AD-HOC MODELS

FOR

TWO-PHOTON DETECTORS

by

ROY SIDNEY BONDURANT

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Signature of Author..
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Certified by..
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Accepted by..
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ABSTRACT

The inconsistencies, both of theoretical and experimental natures, which arise from the current theory of multi-photon optical detection are pointed out. An ad-hoc model for two-photon detection, which has none of these inconsistencies is proposed, and, its performance in optical receivers employing homodyne and heterodyne detection is analyzed. It is found that the two-photon systems have signal-to-noise ratios that are independent of quantum efficiency, but are otherwise identical to the signal-to-noise ratios of the corresponding single-photon systems. It is also shown that the use of time dependent perturbation theory, to characterize multi-photon detectors is not valid when the expected number of photo-electron counts is large. Other methods of calculating the interesting statistical quantities are considered.

Thesis Supervisor: Jeffrey H. Shapiro

Title: Associate Professor of Electrical Engineering

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CHAPTER 1

INTRODUCTION

In the years since the invention of the laser, an extensive body of knowledge, called quantum mechanical communication theory, has been developed. It is the goal of this theory to determine fundamental performance limits on optical communication systems, as dictated by the laws of quantum mechanics. The recourse to quantum mechanics is necessitated by the inability of classical physics to adequately explain phenomena occuring at optical frequencies. Although classical electromagnetic theory may serve quite well to describe the propagation of optical disturbances, it fails to satisfactorily describe the interaction of optical disturbances with material media, i.e. detectors, and we find that in order to fully understand the phenomena that we observe, a quantum mechanical description is required. At optical frequencies, the uncertainty principle plays a significant role in the outcome of experiments. We can no longer assume that noise is independent of the detection process. In many cases, the simple additive white Gaussian noise model must be replaced by more complicated shot-noise models that take into account the discrete nature of light. [1]

The block diagram for an ideal quantum limited optical detection system is shown in Figure 1.1. We first have an ideal photodetector. The purpose of the photodetector is to convert

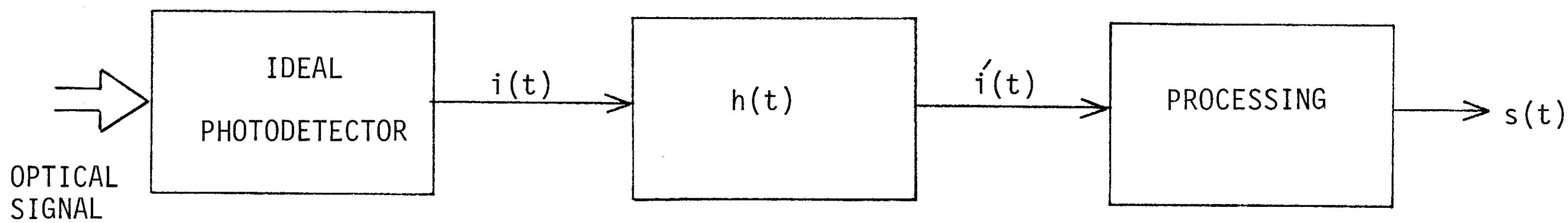

Figure 1.1: Block Diagram of Optical Receiver

optical disturbances into electrical signals. The photodetector is composed of a material from which electrons are easily ejected by the impinging photons. The detector is assumed to be ideal in the sense that its bandwidth is far broader than any other component in the system. The restriction to quantum limited detection system allows us to focus on the fundamental quantum noise process. We can add other noise processes such as dark current, thermal, and background noise later. The second block of the diagram is a filter which incorporates any non-ideal features of a "real" photodetector into whatever external filtering we may see fit to add. After this filter, we may require some additional processing (demodulation, decision/estimation, etc.) before obtaining the desired signal s(t).

Because of the discrete nature of light, a close observation of the output current i(t) reveals it to be a discontinuous signal composed of many separate "lumps" or impulses of electrons that have been ejected from the photodetector. This is shown in Figure 1.2a. The arrival times of these current impulses, which represent the times at which electrons are ejected in response to the absorption of photons, are random. We can thus associate with the output of the detector a counting process N(t). A typical sample function of N(t) is shown in Figure 1.2b. For a given time interval [0,T] if we know the total number of counts in the interval N(T), and all the arrival times $(\tau_1,\tau_2,\ldots,\tau_{N(T)})$ we know the sample function exactly. The current i(t) is a random process whose exact statistics depend upon many things including the quantum mechanical state of

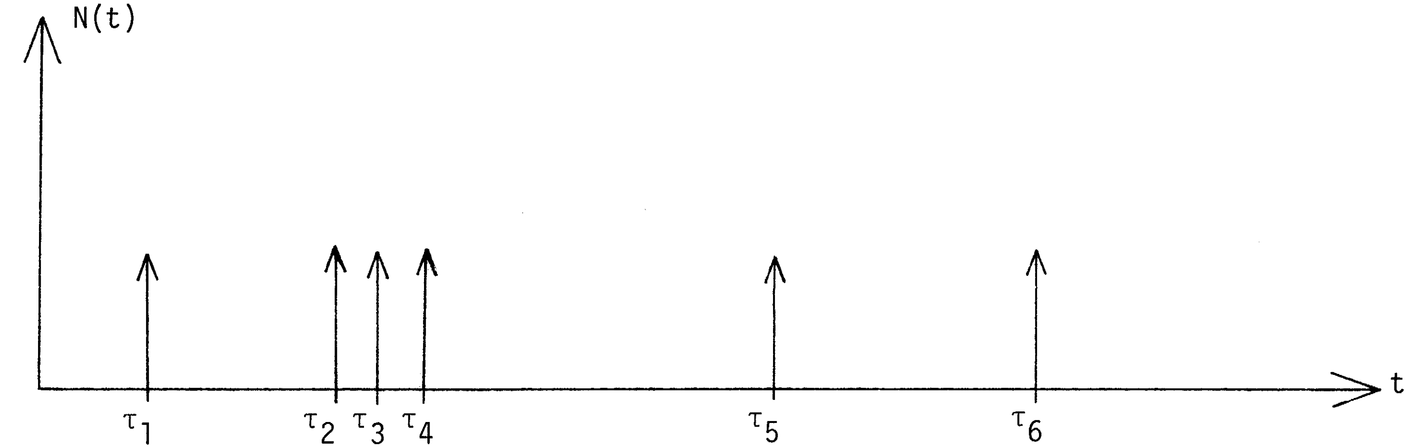

Figure 1.2a: Output of Photodetector

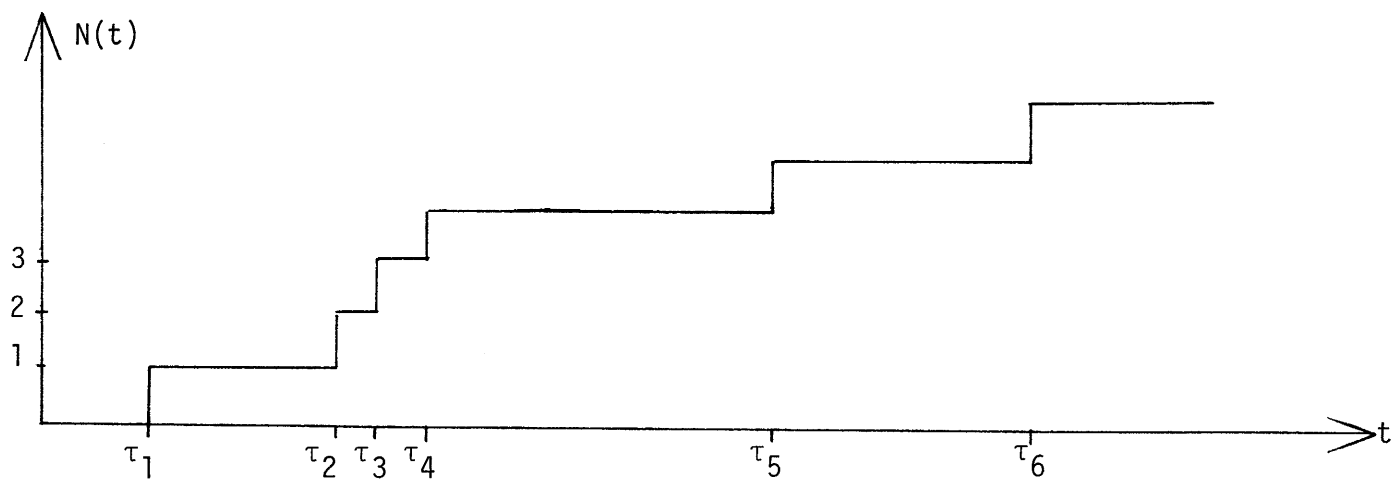

Figure 1.2b: Typical Sample Function of N(t)

the illuminating radiation. Often it is easier to first determine the statistics of N(t) and then relate them to those of i(t) via the relation

$$i(t) = e \frac{dN(t)}{dt} \tag{1.1}$$

where e is the charge of the electron. In the classical formulation, given that we know the statistics of the illuminating radiation, N(t) is always a Poisson process with a rate parameter proportional to the illuminating intensity. In the more general quantum mechanical treatment, N(t) need not be Poisson.

In the past, most work centered on receivers employing single-photon photoemissive detectors, i.e. detectors whose constituent atoms may be ionized by the absorption of a single photon of a given frequency. More recently, there has been an interest in optical systems employing detectors which must absorb more than one photon (multi-photon detectors). There are fundamental questions that arise for these detectors such as what operators can be measured by these devices, as well as more practical ones such as what performance gains, if any, can be realized by these devices if they are used in communication systems. In the remainder of this chapter the results of the theory of single photon detectors will be presented. This will serve as a stepping stone to the theory of multi-photon detectors which is the main concern of this thesis.

The quantum mechanical description of single photon photodetection is now well understood. [2]-[6] We will examine the single photon results as a conceptual point of departure for the remainder of this work, as well as to establih notation.

Field Operators

We will make extensive use of the field operators as described in [4]. Specifically, we will consider a quasimonochromatic, paraxial field illuminating a receiver with entrance pupil A, located in the z = L plane, over a time interval T. The normalized field operator at the receiver is given by the modal expansion

$$E(\overline{x},t) = \sum_{k} a_k \, \phi_k(t,\overline{x}) \tag{1.2}$$

where $\overline{x} = (x,y)$, $\{a_k\}$ are photon annihilation operators, and the $\{\phi_k(t,\overline{x})\}$ form a complete orthonormal set on A x T.

The $\{a_k\}$ obey the following commutation relations

$$[a_k, a_j^{\dagger}] = \delta_{kj} \tag{1.3}$$

$$[a_k, a_j] = 0 \tag{1.4}$$

and the field commutator is therefore

$$[E(t,\overline{x}),E^{\dagger}(t',\overline{x}')] = \delta(\overline{x}-\overline{x}')\ \delta(t-t') \tag{1.5}$$

We shall be interested in taking expectations with respect to arbitrary states of the radiation field. These expectations will be denoted either by brackets < > or by a trace, Tr ρ, where ρ is the density operator of the field. For example, the expected value of the operator $E^{\dagger}(t,\overline{x})\ E(t,\overline{x})$ is represented by

$$\langle E^{\dagger}(t,\overline{x})\ E(t,\overline{x})\rangle$$

or

$$\mathrm{Tr}(\rho E^{\dagger}(t,\overline{x})\ E(t,\overline{x}))$$

<u>Field States</u>

We shall consider certain types of field states that are of interest to communication engineers.

(1) <u>Number States</u>

The number state, denoted $|N\rangle$ for a single mode field, contains exactly N photons. For a multi mode field the notation is $|n_1,n_2,\ldots,n_n,\ldots\rangle$ where n_i is the number of photons in the i^{th} mode.

(2) <u>Coherent States</u>

The coherence state (CS) is denoted by $|\alpha\rangle$. It is the type of state that is produced by conventional light sources, lasers, ideal antennas, etc. [7]. Any optical field state that has thus far been observed in the laboratory has been either a coherent state or a classically random superposition of coherent states. Coherent states generate the well known Poisson statistics associated with optical detection. A multi-mode coherent state is denoted by $|\alpha_1,\alpha_2,\ldots,\alpha_m,\ldots\rangle$ with each α_i associated with the i^{th} mode in (1.2).

