mu^{2}} \frac{e^{-\pi / 2 \gamma \mu}}{k \mp 1 / \mu} \quad(k \rightarrow \pm 1 / \mu),
$$

and, using (2.5), the induced tail in $x<0$ reads

$$
\begin{equation*}
u \sim 2 \pi K \frac{e^{-\pi / 2 \gamma \mu}}{\mu^{2}} \sin \frac{x}{\mu} \quad(x \rightarrow-\infty) . \tag{2.45}
\end{equation*}
$$

This result is in agreement with the previous work (Grimshaw \& Joshi 1995; Pomeau et al. 1988) under the assumption that there is no tail on the right-hand side of the wave ( $x>0$ ). as implied by the choice of the contour $\mathcal{C}$ in (2.5). In case the wave disturbance is taken to be symmetric about $x=0$, a one-parameter family of solutions obtains (Grimshaw \& Joshi 1995; see also Boyd 1991) and the (minimum) tail amplitude is half of that given in (2.45), but the details will not be pursued here. [Unlike the forced wave response discussed in the previous sections, however, here only the symmetric disturbances correspond to perfectly steady solutions of the fifth-order KdV equation (Akylas \& Grimshaw 1992; Grimshaw \& Joshi 1995).]

In recent work, Karpman (1993) and Milewski (1993) suggested that the tails of weakly non-local solitary waves of the fifth-order KdV equation could be determined by linearizing about the (locally confined) KdV solitary-wave solution. In view of the remarks made earlier in this section, it is now clear that this approach is not uniformly valid near the solitary-wave tails.

Finally; it would be interesting to compute the tails of solitary internal waves in fluids of intermediate and large depth (see Chapter 3). As these solitary waves have simplepole singularities (unlike the double poles of KdV-like solitary waves in shallow fluids), the asymptotic analysis of the tails is expected to be similar to that of the sech $x$-forcing discussed in §2.3.


Figlre 2.1 Comparison, for various values of $\mu^{2}=\epsilon$, of asymptotic expression (2.17) (——) for the amplitude of the tail induced by $\operatorname{sech}^{2} x$-forcing against numerical results ( 0 ) and the linear estimate (2.6) (- - - -). All results have been normalized with $-4 \pi C e^{-\pi / 2 \mu} / \mu^{2}(C=0.77)$.
(a)


Figure $2.2 a$ (for caption, see following page)
(b)


Figlre 2.2 Comparison, for various values of $\mu=\epsilon$, of asymptotic expression (2.29) (-) for the tail induced by sech $x$-forcing against numerical results ( 0 ) and the linear estimate (2.30) (-- - -). (a) Tail amplitude: (b) Phase shift in the nonlinear response (2.29) relative to the linear result (2.30).


FIGLRE 2.3 Comparison, for various values of $\mu^{2}=\epsilon$. of asymptotic expression (2.42) (-) for the amplitude of the tail induced by Gaussian forcing against numerical results $(0)$ and the linear estimate (2.6) ( $-\cdots-$ ). The absolute value of the amplitude is plotted.

## CHAPTER 3

## WEAKLY' NON-LOCAL SOLITARY WAVES of a PERTURBED ILW EQCATION

Kubota. Ko \& Dobbs (1978) and Joseph (1977) proposed a nonlinear integraldifferential evolution equation for weakly nonlinear, long internal waves in stratified fluids of finite depth. This model equation accounts for the leading-order nonlinear and dispersive effects on the assumption that the disturbance horizontal lengthscale is much longer than the characteristic lengthscale of the background stratification in the vertical direction. without placing any restriction on the total fluid depth. Accordingly, this equation sometimes is referred to as the intermediate long-wave (ILW) equation: it bridges the gap between the Korteweg-de Viries (KdV) equation (in the shallow-depth limit) and the Benjamin-Davis-Ono (BDO) equation (in the infinite-depth limit).

On the basis of the ILW equation. Joseph (1977) constructed an analytic expression for the shape of solitary waves in fluids of finite depth. This expression covers the entire range from the ${ }^{\text {sech }}{ }^{2}$ ' profile of the KdV soliton in shallow fluids to the algebraic soliton profile of the BDO equation in deep fluids.

In analogy with the previous work on the fifth-order KdV equation in shallow fluids. here we discuss weakly non-local solitary-wave solutions of the ILW equation in the presence of a small third-order-derivative dispersive perturbation. The perturbed ILW equation may be viewed as a model problem for non-local solitary internal waves in fluids of finite depth, and the analysis presented below bears on the numerical work of Vanden-Broeck \& Turner (1992) and the experimental observations of Davis \& Acrivos (1967) in the large-depth limit.

The dispersive term of the ILW equation involves a convolution integral and, in analyzing the solitary-wave tails induced by the higher-order dispersive perturbation, it is more convenient to follow the wavenumber-domain approach presented in Chapter 2, rather than asymptotic matching in the complex plane (see §1.2). Compared with the corresponding model problem in the shallow-depth limit (§2.5), the analysis turns out to be
more involved here because, unlike the KdV soliton, finite-depth solitons have simple-pole singularities in the complex plane. This difference causes a phase modulation of the tails and, as a result, the tail oscillations with the minimum amplitude have a finite phase shift.

### 3.1 Perturbed ILW equation

Before focusing on solitary-wave solutions, we shall briefly comment on the physical significance of the perturbed ILW equation in the context of two-layer flow, following the heuristic approach of Joseph (1977).

The phase speed $c$ of linear sinusoidal waves of wavenumber $k$ at the interface of a two-layer fluid system bounded by rigid walls obeys the dispersion relation
where $\quad d_{1}=\operatorname{coth} k h_{1}-\frac{1}{k h_{1}}, d_{2}=\operatorname{coth} k h_{2}-\frac{1}{k h_{2}}$.
Here $h_{2}$ is the depth of the lower fluid with density $\rho, h_{1}$ is the depth of the upper fluid with density $\rho-\delta \rho(\delta \rho \ll \rho)$, and $h$ denotes the total fluid depth $h_{1}+h_{2}$. The linear-long-wave speed $c_{0}$ is given by

$$
c_{0}^{2}=g \frac{\delta \rho}{\rho} \frac{h_{1} h_{2}}{h}
$$

where $g$ is the gravitational acceleration, and the parameter

$$
\gamma=\frac{T}{g \delta \rho h_{2}^{2}}
$$

measures the relative significance of surface-tension effects, $T$ being the coefficient of surface tension.

In this physical setting, the finite-depth long-wave theory obtains when the disturbance horizontal lengthscale, $l$, is much longer than the depth of one of the two fluid layers, the lower one say ( $h_{2} \ll l$ ); no restriction is placed on the total fluid depth, however, so that $l$ can be comparable to (or even short compared with) the upper-fluid depth $h_{1}$. Under these
conditions. the dispersion relation (3.1) can be approximated by expanding in powers of $k h_{2}$ :

$$
\begin{equation*}
\frac{c}{c_{0}}=1-\frac{1}{2} k h_{2} d_{1}+\frac{1}{2}\left(k h_{2}\right)^{2}\left\{\frac{3}{4} d_{1}^{2}+\gamma-\frac{1}{3}+\frac{d_{1}}{k h_{1}}\right\}+\cdots \tag{3.2}
\end{equation*}
$$

The ILIV equation combines the leading-order dispersive effects-the $O\left(k h_{2}\right)$ term in the above expansion-with the leading-order nonlinear effects, and admits locally confined solitary-wave solutions (Joseph 1977).

The objective here is to examine how the solitary waves of the ILW equation are modified by higher-order dispersive effects. To this end, note that including the $O\left(k h_{2}\right)^{2}$ term in expansion (3.2) allows small-amplitude short waves [of wavelength comparable to $h_{2}$. $\left.k h_{2} \approx 12 /(5+12 \gamma)\right]$ with phase speed equal to the long-wave speed $c_{0}$. Intuitively, since $c_{0}$ is close to the solitary-wave speed, it is expected that these waves will be resonantly excited by a solitary wave. forming oscillatory tails.

In line with this reasoning, for the purpose of constructing a simple model equation to describe weakiy non-local solitary waves in fluids of finite depth, we shall use, rather than (3.1), the simplified dispersion relation (in dimensionless form)
where

$$
\begin{equation*}
c=1+\epsilon G(k)+\frac{1}{2} \epsilon^{2} k^{2} \tag{3.3}
\end{equation*}
$$

Here $\epsilon=h_{2} / l \ll 1$ is the long-wave parameter and $\beta=h_{1} / l$ measures the relative depth of the upper fluid layer. In addition to the dominant dispersive effects accounted for in the ILW equation, the model dispersion relation (3.3) includes an $O\left(\epsilon^{2}\right)$ dispersive term, analogous to the $O\left(k h_{2}\right)^{2}$ term in (3.2), that controls the radiation of short-scale solitary-wave tails with resonant wavenumber $k \sim 1 / \epsilon$.

Combining, then, (3.3) with an $O(\epsilon)$ quadratic (KdV-like) nonlinearity, we arrive at the following integral-differential evolution equation

$$
\begin{gather*}
u_{t}+u_{x}+\epsilon u u_{x}+\epsilon \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \mathrm{d} \xi u(\xi, t) K(x-\xi)-\frac{1}{2} \epsilon^{2} u_{x x x}=0  \tag{3.4}\\
K(x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \mathrm{d} k e^{\mathrm{i} k x} G(k)
\end{gather*}
$$

where

Leaving out the $O\left(\epsilon^{2}\right)$ dispersive perturbation, this is the ILW equation; it reduces to the BDO equation in the deep-fluid limit $(\beta \gg 1)$.

The perturbed ILW equation (3.4) may be viewed as the finite-depth counterpart of the fifth-order KdV equation. Both (3.4) and the fifth-order KdV equation are model equations. however, and it should be noted that their weakly non-local solitary-wave solutions will agree only qualitatively with those of the actual hydrodynamic equations; as the tail amplitude lies beyond all orders of the long-wave expansion, it is necessary to retain all nonlinear and dispersive effects in order to describe the solitary-wave tails accurately. [This is also indicated by the fact that neither (3.2) nor (3.3) are valid approximations to the exact dispersion relation (3.1) for wavelengths comparable to that of the tail oscillations, $\left.k h_{2}=O(1).\right]$ On the other hand, the asymptotic approach presented here in the context of the perturbed ILW equation may also prove useful in dealing with weakly non-local solitary waves in fluids of finite depth.

### 3.2 Wavenumber-domain formulation

We now seek solitary-wave solutions to the perturbed ILW equation (3.4). In a reference frame moving with the wave speed, $X=x-c t$ with $c=1+\epsilon \lambda, \lambda=O(1)$, so that $u(x, t)=u(X)$, after one integration with respect to $X$, (3.4) becomes

$$
\begin{equation*}
-\lambda u+\frac{1}{2} u^{2}+\int_{-\infty}^{\infty} \mathrm{d} \xi u(\xi) K(X-\xi)-\frac{1}{2} \epsilon u_{X X}=0 \tag{3.5}
\end{equation*}
$$

As already remarked, it is anticipated that, in general, this equation admits no locally confined solutions for $\epsilon \neq 0$. Accordingly, we shall assume that

$$
\begin{equation*}
u \rightarrow 0 \quad(X \rightarrow-\infty) \tag{3.6}
\end{equation*}
$$

allowing for a possible wave tail to appear in $X>0$; symmetric solitary-wave solutions will be discussed later (see §3.5).

As remarked earlier, it proves more convenient to formulate the boundary-value problem
(3.5),(3.6) in the wavenumber domain by making use of the Fourier transform.

$$
\widehat{u}(k)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \mathrm{d} X e^{-\mathrm{i} k X} u(X)
$$

so that formally (3.5) is converted to an integral equation for $\widehat{u}(k)$ :

$$
\begin{equation*}
\left[-\lambda+G(k)+\frac{1}{2} \epsilon k^{2}\right] \widehat{u}(k)+\frac{1}{2} \int_{-\infty}^{\infty} \widehat{u}(l) \widehat{u}(k-l) \mathrm{d} l=0 . \tag{3.7}
\end{equation*}
$$

It is important to note that the coefficient of $\hat{u}$ vanishes and hence the above equation is singular at $k= \pm k_{0} \sim \pm\left(\epsilon^{-1}+2 \lambda-\beta^{-1}\right)$, the resonant wavenumber of the tail oscillations according to (3.3). It will be shown that $\hat{u}(k)$ has simple-pole singularities there which cause a tail to appear in $X>0$. Specifically, in inverting the Fourier transform,

$$
\begin{equation*}
u(X)=\int_{\mathcal{C}^{-}} \mathrm{d} k e^{i k X} \widehat{u}(k) \tag{3.8}
\end{equation*}
$$

the contour $\mathcal{C}^{-}$must be indented to pass below all singularities of $\widehat{u}$ on the real $k$-axis so that no tail appears in $X<0$, according to (3.6). On the other hand, owing to the contribution of the poles at $k= \pm k_{0}$, an oscillatory tail of constant amplitude and wavenumber equal to $k_{0}$ appears in $X>0$, and is completely determined once the pole residues are known.

In the following sections, perturbation techniques are used to examine the asymptotic behavior of $\widehat{u}(k)$ near $k= \pm k_{0}$ and thereby compute the solitary-wave tail in $X>0$.

### 3.3 Perturbation analysis

Attention is now focused on (3.7) which forms the basis for the ensuing analysis. We first attempt to expand $\widehat{u}$ and $\lambda$ in powers of $\epsilon$ :

$$
\begin{equation*}
\widehat{u}=\widehat{u}_{0}+\epsilon \widehat{u}_{1}+\cdots, \quad \lambda=\lambda_{0}+\epsilon \lambda_{1}+\cdots ; \tag{3.9a,b}
\end{equation*}
$$

the leading-order solutions,

$$
\begin{equation*}
\widehat{u}_{0}=\frac{\sinh \beta k}{\sinh \alpha k}, \quad \lambda_{0}=\frac{1}{2 \beta}(1-\delta \cot \delta) \tag{3.10a,b}
\end{equation*}
$$

where $\alpha=\pi \beta / \delta(0<\delta<\pi)$, follow directly from the known solitary-wave solution of the ILW equation (Joseph 1977),

$$
\begin{equation*}
u_{0}=\frac{\delta \sin \delta}{\beta}\left(\cosh \frac{\delta X}{\beta}+\cos \delta\right)^{-1} \tag{3.11}
\end{equation*}
$$

In particular, in the deep-fluid limit ( $\beta \gg 1$ ), one has
where

$$
\begin{gather*}
\delta \sim \pi-\frac{2 \pi}{a \beta}+\cdots,  \tag{3.12}\\
a=\frac{\delta \sin \delta}{\beta(1+\cos \delta)}
\end{gather*}
$$

is the solitary-wave peak amplitude, and (3.10a,b) yield

$$
\widehat{u}_{0} \sim \exp \left(-\frac{2}{a}|k|\right), \quad \lambda_{0} \sim \frac{1}{4} a
$$

consistent with the algebraic-soliton solution of the BDO equation,

$$
u_{0} \sim \frac{a}{1+\frac{1}{4} a^{2} X^{2}}
$$

In principle, one may proceed to find higher-order terms in the straightforward expansions ( $3.9 a, b$ ). However, in view of the singularities of (3.7) at $k \sim \pm 1 / \epsilon$, it is expected that the expansion of $\hat{u}$ will become non-uniform when $\epsilon k=O(1)$. Accordingly, guided by previous experience in handling similar difficulties encountered in Chapter 2, we introduce the two-scale expansion

$$
\begin{equation*}
\widehat{u} \sim \frac{\sinh \beta k}{\sinh \alpha k} U(\kappa)+\cdots \tag{3.13}
\end{equation*}
$$

in terms of the scaled wavenumber variable $\kappa=\epsilon k$, with

$$
\begin{equation*}
U(\kappa) \sim 1+\cdots \quad(\kappa \rightarrow 0) \tag{3.14}
\end{equation*}
$$

Substituting (3.13) into (3.7), one then has to leading order

$$
\begin{equation*}
|\kappa|(1-|\kappa|) U(\kappa)-S(\kappa) \int_{-\infty}^{\infty} \mathrm{d} \sigma \frac{U(\sigma) U(\kappa-\sigma)}{S(\sigma) S(\kappa-\sigma)}=0 \tag{3.15}
\end{equation*}
$$

where

$$
S(\kappa)=\frac{\sinh \alpha \kappa / \epsilon}{\sinh \beta \kappa / \epsilon}
$$

Keeping in mind that $U(\kappa)$ is expected to be singular at $\kappa= \pm 1$, it is permissible to replace (3.15) by the simpler integral equation

$$
\begin{equation*}
|\kappa|(1-|\kappa|) U(\kappa)-\operatorname{sgn} \kappa \int_{0}^{\kappa} U(\sigma) U(\kappa-\sigma) \mathrm{d} \sigma=0 \tag{3.16}
\end{equation*}
$$

in the limit $\epsilon \rightarrow 0^{+}$, provided that $\kappa$ is not too close to $\pm 1,(1-|\kappa|) / \epsilon \gg 1$.
Assuming that this condition is met, the local behavior of $U(\kappa)$ in the vicinity of $|\kappa|=1$ can be readily found from (3.16) by dominant balance:

$$
\begin{equation*}
U(\kappa) \sim \frac{C}{(1-|\kappa|)^{3}} \quad(|\kappa| \rightarrow 1,1-|\kappa| \gg \epsilon) . \tag{3.17}
\end{equation*}
$$

To determine the constant $C$, we look for a series solution of (3.16) in the form

$$
\begin{equation*}
U(\kappa)=\sum_{m=0}^{\infty} b_{m}(\kappa)|\kappa|^{m} ; \tag{3.18}
\end{equation*}
$$

upon substitution into (3.16), the coefficients $b_{m}(\kappa)=\sum_{n=0}^{m} d_{n}^{(m)} \ln ^{n}|\kappa|$ satisfy

$$
b_{m}(\kappa)-b_{m-1}(\kappa)-\frac{\operatorname{sgn} \kappa}{|\kappa|^{m+1}} \sum_{p=0}^{m} \int_{0}^{\kappa}|\sigma|^{p}|\kappa-\sigma|^{m-p} b_{p}(\sigma) b_{m-p}(\kappa-\sigma) \mathrm{d} \sigma=0 \quad(m \geq 1)
$$

with $b_{0}=1$ according to (3.14). On the basis of this recurrence relation, the first few $b_{m}$ can be found analytically, viz., $b_{1}=2 \ln |\kappa|, b_{2}=2 \ln ^{2}|\kappa|+5-\pi^{2} / 3$, etc., while for large $m$ it can be verified numerically that

$$
b_{m}(1) \sim \frac{1}{2} C(m+1)(m+2) \quad(m \rightarrow \infty)
$$

with $C=0.034$, consistent with the asymptotic solution (3.17).
The presence of logarithmic terms in (3.18) suggests that the perturbation expansion (3.9a) needs to be revised to include terms of order $\epsilon \ln \epsilon$ and higher, in order for matching of (3.9a) with the two-scale expansion (3.13) to be possible in the region $1 \ll|k| \ll 1 / \epsilon$. However, these logarithmic corrections play no role, at least to leading order, in determining the solitary-wave tail.

It was noted earlier that the integral equation (3.16) is not a valid approximation to (3.15) and hence the asymptotic solution (3.17) fails when $|\kappa|$ is very close to 1 ,
$1-|\kappa|=O(\epsilon)$. As explained in $\S 2.3$, this complication stems from the fact that the solitarywave solutions (3.11) of the ILW equation, unlike those of the KdV equation, have simplepole singularities in the complex plane. In the following section, a local solution, valid when $1-|\kappa|=O(\epsilon)$, is discussed that allows one to obtain the residues of the poles of $U(\kappa)$ at $|\kappa|=1$ and thereby compute the tail oscillations.

### 3.4 Solution near $|\kappa|=1$

When $1-\kappa=O(\epsilon)$, the asymptotic solution (3.17) suggests the following rescaling

$$
\begin{equation*}
U(\kappa)=\frac{\psi(\eta)}{\epsilon^{3}} \tag{3.19}
\end{equation*}
$$

in terms of the 'inner' wavenumber variable $\eta=(1-\kappa) / \epsilon$. Upon substitution of (3.19) into (3.15), it is found that, to leading order, $\psi$ satisfies the linear integral equation

$$
\begin{equation*}
\eta \psi(\eta)-2 \int_{-\infty}^{\infty} \mathrm{d} \rho \exp (\beta-\alpha) \rho \frac{\sinh \beta \rho}{\sinh \alpha \rho} \psi(\eta-\rho)=0 \tag{3.20}
\end{equation*}
$$

subject to the condition

$$
\begin{equation*}
\psi(\eta) \sim \frac{C}{\eta^{3}} \quad(\eta \rightarrow \infty) \tag{3.21}
\end{equation*}
$$

that ensures matching with (3.17).
The integral equation (3.20) is of the convolution type, and the solution is posed in the form

$$
\begin{equation*}
\psi(\eta)=\int_{\Gamma^{-}} e^{-\eta s} F(s) \mathrm{d} s \tag{3.22}
\end{equation*}
$$

where the contour $\Gamma^{-}$extends from $s=0$ to $\infty$ such that $\operatorname{Re} \eta s>0$ with $\operatorname{Im} s<0$. For this choice of $\Gamma^{-}$, the above integral is convergent and, furthermore, $\psi(\eta)$ is analytic in $\operatorname{Im} \eta>0$; hence $\psi(\eta)$ does not contribute to the singularities of $\widehat{u}(k)$ in $\operatorname{Im} k<0$, in accordance with the radiation condition (3.6) that the solitary-wave tail vanishes in $X<0$.

Inserting (3.22) into (3.20), taking into account (3.21), $F(s)$ satisfies the differential equation

$$
\frac{\mathrm{d} F}{\mathrm{~d} s}-\frac{2 \delta \sin \delta}{\beta} \frac{F(s)}{\cos \delta+\cos [\delta(s-\alpha+\beta) / \beta]}=0,
$$

subject to the boundary condition $F(s) \sim \frac{1}{2} C s^{2}(s \rightarrow 0)$. The corresponding solution is

$$
F(s)=2 C \beta^{2} \frac{\sin ^{2} \delta}{\delta^{2}} \frac{\tan ^{2} \frac{\delta s}{2 \beta}}{\left(\cos \delta \tan \frac{\delta s}{2 \beta}+\sin \delta\right)^{2}}
$$

and, from (3.22), it follows that

$$
\begin{equation*}
\psi(\eta) \sim 2 C \beta^{2} \frac{\sin ^{2} \delta}{\delta^{2}} \frac{e^{-2 \mathrm{i} \delta}}{\eta} \quad(\eta \rightarrow 0) \tag{3.23}
\end{equation*}
$$

Hence, $\psi(\eta)$ has a simple-pole singularity at $\eta=0$ which makes the integral in equation (3.20) singular. The proper way to interpret this integral, by deforming the integration path to avoid the singularity at $\rho=\eta$, derives from the radiation condition (3.6). Recall, however, that the solution of $(3.20), \psi(\eta)$, is consistent with this radiation condition in view of the choice of the contour $\Gamma^{-}$in (3.22).

Combining (3.23) with (3.13) and (3.19), it is now clear the $\widehat{u}(k)$ has a simple-pole singularity at $k=k_{0} \sim 1 / \epsilon$ and, working in a similar way near $\kappa=-1$, it is easy to show that the same is true at $k=-k_{0} \sim-1 / \epsilon$ :

$$
\widehat{u} \sim \mp 2 C \beta^{2} \frac{\sin ^{2} \delta}{\delta^{2}} \frac{\exp (\beta-\alpha) / \epsilon}{\epsilon^{3}} \frac{e^{\mp 2 \mathrm{i} \delta}}{k \mp 1 / \epsilon} \quad(k \rightarrow \pm 1 / \epsilon) .
$$

Finally, returning to (3.8), it is seen that these poles contribute an exponentially small oscillatory tail for $X>0$ :

$$
\begin{equation*}
u \sim 8 \pi C \beta^{2} \frac{\sin ^{2} \delta}{\delta^{2}} \frac{\exp (\beta-\alpha) / \epsilon}{\epsilon^{3}} \sin \left(\frac{X}{\epsilon}-2 \delta\right) \quad(X \rightarrow \infty) . \tag{3.24}
\end{equation*}
$$

According to (3.24), the tail oscillations that accompany finite-depth solitons experience a phase shift $-2 \delta$ far from the main peak $(X \rightarrow \infty)$. This shift is related to a phase modulation of the short-wave tail that is caused by the underlying solitary wave, as discussed below.

### 3.5 Symmetric solitary waves

On physical grounds, permanent-wave disturbances that are locally confined on one side but feature an oscillatory tail on the other side cannot be exact steady-state solutions
of (3.4)-they would violate energy conservation as they would imply non-zero energy flux on the side where the oscillations are present and zero energy flux on the other side. Accordingly, the asymmetric solitary-wave disturbances discussed above should be interpreted as asymptotic approximations to slightly unsteady solutions of (3.4), with the main peak being slowly attenuated owing to radiation of small-amplitude short waves. These weakly radiating 'solitary-like' waves may arise from a localized initial disturbance, as has been demonstrated in the context of the fifth-order KdV equation (Benilov, Grimshaw \& Kuznetsova 1993).

On the other hand, non-local solitary waves that are symmetric about $X=0$. $u(X)=u(-X)$, with oscillatory tails on both sides, are qualified for steady-state solutions of (3.4), although they cannot be excited by a localized initial disturbance.

To discuss symmetric solitary waves, in place of (3.8), we propose the asymptotic solution

$$
\begin{equation*}
u(X) \sim \frac{1}{2} \int_{\mathcal{C}^{+}} \mathrm{d} k e^{\mathrm{i} k X_{\widehat{u}}(k)+\frac{1}{2} \int_{\mathcal{C}^{-}} \mathrm{d} k e^{\mathrm{i} k X_{\widehat{u}}(k)}, .,{ }^{2},} \tag{3.25}
\end{equation*}
$$

where the contour $\mathcal{C}^{+}$is deformed to pass above the poles of $\widehat{u}(k)$ at $k \sim \pm 1 / \epsilon$. Furthermore, as noted earlier, the choice of the contour $\Gamma^{ \pm}$in the integral representation (3.22) of the local solution near $\kappa=1$ has to be consistent with the radiation condition implied by the contour $\mathcal{C}^{\ddagger}$; hence $\Gamma^{+}$is defined such that $\operatorname{Im} s>0$ in addition to $\operatorname{Re} \eta s>0$.

Taking into account these modifications and following the same procedure as before, it is straightforward to show that the tail amplitude of symmetric solitary waves is half of the previous result (3.24):

$$
\begin{equation*}
u \sim 4 \pi C \beta^{2} \frac{\sin ^{2} \delta}{\delta^{2}} \frac{\exp (\beta-\alpha) / \epsilon}{\epsilon^{3}} \sin \left(\frac{|X|}{\epsilon}-2 \delta\right) \quad(|X| \rightarrow \infty) \tag{3.26}
\end{equation*}
$$

However, as is also the case for symmetzic solitary-wave solutions of the fifth-order KdV equation (Grimshaw \& Joshi 1995), there is a one-parameter family of symmetric solitary waves. Specifically, to demonstrate this point, we seek a WKB solution of (3.5) in the form of a small-amplitude short-scale wavetrain (with wavenumber equal to the resonant value $k_{0} \sim 1 / \epsilon$ ) riding on the solitary-wave core (approximated by the finite-depth soliton $u_{0}(X)$
given in (3.11)):

$$
u \approx u_{0}(X)+\left\{v(X) e^{i k_{0} X}+\text { c.c. }\right\}
$$

Linearizing (3.5) about $u_{0}(X), v$ satisfies to leading order

$$
v_{X}+2 \mathrm{i} u_{0} v=0
$$

and hence

$$
\begin{gather*}
v \propto e^{-\mathrm{i} \phi(X)},  \tag{3.27a}\\
\phi(X)=4 \tan ^{-1}\left(\tan \frac{\delta}{2} \tanh \frac{\delta X}{2 \beta}\right) . \tag{3.27b}
\end{gather*}
$$

where
Therefore, the short wave experiences a phase modulation owing to the presence of the solitary-wave core.

