The Geometry of the Generic Line Complex

by

Joshua S. Sher

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Author ....
Department of Mathematics
May 1, 1995

Certified by ....
Victor Guillemin
Professor of Mathematics
Thesis Supervisor

Accepted by ....
David Vogan
Chairman, Departmental Committee on Graduate Students

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Abstract

In this paper, we study the Radon transform from $\mathbf{RP}^3$ to the Lagrangian Grassmanian. We use the representation theory of $Sp(4, \mathbf{R})$ to characterize the kernel and the range of the Radon transform. We explicitly construct a Fourier integral operator on $\mathbf{RP}^3$ which picks off the kernel for us and we give a number of descriptions of its associated canonical transformation.

Thesis Supervisor: Victor Guillemin
Title: Professor of Mathematics

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Chapter 0
Introduction

In this paper we study the Radon transform from $\mathbf{RP}^3$ to the Lagrangian Grassmanian. Much of the background material discussed in Chapter 1 can be found in Victor Guillemin's book: *Cosmology in (2 + 1)-Dimensions, Cyclic Models, and Deformations of* $M_{2,1}$.

In Chapter 1 we give a number of descriptions of the Lagrangian Grassmanian and the Radon transform. The most important fact we use is that both $\mathbf{RP}^3$ and the Lagrangian Grassmanian are homogeneous spaces for the ten dimensional group $Sp(4, \mathbf{R})$. Accordingly, we give a $Sp(4, \mathbf{R})$-equivariant description of the Radon transform.

In Chapter 2 we use the representation theory of $Sp(4, \mathbf{R})$ to identify the kernel and range of the Radon transform. Furthermore, we construct geometrically an operator that picks off the kernel for us. Finally, we show that this operator is actually a Fourier integral operator corresponding to an interesting involution of $T^*\mathbf{RP}^3 - 0$ and we give a few descriptions of this involution.

In Appendix 1, we discuss the canonical relation associated to the Radon transform. In Appendix 2, we discuss some facts about the representation theory of $U(2)$ which is the maximal compact subgroup of $Sp(4, \mathbf{R})$.

Chapter 1
Background

Section a:
Some Motivation

Consider the following: Take a function $f(x)$ on $\mathbf{RP}^3$, a metric on $\mathbf{RP}^3$, and a line γ on $\mathbf{RP}^3$ (which you can think of as a great circle on S^3) and form the integral

$$\int_\gamma f(x)ds$$

where ds is the arc length derived from the metric. The space of lines on $\mathbf{RP}^3$ is, as a manifold, $G_{2,4}$ - the Grassmanian of 2-planes in 4-space. Thus the above defines a smooth map

$$R : C^\infty(\mathbf{RP}^3) \to C^\infty(G_{2,4})$$

–called the Radon transform (or the x-ray transform). It is well known that this map is injective, so given the integral of a function over every line we can recover the function.

Since the dimension of $G_{2,4}$ is bigger then the dimension of $\mathbf{RP}^3$, an obvious question is: can we determine a function on $\mathbf{RP}^3$ with less information? To put this more precisely, for what 3 dimensional hypersurfaces X of $G_{2,4}$ is the map $R : C^\infty(\mathbf{RP}^3) \to C^\infty(X)$ injective. We will call such a hypersurface admissible. Gelfand and Graev gave a characterization of these hypersurfaces.

Theorem 1 (Gelfand and Graev) *A hypersurface X is admissible iff X is locally (near a generic point) either:*

a) the set of lines incident to some non-singular curve in $\mathbf{RP}^3$, or

b) the set of lines tangent to some smooth surface in $\mathbf{RP}^3$.

In this paper we will look at the simplest example of a non-admissible complex of lines (i.e. a non-admissible hypersurface in $G_{2,4}$). It is well known that $G_{2,4}$ imbeds in $\mathbf{RP}^5$ via the Plücker imbedding (we will give more details about the Plücker imbedding later on). Consider a generic (we will define this later) degree 1 hypersurface in $\mathbf{RP}^5$. Let L be the intersection of $G_{2,4}$ with this hypersurface. Then L is **NOT** admissible. In this paper we study the kernel and range of the Radon transform from $C^\infty(\mathbf{RP}^3) \to C^\infty(L)$.

Section b:
The Lagrangian Grassmanian

The surface inside $G_{2,4}$ which we are interested in is the Lagrangian Grassmanian (the space of Lagrangian 2-planes in $\mathbf{R}^4$ with respect to some symplectic form). This surface is cut out of $G_{2,4}$ by a linear equation (we will describe this in detail). We will also give two other well known descriptions of the Lagrangian Grassmanian.

Let us recall some basic facts about $G_{2,4}$. Let $\Lambda^n(\mathbf{R}^m)$ be the n'th graded piece of the exterior algebra on $\mathbf{R}^m$. As is well known, $G_{n,m}$ can be imbedded into $P(\Lambda^n(\mathbf{R}^m))$ via the Plücker imbedding which takes the plane spanned by $v_1, \ldots, v_n$ to the multivector $v_1 \wedge \cdots \wedge v_n$. The range of this map consists of the decomposable elements in $\Lambda^n(\mathbf{R}^m)$ which in the case of $\Lambda^2(\mathbf{R}^4)$ has a particularly simple description: $\alpha \in \Lambda^2(\mathbf{R}^4)$ is decomposable iff $\alpha \wedge \alpha = 0$. (Write α as a sum of a minimal number of decomposables. An easy fact is that for such a minimal representation of a two-vector, all the vectors involved are linearly independant. Therefore, every element in $\Lambda^2(\mathbf{R}^4)$ can be written as a sum of two decomposables. If α is not decomposable then write $\alpha = a \wedge b + c \wedge d$ where a, b, c, and d are linearly independant. So, $\alpha \wedge \alpha = 2a \wedge b \wedge c \wedge d$ which is not 0 since a, b, c, and d are linearly independant. If $\alpha = a \wedge b$, then clearly $\alpha \wedge \alpha = 0$.)