(3) Two Photon Coherent State

The two photon coherent state (TCS) is a novel quantum state that has very interesting statistical properties, but has yet to be observed in the laboratory [8]. A single mode TCS is sometimes denoted as $|\beta\rangle_g$ with the g to distinguish it from a coherent state. It can be shown [5] that in certain receiver configurations TCS radiation has the potential to yield much higher signal-to-noise ratios than the same structure using CS radiation. Often we will have a multi-mode field in which there is a single TCS mode, and all the rest are CS. In this instance, we find it convenient to use the density operator notation with the state of the field being represented by ρ where ρ is given by

$$\rho = |\beta\rangle_{g\,g}\langle\beta| \otimes \prod_i |\alpha_i\rangle\langle\alpha_i| \qquad (1.6)$$

Multicoincidence Rates

One of the problems encountered in using quantum mechanics to describe optical detection is to relate the quantum measurement performed by the detector to the random process that we take to be the output of said detector.

One particularly convenient vehicle for this purpose is the so called product density or multicoincidence rate (MCR) [2], [5],[9]. We have the following definition. For a counting process N(t) the m^{th} order MCR is

$$w_m(t_1,t_2,\ldots,t_m) \triangleq \lim_{\Delta t\to 0} Pr[(\prod_{i=1}^{k} (N(t_i+\Delta t) - N(t_i)) = 1)]/\Delta t^k \tag{1.7}$$

Knowledge of the MCRs for all $m \geq 1$ provides complete statistical characterization of N(t). More important to our work however, are the following results derived in [5] concerning just the mean and covariance functions of N(t)

$$E(N(t)) \triangleq m_N(t) = \int_0^t w_1(\tau)\, d\tau \tag{1.8}$$

$$E(N(t) - m_N(t))(N(s) - m_N(s)) \triangleq K_{NN}(t,s)$$

$$= \int_0^{\min t,s} w_1(\tau)d\tau + \int_0^t d\tau \int^s d\tau'[w_2(\tau,\tau')-w_1(\tau)w_1(\tau')] \tag{1.9}$$

Several authors have shown [2],[3], that for a single photon photoemissive detector, the MCRs are given by

$$w_m(t_1,t_2,\ldots,t_m) = \int_A d\bar{x}_1 \int_A d\bar{x}_2 \ldots \int d\bar{x}_m\ w_m(t_1,\bar{x}_1;t_2,\bar{x}_2;\ldots t_m,\bar{x}_m) \tag{1.10}$$

where

$$w_m(t_1,\bar{x}_1;t_2,\bar{x}_2;\ldots t_m,\bar{x}_m) = \eta^m \mathrm{Tr}(\rho E^\dagger(t_1,\bar{x}_1)E^\dagger(t_2,\bar{x}_2)\ldots E^\dagger(t_m,x_m) \times E(t_1,\bar{x}_1)E(t_2,\bar{x}_2)\ldots E(t_m,\bar{x}_m)) \tag{1.11}$$

$0 \leq \eta \leq 1$ is the quantum efficiency of the detector and ρ is the density operator of the field.

We now have a connection between the classical random process output by the detector and the statistics associated with the state of the field and the detection process.

Equations (1.8-1.9) follow directly from a relatively simple, straightforward application of time dependent perturbation theory for atoms in an electromagnetic field. It can also be shown [6] that the operator measured by a single photon photoemissive detector of unity quantum efficiency is

$$\int_0^t d\tau \int_A dx\ E^\dagger(\tau,\bar{x})\ E(\tau,\bar{x}) \tag{1.12}$$

In this work, we will use the term measure in the following sense: a detector whose output is a classical random process N(t) measures an operator O(t) if in all calculations involving statistics of processed versions of N(t), we obtain the same answers by using quantum expectations on the processed versions of O(t) as we do for classical expectations on the processed versions of N(t). For instance, we can replace terms like E(N(t)N(s)) with $\mathrm{Tr}(\rho\, O(t)O(s))$. We will denote this equivalence N(t) ~ O(t). So, in view of (1.12)

$$N(t) \sim \int_0^t d\tau \int_A d\bar{x}\; E^\dagger(\tau,\bar{x})\; E(\tau,\bar{x}) \qquad (1.13)$$

Equation (1.13) follows from (1.10-1.12) in the following way: In [6] it was shown that a device that measured the operator in (1.12) had the same statistics as a counting process N(t) with the MCRs of (1.9-1.10). This was done by noting (1.13) implied the same characteristic functional as a counting process with MCRs given by (1.9-1.10). Since single photon detectors were known to have such MCRs, equation (1.13) was proved.

In the next chapter, we will see now the obvious generalization of the preceding results to the case of multi-photon detectors fails. In Chapter Three we will present a model for the two photon detector that seems, thus far, to be satisfactory from both theoretical and experimental standpoints. We shall examine the performance of this model in homodyne and heterodyne receiver structures in Chapter Four. In Chapter Five, possible reasons for the breakdown of present theory will be examined. In Chapter Six, a brief summary of our results is given, along with suggestions for further work is given.

CHAPTER 2

BREAKDOWN OF THEORY

In this chapter we will examine attempts that have been made to generalize the results presented in the last chapter to the case of multiple photon absorption. As was mentioned earlier, it is desirable to do this from both the standpoint of obtaining possible gains in optical communication performance, as well as gains in understanding of the quantum measurement process.

From [6] the speculation can be made that a k-photon photoemissive detector measures the operator

$$\int_0^t d\tau \int_A d\bar{x}\ E^{\dagger k}(\tau,\bar{x})\ E^k(\tau,\bar{x}) \tag{2.1}$$

This seems to be entirely reasonable since the expressions for the MCRs were thought to be of a form similar to (1.11). Namely, for a k photon process, the m^{th} order MCR would be

$$w_m(t_1,\bar{x}_1;t_2,\bar{x}_2;\ldots t_m,\bar{x}_m) = \eta^m \mathrm{Tr}(\rho E^{\dagger k}(t_1,\bar{x}_1)E^{\dagger k}(t_2,\bar{x}_2)\ldots E^{\dagger k}(t_m,\bar{x}_m)$$
$$\times\ E^k(t_1,\bar{x}_1)E^k(t_2,\bar{x}_2)\ldots E^k(t_m,\bar{x}_m)) \tag{2.2}$$

where $\tilde{\eta}$ is an efficiency factor, $1 \leq m \leq \infty$. We may also supress the space dependency as in (1.10).

For the case of two photon absorption, we have k = 2 and (2.2), (1.8) predicts a value for $m_N(t)$ which varies as the square of the illuminating intensity. When the density operator ρ for the field is a coherent state, (2.2) in conjunction with (1.8), (1.9) yields the following results

$$m_N(t) = \tilde{\eta} \int_0^t d\tau \int_A d\overline{x} |\varepsilon(\tau,\overline{x})|^4 \tag{2.3}$$

$$k_{NN}(t,s) = \tilde{\eta} \int_0^{\min t,s} d\tau \int_A d\overline{x} |\varepsilon(\tau,\overline{x})|^4 \tag{2.4}$$

where $\varepsilon(t,\overline{x}) = \langle E(t,\overline{x}) \rangle$ is the classical field associated with this state.

It would appear that the output of a two photon detector is a Poisson process with a rate dependent upon the square of the illuminating intensity, rather than just the intensity as is the case with single photon detection. This is not surprising, since by making the apriori assumption that the output is Poisson, it can be shown that the MCRs given by (2.2) result [10].

Although the above formulation works satisfactorily when ρ describes a field with a classical analog, and leads to no theoretical or experimental inconsistencies [11], [12], if it is indeed correct, it should hold for a field in an arbitrary state. We will now show that (2.2) leads to serious inconsistencies when ρ describes a field with no classical analog.

Consider a single-mode field in a number state $|N\rangle$. The density operator for this field is

$$\rho = |N\rangle\langle N| \otimes \Pi\ |0\rangle\langle 0| \tag{2.5}$$

For $N \geq 4$, if we perform two photon absorption on the field we find, from (2.2) with $K = 2$ and (1.9), that the variance of the process $N(t)$ is

$$\mathrm{Var}(N(t)) = K_{NN}(t,t) = N(N-1)\ f(t)\ \tilde{\eta}[1+\tilde{\eta}\ f(t)(6-4N)] \tag{2.6}$$

where $\tilde{\eta}$ is the efficiency, and $f(t)$ is

$$f(t) = \int_0^t d\tau \int_A d\overline{x}|\phi_1(\tau,\overline{x})|^4 \tag{2.7}$$

where ϕ_1 is associated with the single non-vacuum mode of the field. We see that Var $N(t)$ will become negative for large values of N, clearly a contradiction. Thus, we are faced with a serious breakdown of the theory unless we allow $\tilde{\eta}$ to depend on the field state. Allowing $\tilde{\eta}$ to vary however, is inconsistent with our notion that a device must measure some operator, and it is the <u>same</u>

operator for any state of the field. A similar calculation for a TCS field also yields negative count variances so we may rest assured that the behavior of (2.6) is not solely due to some peculiarity of number states.

We are thus led to the conclusion that (2.2) is incorrect. We may obtain further, though not as strong, support for this conclusion by a close examination of experimental work. A number of authors have performed two-photon absorption experiments in which they found average detector outputs which did not vary strictly as the square of the illuminating intensity [13],[14]. As an example, Shiga and Immanura [14], found that the average number of photoelectrons obeyed

$$N_e = \beta N_p^2 + \alpha N_p \tag{2.8}$$

where N_e was the number of photoelectrons/sec cm^2, N_p was the photon flux in photon/cm^2 sec, $\beta \simeq 10^{-34}$ and $\alpha \sim 10^{-12}$.

It should be stressed that these experimental results, by themselves, are insufficient grounds for discarding (2.2). Indeed, several papers have offered explanations of the linear term in (2.8), all of which attribute it to a very broad absorption linewidth at the single photon absorption frequency [14],[15]. The linear term might also arise from impurities in the target, or emissions from the substrate upon which the target was deposited.

The results do indicate, however, that we should not assume apriori that the output of a two photon detector will be a Poisson process with a rate dependent upon the square of the illuminating intensity.

At this point, it is instructive to consider the way in which (2.2) was obtained. In a few cases quantum mechanical derivations have been attempted, but only for quantities corresponding to the mean function of a two photon process [16],[17]. In other instances, the form of (2.2) has been postulated on the basis of experiment, or as has been mentioned earlier, by an apriori assumption of Poisson statistics. However, since these experiments involved only classical fields of high intensity, no inconsistency arose. Furthermore, these experiments all measured what amounts to the mean function of the output of the detector.

Since no quantum mechanical derivation for $w_2(t_1,t_2)$, or equivalently, $K_{NN}(t,s)$ for a two photon process is to be found in the literature, the author attempted one, with disappointing results. The expression obtained by using perturbation theory for two photon absorption is identical to (2.2). Under close examination, it appears that we are trying to apply perturbation theory in a regime where it is not valid. Further discussion of this issue along with a brief sketch of the perturbation theory derivation will appear in Chapter 5.

Since we do not yet know what the correct expressions for the mean and covariance functions for multiple photon absorption are, we are free to speculate on possibilities so long as the speculations remain reasonable, and are consistent with available experimental evidence. In this spirit, and restricting ourselves henceforth to two photon processes, we will examine an ad-hoc model proposed by Shapiro that promises to meet the above requirements. Although there is no guarantee that this model is correct, it may, however, give us insight into the operation and behavior of the correct model when it is found. In the next two chapters we will present this model and evaluate its performance in certain optical communication systems.

CHAPTER 3

THE AD-HOC MODEL

In this chapter, we will develop a model for two-photon detectors based on purely ad-hoc considerations. This model, proposed by Shapiro, suffers no contradictions and agrees with the limited available experimental results. For a two-photon detector we will assume

$$N(t) \sim \int_0^t d\tau \int_A d\overline{x} \int_0^\infty d\rho \int_A d\overline{\xi} \;\; \tilde{\eta}\; s(\rho,\overline{\xi}) E^\dagger(\tau,\overline{x}) E(\tau,\overline{x}) E^\dagger(\tau-\rho,\overline{x}-\overline{\xi}) E(\tau-\rho,\overline{x}-\overline{\xi}) \tag{3.1}$$

where $\tilde{\eta}$ is a positive constant and $s(\rho,\overline{\xi})$ is a time-space sensitivity function that will be described later. Physically, this model includes two absorptions; (one for each $E^\dagger E$ pair). The first occurs at the space time point $(\overline{x}-\overline{\xi},\tau-\rho)$ and the second occurs at a later time, and different location $(\overline{x},\tau)$. The model averages over all possible space-time shifts between the absorptions, weighted by the sensitivity function $s(t,\overline{x})$. This allows us to incorporate our expectation that absorptions occurring very far apart in space and/or time have little probability of causing the emission of a photoelectron. Finally, to obtain the total photon count from the detector up until time t, the contributions from all over the detector surface are accumulated by integrating over $\overline{x}$, and the counts are accumulated by integrating over τ. We note that the operator in (3.1) is not in normal order form, and herein lies the difference between this model and the models based on the MCRs of equation (2.2).