Returning now to the symmetric solitary waves constructed in (3.25), one may superpose an exponentially small amount of the short waves considered above,

$$
\begin{equation*}
B \frac{\exp (\beta-\alpha) / \epsilon}{\epsilon^{3}} \cos \left(\frac{X}{\epsilon}-\phi(X)\right) \tag{3.28}
\end{equation*}
$$

where $B$ is an arbitrary real constant. Hence, combining (3.28) with (3.26) and (3.27), the oscillatory tails of symmetric solitary waves take the form

$$
\begin{equation*}
u \sim A \frac{\exp (\beta-\alpha) / \epsilon}{\epsilon^{3}} \sin \left(\frac{|X|}{\epsilon}+\chi\right) \quad(|X| \rightarrow \infty) \tag{3.29}
\end{equation*}
$$

where the amplitude factor $A$ and the phase $\chi$ are given by

$$
A=\left(D^{2}+B^{2}\right)^{1 / 2}, \quad \tan \chi=\frac{B-D \tan 2 \delta}{B \tan 2 \delta+D}
$$

with

$$
D=4 \pi C \beta^{2} \frac{\sin ^{2} \delta}{\delta^{2}}
$$

This is a one-parameter family of symmetric solitary waves characterized by the phase shift $\chi$ of the tail oscillations. In particular, the amplitude factor $A$ attains its minimum value $D$ when the phase shift equals $-2 \delta$.

Finally, in the deep-fluid limit ( $\beta \gg 1$ ), using (3.12), (3.29) reduces to

$$
u \sim A \frac{\exp (-2 / a \epsilon)}{\epsilon^{3}} \sin \left(\frac{|X|}{\epsilon}+\chi\right) \quad(|X| \rightarrow \infty)
$$

where

$$
A=\left(D^{2}+B^{2}\right)^{1 / 2}, \quad \tan \chi=\frac{B}{D}
$$

with

$$
D=16 \pi C / a^{2}
$$

In this limit, the minimum tail amplitude obtains when the phase shift vanishes, as in the fifth-order KdV equation (Grimshaw \& Joshi 1995).

## CHAPTER 4

## FINITE-AMPLITUDE EFFECTS ON STEADY LEE-WAVE PATTERNS

To examine the physical significance of the resonance mechanism described in Chapter 1, here the flow of a continuously stratified fluid over a smooth bottom bump in a channel of finite depth is considered. As will be seen, in the weakly nonlinear-weakly dispersive regime $\epsilon=a / h \ll 1, \mu=h / l \ll 1$ (where $h$ is the channel depth and $a, l$ are the peak amplitude and the width of the obstacle respectively), the parameter $A=\epsilon / \mu^{p}$ (where $p>0$ depends on the obstacle shape) controls the effect of nonlinearity on the steady lee wavetrain that forms downstream of the obstacle for subcritical flow speeds. For $A=O(1)$, when nonlinear and dispersive effects are equally important, the interaction of the long-wave disturbance over the obstacle with the lee wave is fully nonlinear, and the wavenumber-domain approach presented in Chapter 2 is followed to determine the (exponentially small as $\mu \rightarrow 0$ ) lee-wave amplitude. Comparison with numerical results indicates that the asymptotic theory often remains reasonably accurate even for moderately small values of $\mu$ and $\epsilon$, in which case the (formally exponentially small) lee-wave amplitude is greatly enhanced by nonlinearity and can be quite substantial. Moreover, these findings reveal that the range of validity of the classical linear lee-wave theory $(A \ll 1)$ is rather limited.

### 4.1 Introduction

When a stably stratified fluid flows over topography, the internal gravity waves that form downstream are commonly referred to as lee waves; in many respects, they are akin to the more familiar gravity waves induced on the free surface of a homogeneous fluid by a moving disturbance. Lee waves are of fundamental meteorological interest, as they are often present on the lee side of mountains due to the prevailing winds and are believed to play an important part in the development of storms (see, for example, Lilly 1978).

We shall focus on the generation of lee-wave disturbances in the simple setting of
continuously stratified flow over a two-dimensional bottom obstacle in a channel bounded by rigid walls. As in the analogous problem of gravity waves on the free surface of a homogeneous fluid layer. it proves useful to introduce the long-wave parameter $\mu=h / l$ and the nonlinearity parameter $\epsilon=a / h$, where $h$ is the channel depth and $a . l$ are, respectively. the peak amplitude and the characteristic lengthscale of the obstacle in the streamwise direction.

These two parameters along with the Froude number $F=U /\left(h . V_{0}\right) . U$ being the undisturbed flow speed and $\mathcal{V}_{0}$ a characteristic value of the Brunt-Väisälä frequency: define various flow regimes. Specifically, the linear lee-wave regime is obtained when $\epsilon \rightarrow 0, \mu=O(1)$ and has been analyzed extensively in previous work using the linearized equations of motion (see Miles 1969 for a review). In this limit, the induced steady leewave pattern comprises a finite number of internal-wave modes. The wavenumber of each of these modes is such that the phase speed matches the flow speed, and, for this to be possible. it follows from the linear dispersion relation (see, for example, Yih 1979, Ch. 5. §4) that the Froude number has to be subcritical relative to each mode that is excited-the flow speed has to be less than the corresponding long-wave speed. The lee-wave amplitude depends on the specific obstacle shape and may be found by Fourier-transform techniques. In particular. in the hydrostatic limit (when the width of the obstacle is large compared to the channel depth, $\mu \ll 1$ ), dispersive effects are weak and the lee-wave amplitude is exponentially small with respect to $\mu$.

The flow characteristics in the nonlinear regime are more difficult to analyze quantitatively. however, because the governing equations are nonlinear in general when $\epsilon$ is finite, and previous efforts center on two approaches that are valid under particular conditions. Specifically, Long's model is based on the observation that the Euler equations of two-dimensional steady flow become linear for certain density and velocity profiles of the background flow, assuming that the flow remains undisturbed far upstream (Dubreil-Jacotin 1935; Long 1953: Yih 1960). In the case of constant upstream flow speed which is of interest here, these conditions are met for a weakly stratified (Boussinesq) fluid
with uniform stratification (constant Brunt-Väisälä frequency). Long's model has been used primarily to obtain theoretical estimates of the critical obstacle steepness (keeping the other flow parameters fixed) above which density inversions occur and the assumption of steady flow is expected to fail owing to static instability (Miles 1969; Miles \& Huppert 1969). For finite-amplitude obstacles, the validity of Long's hypothesis of no upstream influence has been questioned (Baines 1977), however, and the issue has not been settled completely as yet (see, for example, the recent numerical study by Lamb (1994)).

The second approach has received attention more recently and is valid in the weakly nonlinear regime $(0<\epsilon \ll 1)$ near resonance-when the flow speed is close to the longwave speed of an internal-wave mode in the channel. Under these conditions, the response is dominated by the resonant mode and the evolution of the corresponding amplitude is governed by a forced Korteweg-de Vries (fKdV) equation (Grimshaw \& Smyth 1986). For transcritical flow speeds, strong upstream influence is found in the form of solitary waves that are generated periodically, as in the analogous problem of free-surface flow of a homogeneous fluid near resonance (Akylas 1984; Cole 1985). In the special case of a uniformly stratified Boussinesq fluid, when Long's model applies, the fKdV equation is replaced by an integral-differential evolution equation which reveals that density inversions often appear during the transient development of the response (Grimshaw \& Yi 1991).

We shall concentrate on finite-amplitude lee-wave disturbances for subcritical flow speeds (away from resonant conditions). Our approach is complementary to the works cited above: as in Long's model, attention is confined to steady waves under the assumption of no upstream influence, but the flow may have general (stable) stratification. Moreover, while the asymptotic theory developed here is formally valid in the weakly nonlinear-weakly dispersive regime ( $\epsilon, \mu \ll 1$ ), comparison with numerical solutions indicates that the analytical results often are reasonably accurate for a wide range of moderately small values of $\epsilon$ and $\mu$-sometimes even close to conditions that density inversions are about to occur.

The asymptotic theory is motivated by the results obtained in Chapter 2. The (steady) fKdV model studied there suggests that nonlinear effects can significantly modify the lee-
wave disturbance induced by a long obstacle ( $\mu \ll 1$ ), even though the wave amplitude is exponentially small with respect to $\mu$. More precisely, in the joint limit $\epsilon, \mu \rightarrow 0$, it turns out that the relative importance of finite-amplitude effects is measured by the parameter

$$
\begin{equation*}
A=\frac{\epsilon}{\mu^{p}} \tag{4.1}
\end{equation*}
$$

where $p>0$ depends on the obstacle shape. In particular, the classical linear lee-wave theory is expected to be valid when $A \ll 1$, so its usefulness is limited-the situation is reminiscent of the 'long-wave paradox' in weakly nonlinear shallow-water waves (Ursell 1953).

For $A=O(1)$, when nonlinear and dispersive effects are equally important, the interaction of the long-wave disturbance over the obstacle with the relatively short lee wave downstream is, in fact, fully nonlinear. As a result, it is necessary to account for all nonlinear and dispersive terms in the governing equations in order to determine the lee-wave amplitude; following Chapter 2, this is carried out using techniques of asymptotics 'beyond all orders' (see, for example, Segur, Tanveer \& Levine 1991).

The predictions of the asymptotic theory are compared against numerically computed wave patterns in subcritical stratified flow over two possible obstacle shapes and when the square of the Brunt-Väisälä frequency is either constant or varies linearly with depth. As expected, there is good agreement in the limit $\epsilon, \mu \rightarrow 0$, confirming the validity of the asymptotic theory. More interestingly, however, the asymptotic results often remain reasonably accurate for moderately small values of $\epsilon$ and $\mu$; in this range, the (formally exponentially small) lee-wave amplitude can be quite substantial-in some cases several times larger than the estimate obtained from linear theory-owing to the effect of nonlinearity.

In a recent numerical study of two-layer flow over topography, Belward \& Forbes (1993) also report that the lee-wave amplitude is enhanced owing to nonlinear effects. It would appear that the approach taken here can be extended to discuss layered flows.

### 4.2 Formulation and linear theory

Consider two-dimensional steady flow of an inviscid, incompressible, density-stratified fluid along a channel that is bounded by rigid walls and features a locally confined bottom bump.

We shall use dimensionless variables taking the uniform channel depth $h$ (away from the bottom bump) as the vertical lengthscale, the characteristic width $l$ of the bump as the horizontal lengthscale, and $\mathcal{N}_{0}^{-1}$, the inverse of a characteristic value of the Brunt-Väisälä frequency, as the timescale. Far upstream $(x \rightarrow-\infty)$, the flow velocity is assumed to be uniform, and the density $\rho_{0}(z)$ varies with height $z$ in a prescribed (stable) way. The Brunt-Väisälä frequency $\mathcal{N}(z)$ is then given by

$$
\beta \rho_{0} \mathcal{N}^{2}=-\rho_{0 z},
$$

where $\beta=\mathcal{N}_{0}^{2} h / g$ is the Boussinesq parameter, $g$ being the gravitational acceleration.
For incompressible flow, it is convenient to work with the stream function $\Psi=F z+F \psi$, where $\psi(x, z)$ describes the disturbance induced by the bottom obstacle and $F$ is the Froude number, $F=U /\left(h \mathcal{N}_{0}\right), U$ being the upstream flow speed. In terms of $\Psi$, the horizontal and vertical velocity components are given by $\Psi_{z},-\mu \Psi_{x}(\mu=h / l)$ respectively; thus, incompressibility is automatically satisfied.

Assuming further that all streamlines originate far upstream where the flow remains undisturbed (no upstream influence),

$$
\begin{equation*}
\psi \rightarrow 0 \quad(x \rightarrow-\infty) \tag{4.2}
\end{equation*}
$$

the momentum equations and the equation of mass conservation can be combined, following the procedure outlined in Akylas \& Grimshaw (1992), into a single equation for $\psi$ :

$$
\begin{equation*}
\mu^{2} \psi_{x x}+\psi_{z z}+\mathcal{N}^{2}(z+\psi)\left\{\frac{\psi}{F^{2}}-\beta \psi_{z}-\frac{1}{2} \beta\left(\mu^{2} \psi_{x}^{2}+\psi_{z}^{2}\right)\right\}=0 . \tag{4.3}
\end{equation*}
$$

This equation, which is usually referred to as Long's equation (Long 1953), is valid under the assumptions stated above within the channel $-\infty<x<\infty, \epsilon f(x) \leq z \leq 1$. Here
$z=\epsilon f(x)$ defines the bottom of the channel where the bump $(f(x) \rightarrow 0, x \rightarrow \pm \infty)$ is present, and $\epsilon=a / h, a$ being the peak height of the bump.

Finally, the appropriate boundary conditions, which ensure that the channel walls are streamlines, are

$$
\begin{array}{cc}
\psi=-\epsilon f(x) & (z=\epsilon f(x)) \\
\psi=0 & (z=1) \tag{4.4b}
\end{array}
$$

For a finite-amplitude obstacle, the problem posed by (4.2)-(4.4) is nonlinear in general. Before turning our attention to nonlinear effects, however, we shall briefly review the salient features of the linear response. In the small-amplitude limit $(\epsilon \rightarrow 0), \psi=O(\epsilon)$ so the governing equation (4.3) and the bottom boundary condition (4.4a) may be linearized. The resulting linear boundary-value problem then can be readily solved by Fourier transform,

$$
\begin{equation*}
\psi(x, z)=\int_{\mathcal{C}^{-}} \mathrm{e}^{\mathrm{i} k x} \widehat{\psi}(k, z) \mathrm{d} k, \tag{4.5}
\end{equation*}
$$

indenting the contour $\mathcal{C}^{-}$to pass below all singularities of $\hat{\psi}$ on the real $k$-axis so that (4.2) is met. It turns out that

$$
\begin{equation*}
\widehat{\psi}(k, z)=-\epsilon \widehat{f}(k)\left\{q(z)+\mu^{2} k^{2} \sum_{r=1}^{\infty} \frac{\gamma_{r}}{\kappa_{r}-\mu^{2} k^{2}} \phi_{r}(z)\right\} \tag{4.6}
\end{equation*}
$$

where $\widehat{f}(k)$ is the Fourier transform of the obstacle shape $f(x)$,

$$
\widehat{f}(k)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i} k x} f(x) \mathrm{d} x
$$

The first term on the right-hand side of (4.6) corresponds to the purely hydrostatic $(\mu=0)$ linear response and is given in terms of the solution to the boundary-value problem

$$
\begin{gather*}
\left(\rho_{0} q_{z}\right)_{z}+\frac{\rho_{0}}{F^{2}} \mathcal{N}^{2} q=0 \quad(0 \leq z \leq 1),  \tag{4.7a}\\
q(0)=1, \quad q(1)=0 . \tag{4.76}
\end{gather*}
$$

The term including the infinite sum in expression (4.6) accounts for dispersive corrections to the hydrostatic response owing to the internal-wave modes $\left\{\phi_{r}(z), \kappa_{r}\right\}(r=1,2, \ldots)$. This orthogonal and complete set of modes are defined by the eigenvalue problem

$$
\begin{equation*}
\left(\rho_{0} \phi_{r z}\right)_{z}+\rho_{0}\left\{\frac{\mathcal{N}^{2}}{F^{2}}-\kappa_{r}\right\} \phi_{r}=0 \quad(0 \leq z \leq 1) \tag{4.8a}
\end{equation*}
$$

$$
\begin{equation*}
\phi_{r}=0 \quad(z=0,1) \tag{4.8b}
\end{equation*}
$$

and satisfy the orthogonality relation

$$
\begin{equation*}
\int_{0}^{1} \rho_{0} \phi_{r} \phi_{s} \mathrm{~d} z=\delta_{r s} \tag{4.9}
\end{equation*}
$$

$\delta_{r s}$ being the Kronecker delta. In particular, the constants $\gamma_{r}$ in (4.6) are the participation factors of these modes:

$$
\gamma_{r}=\int_{0}^{1} \rho_{0} q \phi_{r} \mathrm{~d} z
$$

It is clear from (4.6) that $\widehat{\psi}$ has simple-pole singularities on the real $k$-axis at $k= \pm \kappa_{r}^{\frac{1}{2}} / \mu \equiv \pm k_{r}$ when $\kappa_{r}>0$ for some $r$, and, upon evaluating the contour integral in (4.5), these singularities contribute lee waves downstream ( $x>0$ ). The sign of $\kappa_{r}$ in turn depends on whether the Froude number $F$ is subcritical or supercritical relative to $c_{r}$, the corresponding linear long-wave speed. These are the speeds of the linear long-wave modes $\left\{f_{s}(z), c_{s}\right\}(s=1,2, \ldots)$ (with $\left.c_{1}>c_{2}>\cdots>0\right)$ defined by the eigenvalue problem

$$
\begin{gather*}
\left(\rho_{0} f_{s z}\right)_{z}+\frac{\rho_{0}}{c_{s}^{2}} \mathcal{N}^{2} f_{s}=0 \quad(0 \leq z \leq 1)  \tag{4.10a}\\
f_{s}=0 \quad(z=0,1) \tag{4.10b}
\end{gather*}
$$

If now $c_{N+1}<F \leq c_{N}$ for some $N \geq 1$, it follows that $\kappa_{1}>\kappa_{2}>\cdots>\kappa_{N} \geq 0>\kappa_{N+1}>\cdots$ and, in particular, $\kappa_{N}=0$ for $F=c_{N}$ (Yih 1979, Ch. 5, §4.1.3). Hence, lee waves are excited only when $F$ is subcritical relative to at least $c_{1}$, the highest of the linear-long-wave speeds. This is easy to understand physically because the phase speed of each internal-wave mode is known to decrease with wavenumber (Yih 1979, Ch. 5, §4.1.4), so stationary (in the frame of the obstacle) lee waves are not possible if the flow speed is supercritical relative to all linear-long-wave speeds.

We also remark here that the steady linear response is singular when the flow speed is critical with respect to a long-wave speed-the inhomogeneous problem (4.7) has no solution since, from (4.10), the long-wave mode $f_{N}(z)$ is a non-trivial homogeneous solution when $F=c_{N}\left(\kappa_{N}=0\right)$. This is the resonant case where, as mentioned earlier, the assumptions of steady flow and no upstream influence fail in a finite range of transcritical flow speeds
(Grimshaw \& Smyth 1986: Grimshaw \& Yi 1991). As our interest centers on steady leewave disturbances, we shall take the flow speed to be subcritical relative to at least one (and not close to any) linear long-wave speed, viz. $c_{N+1}<F<c_{N}$ with $N \geq 1$.

Returning then to (4.5) and (4.6), we conclude that the downstream response consists of a finite number of lee waves:

$$
\psi \sim-2 \pi \epsilon \sum_{r=1}^{N} \gamma_{r} k_{r} \hat{f}\left(k_{r}\right) \phi_{r}(z) \sin k_{r} x \quad(x \rightarrow \infty),
$$

taking the obstacle to be symmetric, $f(x)=f(-x)$. If, furthermore, the obstacle is long relative to the channel depth ( $\mu \ll 1$ ), the induced lee waves have short wavelength relative to the obstacle length (the lee wavelength is comparable to the channel depth) and exponentially small amplitude (the Fourier transform of a smooth obstacle profile $f(x)$ is exponentially small for large wavenumbers, $k_{r} \gg 1$ ). Accordingly, in this nearly hydrostatic regime, the lee wave of mode $N$ which has the relatively longer wavelength $\left(k_{N}<k_{N-1}<\cdots<k_{1}\right)$ dominates:

$$
\begin{equation*}
\psi \sim-2 \pi \epsilon_{\neq}{ }_{\mathcal{N}} k_{N} \widehat{f}\left(k_{N}\right) \phi_{N}(z) \sin k_{N} x \quad(x \rightarrow \infty ; \mu \ll 1) . \tag{4.11}
\end{equation*}
$$

However, we shall demonstrate in the ensuing discussion that the validity of this expression for the induced lee-wave disturbance is severely limited by finite-amplitude effects. Specifically, it follows from the asymptotic theory developed below that, close to the hydrostatic limit ( $\mu \ll 1$ ), the relative importance of nonlinearity is controlled by the parameter $A$ defined in (4.1); the linear, nearly hydrostatic approximation (4.11) is valid only when $\epsilon \ll \mu^{p}$, where $p$ depends on the particular obstacle shape.

### 4.3 Asymptotic theory

Attention is now focused on steady lee-wave disturbances in the weakly nonlinear-weakly dispersive regime ( $\epsilon, \mu \ll 1$ ). As indicated by the linear solution sketched in $\S 4.2$, lee waves in the physical domain go hand in hand with simple-pole singularities of $\widehat{\psi}(k, z)$ on the real $k$-axis in the wavenumber domain, and the lee-wave amplitude is completely determined by
the corresponding residues. Accordingly, following the perturbation procedure presented in Chapter 2, we shall compute the residues of $\hat{\psi}(k, z)$ at the dominant lee wavenumber $k= \pm k_{N}$ asymptotically for $\epsilon, \mu \ll 1$.

To this end, returning to the full nonlinear governing equation (4.3) and taking Fourier transform formally, it is found that $\hat{\psi}$ satisfies
where

$$
\begin{align*}
\hat{\psi}_{z z} & -\beta \mathcal{N}^{2} \widehat{\psi}_{z}+\left(\frac{\mathcal{N}^{2}}{F^{2}}-\mu^{2} k^{2}\right) \hat{\psi}-\frac{1}{2} \beta \mathcal{N}^{2}\left(\widehat{\psi_{z}^{2}}+\mu^{2} \widehat{\psi_{x}^{2}}\right) \\
& +\sum_{j=1}^{\infty} M_{j}\left\{\frac{\psi^{j+1}}{F^{2}}-\beta \widehat{\psi^{j} \psi_{z}}-\frac{1}{2} \beta\left(\widehat{\psi^{j} \psi_{z}^{2}}+\mu^{2} \widehat{\psi^{j} \psi_{x}^{2}}\right)\right\}=0, \tag{4.12}
\end{align*}
$$

with

$$
\begin{aligned}
\mathcal{N}^{2}(z+\psi) & =\mathcal{N}^{2}(z)+\sum_{j=1}^{\infty} M_{j}(z) \psi^{j} \\
M_{j} & \equiv \frac{1}{j!} \frac{\mathrm{d}^{j}}{\mathrm{~d} z^{j}} \mathcal{N}^{2}(z)
\end{aligned}
$$

Similarly, the boundary conditions (4.4) (after expanding (4.4a) about $z=0$ ) transform to

$$
\begin{gather*}
\widehat{\psi}+\sum_{j=1}^{\infty} \frac{\epsilon^{j}}{j^{j}} \frac{\partial^{\mathfrak{j} \psi}}{\partial z^{j}} f^{j}=-\epsilon \widehat{f} \quad(z=0),  \tag{4.13a}\\
\hat{\psi}=0 \quad(z=1) \tag{4.13b}
\end{gather*}
$$

Expanding then $\hat{\psi}$ in powers of $\epsilon$ and $\mu^{2}$, it follows from (4.12),(4.13) that

$$
\begin{align*}
\widehat{\psi}(k, z)= & -\epsilon \widehat{f}(k)\left\{q(z)+\mu^{2} k^{2} \sum_{r=1}^{\infty} \frac{\gamma_{r}}{\kappa_{r}} \phi_{r}(z)+\cdots\right\} \\
& +\epsilon^{2} \widehat{f^{2}}(k)\left\{q^{\prime}(0) q(z)+\sum_{r=1}^{\infty} \frac{\delta_{r}}{\kappa_{r}} \phi_{r}(z)\right\}+\cdots \tag{4.14}
\end{align*}
$$

where

$$
\delta_{r}=-\int_{0}^{1} \rho_{0} \phi_{\cdot}\left(\frac{M_{1}}{F^{2}} q^{2}-\beta M_{1} q q^{\prime}-\frac{1}{2} \beta \mathcal{N}^{2} q^{\prime 2}\right) \mathrm{d} z .
$$

The $O(\epsilon)$ term in this expansion corresponds to the linear hydrostatic solution; the rest of the terms are the leading-order nonlinear and dispersive corrections.

In view of the poles at $k= \pm k_{\tau}$ of the linear dispersive solution (4.6), it is expected that expansion (4.14) will become disordered when $k=O(1 / \mu)$ due to dispersive effects.

There is a second non-uniformity in (4.14), however, that derives from nonlinear effects: since $f^{2}(x)$ is generally steeper than $f(x)$ in the physical domain, $\widehat{f^{2}}(k)$ goes to zero less rapidly than $\hat{f}(k)$ as $|k| \rightarrow \infty$ in the wavenumber domain.

To be specific, for the obstacle profile $f(x)=\operatorname{sech}^{p} x$, where $p$ is a positive integer,

$$
\begin{gathered}
\hat{f}(k)=O\left(|k|^{p-1} \mathrm{e}^{-\pi|k| / 2}\right), \quad \widehat{f^{2}}(k)=O\left(|k|^{2 p-1} \mathrm{e}^{-\pi|k| / 2}\right) \quad(|k| \rightarrow \infty) ; \\
\frac{\widehat{f^{2}}(k)}{\hat{f}(k)}=O\left(|k|^{p}\right) \quad(|k| \rightarrow \infty),
\end{gathered}
$$

hence
resulting in a disordering of (4.14) when $k=O\left(\epsilon^{-1 / p}\right)$. For this class of obstacles, therefore, it is now clear that the classical linear lee-wave theory is valid-nonlinearity does not affect the residues of the poles of $\hat{\psi}(k, z)$ at $k= \pm k_{N}$-only in case $\epsilon \ll \mu^{p}$; in terms of the parameter $A=\epsilon / \mu^{p}$, this condition is met when $A \ll 1$. Note, however, that $p$ is also the order of the pole singularities (closest to the real axis) of the obstacle shape $f(x)=\operatorname{sech}^{p} x$ in the complex plane; with this interpretation of $p$, the parameter $A$ can be defined for any smooth, locally confined obstacle profile with pole singularities in the finite complex plane ${ }^{1}$, and the above conclusion regarding the role of nonlinear effects is generally valid.

The following discussion focuses on two particular obstacle shapes, (i) $f(x)=\operatorname{sech}^{2} x$ and (ii) $f(x)=\operatorname{sech} x$, assuming that $A=O(1)$ so that nonlinear and dispersive effects are equally important. As indicated in Chapter 2, these two examples typify obstacle profiles with pole singularities in the finite complex plane.

The asymptotic analysis parallels that of the model problem studied in Chapter 2. However, the technical details turn out to be quite more involved here because the governing equation (4.3) is a partial differential equation and, generally, the nonlinear terms in (4.3) and in the boundary condition (4.4a) are higher than quadratic. In this section, we shall only outline the main steps in the analysis and give the final results; details of the derivation may be found in Appendices A and B.

[^0]$$
\text { 4.3.1 Case (i): } f(x)=\operatorname{sech}^{2} x
$$

For this obstacle profile.

$$
\hat{f}(k)=\frac{1}{2} k \operatorname{csch} \frac{\pi k}{2}, \quad \widehat{f^{2}}(k)=\frac{1}{12} k\left(k^{2}+4\right) \operatorname{csch} \frac{\pi k}{2} .
$$

and $A=\epsilon / \mu^{2}$. The straightforward expansion (4.14) becomes disordered when $|k|=O(1 / \mu)=O\left(\epsilon^{-\frac{1}{2}}\right)$. To remove this non-uniformity, we replace (4.14) with the two-scale expansion

$$
\begin{equation*}
\hat{\psi}=\mu \operatorname{csch} \frac{\pi k}{2} \Phi(\kappa, z)+\cdots \tag{4.15}
\end{equation*}
$$

in terms of the scaled wavenumber variable $\kappa=\mu k$, with

$$
\begin{equation*}
\Phi \sim-\frac{1}{2} A\left\{\kappa q(z)-\kappa^{3}\left[\frac{1}{6} \cdot A q^{\prime}(0) q(z)+\sum_{r=1}^{\infty} \frac{1}{\kappa_{r}}\left(\frac{1}{6} A \delta_{r}-\gamma_{r}\right) \phi_{r}(z)\right]+\cdots\right\} \quad(\kappa \rightarrow 0) \tag{4.16}
\end{equation*}
$$

so that (4.15) is consistent with (4.14) when $1 \ll|k| \ll 1 / \mu$.
As already remarked, $\widehat{\psi}(k, z)$, and hence $\Phi(\kappa, z)$, are expected to have simple-pole singularities on the real $k$-axis at the lee wavenumbers $k= \pm k_{r}\left(\kappa= \pm \kappa_{T}^{\frac{1}{2}}\right)(r=1, \ldots, N)$. The dominant contribution to the lee-wave amplitude, however, comes from the residues of the poles that are closest to the origin. Accordingly, focusing attention on $\Phi(\kappa, z)$, we wish to compute the residues of the poles at $\kappa= \pm \kappa_{N}^{\frac{1}{2}}$ asymptotically as $\mu \rightarrow 0$ for $A=O(1)$.