Let $(x_1, \cdots, x_4)$ be a basis for $\mathbf{R}^4$, then $(x_1 \wedge x_2, \cdots, x_3 \wedge x_4)$ is a basis for $\Lambda^2(\mathbf{R}^4)$ which is $\mathbf{R}^6$. Denote a general element $\alpha \in \Lambda^2(\mathbf{R}^4)$ by

$$\alpha = \sum_{i<j} \gamma_{ij} x_i \wedge x_j.$$

Define the function $Q(\alpha)$ by the equation $\alpha \wedge \alpha = Q(\alpha) x_1 \wedge x_2 \wedge x_3 \wedge x_4$. Q is a quadratic form on $\mathbf{R}^6$ with signature (3, 3). Furthermore, the set $Q = 0$ is independent of our choice of basis and is projectively invariant so it cuts out a hypersurface in $\mathbf{RP}^5$. This hypersurface is $G_{2,4}$.

In our present coordinates we can write $Q(\alpha) = \gamma_{12}\gamma_{34} - \gamma_{13}\gamma_{24} + \gamma_{14}\gamma_{23}$. Let us change coordinates to exhibit the (3, 3) signature of this quadratic form. Let

$$v_1 = (\gamma_{12} + \gamma_{34})/\sqrt{2} \qquad v_2 = (\gamma_{14} + \gamma_{23})/\sqrt{2}$$

$$v_3 = (\gamma_{13} - \gamma_{24})/\sqrt{2} \qquad v_4 = (\gamma_{12} - \gamma_{34})/\sqrt{2}$$

$$v_5 = (\gamma_{14} - \gamma_{23})/\sqrt{2} \qquad v_6 = (\gamma_{13} + \gamma_{24})/\sqrt{2}.$$

In these coordinates:

$$Q(\alpha) = v_1^2 + v_2^2 + v_3^2 - v_4^2 - v_5^2 - v_6^2.$$

Now let ω be a symplectic form on $\mathbf{R}^4$. We can define a linear functional, l, on $\mathbf{R}^6$ by $l(x \wedge y) = \omega(x, y)$ and extend this to all of $\mathbf{R}^6$ linearly. A 2-plane spanned by q_1 and q_2 is Lagrangian iff $\omega(q_1, q_2) = 0$. Therefore, the Lagrangian Grassmanian, which is by definition the space of Lagrangian 2-planes in $\mathbf{R}^4$, sits inside $G_{2,4}$ as a hyperplane corresponding to the additional equation $l = 0$. Furthermore, if we choose coordinates on $\mathbf{R}^4$ so that ω is in canonical form then our equation becomes $v_1 = 0$. We will denote this space L. Note that arbitrary linear functionals on $\mathbf{R}^6$, by the reverse procedure, define two-forms on $\mathbf{R}^4$. So generic linear functionals define non-degenerate two-forms (symplectic forms) on $\mathbf{R}^4$. For this reason, L is sometimes called the generic line complex.

We proceed by giving another description of L due to Veblen. Note that restricted to L, our quadratic form is:

$$Q = v_2^2 + v_3^2 - v_4^2 - v_5^2 - v_6^2.$$

So for $\alpha \in L$:

$$v_2^2 + v_3^2 - v_4^2 - v_5^2 - v_6^2 = 0.$$

Furthermore, L is contained inside $P(v_2, \cdots, v_6) = \mathbf{RP}^5$ which is double covered by S^5 (which we will choose to have radius 2). So points on the double cover of L satisfy the equation:

$$v_2^2 + v_3^2 + v_4^2 + v_5^2 + v_6^2 = 2.$$

So combining these equations, we see that L is double covered by $v_2^2 + v_3^2 = 1$ and $v_4^2 + v_5^2 + v_6^2 = 1$. In other words, L is double covered by $S^2 \times S^1$. Furthermore, Q defines the metric $(dx)^2 - (d\theta)^2$ on $S^2 \times S^1$.

Let $(dx_1^2 + dx_2^2 - dt^2)$ be the usual Minkowski $(2,1)$ metric on $\mathbf{R}^3$. Notice that $S^2 \times S^1$ with the metric $(dx)^2 - (d\theta)^2$ is a compactification of Minkowski $(2,1)$ space. On this space, the involution $(x,\theta) \to (-x,-\theta)$ preserves the conformal structure of $S^2 \times S^1$ and thus $L = (S^2 \times S^1)/((x,\theta) \sim (-x,-\theta))$ is a compactification of Minkowski $(2,1)$ space possessing the same conformal structure. Therefore, we will sometimes refer to L as compactified Minkowski space, denoted $M_{2,1}$, and use the fact that $T^*M_{2,1}$ decomposes into timelike, lightlike, and spacelike regions.

One fact that we will refer to later on, is that L can be viewed as one component of the boundary of the Siegel domain (called the Shilov boundary). The Siegel domain consists of two by two symmetric matrices of the form: $A + iB$, where B is positive definite, so it has a complex structure and therefore we have a notion of holomorphic functions on the Siegel domain.

The final fact about L that we will need is the following: notice that each lightlike line in L intersects the plane $v_2 = 0$ in exactly one point and thus the space of all lightlike lines is $\mathbf{RP}^3$.

Section c:
Double Fibrations

A general framework exists for dealing with Radon transforms–that of double fibrations.

Definition 1 *Let X, Y, and Z be manifolds. A double fibration is a diagram*

$$\begin{tikzcd} & Z \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & \\ X & & Y \end{tikzcd}$$

such that the map $\rho_1 \times \rho_2 : Z \to X \times Y$ is a proper differentiable imbedding of Z into $X \times Y$.

Let $F_x = \rho_1^{-1}(x)$ and $G_y = \rho_2^{-1}(y)$ be the fibers above x and y. Since $\rho_1 \times \rho_2$ is a proper differentiable imbedding of Z into $X \times Y$, $(\rho_1 \times \rho_2)(F_x)$ and $(\rho_1 \times \rho_2)(G_y)$ are submanifolds of $\{x\} \times Y$ and $X \times \{y\}$. Thus, we can view the F_x's as a smooth family of submanifolds of Y and G_y's as a smooth family of submanifolds of X. Z is called the incidence relation since, viewed as a submanifold of $X \times Y$, it consists of pairs (x, y) such that $x \in G_y$ (and equivalently $y \in F_x$).

