No-arbitrage bounds on American Put Options with a single maturity

by

Premal Shah
B. Tech. & M. Tech., Electrical Engineering (2003),
Indian Institute of Technology (IIT) Bombay, Mumbai

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Author ....
Sloan School of Management
May 22, 2006

Certified by ....
Dimitris J. Bertsimas
Boeing Professor of Operations Research, Sloan School of Management
Thesis Supervisor

Accepted by ....
James B. Orlin
Edward Pennell Brooks Professor of Operations Research, Codirector,
Operations Research Center

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Abstract

We consider in this thesis the problem of pricing American Put Options in a model-free framework where we do not make any assumptions about the price dynamics of the underlying except those implied by the no-arbitrage conditions. Our goal is to obtain bounds on the price of an American put option with a given strike and maturity directly from the prices of other American put options with the same maturity but different strikes and the current price of the underlying. We proceed by first investigating the structural properties of the price curve of American Put Options of a fixed maturity and derive necessary and sufficient conditions that strike - price pairs of these options must satisfy in order to exclude arbitrage. Using these conditions, we can find tight bounds on the price of the option of interest by solving a very tractable Linear Programming Problem. We then apply the methods developed to real market data. We observe that the quality of bounds that we obtain compares well with the quoted bid-ask spreads in most cases.

Thesis Supervisor: Dimitris J. Bertsimas
Title: Boeing Professor of Operations Research, Sloan School of Management

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Chapter 1

Introduction

Derivative securities or derivatives are securities whose payoffs are linked to price(s) of other asset(s) called the underlying. Derivatives facilitate hedging, speculating and other investment objectives and help to make the financial markets richer or in somewhat loose terms 'more complete'. Because of the contractual nature of a derivative, which makes its payoff a function of the price(s) of the underlying, it's own price, in principle, can be determined if the price dynamics of the underlying asset(s) are known. Derivatives such as options are actively traded in many markets, and as such, pricing of derivatives is a problem of significant practical interest.

1.1 Background

The landmark Black and Scholes option pricing formula was probably the first widely noted attempt to price a derivative security. The formula that first appeared in [1] priced a European option (both calls and puts) on a common stock and the approach was based on modeling the price dynamics of the underlying and then inferring the price of the security from this model within the no-arbitrage framework. Prices of the underlying asset in the Black-Scholes framework are assumed to follow a log-normal Brownian Motion. Under this assumption, the price-dynamics of the asset are completely specified by two parameters, the drift and the volatility of stock returns. Note that, the price dynamics are not known but assumed to be log-normal in a quite adhoc fashion. These assumptions might be reasonable to a certain extent but are far from accurate. In fact, practitioners make many adjustments to the basic Black-Scholes model such as using different volatilities for different strikes, before applying it to price options in real markets. Other models of stock price dynamics that refine the somewhat simplistic assumptions of the Black-Scholes model to better accommodate the observed prices of the securities in the market have also been proposed. See [2] for a survey of some of the more commonly used models and their efficacy in practice. Amongst these are mean-reverting, stochastic volatility models and others. In general, these more sophisticated models have more parameters (rather than the two present in the Black-Scholes model) and make it possible to incorporate more of the information conveyed by the market. This is because, to price a derivative security using a price dynamics model, one would follow a two-step process. The first step involves calibrating the assumed price dynamics model on the underlying. This is done by attuning the parameters of the model (the degrees of freedom) so as to make its predictions fit observed prices of highly liquid derivative securities linked to the underlying. One then uses the calibrated model to price not-so-liquid derivatives of interest. Although this approach gives a single value price for the security of interest, the assumptions that are implicitly made are difficult to justify or verify. Pricing of securities in markets is always relative, wherein one takes the prices of certain instruments as given to infer or estimate the value of others. A price dynamics model provides a framework to interpret and re-use the information provided by market signals (prices of reliable or liquid securities). However, it does not transparently reflect this information about asset prices that is revealed through the market observed prices, (and is therefore more trustworthy,) but combines it with that arising out of imposing a certain price dynamics belief model on the asset prices (which is always suspect). Moreover, as it is difficult to isolate the degree of individual impact of these two factors that are used to derive the final price, the model based approach, in a sense, 'contaminates' information conveyed by the market.