Following the procedure described in Appendix A, upon substituting (4.15) in (4.12), (4.13), it is found that, to leading order, $\Phi$ satisfies a Volterra integral-differential boundaryvalue problem. This problem is singular at $\kappa= \pm \kappa_{N}^{\frac{1}{2}}$ and posing the solution as a power series in $\kappa$ consistent with (4.16), one may deduce the local behavior of $\Phi(\kappa, z)$ near these singularities:

$$
\begin{equation*}
\Phi \sim-C_{N} \frac{\kappa}{\kappa^{2}-\kappa_{N}} \phi_{N}(z) \quad\left(\kappa \rightarrow \pm \kappa_{N}^{\frac{1}{2}}\right) \tag{4.17}
\end{equation*}
$$

where $\phi_{N}(z)$ is the mode shape corresponding to the dominant lee-wave mode and $C_{N}$ is a constant that can be determined numerically from an infinite sequence of boundary-value problems.

Combining (4.17) with (4.15), it is now seen that $\hat{\psi}(k, z)$ has simple-pole singularities at $k= \pm k_{N}$ and the corresponding residues are known:

$$
\hat{v} \sim \mp C_{N} \frac{\exp \left(-\frac{1}{2} \pi k_{N}\right)}{k \mp k_{N}} \phi_{N}(z) \quad\left(k \rightarrow \pm k_{N}\right) .
$$

Hence, returning to (4.5) and evaluating the contour integral, these poles contribute a lee wave with wavenumber $k_{N}$ for $x>0$ :

$$
\begin{equation*}
\psi \sim 4 \pi C_{N} \exp \left(-\frac{1}{2} \pi k_{N}\right) \phi_{N}(z) \sin k_{N} x \quad(x \rightarrow \infty) \tag{4.18}
\end{equation*}
$$

As expected, the lee-wave amplitude is exponentially small $\left(k_{N}=\kappa_{N}^{\frac{1}{2}} / \mu \gg 1\right)$. In fact, comparing (4.18) with the linear result (4.11) [using $\widehat{f}\left(k_{N}\right) \sim k_{N} \exp \left(-\frac{1}{2} \pi k_{N}\right)$ in this case], it is seen that the effect of finite amplitude shows up solely in the value of the constant $C_{N}$; in order to assess the significance of finite-amplitude effects, one needs to know how $C_{N}$ depends on the parameter $A$. This question is addressed in $\S 4.5$ by computing $C_{N}$ numerically for specific examples.
4.3.2 Case (ii): $f(x)=\operatorname{sech} x$

In this case,

$$
\widehat{f}(k)=\frac{1}{2} \operatorname{sech} \frac{\pi k}{2}, \quad \widehat{f^{2}}(k)=\frac{1}{2} k \operatorname{csch} \frac{\pi k}{2}
$$

and $A=\epsilon / \mu$. The straightforward expansion (4.14) becomes disordered when $k=O(1 / \mu)=O(1 / \epsilon)$, and, proc巨eding as in case (i), we propose the uniformly valid expansion

$$
\begin{equation*}
\widehat{\psi}=\mu \operatorname{sech} \frac{\pi k}{2} \Phi(\kappa, z)+\cdots \tag{4.19}
\end{equation*}
$$

with

$$
\begin{equation*}
\Phi(\kappa, z) \sim-\frac{1}{2} A q(z)+\frac{1}{2} A^{2}|\kappa|\left\{q^{\prime}(0) q(z)+\sum_{r=1}^{\infty} \frac{\delta_{r}}{\gamma_{r}} \phi_{r}(z)\right\}+\cdots \quad(\kappa \rightarrow 0) \tag{4.20}
\end{equation*}
$$

Again, we expect that $\Phi(\kappa, z)$ has simple-pole singularities at $\kappa= \pm \kappa_{N}^{\frac{1}{2}}$ and the goal is to compute the corresponding residues. Here, however, the (approximate) boundary-value problem that governs $\Phi(\kappa, z)$ to leading order for $|\kappa|=O(1)$ is not valid when $\kappa$ is very close
to these singularities $\left(\kappa_{N}^{\frac{1}{2}}-|\kappa| \leq O(\mu)\right)$ and a local analysis is needed there. As explained in Chapter 2 (see also Chapter 3), this complication arises from the fact that $\Phi=O(1)$ as $\kappa \rightarrow 0$; this in turn is a consequence of the simple-pole singularities (closest to the real $x$-axis) of sech $x$ which determine the dominant behavior of $\hat{f}(k)$ as $|k| \rightarrow \infty$.

Specifically, in the 'intermediate' region $\mu \ll \kappa_{N}^{\frac{1}{2}}-|\kappa| \ll 1$, where $\kappa$ is close but not too close to $\pm \kappa_{N}^{\frac{1}{2}}$, the leading-order behavior of $\Phi(\kappa, z)$ turns out to be (see Appendix B)

$$
\begin{equation*}
\Phi(\kappa, z) \sim \frac{C_{N} \phi_{N}(z)}{\left(\kappa_{N}^{\frac{1}{2}}-|\kappa|\right)^{\alpha}} \quad\left(\mu \ll \kappa_{N}^{\frac{1}{2}}-|\kappa| \ll 1\right), \tag{4.21}
\end{equation*}
$$

where $C_{N}$ is a constant to be determined numerically, and

$$
\begin{equation*}
\alpha=1-\frac{A}{2 \kappa_{N}^{\frac{1}{2}}}\left(\nu+\kappa_{N} \gamma_{N} \phi_{N}^{\prime}(0)\right) \tag{4.22}
\end{equation*}
$$

with
and

$$
\nu=\int_{0}^{1} \rho_{0} \phi_{N}\left\{\beta \mathcal{N}^{2} q^{\prime} \phi_{N}^{\prime}-2 \frac{M_{1}}{F^{2}} q \phi_{N}+\beta M_{1}\left(q \phi_{N}\right)^{\prime}\right\} \mathrm{d} z
$$

$$
\gamma_{N}=\int_{0}^{1} \rho_{0} q \phi_{N} \mathrm{~d} z
$$

Note that, according to (4.21), the singularities of $\Phi(\kappa, z)$ at $\kappa= \pm \kappa_{N}^{\frac{1}{2}}$ are not simple poles (save in the linear limit $A \rightarrow 0$ ), contrary to the fact that the induced lee wave has constant amplitude as $x \rightarrow \infty$; this is an indication that the asymptotic behavior (4.21) breaks down in the immediate vicinity of $\kappa= \pm \kappa_{N}^{\frac{1}{2}}, \kappa_{N}^{\frac{1}{2}}-|\kappa|=O(\mu)$. Also, the argument used in Appendix B to obtain (4.21) is not strictly valid in case $\alpha<1$-a different theoretical approach is needed but this matter is not pursued here. The physical significance of this condition is discussed in $\S 4.5$ for specific examples.

The correct structure of $\Phi(\kappa, z)$ at $\kappa= \pm \kappa_{N}^{\frac{1}{2}}$ can be obtained using a matchedasymptotics procedure in terms of the 'inner' variable $\eta=\left(\kappa_{N}^{\frac{1}{2}}-|\kappa|\right) / \mu$ (see Appendix B). It transpires that $\Phi(\kappa, z)$ has simple-pole singularities at $\kappa= \pm \kappa_{N}^{\frac{1}{2}}$ and the corresponding residues are completely determined in terms of the constant $C_{N}$ in (4.21). In view of (4.19), one then has

$$
\begin{equation*}
\widehat{\psi} \sim \mp C_{N} \frac{2^{\alpha}}{\Gamma(\alpha)} \mu^{1-\alpha} \exp \left(\mp \mathrm{i} \frac{1}{2} \pi(\alpha-1)\right) \frac{\exp \left(-\frac{1}{2} \pi k_{N}\right)}{k \mp k_{N}} \phi_{N}(z) \quad\left(k \rightarrow \pm k_{N}\right) \tag{4.23}
\end{equation*}
$$

where $\Gamma(\alpha)$ denotes the gamma function.
Therefore, returning to (4.5) and accounting for the contribution of the poles of $\hat{\psi}$ at $k= \pm k_{N}$ according to (4.23), the induced lee wave downstream takes the form

$$
\begin{equation*}
\psi \sim 4 \pi C_{N} \frac{2^{\alpha}}{\Gamma(\alpha)} \mu^{1-\alpha} \exp \left(-\frac{1}{2} \pi k_{N}\right) \phi_{N}(z) \sin \left(k_{N} x-\frac{1}{2}(\alpha-1) \pi\right) \quad(x \rightarrow \infty) \tag{4.24}
\end{equation*}
$$

Comparing the above expression with the corresponding linear result (4.11) [with $\left.\widehat{f}\left(k_{N}\right) \sim \exp \left(-\frac{1}{2} \pi k_{N}\right)\right]$,

$$
\psi \sim-2 \pi A \gamma_{N} \kappa_{N}^{\frac{1}{2}} \exp \left(-\frac{1}{2} \pi k_{N}\right) \phi_{N}(z) \sin k_{N} x
$$

note that, apart from the value of $C_{N}$, here nonlinearity affects both the order of magnitude of the amplitude and the phase of the lee waves through the value of $\alpha$ which depends on the background flow and the nonlinearity parameter $A$ according to (4.22). These additional nonlinear effects arise from the fact that the singularities (closest to the real axis) of the obstacle profile $f(x)=\operatorname{sech} x$ are simple poles so $\Phi(\kappa, z)=O(1)(\kappa \rightarrow 0)$; this necessitates the local analysis near $\kappa=\kappa_{N}^{\frac{1}{2}}$ (see Appendix B), which determines the lee-wave amplitude and the phase shift that the lee wavetrain experiences owing to its interaction with the long wave above the topography. Similar effects are to be expected generally for lee waves induced by obstacle profiles with simple-pole singularities as, for example, the algebraic obstacle or 'Witch of Agnesi' (see Chapters 2 and 3).

### 4.4 Numerical solution

The asymptotic results (4.18) and (4.24) suggest that, for $A=O(1)$, nonlinearity may seriously affect the lee-wave pattern induced by a long smooth obstacle. On the other hand, the asymptotic theory is formally valid in the weakly nonlinear-weakly dispersive limit $(\epsilon, \mu \ll 1)$ where lee waves have exponentially small amplitude and form only a small portion of the overall disturbance. From a more practical point of view, therefore, it would be of interest to know whether nonlinear effects play an equally important part in the parameter range where the lee-wave amplitude is more substantial. To address this issue,
we shall resort to a numerical procedure for calculating steady, finite-amplitude lee wates which also provides independent confirmation of the asymptotic theory.

One of the complications in the numerical treatment of the nonlinear lee-wave problem (4.3),(4.4) arises from the lower boundary condition (4.4a) that holds on the bottom bump. Although this condition can be implemented through a topography-following coordinate transformation (see, for example, Lilly \& Klemp 1979), we find it more convenient to work in $0 \leq z \leq 1$ by applying the 'fictitious' condition

$$
\begin{equation*}
\psi(x, 0)=-\epsilon \bar{f}(x) \tag{4.25}
\end{equation*}
$$

at $z=0$, rather than the actual nonlinear condition (4.4a) at $z=\epsilon f(x)$ (the same device was used by Laprise \& Peltier 1989). Of course, $\bar{f}(x)$ has to be such that (4.4a) is also satisfied and this is achieved via an iterative procedure: denoting by $\bar{f}^{(n)}(x)$ the estimate of $\bar{f}(x)$ and $\psi^{(n)}(x, z)$ the corresponding estimate of $\psi(x, z)$ after $n$ iterations, the next iteration for determining $\psi^{(n+1)}$ proceeds using

$$
\bar{f}^{(n+1)}(x)=\bar{f}^{(n)}(x)+f(x)+\frac{1}{\epsilon} \psi^{(n)}(x, \epsilon f(x)),
$$

where $f(x)$ is the obstacle profile. [The iteration scheme is initiated by choosing $\bar{f}^{(1)}(x)=f(x)$ in which case (4.25) reduces to the linear boundary condition.]

Attention now is focused on the solution of equation (4.3) in $0 \leq z \leq 1,-\infty<x<\infty$, subject to the upstream condition (4.2) and the boundary conditions (4.4b) and (4.25), bearing in mind that $\bar{f}(x)$ and $\psi(x, z)$ are intermediate estimates in the above iteration scheme. Mathematically, the governing equation (4.3) is an elliptic partial differential equation while the boundary conditions are of the parabolic type in the sense that a downstream condition cannot be specified a priori. For this reason, numerical-instability problems are to be expected if one attempts to determine the lee waves induced downstream by solving (4.3) through a marching scheme, starting from far upstream.

To circumvent this difficulty, in analogy with the linearized response (4.6), we expand $\psi(x, z)$ in terms of the linear hydrostatic response $q(z)$ (see (4.7)) and the linear internal-
wave modes defined in (4.8):

$$
\begin{equation*}
\psi(x, z)=-\epsilon \bar{f}(x) q(z)+\sum_{r=1}^{\infty} a_{r}(x) \phi_{r}(z) \tag{4.26}
\end{equation*}
$$

Hence, the boundary conditions (4.4b) and (4.25) are automatically met, and substituting (4.26) into (4.3), making use of the orthogonality relation (4.9), yields an (infinite) system of coupled nonlinear ordinary differential equations for the modal amplitudes $a_{r}(x)(r=1,2, \ldots)$.

In this system, the equations for the amplitudes of the modes that contribute lee waves downstream ( $r=1, \ldots, N$ ) are solved numerically by a standard fourth-order Runge-Kutta marching scheme starting far upstream; the equations for the remaining evanescent modes are treated as boundary-value problems by imposing the downstream conditions

$$
a_{r} \rightarrow 0 \quad(x \rightarrow \infty ; r=N+1, N+2, \ldots)
$$

Of course, since the system of equations at hand is nonlinear and coupled, it is necessary to use an (inner-level) iteration procedure in order to solve for the modal amplitudes. However, in the examples discussed below the nonlinear coupling among the modes happens to be relatively weak, so typically 5-10 Jacobi iterations are sufficient to obtain converged solutions for $a_{r}$. After these amplitudes are found, the 'fictitious' obstacle profile $\bar{f}(x)$ is updated as already described, and the procedure is repeated until the exact lower boundary condition (4.4a) is met.

Finally, in implementing this numerical procedure, it is necessary to use a finite computational domain in the streamwise direction and to truncate the modal expansion (4.26) to a finite number of modes. The size of the computational domain is chosen such that the height of the obstacle at the boundaries is small compared to the lee-wave amplitude. Also, in the examples discussed below, typically, using 8 modes is sufficient to compute lee waves with $O\left(10^{-10}\right)$ amplitude as long as the grid size (used in computing the modes $\phi_{r}(z)$ and the modal amplitudes $\left.a_{r}(x)\right)$ is also made fine enough to be consistent with this accuracy.

### 4.5 Results and discussion

In the following discussion, the density stratification is taken to be such that $\beta=0$ (Boussinesq approximation, $\rho_{0}=1$ ) and $\mathcal{N}^{\boldsymbol{2}}(z)$ varies linearly with height:

$$
\mathcal{N}^{2}(z)=1+b z \quad(0 \leq z \leq 1)
$$

where $b \geq 0$ is a constant, so $M_{j} \equiv 0(j \geq 2)$ in (4.12) and the problem simplifies considerably. Moreover, it is assumed that $c_{2}<F<c_{1}$ so lee waves of the first mode only are excited ( $N=1$ ).

Under these conditions, we shall present asymptotic and numerical results for lee waves induced by obstacles having either a $\operatorname{sech}^{2} x$ or $\operatorname{sech} x$ profile, and for two particular examples of density stratification, namely $b=0$ and $b=1$.

### 4.5.1 Uniform stratification

When $b=0$, we are dealing with a uniformly stratified Boussinesq fluid (constant $\mathcal{N}, \beta=0$ ) in which case Long's model applies and the governing equation (4.3) is linear (of course the bottom boundary condition (4.4a) still is nonlinear). Furthermore, the eigenvalue problems (4.8),(4.10) and the boundary-value problem (4.7) for the hydrostatic response can be solved in closed form:

$$
\begin{array}{ll}
\kappa_{r}=\frac{1}{F^{2}}-r^{2} \pi^{2}, & \phi_{r}(z)=\sqrt{2} \sin r \pi z \\
c_{s}=\frac{1}{s \pi}, & f_{s}(z)=\sqrt{2} \sin s \pi z
\end{array}(s=1,2, \ldots) ;
$$

and

$$
\begin{gather*}
q(z)=\frac{\sin (1-z) / F}{\sin (1 / F)} . \\
\alpha=1+\frac{\pi^{2}}{\sqrt{\kappa_{1}}} A \tag{4.27}
\end{gather*}
$$

and $\alpha>1$ always. According to the criterion given in Grimshaw \& Yi (1991), nonresonant subcritical flow (relative to $c_{1}$ ) occurs when $c_{2}+\epsilon \Delta_{2}<F<c_{1}-\epsilon \Delta_{1}$ with $\Delta_{1}=\frac{\sqrt{6}}{2} \pi^{-\frac{5}{2}}=0.07, \Delta_{2}=\frac{\sqrt{3}}{8} \pi^{-\frac{5}{2}}=0.01$. We choose $1 / F^{2}=\pi^{2}+2$
( $F=0.2903=c_{1}-0.028$ ); this value of the flow speed is within the non-resonant range for the values of $\epsilon$ considered here ( $\epsilon<0.1$ ), and the assumption of steady flow is expected to hold.

As noted in $\$ 4.3$, the asymptotic expressions (4.18) and (4.24) for the lee wavetrain downstream depend on the constant $C_{1}(N=1)$ which needs to be determined numerically: Specifically, the value of $C_{1}$ can be estimated from the coefficients $b_{m}(z)$ of the power series (A3) (for $f=\operatorname{sech}^{2} x$ ) and (B11) (for $f=\operatorname{sech} x$ ) by computing the projection of $b_{m}(z)$ on $\phi_{1}(z)$ as $m \rightarrow \infty$. Note that for $f=\operatorname{sech}^{2} x, b_{m}(z)(m \geq 2)$ satisfy the sequence of boundaryvalue problems (A4),(A5), while for $f=\operatorname{sech} x$, an analogous sequence is obtained upon substitution of (B11) into (B1),(B2). In solving for $b_{m}(z)$ numerically, it is important to note that, as $m$ increases, high-order derivatives of $b_{1}, b_{2}, \ldots$ enter the inhomogeneous boundary condition (A5a); these need to be evaluated accurately to avoid introducing significant error in estimating $C_{1}$. [The numerical procedure for determining $C_{1}$ is illustrated in Appendix C.]

Table 4.1 lists values of the constant $C_{1}$ for certain values of $A$. It is worth noting that, as the nonlinearity parameter $A$ is increased, $C_{1}$ increases quite more rapidly than linear theory would imply; for example, doubling the obstacle height can result in an order-ofmagnitude increase of the lee-wave amplitude. This suggests that the range of validity of the linear theory is limited.

We now turn to a discussion of results from fully numerical computation of lee waves. In the case of uniformly stratified flow, as noted earlier, the governing equation is linear. Hence, the equations for the modal amplitudes $a_{\tau}(x)$ in (4.26) are uncoupled and can be readily integrated without iteration. Figures 4.1 and $4.2(a)$, respectively, show plots of the computed lee-wave amplitude (defined as the amplitude of $a_{1}(x)$ as $x \rightarrow \infty$ ) against the dispersion parameter $\mu$ (keeping $A$ fixed) for the two obstacle shapes $f(x)=\operatorname{sech}^{2} x$ and $f(x)=\operatorname{sech} x$. Figure $4.2(b)$ shows the lee-wave phase shift, caused by nonlinearity, as a function of $\mu$ for $f(x)=\operatorname{sech} x$. In these plots, the predictions of the asymptotic theory (viz., (4.18) and (4.24) respectively) are put together for comparison. Also, to bring out the
effect of finite amplitude, several values of the parameter $A$ are considered and the results are normalized by the corresponding linear result (4.11).

According to Figures 4.1 and 4.2, there is reasonably good agreement between the asymptotic and numerical results for a wide range of values of $\mu$, well beyond the weakly nonlinear, nearly hydrostatic regime $(\epsilon, \mu \ll 1)$ where the theory is formally valid: in fact, the flows corresponding to the largest values of $\mu$ for which results are plotted in Figures 4.1 and 4.2, are quite nonlinear as regions of closed streamlines are about to appear (when this happens. Long's equation (4.3) is invalidated because not all streamlines originate far upstream). Moreover, as suggested by the asymptotic theory, the effect of finite amplitude results in significant amplification of the lee waves, even for values of $A$ as small as 0.1.

Comparing Figure $4.2(a)$ with Figure 4.1 , it is clear that, in the limit $\mu \rightarrow 0$, finiteamplitude effects are relatively stronger for $f=\operatorname{sech} x$; the ratio of the nonlinear to the linear lee-wave amplitude tends to $\infty$ as $\mu \rightarrow 0$ in Figure 4.2(a) because the value of $\alpha$ in (4.24) is always greater than 1 as noted in (4.27). The actual lee-wave amplitude, however, is negligibly small for both $f=\operatorname{sech}^{2} x$ and $f=\operatorname{sech} x$ in this limit, and the difference between these two obstacle profiles has no serious physical consequences.

On the other hand, when $\mu$ is not very small, the lee-wave amplitude can be substantial and, physically, the role of nonlinearity in enhancing the lee waves induced by both obstacle shapes is of far greater significance. This point is clearly demonstrated in Figure 4.3 which shows streamlines ${ }^{2}$ of the flow field induced by the obstacle $f(x)=\operatorname{sech}^{2} x$ for $\mu=0.3$, $A=0.4(\epsilon=0.036)$, as predicted by the linearized equations of motion (Figure 4.3(a)) and the full nonlinear equations (Figure 4.3(b)). Even though the value of $\epsilon$ is quite small, linear theory grossly underestimates the lee waves downstream. On the basis of the asymptotic theory, this dramatic difference can be attributed to the fact that, for $A=0.4$, the constant $C_{1}$ in (4.18) is about 7 times larger than the corresponding linear $(A \rightarrow 0)$ value (see

[^1]Table 4.1).
We also remark that, unlike their completely different behavior in the limit $\mu \rightarrow 0$, the lee waves induced by the two obstacle profiles considered here behave similarly for moderately small values of $\mu$, when the lee-wave amplitude is appreciable. To illustrate, for $\mu=0.3 \pi / 2=0.47, \epsilon=0.036(A=0.076)$, the profile $f(x)=\operatorname{sech} x$ closely resembles $f(x)=\operatorname{sech}^{2} x$ for $\mu=0.3, \epsilon=0.036(A=0.4)$-both have the same peak amplitude and enclose the same area. Under these conditions, the two obstacle profiles give rise to very similar lee-wave patterns; in both cases the nonlinear response is several times larger than the linear one, as illustrated in Figure 4.3 for $f(x)=\operatorname{sech}^{2} x$.

### 4.5.2 Linearly varying $\mathcal{N}^{2}$

In the case that $\mathcal{N}^{2}(z)$ varies linearly with depth $(b=1)$, it is necessary to solve the problems (4.7),(4.8) and (4.10) numerically; moreover, depending on the flow speed, the value of $\alpha$ in (4.24) can be less than 1.

From (4.10), we find $c_{1}=0.391, c_{2}=0.194$, and as a first choice of flow speed we take $F=0.25$. According to Grimshaw \& Smyth (1986), this value of $F$ is within the non-resonant range $c_{2}+\epsilon^{\frac{1}{2}} \Delta_{2}<F<c_{1}-\epsilon^{\frac{1}{2}} \Delta_{1}\left(\Delta_{1}=0.22, \Delta_{2}=0.03\right)$ for $\epsilon<0.2$. The eigenvalue problem (4.8) then yields $\kappa_{1}=14.4$ and one has $\alpha=1+0.12 \mathrm{~A}$ from (4.22). Computed values of the constant $C_{1}$ under these flow conditions are listed in Table 4.2, for the two obstacle shapes under consideration and certain values of the nonlinearity parameter $A$. As in the case of uniform stratification, $C_{1}$ increases with $A$ more rapidly than linear theory would predict so nonlinearity is expected to amplify the lee waves.

When the stratification is not uniform, there is an additional technical complication in computing finite-amplitude lee waves because the governing equation (4.3) is nonlinear and it becomes necessary to use iteration in solving for the modal amplitudes $a_{r}(x)$ in (4.26). Nevertheless, as shown in Figures 4.4 and 4.5 for $F=0.25$, the results are qualitatively similar to the case of constant $\mathcal{N}$ discussed above (see Figures 4.1, 4.2(a)). Again nonlinearity increases the lee-wave amplitude significantly, although not as dramatically as in the previous case, consistent with the asymptotic theory. Also, for $f(x)=\operatorname{sech}^{2} x$,
as $A$ is increased, the validity of the asymptotic results seems to be limited to a relatively narrow region close to $\mu=0$.

Finally, we consider the flow speed $F=0.3162\left(F^{2}=0.1\right)$ which also lies within the subcritical non-resonant range for $\epsilon<0.1$, say. For this value of $F$, (4.8) yields $\kappa_{1}=5.24$ and $\alpha=1-0.85 . A$ according to (4.22). Hence $\alpha<1$ and, even though the asymptotic analysis in $\S 4.3 .2$ that leads to (4.24) for $f(x)=\operatorname{sech} x$ is not strictly valid under these flow conditions (see Appendix B), we suspect that here nonlinearity could diminish the lee-wave amplitude. As shown in Figure 4.6, this hypothesis seems to be supported by numerical computations only when $\mu$ and $\epsilon$ are quite small and the lee-wave amplitude is negligible. For larger values of $\mu$ and $\epsilon$, the computed lee-wave amplitudes still are somewhat greater than those predicted by linear theory, but the effect of nonlinearity is not as pronounced as in the cases discussed earlier where $\alpha>1$.

### 4.6 Summary

We have studied the effect of nonlinearity on steady lee-wave patterns induced by subcritical stratified flow over smooth topography. For a given topography shape, the relative significance of nonlinearity in comparison with dispersion is measured by the parameter $A$ defined in (4.1), and the classical linear lee-wave theory is valid only when $A \rightarrow 0$. For $A=O(1)$, when weak nonlinear and dispersive effects are equally important, the amplitude of the induced lee wave, even though it is exponentially small, generally is determined by a fully nonlinear mechanism. In this regime, we have developed an asymptotic theory using asymptotics beyond all orders, which reveals that nonlinearity can enhance the lee-wave amplitude dramatically. We have also carried out numerical computations to confirm the predictions of the asymptotic theory. Our computations, in addition, indicate that the asymptotic results remain reasonably accurate for a wide range of flow conditions beyond the formal region of validity of the theory. Under such conditions, the (formally exponentially small) lee-wave amplitude is substantial and can be
significantly larger. sometimes by an order of magnitude, than the estimate obtained on the basis of linear lee-wave theory.