By choosing a metric on X, we can define our Radon transform: $R : C^\infty(X) \to C^\infty(Y)$ by

$$Rf(y) = \int_{G_y} f(x)dx.$$

There is a microlocal analog to our double fibration diagram. Since Z imbeds in $X \times Y$ as a submanifold we can consider its conormal bundle $N^*Z \subset T^*(X \times Y)$. (Let ι be the inclusion of a manifold M into a manifold P. N^*M is the kernel of ι^* and is a Lagrangian submanifold of T^*P.) Since $T^*(X \times Y) \cong T^*X \times T^*Y$, we get projections $\pi_1 : N^*Z \to T^*X$ and $\pi_2 : N^*Z \to T^*Y$. Deleting the zero sections, we get the following diagram:

$$\begin{tikzcd} & N^*Z - 0 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ T^*X - 0 & & T^*Y - 0 \end{tikzcd}$$

Since $N^*Z - 0$ is a Lagrangian submanifold of $T^*(X \times Y)$ this diagram defines a canonical relation between $T^*X - 0$ and $T^*Y - 0$. Assuming that we can pick non-zero densities on Y, and Z (which is possible for the manifolds that we will look at), then we can identify functions with densities. Let d be the density on Z. The Radon transform is the map:

$$f \to \rho_{2*}((\rho_1^* f)d).$$

Assuming that the fibers of ρ_2 are compact, then ρ_{2*} makes sense since we can push forward densities under submersions by integrating over the fiber. From this perspective, our Radon transform is a Fourier integral operator (of order -1/2) with the above diagram as its canonical relation.

In the situation that we are interested in, $X = \mathbf{RP}^3$, $Y = L$ and $Z = \{(x,l) | x \subset l\}$ where x and l are viewed as lines and planes in $\mathbf{R}^4$. The fibers that we are integrating over are circles. The double fibration associates to every point $x \in \mathbf{RP}^3$ a lightlike geodesic in L and to every point $l \in L$ a "Lagrangian" geodesic in $\mathbf{RP}^3$. This gives us the canonical relation:

$$\begin{tikzcd} & N^*Z - 0 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ T^*\mathbf{RP}^3 - 0 & & T^*L - 0 \end{tikzcd}$$

Let us now summarize some facts about the maps π_1 and π_2 (for details see the appendix). Recall that every fiber of $T^*L - 0$ decomposes into a spacelike, a lightlike, and a timelike subspace and thus $T^*L - 0$ globally decomposes into spacelike, timelike, and lightlike regions. Let Σ denote the lightlike region. Let $\Gamma = \pi_2^{-1}(\Sigma)$. Let $\Upsilon = \pi_1(\Gamma)$. The map π_1 is 1 to 1 on the complement of Γ in $N^*Z - 0$ and the map π_2 is 2 to 1 on the same set. From this data, there exists a smooth involution of $N^*Z - 0$ which is the identity on Γ and switches the preimages of π_2 on the complement. Furthermore, since π_1 is 1 to 1 off Γ we get a corresponding smooth involution of $T^*\mathbf{RP}^3 - 0$ with fixed point set equal to Υ. We denote this involution ι and notice that ι is a canonical transformation. Let ι^* be the corresponding involution on functions. (We will construct this operator in chapter 2.) Note that by construction ι preserves R's canonical relation, and, therefore, $R \circ \iota^* = R$. Therefore, it is clear that Range $(1 - \iota^*) \subset$ Kernel R. We will, in fact, show that Range $(1 - \iota^*) =$ Kernel R and give an interesting description of ι.

Section d:
Symmetry

Recall that our two spaces, $\mathbf{RP}^3$ and L can be viewed as homogeneous spaces for $G = Sp(4, \mathbf{R})$ (the symplectic group on $\mathbf{R}^4$). Choose a Lagrangian plane, l, in $\mathbf{R}^4$ and a line, x, in l. Let $P =$ the stabilizer of x in G and let $Q =$ the stabilizer of l in

G. These are the two non-conjugate parabolics in G. In this notation, $\mathbf{RP}^3 = G/P$, $L = G/Q$ and $Z = G/(P \cap Q)$.

To take advantage of this symmetry we need to recast our Radon transform in a G-equivariant way. Let L_1 be the bundle over $\mathbf{RP}^3$ whose sections are smooth functions $f(x)$ on $\mathbf{R}^4 - 0$ that are homogeneous of degree -2 (in other words it is the bundle $\mathcal{O}(2)$). Let L_2 be the bundle (over L) $(|\Omega|(V_l))^*$ where V_l is the Lagrangian plane in $\mathbf{R}^4$ corresponding to the point $l \in L$ and $|\Omega|(V_l)$ denotes densities on V_l. (Note: these are not densities on L.) Recall that a density δ is a volume form Ω and an orientation $\pm$ with the equivalence relation $(\Omega, \pm) \sim (-\Omega, \mp)$. Thus under linear change of coordinates densities transform by $\frac{1}{|det|}$ and thus duals of densities transform by $|det|$. We can represent our two parabolics as the matrices in $Sp(4, \mathbf{R})$ of the following form:

$$p = \begin{pmatrix} a & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \quad q = \begin{pmatrix} A & * \\ 0 & * \end{pmatrix}$$

where the blocks in q are two by two. These bundles are induced from one-dimensional representations of P and Q:

$$\begin{array}{ccc} L_1 = G \times_P \mathbf{R} & \qquad & L_2 = G \times_Q \mathbf{R} \\ \big\downarrow & & \big\downarrow \\ G/P & & G/Q \end{array}$$

where the respective characters of the two parabolics are $\chi_1(p) = a^2$ and $\chi_2(q) = |det(A)|$. Notice that both characters are trivial on their maximal compact subgroups so that we can trivialize both bundles and view sections as functions on the base.