Another approach to derivatives pricing that has gained popularity recently, is the model-free approach - i.e., one which does away with making any assumptions on the price dynamics of the underlying altogether. The only assumption made is that of no-arbitrage, a fairly robust assumption in the context of derivative pricing. In their seminal work, Cox and Ross [3] and Harrison and Kreps [4], show that the condition of no-arbitrage is in fact equivalent to that of the existence of a probability measure (called the Equivalent Martingale Measure or the Risk Neutral Measure), with respect to which the discounted asset prices are martingales. In the model-free approach, one is interested in finding the maximum and minimum price that a derivative security of interest might take without creating an arbitrage opportunity. While this approach typically would not give a single number for the asset price (unless the markets are complete), it faithfully represents the market implied information about the security price. It also gives an indication about how deep the market is in that security through the size of the price-bracketing interval. Another advantage is that this approach is also often useful to identify real-arbitrage opportunities if they exist in the market, unlike the price-model based approach.

The work in this thesis is concerned with coming up with a model-free approach to price a practically important class of derivative securities - the American Put Options. American Put Options are actively traded on different exchanges for a variety of asset classes. However, pricing an American Put Option is a difficult problem even while assuming a price dynamics model for the underlying. This is because unlike its call counterpart, the American Call[1], an early exercise can indeed be optimal for an American Put, and hence the price of an American Put should be and is typically more than that of a European Put Option on the same asset and with the same strike and maturity. As a result there is no simple or 'closed form formula' known for pricing the American Put Option for any of the commonly used price dynamics models, though this can be done numerically for a given model, by solving a conceptually simple but computationally unattractive Dynamic Programming Problem.

[1]Assuming the stock is non-dividend paying, conditional Jensen's inequality directly leads to the conclusion that terminal exercise policy for an American Call is always Optimal.

1.2 Related Work

Much work about pricing American Put Options has been devoted to the problem of obtaining good estimates of the price/value of an American Put Option efficiently. An indicative, but by no means exhaustive sample can be obtained through a study of [5, 6, 7, 8, 9, 10]. For a survey and comparison of various methods, refer to [11]. In this context, deriving bounds on the prices of an American Put Option can be useful not only in terms of the advantages that the model-free pricing method has to offer, but also as a computational scheme for pricing an American Put Option, provided the algorithms to derive the price bounds on American Put Option turn out to be efficient.

The idea of relaxing distributional assumptions on asset prices is indeed not new. Lo [12] derived bounds on the price of European Options given the mean and the variance of the underlying stock, under the risk neutral price measure. Grundy [13] generalized this approach to the case when the first and the kth moments of the stock price are known. Later, Bertsimas and Popescu [14] derived tight bounds on the price of a European Call Option, given prices of other options on the same stock, with the same maturity, using a convex optimization approach. In [15] and [16], Bertsimas and Bushueva derived tight bounds on the prices of European Options, given prices of other European Options with possibly different maturities. D' Aspremont and El Ghaoui [17] solved the problem of finding the bounds on the price of a European basket call option, given prices of other similar baskets. All the above works however address the problem of finding bounds on the price of a European-style option.