In the present study, we have considered steady flow, excluding the pussibility of upstream influence. We expect these conditions to hold as long as the flow speed is not close to critical and no wave breaking (density inversions) occurs during the transient developinent of the flow (Grimshaw \& Smyth 1986; Grimshaw \& Yi 1991; Lamb 1994). Furthermore, the inviscid model used here is expected to be relevant as long as no flow separation occurs. In spite of these limitations of the theory, the effect of nonlinearity on the induced lee waves is very pronounced even when the topography has small amplitude, and should be noticeable in experiments.

Finally, it would be worth exploring finite-amplitude effects on three-dimensional leewave patterns, induced by stratified flow over topography that depends on both the streamwise and the spanwise directions. In this case, a continuous distribution of wavenumbers are excited, and nonlinearity may influence certain portions of the wave pattern more seriously than others.

| $f(x)=\operatorname{sech}^{2} x$ |  |  | $f(x)=\operatorname{sech} x$ |  |
| :---: | :---: | :---: | :---: | :---: |
| $A$ | $C_{1}$ | $A$ | $C_{1}$ |  |
| $A \rightarrow 0$ | $2^{-\frac{1}{2}} \pi A$ | $A \rightarrow 0$ | $\frac{1}{4} \pi A$ |  |
| 0.2 | 1.3 | 0.1 | 0.20 |  |
| 0.4 | 6.3 | 0.2 | 0.99 |  |
| 0.8 | 52 |  |  |  |

Table 4.1 Computed values of the constant $C_{1}$ appearing in the asymptotic results (4.18) and (4.24) for certain values of the parameter $A$ in the case of uniformly stratified flow of a Boussinesq fluid ( $\mathcal{N}=1, \beta=0$ ) over the two obstacle shapes $f(x)=\operatorname{sech}^{2} x$ and $f(x)=\operatorname{sech} x$. The Froude number $F=0.2903$.


Table 4.2 Computed values of the constant $C_{1}$ appearing in the asymptotic results (4.18) and (4.24) for certain values of the parameter $A$ in the case of stratified flow of a Boussinesq fluid $(\beta=0)$ with $\mathcal{N}^{2}=1+z$ over the two obstacle shapes $f(x)=\operatorname{sech}^{2} x$ and $f(x)=\operatorname{sech} x$. The Froude number $F=0.25$.


Figure 4.1 The lee-wave amplitude (normalized with the result of linear theory (4.11)) as a function of the dispersion parameter $\mu$ for uniformly stratified flow of a Boussinesq fluid $(V=1,3=0)$ over the obstacle $f(x)=\operatorname{sech}^{2} x$, and for certain values of the parameter $A$; (——): asymptotic results (4.18); (o): numerical results. The Froude number $F=0.2903$.
(a)


Figure $4.2 a$ (for caption, see following page)
(b)


Figure 4.2 Asymptotic (—) and numerical ( 0 ) results as a function of the dispersion parameter $\mu$ for lee waves induced by uniformly stratified flow of a Boussinesq fluid $\left(. V^{\prime}=1, \beta=0\right)$ over the obstacle $f(x)=\operatorname{sech} x$, and for certain values of the parameter $A$; (a) the lee-wave amplitude (normalized with the result of linear theory (4.11)); (b) the lee-wave phase shift. The Froude number $F=0.2903$.

(a)
2

Figure 4.3 Streamlines of uniformly stratified flow of a Bonssinesy fluid $(\mathcal{N}=1, \beta=0)$
(b)
2


Figure 4.4 The lee-wave amplitude (normalized with the result of linear theory (4.11)) as a function of the dispersion parameter $\mu$ for stratified flow of a Boussinesq fluid ( $\beta=0$ ) with $\mathcal{V}^{-2}=1+z$ over the obstacle $f(x)=\operatorname{sech}^{2} x$, and for certain values of the parameter $A$; (-) : asymptotic results (4.18); (o): numerical results. The Froude number $F=0.25$.


Figire 4.5 The lee-wave amplitude (normalized with the result of linear theory (4.11)) as a function of the dispersion parameter $\mu$ for stratified flow of a Boussinesq fluid ( $\beta=0$ ) with $.^{-2}=1+z$ over the obstacle $f(x)=\operatorname{sech} x$, and for certain values of the parameter $A$ : $(-)$ : asymptotic results (4.24); (o): numerical results. The Froude number $F=0.2 \overline{5}$.


Figure 4.6 Numerically computed values of the lee-wave amplitude (normalized with the result of linear theory (4.11)) as a function of the dispersion parameter $\mu$ for stratified flow of a Boussinesq fluid ( $\beta=0$ ) with $\mathcal{N}^{2}=1+z$ over the obstacle $f(x)=\operatorname{sech} x$, and for certain values of the parameter $A$. The Froude number $F=0.3162$.

## CHAPTER 5

## WEAKLY NON-LOCAL GRAVITY-CAPILLARY SOLITARY WAVES

There is evidence from previous analytical work based on the fifth-order KdV equation and from numerical computations that gravity-capillary solitary waves in a liquid layer are non-local-they feature oscillatory tails of constant amplitude-when the Bond number $\tau$ is less than $1 / 3$. Here, the full gravity-capillary wave problem is examined and these tails are calculated asymptotically in the weakly nonlinear regime. For given values of $\tau$ ( $0<\tau<1 / 3$ ) and Froude number $F$ slightly greater than 1 , there exists a one-parameter family of weakly non-local solitary waves characterized by the phase shift of the tails relative to the main peak. The tail amplitude depends on the phase shift and is exponentially small with respect to the wave peak amplitude. Predictions of the asymptotic theory are confirmed by numerical computations using a spectral method.

### 5.1 Introduction

In the course of computing periodic waves on the surface of a fluid of finite depth, Hunter \& Vanden-Broeck (1983) noted that, in the presence of weak surface tension (Bond number $\tau<1 / 3$ ), gravity-capillary solitary waves are not entirely locally confined: as the wavelength is increased, periodic-wave profiles approach a solitary-like main hump that is accompanied by capillary ripples forming oscillatory tails of small, but constant, amplitude. In more recent numerical work, Vanden-Broeck (1991) confirmed these results by computing periodic waves of longer wavelength that approximate non-local gravity-capillary solitary waves more closely. For a given value of $\tau(0<\tau<1 / 3)$, there is in fact a two-parameter family of elevation solitary waves with oscillatory tails; moving along each solution branch, the tail amplitude varies considerably and its minimum value is found to be equal to zero, at least within numerical accuracy. These numerical results are also supported by existence proofs of solitary waves with capillary ripples at infinity (Beale 1991; Sun 1991).

On the other hand, the Korteweg-de Vries (KdV) equation predicts that. for $0<\tau<1 / 3$, there exist elevation gravity-capillary solitary waves free of tails. This discrepancy is explained by recalling that the KdV equation is a model equation for weakly nonlinear, long waves; it is derived from the full water-wave problem via a long-wave expansion and cannot account for the presence of short-scale ripples at the solitary-wave tails.

A simple way to understand how non-local solitary waves arise is by perturbing the KdV equation with a small fifth-order-derivative term representing higher-order dispersive effects (Hunter \& Scheurle 1988). The dispersion relation of this fifth-order KdV equation allows relatively short waves that propagate with the same phase speed as a KdV solitary wave and form short-scale oscillatory tails, similar to the capillary ripples noted by Hunter \& Vanden-Broeck (1991). However, when the higher-order dispersive term of the fifth-order KdV equation goes to zero, the amplitude of the oscillations at the solitary-wave tails turns out to be exponentially small with respect to the solitary-wave peak amplitude. In this limit, it is necessary to use techniques of exponential asymptotics in order to calculate the tail amplitude (see Segur, Tanveer \& Levine 1991 for a collection of recent articles on exponential asymptotics). Specifically, Pomeau, Ramani \& Grammaticos (1988) followed a nonlinear WKB method using asymptotic matching in the complex plane combined with Borel summation. Through a similar approach, Grimshaw \& Joshi (1995) pointed out that symmetric non-local solitary waves of the fifth-order KdV equation form a one-parameter solution family characterized by the phase shift of the oscillations at the tails; the tail amplitude is a function of the phase shift and cannot be made to vanish, consisient with the numerical results of Boyd (1991).

In calculating the amplitude of the solitary-wave tails asymptotically, it is interesting that all dispersive and nonlinear terms of the fifth-order KdV equation are equally important. In the WKB approach (Pomeau et al. 1988; Grimshaw \& Joshi 1995), this becomes apparent when one considers the 'inner' problem, near each of the singularities of the regular perturbation expansion in the complex plane, where all these terms enter at
the same level. Equivalently, in the wavenumber-domain approach presented in Chapter 2. the tail amplitude is determined by the residues of the poles on the real axis of the Fourier transform of the wave profile, and all dispersive and nonlinear terms contribute to these residues.

The fact that the tails of non-local solitary waves are affected by dispersive and nonlinear terms of all orders has a direct bearing on gravity-capillary solitary waves: the tail amplitude cannot be determined accurately using model equations, such as the fifth-order KdV equation, that are truncations of the full water-wave problem. This fully nonlinear mechanism is also important in other physical problems that involve long waves interacting with short-scale oscillations; examples include non-local internal solitary waves of mode greater than one (Akylas \& Grimshaw 1992) and short-scale lee waves in subcritical stratified flow over extended topography (see Chapter 4).

In light of the remarks made above, we shall calculate the oscillatory tails of smallamplitude gravity-capillary solitary waves on the basis of the full water-wave equations. Compared with the analysis of the fifth-order KdV equation (Pomeau et al. 1988; Grimshaw \& Joshi 1995: see also §2.5), our task is complicated by the free-surface nonlinearity which has to be retained to all orders and by the dependence of the flow on the vertical direction. While it is possible to handle these difficulties by a matching procedure in the complex plane, we find it most economical to avoid matching by working in the wavenumber domain following Chapter 2.

For given values of $\tau(0<\tau<1 / 3)$ and Froude number $F$ slightly greater than 1, there exists a one-parameter family of weakly non-local symmetric solitary waves, consistent with the previous analytical and numerical work. The tail amplitude depends on the phase shift of the oscillations at the tails relative to the main solitary-wave peak; the minimum value of the amplitude, however, cannot be made zero. The predictions of the asymptotic analysis are confirmed by computing weakly non-local solitary waves numerically. For this purpose, a numerical technique that can capture the exponentially small amplitude of the tails is used, adapting to the water-wave problem the spectral method devised by Boyd (1991) for
the fifth-order KdV equation.

### 5.2 Asymptotic analysis: overview

Consider the classical problem of gravity-capillary waves on water of finite uniform depth $h$. As we are interested in traveling waves of permanent form, we choose a reference frame moving with the wave speed $c$ so the flow is steady. Dimensionless variables are used throughout. Consistent with the standard long-wave theory (see, for example, Whitham 1974), we scale the horizontal ( $x$ ) coordinate with $l$, the characteristic width of the solitarywave core, while the vertical $(z)$ coordinate is scaled with $h$, and we use the linear-long-wave speed $c_{0}=\sqrt{g h}$ ( $g$ is the gravitational acceleration) as a characteristic velocity. This defines two parameters, the Froude number $F=c / c_{0}$ and the long-wave parameter $\epsilon=h / l$.

In terms of the velocity potential $-F x+\phi$, where $\phi(x, z)$ describes the wave disturbance, and the free-surface elevation $z=\eta(x)$, the problem reads (in dimensionless variables)

$$
\begin{gather*}
\epsilon^{2} \phi_{x x}+\phi_{z z}=0 \quad(-\infty<x<\infty,-1 \leq z \leq \eta),  \tag{5.1}\\
-F \eta_{x}+\phi_{x} \eta_{x}=\frac{\phi_{z}}{\epsilon^{2}} \quad(z=\eta),  \tag{5.2}\\
-F \phi_{x}+\eta+\frac{1}{2}\left(\phi_{x}^{2}+\frac{1}{\epsilon^{2}} \phi_{z}^{2}\right)=\tau \epsilon^{2} \frac{\eta_{x x}}{\left(1+\epsilon^{2} \eta_{x}^{2}\right)^{3 / 2}} \quad(z=\eta),  \tag{5.3}\\
\phi_{z}=0 \quad(z=-1) . \tag{5.4}
\end{gather*}
$$

Here $\tau=T /\left(\rho g h^{2}\right)$ is the Bond number, $T$ being the coefficient of surface tension and $\rho$ the fluid density.

The KdV equation is based on the assumptions that the horizontal lengthscale of the disturbance is long relative to the fluid depth so $\epsilon \ll 1$ and that the disturbance amplitude is weak. To balance dispersive with nonlinear effects, $\phi$ and $\eta$ are taken to be $O\left(\epsilon^{2}\right)$ and are expanded as

$$
\begin{align*}
\phi & =\epsilon^{2}\left(\phi_{0}+\epsilon^{2} \phi_{1}+\cdots\right),  \tag{5.5a}\\
\eta & =\epsilon^{2}\left(\eta_{0}+\epsilon^{2} \eta_{1}+\cdots\right) . \tag{5.5b}
\end{align*}
$$

Also, consistent with these assumptions, the wave speed is close to the linear-long-wave speed:

$$
\begin{equation*}
F=1+\epsilon^{2} F_{1}+\cdots . \tag{5.5c}
\end{equation*}
$$

Substituting expansions (5.5) into (5.1)-(5.4), it follows that

$$
\phi_{0 x}=\eta_{0}
$$

and $\eta_{0}$ satisfies the steady KdV equation

$$
2 F_{1} \eta_{0 x}-3 \eta_{0} \eta_{0 x}+\left(\tau-\frac{1}{3}\right) \eta_{0 x x x}=0
$$

The KdV solitary wave is given by

$$
\begin{equation*}
\eta_{0}=a \operatorname{sech}^{2} x \tag{5.6}
\end{equation*}
$$

with

$$
a=2 F_{1}=4\left(\frac{1}{3}-\tau\right) .
$$

For $0 \leq \tau<1 / 3$, this is a locally confined wave of elevation ( $a>0$ ) moving at slightly supercritical speed ( $F>1$ ).

One may proceed to determine higher-order terms in expansions (5.5). These corrections to the KdV solitary wave remain locally confined to all orders, however, and give no evidence of oscillatory tails. The reason is that the tail amplitude is exponentially small with respect to $\epsilon$, so the tails cannot be captured by the standard long-wave expansions (5.5).

To remedy this difficulty, rather than matched asymptotics in the complex plane (Pomeau et al. 1988; Grimshaw \& Joshi 1995; Akylas \& Grimshaw 1992), we shall follow the wavenumber-domain approach presented in Chapter 2. Our approach makes use of the fact that, owing to the presence of oscillatory tails, the Fourier transform of the wave profile,

$$
\widehat{\eta}(k)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \eta(x) e^{-\mathrm{i} k x} \mathrm{~d} x,
$$

is expected to have simple-pole singularities at $k= \pm k_{*}$, where $k_{*}$ is the wavenumber of the ripples that form the tails. Since the tails must have the same phase speed as the solitarywave core and their amplitude is small, $k_{*}$ may be deduced from the linear gravity-capillary
dispersion relation:

$$
\begin{equation*}
F^{2} \epsilon k_{*}=\left(1+\tau \epsilon^{2} k_{*}^{2}\right) \tanh \epsilon k_{*} \tag{5.7}
\end{equation*}
$$

In particular, in the small-amplitude limit when $F$ is close to $1, k_{*} \sim k_{0}$ with

$$
\begin{equation*}
\epsilon k_{0}=\left(1+\tau \epsilon^{2} k_{0}^{2}\right) \tanh \epsilon k_{0} \tag{5.8}
\end{equation*}
$$

Note that (5.8) has nonzero roots, and hence oscillatory tails are possible, only in the low-surface-tension regime. $0<\tau<1 / 3$. The goal then is to calculate the residues of $\hat{\eta}(k)$ at the poles $k= \pm k_{0}$ : once these residues are known, the tails can be readily determined by inverting the Fourier transform along a suitably chosen path ( $\S 5.4$ ).

As noted above, the long-wave expansions (5.5) do not develop non-uniformities on the real axis that would hint at the presence of tails. On the other hand, this is not the case for the corresponding expansions in the wavenumber domain. Specifically, from (5.6),

$$
\widehat{\eta}_{0}=\frac{1}{2} a k \operatorname{csch} \frac{1}{2} \pi k
$$

and one may show that, in the limit $|k| \rightarrow \infty$,

$$
\hat{\eta}_{n}=O\left(k^{2 n+1} \exp \left(-\frac{1}{2} \pi|k|\right)\right) \quad(n=1,2, \ldots)
$$

Hence, expansion ( $5.5 b$ ) becomes disordered in the wavenumber domain when $\epsilon k=O(1)$ and the same is true for ( $5.5 a$ ). This suggests the two-scale expansions

$$
\begin{gather*}
\widehat{\eta}(k) \sim \epsilon \operatorname{csch} \frac{1}{2} \pi k A(\kappa)+\cdots,  \tag{5.9a}\\
\hat{\phi}(k, z) \sim-\mathrm{i} \epsilon^{2} \operatorname{csch} \frac{1}{2} \pi k Q(\kappa, z)+\cdots, \tag{5.9b}
\end{gather*}
$$

in terms of the scaled wavenumber variable $\kappa=\epsilon \boldsymbol{k}$, with

$$
\begin{array}{ll}
A(\kappa) \sim \frac{1}{2} a \kappa+\cdots & (\kappa \rightarrow 0), \\
Q(\kappa, z) \sim \frac{1}{2} a+\cdots & (\kappa \rightarrow 0) \tag{5.10b}
\end{array}
$$

In view of (5.9), the simple-pole singularities of $\hat{\eta}(k)$ and $\widehat{\phi}(k, z)$ that are expected to occur on the real $k$-axis at the wavenumber of the tail oscillations $k= \pm k$ translate into simple poles of $A(\kappa)$ and $Q(\kappa, z)$ at $\kappa= \pm \kappa * \approx \pm \kappa_{0}$ on the real $\kappa$-axis; for the purpose of determining the solitary-wave tails, we shall now compute the residues of these poles.

### 5.3 Pole residues

The residues of the poles at $\kappa= \pm \kappa$. are obtained by first deriving (approximate) equations for $A(\kappa)$ and $Q(\kappa, z)$ and then seeking solutions to these equations in the form of power series in the $\kappa$-plane.

We begin by transforming the governing equations (5.1)-(5.4) into the wavenumber domain, taking Fourier transform formally. Laplace's equation (5.1) and the bottom boundary condition (5.4) are linear and present no difficulty. The free-surface conditions (5.2). (5.3) are nonlinear. however, and when expanded in Taylor series about the undisturbed water level $z=0$. they give rise to infinite series of multiple convolution integrals in the wavenumber domain; all these terms need to be retained in computing the pole residues.

Substituting then the proposed expansions (5.9) into the equations governing $\bar{\eta}(k)$ and $\hat{o}(k . z)$. it is found (see Appendix D for details) that, to leading order. $A(\kappa)$ and $Q(\kappa . z)$ satisfy the following integral-differential boundary-value problem:

$$
\begin{gather*}
Q_{z z}-\kappa^{2} Q=0 \quad(-1 \leq z \leq 0)  \tag{5.11}\\
\kappa . A-Q_{z}(\kappa .0)=\sum_{j=1}^{\infty} \frac{2^{j}}{j^{\prime}} \int_{0}^{\kappa} \mathrm{d} \lambda A^{(0)}(\kappa-\lambda) \frac{\partial^{j+1} Q}{\partial z^{j+1}}(\lambda, 0) \\
+\sum_{j=0}^{\infty} \frac{2^{j+1}}{(j+1)!} \int_{0}^{\kappa} \mathrm{d} \lambda(\kappa-\lambda) \lambda .4^{(0+1)}(\kappa-\lambda) \frac{\partial^{\prime} Q}{\partial z^{j}}(\lambda, 0) .  \tag{5.12}\\
\left(1+\tau \kappa^{2}\right) A-\kappa Q(\kappa, 0)= \\
\int_{0}^{\kappa} \mathrm{d} \lambda\left\{Q_{z}(\kappa-\lambda, 0) Q_{z}(\lambda, 0)-\lambda(\kappa-\lambda) Q(\kappa-\lambda, 0) Q(\lambda, 0)\right\} \\
+\sum_{j=1}^{\infty}\left\{\frac{2^{j}}{j^{j}} \int_{0}^{\kappa} \mathrm{d} \lambda . A^{(\jmath)}(\kappa-\lambda) \lambda \frac{\partial^{J} Q}{\partial z^{j}}(\lambda, 0)\right. \\
+2^{j} \sum_{m=0}^{j} \frac{1}{m!(j-m)!} \int_{0}^{\kappa} \mathrm{d} \lambda\left[A^{(j)}(\kappa-\lambda)\right.  \tag{5.13}\\
\times\left(\int_{0}^{\lambda} \mathrm{d} \lambda_{1} \frac{\partial^{m+1} Q}{\partial z^{m+1}}\left(\lambda-\lambda_{1}, 0\right) \frac{\partial^{j-m+1} Q}{\partial z^{j-m+1}}\left(\lambda_{1}, 0\right)\right. \\
\left.\left.\left.-\int_{0}^{\lambda} \mathrm{d} \lambda_{1}\left(\lambda-\lambda_{1}\right) \lambda_{1} \frac{\partial^{m} Q}{\partial z^{m}}\left(\lambda-\lambda_{1}, 0\right) \frac{\partial^{j-m} Q}{\partial z^{j-m}}\left(\lambda_{1}, 0\right)\right)\right]\right\} \\
-\tau \sum_{j=1}^{\infty} \frac{2^{j}}{j!}(2 j+1)!!\int_{0}^{\kappa} \mathrm{d} \lambda(\kappa-\lambda)^{2} A(\kappa-\lambda) B^{(2 j)}(\lambda),
\end{gather*}
$$

$$
\begin{equation*}
Q_{z}=0 \quad(z=-1) \tag{5.14}
\end{equation*}
$$

In expressing the multiple convolution sums above that derive from the nonlinear terms in the free-surface conditions, we have used the following short-hand notation

$$
\begin{array}{rlrl}
A^{(j)}(\kappa) & =\int_{0}^{\kappa} \mathrm{d} \lambda A(\kappa-\lambda) A^{(j-1)}(\lambda), & A^{(1)}(\kappa) \equiv A(\kappa) ; \\
B^{(j)}(\kappa) & =\int_{0}^{\kappa} \mathrm{d} \lambda B^{(1)}(\kappa-\lambda) B^{(j-1)}(\lambda), & B^{(1)}(\kappa) \equiv \kappa A(\kappa) ; \\
(2 j+1)!! & =1 \cdot 3 \cdot 5 \ldots(2 j+1) & &
\end{array}
$$

Keeping in mind that $A(\kappa), Q(\kappa, z)$ are expected to have simple-pole singularities at $\kappa= \pm \kappa_{0}$ ( $\kappa_{*}=\kappa_{0}$ to leading order in $\epsilon$ according to (5.7), (5.8)), the solution of the problem (5.11)-(5.14) is posed as power series in $\kappa$ :

$$
\begin{align*}
A & =\sum_{r=0}^{\infty} a_{r} \frac{\kappa^{2 r+1}}{\kappa_{0}^{2 r+2}}  \tag{5.15a}\\
Q & =\sum_{r=0}^{\infty} q_{r}(z) \frac{\kappa^{2 r}}{\kappa_{0}^{2 r+1}} \tag{5.15b}
\end{align*}
$$

with $a_{0}=a \kappa_{0}^{2} / 2$ and $q_{0}=a \kappa_{0} / 2$, consistent with the matching conditions (5.10). Substituting these power series into the governing equations (5.11)-(5.14), after some algebra, it is found that $q_{r}(z)$ and $a_{r}(r \geq 1)$ are determined from the sequence of boundaryvalue problems

$$
\begin{gather*}
\frac{\mathrm{d}^{2} q_{r}}{\mathrm{~d} z^{2}}=\kappa_{0}^{2} q_{r-1} \quad(-1 \leq z \leq 0)  \tag{5.16}\\
\frac{\mathrm{d} q_{r}}{\mathrm{~d} z}=\kappa_{0} a_{r-1}-P_{r} \quad(z=0)  \tag{5.17a}\\
\frac{\mathrm{d} q_{r}}{\mathrm{~d} z}=0 \quad(z=-1) \tag{5.17b}
\end{gather*}
$$

and

$$
\begin{equation*}
a_{r}+\tau \kappa_{0}^{2} a_{r-1}=\kappa_{0} q_{r}-S_{r} \quad(z=0) \tag{5.18}
\end{equation*}
$$

where $P_{r}$ and $S_{r}$ are complicated convolution sums involving $q_{0}, \ldots, q_{r-1}$ and $a_{0}, \ldots, a_{r-2}$. The explicit expressions for $P_{r}$ and $S_{r}$ are given in Appendix D.

The location of the singularities of $A(\kappa)$ and $Q(\kappa, z)$ (closest to the origin) in the $\kappa$-plane determines the radius of convergence of the series (5.15). The convergence of these series in turn is controlled by the behavior of the coefficients $a_{r}$ and $q_{r}(z)$ as $r \rightarrow \infty$. However,
in this limit, $P_{r}$ and $S_{r}$ become subdominant and (5.16)-(5.18) can be readily solved for $a_{r}$ and $q_{r}(z)$. Specifically, taking into account the kinematic constraint ( 5.8 ) that defines $\kappa_{0}$. one has

$$
a_{r} \sim a_{\infty}, \quad q_{r}(z) \sim a_{\infty} \frac{\cosh \kappa_{0}(z+1)}{\sinh \kappa_{0}} \quad(r \rightarrow \infty)
$$

where $a_{\infty}$ is a constant. Returning then to the power-series solutions (5.15), it is verified that $A(\kappa)$ and $Q(\kappa, z)$ indeed have simple-pole singularities at $\kappa= \pm \kappa_{0}$ and the corresponding residues are known in terms of $a_{\infty}$ :
with

$$
\begin{gather*}
A \sim \frac{C}{\kappa \mp \kappa_{0}}, \quad Q \sim \pm \frac{C f_{0}(z)}{\kappa \mp \kappa_{0}} \quad\left(\kappa \rightarrow \pm \kappa_{0}\right) .  \tag{5.19}\\
f_{0}(z)=\frac{\cosh \kappa_{0}(z+1)}{\sinh \kappa_{0}}
\end{gather*}
$$

and $C=-a_{\infty} / 2$. The constant $C$. however, cannot be determined by asymptotic analysis alone; to compute $C$, it is necessary to solve the infinite sequence of boundaryvalue problems (5.16)-(5.18) numerically (see §5.6).

Finally, combining (5.9) with (5.19), the residues of the poles of $\hat{\eta}(k)$ and $\widehat{\phi}(k, z)$ at $k \sim \pm k_{0}$ are known

$$
\begin{array}{cc}
\widehat{\eta}(k) \sim \pm 2 C \exp \left(-\frac{1}{2} \pi k_{0}\right) \frac{1}{k \mp k_{0}} & \left(k \rightarrow \pm k_{0}\right), \\
\hat{\phi}(k, z) \sim-2 \mathrm{i} \epsilon C \exp \left(-\frac{1}{2} \pi k_{0}\right) \frac{f_{0}(z)}{k \mp k_{0}} & \left(k \rightarrow \pm k_{0}\right) . \tag{5.20b}
\end{array}
$$

### 5.4 Solitary-wave tails

Having computed the pole residues at $k \sim \pm k_{0}$, we are now prepared to discuss the tails of non-local gravity-capillary solitary waves. We shall consider waves that are symmetric about $x=0, \eta(x)=\eta(-x)$; such waves can have permanent form on energetic grounds (Grimshaw \& Joshi 1995; Akylas \& Grimshaw 1992), and are consistent with previous numerical work (Hunter \& Vanden-Broeck 1983; Vanden-Broeck 1991; Boyd 1991).