We now define our map $R : \Gamma(L_1) \to \Gamma(L_2)$. Let h_l be a generator for $H_1(V_l - 0)$ (i.e. a circle). Let $\frac{d}{dr}$ represent the radial vector field on V_l. Let $\delta = (\Omega, \pm)$ be a density on $V_l - 0$. Let $f(x)$ be a section of L_1. On $V_l - 0$ we can write $f(x) = \frac{f(\theta)}{r^2}$

and $\Omega = dx \wedge dy$ where (x, y) are a basis for V_l. Therefore,

$$f(x)\Omega(\frac{d}{dr}, \cdot) = f(x)(xdy - ydx) = f(\theta)d\theta$$

on $(V_l - 0)$ which is closed, so it defines a cohomology class. Our orientation $\pm$ gives us a pairing between homology and cohomology. $Rf(l)$ is a dual to a density. To define $Rf(l)$, we define how it pairs with the density δ. We define:

$$< Rf(l), \delta > = < h_l, f(\theta)d\theta >$$

where the pairing on the right is the pairing between homology and cohomology.

We now show that this map is G-equivariant. It is enough to check this when $g \in Q$ and $l = eQ$ is the identity coset (the general case follows since the compact part of G only translates the base $L = G/Q$). Let A be the two by two block in Q which acts on the plane V_l. Note that because g is a linear map $g^{-1}h_l = \pm h_l$ in homology depending on the sign of $det(A)$. This map is G-equivariant because:

$$< Rgf(l), \delta > = < h_l, f(g^{-1}x)(xdy - ydx) >$$

which equals by a change of variables:

$$< g^{-1}h_l, f(x)g^*(xdy - ydx) > =$$

$$< g^{-1}h_l, f(x)det(A)(xdy - ydx) >$$

and since $g^{-1}h_l = \pm h_l$ in homology depending on the sign of $det(A)$, this equals:

$$|det(A)| < h_l, f(x)(xdy - ydx) > =$$

$$< gRf(l), \delta >$$

since g acts on duals to densities by $|det(A)|$.

Notice that one generator for homology in $V_l - 0$ is precisely the great circle that we want to integrate around. Therefore, viewing sections of these bundles as functions, this map is the Radon transform.

There is one more fact that we will need. Recall our canonical relation:

$$\begin{tikzcd}
 & N^*Z - 0 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
T^*\mathbf{RP}^3 - 0 & & T^*L - 0
\end{tikzcd}$$

Lemma 1 *This canonical relation is G-equivariant.*

Proof: Let $i : Z \to X \times Y$ be the inclusion. Then we get a corresponding map $i^* : T^*(X \times Y) \to T^*Z$, where $N^*Z = \text{Kernel } i^*$. Let $g \in G$. It is obvious that $x \in l$ iff $gx \in gl$. So, $gi = ig$, and therefore $gi^* = i^*g$. So Kernel $gi^* =$ Kernel i^*g. $\square$

Corollary 1 *The involution ι is G-equivariant.*

Proof: ι switches the two points with the same image under π_2. Let z_1 and z_2 be two points on $T^*\mathbf{RP}^3 - 0$ such that $\pi_2\pi_1^{-1}(z_1) = \pi_2\pi_1^{-1}(z_2)$. Therefore,

$$\pi_2\pi_1^{-1}g(z_1) = g\pi_2\pi_1^{-1}(z_1) = g\pi_2\pi_1^{-1}(z_2) = \pi_2\pi_1^{-1}g(z_2).$$

Thus, $\iota(gz_1) = gz_2$. $\square$

Chapter 2:
The Kernel and Range of the Radon Transform

Section a:
Some representation theory

Recall that our two spaces, $\mathbf{RP}^3$ and L, can be viewed as homogeneous spaces for $G = Sp(4, \mathbf{R})$. Choose a Lagrangian plane, l, in $\mathbf{R}^4$ and a line, x, in l. Let P be the stabilizer of x in G and let Q be the stabilizer of l in G. These are the two non-conjugate parabolics in G. Furthermore, $\mathbf{RP}^3 = G/P$ and $L = G/Q$. Also note that when P is a parabolic subgroup then $G/P = K/(P \cap K)$ where K is the maximal compact subgroup of G.

In our case, $K = U(2)$ and $K \cap P$ is a direct product of a S^1 with $\mathbf{Z}/2$ and $K \cap Q$ is an $O(2)$. We are interested in how our line bundles over $\mathbf{RP}^3$ and L decompose into irreducible representations of G but first let us look at the $U(2)$ story. By the Peter-Weyl Theorem, $L^2(K) = \oplus(V_\lambda \otimes V_\lambda^*)$ where λ ranges over all the representations of K. Similarly, $L^2(K/H) = \oplus(V_\lambda^H \otimes V_\lambda^*)$. In other words, each K representation occurs with multiplicity equal to the number of H invariants in V_λ. Notice that the characters of both parabolics are trivial on their maximal compact subgroups, so sections of our bundles can be regarded as functions on the base. Therefore, it is easy to characterize which $U(2)$ representations occur as sections of our line bundles. We will summarize the results here (for details see the appendix). From standard results about the representations of $U(2)$ we recall that any representation of $U(2)$ can be written as $S^k \otimes det^l$ where S^k is the standard representation of $U(2)$ on degree k polynomials in two variables and det is the determinate representation.

In this notation, the $U(2)$ representations that occur in $\Gamma(L_1)$ (L_1 is the line bundle over $\mathbf{RP}^3$) are of the form $(2n, -a)$ where $n \geq 0$ and $0 \leq a \leq 2n$. With respect to the $Sp(4, \mathbf{R})$ action, there are only two irreducible components of $\Gamma(L_1)$: a odd and a even (we will call these O and E), and $\Gamma(L_1)$ splits into a direct sum of these two irreducibles. (We will describe these components fully in the next section.) We represent the decomposition of $\Gamma(L_1)$ with respect to the $Sp(4, \mathbf{R})$ action with the following diagram:

$$\begin{tikzcd}[column sep=small] & \Gamma(L_1) \arrow[dl] \arrow[dr] & \\ O \arrow[dr] & & E \arrow[dl] \\ & 0 & \end{tikzcd}$$

Paths from the top to the bottom represent maximal chains of G-submodules so the quotients are irreducible. (Note: Maximal chains of G-submodules have the Jordan-Hölder property. That is, the set of quotients and their multiplicities are the same for every maximal chain. More generally, if G is any connected semisimple Lie group then the representations of G arising from parabolic induction have this property. See [Knapp] p.373 for details.)