We must put (3.1) into normal order form in order to work with it. To do so, we use (1.5) to move all the adjoint field operators to the left of all the field operators. After doing this we find

$$N(t) \sim \int_0^t d\tau \int_A d\overline{x} \int_0^\infty d\rho \int_A d\overline{\xi}\ \tilde{\eta}\ s(\overline{\rho},\overline{\xi}) E^\dagger(\tau-\rho,\overline{x}-\overline{\xi}) E^\dagger(\tau,\overline{x}) E(\tau-\rho,\overline{x}-\overline{\xi}) E(\tau,\overline{x})$$
$$x\ \tilde{\eta}\ s(0,\overline{0}) \int_0^t d\tau \int_A d\overline{x}\ E^\dagger(\tau,\overline{x}) E(\tau,\overline{x}) \qquad (3.2)$$

where $s(0,\overline{0})$ is the sensitivity function evaluated at $t = 0$, $\overline{x} = (0,0)$. Now we being to see how this model can account for the linear component in the detector output as described in the last chapter.

Before proceeding further we must place some constraints upon the sensitivity function. We will assume that $s(t,\overline{x})$ describes a joint probability distribution upon the space-time separations that give rise to an emission. That is to say, $s(\rho,\overline{\xi})\ d\rho d\overline{\xi}$ is the probability that two absorptions separated by ρ seconds in time and by vector $\overline{\xi}$ in space will give rise to an emission. This implies that $s(t,\overline{x}) \geq 0$ and

$$\int_0^{\infty} dt \int_A d\bar{x}\; s(t,\bar{x}) = 1$$

We will also assume that the space-time dependence of $s(t,\bar{x})$ is separable, i.e.,

$$s(t,\bar{x}) = \begin{cases} S'(t)\; S''(\bar{x}) & 0 \leq t < \infty, \quad \bar{x} \in A \\ 0 & \text{otherwise} \end{cases}$$

with

$$\int_0^{\infty} S'(t)dt = \int_A d\bar{x}\; S''(\bar{x}) = 1$$

This amounts to statistical independence of the space and time components of S, which cannot generally be true. Indeed, if the spatial separation of absorptions is greater than the speed of light times the temporal separation, no emission can take place at the instant of the second absorption. The above difficulty is resolved, however, by previous approximations made in obtaining the commutator (1.5), which have washed out the field causality. As a result though, our model will be invalid for extremely small values of t.

A further assumption on the properties of $s(t,\bar{x})$ is that it is much "narrower" (i.e. has much greater bandwidth) than any modulation on the field. This enables us to consider $s(t,\bar{x})$ as an impulse which will greatly simplify the calculations that follow.

A final assumption deals with the value of $s(0,\overline{0})$. In view of the experimental results discussed in the last chapter we will assume $s(0,\overline{0})$ to be non-zero.

The parameter $\tilde{\eta}$ in the model corresponds roughly to the quantum efficiency, although unlike the single photon case, $\tilde{\eta}$ is not dimensionless, but rather has the dimensions of (cm^2sec). The exact physical interpretation of $\tilde{\eta}$ is still unclear, and though we shall refer to it as an efficiency factor, there is no reason that prevents $\tilde{\eta}$ from taking on values greater than 1.

We must take care that all operators are in normal order form before performing any integrations, especially those involving $s(t,\overline{x})$. Failure to do so can result in the appearance of infinities in the results. This is because $s(t,\overline{x})$ is only approximately impulsive, whereas the use of the commutator relation (1.5) introduces true impulses. For example, let us compute $m_N(t)$, the mean detector count, in two ways. Using the normal ordered form (3.2), performing the integrations on ρ and $\overline{\xi}$ and taking quantum expectations we obtain

$$m_N(t) = \tilde{\eta} \int_0^t d\tau \int_A d\overline{x} \langle E^{\dagger^2}(\tau,\overline{x}) E^2(\tau,\overline{x}) \rangle + \tilde{\eta}\, s(0,\overline{0}) \int_0^t d\tau \int_A d\overline{x} \langle E^{\dagger}(\tau,\overline{x}) E(\tau,x) \rangle \tag{3.3}$$

where we have made the approximation $s(t,\overline{x}) \approx \delta(t)\delta(\overline{x})$. Were we to make the same approximation in (3.1) without first putting it into normal order form we would obtain

$$N(t) \sim \tilde{\eta} \int_o^t d\tau \int_A d\bar{x}\; E^\dagger(\tau,\bar{x})E(\tau,\bar{x})E^\dagger(\tau,\bar{x})E(\tau,\bar{x}) \qquad (3.4)$$

Were we now to put this into normal order form and take quantum expectations we would have

$$m_N(t) = \tilde{\eta}\int_o^t d\tau\int_A d\bar{x}\langle E^{\dagger^2}(\tau,\bar{x})E^2(\tau,\bar{x})\rangle + \tilde{\eta}\int_o^t d\tau\int_A d\bar{x}\langle E^\dagger(\tau,\bar{x})E(\tau,\bar{x})\rangle\delta(\bar{0})\delta(0) \qquad (3.5)$$

Henceforth the approach that leads to (3.3) will be used exclusively.

It is interesting to see how $m_N(t)$ predicted in (3.3) compares with what we know experimentally about two-photon absorption. Let us consider the field to be in a single mode coherent state described by the density operator

$$\rho = |N^{\frac{1}{2}}\rangle\langle N^{\frac{1}{2}}| \otimes \Pi|0\rangle\langle 0| \qquad (3.6)$$

The observation interval is T seconds long and the detector area is A (cm^2). Thus, the operator for the field mode of interest can be taken to be

$$E(t,\bar{x}) = \frac{a}{\sqrt{TA}} e^{-j\omega t} \tag{3.7}$$

Using (3.6) and (3.7) in (3.3), and defining

$$N_e \equiv \frac{m_N(t)}{TA} \tag{3.8}$$

to be the average number of remitted photoelectrons/sec cm^2, and

$$N_p \equiv N/AT \tag{3.9}$$

to be the incident photon flux in photons/sec cm^2, we obtain

$$N_e = \tilde{\eta}\, N_p^2 + \tilde{\eta}\, s(0,\bar{0})\, N_p \tag{3.10}$$

Thus we see that our model predicts a photocurrent that has both a linear and quadratic component. For high input intensities the quadratic term dominates and we have

$$N_e \approx \tilde{\eta}\, N_p^2 \tag{3.11}$$

which is the usual result for two-photon absorption, both experimentally and theoretically.

On the other hand, if N_p is not sufficiently large, the linear term will dominate. Thus, our model predicts an average photocurrent of the form observed by Shiga, Immanura, and others as discussed in Chapter 2. Finally, let us push this model to the limit by postulating an explicit form for $s(t,\overline{x})$ so that theory may be compared quantitatively with the experimental values given in the last chapter.

Many atomic systems exhibit an exponential time behavior with time constants on the order of 10^{-8} seconds, though this may vary by several orders of magnitude. Thus, we shall suppose

$$S'(t) = \begin{cases} \frac{1}{\tau} \ e^{-t/\tau} & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.12}$$

with $\tau = 10^{-8}$ sec.

The spatial extent of atomic systems is on the order of a few angstroms (1 Å = 10^{-8} cm). For simplicity, we will assume the atom or molecule doing the absorbing to be symmetric with an effective radius of say 3 Å. We shall take for $S''(\overline{x})$

$$S''(\overline{x}) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2} \tag{3.13}$$

with $\sigma = 10^{-8}$ cm. Hence, we have the space-time sensitivity function

$$s(t,\overline{x}) = (2\pi\sigma^2\tau)^{-1}\ e^{-t/\tau}\ e^{-(x^2+y^2)/2\sigma^2} \tag{3.14}$$

where

$$\begin{aligned} s(0,\overline{0}) &= (2\pi\sigma^2\tau)^{-1} \\ &= 1.6 \times 10^{23}\ (\text{sec cm}^2)^{-1} \end{aligned} \tag{3.15}$$

In Chapter 2, we presented experimentally results that would require the following values for the parameters in our model (see (2.8))

$$\begin{aligned} \tilde{\eta} &= 10^{-34}\ (\text{sec cm}^2)^{+1} \\ \tilde{\eta}\ s(0,\overline{0}) &= 10^{-12}\ (\text{sec cm}^2)^{-1} \end{aligned} \tag{3.16}$$

Our postulated form for $s(0,\overline{0})$ gives us a value for $s(0,\overline{0})$ of 10^{23}. Using the second equation in (3.16) we find that to obtain these numbers with our model, we must make $\tilde{\eta}$ equal to approximately 10^{-35} (sec cm^2). Under this assumption our simple-minded model for $s(t,\overline{x})$ gives results that are consistent with experiment to within an order of magnitude. It should be emphasized that these numbers have little value other than to give encouragement. The numbers are highly sensitive to small changes in our assumptions. Moreover more accurate forms for $S'(t)$ and $S''(\overline{x})$ could lead to substantial disagreement with experiment. This is especially true since our model provides no interpretation for $\tilde{\eta}$, and hence no way to estimate its value. We shall now turn to calculating $K_{NN}(t,s)$.

We have that

$$K_{NN}(t,s) = E(N(t)\ N(s)) - E(N(t))\ E(N(s)) \qquad (3.17)$$

so, the first step to take is to normal order the operator corresponding to N(t) N(s). Using (3.2) we find the operator corresponding to N(t) N(s) is

$$\int_0^t d\tau \int_0^s d\tau' \int_A d\overline{x} \int_A d\overline{x}' \int_0^\infty d\rho \int_0^\infty d\rho' \int_A d\overline{\xi} \int_A d\overline{\xi}'\ \tilde{\eta}^2\ s(\rho,\overline{\xi}) s(\rho',\overline{\xi}') E^\dagger(\tau-\rho,\overline{x}-\overline{\xi}) E^\dagger(\tau,\overline{x})$$
$$\times\ E(\tau-\rho,\overline{x}-\overline{\xi}) E(\tau,\overline{x}) E^\dagger(\tau'-\rho',\overline{x}'-\overline{\xi}') E^\dagger(\tau',\overline{x}') E(\tau'-\rho',\overline{x}'-\overline{\xi}') E(\tau',\overline{x}')$$
$$+\ \tilde{\eta}\ s(0,\overline{0}) \int_0^t d\tau \int_A d\overline{x} \int_0^s d\tau' \int_A d\overline{x}' \int_0^\infty d\rho' \int_A d\overline{\xi}'\ s(\rho',\overline{\xi}') E^\dagger(\tau,\overline{x}) E(\tau,\overline{x})$$
$$\times\ E^\dagger(\tau'-\rho',\overline{x}'-\overline{\xi}') E^\dagger(\tau',\overline{x}') E(\tau'-\rho',\overline{x}'-\overline{\xi}') E(\tau',\overline{x}')$$
$$+\ \tilde{\eta}\ s(0,\overline{0}) \int_0^t d\tau \int_A d\overline{x} \int_0^s d\tau' \int_A d\overline{x}' \int_0^\infty d\rho' \int_A d\overline{\xi}\ \ s(\rho,\overline{\xi}) E^\dagger(\tau-\rho,\overline{x}-\overline{\xi}) E^\dagger(\tau,\overline{x})$$
$$\times\ E(\tau-\rho,\overline{x}-\overline{\xi}) E(\tau,\overline{x}) E^\dagger(\tau',\overline{x}') E(\tau',\overline{x}')$$
$$+\ \tilde{\eta}^2\ s^2(0,\overline{0}) \int_0^t d\tau \int_0^s d\tau' \int_A d\overline{x} \int_A d\overline{x}'\ E^\dagger(\tau,\overline{x}) E(\tau,\overline{x}) E^\dagger(\tau',\overline{x}') E(\tau',\overline{x}') \qquad (3.18)$$