To ensure symmetry, the inverse Fourier transform is taken as

$$
\eta(x) \sim \frac{1}{2} \int_{\mathcal{C}^{+}} \hat{\eta}(k) e^{\mathrm{i} k x} \mathrm{~d} k+\frac{1}{2} \int_{\mathcal{C}^{-}} \widehat{\eta}(k) e^{\mathrm{i} k x} \mathrm{~d} k,
$$

where the contour $\mathcal{C}^{+}\left(\mathcal{C}^{-}\right)$extends from $k=-\infty$ to $\infty$ and passes above (below) the poles at $k \sim \pm k_{0}$. The contributions from these poles then amount to symmetric oscillatory tails:

$$
\begin{equation*}
\eta(x) \sim-4 \pi C \exp \left(-\frac{1}{2} \pi k_{0}\right) \sin k_{0}|x| \quad(x \rightarrow \pm \infty) \tag{5.21a}
\end{equation*}
$$

and, similarly,

$$
\begin{equation*}
\phi(x, z) \sim \pm 4 \pi \epsilon C \exp \left(-\frac{1}{2} \pi k_{0}\right) f_{0}(z) \cos k_{0} x \quad(x \rightarrow \pm \infty) \tag{5.21b}
\end{equation*}
$$

Recall that $k_{0}=\kappa_{0} / \epsilon(0<\epsilon \ll 1)$ so the tail amplitude is exponentially small, as expected.
It is important to note that one may superpose an arbitrary (but exponentially small) component of $\cos k_{0} x$ to the asymptotic expression (5.21a) without violating the symmetry property $\eta(x)=\eta(-x)$. This results in a phase shift, $\chi$, of the tail oscillations relative to the main solitary-wave peak:

$$
\begin{equation*}
\eta(x) \sim-\frac{4 \pi C}{\cos \chi} \exp \left(-\frac{\pi \kappa_{0}}{2 \epsilon}\right) \sin \left(\frac{\kappa_{0}}{\epsilon}|x|+\chi\right) \quad(x \rightarrow \pm \infty) \tag{5.22a}
\end{equation*}
$$

and, similarly,

$$
\begin{equation*}
\phi(x, z) \sim \pm \frac{4 \pi \epsilon C}{\cos \chi} \exp \left(-\frac{\pi \kappa_{0}}{2 \epsilon}\right) f_{0}(z) \cos \left(\frac{\kappa_{0}}{\epsilon}|x|+\chi\right) \quad(x \rightarrow \pm \infty) \tag{5.22b}
\end{equation*}
$$

Hence, there is a one-parameter family of symmetric non-local solitary waves characterized by the phase shift $\chi$. The minimum amplitude of the oscillatory tails occurs when $\chi=0$, and the tail amplitude can become quite large as $\chi$ approaches $\pi / 2$. A similar resonance phenomenon was also noted by Boyd (1991) in his numerical study of the fifth-order KdV equation.

For the purpose of comparing the predictions of the asymptotic analysis with numerical results (see §5.6), it will prove useful to obtain higher-order corrections to expressions ( $\mathbf{5} .22$ ) for the oscillatory tails. Following Grimshaw \& Joshi (1995), we may refine (5.22) by
looking for a WKB-type solution of (5.1)-(5.4) in the form of a small-amplitude short-scale wavetrain riding on the solitary-wave core:

$$
\begin{align*}
\eta & \approx \epsilon^{2} \eta_{0}(x)+\left\{\xi(x) e^{\mathrm{i} \kappa . x / \epsilon}+\text { c.c. }\right\}  \tag{5.23a}\\
\phi & \approx \epsilon^{2} \phi_{0}(x)+\left\{\psi(x, z) e^{\mathrm{i} \kappa . x / \epsilon}+\text { c.c. }\right\} \tag{5.23b}
\end{align*}
$$

In the above expressions, the solitary-wave core is approximated by the KdV solitary wave (5.6) and the wavenumber of the short wave is $k_{*}=\kappa_{*} / \epsilon$ according to the exact resonance condition (5.7).

Substituting (5.23) into (5.1)-(5.4), linearizing about $\epsilon^{2} \phi_{0}$ and $\epsilon^{2} \eta_{0}$, and expanding $\xi, \psi$ as

$$
\begin{gather*}
\xi=\left\{\xi_{0}(x)+\epsilon^{2} \xi_{1}(x)+\cdots\right\} \exp \left\{\mathrm{i} \epsilon\left[\theta_{0}(x)+\epsilon^{2} \theta_{1}(x)+\cdots\right]\right\},  \tag{5.24a}\\
\psi=\left\{\psi_{0}(x, z)+\epsilon^{2} \psi_{1}(x, z)+\cdots\right\} \exp \left\{i \epsilon\left[\theta_{0}(x)+\epsilon^{2} \theta_{1}(x)+\cdots\right]\right\}, \tag{5.24b}
\end{gather*}
$$

it is found that $\xi_{0}$ is constant while
with

$$
\begin{gathered}
\psi_{0}=-\mathrm{i} \epsilon F \xi_{0} f_{*}(z) \\
f_{*}(z)=\frac{\cosh \kappa_{*}(z+1)}{\sinh \kappa_{*}}
\end{gathered}
$$

Furthermore,

$$
\frac{\mathrm{d} \theta_{0}}{\mathrm{~d} x}=-D \eta_{0}
$$

where

$$
D=\frac{\kappa_{*}^{2} F^{2}\left(\operatorname{coth} \kappa_{*}-\tanh \kappa_{*}\right)+2 \kappa_{*} F}{\kappa_{*} F^{2}\left(\operatorname{coth} \kappa_{*}-\tanh \kappa_{*}\right)+2 \tau \kappa_{*} \tanh \kappa_{*}-F^{2}} .
$$

Hence,

$$
\begin{equation*}
\theta_{0}=-a D \tanh x \tag{5.25}
\end{equation*}
$$

Combining then (5.23) with (5.24), taking into account (5.25), the leading-order expressions (5.22) for the solitary-wave tails are replaced with

$$
\begin{gather*}
\eta(x) \sim-\frac{4 \pi C}{\cos \chi} \exp \left(-\frac{\pi \kappa_{*}}{2 \epsilon}\right) \sin \left(\frac{\kappa_{*}}{\epsilon}|x|+\chi-\epsilon a D\right) \quad(x \rightarrow \pm \infty),  \tag{5.26a}\\
\phi(x, z) \sim \pm \frac{4 \pi \epsilon C}{\cos \chi} \exp \left(-\frac{\pi \kappa_{*}}{2 \epsilon}\right) f_{*}(z) \cos \left(\frac{\kappa_{*}}{\epsilon}|x|+\chi-\epsilon a D\right) \quad(x \rightarrow \pm \infty) \tag{5.26b}
\end{gather*}
$$

Apart from using the more accurate value $k_{*}=\kappa_{*} / \epsilon$ rather than $k_{0}$ for the tail wavenumber, the improved expressions (5.26) account for an $O(\epsilon)$ phase shift that the tail oscillations experience as a result of their modulation by the solitary-wave core. Accordingly. the tail amplitude is now expected to reach its maximum value when the (total) phase shift of the tail oscillations relative to the main peak approaches $\pi / 2-\epsilon a D$.

### 5.5 Numerical solution

As shown above, weakly nonlinear gravity-capillary solitary waves feature oscillatory tails with exponentially small amplitude. Therefore, computation of these tails in the regime that the asymptotic results (5.26) are valid demands numerical methods with exceedingly high accuracy. For this purpose, we shall adapt to the gravity-capillary water-wave problem (5.1)-(5.4) the spectral method devised by Boyd (1991) for computing non-local solitary waves of the fifth-order KdV equation.

The solution of Laplace's equation (5.1) consistent with the bottom condition (5.4) is expressed as a power series

$$
\begin{equation*}
\phi=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2 n)!} \epsilon^{2 n} \frac{\mathrm{~d}^{2 n} G}{\mathrm{~d} x^{2 n}}(z+1)^{2 n} \tag{5.27}
\end{equation*}
$$

where $G(x)$ is a new unknown replacing $\phi(x, z)$. Thus, the vertical structure of the flow is determined formally, and substituting (5.27) into the free-surface conditions (5.2) and (5.3) yields two coupled nonlinear ordinary differential equations governing $\mathrm{d} G / \mathrm{d} x$ and $\eta$. These two equations are exact and form the basis of our numerical procedure for computing non-local solitary waves.

For symmetric waves, both $\mathrm{d} G / \mathrm{d} x$ and $\eta$ are even functions of $x$ so it suffices to solve only in $x \geq 0$. Following Boyd (1991), $\eta$ and $\mathrm{d} G / \mathrm{d} x$ are approximated by

$$
\begin{aligned}
\eta & =\sum_{m=1}^{M-1} a_{m} T_{2 m}(x)+a_{M} \phi_{\mathrm{rad}}(x) \\
\frac{\mathrm{d} G}{\mathrm{~d} x} & =\sum_{m=1}^{M-1} b_{m} T_{2 m}(x)+b_{M} \phi_{\mathrm{rad}}(x)
\end{aligned}
$$

where

$$
T_{2 m}(x)=\cos \left(2 m \cot ^{-1} x\right)-1
$$

are even Chebyshev polynomials, and

$$
\phi_{\mathrm{rad}}=\frac{1}{2}(1+\tanh x) \sin \left(\frac{\kappa_{*}}{\epsilon} x+\theta\right)-\frac{1}{2}(1-\tanh x) \sin \left(\frac{\kappa_{*}}{\epsilon} x-\theta\right)
$$

is a radiation basis function used to approximate the oscillatory tails, $\theta$ being a phase constant that one can specify. Note that the Chebyshev polynomials are locally confined, $T_{2 m}(x) \rightarrow 0$ as $x \rightarrow \pm \infty$, and the choice of $\phi_{\mathrm{rad}}$ is guided by the asymptotic expressions (5.26) for the tails (the factors $1 \pm \tanh x$ are smoothed step functions).

In terms of these spectral expansions, the equations governing $\eta$ and $\mathrm{d} G / \mathrm{d} x$ are then applied at the $M$ collocation points

$$
x_{m}=\cot \frac{(2 m-1) \pi}{4 M} \quad(m=1,2, \cdots, M)
$$

Thus, a system of $2 M$ nonlinear algebraic equations for the $2 M$ unknowns $a_{m}, b_{m}$ ( $m=1, \cdots, M$ ) is obtained, and is solved by Newton's method.

In implementing the spectral method outlined above, expansion (5.27) is truncated to a finite number of terms $N$, say. As $N$ increases, however, it becomes necessary to evaluate derivatives of the basis functions to very high orders and, in order to retain accuracy, this task is carried out analytically with the help of the symbolic manipulator Maple V. The numerical results reported in the next section are obtained using $N=130$ and $M=20$; this resolution allows us to compute with confidence oscillatory tails with $a_{M}, b_{M} \geq O\left(10^{-13}\right)$.

### 5.6 Results and discussion

Before making a comparison between numerical and asymptotic results, it is necessary to determine the constant $C$ that enters in the asymptotic expressions (5.26). As noted in $\S 5.3, C$ can be computed numerically from the sequence of boundary-value problems (5.16)-(5.18). However, the expressions for $P_{r}$ and $S_{r}$ (see Appendix D) on the righthand sides of (5.17a) and (5.18), respectively, involve high-order derivatives of $q_{r}(z)$; these derivatives need to be evaluated accurately, otherwise the value of $C$ may even fail to
converge as $r \rightarrow \infty$. One way to handle this difficulty is by expanding $q_{r}(z)(r=1,2, \ldots)$ in power series in $z$ and determining the unknown coefficients by substitution into ( 5.16 ) $-(5.18)$ (see Appendix E for details).

Computed values of $C$ are shown in Figure 5.1 for several values of $\tau<1 / 3$. Note that $C$ grows monotonically as $\tau$ is decreased below the critical value $1 / 3$ so the tail amplitude, however small it may be, is not identically zero. On the other hand, even though $C$ grows exponentially, the (minimum) tail amplitude actually decreases as surface tension is decreased because the ripples at the tails get shorter ( $\kappa_{*}$ increases as $\tau$ is decreased according to (5.7)) and the exponentially small factor in (5.26) offsets the growth of $C$. This seems reasonable, given that the tails are expected to disappear completely as $\tau \rightarrow 0$.

We are now prepared to compare the asymptotic results (5.26) with fully numerical computations. According to (5.26), the minimum tail amplitude occurs when $\chi=0$ in which case the total phase shift $\theta=\theta_{\min }=-\epsilon a D$ and the minimum tail amplitude $A_{\text {min }}=-4 \pi C \exp \left(-\pi \kappa_{*} / 2 \epsilon\right)$. These predictions are compared against numerical results in Figure 5.2 for various values of the Froude number $F$ and $\tau=0.3$. Overall, the agreement is reasonably good and, as expected, becomes better as the linear limit $(\epsilon \rightarrow 0, F \rightarrow 1)$ is approached. However, the predictions of the asymptotic theory for the phase shift of the tail oscillations are more accurate than those for the tail amplitude. The variation of the tail amplitude with the phase shift is plotted in Figure 5.3 for $\tau=0.3$ and $F=1.0008$. Again, there is reasonably good agreement between asymptotic and numerical results, and the asymptotic theory provides a fairly accurate estimate of the critical phase shift $\theta_{\text {crit }}=\pi / 2-\epsilon a D$, at which the resonance phenomenon mentioned in $\S 5.4$ occurs.

Finally, we remark that our numerical computations have focused on gravity-capillary solitary waves in the small-amplitude limit, where comparison with the asymptotic results can be made. In this regime, the tail amplitude is exponentially small with respect to the peak solitary-wave amplitude, and the spectral technique is tailored to capture such smallamplitude tails. On the other hand, the approach of Hunter \& Vanden-Broeck (1991) may be preferable for computing non-local gravity-capillary solitary waves of large amplitude,
when the oscillatory tails can form a significant part of the disturbance: as the value of $\epsilon$ is increased beyond 0.1 or so, the number of terms that need to be kept in expansion (5.27) increases rapidly and the size of the computer code becomes difficult to handle.


Figure 5.1 Computed values of the constant $C$ appeariag in the asymptotic expressions (5.26) for various values of the Bond number $\tau$.
(a)


Figure $5.2 a$ (for caption, see following page)


Figure 5.2 Comparison between asymptotic (-) and numerical (o) results for various values of the Froude number $F ;(a)$ the minimum tail amplitude normalized with the results of the asymptotic theory ( $a_{M, \min } / A_{\text {min }}$ ); b) the phase shift of the tail oscillations corresponding to the minimum tail amplitude. The Bond number $\tau=0.3$.


Phase Shift

Figure 5.3 The tail amplitude as a function of the phase shift; (-): asymptotic results ( 5.26 ); ( 0 ): numerical results ( $a_{M}$ ). The Bond number $\tau=0.3$ and the Froude number $F=1.0008(\epsilon=0.1095)$.

## CHAPTER 6

## ASYMMETRIC SOLITARY WAVES

### 6.1 Introduction

Previous theoretical studies of two-dimensional gravity-capillary solitary waves have dealt almost exclusively with symmetric waves. In fact, in the absence of surface tension, Craig \& Sternberg (1988) proved that gravity solitary waves traveling at supercritical speeds on the free surface of a finite-depth fluid layer can only be symmetric (see also Craig \& Sternberg 1992). Asymmetric waves, however, are often encountered in the field and in laboratory experiments (see, for example, Zhang 1995), and in this chapter we shall examine the possibility of asymmetric solitary waves.

To motivate our approach, we shall briefly review some recent work concerning gravitycapillary solitary waves. Longuet-Higgins (1988) first predicted the existence of solitary waves on the surface of a deep fluid, under the combined action of gravity and surface tension. This was a somewhat surprising result since it is well known that gravity solitary waves cannot exist in deep water. However, the origin of these sclitary waves is different from that of the well known solitary waves of the Korteweg-de Vries (KdV) type. KdV-type solitary waves bifurcate from infinitesimal long waves when the phase speed $c$ is equal to the value corresponding to the local extremum of the dispersion relation $c(k)$ at $k=0$, where $k$ denotes the wavenumber. The new solitary waves, on the other hand, bifurcate when the phase speed $c$ is equal to the value corresponding to the minimum of the dispersion relation at $k=k_{m}$, with $k_{m} \neq 0$. Such a minimum is present only if the effects of surface tension are taken into account.

In a follow-up paper, Longuet-Higgins (1989) computed numerically some of these new solitary waves. The waves he computed are all symmetric and actually are part of the socalled depression branch (because the level of the free surface at the point of symmetry is below the free-surface level at infinity). Subsequently, Iooss \& Kirchgässner (1990) provided
a proof of existence of this new type of solitary waves in finite-depth fluids. The existence proof of deep-water solitary waves is somewhat more involved and was provided recently by Kirrmann \& looss (199i). However, the rigorous work can only establish the existence of two branches of small-amplitude symmetric solitary waves, the elevation and the depression branches; it is unclear whether asymmetric solitary waves exist at infinitesimal amplitude or even at finite amplitude.

Vanden-Broeck \& Dias (1992) followed the symmetric solitary-wave branches numerically to compute large-amplitude waves in infinite depth. Dias \& Iooss (1993) extended the analysis to higher order and favorably compared their results with the numerical results. Numerical results in finite depth were reported by Dias, Menasce \& Vanden-Broeck (1995).

A physical interpretation of these new solitary waves in terms of modulated wavepackets was independently provided by Akylas (1993) and Longuet-Higgins (1993). They showed that these waves correspond to stationary solutions of the nonlinear Schrödinger (NLS) equation that governs slow modulations in space and in time of gravity-capillary waves. It is well known that the NLS equation admits solutions in the form of wavepackets characterized by two lengthscales, the characteristic length of the envelope and the wavelength of the oscillations inside the envelope. The envelope travels at the group velocity while the oscillations travel at the phase velocity. At the minimum of the dispersion relation, where the phase velocity is equal to the group velocity, it is therefore natural to obtain localized wavepackets of permanent form that appear to be solitary waves with oscillatory outskirts. Moreover, this interpretation suggests that asymmetric solitary waves may also be possible, as the wave crests may be shifted (by an arbitrary amount) relative to the wave envelope.

It is important to emphasize here that, as far as small-amplitude asymmetric solitary waves are concerned, there seems to be a gap between the rigorous results and the asymptotic results. As will be discussed below, the resolution of this difficulty involves exponentially small terms that lie beyond all orders of the straightforward two-scale
expansion. The presence of exponentially small terms has already allowed the resolution of apparent contradictions in the classical KdV-type solitary internal wave and gravitycapillary solitary wave problems (Akylas \& Grimshaw 1992; Yang \& Akylas 1996b).

A model equation that has become popular as a starting basis to understand complex nonlinear wave phenomena is the fifth-order KdV equation, which has been derived by several authors in various contexts (see, for example, Zufiria 1987; Hunt \& Wadee 1991). In the weakly nonlinear theory of gravity-capillary long waves in fluids of finite depth, this equation is relevant when the Froude number is close to 1 and the Bond number close to $1 / 3$ (Zufiria 1987). The fifth-order KdV equation exhibits the bifurcations of the KdV-type solitary waves and the solitary wavepackets discussed above. Accordingly, we shall examine the possibility of asymmetric solitary waves on the basis of the fifth-order KdV equation.

In a recent study, Grimshaw, Malomed \& Benilov (1994) constructed symmetric solitary wavepacket solutions of the fifth-order KdV equation. They showed that higherorder nonlinear and dispersive effects become important and, as a result, the envelope actually is governed by a singularly perturbed NLS equation. Although not emphasized in their paper, the asymptotic theory suggests that asymmetric solitary wavepackets also are admissible solutions of the fifth-order KdV equation in accordance with the remarks made above.

On the other hand, some asymmetric solutions of the fifth-order KdV equation have been computed numerically by Zufiria (1987) and Buffoni, Champneys \& Toland (1995). Zufiria (1987) was mostly interested in periodic waves. As he followed branches of symmetric periodic waves, he discovered symmetry-breaking bifurcations. By making the period tend to infinity, he was the first to our knowledge to find asymmetric solitary waves numerically. Treating the fifth-order KdV equation as a dynamical system, Champneys \& Toland (1993) and more recently Buffoni et al. (1995) proved the existence of an infinity of homoclinic orbits corresponding to symmetric solitary waves and the presence of spatial chaos. By following these branches of homoclinic orbits numerically, they also found bifurcations into branches of asymmetric homoclinic orbits. The asymmetric solitary waves reported in

Buffoni et al. (1995) may be viewed geometrically as several localized wavepackets pieced together, and are called multi-modal solitary waves. The numerical results of Buffoni et al. (1995) also suggest that multi-modal solitary waves only exist at finite amplitude. In this chapter we wish to obtain, through asymptotic analysis, a coherent and more intuitive understanding of the analytical and numerical results cited above.

It is interesting to note that Grimshaw (1995) considered a perturbed NLS equation and showed that, owing to higher-order dispersive effects, the envelope-soliton solution of the NLS equation becomes non-local by the radiation of short-scale oscillatory waves of exponentially small amplitude (see also Wai, Chen \& Lee 1990). This observation hints at the presence of exponentially small terms that turn out to be relevant to the construction of multi-modal solitary waves (see $\S 6.5$ ). On the other hand, the NLS equation is derived on the assumption that the envelope is slowly varying in space, and the physical significance of the short-scale oscillatory waves radiated by an NLS solitary wave is not clear.

In order to clarify the origin of exponentially small terms in asymmetric solitary-wave solutions of the fifth-order KdV equation, the asymptotic theory of Grimshaw et al. (1994) is re-examined in §6.3. Note that, since these terms lie beyond all orders of the straightforward two-scale expansion, calculating them requires the techniques of exponential asymptotics (see, for example, Segur, Tanveer \& Levine 1991). In $\S 6.4$, the wavenumber-domain (Fouriertransform) approach presented in Chapter 2 is followed to calculate these terms.

To interpret these exponentially small terms physically, however, one has to take into account not only the fundamental oscillations, but also the induced mean term and all higher harmonics. It turns out that a small-amplitude wavepacket can be localized only when it is symmetric, consistent with the rigorous results (Iooss \& Kirchgässner 1990; Kirrmann \& Iooss 1995). When it is asymmetric, growing (in space) oscillations will inevitably arise on either or both sides of the wavepacket. In that case nonlinear effects will come into play and prevent the tails of the wavepacket from growing indefinitely, giving rise to new wavepackets. If the oscillations inside the envelope have just the right phase, it is possible to have localized solutions consisting of two or more wavepackets, i.e. multi-modal solitary
waves in the terminology of Buffoni et al. (1995), otherwise spatial chaos sets in. The multimodal solitary waves may be symmetric or asymmetric and exist only at finite amplitude. consistent with the numerical results of Buffoni et al. (1995).

It is also found that the asymmetric and the symmetric branches of multi-modal solitary waves intersect in a bifurcation diagram, so it is appropriate to attribute the existence of asymmetric (multi-modal) solitary waves to a symmetry-breaking bifurcation (Zufiria 1987). The validity of the asymptotic theory presented here is also confirmed by numerica: results.

### 6.2 Formulation

We shall investigate the possibility of asymmetric solitary waves based on the fifthorder KdV equation. This model equation arises in the weakly nonlinear theory of gravitycapillary long waves in fluids of finite depth when the Froude number is close to 1 and the Bond number close to $1 / 3$ (see, for example, Zufiria 1987).

As we are interested in waves of permanent form, the fifth-order KdV equation can be integrated once to give a fourth-order differential equation, which in turn may be manipulated to the following standard form

$$
\begin{equation*}
-c u+3 u^{2}+u_{x x}+u_{x x x x}=0 \quad(-\infty<x<\infty) \tag{6.1}
\end{equation*}
$$

In the above equation, $c$ and $u(x)$ are the wave speed and the wave profile, respectively. For solitary-wave solutions of (6.1), it is appropriate to impose the boundary conditions

$$
\begin{equation*}
u \rightarrow 0 \quad(x \rightarrow \pm \infty) \tag{6.2}
\end{equation*}
$$

The linearized version of (6.1) admits sinusoidal solutions of wavenumber $k$ and phase speed $c=-k^{2}+k^{4}$. It can be readily shown that $c(k)$ attains its minimum value $c_{m}=-1 / 4$ at $k=k_{m}=1 / \sqrt{2}$. Here we are interested in wavepacket solutions near the minimum of the dispersion curve, so $c$ will be taken to be close to $c_{m}$,

$$
\begin{equation*}
c=c_{m}-2 \epsilon^{2} \tag{6.3}
\end{equation*}
$$

where the parameter $\epsilon$ is assumed to be small $(0<\epsilon \ll 1)$.
In view of (6.2), $u(x)$ is small in the far field ( $x \rightarrow \pm \infty$ ), and hence (6.1) may be linearized there. The linearized version of (6.1) has four independent solutions:

$$
e^{ \pm \epsilon \gamma x} \cos k_{c} x \quad \text { and } \quad e^{ \pm \epsilon \gamma x} \sin k_{c} x,
$$

where

$$
\begin{gather*}
k_{c}=\frac{1}{2}\left(1+\left(1+8 \epsilon^{2}\right)^{1 / 2}\right)^{1 / 2}=k_{m}\left(1+\epsilon^{2}+\cdots\right),  \tag{6.4a}\\
\gamma=\frac{1}{2 \epsilon}\left(-1+\left(1+8 \epsilon^{2}\right)^{1 / 2}\right)^{1 / 2}=1-\epsilon^{2}+\cdots \tag{6.4b}
\end{gather*}
$$

Enforcing conditions (6.2) amounts to ruling out the two non-localized solutions, proportional to $e^{-\epsilon \gamma x}$ for $x<0$ and to $e^{\epsilon \gamma x}$ for $x>0$, and the admissible far-field solutions can be combined into the following form

$$
\begin{equation*}
u \sim a_{ \pm} e^{-\epsilon \gamma|x|} \cos \left(k_{c} x+\phi_{ \pm}\right) \quad(x \rightarrow \pm \infty) . \tag{6.5}
\end{equation*}
$$

Note, however, that (6.1) is invariant under a translation of the origin. Therefore, it is legitimate to specify the amplitude parameter $a_{-}$arbitrarily as this amounts to fixing the location of the origin $x=0$; the phase constant $\phi_{-}$then is the only free parameter far upstream ( $x \rightarrow-\infty$ ). If now (6.1) is thought of as a propagation (marching) problem with known upstream conditions, it would seem difficult in general to eliminate the two non-localized solutions far downstream $(x \rightarrow \infty)$ with only one free upstream parameter ( $\phi_{-}$), unless $u(x)$ is assumed to be symmetric with respect to some point. In fact, this is the reason why it is not clear immediately whether asymmetric solitary-wave solutions of (6.1) are possible.

### 6.3 Straightforward two-scale expansion

The solitary wavepacket solution of (6.1) was constructed by Grimshaw et al. (1994) to leading order. To gain some insights into the possibility of asymmetric solitary wavepackets, let us briefly re-examine their asymptotic theory.