The $U(2)$ representations in $\Gamma(L_2)$ are of the form $(2n, -2a)$ where $n \geq 0$ and a is any integer. The $Sp(4, \mathbf{R})$-submodule structure looks like:

$$\begin{tikzcd}[column sep=small] & \Gamma(L_2) \arrow[dl] \arrow[dr] & \\ H \arrow[dr] & & AH \arrow[dl] \\ & S \arrow[d] & \\ & 0 & \end{tikzcd}$$

H and AH represent functions with holomorphic and anti-holomorphic boundary data on the Shilov Boundary (L) of the Siegel domain. S consists of the $U(2)$-representations $n \geq a \geq 0$ and is the only irreducible G-submodule of $\Gamma(L_2)$. Note that these are the same $U(2)$-representations that occured in E. Thinking of L as compactified Minkowski space, these representations correspond to functions which live microlocally in the spacelike region and its boundary the lightlike region. (If we consider the hyperbolic Laplace operator Δ on $M_{2,1}$ corresponding to the $(2,1)$ metric then S is spanned by the functions with non-negative eigenvalues.) The other two irreducible quotients in this diagram correspond to the positive and negative timelike regions. Thus, the G structure of this bundle, by dividing the timelike space into two pieces, represents the fact that compactified Minkowski space is a causal model of the universe. For more details on causality see [Hawking-Ellis].

From all this it is now trivial to identify the kernel and range of our Radon transform. By Schur's Lemma, any G-equivariant map must map irreducibles back onto themselves or to 0. Thus, it is clear that the component O (the representations $(2n, -a)$ with a odd) is in the kernel of our map since it does not occur as a $Sp(4, \mathbf{R})$ representation in $\Gamma(L_2)$. Furthermore, the fact that there exist functions on $\mathbf{RP}^3$ that are not in the kernel of our map implies that the other component of $\Gamma(L_1)$ is not in the kernel. Thus, the range of our map is the component S, which consists of the representations $(2n, -2a)$ with $n \geq a \geq 0$. In other words, it is the spacelike and lightlike functions. This corresponds to the following observation. Consider our canonical relation:

$$\begin{tikzcd}[column sep=small] & N^*Z - 0 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ T^*\mathbf{RP}^3 - 0 & & T^*L - 0 \end{tikzcd}$$

The range of π_2 is the spacelike and lightlike regions.

To summarize, let $2n \geq a \geq 0$. We have proven the following theorem:

Theorem 2 *The kernel of the Radon transform consists of the $U(2)$-representations $(2n, -a)$ with a odd and the range consists of the $U(2)$-representations $(2n, -a)$ with a even.*

Section b:
The Symplectic Fourier Transform

In the previous section we saw that $\Gamma(L_1)$ splits as a direct sum of two irreducible components called O and E (for odd and even). Let σ be the map from $\Gamma(L_1)$ to $\Gamma(L_1)$ defined by $\sigma(f) = -f$ if f is in O, and $\sigma(f) = f$ if f is in E (you extend this linearly to any function). It is clear that $\sigma^2 = 1$ and that Kernel (R) = Range $(1-\sigma)$. In this section, we construct σ geometrically, using the fact that since $\Gamma(L_1)$ has only two $Sp(4, \mathbf{R})$ irreducible pieces, *there is, up to sign, only one non-trivial $Sp(4, \mathbf{R})$-equivariant involution of* $\Gamma(L_1)$.

The involution that we are interested in is the symplectic Fourier transform which we define as follows: Let ω be a symplectic form on $\mathbf{R}^4$. Let g be the standard metric. Let J be the complex structure on $\mathbf{R}^4$ such that $g = \omega \circ J^{-1}$. We define the symplectic Fourier transform, SF, from $\mathcal{S}'(\mathbf{R}^4) \to \mathcal{S}'(\mathbf{R}^4)$ (where $\mathcal{S}'$ denotes the Schwartz space) as follows:

$$SF(f)(y) = \frac{1}{2\pi} \int_{x \in \mathbf{R}^4} e^{i\omega(x,y)} f(x) \omega^2.$$

Note that SF is related to the usual Fourier transform, FT , by $SF = FT \circ J^* = J^* \circ FT$ where $J^* f(x) = f(Jx)$. Furthermore, note that $FT^2 f(x) = f(-x)$ and that $J^{*2} f(x) = f(-x)$, so that on even functions FT, J, and SF are all involutions.

We now recall some facts about the usual Fourier transform. Let f(x) be a smooth section of L_1. That is, $f(x)$ is homogeneous of degree -2 on $\mathbf{R}^4$ and C^∞ on $\mathbf{R}^4 - 0$. The section $f(x)$ is of tempered growth and is locally integrable so it is a tempered distribution and thus has a Fourier transform, $FT(f)$. Furthermore, the fact that $|y|^{2k} FT(f)(y) = FT(\triangle^k f)(y)$ and that f is C^∞ on $\mathbf{R}^4 - 0$ implies that $FT(f)$ is also C^∞ on $\mathbf{R}^4 - 0$. Finally, note that FT maps homogeneous distributions of degree k on $\mathbf{R}^n$ isomorphically to the homogeneous distributions of degree $-n-k$. Combining this with the fact that $FT^2 =$ identity on even functions we get, for $n = 4$, that FT is an involution of homogeneous distributions of degree -2 and thus of $\Gamma(L_1)$ (since it preserves smoothness).

Since J^* also preserves the degree of homogeneity, we can conclude that SF is also an involution of $\Gamma(L_1)$ which is, by construction, $Sp(4, \mathbf{R})$-equivariant and not $\pm$

identity. (Note that this fact implies the representation theory fact that $\Gamma(L_1)$ splits as a direct sum of two representations.) Therefore, $SF = \pm\sigma$. Furthermore, it is easy to check (by applying SF and σ to one function) that:

Theorem 3 $SF = \sigma$

In the next section, we will show that σ is actually a 0'th order classical Fourier integral operator on $\mathbf{RP}^3$ and we describe and interpret its canonical transformation. First, let us describe σ in more detail and thus calculate explicitly the kernel of our Radon transform.