Using (1.5) to normal order each of the four terms, and evaluating the integrals as discussed previously, we obtain

$$\tilde{\eta}^2 \int_0^\infty d\rho \int_A d\overline{\xi}\ s^2(\rho,\overline{\xi}) \int_0^{\min t,s} d\tau \int_A d\overline{x}\ E^{\dagger^2}(\tau,\overline{x})\ E^2(\tau,\overline{x})$$

$$+\ \tilde{\eta}^2 \int_0^t d\tau \int^s d\tau' \int_A d\overline{x} \int_A d\overline{x}'\ E^{\dagger^2}(\tau,\overline{x})E^{\dagger^2}(\tau',\overline{x}')E^2(\tau,\overline{x})E^2(\tau',\overline{x}')$$

$$+\ 4\tilde{\eta}^2 \int^{\min t,s} d\tau \int_A d\overline{x}\ E^{\dagger^3}(\tau,\overline{x})\ E^3(\tau,\overline{x}) \quad \text{(from the first term of (3.18))}$$

$$+\ 2\tilde{\eta}^2\ s(0,\overline{0}) \int_0^{\min t,s} d\tau \int_A d\overline{x}\ E^{\dagger^2}(\tau,\overline{x})\ E^2(\tau,\overline{x})$$

$$+\ \tilde{\eta}^2\ s(0,\overline{0}) \int_0^t d\tau \int^s d\tau' \int_A d\overline{x} \int_A d\overline{x}'\ E^{\dagger}(\tau,\overline{x})E^{\dagger^2}(\tau',\overline{x}')E(\tau,\overline{x})E^2(\tau',\overline{x}')$$

(from the second term)

$$+\ 2\tilde{\eta}^2\ s(0,\overline{0}) \int_0^{\min t,s} d\tau \int_A d\overline{x}\ E^{\dagger^2}(\tau,\overline{x})E^2(\tau,\overline{x})$$

$$+\ \tilde{\eta}^2\ s(0,\overline{0}) \int_0^t d\tau \int^s d\tau' \int_A d\overline{x} \int_A d\overline{x}'\ E^{\dagger^2}(\tau,\overline{x})E^2(\tau',\overline{x}')E^2(\tau,\overline{x})E(\tau',\overline{x}')$$

(from the third term)

$$+\ \tilde{\eta}^2\ s^2(0,\overline{0}) \int_0^t d\tau \int^s d\tau' \int_A d\overline{x} \int_A d\overline{x}'\ E^{\dagger}(\tau,\overline{x})E^{\dagger}(\tau,\overline{x})\ E(\tau,\overline{x})E(\tau',\overline{x}')$$

$$+\ \tilde{\eta}^2\ s^2(0,\overline{0}) \int_0^{\min t,s} d\tau \int_A d\overline{x}\ E^{\dagger}(\tau,\overline{x})E(\tau,\overline{x})$$

(from the last term)

If we now perform quantum expectations and subtract off E(N(t) E(N(s)) (using (3.3)) we obtain

$$K_{NN}(t,s) = \tilde{\eta}^2 \left(4\ s(0,\bar{0}) + \int_0^\infty d\rho \int_A d\bar{\xi}\ s^2(\rho,\bar{\xi})\right) \int_0^{\min t,s} d\tau \int_A d\bar{x} \langle E^{\dagger^2}(\tau,\bar{x}) E^2(\tau,\bar{x})\rangle$$

$$+ \tilde{\eta}^2 \int^t d\tau \int^s d\tau' \int_A d\bar{x} \int_A d\bar{x}' (\langle E^{\dagger^2}(\tau,\bar{x}) E^{\dagger^2}(\tau',\bar{x}') E^2(\tau,\bar{x}) E^2(\tau',\bar{x}')\rangle$$

$$- \langle E^{\dagger^2}(\tau,\bar{x}) E^2(\tau,\bar{x})\rangle\ E^{\dagger^2}(\tau',\bar{x}') E^2(\tau',\bar{x}')\rangle)$$

$$+ \tilde{\eta}^2\ s^2(0,\bar{0}) \left(\int^{\min t,s} d\tau \int d\bar{x} \langle E^\dagger(\tau,\bar{x}) E(\tau,\bar{x})\rangle + \int^t d\tau \int^s d\tau \int_A d\bar{x} \int_A d\bar{x}'\right.$$

$$\times\ (\langle E^\dagger(\tau,\bar{x}) E^\dagger(\tau,\bar{x}) E(\tau,\bar{x}) E(\tau',\bar{x}')\rangle$$

$$\left. - \langle E^\dagger(\tau,\bar{x}) E(\tau,\bar{x})\rangle \langle E^\dagger(\tau',\bar{x}') E(\tau',\bar{x}')\rangle)\right)$$

$$+ 4\tilde{\eta}^2 \int_0^{\min t,s} d\tau \int_A d\bar{x} \langle E^{\dagger^3}(\tau,\bar{x}) E^3(\tau,\bar{x})\rangle$$

$$+ \tilde{\eta}^2\ s(0,\bar{0}) \left(\int^t d\tau \int^s d\tau' \int_A d\bar{x} \int_A d\bar{x}' \langle E^{\dagger^2}(\tau,\bar{x}) E^\dagger(\tau',\bar{x}') E^2(\tau,\bar{x}) E(\tau',\bar{x}')\rangle\right.$$

$$\left. - \langle E^{\dagger^2}(\tau,\bar{x}) E^2(\tau,\bar{x})\rangle \langle E^\dagger(\tau',\bar{x}') E(\tau',\bar{x}')\rangle\right)$$

$$+ \tilde{\eta}^2\ s(0,\bar{0}) \int_0^t d\tau \int^s d\tau' \int_A d\bar{x} \int_A d\bar{x}' \langle E^\dagger(\tau,\bar{x}) E^{\dagger^2}(\tau',\bar{x}') E(\tau,\bar{x}) E^2(\tau,\bar{x})\rangle$$

$$- \langle E^\dagger(\tau,\bar{x}) E(\tau,\bar{x})\rangle \langle E^{\dagger^2}(\tau',\bar{x}') E^2(\tau',\bar{x}')\rangle \qquad (3.19)$$

Although at first (3.19) may seem very complicated, upon close examination, it presents an interesting structure. The first two terms are essentially what one would expect for the covariance function using the old model (i.e. equation (2.1) with $k = 2$). The second two terms essentially give the covariance function for a single photon device (i.e. equation (1.9)). The remaining terms arise from the interaction of the first and second order processes.

We may now use (3.19) to calculate the variance for a single mode field in a number state. (This case gave a negative variance using the MCR model (2.2), as shown in Chapter 2.)

Using (2.5) in (3.19) we find for $N \geq 4$

$$\mathrm{Var}\, N(t) = K_{NN}(T,T)$$

$$= N(n-1)\tilde{\eta}^2 \; f(T)\left[\int_0^\infty d\tau \int_A d\bar{x}\; s^2(\tau,\bar{x}) + 6f(T) - 8f'(T)/f(T) \right.$$

$$\left. + 4N\left(\frac{f'(T)}{f(T)} - f(T)\right)\right] \tag{3.20}$$

where $f(T)$ is given by (2.7) and $f'(T) = \int_0^T d\tau \int_A d\bar{x} |\phi_1(\tau,\bar{x})|^6$. Since $\int_0^T d\tau \int_A d\bar{x}\; s^2(\tau,\bar{x}) >> 1$, the only way, in general, to have a negative variance is for the last term in (3.20) to be negative. It is easy to show that this can never be.

We must show that

$$f'(T) \geq f^2(T)$$

or

$$\left(\int_0^T d\tau \int_A d\bar{x}\, |\phi_1(\tau,\bar{x})|^4\right)^2 \leq \int_0^T d\tau \int_A d\bar{x}\, |\phi_1(\tau,x)|^6 \tag{3.21}$$

The left hand side of the inequality can be rewritten as

$$\left(\int_0^T d\tau \int_A d\bar{x}\, |\phi_1(\tau,\bar{x})|^3 \; |\phi_1(\tau,\bar{x})|\right)^2 \tag{3.22}$$

and we have from the Schwarz inequality

$$\left(\int_0^T d\tau \int d\bar{x}\, |\phi_1(\tau,\bar{x})|^3 |\phi_1(\tau,\bar{x})|^2\right)^2 \leq \int_0^T d\tau \int d\bar{x}\, |\phi_1(\tau,\bar{x})|^6 \int_0^T d\tau \int_A d\bar{x}\, |\phi_1(\tau,\bar{x})|^2 \tag{3.23}$$

The last factor in (3.23) is unity since the ϕ's are orthonormal. Thus we have

$$\left(\int_0^T d\tau \int d\bar{x}\, |\phi_1(\tau,\bar{x})|^3 |\phi_1(\tau,\bar{x})|^2\right)^2 \leq \int_0^T d\tau \int_A d\bar{x}\, |\phi_1(\tau,\bar{x})|^6 \tag{3.24}$$

thereby proving (3.21).

In other words, with the ad-hoc model (3.1) we do not predict negative variances for large values of N in number-state field detection.

So far we have shown that the model as described by (3.1) agrees with experimental data, and is well behaved. In the next chapter, we will use this model to evaluate two-photon detection performance in various communication systems. Before doing this, we will recast our results into a more useful form. In Chapter 1, we noted that the actual current output by a photodetector is proportional to the derivative of the associated counting process (see equation (1.1)). The mean and covariance functions of the current i(t), using (1.1) are given by

$$\overline{i(t)} = e \frac{d}{dt} m_N(t) \tag{3.25}$$

$$K_{ii}(t,s) = e^2 \frac{\partial^2}{\partial t \partial s} K_{NN}(t,s) \tag{3.26}$$

Applying (3.24-3.25) to (3.19) and (3.3) we find

$$\overline{i(t)} = e\tilde{\eta} \int_A d\overline{x} \langle E^{\dagger^2}(t,\overline{x}) E^2(t,\overline{x}) \rangle + e\tilde{\eta}\ s(0,\overline{0}) \int_A d\overline{x} \langle E(t,\overline{x}) E(t,\overline{x}) \rangle \tag{3.27}$$

$$
\begin{aligned}
K_{ii}(t,s) = {} & e^2\tilde{\eta}^2\left(4\ s(0,\overline{0})+\int_0^\infty d\tau\int_A d\overline{x}\ s^2(\tau,\overline{x})\right)\int_A d\overline{x}<E^{\dagger^2}(t,\overline{x})E^2(t,x)>\delta(t-s) \\
& + e^2\tilde{\eta}^2\int_A d\overline{x}\int_A d\overline{x}'(<E^{\dagger^2}(t,\overline{x})E^{\dagger^2}(s,\overline{x}')E^2(t,\overline{x})E^2(s,\overline{x})>-<E^{\dagger^2}(t,\overline{x})E\ (t,\overline{x})> \\
& \qquad \times <E^{\dagger^2}(s,\overline{x}')E^2(s,\overline{x}')>) \\
& + e^2\tilde{\eta}^2\ s(0,\overline{0})\int_A d\overline{x}<E^{\dagger}(t,\overline{x})E(t,\overline{x})>\delta(t-s) \\
& + e^2\tilde{\eta}^2\ s(0,\overline{0})\int_A d\overline{x}\int_A d\overline{x}'(<E^{\dagger}(t,\overline{x})E^{\dagger}(s,\overline{x}')E(t,\overline{x})E(s,\overline{x}')>-<E^{\dagger}(t,\overline{x})E(t,\overline{x})> \\
& \qquad \times <E^{\dagger}(s,\overline{x}')E(s,\overline{x}')>) \\
& + 4e^2\tilde{\eta}^2\int_A dx<E^{\dagger^3}(t,\overline{x})E^3(t,\overline{x})>\delta(t-s) \\
& + e^2\tilde{\eta}^2\ s(0,\overline{0})\int_A d\overline{x}\int_A d\overline{x}'<E^{\dagger^2}(t,\overline{x})E^{\dagger}(s,\overline{x}')E^2(t,\overline{x})E(s,\overline{x}')>-<E^{\dagger^2}(t,\overline{x})E^2(t,\overline{x}')> \\
& \qquad \times <E^{\dagger}(s,\overline{x}')E(s,\overline{x}')> \\
& + e^2\tilde{\eta}^2\ s(0,\overline{0})\int_A d\overline{x}\int d\overline{x}'<E^{\dagger}(t,\overline{x})E^{\dagger^2}(s,\overline{x}')E(t,\overline{x})E^2(s,\overline{x}')>-<E^{\dagger}(t,\overline{x})E(t,\overline{x})> \\
& \qquad \times <E^{\dagger^2}(s,\overline{x}')E^2(s,\overline{x}')>
\end{aligned}
\tag{3.28}
$$

In the next chapter, we will deal with systems using fields of high intensity, thus we will not need to retain all the terms in the above two equations.