In view of (6.5), the envelope and the oscillations inside the envelope have very different lengthscales in the limit $\epsilon \rightarrow 0$. It is then assumed that $u$ depends on both $x$ and the slow variable $\mathrm{X}=\epsilon x$, so that formally (6.1) becomes

$$
\begin{align*}
& -c u+u_{x x}+u_{x x x x}+3 u^{2}+2 \epsilon u_{x X}+4 \epsilon u_{x x x X} \\
& \quad+\epsilon^{2} u_{X X}+6 \epsilon^{2} u_{x x X X}+4 \epsilon^{3} u_{x X X X}+\epsilon^{4} u_{X X X X}=0 . \tag{6.6}
\end{align*}
$$

The analysis proceeds by seeking an asymptotic solution of (6.6) in the form

$$
\begin{equation*}
u=\epsilon\left[A(X) e^{i k_{c} x}+\text { c.c. }\right]+\epsilon^{2}\left[A_{2}(X) e^{2 k_{c} x}+\text { c.c. }+A_{0}(X)\right]+\cdots \tag{6.7}
\end{equation*}
$$

Upon substitution of (6.7) into (6.6), it is found that $A_{2}$ and $A_{0}$ are related to $A$ by

$$
A_{2}=\frac{3}{c+4 k_{c}^{2}-16 k_{c}^{4}} A^{2}+\frac{24 \mathrm{i} \epsilon k_{c}\left(1-8 k_{c}^{2}\right)}{\left(c+4 k_{c}^{2}-16 k_{c}^{4}\right)^{2}} A A_{X}+O\left(\epsilon^{2}\right)
$$

and

$$
A_{0}=\frac{6}{c}|A|^{2}+O\left(\epsilon^{2}\right)
$$

respectively, and $A$ is governed by the perturbed NLS equation

$$
\begin{equation*}
A-A_{X X}-76|A|^{2} A-2 \mathrm{i} \epsilon k_{m}\left(A_{X}-A_{X X X}\right)-\frac{128}{3} \mathrm{i} \epsilon k_{m}|A|^{2} A_{X}+O\left(\epsilon^{2}\right)=0 \tag{6.8}
\end{equation*}
$$

The solution of (6.8) can be posed as

$$
\begin{equation*}
A(X)=S(X) e^{\mathrm{i} \phi(X)} \tag{6.9}
\end{equation*}
$$

where both $S(X)$ and $\phi(X)$ are real-valued functions. It is straightforward to show that

$$
S=\frac{1}{\sqrt{38}} \operatorname{sech} X+O\left(\epsilon^{2}\right), \quad \phi=\phi_{0}-\frac{187}{57 \sqrt{2}} \epsilon \tanh X+O\left(\epsilon^{2}\right)
$$

where $\phi_{0}$ is an arbitrary phase constant.
Combining (6.7) with (6.9), and expanding the resulting expression in powers of $\epsilon$, yields

$$
\begin{align*}
& u=\sqrt{\frac{2}{19}} \epsilon \cos \left(k_{m} x+\phi_{0}\right) \operatorname{sech} X \\
&+ \epsilon^{2}\left\{\frac{187}{57 \sqrt{19}} \sin \left(k_{m} x+\phi_{0}\right) \operatorname{sech} X \tanh X\right.  \tag{6.10}\\
&\left.-\frac{4}{19}\left[3+\frac{1}{3} \cos \left(2 k_{m} x+2 \phi_{0}\right)\right] \operatorname{sech}^{2} X\right\}+O\left(\epsilon^{3}\right) .
\end{align*}
$$

The wavepacket solution (6.10) is locally confined, and is symmetric with respect to $x=0$ if the the phase constant $\phi_{0}=0$ or $\pi ; \phi_{0}=0$ corresponds to elevation waves while $\phi_{0}=\pi$ corresponds to depression waves.

Grimshaw et al. (1994) did not discuss the possibility of asymmetric solitary waves. However, for values of $\phi_{0}$ other than 0 and $\pi$, the solitary wave described by (6.10) is asymmetric, and it seems that asymmetric solitary waves may exist at infinitesimal amplitude. On the other hand, the numerical results of Zufiria (1987) and Buffoni et al. (1995) suggest that, contrary to this asymptotic result, asymmetric solitary waves only exist at finite amplitude. In principle, one can extend the straightforward expansion (6.7) to higher orders. These higher-order corrections, however, would not rule out the possibility of small-amplitude asymmetric solitary waves.

A recent paper of Grimshaw (1995) shed some light on the resolution of the contradiction between the asymptotic and the numerical results. He considered a perturbed NLS equation analogous to (6.8) and, using techniques of exponential asymptotics, found weakly non-local solutions with short-scale oscillatory waves of exponentially small amplitude. This hints at the presence of exponentially small terms in the present problem. In fact, taking Fourier transform with respect to the slow variable $X$,

$$
\widehat{u}(x, K)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} u(x, X) e^{-\mathrm{i} K X} \mathrm{~d} X,
$$

the straightforward expansion (6.10) yields

$$
\begin{align*}
\widehat{u}= & \epsilon \operatorname{sech} \frac{\pi K}{2}\left\{\frac{1}{\sqrt{38}} \cos \left(k_{m} x+\phi_{0}\right)-\epsilon K\left[\frac{187 \mathrm{i}}{114 \sqrt{19}} \sin \left(k_{m} x+\phi_{0}\right)\right.\right. \\
& \left.\left.-\frac{2}{19}\left(3+\frac{1}{3} \cos \left(2 k_{m} x+2 \phi_{0}\right)\right) \operatorname{coth} \frac{\pi K}{2}\right]+\cdots\right\}, \tag{6.11}
\end{align*}
$$

which becomes disordered when $K=O(1 / \epsilon)$. Therefore, singularities of exponentially small strength (the common factor sech $\frac{\pi K}{2}$ in (6.11) is exponentially small for large $K$ ) are expected to be present in the Fourier (wavenumber) domain. These singularities also reflect the presence of exponentially small terms in the physical domain.

Note, however, that (6.8) is derived on the assumption that the envelope $A$ depends
only on the slow variable $X$. When $A$ involves short-scale oscillations depending on $X / \epsilon$. (6.8) is no longer a valid physical model and one has to return to the original equation (6.6) to interpret these oscillations.

### 6.4 Asymptotics beyond all orders

The Fourier-transform (wavenumber-domain) approach presented in Chapter 2 will be followed here to calculate exponentially small terms that lie beyond all orders of the straightforward expansion (6.10). In preparation for the ensuing analysis, (6.6) is converted to an integral-differential equation for $\widehat{u}(x, K)$ :

$$
\begin{align*}
& \left(-c-\epsilon^{2} K^{2}+\epsilon^{4} K^{4}\right) \widehat{u}+2 \mathrm{i} \epsilon K\left(1-2 \epsilon^{2} K^{2}\right) \widehat{u}_{x} \\
& \quad+\left(1-6 \epsilon^{2} K^{2}\right) \widehat{u}_{x x}+4 \mathrm{i} \epsilon K \widehat{u}_{x x x}+\widehat{u}_{x x x x}+3 \widehat{u^{2}}=0 \tag{6.12}
\end{align*}
$$

The disordering of (6.11) when $K=O(1 / \epsilon)$ suggests the two-scale expression

$$
\begin{equation*}
\widehat{u}=\epsilon \operatorname{sech} \frac{\pi K}{2} U(x, \kappa), \tag{6.13}
\end{equation*}
$$

in terms of the stretched wavenumber variable $\kappa=\epsilon K$, with

$$
\begin{align*}
U(x, \kappa) \sim & \frac{1}{\sqrt{38}} \cos \left(k_{m} x+\phi_{0}\right)-\frac{187 \mathrm{i}}{114 \sqrt{19}} \kappa \sin \left(k_{m} x+\phi_{0}\right) \\
& -\frac{2}{19}|\kappa|\left[3+\frac{1}{3} \cos \left(2 k_{m} x+2 \phi_{0}\right)\right]+\cdots \quad(\kappa \rightarrow 0) . \tag{6.14}
\end{align*}
$$

Substituting then (6.13) into (6.12), the equation governing $U(x, \kappa)$ is found to be

$$
\begin{align*}
& \left(-c-\kappa^{2}+\kappa^{4}\right) U+2 \mathrm{i} \kappa\left(1-2 \kappa^{2}\right) U_{x}+\left(1-6 \kappa^{2}\right) U_{x x}+4 \mathrm{i} \kappa U_{x x x} \\
& \quad+U_{x x x x}+3 \cosh \frac{\pi \kappa}{2 \epsilon} \int_{-\infty}^{\infty} \frac{U(\lambda) U(\kappa-\lambda)}{\cosh \pi \lambda / 2 \epsilon \cosh \pi(\kappa-\lambda) / 2 \epsilon} \mathrm{~d} \lambda=0 . \tag{6.15}
\end{align*}
$$

The solution of (6.15) can be posed as a Fourier series

$$
\begin{equation*}
U(x, \kappa)=\sum_{n=-\infty}^{\infty} \mathcal{A}_{n}(\kappa) e^{\mathrm{i} n \theta} \tag{6.16}
\end{equation*}
$$

in terms of the total phase $\theta(x)=k_{c} x+\phi_{0}$, with

$$
\begin{align*}
\mathcal{A}_{0} & \sim \frac{6}{19}|\kappa|+\cdots \quad(\kappa \rightarrow 0),  \tag{6.17a}\\
\mathcal{A}_{ \pm 1} & \sim \frac{1}{2 \sqrt{38}} \mp \frac{187}{228 \sqrt{19}} \kappa+\cdots \quad(\kappa \rightarrow 0),  \tag{6.17b}\\
\mathcal{A}_{ \pm 2} & \sim-\frac{1}{57}|\kappa|+\cdots \quad(\kappa \rightarrow 0), \tag{6.17c}
\end{align*}
$$

etc. Upon substitution of (6.16) into (6.15), it is found that $\mathcal{A}_{n}(n=0, \pm 1, \pm 2, \cdots)$ are governed by a system of coupled integral equations

$$
\begin{align*}
& \left\{-c-\kappa^{2}+\kappa^{4}-2 n k_{c} \kappa\left(1-2 \kappa^{2}\right)-n^{2} k_{c}^{2}\left(1-6 \kappa^{2}\right)+4 n^{3} k_{c}^{3} \kappa+n^{4} k_{c}^{4}\right\} \mathcal{A}_{n} \\
& +3 \sum_{p=-\infty}^{\infty} \cosh \frac{\pi \kappa}{2 \epsilon} \int_{-\infty}^{\infty} \frac{\mathcal{A}_{p}(\lambda) \mathcal{A}_{n-p}(\kappa-\lambda)}{\cosh \pi \lambda / 2 \epsilon \cosh \pi(\kappa-\lambda) / 2 \epsilon} \mathrm{~d} \lambda=0 \quad(n=0, \pm 1, \pm 2, \cdots) . \tag{6.18}
\end{align*}
$$

However, in the limit $\epsilon \rightarrow 0$, the main contribution to the convolution integrals above comes from the range $0<\lambda<\kappa(\kappa>0), \kappa<\lambda<0(\kappa<0)$, provided that $\kappa$ is not too close to $\pm k_{m}\left(k_{m}-|\kappa| \gg O(\epsilon)\right.$; see below). Also, since $u$ is real, it can be readily shown that $\mathcal{A}_{n}(-\kappa)=\mathcal{A}_{-n}(\kappa)$ on the real $\kappa$-axis; it suffices then to consider only $\mathcal{A}_{n}(n \geq 0)$. Therefore, taking into account (6.3) and (6.4a), (6.18) simplifies to

$$
\begin{align*}
& \left\{\kappa+(n+1) k_{m}\right\}^{2}\left\{\kappa+(n-1) k_{m}\right\}^{2} \mathcal{A}_{n}+6 \sum_{p=0}^{n} \operatorname{sgn} \kappa \int_{0}^{\kappa} \mathcal{A}_{p}(\lambda) \mathcal{A}_{n-p}(\kappa-\lambda) \mathrm{d} \lambda \\
& \quad+12 \sum_{p=1}^{\infty} \operatorname{sgn} \kappa \int_{0}^{\kappa} \mathcal{A}_{p}(-\lambda) \mathcal{A}_{n+p}(\kappa-\lambda) \mathrm{d} \lambda=0 \quad(n \geq 0) \tag{6.19}
\end{align*}
$$

in the limit $\epsilon \rightarrow 0$. Note that the coefficient of $\mathcal{A}_{n}$ above vanishes when $\kappa=-(n \pm 1) k_{m}$, and $\mathcal{A}_{n}(\kappa)$ would appear to be singular there. The origin $\kappa=0$, however, is a regular point in view of (6.17). The singularities closest to the origin are located at $\kappa= \pm k_{m}$, and make the dominant contribution to the solution of $u$.

Let us focus for now on the singularities at $\kappa=-k_{m}$. By dominant balance, it may be deduced from (6.19) and (6.17) (see Appendix F) that

$$
\begin{align*}
& \mathcal{A}_{0} \sim \frac{C_{1}}{\left(\kappa+k_{m}\right)^{4}}+\frac{C_{2}}{\left(\kappa+k_{m}\right)^{3}}+\cdots \quad\left(\kappa \rightarrow-k_{m}\right),  \tag{6.20a}\\
& \mathcal{A}_{2} \sim \frac{C_{1}}{\left(\kappa+k_{m}\right)^{4}}-\frac{C_{2}}{\left(\kappa+k_{m}\right)^{3}}+\cdots \quad\left(\kappa \rightarrow-k_{m}\right), \tag{6.20b}
\end{align*}
$$

and all other harmonics are less singular at $\kappa=-k_{m}$. The constants $C_{1}$ and $C_{2}$, however, cannot be determined by asymptotic analysis alone. Following the procedure described in Appendix $F$, they are calculated numerically to be $C_{1}=-0.011$ and $C_{2}=-0.0067$.

However, in view of (6.17) and (6.20), it is not permissible to approximate the integrals in (6.18) by those in (6.19) if $\kappa$ is close to $\pm k_{m}\left(k_{m}-|\kappa| \leq O(\epsilon)\right)$, and therefore (6.20) is not valid in the immediate vicinity of $\kappa=-k_{m}$. Accordingly, to obtain the structure of $\mathcal{A}_{n}(\kappa)$ (and hence $\widehat{u}(x, K)$ ) in the neighborhood of $\kappa=-k_{m}\left(K=-k_{m} / \epsilon\right)$, we return to (6.18) and use a more accurate approximation to the integral terms. Specifically, the asymptotic behaviors (6.20) suggest the following rescaling

$$
\begin{equation*}
\mathcal{A}_{0}=\frac{1}{\epsilon^{4}} \Phi_{0}(\sigma), \quad \mathcal{A}_{2}=\frac{1}{\epsilon^{4}} \Phi_{2}(\sigma) \tag{6.21}
\end{equation*}
$$

in terms of the 'inner' wavenumber variable $\sigma=\left(\kappa+k_{m}\right) / \epsilon$. Substituting then (6.21) into (6.18), it is found that, to leading order, $\Phi_{0}$ and $\Phi_{2}$ are equal ( $\Phi_{0} \sim \Phi_{2} \equiv \Phi$ ) and satisfy the linear integral equation ${ }^{1}$,

$$
\begin{align*}
& \left(\sigma^{2}+1\right) \Phi(\sigma)-\int_{-\infty}^{\infty} \mathrm{d} l l e^{-\pi l / 2} \operatorname{csch} \frac{\pi l}{2} \Phi(\sigma-l) \\
& \quad-\int_{-\infty}^{\infty} \mathrm{d} l e^{-\pi l / 2} \operatorname{sech} \frac{\pi l}{2} \int_{-\infty}^{\infty} \mathrm{d} l_{1} e^{-\pi l_{1} / 2} \operatorname{sech} \frac{\pi l_{1}}{2} \Phi\left(\sigma-l-l_{1}\right)=0 . \tag{6.22}
\end{align*}
$$

Furthermore, to be consistent with (6.20), the matching condition

$$
\begin{equation*}
\Phi \sim \frac{C_{1}}{\sigma^{4}} \quad(\sigma \rightarrow \infty) \tag{6.23}
\end{equation*}
$$

is imposed.

[^2]Guided by previous experience (see Chapter 2), the solution of $\Phi(\sigma)$ is posed as an integral transform:

$$
\begin{equation*}
\Phi(\sigma)=\int_{\mathcal{L}} e^{-\eta \sigma} \Psi(\eta) \mathrm{d} \eta \tag{6.24}
\end{equation*}
$$

where the contour $\mathcal{L}$ extends from $\eta=0$ to $\infty$ with $\operatorname{Re} \eta \sigma>0$. Proceeding then formally, (6.22) is transformed into a second-order differential equation for $\Psi(\eta)$ :

$$
\begin{equation*}
\frac{\mathrm{d}^{2} \Psi}{\mathrm{~d} \eta^{2}}+\left(1-\frac{6}{\sin ^{2} \eta}\right) \Psi=0 \tag{6.25}
\end{equation*}
$$

where, in view of ( 6.23 ), $\Psi(\eta) \sim \frac{1}{6} C_{1} \eta^{3}(\eta \rightarrow 0)$.
The solution of this initial-value problem is

$$
\Psi(\eta)=\frac{5}{12} C_{1}\left(\frac{2}{\sin \eta}+\frac{\cos ^{2} \eta}{\sin \eta}-\frac{3 \eta \cos \eta}{\sin ^{2} \eta}\right)
$$

and, returning to (6.24), it is concluded that $\Phi(\sigma)$ has a pair of simple-pole singularities at $\sigma= \pm \mathrm{i}:$

$$
\begin{equation*}
\Phi(\sigma) \sim \mp \frac{\mathrm{i} D}{\sigma \mp \mathrm{i}} \quad(\sigma \rightarrow \pm \mathrm{i}) \tag{6.26}
\end{equation*}
$$

where $D=-5 C_{1} / 24=0.0023$. Finally, combining (6.26) with (6.13), (6.16) and (6.21), it is now clear that $\widehat{u}(x, K)$ has a pair of simple-pole singularities at $K=-k_{m} / \epsilon \pm \mathrm{i}$ :

$$
\begin{equation*}
\widehat{u} \sim \frac{2 D}{\epsilon^{3}} e^{-\pi k_{m} / 2 \epsilon} \frac{1+e^{2 \mathrm{i} \theta}}{K+k_{m} / \epsilon \mp \mathrm{i}} \quad\left(K \rightarrow-\frac{k_{m}}{\epsilon} \pm \mathrm{i}\right) . \tag{6.27a}
\end{equation*}
$$

Furthermore, since $u(x, X)$ is real, it can be readily deduced that

$$
\begin{equation*}
\widehat{u} \sim-\frac{2 D}{\epsilon^{3}} e^{-\pi k_{m} / 2 \epsilon} \frac{1+e^{-2 \mathrm{i} \theta}}{K-k_{m} / \epsilon \mp \mathrm{i}} \quad\left(K \rightarrow \frac{k_{m}}{\epsilon} \pm \mathrm{i}\right) . \tag{6.27b}
\end{equation*}
$$

Also, in view of (6.5), the precise locations of the simple-pole singularities are at $K=k_{c} / \epsilon \pm \mathrm{i} \gamma,-k_{c} / \epsilon \pm \mathrm{i} \gamma$, and (6.27) may be refined heuristically to be

$$
\begin{equation*}
\widehat{u} \sim \frac{2 D}{\epsilon^{3}} e^{-\pi k_{c} / 2 \epsilon} \frac{1+e^{2 \mathrm{i} \theta}}{K+k_{c} / \epsilon \mp \mathrm{i} \gamma} \quad\left(K \rightarrow-\frac{k_{c}}{\epsilon} \pm \mathrm{i} \gamma\right) \tag{6.28a}
\end{equation*}
$$

$$
\begin{equation*}
\widehat{u} \sim-\frac{2 D}{\epsilon^{3}} e^{-\pi k_{c} / 2 \epsilon} \frac{1+e^{-2 \mathrm{i} \theta}}{K-k_{c} / \epsilon \mp \mathrm{i} \gamma} \quad\left(K \rightarrow \frac{k_{c}}{\epsilon} \pm \mathrm{i} \gamma\right) . \tag{6.28b}
\end{equation*}
$$

As remarked earlier, the amplitude parameter $a_{-}$in (6.5) can be arbitrarily specified as this fixes the origin $x=0$. For convenience, we shall fix the value of $a_{-}$to be as predicted by the straightforward expansion (6.10). Accordingly, in taking the inverse Fourier transform.

$$
u(x, X)=\int_{\mathcal{C}} u(x, K) e^{\mathrm{i} K X} \mathrm{~d} K
$$

the contour $\mathcal{C}$ is indented to pass below all simple-pole singularities below the real axis. It then transpires that small-amplitude wavepackets do not remain localized in general due to the radiation of oscillatory waves of exponentially small but growing (in space) amplitude:

$$
\begin{align*}
u \sim & -\frac{16 \pi D}{\epsilon^{3}} \exp \left(-\frac{\pi k_{c}}{2 \epsilon}\right) \sin \phi_{0} e^{\epsilon \gamma x} \cos \left(k_{c} x+\phi_{0}\right)  \tag{6.29}\\
& +\sqrt{\frac{8}{19}} \epsilon e^{-\epsilon \gamma x} \cos \left(k_{c} x+\phi_{0}\right) \quad(x \rightarrow \infty) .
\end{align*}
$$

However, symmetric wavepackets with $\phi_{0}=0$ or $\pi$ are locally confined-since $\sin \phi_{0}=0$ when $\phi_{0}=0, \pi$, the growing oscillatory wave in (6.29) is suppressed. We also remark that higher-order corrections to (6.29) are not expected to alter these conclusions for smallamplitude symmetric solitary wavepackets, consistent with the rigorous work of Iooss \& Kirchgässner (1990) and Kirrmann \& Iooss (1995).

Of course, the asymptotic result (6.29) is valid only when $u$ remains small; as $u$ grows, nonlinear effects will come into play and prevent the wave disturbance from growing indefinitely. As a result, a second wavepacket arises. An interesting and important question then is whether the wave disturbance would decay to zero after two (or more) wavepackets and become a multi-modal solitary wave (Buffoni et al. 1995). This question is addressed in the next section.

## 6.5

 Bi-modal solitary wavesThe construction of bi-modal solitary waves involves piecing together smoothly two asymmetric wavepackets. Here we will show that such connection can be made and, moreover, a bi-modal solitary wave as a whole can be symmetric or asymmetric.

Specifically, suppose that far upstream

$$
\begin{equation*}
u \sim \sqrt{\frac{8}{19}} \epsilon e^{\epsilon \gamma x} \cos \left(k_{c} x+\phi_{-}\right) \quad(x \rightarrow-\infty) \tag{6.30a}
\end{equation*}
$$

then, the asymptotic results (6.29) yields

$$
\begin{align*}
u \sim & -\frac{16 \pi D}{\epsilon^{3}} \exp \left(-\frac{\pi k_{c}}{2 \epsilon}\right) \sin \phi_{-} e^{\epsilon \gamma x} \cos \left(k_{c} x+\phi_{-}\right) \\
& +\sqrt{\frac{8}{19}} \epsilon e^{-\epsilon \gamma x} \cos \left(k_{c} x+\phi_{-}\right) \quad\left(O\left(\frac{1}{\epsilon}\right) \ll x \ll O\left(\frac{1}{\epsilon^{2}}\right)\right) . \tag{6.30b}
\end{align*}
$$

Note also that, in addition to being invariant under a translation of the origin, the fifthorder KdV equation (6.1) preserves its form under the change of coordinate $x \rightarrow-x$. It is therefore legitimate to read (6.30) from the opposite direction and to translate the whole solution by a distance $L$. Specifically, if

$$
\begin{align*}
u \sim & \frac{16 \pi D}{\epsilon^{3}} \exp \left(-\frac{\pi k_{c}}{2 \epsilon}\right) \sin \phi_{+} e^{-\epsilon \gamma(x-L)} \cos \left(k_{c} x-k_{c} L+\phi_{+}\right) \\
& +\sqrt{\frac{8}{19}} \epsilon e^{\epsilon \gamma(x-L)} \cos \left(k_{c} x-k_{c} L+\phi_{+}\right)\left(O\left(\frac{1}{\epsilon}\right) \ll L-x \ll O\left(\frac{1}{\epsilon^{2}}\right)\right), \tag{6.31a}
\end{align*}
$$

it is expected that

$$
\begin{equation*}
u \sim \sqrt{\frac{8}{19}} \epsilon e^{-\epsilon \gamma(x-L)} \cos \left(k_{c} x-k_{c} L+\phi_{+}\right) \quad(x \rightarrow \infty) \tag{6.31b}
\end{equation*}
$$

Hence, bi-modal solitary waves can be constructed if there exist some $L, \phi_{-}$, and $\phi_{+}$such that ( $6.30 b$ ) perfectly matches (6.31a). In that case, the parameter $L$ in (6.31) may be viewed as the distance between the centers of the two wavepackets that make up a bi-modal solitary wave. Without any loss of generality, it will be assumed that $-\pi<\phi_{ \pm}<\pi$ in the discussion below.

For perfect matching of $(6.30 b)$ with $(6.31 a)$, it is necessary that

$$
\begin{align*}
\left|\sin \phi_{-}\right|=\left|\sin \phi_{+}\right| & =\frac{\epsilon^{4}}{4 \sqrt{38} \pi D} \exp \left(\frac{\pi k_{c}}{2 \epsilon}-\epsilon \gamma L\right),  \tag{6.32a}\\
\phi_{+}+\chi_{+} & =k_{c} L+\phi_{-}+2 m_{1} \pi  \tag{6.32b}\\
\phi_{+}-\chi_{-} & =k_{c} L+\phi_{-}+2 m_{2} \pi \tag{6.32c}
\end{align*}
$$

where $\chi_{+}=0(\pi)$ if $0<\phi_{+}<\pi\left(-\pi<\phi_{+}<0\right), \chi_{-}=\pi(0)$ if $0<\phi_{-}<\pi\left(-\pi<\phi_{-}<0\right)$, and $m_{1}, m_{2}$ are integers. Subtracting (6.32c) from (6.32b) yields

$$
\chi_{+}+\chi_{-}=2\left(m_{1}-m_{2}\right) \pi
$$

Therefore, the phase constants $\phi_{+}$and $\dot{\phi}_{-}$must have opposite signs, and there are two possibilities to satisfy the requirement (6.32a): (i) $\phi_{+}=-\phi_{-}$; (ii) $\phi_{+}=\phi_{-}+\pi$ (if $-\pi<\phi_{-}<0$ ) or $\phi_{+}=\phi_{-} \pi$ (if $0<\phi_{-}<\pi$ ).

In view of ( $6.30 a$ ) and (6.31b), case (i) corresponds to waves that are symmetric with respect to $x=L / 2$. Also, in this case ( $6.32 b$ ) gives

$$
\begin{equation*}
k_{c} L=m \pi-2 \phi_{-}+(-1)^{m} \pi \tag{6.33}
\end{equation*}
$$

where $m$ is an integer, and the last term is inserted for the purpose of consistent labeling of solution branches. Even and odd values of the integer $m$ correspond to positive and negative values of the phase constant $\phi_{-}$, respectively. Equations (6.32a) and (6.33) can be combined to yield an algebraic equation for $\phi_{-}$:

$$
\begin{equation*}
\left|\sin \phi_{-}\right|=\frac{\epsilon^{4}}{4 \sqrt{38} \pi D} \exp \left(\frac{\pi k_{c}}{2 \epsilon}-\frac{\epsilon \gamma\left(m+(-1)^{m}\right) \pi}{k_{c}}+\frac{2 \epsilon \gamma \phi_{-}}{k_{c}}\right) \tag{6.34}
\end{equation*}
$$

On the other hand, case (ii) corresponds to asymmetric waves in general, and in $\mathrm{t}^{\mathrm{r}}$. s case the distance $L$ is given by

$$
\begin{equation*}
k_{c} L=m \pi \tag{6.35}
\end{equation*}
$$

where the integer $m$ will be assigned the same value as that in (6.33) for labeling purposes. Clearly, the integer $m$ is related to the number of periods of oscillations between the centers of the two wavepackets that make up a bi-modal solitary wave. Upon substitution of (6.35) into ( $6.32 a$ ), it is found that the phase constant $\phi_{-}$satisfies

$$
\begin{equation*}
\left|\sin \phi_{-}\right|=\frac{\epsilon^{4}}{4 \sqrt{38} \pi D} \exp \left(\frac{\pi k_{c}}{2 \varepsilon}-\frac{\epsilon \gamma m \pi}{k_{c}}\right) \tag{6.36}
\end{equation*}
$$

in this case.
Since $\left|\sin \phi_{-}\right|$has to be less than 1 , it can be readily seen from (6.34) and (6.36) that, for fixed values of $\epsilon$, the integer $m$ has to exceed a certain minimum value. Also, for fixed values of $m,(6.34)$ and (6.36) have solutions only if $\epsilon$ is greater than a certain minimum value. In this sense, bi-modal solitary waves (and multi-modal solitary waves, in general) exist only at finite amplitude, consistent with the numerical results of Buffoni et al. (1995). Recall that the distance $L$ between the centers of the two wavepackets of a bi-modal solitary wave is related to the integer $m$. Also, the distance $L$ increases with decreasing $\epsilon$. [In the extreme case $\epsilon=0$, it takes forever for the second wavepacket to show up.] Therefore, the fact that bi-modal solitary waves exist at finite amplitude can be understood as a consequence of geometrical compatibility.