Let H^k denote the degree k homogeneous harmonic polynomials on $\mathbf{R}^4$. The restriction of the H^k's to S^3 gives a basis for $L^2(S^3)$ and thus the even harmonic polynomials (k even) gives a basis for $L^2(\mathbf{RP}^3)$. Therefore, any homogeneous function of degree -2 can be uniquely represented as a sum $\Sigma \frac{h_{2k}}{r^{2k+2}}$ where $h_{2k} \in H^{2k}$.

Definition 2 *Let* $F^{2k} = \frac{H^{2k}}{r^{2k+2}}$.

If $f \in F^{2k}$ then $FT(f) = (-1)^k f$ (see [Stein], p. 73, for details about the Fourier transform of homogeneous distributions). Finally, J is an isometry so it preserves r^{2k+2} and it maps H^{2k} back onto H^{2k} with eigenvalues ± 1 (since $J^{*2}f(x) = f(-x) = f(x)$ because f is even).

Definition 3 *Let* $F^{2k,s}$*, where* $s = 0$ *or* $s = 1$*, denote the* $(-1)^s$ *eigenspace for* J^* *in* F^{2k}.

Recall that $\sigma = FT \circ J^*$. Thus σ acts on $F^{2k,s}$ by $(-1)^{k+s}$. Therefore, our kernel consists of "half" of each F^{2k} consisting of the $F^{2k,s}$ where k+s is odd. These correspond to the representations $(2k, -a)$ where a is odd.

We summarize with the following theorem:

Theorem 4 $Kernel(R) = Range(1 - \sigma) = \oplus F^{2k,s}$ *where k+s is odd.*

Furthermore, note that the map σ, by construction, depended only on the symplectic form ω and not on the complex structure J. In the next section we show that σ actually lives as a FIO on $\mathbf{RP}^3$ and we give another canonical description of it.

Section c:
The Canonical Involution on $T^*\mathbf{RP}^3 - 0$

Recall that eigenvalues for the Laplace operator on S^3 are $n(n+2)$ with eigenspace H^n. Thus, the eigenvalues on $\mathbf{RP}^3$ are $4k(k+1)$ where $n = 2k$ are the even eigenspaces. We renormalize the Laplacian so that geodesic flow has period 2π. Under this renormalization, the Laplacian, Δ, has eigenvalue $k(k+1)$ on H^{2k}. Let $A = \sqrt{\Delta + \frac{1}{4}} - \frac{1}{2}$. Notice that A has eigenvalue k on H^{2k} and has the same symbol as $\sqrt{\Delta}$. Since A is self-adjoint and elliptic, $e^{i\pi A}$ is a unitary 0'th order FIO with symbol equal to Hamiltonian flow for period π generated by the symbol of A. (see Theorem 1.1 in [Duistermaat-Guillemin]) In other words, $e^{i\pi A}$'s corresponding canonical transformation of $T^*\mathbf{RP}^3$ is geodesic flow for period π. Furthermore, $e^{i\pi A}$ has eigenvalue $(-1)^k$ on H^{2k} so if we identify functions on $\mathbf{RP}^3$ with sections of L_1 we see that $e^{i\pi A}$ is the same as the Fourier transform, FT. Notice that J preserves lines on $\mathbf{R}^4$ and thus acts on $\mathbf{RP}^3$. Therefore, we have proven:

Theorem 5 *$\sigma = e^{i\pi A} \circ J^*$ is a 0'th order unitary FIO associated with the canonical transformation $I =$ (Geodesic Flow for period π) $\circ J^*$.*

We proceed by giving a nice description of I. First let us recall the following description of $T^*\mathbf{RP}^3 - 0$. Let $M = T^*(\mathbf{R}^4 - 0) - 0 \cong (\mathbf{R}^4 - 0) \times (\mathbf{R}^{4*} - 0)$. Consider the action of $\mathbf{R}^*$ on M defined by $(v, \xi) \to (\lambda v, \frac{1}{\lambda}\xi)$. Since the function $< v, \xi >$ is non-constant and is preserved by this action, it is the moment map for this action. Let us do symplectic reduction at the 0 level set of the moment map. What we get is pairs

$$\{v \otimes_{\mathbf{R}^*} \xi \mid v \neq 0,\ \xi \neq 0,\ and\ < v, \xi >= 0\}.$$

Note that we can view this space as maps $A : \mathbf{R}^4 \to \mathbf{R}^4$ where $A = \xi(\cdot)v$ and that this space is precisely $T^*\mathbf{RP}^3 - 0$. Alternatively, we can use the symplectic form ω on $\mathbf{R}^4$ to represent $\mathbf{RP}^3$ as pairs

$$\{(x \otimes_{\mathbf{R}^*} v) \in (\mathbf{R}^4 - 0) \otimes_{\mathbf{R}^*} (\mathbf{R}^4 - 0) \mid \omega(x, v) = 0\}.$$

Note that $M//\mathbf{R}^* \cong T^*\mathbf{RP}^3$, so $T^*\mathbf{RP}^3$ acquires its symplectic structure from this reduction.

Using our symplectic form, ω, on $\mathbf{R}^4$ we can identify $(\mathbf{R}^4 - 0) \times (\mathbf{R}^{4*} - 0)$ with $(\mathbf{R}^4 - 0) \times (\mathbf{R}^4 - 0)$. Consider the following involution, i, on $(\mathbf{R}^4 - 0) \times (\mathbf{R}^4 - 0)$ defined by:

$$i : (x, v) \to (v, x).$$

Notice that this involution maps $\mathbf{R}^*$ orbits to $\mathbf{R}^*$ orbits.