CHAPTER 4

COMMUNICATION PERFORMANCE OF THE TWO-PHOTON PHOTODETECTOR

In this chapter we will examine the near-field performance of optical communication systems which use two-photon detectors. Our analysis presumes the validity of the detector model developed in the last chapter. We shall consider both homodyne and heterodyne reception configurations for systems that employ either CS or TCS transmitters.

Consider a simple analog communication scheme in which a real-valued random variable m, with density function p(m), is transmitted via linear modulation of a single mode field. That is, to transmit m we place the mode associated with operator a_1 in (1.2) into state ρ_m so that the density operator for the entire signal field is ρ_s where

$$\rho_s = \rho_m \otimes \Pi|0\rangle\langle 0| \qquad (4.1)$$

The linear modulation constraint requires

$$\mathrm{Tr}(\rho_m a_1) = Km \qquad (4.2)$$

where K is a positive constant. To permit fair comparison between various different systems we constrain the average energy of the transmitted signal to be less than or equal to N_s:

$$\int dm\ p(m)\ Tr(\rho_m\ a_1^\dagger a_1) \leq N_s \tag{4.3}$$

The performance measure we will employ is the average signal-to-noise ratio (SNR) defined by

$$SNR \equiv \frac{\int dm\ p(m)(E(y|m))^2}{\int dm\ p(m)\ var(y|m)} \tag{4.4}$$

where y is the receiver output. This formulation is identical to that employed in [5].

The general homodyne/heterodyne receiver structure, using a two-photon detector, is shown in Fig. 4.1. The signal field is combined with the local oscillator field via a beamsplitter of intensity transmission ε. Thus, the field operator for the total field falling on the detector is

$$E_D(t,\overline{x}) = \varepsilon^{\frac{1}{2}}\ E_S(t,\overline{x}) + (1-\varepsilon)^{\frac{1}{2}}\ E_L(t,\overline{x}) \tag{4.5}$$

In this analysis we shall take both the signal and local oscillator fields to be single mode, normally incident plane waves with uniform spatial variation over the receiving aperture. Both E_S and E_L have modal expansions given by (1.2). To differentiate between

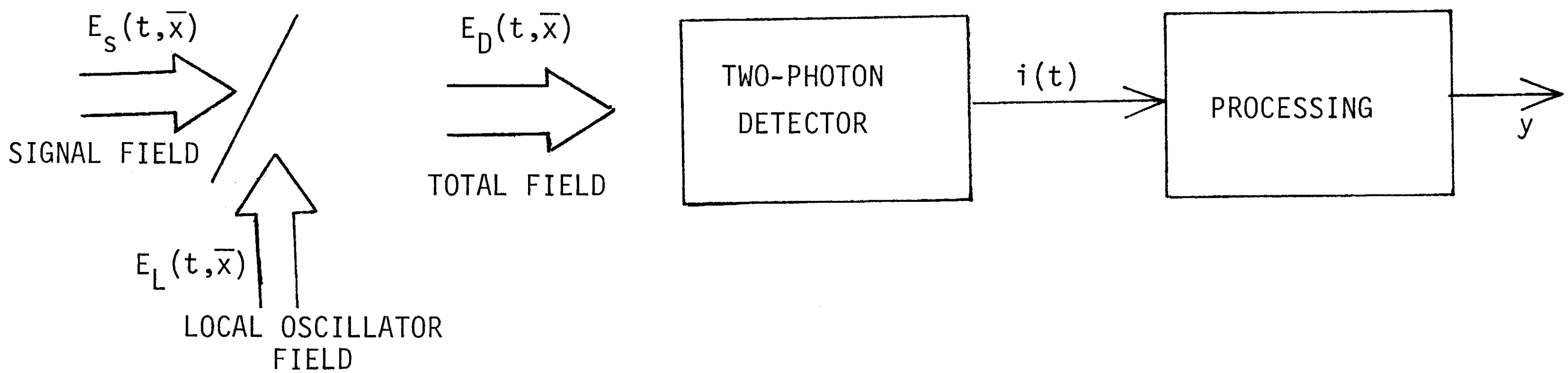

Figure 4.1: Block Diagram of Homodyne/Heterodyne Optical Receivers

the a_1 mode corresponding to E_S and the a_1 mode corresponding to E_L, we shall henceforth refer to the a_1 mode of E_S as a_S, and the a_1 mode of E_L as a_L. Thus we may write (4.4) explicitly as

$$E_D(t,\overline{x}) = \frac{e^{-j\omega_0 t}}{\sqrt{AT}} (\varepsilon^{\frac{1}{2}} a_S + (1-\varepsilon)^{\frac{1}{2}} a_L e^{j\omega_{if} t}) + \text{V.S.M.'s} \quad (4.6)$$

for $\overline{x} \varepsilon A$, $0 \leq t \leq T$, The V.S.M.'s are vacuum state modes that will give no contribution to our final results. A is detector aperture, and T is the observation interval. The density operator for the local oscillator, ρ_L, is given by

$$\rho_L = |N_L^{\frac{1}{2}}><N_L^{\frac{1}{2}}| \otimes \Pi|0><0| \quad (4.7)$$

where $|N_L^{\frac{1}{2}}>$ is a coherent state. The density operator for the total field is therefore

$$\rho_D = \rho_S \otimes \rho_L \quad (4.8)$$

The optical carrier frequency ω_0 is much greater than the intermediate frequency ω_{1f} for both heterodyne and helerodyne reception; in the case of homodyne detector $\omega_{1f} = 0$. In both cases, we will assume N_L to be much greater than the average signal energy $<a_S^{\dagger}a_S>$, and ultimately we will allow N_L to become infinite.

With this in mind, we can see that the quantities of interest in (3.27) and (3.28) can be simplified by keeping only the highest order terms. For the mean and covariance functions of the detector output we have therefore

$$\overline{i(t)} = e\tilde{\eta} \int_A d\overline{x} \langle E_D^{\dagger 2}(t,\overline{x})\ E_D^2(t,\overline{x})\rangle \tag{4.9}$$

$$\begin{aligned} K_{ii}(t,s) = e^2\tilde{\eta}^2 \int_A d\overline{x} \int_A d\overline{x}' [&\langle E_D^{\dagger 2}(t,x) E_D^{\dagger 2}(s,\overline{x}') E_D^2(t,\overline{x}) E_D^2(s,\overline{x}')\rangle \\ &- \langle E_D^{\dagger 2}(t,\overline{x}) E_D^2(t,\overline{x})\rangle\langle E_D^{\dagger 2}(s,\overline{x}') E_D^2(s,\overline{x}')\rangle] \\ &+ 4\tilde{\eta}^2 e^2 \int_A d\overline{x} \langle E_D^{\dagger 3}(t,\overline{x}) E_D^3(t,\overline{x})\rangle\ \delta(t-s) \end{aligned} \tag{4.10}$$

where we must remember that both of the above quantities are expectations conditioned upon knowing m.

In view of the uniform spatial dependence of E_D, we can achieve a notational simplication by performing the integrations in (4.9)-(4.10) and defining

$$\eta \equiv \tilde{\eta}/A_D T \tag{4.11}$$

and

$$E(t) \equiv e^{-j\omega_o t}(\varepsilon^{\frac{1}{2}} a_s + (1-\varepsilon)^{\frac{1}{2}}\ a_L e^{j\omega_{if} t}) \tag{4.12}$$

We then obtain

$$\overline{i(t)} = \frac{e\eta}{T} \langle E^{\dagger^2}(t)E^2(t)\rangle \qquad (4.13)$$

$$K_{ii}(t,s) = \frac{e^2\eta^2}{T}\Big[\langle E^{\dagger^2}(t)E^{\dagger^2}(s)E^2(t)E^2(s)\rangle - \langle E^{\dagger^2}(t)E^2(t)\rangle\langle E^{\dagger^2}(s)E^2(s)\rangle$$

$$- 4 \langle E^{\dagger^3}(t)E^3(t)\rangle T\delta(t-s)\Big] \qquad (4.14)$$

The constant η is now dimensionless and appears in the expressions in the same way a quantum efficiency would, although as mentioned before, the validity of this interpretation is still an open question. We have kept certain third order terms and left out others from the expression for $K_{ii}(t,s)$ as given by (3.28). It will turn out that only terms proportional to N_L^3 will survive, and the contributions from the neglected terms of third order in E_D in (3.28) will all cancel at this order in N_L and the remaining parts will all be negligible as $N_L \to \infty$.

Writing out (4.13) explicitly, by substituting (4.11)-(4.12) into (4.13), we obtain

$$i(t) = \frac{e\tilde{\eta}}{T}\Big[e^2\langle a_s^{\dagger^2} a_s^2\rangle + \varepsilon(1-\varepsilon)\langle a_s^{\dagger^2}\rangle N_L e^{2j\omega_{if}t}$$

$$+ 2\varepsilon^{3/2}(1-\varepsilon)^{\frac{1}{2}}\langle a_s^{\dagger^2} a_s\rangle N_L^{\frac{1}{2}} e^{j\omega_{if}t} + \varepsilon(1-\varepsilon)\langle a_s^2\rangle N_L e^{-j\omega_{if}t}$$

$$+ (1-\varepsilon)^2 N_L^2 + 2\varepsilon^{\frac{1}{2}}\langle a_s\rangle N_L^{3/2} e^{-j\omega_{if}t} + 2\varepsilon^{3/2}(1-\varepsilon)^{\frac{1}{2}}\langle a_s^{\dagger} a_s^2\rangle N_L^{\frac{1}{2}} e^{-j\omega_{if}t}$$

$$+ 2\varepsilon^{\frac{1}{2}}(1-\varepsilon)^{3/2}\langle a_s^{\dagger}\rangle N_L^{3/2} e^{j\omega_{if}t} + 4\varepsilon(1-\varepsilon)\langle a_s^{\dagger} a_s\rangle N_L\Big] \qquad (4.15)$$

Keeping only the highest order terms in (4.15) we find

$$\overline{i(t)} = \frac{e\eta}{T} N_L^2(1-\varepsilon)^2 \left[1 + \frac{4\varepsilon^{\frac{1}{2}}}{((1-\varepsilon)N_L)^{\frac{1}{2}}} \left(<a_s^\dagger> \frac{e^{j\omega_{if}t}}{2} + <a_s> \frac{e^{j\omega_{if}t}}{2} \right) \right] \tag{4.16}$$

In the case of homodyne detection $\omega_{if} = 0$ and we have

$$\overline{i(t)} = \frac{e\eta}{T} N_L^2(1-\varepsilon)^2 \left[1 + 4 \left[\frac{\varepsilon}{(1-\varepsilon)N_L} \right]^{\frac{1}{2}} \frac{<a_s^\dagger + a_s>}{2} \right] \tag{4.17}$$

In order to evaluate (4.14) we must substitute in (4.12) and expand. If we keep the highest order terms, we find only terms proportional to N_L^3 survive. The terms proportional to N_L^4 and $N_L^{7/2}$ all cancel, and terms proportional to powers of N_L less than 3 are negligible when N_L is large. After much tedious algebra we obtain

$$K_{ii}(t,s) = \frac{e^2\eta^2}{T^2} 4(1-\varepsilon)^3 N_L^3 \Big[T\,\delta(t-s) + 2\varepsilon[(<a_s^\dagger a_s> - <a_s^\dagger><a_s>)\cos\omega_{if}(t-s)]$$
$$+ \varepsilon \Big[(<a_s^{\dagger 2}> - <a_s^\dagger>^2) e^{j\omega_{if}(t+s)} + (<a_s^2> - <a_s>^2) e^{j\omega_{if}(t+s)} \Big] \Big] \tag{4.18}$$

In the case of homodyne detection (4.18) simplifies to

$$K_{ii}(t,s) = \eta^2 \frac{e^2}{T^2} 4(1-\varepsilon)^3 N_L^3 \Big[T\ \delta(t-s) + 2\varepsilon(\langle a_s^\dagger a_s \rangle - \langle a_s^\dagger \rangle \langle a_s \rangle)$$

$$+ \varepsilon(\langle a_s^{\dagger 2} \rangle - \langle a_s^\dagger \rangle^2) + \varepsilon(\langle a_s^2 \rangle - \langle a_s \rangle^2) \Big] \tag{4.19}$$

We will now consider homodyne and heterodyne detection individually.