The minimum amplitudes for symmetric and asymmetric waves corresponding to the same $m$ are different, and can be made arbitrarily small by increasing $m$. Note also that, at the minimum amplitude, $\phi_{ \pm}= \pm \pi / 2$ or $\mp \pi / 2$, and therefore branches of asymmetric waves intersect branches of symmetric waves in a bifurcation diagram. It is therefore appropriate to attribute the existence of asymmetric solitary waves to a symmetry-breaking bifurcation (Zufiria 1987).

It is also worth mentioning that multi-modal solitary waves consisting of more than two wavepackets also can be constructed on the basis of the asymptotic theory presented here.

### 6.6 Numerical results and discussion

In order to confirm the validity of our asymptotic theory and to calculate large-amplitude waves, the fifth-order KdV equation (6.1) is solved numerically by the standard fourth-order Runge-Kutta method. However, due to numerical instability, (6.1) actually cannot be solved as a one-way marching problem. Instead, the computational domain is truncated at some large distance, say at $x= \pm x_{\infty}$, and, using starting conditions of the form (6.5), (6.1) is integrated from both $x=-x_{\infty}$ and $x=x_{\infty}$ to some matching point in between (say, $x=0$ ). As remarked earlier, (6.1) is translationally invariant, so $a_{-}$can be specified arbitrarily. Here $a_{-}=\sqrt{\frac{8}{19}} \epsilon$ is used in numerical computations. [Accordingly, the asymptotic result (6.31b) predicts that $a_{+} \sim e^{\epsilon \gamma} a_{-}$.] We then have three free parameters ( $\phi_{ \pm}$and $a_{+}$), while there are four quantities ( $u, u_{x}, u_{x x}$, and $u_{x x x}$ ) to be matched at the matching point $x=0$. Naturally, mismatches at the matching point are present in general, and a solution is found when the mismatches can be eliminated. It involves some searching in the parameter space to find a solution.

After a solution is found, the whole branch on which the solution lies can be traced by slowly varying the parameters. Solution branches are followed to estimate the minimum amplitude of bi-modal solitary waves. Figure $6.1(a)$ shows the numericaliy estimated minimum amplitude of symmetric branches corresponding to various values of $m$. The predictions of the asymptotic theory can be solved numerically from (6.34), and are also plotted in Figure 6.1(a) for comparison. It is seen that the asymptotic theory is valid for la-ge $m$, where the minimum amplitude is small. However, the asymptotic theory remains reasonably accurate even when $m$ is not all that large. The difference between the minimum amplitudes of symmetric and asymmetric waves is shown in Figure 6.1(b) for various values of $m$. Note that the difference is very small, but is never zero. In Figure 6.1(b), asymptotic and numerical results exhibit opposite trends as $m$ decreases and, not unexpectedly, the asymptotic theory fails when $m$ is small.

Several solution branches were also followed to larger amplitude. In Figures 6.2 and 6.3, symmetric and asymmetric branches corresponding to $m=19$ and $m=20$, respectively,
are shown. The calculated upstream phase constant $\phi_{-}$and the corresponding downstream amplitude parameter $a_{+}$are plotted against the wave speed $c$ in these figures. It is clearly seen that the asymmetric branches intersect the symmetric branches near the turning points of these branches, consistent with the numerical results of Buffoni et al. (1995). Examples of symmetric and asymmetric bi-modal solitary waves are shown in Figures 6.4 and 6.5, respectively.

Finally, we remark that, as exponentially small terms are involved, all nonlinear and dispersive effects are equally important. Therefore, the predictions of the fifth-order KdV equation are not expected to agree quantitatively with those of the full water-wave equations. However, in light of the asymptotic theory presented here, it is expected that asymmetric multi-modal solitary waves also can be constructed in the water-wave problem provided that the dispersion curve has a minimum point, which happens to be the case in the presence of surface tension. Such waves are expected to exist at finite amplitude.
(a)


Figure $6.1 a$ (for caption, see following page)
(b)


Figcre 6.1 Comparison, for various values of $m$, of the predictions of the asymptotic results (6.34),(6.36) (——) for the minimum amplitudes of bi-modal solitary waves against numerical results (o). (a) Minimum amplitude of symmetric waves; (b) Difference between the minimum amplitudes of asymmetric and symmetric waves.
(a)


Figure $6.2 a$ (for caption, see following page)
(b)


Figure 6.2 Comparison, for various values of $c$, of the predictions of the asymptotic results (6.34),(6.36) ( - : symmetric waves; - - -: asymmetric waves) for bi-modal solitary waves corresponding to $m=19$ against numerical results ( 0 : symmetric waves; $\circ$ : asymmetric waves). (a) Upstream phase constant, $\phi_{-}$; (b) Downstream amplitude parameter, $a_{+}$.
(a)


Figure 6.3a (for caption, see following page)
(b)


Figcre 6.3 Comparison, for various values of $c$, of the predictions of the asymptotic results (6.34).(6.36) (-: symmetric waves; -- --: asymmetric waves) for bi-modal solitary waves corresponding to $m=20$ against numerical results (o: symmetric waves; $০$ : asymmetric waves). (a) Upstream phase constant, $\phi_{-}$; (b) Downstream amplitude parameter, $a_{+}$.

Figunt 6.5 Numerically computed asymumetris bi-modal solitary wave corresponding ${ }^{\prime \prime}$
$m=20 ; r=-0.2614, \phi . .=-0.761 \pi$.

## CHAPTER 7

## CONCLUSIONS

We have studied a number of hydrodynamic wave problems where coupling of longwave disturbances with short-scale oscillations is important, and we have identified a fully nonlinear coupling mechanism that obtains when the long-wave speed matches the shortwave phase speed. As explained in §1.1, the short-wave amplitude is exponentially small (with respect to the long-wave amplitude) in the weakly nonlinear-weakly dispersive regime and lies beyond all orders of the usual expansion procedures. For this reason, in order to understand the long-short wave interaction theoretically, it is necessary to use techniques of exponential asymptotics. Also, in computing large-amplitude disturbances, the dual demand of high accuracy and capability of resolving coexisting waves of very different lengthscales often renders numerical computations expensive and high-performance numerical methods are needed.

A perturbation procedure in the wavenumber domain for calculating the exponentially small short-wave amplitude asymptotically was presented in Chapter 2. Using the forced KdV (fKdV) equation as a simple model, it was demonstrated that the wavenumber-domain approach is very efficient and robust compared with other existing approaches and, more importantly, the short-wave tail is sensitive to the details of the long-wave profile. This interesting result prompted us to study the generation of lee waves by stratified flow over topography in Chapter 4.

In the study of steady lee-wave patterns in subcritical stratified flow over extended topography, it was found that the importance of nonlinear effects (relative to dispersive effects) is measured by a parameter $A$ which depends on the obstacle shape (see (4.1)). For $A=O(1)$, when weak nonlinear and dispersive effects are equally important, the (formally exponentially small) lee-wave amplitude can be dramatically enhanced by nonlinearity and thus becomes substantial. This somewhat surprising conclusion was vividly illustrated by the streamline patterns shown in Figure 4.3. It would be interesting to verify the results of
this study experimentally.
The theoretical study of lee waves can be extended in at least two directions. First. it is worth exploring finite-amplitude effects on three-dimensional lee-wave patterns, induced by stratified flow over topography that depends on both the streamwise and the spanwise directions. In this case, a continuous distribution of wavenumbers are excited, and it is expected that nonlinearity may influence certain portions of the wave pattern more seriously than others.

It would also be of interest to examine the influence of lee waves on the stability of transcritical (resonant) stratified flow over topography. Previous studies of resonant flow over topography often assumed that the obstacle is long and thereby neglected the presence of lee waves as the lee-wave amplitude is expected to be small in that case. However, in light of the results obtained in Chapter 4, this assumption may not always be appropriate. Physically, the presence of lee waves provides an additional source of (radiation) damping; it is therefore expected that lee waves will make the flow more stable.

Weakly non-local solitary-wave solutions of a perturbed ILW equation were discussed in Chapter 3. This study demonstrated once again the robustness of the wavenumber-domain approach proposed in the present thesis and shed light on the analysis of non-local solitary internal waves in fluids of finite or large depth. These waves are important in undersea navigation and in the design of deep-water offshore facilities (see, for example, Osborne \& Burch 1980). It is possible to study these waves by adapting the analysis of the perturbed ILW equation to the full stratified flow equations. In fact, our asymptotic analysis of weakly non-local gravity-capillary solitary waves (Chapter 5) was guided by that of the fifth-order KdV equation (§2.5). Non-local solitary waves may also arise in layered fluids, and can be studied by modifying the asymptotic theory presented in Chapter 5.

Finally, in Chapter 6, the possibility of small-amplitude asymmetric solitary waves was examined based on the fifth-order KdV equation. In general, such waves exist and are related to envelope solitons with stationary crests near the minimum of the dispersion curve, where the phase velocity is equal to the group velocity. Both symmetric and asymmetric
multi-modal wavepackets were found to exist at finite amplitude. This work emphasizes that the techniques developed in this thesis are of rather wide applicability and, moreover. that exponentially small terms can be surprisingly important-they affect the very existence of asymmetric solitary waves. A natural extension of this work is to construct asymmetric solitary waves in specific physical systems based on the exact governing equations. In particular, it is expected that asymmetric solitary waves can be found in the water-wave problem provided that the dispersion curve has a minimum point, which happens to be the case in the presence of surface tension.

## APPENDIX A

## ASYMPTOTIC ANALYSIS FOR $f(x)=\operatorname{sech}^{2} x$ in §4.3.1

Here we give details of the asymptotic analysis that leads to expression (4.18) for the lee wave induced by the topography $f(x)=\operatorname{sech}^{2} x$.

Substituting (4.15) into (4.12), it is found that. to leading order, $\Phi$ satisfies the integraldifferential equation

$$
\begin{align*}
\Phi_{2 z}- & \beta \mathcal{N}^{2} \Phi_{z}+\left(\frac{\mathcal{N}^{2}}{F^{2}}-\kappa^{2}\right) \Phi= \\
& \beta \mathcal{N}^{2}\left\{\int_{0}^{\kappa} \mathrm{d} \lambda\left[\Phi_{z}(\kappa-\lambda, z) \Phi_{z}(\lambda, z)-\lambda(\kappa-\lambda) \Phi(\kappa-\lambda, z) \Phi(\lambda, z)\right]\right\} \\
& -\sum_{j=1}^{\infty} 2^{2} M_{j}\left\{\frac{Y_{j+1}}{F^{2}}-\beta \int_{0}^{\kappa} \mathrm{d} \lambda Y_{j}(\kappa-\lambda, z) \int_{0}^{\lambda} \mathrm{d} \lambda_{1} \Phi_{z}\left(\lambda-\lambda_{1}, z\right) \Phi_{z}\left(\lambda_{1}, z\right)\right. \\
& \left.-\frac{\beta}{j+1}\left(Y_{j+1}\right)_{z}+\beta \int_{0}^{\kappa} \mathrm{d} \lambda Y_{j}(\kappa-\lambda, z) \int_{0}^{\lambda} \mathrm{d} \lambda_{1}\left(\lambda-\lambda_{1}\right) \lambda_{1} \Phi\left(\lambda-\lambda_{1}, z\right) \Phi(\lambda, z)\right\}, \tag{A1}
\end{align*}
$$

with the notation

$$
\begin{aligned}
Y_{J}(\kappa, z)= & \int_{0}^{\kappa} \mathrm{d} \lambda_{1} \Phi\left(\kappa-\lambda_{1}, z\right) \int_{0}^{\lambda_{1}} \mathrm{~d} \lambda_{2} \Phi\left(\lambda_{1}-\lambda_{2}, z\right) \\
& \cdots \int_{0}^{\lambda_{j-2}} \mathrm{~d} \lambda_{j-1} \Phi\left(\lambda_{j-2}-\lambda_{j-1}, z\right) \Phi\left(\lambda_{j-1}, z\right) .
\end{aligned}
$$

Moreover, the boundary conditions (4.13) yield

$$
\begin{gather*}
\Phi(\kappa, 0)+\sum_{j=1}^{\infty} \frac{A^{j}}{j!(2 j-1)!} \int_{0}^{\kappa} \mathrm{d} \lambda \frac{\partial^{j} \Phi}{\partial z^{j}}(\kappa-\lambda, 0) \lambda^{2 j-1}=-\frac{1}{2} A \kappa,  \tag{A2a}\\
\Phi(\kappa, 1)=0 . \tag{A2b}
\end{gather*}
$$

The solution of this problem is posed as a power series,

$$
\begin{equation*}
\Phi(\kappa, z)=\sum_{m=1}^{\infty} b_{m}(z) \kappa^{2 m-1} \tag{A3}
\end{equation*}
$$

with $b_{1}=-\frac{1}{2} A q(z)$ according to (4.16). The coefficients $b_{m}(m \geq 2)$ then can be determined
successively from the infinite sequence of boundary-value problems

$$
\begin{align*}
& b_{m}^{\prime \prime}-3 \mathcal{N}^{2} b_{m}^{\prime}+\frac{\mathcal{N}^{2}}{F^{2}} b_{m}= b_{m-1}+\beta \mathcal{N}^{2}\left(L_{m}-S_{m}\right) \\
&-\sum_{j=1}^{m-1} 2^{j} M_{j}\left\{\frac{P_{m}^{(j+1)}}{F^{2}}-\frac{\beta}{j+1}\left(P_{m}^{(j+1)}\right)^{\prime}\right\}  \tag{A4}\\
&-\beta \sum_{j=1}^{m-3} 2^{j} M_{j} Q_{m}^{(j)}+\beta \sum_{j=1}^{m-2} 2^{j} M_{j} R_{m}^{(j)} \quad(0 \leq z \leq 1) \\
& b_{m}(0)=-\sum_{j=1}^{m-1} A^{j} \frac{(2 m-2 j-1)!}{j!(2 m-1)!} \frac{\mathrm{d}^{j} b_{m-j}}{\mathrm{~d} z^{j}}(0), \quad b_{m}(1)=0 \tag{A5a,b}
\end{align*}
$$

The expressions for $L_{m}, S_{m}, P_{m}^{(j)}, Q_{m}^{(j)}$ and $R_{m}^{(j)}$ above contain complicated convolution sums involving $b_{1}, \ldots, b_{m-1}$; they arise from the multiple convolution integrals in (A1) and are given by

$$
\begin{gathered}
L_{m}=\sum_{s=1}^{m-1} \frac{(2 m-2 s-1)!(2 s-1)!}{(2 m-1)!} b_{m-s}^{\prime} b_{s}^{\prime} ; \\
S_{m}=\sum_{s=1}^{m-2} \frac{(2 m-2 s-2)!(2 s)!}{(2 m-1)!} b_{m-s-1} b_{s} \quad(m \geq 3), \quad S_{2}=0 ; \\
S_{m}=\sum_{s=1}^{m-2} \frac{(2 m-2 s-2)!(2 s)!}{(2 m-1)!} b_{m-s-1} b_{s} \quad(m \geq 3), \quad S_{2}=0 ; \\
P_{m}^{(1)}=b_{m}, \quad P_{m}^{(j+1)}=\sum_{s=j}^{m-1} \frac{(2 m-2 s-1)!(2 s-1)!}{(2 m-1)!} b_{m-s} P_{s}^{(j)} ; \\
Q_{m}^{(j)}=\sum_{s=j}^{m-3} \frac{(2 m-2 s-1)!(2 s-1)!}{(2 m-1)!} S_{m-s} P_{s}^{(j)}(m \geq 4), \quad Q_{2}^{(j)}=Q_{3}^{(j)}=0 ; \\
R_{m}^{(j)}=\sum_{s=j}^{m-2} \frac{(2 m-2 s-1)!(2 s-1)!}{(2 m-1)!} L_{m-s} P_{s}^{(j)}(m \geq 3), \quad R_{2}^{(j)}=0 .
\end{gathered}
$$

In particular, for $m=2$, (A4),(A5) give

$$
\begin{gathered}
b_{2}^{\prime \prime}-\beta \mathcal{N}^{-2} b_{2}^{\prime}+\frac{\mathcal{N}^{2}}{F^{2}} b_{2}=b_{1}+\frac{1}{6} \beta \mathcal{N}^{2} b_{1}^{\prime 2}+\frac{1}{3} M_{1}\left(\beta b_{1} b_{1}^{\prime}-\frac{b_{1}^{2}}{F^{2}}\right) \quad(0 \leq z \leq 1) \\
b_{2}(0)=-\frac{1}{6} A b_{1}^{\prime}(0), \quad b_{2}(1)=0
\end{gathered}
$$

the solution is

$$
b_{2}=\frac{1}{12} A^{2} q^{\prime}(0) q(z)+\frac{1}{2} A \sum_{r=1}^{\infty} \frac{1}{\kappa_{r}}\left(\frac{1}{6} A \delta_{r}-\gamma_{r}\right) \phi_{r}(z)
$$

consistent with (4.16).
Although, in principle, one may proceed to determine $b_{m}(z)$ for $m>2$, it is the asymptotic behavior of $b_{m}(z)$ as $m \rightarrow \infty$ that controls the convergence of the power series (A3) and, hence, the singularities of $\Phi(\kappa, z)$ in the $\kappa$-plane. In the limit $m \rightarrow \infty$, the boundary-value problem (A4),(A5) in fact simplifies significantly because the contribution of the convolution sums in (A4) is subdominant,

$$
b_{m}^{\prime \prime}-\beta \mathcal{N}^{2} b_{m}^{\prime}+\frac{\mathcal{N}^{2}}{F^{2}} b_{m} \sim b_{m-1} \quad(m \rightarrow \infty)
$$

and the boundary condition (A5a) becomes effectively homogeneous:

$$
b_{m}(0) \rightarrow 0 \quad(m \rightarrow \infty)
$$

Hence, $b_{m}(z)$ may be expanded in terms of the lee-wave modes defined by the eigenvalue problem (4.8):

$$
b_{m}(z) \sim \sum_{r=1}^{\infty} g_{m r} \phi_{r}(z) \quad(m \rightarrow \infty)
$$

The coefficients $g_{m r}$ obey the simple recurrence relation

$$
\kappa_{r} g_{m r} \sim g_{m-1 r}
$$

$\kappa_{r}$ being the corresponding eigenvalues, so that
and

$$
\begin{gather*}
g_{m r} \sim C_{r} \kappa_{r}^{-m} \\
b_{m} \sim \sum_{r=1}^{\infty} C_{r} \kappa_{r}^{-m} \phi_{r}(z) \quad(m \rightarrow \infty), \tag{A6}
\end{gather*}
$$

where $C_{r}$ are certain constants that depend on $b_{1}(z)$ and cannot be determined by asymptotic analysis alone.

Combining (A6) with (A3), it is now clear that $\Phi(\kappa, z)$ has simple-pole singularities at $\kappa= \pm \kappa_{N}^{\frac{1}{2}}$ :

$$
\Phi \sim-C_{N} \frac{\kappa}{\kappa^{2}-\kappa_{N}} \phi_{N}(z) \quad\left(\kappa \rightarrow \pm \kappa_{N}^{\frac{1}{2}}\right) .
$$

## APPENDIX B

## ASYMPTOTIC ANALYSIS FOR $f(x)=\operatorname{sech} x$ in §4.3.2

Here we give details of the asymptotic analysis that leads to expression (4.24) for the lee wave induced by the topography $f(x)=\operatorname{sech} x$.

Assuming for now that $\kappa$ is not too close to $\pm \kappa_{N}^{\frac{1}{2}}$, upon substituting (4.19) into (4.12),(4.13), $\Phi(\kappa, z)$ satisfies to leading order

$$
\begin{align*}
\Phi_{z z}- & \beta \mathcal{N}^{2} \Phi_{z}+\left(\frac{\mathcal{N}^{2}}{F^{2}}-\kappa^{2}\right) \Phi= \\
& \beta \mathcal{N}^{2} \operatorname{sgn} \kappa\left\{\int_{0}^{\kappa} \mathrm{d} \lambda\left[\Phi_{z}(\kappa-\lambda, z) \Phi_{z}(\lambda, z)-\lambda(\kappa-\lambda) \Phi(\kappa-\lambda, z) \Phi(\lambda, z)\right]\right\} \\
& -\sum_{j=1}^{\infty} 2^{j} M_{j}\left\{\frac{\tilde{Y}_{j+1}}{F^{2}}-\beta \int_{0}^{\kappa} \mathrm{d} \lambda \tilde{Y}_{j}(\kappa-\lambda, z) \int_{0}^{\lambda} \mathrm{d} \lambda_{1} \Phi_{z}\left(\lambda-\lambda_{1}, z\right) \Phi_{z}\left(\lambda_{1}, z\right)\right. \\
& \left.-\frac{\beta}{j+1}\left(\tilde{Y}_{J+1}\right)_{z}+\beta \int_{0}^{\kappa} \mathrm{d} \lambda \tilde{Y}_{j}(\kappa-\lambda, z) \int_{0}^{\lambda} \mathrm{d} \lambda_{1}\left(\lambda-\lambda_{1}\right) \lambda_{1} \Phi\left(\lambda-\lambda_{1}, z\right) \Phi\left(\lambda_{1}, z\right)\right\} \tag{B1}
\end{align*}
$$

subject to the boundary conditions

$$
\begin{gather*}
\Phi(\kappa, 0)+\sum_{j=1}^{\infty} \frac{A^{j}(\operatorname{sgn} \kappa)^{j}}{j!(j-1)!} \int_{0}^{\kappa} \mathrm{d} \lambda \frac{\partial^{j} \Phi}{\partial z^{j}}(\kappa-\lambda, 0) \lambda^{j-1}=-\frac{1}{2} \cdot A,  \tag{B2a}\\
\Phi(\kappa, 1)=0, \tag{B2b}
\end{gather*}
$$

where $\tilde{Y}_{j}(\kappa, z)=(\operatorname{sgn} \kappa)^{j-1} Y_{j}(\kappa, z)$ in terms of the $Y_{j}(\kappa, z)$ appearing in (A1).
In preparation for the local analysis that becomes necessary close to $\pm \kappa_{N}^{\frac{1}{2}}$, next we examine the asymptotic behavior of $\Phi(\kappa, z)$ in the 'intermediate' region where $\kappa$ is close to $\pm \kappa_{N}^{\frac{1}{2}}$ but (B1),(B2) still hold, $\mu \ll \kappa_{N}^{\frac{1}{2}}-\mid \kappa_{1}^{!} \ll 1$. Specifically, suppose

$$
\begin{equation*}
\Phi(\kappa, z) \sim \frac{C_{N} \phi_{N}(z)}{\left(\kappa_{N}^{\frac{1}{2}}-|\kappa|\right)^{\alpha}}+\frac{C_{0} q(z)}{\left(\kappa_{N}^{\frac{1}{2}}-|\kappa|\right)^{\alpha-1}}+\cdots \quad\left(\mu \ll \kappa_{N}^{\frac{1}{2}}-|\kappa| \ll 1\right) \tag{B3}
\end{equation*}
$$

where $C_{N}, C_{0}$ and $\alpha$ are to be determined. Then, by dominant balance, it follows from (B1) and (B2a) respectively that

$$
\begin{equation*}
2 C_{N} \kappa_{N}^{\frac{1}{2}}-C_{0} \kappa_{N} \gamma_{N}=\frac{A \nu}{1-\alpha} C_{N}, \quad C_{0}=\frac{A \phi_{N}^{\prime}(0)}{1-\alpha} C_{N} \tag{B4a,b}
\end{equation*}
$$

where

Hence.

$$
\nu=\int_{0}^{1} \rho_{0} \phi_{N}\left\{\beta \mathcal{N}^{2} q^{\prime} \phi_{N}^{\prime}-2 \frac{M M_{1}}{F^{2}} q \phi_{N}+\beta M_{1}\left(q \phi_{N}\right)^{\prime}\right\} \mathrm{d} z
$$

$$
\begin{equation*}
\alpha=1-\frac{A}{2 \kappa_{N}^{\frac{1}{2}}}\left(\nu+\kappa_{N} \gamma_{N} \phi_{N}^{\prime}(0)\right), \tag{B5}
\end{equation*}
$$

and, on the basis of (B1),(B2), the singularities of $\Phi(\kappa, z)$ at $\kappa= \pm \kappa_{N}^{\frac{1}{2}}$ are not simple poles (save in the linear limit $A \rightarrow 0$ ), contrary to the fact that the induced lee wave has constant amplitude as $x \rightarrow \infty$; this is an indication that the leading-order problem (B1),(B2) for $\Phi$ breaks down in the immediate vicinity of $\kappa= \pm \kappa_{N}^{\frac{1}{2}}, \kappa_{N}^{\frac{1}{2}}-|\kappa|=O(\mu)$, and the asymptotic behavior (B3) is not valid there. Also note that the dominant-balance argument used above to obtain (B4) is not valid in case $\alpha<1$-a different theoretical approach is needed but this matter is not pursued here.

To obtain the structure of $\hat{\psi}(k, z)$ close to $k= \pm k_{N}$ (assuming $\alpha>1$ ), we return to the (exact) boundary-value problem (4.12),(4.13) and approximate the terms involving convolution integrals more accurately. Specifically, near $\kappa=\kappa_{N}^{\frac{1}{2}}$, (B3) suggests the rescaling

$$
\begin{equation*}
\Phi(\kappa, z)=\frac{v(\eta)}{\mu^{\alpha}}\left\{\phi_{N}(z)+\mu \frac{C_{0}}{C_{N}} \eta q(z)\right\} \tag{B6}
\end{equation*}
$$

in terms of the 'inner' variable $\eta=\left(\kappa_{N}^{\frac{1}{2}}-\kappa\right) / \mu$. Combining then (B6) with (4.19), taking into account (B4), it follows from (4.12),(4.13) that, to leading order, $v(\eta)$ satisfies the (singular) integral equation

$$
\begin{equation*}
\eta v(\eta)-\frac{1}{2}(\alpha-1) \int_{-\infty}^{\infty} \mathrm{d} \sigma \mathrm{e}^{-\frac{1}{2} \pi \sigma} \operatorname{sech} \frac{\pi \sigma}{2} v(\eta-\sigma)=0 \tag{B7a}
\end{equation*}
$$

subject to the condition

$$
\begin{equation*}
v(\eta) \sim \frac{C_{N}}{\eta^{\alpha}} \quad(\eta \rightarrow \infty) \tag{B7b}
\end{equation*}
$$

that ensures matching with (B3).
This linear problem can be readily solved via integral transform (see $\S 2.3$ ). Briefly, the solution is posed as

$$
\begin{equation*}
v(\eta)=\int_{\mathcal{L}} \mathrm{e}^{-\eta s} V(s) \mathrm{d} s \tag{B8}
\end{equation*}
$$

where the contour $\mathcal{L}$ extends from $s=0$ to $\propto$ with $\operatorname{Re} \eta s>0$ and $\operatorname{Im} s<0$. With this choice of $\mathcal{L}, v(\eta)$ is analytic in $\operatorname{Im} \eta>0$ and, hence, does not contribute to the singularities of $\hat{\psi}(k, z)$ in $\operatorname{Im} k<0$, consistent with the upstream condition (4.2) that the excited lee waves vanish in $x<0$. Upon substitution of (B8) into (B7a), $V(s)$ satisfies the differential equation

$$
\frac{d V}{d s}-(\alpha-1) \frac{V}{\sin s}=0
$$

and, in view of the matching condition ( $B 7 b$ ),

$$
V(s) \sim \frac{C_{N}}{\Gamma(\alpha)} s^{a-1} \quad(s \rightarrow 0)
$$

where $\Gamma(\alpha)$ denotes the gamma function. The appropriate solution is

$$
V(s)=C_{N} \frac{2^{\alpha-1}}{\Gamma(\alpha)}\left(\tan \frac{s}{2}\right)^{\alpha-1}
$$

and, using (B8), it is seen that $v(\eta)$ has a simple-pole singularity at $\eta=0$ :

$$
\begin{equation*}
v(\eta) \sim \frac{(-2 \mathrm{i})^{\alpha-1}}{\Gamma(\alpha)} \frac{C_{N}}{\eta} \quad(\eta \rightarrow 0) \tag{B9}
\end{equation*}
$$

Finally, combining (B9) with (4.19) and (B6), we have established that $\hat{\psi}(k, z)$ has a simple-pole singularity at $k=k_{N}$ and, by very similar reasoning, the same is true at $k=-k_{N}$ :

$$
\begin{equation*}
\hat{\psi} \sim \mp C_{N} \frac{2^{\alpha}}{\Gamma(\alpha)} \mu^{1-\alpha} \exp \left(\mp \mathrm{i} \frac{1}{2} \pi(\alpha-1)\right) \frac{\exp \left(-\frac{1}{2} \pi k_{N}\right)}{k \mp k_{N}} \phi_{N}(z) \quad\left(k \rightarrow \pm k_{N}\right) \tag{B10}
\end{equation*}
$$

According to ( B 10 ), the residues of the poles of $\hat{\psi}(k, z)$ at $k= \pm k_{N}$, and hence the lee-wave amplitude, depend on the constant $C_{N}$, which has to be determined numerically. For this purpose, the solution to the problem (B1),(B2) is posed as a power series,

$$
\begin{equation*}
\Phi(\kappa, z)=\sum_{m=1}^{\infty} b_{m}(z)|\kappa|^{m-1} \tag{B11}
\end{equation*}
$$

with $b_{1}=-\frac{1}{2} A q(z)$ according to (4.20), and, upon substitution into (B1),(B2), one has an infinite sequence of boundary-value problems, analogous to (A4),(A5), for $b_{m}(z)(m \geq 2)$. The value of $C_{N}$ is then found from integrating these problems numerically making use of the fact that, in view of (B3), the projection of $b_{m}(z)$ on $\phi_{N}(z)$ behaves like

$$
C_{N} \kappa_{N}^{-\frac{1}{2}(\alpha+m-1)} \frac{\alpha(\alpha+1) \cdots(\alpha+m-2)}{(m-1)!}
$$

as $m \rightarrow \infty$.