Lemma 2 *i is a canonical transformation of $T^*(\mathbf{R}^4)$.*

Proof: Let ω_M be the canonical symplectic form on $T^*(\mathbf{R}^4)$ which is identified with $T(\mathbf{R}^4)$. Let $(x_1, v_1), (x_2, v_2)$ be a pair of tangent vectors at (x, v). We note that for the canonical form:

$$\omega_M((x_1, v_1), (x_2, v_2)) = \omega_M((x_1, 0)(0, v_2)) + \omega_M((v_1, 0), (0, x_2))$$

and that

$$\omega_M((x, 0), (0, y)) = \omega((0, x), (y, 0)).$$

Combining these two facts we get,

$$i^*\omega_M((x_1, v_1), (x_2, v_2)) = \omega_M((v_1, x_1), (v_2, x_2)) = \omega_M((x_1, v_1), (x_2, v_2)).\Box$$

Since i maps $\mathbf{R}^*$ orbits to $\mathbf{R}^*$ orbits, we get a corresponding involution of $T^*\mathbf{RP}^3 - 0$. If we represent an element in $T^*\mathbf{RP}^3 - 0$ by the matrix $\omega(x, \cdot)v$, then our involution maps:

$$\omega(x, \cdot)v \to \omega(v, \cdot)x.$$

For this reason, we will call this involution the symplectic transpose, or ST for short. Since $T^*\mathbf{RP}^3 - 0$ is the symplectic reduction of $T^*(\mathbf{R}^4 - 0)$, we get as a corollary of the previous lemma:

Corollary 2 *ST is a canonical transformation.*

Recall that we had two other involutions of $T^*\mathbf{RP}^3 - 0$: $I =$ (Geodesic Flow for period π) $\circ\, J^*$ and ι which came from the canonical relation. Also notice that ST is also G-equivariant (since G preserves ω). We now show that these involutions are all the same.

Theorem 6 $I = \check{S}T = \iota$.

Proof: Geodesic flow for period π on $T^*\mathbf{RP}^3 - 0$ is the same as geodesic flow for period $\pi/2$ on $T^*S^3 - 0$, so it maps $T^*\mathbf{RP}^3 - 0$, viewed as matrices, by $g(v,\cdot)x \to -g(x,\cdot)v$. In other words, it is the map $A \to -A^t$. J^* maps $(x,\xi) \to (Jx, \xi \circ J^t)$. In other words, $J^*(A) = JAJ^t$. Combining these,

$$I(A) = -JAJ^t = JAJ.$$

On the other hand,

$$\omega(x,\cdot)y = g(x, J\cdot)y = g(J^t x, \cdot)y$$

and

$$I(g(J^t x,\cdot)y) = g(y, J\cdot)JJ^t x = g(y, J\cdot)x = \omega(y,\cdot)x.$$

Therefore, $ST = I$.

For the other equality, notice that since $R\sigma = R$, σ's canonical relation must preserve R's canonical relation. That is, I preserves $N^*Z - 0$ and thus is either ι or the identity and it is clearly not the identity. □

Our final interesting observation is the following description of the fixed point set of the involution I. The fixed point set of this involution are the elements $\omega(\lambda x,\cdot)x$ where $\lambda \in \mathbf{R}^*$. This is the four dimensional submanifold, Λ, which are locally multiples of the one-form $\alpha_x = \omega(v(x),\cdot)$ where $v(x)$ is any local section of the tautological bundle over $\mathbf{RP}^3$. Notice that α_x determines the usual contact structure on $\mathbf{RP}^3$ (its kernel is precisely the hyperplanes orthogonal to the line x with respect to the symplectic form). Furthermore, notice that our Lagrangian lines are Legendre curves for this contact structure. Finally, notice that the characteristics for this contact structure are precisely the fibers of the Hopf fibration $\pi : \mathbf{RP}^3 \to S^2$ and thus the contact structure on $\mathbf{RP}^3$ is the same as the contact structure we get by pulling back a symplectic form from S^2 to $\mathbf{RP}^3$.

This contact structure is actually central to why L is not admissible. Recall Gelfand and Graev's characterization of admissible hypersurfaces. A hypersurface X is admissible iff X is locally (near a generic point) either: a) the set of lines incident to some non-singular curve in $\mathbf{RP}^3$, or b) the set of lines tangent to some smooth surface in $\mathbf{RP}^3$. At every point in $\mathbf{RP}^3$ there are no lines in the contact direction, so condition a clearly fails. Furthermore, contact structures are not integrable, so condition b fails. Therefore, L is not admissible.

We conclude with some conjectures. Let Y be a generic degree 1 hypersurface in $G_{2,2n}$. Notice that a generic linear functional on $G_{2,2n}$ defines a symplectic form on R^{2n} and that Y is the space of isotropic two-planes in R^{2n} with respect to that symplectic form and thus it is also a homogeneous space for $Sp(2n, \mathbf{R})$. Again we can define a Radon transform $R : C^\infty(\mathbf{RP}^{2n-1}) \to C^\infty(Y)$ by integrating over the the corresponding lines in $\mathbf{RP}^{2n-1}$. Notice that the involution ST generalizes to $\mathbf{RP}^{2n-1}$ and its fixed point set defines the usual contact structure.

Conjecture 1 *Let Y be a generic degree 1 hypersurface in $G_{2,2n}$. Let $R : C^\infty(\mathbf{RP}^{2n-1}) \to C^\infty(Y)$ be the Radon transform. Then Kernel $R =$ Range $(1 - \sigma)$ where σ is the involution on functions associated with the symplectic transpose.*

More generally, we conjecture:

Conjecture 2 *Let Y be the space of isotropic k-planes ($k > 1$) in $\mathbf{R}^{2n}$. Let $R : C^\infty(\mathbf{RP}^{2n-1}) \to C^\infty(Y)$ be the Radon transform. Then Kernel $R =$ Range $(1 - \sigma)$.*

Finally, consider the Radon transform $R : C^\infty(\mathbf{RP}^{2n-1}) \to C^\infty(G_{n,2n})$. Let SP be the involution on $G_{n,2n}$ that maps each plane to its orthogonal plane with respect to the symplectic form. Notice that restricted to the Lagrangian Grassmanian SP is trivial. From this we get the associated involution on functions SP^*.

Conjecture 3 *The Radon transform intertwines σ and SP^*. That is, $R\sigma = SP^*R$.*

The geometric meaning of this conjecture is that the integral of a function over all Lagrangian planes is sufficient to determine the average of the integrals over any two perpendiclar (via the symplectic form) planes. Therefore, we hope to explicitly extend a function g on L to a SP^* invariant function on $G_{n,2n}$ and use this extension to "invert" our Radon transform. That is, this procedure would give us the unique σ invariant function f such that $Rf = g$.