Homodyne Detection

For homodyne detection, the block labeled "Processing" in Fig. 4.1 is shown in detail in Fig. 4.2. The first step is the removal of the constant bias term, due to the local oscillator, in (4.17). This is done by letting B in Fig. 4.2 equal $\frac{e\eta}{T} N_L^2(1-\varepsilon)^2$. This gives

$$\overline{i(t)} = \frac{e\eta}{T} N_L^2 (1-\varepsilon)^2 \left[4 \left[\frac{\varepsilon}{(1-\varepsilon)N_L} \right]^{\frac{1}{2}} \frac{\langle a_s^\dagger + a_s \rangle}{2} \right] \tag{4.20}$$

where we are still conditioned on m. We now integrate over the observation interval to obtain

$$E(y|m) = \frac{\langle a_s^\dagger + a_s \rangle}{2} = Km \tag{4.21}$$

where we have set the scaling factor H_o in Fig. 4.2 to be

$$H_o = (4\ e\eta(1-\varepsilon)^{3/2}\ \varepsilon^{\frac{1}{2}}\ N_L^{3/2})^{-1} \tag{4.22}$$

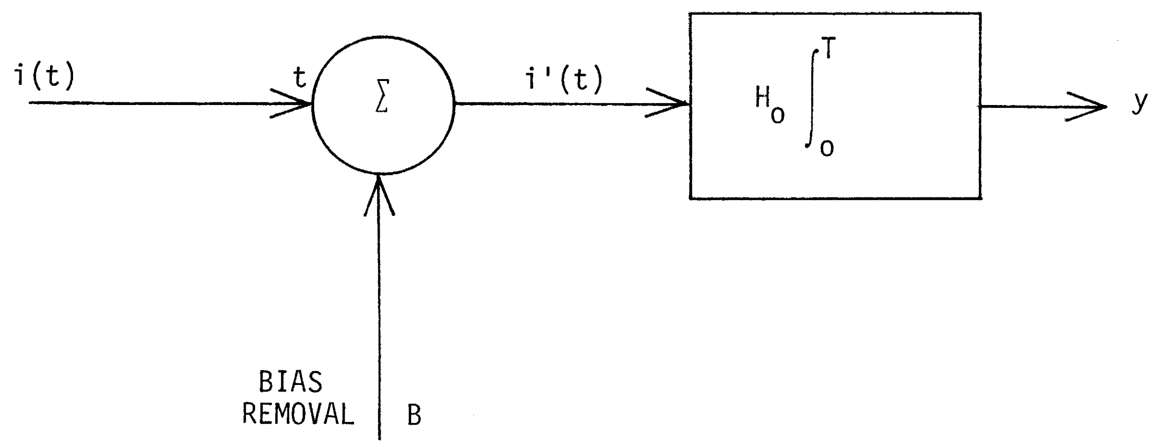

Figure 4.2: Homodyne Processing

The conditional variance can be obtained by noting that $K_{ii}(t,s) = K_{i'i'}(t,s)$ when we know m, and thus

$$\mathrm{Var}(y|m) = H_0^2 \int_0^t dt \int_0^t ds\ K_{i'i'}(t,s)$$

$$= \eta^2 e^2\ 4(1-\varepsilon)^3 N_L^3 H_0^2 (1+2\varepsilon[\langle a_s^\dagger a_s\rangle - \langle a_s^\dagger\rangle\langle a_s\rangle] + \varepsilon(\langle a_s^{\dagger 2}\rangle - \langle a_s^\dagger\rangle^2)$$
$$+ \varepsilon(\langle a_s^2\rangle - \langle a_s\rangle^2))$$

$$= \frac{1}{4}\left[2(\langle a_s^\dagger a_s\rangle - \langle a_s^\dagger\rangle\langle a_s\rangle) + \langle a_s^{\dagger 2}\rangle - \langle a_s^\dagger\rangle^2 + \langle a_s^2\rangle - \langle a_s\rangle^2 + 1\right] + \frac{1-\varepsilon}{4\varepsilon} \tag{4.23}$$

where we have used (4.19) and (4.22). If we define the operator a_{s_1} to be

$$a_{s_1} = \frac{a_s^\dagger + a_s}{2} \tag{4.24}$$

we can rewrite (4.21) and (4.23) in terms of a_{s_1} as

$$E(y|m) = \langle a_{s_1}\rangle \tag{4.25}$$

$$\mathrm{Var}(y|m) = \langle \Delta a_{s_1}^2\rangle + \frac{1-\varepsilon}{4\varepsilon} \tag{4.26}$$

where $\Delta a_{s_1} = a_{s_1} - \langle a_{s_1}\rangle$.

Note that these results have been left in terms of expectations with respect to arbitrary field states. From the above and (4.4) we find

$$\mathrm{SNR}_{H_o} = \frac{\int dm\ p(m)\ \langle a_{s_1} \rangle^2}{\int dm\ p(m)\ \langle \Delta a_{s_1}^2 \rangle + \frac{1-\varepsilon}{4\varepsilon}} \qquad (4.27)$$

In [5] the SNR for near-field homodyne detection employing a single-photon detector of quantum efficiency η was found to be (in our notation)

$$\mathrm{SNR}_{H_o} = \frac{\int dm\ p(m)\ \langle a_{s_1} \rangle^2}{\int dm\ p(m)\ \langle \Delta a_{s_1}^2 \rangle + \frac{1-\eta\varepsilon}{4\eta\varepsilon}} \qquad (4.28)$$

which is identical to (4.27) except for the appearance of the quantum efficiency in (4.28). Thus we note the potential for the two-photon device to significantly outperform single-photon devices. For unity beamsplitter transmission, the ratio of the SNR for the two-photon detector to the SNR of the single photon detector is

$$\frac{\mathrm{SNR}_{\text{Two-photon}}}{\mathrm{SNR}_{\text{One-photon}}} = 1 + \frac{1}{4\sigma^2}\left(\frac{1}{\eta} - 1\right) \qquad (4.29)$$

where $\sigma^2 = \int dm\ p(m) \langle \Delta a_{s_1}^2 \rangle$. The right-hand side of (4.29) is always greater than or equal to 1, and is much greater than 1 for $\eta \ll 1$.

This rather surprising result can be seen "physically" from the following argument based on dimensional analysis. In the single photon case the output of the detector is associated with a counting process with a mean function proportional to η and a covariance function consisting of two terms, one of which is proportional to η and the other to η^2 (see (1.8) and (1.9)). The SNR is the ratio of the squared mean of the output divided by the variance. If we divide both numerator and denominator by η^2, we see that this still leaves one term in the denominator with a $1/\eta$ dependency. In our two-photon model the dominant noise contributions are proportional to η^2 while the mean function is still proportional to η. When we form the ratio of the squared mean to the variance, the powers of η are equal in both the numerator and denominator and their effects canal. (We remind the reader that in the two-photon case the quantity η may not have the precise interpretation of an efficiency, but its position in the equations suggests we treat it as such.)

In the limit of unity beamsplitter transmission (4.27) reduces to

$$SNR_{H_0} = \frac{\int dm\ p(m)\ \langle a_{s_1} \rangle^2}{\int dm\ p(m)\ \langle \Delta a_{s_1}^2 \rangle} \tag{4.30}$$

In [5], (4.30) was maximized subject to the constraints set forth earlier with the following results. For homodyne detection the maximum attainable SNR using CS radiation is

$$\max \text{SNR}_{CS} = 4N_s \tag{4.31}$$

and the maximum attainable SNR using any field state is

$$\max \text{SNR}_{TCS} = 4N_s(N_s + 1) \tag{4.32}$$

which is realized with a TCS transmitter.

Heterodyne Detection

For heterodyne detection, the block labeled "Processing" is shown in Fig. 4.3. The first step here involves translating the signals of interest down to baseband through multiplication by $\cos \omega_{if}t$. We then integrate over the observation interval (low pass filter) to obtain the output y.

Using (4.16) in the above scheme we obtain

$$\varepsilon(y|m) = 2e\eta\ N_L^2(1-\varepsilon)^2 H_1 (\varepsilon/(1-\varepsilon)N_L)^{\frac{1}{2}} \frac{\langle a_s^\dagger + a_s \rangle}{2} \tag{4.33}$$

which can be written as

$$E(y|m) = \frac{\langle a_s^\dagger + a_s \rangle}{2} = Km \tag{4.34}$$

when we set the scale factor H_1 to be

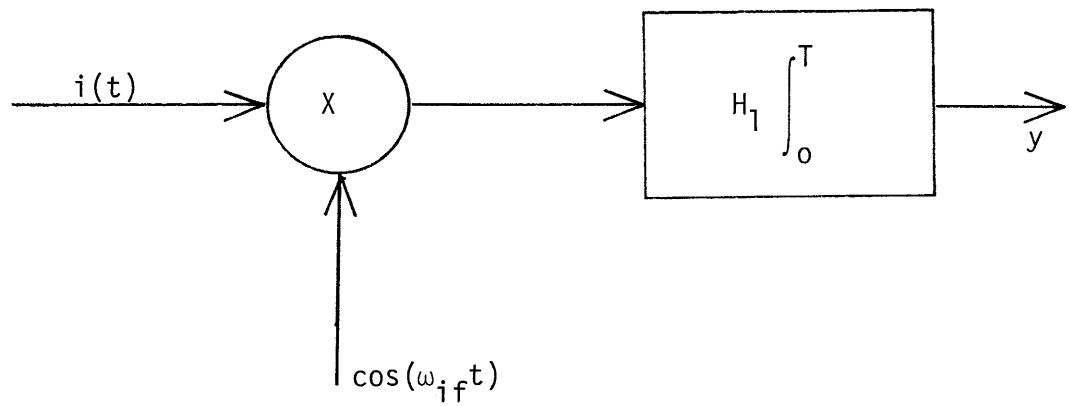

Figure 4.3: Heterodyne Processing

$$H_1 = (2e\eta\ N_L^{3/2}(1-\varepsilon)^{3/2}\ \varepsilon^{\frac{1}{2}})^{-1} \tag{4.35}$$

we can calculate the conditional variance from

$$\mathrm{Var}(y|m) = H_1^2 \int_0^t dt \int_0^t ds \cos\omega_{if}t \cos\omega_{if}s\ K_{ii}(t,s) \tag{4.36}$$

Using (4.18) and (4.35) in the above expression we obtain

$$\mathrm{Var}(y|m) = \frac{1}{\varepsilon}\left\{\frac{1}{2} + \frac{\varepsilon}{2}\left[<a_s^\dagger a_s>-<a_s^\dagger><a_s> + \frac{<a_s^{\dagger 2}>-<a_s^\dagger>^2}{2} + \frac{<a_s^2>-<a_s>^2}{2}\right]\right\} \tag{4.37}$$

Using (4.24) we can rewrite (4.34) and (4.37) as

$$E(y|m) = <a_{s_1}> \tag{4.38}$$

$$\mathrm{Var}(y|m) = \frac{1-\varepsilon}{2\varepsilon} + <\Delta a_{s_1}^2> + \frac{1}{4} \tag{4.39}$$

and thus, the SNR is

$$\mathrm{SNR}_{He} = \frac{\int dm\ p(m)\ <a_{s_1}>^2}{\frac{1-\varepsilon}{2\varepsilon} + \int dm\ p(m)\ <\Delta a_{s_1}^2> + \frac{1}{4}} \tag{4.40}$$

Equation (4.40) should be compared with the result for the single photon detector from [5]:

$$\mathrm{SNR}_{He} = \frac{\int dm\ p(m)\ \langle a_{s_1} \rangle^2}{\frac{1-\eta\varepsilon}{2\eta\varepsilon} + \int dm\ p(m)\ \langle \Delta a_{s_1}^2 \rangle + \frac{1}{4}} \tag{4.41}$$

We again see the independence of the two-photon system to quantum efficiency. For the same transmitter used in the homodyne analysis, the maximum SNR's for CS and TCS radiation in the limit of $\varepsilon \to 1$, are

$$\max \mathrm{SNR}_{CS} = 2N_s \tag{4.42}$$

$$\max \mathrm{SNR}_{TCS} = 2N_s \tag{4.43}$$

Optimizing the transmitter field state for heterodyne detection will produce at most a 3 db SNR improvement over (4.43) as argued in [3].