## APPENDIX C DETERMINATION OF THE CONSTANT $C_{1}$ <br> APPEARING IN (4.18) AND (4.24)

Here we present a numerical procedure for determining the constant $C_{1}$ that appears in the asymptotic results (4.18) and (4.24). As remarked in §4.5, the constant $C_{1}$ has to be estimated from the coefficients $b_{m}(z)$ of the power series (A3) (for $f=\operatorname{sech}^{2} x$ ) and (B11) (for $f=\operatorname{sech} x$ ) by computing the projection of $b_{m}(z)$ on $\phi_{1}(z)$ as $m \rightarrow \infty$.

To explain the idea of the numerical procedure, we shall only consider the case $f=\operatorname{sech}^{2} x ; b_{m}(z)(m \geq 2)$ then satisfy the sequence of boundary-value problems (A4),(A5). However, difficulties arise in solving for $b_{m}(z)$ numerically because, as $m$ increases, higherorder derivatives of $b_{1}, b_{2}, \ldots$ enter the inhomogeneous boundary condition ( $\mathrm{A} 5 a$ ); these derivatives need to be evaluated accurately to avoid introducing significant error in estimating $C_{1}$. To this end, the solution of $b_{m}(z)$ is approximated by the power series

$$
\begin{equation*}
b_{m}(z)=\sum_{r=0}^{M} B_{m, r} z^{r} \tag{C1}
\end{equation*}
$$

and the accuracy of the approximation can be improved by increasing $M$. For the examples discussed in $\S 4.5$, it suffices to set $M=64$.

The task now is to calculate the coefficients $B_{m, r}$ in ( Cl ). The equations governing $B_{m, r}$ are derived by substituting (C1) into (A4),(A5) and collecting terms of like powers of $z$. For $\beta=0$ and $\mathcal{N}^{2}=1+b z$, we obtain

$$
\begin{gather*}
(r+1)(r+2) B_{m, r+2}+\frac{1}{F^{2}}\left(B_{m, r}+b d_{r} B_{m, r-1}\right) \\
=B_{m-1, r}-\frac{2 b}{F^{2}} \sum_{s=1}^{m-1} \sum_{j=0}^{r} \frac{(2 m-2 s-1)!(2 s-1)!}{(2 m-1)!} B_{m-s, j} B_{s, r-j},  \tag{C2}\\
B_{m, 0}=-\sum_{j=1}^{m-1} \frac{(2 m-2 j-1)!}{(2 m-1)!} A^{j} B_{m-j, j}, \tag{C3}
\end{gather*}
$$

and

$$
\begin{equation*}
\sum_{r=0}^{M} B_{m, r}=0 \tag{C4}
\end{equation*}
$$

with $d_{0}=0, d_{r}=1(r \geq 1)$.
The solution procedure starts with the known coefficients $B_{1, r}(r=0,1, \ldots, M)$. For $m \geq 2, B_{m, 0}$ can be calculated directly from (C3). However, a tentative value of $B_{m, 1}$ needs to be guessed, so that $B_{m, r}(r=2, \ldots, M)$ can be calculated from (C2). The value of $B_{m, 1}$ is updated iteratively, using Newton's method, such that (C4) is satisfied, and then the value of $C_{1}$ can be obtained by projecting $b_{m}$ (approximated by ( C 1 )) on $\phi_{1}(z)$.

For $f=\operatorname{sech} x$, an sequence of boundary-value problems analogous to (A4),(A5) is obtained upon substitution of (B11) into (B1),(B2), and can be very similarly handled by the procedure described above.

## APPENDIX D

## DERIVATION OF (5.11)-(5.14)

Here we outline the derivation of the integral-differential boundary-value problem (5.11)-(5.14), and give the explicit expressions for $P_{r}$ and $S_{r}$ that appear in (5.17a) and (5.18) respectively.

The equations governing $\hat{\eta}(k)$ and $\hat{\phi}(k, z)$ are obtained by taking Fourier transform of (5.1)-(5.4) (the nonlinear free-surface conditions (5.2) and (5.3), however, need to be expanded in Taylor series about $z=0$ first) formally. The results are

$$
\begin{gather*}
\hat{\phi}_{z z}-\epsilon^{2} k^{2} \hat{\phi}=0 \quad(-1 \leq z \leq 0),  \tag{D1}\\
-\mathrm{i} k F \hat{\eta}+\widehat{\phi_{x} \eta_{x}}=\frac{\hat{\phi}_{z}}{\epsilon^{2}}+\sum_{j=1}^{\infty} \mathcal{F}\left\{\frac{1}{\epsilon^{2} j!} \eta^{j} \frac{\partial^{j+1} \phi}{\partial z^{j+1}}-\frac{1}{(j+1)!}\left(\eta^{j+1}\right)_{x} \frac{\partial^{j} \phi_{x}}{\partial z^{j}}\right\} \quad(z=0),  \tag{D2}\\
-\mathrm{i} k F \hat{\phi}+\widehat{\eta}+\frac{1}{2}\left(\widehat{\phi_{x}^{2}}+\frac{1}{\epsilon^{2}} \widehat{\phi_{z}^{2}}\right)=-\tau \epsilon^{2} k^{2} \widehat{\eta} \\
+\sum_{j=1}^{\infty} \mathcal{F}\left\{\frac{(-1)^{j}(2 j+1)!!}{2^{j} j!} \tau \epsilon^{2 j+2} \eta_{x x} \eta_{x}^{2 j}+\frac{F}{j!} \eta^{j} \frac{\partial^{j} \phi_{x}}{\partial z^{j}}\right.  \tag{D3}\\
\left.-\frac{1}{2} \sum_{m=0}^{j} \frac{1}{m!(j-m)!} \eta^{j}\left(\frac{\partial^{m} \phi_{x}}{\partial z^{m}} \frac{\partial^{j-m} \phi_{x}}{\partial z^{j-m}}+\frac{1}{\epsilon^{2}} \frac{\partial^{m+1} \phi}{\partial z^{m+1}} \frac{\partial^{j-m+1} \phi}{\partial z^{j-m+1}}\right)\right\} \quad(z=0), \\
\hat{\phi}_{z}=0 \quad(z=-1), \tag{D4}
\end{gather*}
$$

where both $\hat{\eta}$ and $\mathcal{F}\{\eta\}$ denote the Fourier transform of $\eta$, and the short-hand notation $(2 j+1)!!=1 \cdot 3 \cdot 5 \ldots(2 j+1)$ is used.

We then substitute the expansions (5.9) and (5.5c) into (D1)-(D4), and simplify the resulting equations in light of the model problem discussed in §2.2. After some tedions but straightforward algebra, it can be verified that, to leading order, $A(\kappa)$ and $Q(\kappa, z)$ satisfy (5.11)-(5.14).

The problem (5.11)-(5.14) can be solved by posing the solutions of $A$ and $Q$ as power series in $\kappa$ (5.15). The resulting infinite sequence of boundary-value problems (5.16)-(5.18)
involve complicated convolution sums $P_{r}$ and $S_{r}$, whose explicit expressions are given below in terms of the coefficients $a_{r}$ and $q_{r}$ in (5.15):

$$
\begin{aligned}
P_{r}= & \sum_{j=1}^{r-1} \sum_{s=1}^{r-j} \frac{2^{j}(2 r-2 s-1)!(2 s)!}{j!(2 r)!} a_{r-s-1}^{(j)} \frac{d^{j+1} q_{s}}{d z^{j+1}}(0) \\
& +\kappa_{0}^{2} \sum_{j=0}^{r-2} \sum_{s=0}^{r-j-2} \frac{2^{j+1}(2 r-2 s-2)!(2 s+1)!}{(j+1)!(2 r)!} a_{r-s-2}^{(j+1)} \frac{d^{j} q_{s}}{d z^{j}}(0), \\
S_{r}= & \kappa_{0}^{2} \sum_{s=0}^{r-1} \frac{(2 r-2 s-1)!(2 s+1)!}{(2 r+1)!} q_{r-s-1}(0) q_{s}(0) \\
& -\sum_{s=1}^{r-1} \frac{(2 r-2 s)!(2 s)!}{(2 r+1)!} \frac{d q_{r-s}}{d z}(0) \frac{d q_{s}}{d z}(0) \\
& -\kappa_{0} \sum_{j=1}^{r-1} \sum_{s=1}^{r-j} \frac{2^{j}(2 r-2 s-1)!(2 s+1)!}{j!(2 r+1)!} a_{r-s-1}^{(j)} \frac{d^{j} q_{s}}{d z^{j}}(0) \\
+ & \tau \sum_{j=1}^{(r-1) / 3} \sum_{s=0}^{r-3 j-1} \frac{2^{j}(2 j+1)!!\kappa_{0}^{2 j+2}(2 r-2 s-3)!(2 s+3)!}{j^{j}!(2 r+1)!} b_{r-s-2}^{(2 j)} a_{s} \\
& +\kappa_{0}^{2} \sum_{j=1}^{r-1} \sum_{s=1}^{r-j} \sum_{m=0}^{j} \frac{2^{j}(2 r-2 s-1)!(2 s+1)!}{m!(j-m)!(2 r+1)!} a_{r-s-1}^{(j)} T_{s}^{(j m)}(0) \\
& -\sum_{j=1}^{r-2} \sum_{s=2}^{r-j} \sum_{m=0}^{j} \frac{2^{j}(2 r-2 s-1)!(2 s+1)!}{m!(j-m)!(2 r+1)!} a_{r-s-1}^{(j)} R_{s}^{(j m)}(0),
\end{aligned}
$$

where

$$
\begin{aligned}
a_{r}^{(j+1)} & =\sum_{s=0}^{r-j} \frac{(2 r-2 s-1)!(2 s+1)!}{(2 r+1)!} a_{r-s-1}^{(j)} a_{s}, \quad a_{r}^{(1)}=a_{r} ; \\
b_{r}^{(2 j+2)} & =\sum_{s=0}^{r-3 j-2} \sum_{m=0}^{r-s-3 j-2} \frac{(2 s+2)!(2 m+2)!(2 r-2 s-2 m-5)!}{(2 r+1)!} b_{r-s-m-3}^{(2 j)} a_{m} a_{s}, \\
b_{r}^{(2)} & =\sum_{s=0}^{r-2} \frac{(2 r-2 s-2)!(2 s+2)!}{(2 r+1)!} a_{r-s-2} a_{s} ; \\
T_{r}^{(j m)}(z) & =\sum_{s=0}^{r-1} \frac{(2 r-2 s-1)!(2 s+1)!}{(2 r+1)!} \frac{d^{m} q_{r-s-1}}{d z^{m}} \frac{d^{j-m} q_{s}}{d z^{j-m}}
\end{aligned}
$$

$$
R_{r}^{(\jmath m)}(z)=\sum_{s=1}^{r-1} \frac{(2 r-2 s)!(2 s)!}{(2 r+1)!} \frac{d^{m+1} q_{r-s}}{d z^{m+1}} \frac{d^{j-m+1} q_{s}}{d z^{j-m+1}}
$$

## APPENDIX E

## DETERMINATION OF THE CONSTANT C APPEARING IN (5.26)

Here we present a numerical procedure for determining the constant $C$ that appears in the asymptotic expressions (5.26). As noted in $\S 5.3, C$ can be computed numerically from the sequence of boundary-value problems (5.16)-(5.18). However, the expressions for $P_{r}$ and $S_{r}$ (see Appendix D) on the right-hand sides of (5.17a) and (5.18), respectively, involve high-order derivatives of $q_{r}(z)$; these derivatives need to be evaluated accurately, otherwise the value of $C$ may even fail to converge as $r \rightarrow \infty$. To handle this difficulty, we construct numerically the exact solutions of $q_{r}(z)(r \geq 2)$, which may be written as power series:

$$
\begin{equation*}
q_{r}(z)=\sum_{j=0}^{2 r} q_{j}^{(r)} z^{j} \tag{E1}
\end{equation*}
$$

with $q_{0}^{(0)}=q_{0}(z)=a \kappa_{0} / 2$.
The ordinary differential equation (5.16) and the bottom boundary condition (5.17b) then give

$$
\begin{equation*}
q_{j}^{(r)}=\frac{\kappa_{0}^{2}}{j(j-1)} q_{j-2}^{(r-1)} \quad(j=2,3, \ldots, 2 r) \tag{E2a}
\end{equation*}
$$

and

$$
\begin{equation*}
q_{1}^{(r)}=\sum_{j=2}^{2 r}(-1)^{j} j q_{j}^{(r)} \tag{E2b}
\end{equation*}
$$

respectively. Note, however, that the boundary conditions (5.17) specify only the first derivative of $q_{r}$, and hence cannot fix the value of $q_{0}^{(r)}$.

To derive the constraint that determines $q_{0}^{(r)}$, the differential equation (5.16) is integrated from $z=-1$ to 0 ; making use of the boundary conditions (5.17), and replacing by $r+2$ the index $r$ in the resulting expression, we obtain

$$
\begin{equation*}
\kappa_{0}^{2} \int_{-1}^{0} q_{r+1} \mathrm{~d} z=\kappa_{0} a_{r+1}-P_{r+2} . \tag{E3}
\end{equation*}
$$

Substituting then (E1) into (E3) and (5.18) (with the index r replaced by $r+1$ ), yields

$$
\begin{equation*}
\kappa_{0} q_{0}^{(r+1)}-a_{r+1}=-\frac{1}{\kappa_{0}} P_{r+2}-\kappa_{0} \sum_{j=1}^{2 r+2} \frac{(-1)^{j}}{j+1} q_{j}^{(r+1)} \tag{E4a}
\end{equation*}
$$

and

$$
\begin{equation*}
\kappa_{0} q_{0}^{(r+1)}-a_{r+1}=S_{r+1}+\tau \kappa_{0}^{2} a_{r}, \tag{E4b}
\end{equation*}
$$

respectively.
In view of the explicit expressions for $P_{r}$ and $S_{r}$ given in Appendix D , the unknown quantities involved on the right-hand sides of (E4a) and (E4b) are $a_{r}, q_{0}^{(r)}$, and $q_{j}^{(r+1)}$ $(j=1, \ldots, 2 r+2)$. Therefore, the linear system (E4) for $q_{0}^{(r+1)}$ and $a_{r+1}$ is consistent only if

$$
\begin{equation*}
\kappa_{0} S_{r+1}+P_{r+2}+\tau \kappa_{0}^{3} a_{r}+\kappa_{0}^{2} \sum_{j=1}^{2 r+2} \frac{(-1)^{j}}{j+1} q_{j}^{(r+1)}=0 \tag{E5}
\end{equation*}
$$

Equation (E5) is the constraint we are seeking, and involves the unknowns $a_{r}, q_{0}^{(r)}$ (implicitly), and $q_{j}^{(r+1)}(j=1, \ldots, 2 r+2)$.

Returning now to the solution of $q_{0}^{(r)}$ and $a_{r}$. A convenient procedure is to guess the value of $q_{0}^{(r)}$, based on which $a_{r}$ and $q_{j}^{(r+1)}(j=1, \ldots, 2 r+2)$ can be calculated from (5.18) and (E2), respectively. The value of $q_{0}^{(r)}$ is updated iteratively, using Newton's method, such that (E5) is satisfied. It turns out that, consistent with (5.19), $a_{r} \rightarrow a_{\infty}$, $q_{0}^{(r)} \rightarrow a_{\infty} \operatorname{coth} \kappa_{0}(r \rightarrow \infty)$, where $a_{\infty}=-2 C$ by definition.

## APPENDIX F

## DOMINANT-BALANCE ANALYSIS LEADING TO (6.20)

Here we outline the dominant-balance analysis that leads to the asymptotic expressions (6.20). A numerical procedure for determining the constants $C_{1}$ and $C_{2}$ that appear in (6.20) is also presented below.

As remarked in $\S 6.4$, the coefficient of $\mathcal{A}_{n}(\kappa)$ in (6.19) vanishes when $\kappa=-(n \pm 1) k_{m}$, where all harmonics $\mathcal{A}_{n}(\kappa)(n \geq 0)$ are expected to be singular due to nonlinear coupling presented by the convolution integrals in (6.19). Since the origin $\kappa=0$ actually is a regular point in view of (6.17), the singularities of $\mathcal{A}_{\boldsymbol{n}}(\kappa)(n \geq 0)$ closest to the origin are located at $\kappa= \pm k_{m}$. These singularities make the dominant contribution to the solution of $u$.

Attention will be focused on the singularities of $\mathcal{A}_{n}(\kappa)(n \geq 0)$ at $\kappa=-k_{m}$, where $\mathcal{A}_{0}(\kappa)$ and $\mathcal{A}_{2}(\kappa)$ are expected to be the most singular harmonics among all (as their coefficients in (6.19) vanish when $\kappa=-k_{m}$ ). Note, however, that through the last sum of convolution integrals in (6.19) the singularities of some $\mathcal{A}_{n}(\kappa)(n \geq 0)$ at $\kappa=k_{m}$ also participate in the dominant balance of (6.19) as $\kappa \rightarrow-k_{m}$. Accordingly, let us assume that

$$
\begin{align*}
& \mathcal{A}_{0} \sim \frac{C^{(0)}}{\left(k_{m} \mp \kappa\right)^{N}} \quad\left(\kappa \rightarrow \pm k_{m}\right)  \tag{F1a}\\
& \mathcal{A}_{2} \sim \frac{C^{(2)}}{\left(k_{m}+\kappa\right)^{N}} \quad\left(\kappa \rightarrow-k_{m}\right) \tag{F1b}
\end{align*}
$$

where the exponent $N$ is to be determined. It can then be deduced from (6.19), taking into account (6.17), that

$$
\begin{array}{ll}
\mathcal{A}_{1} & \sim \frac{C^{( \pm 1)}}{\left(k_{m} \mp \kappa\right)^{N-1}} \\
\mathcal{A}_{3} \sim \frac{C^{(3)}}{\left(k_{m}+\kappa\right)^{N-1}} & \left(\kappa \rightarrow \pm k_{m}\right), \\
\end{array}
$$

and the constants $C^{(m)}(m=0, \pm 1,2,3)$ are related by

$$
\begin{equation*}
2 C^{(0)}=\frac{1}{(N-1)(N-2)}\left(\frac{72}{19} C^{(0)}+\frac{4}{19} C^{(2)}\right)-\frac{1}{N-2} \frac{6}{\sqrt{38}}\left(C^{(-1)}+C^{(1)}\right), \tag{F2}
\end{equation*}
$$

$$
\begin{gather*}
2 C^{(2)}=\frac{1}{(N-1)(N-2)}\left(\frac{72}{19} C^{(2)}+\frac{4}{19} C^{(0)}\right)-\frac{1}{N-2} \frac{6}{\sqrt{38}}\left(C^{(-1)}+C^{(3)}\right),  \tag{F3}\\
\frac{1}{4} C^{(-1)}=-\frac{6}{\sqrt{38}} \frac{C^{(0)}+C^{(2)}}{N-1},  \tag{F4}\\
\frac{9}{4} C^{(i)}=-\frac{6}{\sqrt{38}} \frac{C^{(0)}}{N-1},  \tag{F5}\\
\frac{9}{4} C^{(3)}=-\frac{6}{\sqrt{38}} \frac{C^{(2)}}{N-1} . \tag{F6}
\end{gather*}
$$

Other harmonics are less singular at $\kappa= \pm k_{m}$, and do not participate in the dominant balance of (6.19) as $\kappa \rightarrow-k_{m}$.

Substituting then (F4)-(F6) into (F2) and (F3) yields

$$
\begin{aligned}
& C^{(0)}=\frac{4 C^{(0)}+2 C^{(2)}}{(N-1)(N-2)} \\
& C^{(2)}=\frac{4 C^{(2)}+2 C^{(0)}}{(N-1)(N-2)}
\end{aligned}
$$

Consistency between the above two relations requires that $C^{(0)}= \pm C^{(2)}$, and the exponent $N$ is determined accordingly to be $N=4$ if $C^{(0)}=C^{(2)}$ ( $\equiv C_{1}$ that appears in (6.20)) and $N=3$ if $C^{(0)}=-C^{(2)}\left(\equiv C_{2}\right.$ in (6.20)). It is now clear that the asymptotic expressions (6.20) result from combining these two possibilities linearly.

To determine the constants $C_{1}$ and $C_{2}$ that appear in (6.20), we look for series solutions of $\mathcal{A}_{n}(\kappa)(n \geq 0)$ in the form

$$
\begin{gather*}
\mathcal{A}_{0}=\sum_{p=2}^{\infty} b_{0, p}|\kappa|^{p-1},  \tag{F7a}\\
\mathcal{A}_{n}=\left\{\begin{array}{ll}
\sum_{p=n}^{\infty} b_{n, p} \kappa^{p-1} & (\kappa>0) \\
\sum_{p=n}^{\infty} \tilde{b}_{n, p} \kappa^{p-1} & (\kappa<0)
\end{array} \quad(n \geq 1),\right. \tag{F7b}
\end{gather*}
$$

with $b_{1,1}=\widetilde{b}_{1,1}=\frac{1}{2 \sqrt{38}}, b_{0,2}=-\frac{6}{19},-b_{2,2}=\tilde{b}_{2,2}=\frac{1}{57}, b_{1,2}=\tilde{b}_{1,2}=-\frac{187}{228 \sqrt{19}}$, etc. Upon substitution of (F7) into (6.19), it is found that the coefficients $b_{n, p}, \widetilde{b}_{n, p}$ ( $n \geq 0, p \geq n$ ) satisfy the recurrence equations

$$
\begin{align*}
& b_{0, p-4}-b_{0, p-2}+\frac{1}{4} b_{0, p}+6 \sum_{q=2}^{p-2} \frac{(p-q-1)!(q-1)!}{(p-1)!} b_{0, p-q} b_{0, q} \\
& +12 \sum_{r=1}^{[p / 2]} \sum_{q=r}^{p-r} \frac{(-1)^{q-1}(p-q-1)!(q-1)!}{(p-1)!} b_{r, p-q} \tilde{b}_{r, q}=0 \tag{F8}
\end{align*}
$$

$$
\begin{align*}
& b_{n, p-4}+4 n k_{m} b_{n, p-3}+\left(3 n^{2}-1\right) b_{n, p-2}+2 k_{m} n\left(n^{2}-1\right) b_{n, p-1} \\
&+\frac{1}{4}\left(n^{2}-1\right)^{2} b_{n, p}+12 \sum_{q=2}^{p-n} \frac{(p-q-1)!(q-1)!}{(p-1)!} b_{n, p-q} b_{0, q} \\
&+6 \sum_{r=1}^{n-1} \sum_{q=n-r}^{p-r} \frac{(p-q-1)!(q-1)!}{(p-1)!} b_{r, p-q} b_{n-r, q}  \tag{F9}\\
&+12 \sum_{r=1}^{[(p-n) / 2]} \sum_{q=n+r}^{p-r} \frac{(-1)^{p-q-1}(p-q-1)!(q-1)!}{(p-1)!} \tilde{b}_{r, p-q} b_{n+r, q}=0, \\
& \tilde{b}_{n, p-4}+4 n k_{m} \tilde{b}_{n, p-3}+\left(3 n^{2}-1\right) \tilde{b}_{n, p-2}+2 k_{m} n\left(n^{2}-1\right) \tilde{b}_{n, p-1} \\
&+\frac{1}{4}\left(n^{2}-1\right)^{2} \tilde{b}_{n, p}-12 \sum_{q=2}^{p-n} \frac{(-1)^{q-1}(p-q-1)!(q-1)!}{(p-1)!} \tilde{b}_{n, p-q} b_{0, q} \\
&-6 \sum_{r=1}^{n-1} \sum_{q=n-r}^{p-r} \frac{(p-q-1)!(q-1)!}{(p-1)!} \tilde{b}_{r, p-q} \tilde{b}_{n-r, q}  \tag{F10}\\
&-12 \sum_{r=1}^{[(p-n) / 2]} \sum_{q=n+r}^{p-r} \frac{(-1)^{p-q-1}(p-q-1)!(q-1)!}{(p-1)!} b_{r, p-q} \tilde{b}_{n+r, q}=0,
\end{align*}
$$

where $[p]$ is the short-hand notation for the largest integer less than $p$.
Upon examination of (F8)-(F10), it can be seen that, for a given value of $p$, the evaluation of $b_{n, p}, \tilde{b}_{n, p}(n=3,4, \ldots, p)$ from (F9),(F10) only involves quantities that are already known. However, the coefficients $b_{1, p}, \tilde{b}_{1, p} b_{2, p+1}, \tilde{b}_{2, p+1}$ and $b_{0, p+1}$ are governed by a system of coupled linear equations that can be readily derived from (F8)-(F10). It is easy to invert the $5 \times 5$ linear system and, once the coefficients $b_{1, p}, \tilde{b}_{1, p} b_{2, p+1}, \tilde{b}_{2, p+1}$ and $b_{0, p+1}$ are determined, one can move on to next integral value of $p$.

It turns out that $b_{0,2 p+1}=0$ and, consistent with (6.20),

$$
\begin{gathered}
b_{0,2 p} \sim C_{1} \frac{\sqrt{2}}{3} 2^{p+4} p(p+1)(p+2)+C_{2} 2^{p+3} p(p+1), \\
\widetilde{b}_{2, p} \sim C_{1} \frac{(-\sqrt{2})^{p+1}}{3} p(p+1)(p+2)+C_{2}(-\sqrt{2})^{p} p(p+1),
\end{gathered}
$$

with $C_{1}=-0.011, C_{2}=-0.0067$.

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[^0]:    ${ }^{1}$ This excludes, for example, a Gaussian obstacle shape which has no singularities in the finite complex plane and requires separate treatment (see §2.4).

[^1]:    ${ }^{2}$ The streamline pattern shown in Figure 4.3(a) satisfies the linearized version of the boundary condition (4.4a) that is applied on $z=0$, so the first streamline above $z=0$ does not coincide with the obstacle profile exactly. On the other hand, the nonlinear response shown in Figure 4.3(b) satisfies the nonlinear boundary condition (4.4a) on $z=\epsilon f(x)$, and the first streamline above $z=0$ coincides with the obstacle profile.

[^2]:    ${ }^{1}$ Note, however, that the first and the third harmonics also participate in deriving (6.22) (see Appendix F).