Appendix 1

The Geometry of the Canonical Relation

We begin by noting the following: $(x, \xi, l, \eta) \in N^*Z$ iff $\xi \in N^*_x F_l$ and symmetrically iff $\eta \in N^*_l G_x$. This is the microlocal analog of the symmetry of the double fibration. So in particular, this gives us an isomorphism $N^*_x F_l \cong N^*_l G_x$ (thus for a fixed $\{(x, l) | x \in l\}$ choosing ξ determines η and visa versa). Also recall that the G_x's are the lightlike geodesics in L.

Proposition 1 (Guillemin) *π_2 is one to one above the lightlike region and is two to one above the spacelike region.*

Proof: Recall that the G_x's are lightlike S^1's. So let v be the tangent vector to G_x at l, then $\eta \in N^*_l G_x$ iff $< v, \eta >= 0$.

Case 1: η is lightlike. It is easy to check that there is a unique lightlike tangent direction which η annihilates, so there is a unique x corresponding to that direction and by our remark above, the data (x, l, η) uniquely determines ξ. Thus π_2 is one to one. (Note: If we use our $(2, 1)$ metric to relate vectors and covectors then the only lightlike vectors orthogonal to a given one point along the given one.)

Case 2: η is spacelike. There are two lightlike directions orthogonal to η. (We can choose coordinates such that η is dual to the vector $\{(z, 0, 1) | z^2 > 1\}$ via the metric. Then the directions along $(1, y, z)$ are orthogonal to η. These are lightlike iff $y = \pm\sqrt{z^2 - 1}$.) Thus there are two different x's which are conormal to η at l.

Case 3: η is timelike. There are no lightlike directions conormal to η. (A general timelike vector is of the form $\{(1, 0, z) | z^2 > 1\}$. The vectors orthogonal to this point along $(z, y, 1)$ and since $z^2 > 1$ these are spacelike.) $\square$

Let Σ denote the lightlike region in $T^*L - 0$ and let $\Gamma = \pi_2^{-1}(\Sigma)$.

Proposition 2 *π_1 is one to one generically.*

Proof: Let $\pi : \mathbf{R}^4 - 0 \to \mathbf{RP}^3$. Given $x \in \mathbf{RP}^3$, G_x is the S^1 of Lagrangian lines through x. Let $(x, \xi) \in T^*\mathbf{RP}^3 - 0$. Let G_l be a Lagrangian line through x. Let v be its tangent vector at x. Using the symplectic form on $\mathbf{RP}^3$, we can identify v with a covector ζ. Let x_1 be a point on x. Using $d\pi^*_{x_1}$ we can identify ζ with a covector ς on $\mathbf{R}^4$ such that $<\varsigma, x_1>=0$. Note that different x_1's give you different ς's but they only differ by a scalar so they annihilate the same vectors. Using the symplectic form on $\mathbf{R}^4$ we write (x,ξ) as $\omega(w,\cdot)x_1$. Conormality in this notation means $<\varsigma, w>=0$ which is an additional condition on ς if $w \notin x$. Thus in this case, there is only one conormal Lagrangian direction. □

Note that generically in the previous proposition means off the contact direction. Furthermore, a general fact is that the critical points of π_1 are the same as the critical points of π_2. Therefore we get as a corollary:

Corollary 3 *π_1 is one to one off Γ.*

Finally, we recall our involution ι which is defined off of $\Upsilon = \pi_1(\Gamma)$ by $\iota(z) = w$ if $z \neq w$ and $\pi_2\pi_1^{-1}(z) = \pi_2\pi_1^{-1}(w)$.

Proposition 3 *ι is a canonical transformation.*

Proof: Since ι is smooth, to show this we only need to show this locally off of Υ, but in a local neighborhood of Υ we can choose an "inverse" to π_2 so that $\iota = \pi_1\pi_2^{-1}\pi_2\pi_1^{-1}$. Since N^*Z is a canonical relation, locally both $\pi_1\pi_2^{-1}$ and $\pi_2\pi_1^{-1}$ are canonical transformations, so ι is as well. □

Appendix 2
The $U(2)$ representations

$U(2)$ is the group of complex 2 by 2 unitary matrices. It acts on $\mathbf{R}^4$, thought of as $\mathbf{C}^2$ in the obvious way. Let K_1 be the stabilizer in $U(2)$ of the real line $\{(x,0)|x \in \mathbf{R}\}$ and let K_2 be the stabilizer of the real plane $\{(x,y)|x,y \in \mathbf{R}\}$. Then as matrices:

$$K_1 = \begin{pmatrix} \pm 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} \quad K_2 = \begin{pmatrix} \pm\cos\theta & \pm\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$\mathbf{RP}^3 \cong U(2)/K_1$ and $L \cong U(2)/K_2$. As we previously remarked, the $U(2)$ representations on sections of the induced bundles considered in this paper are the same as on functions on the base. Therefore, these representations are the K_1 and K_2 invariants in $S^k \otimes det^l$.

K_1 is a direct product of $\mathbf{Z}/2$ with an S^1.

$$K_1 = \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

The S^1 maps:

$$x^{k-p}y^p \to e^{i(p+l)\theta}x^{k-p}y^p$$

so the only S^1 invariants occur when $l = -p \geq -k$. The $\mathbf{Z}/2$ maps:

$$x^{k-p}y^p \to (-1)^{k+l-p}x^{k-p}y^p$$

which equals $(-1)^k x^{k-p}y^p$ when $l = -p$. So $x^{k-p}y^p$ is also an invariant for $\mathbf{Z}/2$ when k is even. Therefore, $S^k \otimes det^l$ contains K_1 invariants iff

$$(k,l) = (2n,-a) \text{ } where \ n \geq 0 \ and \ 0 \leq a \leq 2n.$$

K_2 is a semi-direct product of another $\mathbf{Z}/2$ with another S^1.

$$K_2 = \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

We can conjugate the S^1 into

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

The only invariants of this S^1 are $x^p y^p$, so k is even. The generator for the $\mathbf{Z}/2$ gets conjugated into

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Under this element,

$$x^p y^p \to (-1)^l x^p y^p$$

so l must also be even. Therefore, $S^k \otimes det^l$ contains K_2 invariants iff

$$(k, l) = (2n, -2z) \textit{ where } n \geq 0.$$

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