We have seen that by using the detector described by (3.1) in homodyne/heterodyne receiving systems, we can obtain signal-to-noise ratios that are independent of the detector efficiency factor, but otherwise equivalent to the performance limits of single-photon devices. The freedom from quantum efficiency dependence would make two-photon devices superior to single photon devices at operating wavelengths where the quantum efficiency of the single photon device becomes substantially less than unity or, in circumstances where the quantity $\int dm\ p(m) \langle \Delta a_{s_1}^2 \rangle$ is much less than 1 (see equation (4.29)). Physically, the η independence arises from the nature of the dominant noise contribution and exploiting this might be a useful way of verifying the validity of our model.

CHAPTER 5

THE FAILURE OF PERTURBATION THEORY

As was mentioned in Chapter 2, the incorrectness of (2.2) is probably due to an improper application of time-dependent perturbation theory. In this chapter we will attempt to clarify this idea as well as explore some possible alternatives to the perturbative approach. This is necessary because, as the title of this thesis indicates, the model that has been described thus far is purely ad-hoc in nature. If it is indeed correct, it should be possible to derive it starting from fundamental physical principles.

In deriving expressions for the MCR's via time dependent perturbation theory the problem to be solved is the following: given the state of the system (field plus atom) at time $t = t_0$, what is the state at $t = t_1$ when the system is acted upon by an interaction Hamiltonian $H_I(t)$? The solution is well known and can be found in any text on quantum mechanics. The argument will be sketched below.

Consider a system whose state at $t = t_0$ is $|\psi(t_0)\rangle$, and that is acted upon by an interaction Hamiltonian $H_I(t)$. The state at a later time $t = t_1$ is then $|\psi(t_1)\rangle$ given by

$$|\psi(t_1)\rangle = U(t_1,t_0)|\psi(t_0)\rangle \qquad (5.1)$$

where $U(t_1,t_0)$ is the state-evolution operator

$$U(t_1,t_0) = 1 + \frac{1}{i\hbar}\int_{t_0}^{t_1} dt' H_I(t') + \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^{t_1} dt' \int_{t_0}^{t'} dt'' H_I(t') H_I(t'') + \ldots \tag{5.2}$$

The series expansion is necessary since the Hamiltonian operators do not, in general, commute at different times. The probability of finding the system in a specified state $|\phi\rangle$ at $t = t_1$ is then $|\langle\phi|\psi(t_1)\rangle|^2$ or

$$|\langle\phi|\mu(t_1,t_0)|\psi(t_0)\rangle|^2 \tag{5.3}$$

In first order theory, it is assumed that the term of first order in $H_I(t)$ in (5.2) provides the major contribution to the probability in (5.3), and all the rest are neglected. For the case of photon absorption by an atom, the interaction Hamiltonian is

$$H_I(t) = -e\ q(t)\ \mathcal{E}(t,\bar{r}) \tag{5.4}$$

In this expression $\bar{r}$ is the position of the nucleus, $q(t)$ is the position operator of the electron relative to the nucleus, e is the charge of the electron and $\mathcal{E}(t,\bar{r})$ is the field operator

$$\mathcal{E}(t,\bar{r}) = E(t,\bar{r}) + E^{\dagger}(t,\bar{r}) \tag{5.5}$$

From this point, several authors have gone on to develop a successful theory of single-photon absorption. By successful, we mean a theory that does not result in any inconsistencies, and one that agrees well with experimental results. We will examine how to extend this development to the case of two-photon absorption.

To deal with two-photon absorption, we must use second-order theory. Thus, we are assuming that the third term in (5.2) is small and that any terms beyond that are inconsequential.

Our state vector must jointly describe the state of the atom and the field. At $t = t_0$ the field will be in some initial state, and the atom will be in its ground state. Thus, at $t = t_0$ the state vector will be denoted as

$$|i,g\rangle \tag{5.6}$$

where the i corresponds to the initial state of the field and g corresponds to the ground state of the atom. At $t = t_1$, the state vector will be denoted as

$$|f,e\rangle \tag{5.7}$$

where f corresponds to a final state of the field and e corresponds to an excited atomic state.

Before proceeding further, it will be helpful to rewrite some of the above in a form that is convenient for generalization to systems of more than one atom. In photocounting experiments, what is done in essence is to open a shutter in front of the atom for $t_1 - t_0$ seconds and then close it. This is effectively the same procedure as turning on the interaction Hamiltonian at $t = t_0$ and turning it off at $t = t_1$. Thus, we may rewrite (5.4) as

$$H_I(t) = -e\ q(t)\ \mathcal{E}(t,\bar{r})(\mu(t-t_0) - \mu(t-t_1)) \qquad (5.8)$$

where $\mu(t)$ is the unit step function. We may now let the integrals in (5.2) range from 0 to ∞

$$\mu(t_1,t_0) = 1 + \frac{1}{i\hbar}\int_0^\infty dt'\ H_I(t) + \left(\frac{1}{i\hbar}\right)^2 \int_0^\infty dt' \int_0^{t'} dt'' H_I(t')H_I(t'') + \ldots \qquad (5.9)$$

where the integration limits are now implicitly contained in the Hamiltonians.

The initial and final atomic states are orthogonal, and since we are interested only in two-photon absorption, we will restrict ourselves to quasimonochromatic input fields (initial field states) whose photons possess half the energy necessary to ionize the atom. Thus, the only contributing term in (5.9) is the third [16], and we must calculate the quantity

$$<f,e|\left(\frac{1}{i\hbar}\right)^2 \int_0^\infty dt' \int_0^{t'} dt''\ H_I(t')\ H_I(t'')|ig> \qquad (5.10)$$

Substituting (5.8) and (5.5) into the above, we obtain

$$<f,e|\left(\frac{e}{i\hbar}\right)^2\int_0^\infty dt'\int_0^{t'} dt''\ q(t')(E^\dagger(t,\bar{r})+E(t,\bar{r}))q(t'')(E^\dagger(t'',\bar{r})+E(t'',\bar{r}))$$

$$\times\ (\mu(t'-t_0)-\mu(t'-t_1)(\mu(t''-t_0)-\mu(t''-t_1))|ig> \tag{5.11}$$

The field operators act only on that part of the state vector corresponding to the field, and the atomic operators act only upon that part of the vector corresponding to the atom. Thus, we can rewrite (5.11) as

$$\left(\frac{e}{i\hbar}\right)^2\int_0^\infty dt'\int_0^\infty dt''<e|q(t')q(t'')|g><f|E^\dagger(t',\bar{r})+E(t',\bar{r})\ E^\dagger(t'',\bar{r})+E(t'',\bar{r})|i>$$

$$\times\ \mu(t'-t'')(\mu(t-t_0)-\mu(t-t_1'))(\mu(t-t_0)-\mu(t-t_1)) \tag{5.12}$$

where the use of the step function $\mu(t'-t'')$ allows us to let both the t' and t" integrals have the same limits.

In the interaction picture that we are using, the time dependent position operators q(t), are given by

$$q(t) = e^{\frac{iH_{at}t}{\hbar}}\ q(0)\ e^{-\frac{iH_{at}t}{\hbar}} \tag{5.13}$$

where H_{at} is the unperturbed atomic Hamiltonian, and q(0) is the position operator at t = 0. Both |e> and |g> are eigenstates of

H_{at}, and we will use the convention $H_{at}|g\rangle = 0$ and $H_{at}|e\rangle = e|e\rangle$.

If we substitute (5.13) into (5.12) and also employ the identity operator

$$I = \sum_j |j\rangle\langle j| \tag{5.14}$$

where the $\{|j\rangle\}$ comprise the complete set of eigenvectors of H_{at}, we obtain

$$\left(\frac{e}{i\hbar}\right)^2 \int_0^\infty dt' \int_0^\infty dt'' \; e^{i\omega_e t'} \sum_j q_{ej} q_{jg} e^{-i\omega_j (t'-t'')} \langle f|(E^\dagger(t',\bar{r})+E(t',\bar{r}))$$
$$\times\ (E^\dagger(t'',\bar{r})+E(t'',\bar{r})|i\rangle \mu(t'-t'')(\mu(t'-t_0)-\mu(t'-t_1))$$
$$\times\ (\mu(t''-t_0)-\mu(t''-t_1)) \tag{5.15}$$

where $q_{\ell,m} = \langle \ell|q(0)|m\rangle$, and $\omega_j = j/\hbar$.

In general, there will be many possible final states of the field. We are interested only in those final states $|f\rangle$ that can be reached by the annihilation of two-photons with no reradiation and/or reabsorption. We will have to sum over all appropriate states when we calculate the magnitude squared of (5.15). This sum can be extended over a complete set of states if we first notice that those terms of interest will come only from that part of the field operator expectation in (5.15) that has only the two positive frequency parts of the field. Thus, we keep only the term corresponding to two annihilations (i.e. $\langle f|E(t',\overline{r})E(t'',\overline{r})|i\rangle$) in (5.15).

We also do not, in general, have exact knowledge of the initial field state so we must average over all such states through use of a density operator ρ. Keeping all this in mind, we find that the probability of going from the ground state to an excited state in t_1-t_o seconds is via (5.3)

$$\begin{aligned}
Pr_{g\to e}(t_1,t_o) = \left(\frac{e}{\hbar}\right)^4 \int_0^\infty dt' \int_0^\infty dt'' \int_0^\infty ds' \int_0^\infty ds''\ e^{i\omega_e(t'-s')} \sum_j q_{ej}q_{jg} e^{-i\omega_j(t'-t'')} \\
\times\ \mu(t'-t'') \sum_r q^*_{er}q^*_{eg} e^{i\omega_r(s'-s'')} \mu(s'-s'') \\
\times\ Tr(\rho E^\dagger(s',\overline{r})E^\dagger(s'',\overline{r})E(t',\overline{r})E(t'',\overline{r})) \\
\times\ (\mu(t'-t_o)-\mu(t'-t_1))(\mu(t''-t_o)-\mu(t''-t_1)) \\
\times\ (\mu(s'-t_o)+\mu(s'-t_1))(\mu(s''-t_o)-\mu(s''-t_1)) \qquad (5.16)
\end{aligned}$$

Now, let us consider the factor in (5.16)

$$\left(\frac{e}{\hbar}\right)^2 \sum_j q_{ej}q_{jg}\ e^{-i\omega_j(t'-t'')} \mu(t'-t'') \qquad (5.17)$$

If we use the approximation for a step function

$$\mu(t) \approx \frac{i}{2\pi}\int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega t}}{\omega+i\varepsilon} \qquad \varepsilon \ll 1 \;, \tag{5.18}$$

we can rewrite (5.17) as

$$\int \frac{d\omega_1}{2\pi}\, g_e(\omega_1)\; e^{-i\omega_1(t'-t'')} \tag{5.19}$$

where

$$g_e(\omega_1) = i\left(\frac{e}{\hbar}\right)^2 \sum_j q_{ej}q_{jg}\, \frac{1}{\omega_1-\omega_j+i\varepsilon}$$

and we have used the change of variables $\omega_1 = \omega + \omega_j$. Thus (5.16) becomes

$$\begin{aligned} Pr_{g\to e}(t_1,t_o) = &\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi}\int_0^{\infty} dt' \int_0^{\infty} dt'' \int_0^{\infty} ds' \int_0^{\infty} ds''\; e^{-i\omega_1(t'-t'')} \\ &\times e^{i\omega_2(s'-s'')}\; e^{i\omega_e(t'-s')}\; g_e(\omega_1)g_e^*(\omega_2) \\ &\times \mathrm{Tr}(\rho E^\dagger(s',\bar{r})E^\dagger(s'',\bar{r})E(t',\bar{r})E(t'',\bar{r})) \\ &\times (\mu(t'-t_o)-\mu(t'-t_1))(\mu(t''-t_o)-\mu(t''-t_1)) \\ &\times (\mu(s'-t_o)-\mu(s'-t_1))(\mu(s''-t_o)-\mu(s''-t_1)) \end{aligned} \tag{5.20}$$

In most detectors, the excited states approximate or form a continuum. Thus we will average over all excited states $|e\rangle$ with a weight P(e) which is the probability that an electron in excited state $|e>$ is registered by the post detection counter [3]. The probability of detecting two photons, or equivalently, of getting a count in $t_1 - t_o$ seconds is therefore

$$\Pr[N(t_1) - N(t_o) = 1] = \sum_e P(e)\, P_{g\to e}(t_1, t_o) \tag{5.21}$$

Following Glauber [3] we introduce a sensitivity function

$$S(\omega_3) \equiv 2\pi \sum_e P(e)\, g_e(\omega_1)\, g_e^*(\omega_2)\, \delta(\omega_3 - \omega_e) \tag{5.22}$$

Using (5.22) and (5.20), we rewrite (5.21) as

$$\Pr[N(t_1)-N(t_o)=1] = \int_{-\infty}^{\infty} \frac{d\omega_3}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \int_0^{\infty} dt' \int_0^{\infty} dt'' \int_0^{\infty} ds' \int_0^{\infty} ds''$$

$$\times\, e^{i\omega_3(t'-s')}\, e^{i\omega_2(s'-s'')}\, e^{-i\omega_1(t'-t'')}\, \mathrm{Tr}(\rho E^\dagger(s',\bar{r}) E^\dagger(s'',\bar{r}) E(t',\bar{r}) E(t'',\bar{r}))$$

$$\times\, (\mu(t'-t_o)-\mu(t'-t_1))(\mu(t''-t_o)-\mu(t''-t_1))(\mu(s'-t_o)-\mu(s'-t_1))$$

$$\times\, (\mu(s''-t_o)-\mu(s''-t_1)) \tag{5.23}$$

We now assume that the detector is extremely broadband relative to the bandwidth of the incident field and so take $s(\omega_3)$ to be approximately constant

$$s(\omega_3) \approx \eta \tag{5.24}$$

We can now remove it from the integral and perform the ω_3 integration obtaining an impulse. We can similarly find impulses from the other ω integrals. Integrating out the impulses leads us to our final result

$$Pr[N(t_1)-N(t_0)=1] = \eta\int_0^\infty dt' \; Tr[\rho E^{2\dagger}(t',\bar{r})E^2(t',\bar{r})](\mu(t'-t_0)-\mu(t'-t_1))$$

By adjusting the limits of integration, we can remove the step functions from the above expression and obtain

$$Pr[N(t_1)-N(t_0)=1] = \eta\int_{t_0}^{t_1} dt' \; Tr(\rho E^{\dagger^2}(t',\bar{r})E^2(t',\bar{r})) \tag{5.25}$$

We are interested in times such that $t_1 = t_0 + \delta$ where δ is tending towards zero. Passing to this limit, and dividing both sides of (5.25) by δ we obtain

$$\lim_{\delta\to 0} Pr(N(t_0+\delta)-N(t_0)=1)/\delta = \lim_{\delta\to 0} \eta\int_{t_0}^{t_0+\delta} dt' Tr[\rho E^{\dagger^2}(t',\bar{r})E^2(t',\bar{r})]/\delta \tag{5.26}$$

The left hand side of (5.26) is just the first order MCR for the process N(t) (by (1.11)) and the right side is the derivative of the integral. Thus

$$\omega_1(t,\bar{r}) = \eta \ \mathrm{Tr}(\rho \ E^{\dagger^2}(t,\bar{r})E^2(t,\bar{r})) \tag{5.27}$$

If we take $E(t,\bar{r})$ to consist of paraxial plane waves as in [4], the z-dependence of the field will be of the form $e^{jk_z z}$ and (5.27) reduces to

$$\omega_1(t,\bar{x}) = \eta \ \mathrm{Tr}(\rho \ E^{\dagger^2}(t,\bar{x})E \ (t,\bar{x})) \tag{5.28}$$

To calculate higher order MCRs, we must add more atoms to the detector, and go to higher order perturbation theory. For instance to calculate $\omega_2(t_1,t_2)$ our Hamiltonian would be

$$H_I(t) = -e[q_1(t)\varepsilon(t,\bar{r}_1)(\mu(t-t_1)-\mu(t-(t_1+\delta)))+(\mu(t-t_2)-\mu(t-(t_2+\delta))) \times q_2 \ \varepsilon(t,\bar{r}_2)]$$

and we would have to use the fourth-order term in (5.9). The manipulations are the same; there are just four times as many and ultimately we obtain

$$\omega_2(t_1,\bar{x}_1;t_2,\bar{x}_2) = \eta^2\mathrm{Tr}(\rho \ E^{\dagger^2}(t_1,\bar{x}_1)E^{\dagger^2}(t_2,\bar{x}_2)E^2(t_1,\bar{x}_1)E^2(t_2,\bar{x}_2))$$

The above derivation closely parallels those of Glauber [3] and Mollow [16] and the reader is referred to these papers for further details.

It would appear from the preceding development that we have a derivation of the MCRs for two-photon absorption. Yet, as was seen in Chapter 2, these MCRs cannot be correct. We will now examine certain inconsistencies associated with the perturbative approach.

Consider the average number of counts one registers over the interval (0,t). By (1.8), the expected number of counts is

$$E(N(t)) = \int_0^t \omega_1(\tau)d\tau$$

but, by (5.27) and (5.26) E(N(t)) is

$$E(N(t)) = \eta \int_0^t dt' \; Tr(\rho \; E^{\dagger^2}(t,\overline{r}),E^2(t,\overline{r})) \qquad (5.29)$$

and by (5.25) we have

$$E(N(t)) = Pr(N(t) - N(0) = 1) \qquad (5.30)$$

As we increase the field amplitude, the expected number of counts over a finite interval will also increase and can easily be much greater than 1. On the other hand, from (5.30) we see E(N(t)) should never exceed 1, since a probability is always less than or equal to one. We do note however, that in the limit of small E(N(t)) or equivalently small field amplitudes, $E(N(t)) \simeq Pr(N(t) = 1)$ and in this case the results of (5.30) are consistent. Thus it would seem, that when the field amplitudes become large, so that the expected number of counts >>1, perturbation theory is no longer a valid means for determining the probability of absorption. As further support we recall the negative variance behavior seen in Chapter 2 resulted when the number of photons in the field, and hence the amplitude of the field became large. It is probably incorrect to assign all of our problems to the breakdown of perturbation theory for strong fields, because the same inconsistencies are present in the derivation of the MCRs for the single-photon case, yet the single-photon results (equation (1.11)) work well in all applications thus far encountered. Whether the apparent correctness of the single photon perturbation results is merely fortuitous or has deeper significance is still an open question.

It would seem that the only way to correctly determine the two-photon absorption MCRs is by employing some method other than perturbation theory to solve Schrödinger's equation for the atom-field system. A number of authors have done work towards this end [18],[19],[20]. Of these studies, only one was specifically directed at multi-photon absorption [18], the other two consider only single-photon absorption. Unfortunately, the work on multi-photon absorption, as it now stands, is not valid for small k (i.e. $k = 2$; two-photon absorption). The other two methods give results which agree with the perturbation analysis. Of these two the method used in [20] seems to be the most promising as far as generalization to two-photon absorption is concerned. The density matrix approach is used, with no approximations other than to assume that the number of atoms in the detector is large, to derive the counting distribution. The method also predicts a time varying quantum efficiency of the form $\eta(t) = \eta(1-e^{-\gamma t})$ where the parameter γ is proportional to the number of atoms in the detector, and the atomic transition rate atoms from the ground state to an excited state. If this method could be generalized to calculate the counting distribution for two-photon absorption, then we would have a check for the mean and covariance functions predicted by the model developed in Chapter 3.

There is another possible approach to the problem, and that is to solve (5.9) diagramatically. Usually, one resorts to perturbation theory when the expansion in (5.9) is too complicated to reduce, as is almost always the case. If we could determine a closed form solution for $U(t_1,t_0)$ we could use it freely, since the infinite series in (5.9) is the exact solution.

There is a graphical approach to solving (5.9) which uses devices known as Feynman Diagrams. The diagrams are essentially "pictures" of each term in (5.9). Each term may have more than one diagram associated with it since there may be more than one process contributing to it. For instance, there may be many different physical processes involving reradiation, and reabsorption which have the same effect we are looking for, namely the net loss of two photons from the field, and the atom in an excited state. Perturbation theory considers only the simplest (lowest order) way for this to occur. The Feynman diagrams are in unique correspondence to the processes they describe. Thus, if we know the diagrammatic expansion of (5.9) for the case of the photo-electric effect we could see immediately if there were any higher order terms of interest, selectively sum over the appropriate terms and obtain an expression for $U(t_1,t_0)$ that is more complete than the perturbative approximation. An infinite number of such higher-order processes might in fact contribute significantly. For further information, the reader is referred to one of the texts on the many-body problem, and especially [21] as an introduction to Feynman diagrams.

CHAPTER 6

SUMMARY AND CONCLUSIONS

We have seen that the natural generalization of the multicoincidence rates to two-photon absorption leads to inconsistent results. Specifically, the use of the expression

$$w_m(t_1,\bar{x}_1;t_2,\bar{x}_2;\ldots t_m,\bar{x}_m) = \tilde{\eta}^m \, \mathrm{Tr}(\rho \, E^{\dagger^2}(t_1,\bar{x}_1)E^{\dagger^2}(t_2,\bar{x}_2)\ldots E^{\dagger^2}(t_m,\bar{x}_m) \times E^2(t_1,\bar{x}_1)E^2(t_2,\bar{x}_2)\ldots E^2(t_m,\bar{x}_m)) \tag{6.1}$$

for the m^{th} order multicoincidence rate for a two-photon detector leads to a negative count variance (see equation (2.6)) when ρ describes a single mode field in a number state. Moreover, when ρ describes a classical field, although we get no theoretical breakdowns, the results do not always agree with available experimental evidence. We have noted that similar inconsistencies arise when TCS radiation is considered, and we have seen that an attempt to derive the correct MCRs via perturbation theory leads back to (6.1).

At this point, a model for two-photon detection was proposed on a purely ad-hoc basis. This model assumes that the operator measured by a two-photon detector is

$$\int_0^t d\tau \int_A d\overline{x} \int_0^\infty d\rho \int_A d\overline{\xi}\ \tilde{\eta}\ s(\rho,\overline{\xi})\ E^\dagger(\tau,\overline{x})E(\tau,\overline{x})E^\dagger(\tau-\rho,\overline{x}-\overline{\xi})E(\tau-\rho,\overline{x}-\overline{\xi}) \tag{6.2}$$

Use of the above expression yields results which differ significantly from those obtained from (6.1). They behave properly, and can be made to agree with all available experimental results.

When the model in (6.2) is applied to optical communication systems employing other homodyne or heterodyne receivers we find that the performance is equivalent to that of single-photon systems except that the two-photon systems have signal-to-noise ratios that are independent of quantum efficiency. Thus, a two-photon receiver has the potential for significantly better performance at those wavelengths where high quantum efficiency single-photon detectors cannot be fabricated.

In the last chapter we saw why perturbation theory gives inconsistent results. We discovered that the perturbation approximation becomes invalid when $E(N(t)) >> 1$, even though this approach applied to single-photon devices gives perfectly usable results with no apparent inconsistencies. We mentioned several alternatives to perturbation theory that might be employed in deriving counting statistics and/or multi-coincidence rates for multi-photon detectors.

At this time, the question of what the correct MCRs are is still unanswered. Further research into this area could be in one of two directions. First, there should be some work towards checking experimentally the validity of the model presented here. A simple photo-counting experiment which measured the dependence of the count variance upon intensity would go a long way towards establishing the validity of this model. A more elaborate experiment could be undertaken to verify the quantum efficiency independence predicted for homodyne receivers.

Regardless of whether this model is a good one, work should be done to establish the interesting statistical quantities from fundamental physical principles, because only after we have done this will we have a thorough understanding of two-photon absorption.

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