1.3 Outline

In the rest of the thesis, we exposit the problem of deriving bounds on the price of an American Put Option with a given strike and maturity using only the no-arbitrage conditions. We consider a setting where the market information set consists just the current price of the underlying and prices of other American Put Options on it that have the same maturity but different strikes. Our method is essentially non-parametric i.e., it does not refer to or make use of any quantities such as 'implied volatilities'. In Chapter 2, we outline the discrete time model used in this paper. In Chapter 3, we derive the necessary conditions on the prices of American Put Options. In Chapter 4, we derive the main result of this paper, Theorem 4.2, about the necessary and sufficient conditions that American Put Option Prices must satisfy. We then use this theorem to develop algorithms to find tight bounds on the price of an American Put Option, given the price of other Put Options with the same maturity in Chapter 5. We illustrate our results with numerical examples in Chapter 6 and conclude with remarks on future directions in Chapter 7. The problem addressed in this thesis - using no arbitrage conditions to derive bounds on prices of an American Put Option has dual objectives

- Using a more robust framework for pricing American Put Options that uses only information implied by the markets.
- To explore if there are efficient procedures to compute bounds on price of an American Put Options, which have been otherwise computationally demanding to valuate.

Chapter 2

Model

We consider a discrete time model with periods $1, 2, 3, \ldots$, where all price changes occur at period boundaries (ends). Without loss of generality, we assume the period length to be 1. An American Put Option, parametrized by an exercise strike K and a maturity T, is a security that allows (but does not require) the holder to sell one share of an underlying asset, (say stock,) at a price K at any time $t \leq T$, irrespective of its prevailing market price. Let S_t denote the price of the asset at time t. Let $r > 0$ be the risk free rate for one period. Then $\beta \triangleq \frac{1}{1+r} < 1$[1] is the one period discount factor (assumed to be constant[2]). We will also assume that the underlying does not pay dividends. Let $\mathbb{Q}$ denote a risk-neutral measure, i.e., any measure under which the discounted stock price process, $\beta^t S_t$, is a martingale. Let τ denote an exercise policy for an American Put Option. To be a valid exercise policy, τ must be a stopping-rule, (note that this constraint can be enforced even without specifying a measure for the price process). For notational convenience we require that the option is always struck, i.e., $\tau \leq T$ and upon exercise at time t, the payoff is $(K - S_t)^+$. The value or the yield of the option with strike K, under an exercise policy τ, will be denoted by $A_T(K, \tau)$.

[1]We shall assume that $r > 0$, as we would expect in reality. If $r = 0$ or $r < 0$ then $\beta \geq 1$ and $\mathbb{E}^{\mathbb{Q}}[\beta^T (K - S_T)^+] \geq \mathbb{E}^{\mathbb{Q}}[\beta^t (K - S_t)^+]$ $\forall t \leq T$. The inequality holds even when we replace t by a stopping time. τ. Hence exercising at T is always optimal. The problem is then the same as that of pricing a European Put Option, which has been studied before.

[2]This assumption is wlog, as one can otherwise construct periods in such a way that discounting over each period is the same.

Given a risk-neutral measure $\mathbb{Q}$, we then have,

$$A_T(K,\tau) \quad = \quad \mathbb{E}^{\mathbb{Q}}[\beta^\tau(K - S_\tau)^+].$$

The value/price of the American Put Option with strike K and maturity T, is then given by

$$A_T(K) \quad = \quad \sup_{\tau} \mathbb{E}^{\mathbb{Q}}[\beta^\tau(K - S_\tau)^+].$$

The same notation, $A_T(\cdot)$, will also be used to denote the value of the option, the meaning of which would be clear from the context - when τ is not explicitly stated as an argument. The set of optimal policies is defined as

$$\{\tau_T^*(K)\} \quad = \quad \arg\max_{\tau \leq T} \mathbb{E}^{\mathbb{Q}}[\beta^\tau(K - S_\tau)^+].$$

Apriori, it is not known if an optimal exercise policy exists. Thus $\{\tau_T^*(K)\}$ may as well be empty. Since all price-changes occur at period boundaries and $\beta < 1$, if $\{\tau_T^*(K)\}$ is non-empty, then there always exists an optimal exercise policy $\tau_T^*(K)$ in which one will exercise with a positive probability only at times in $\{0, 1, 2, \ldots, T\}$. Hence, we restrict ourselves to consider only those exercise policies that allow striking in $\{0, 1, 2, \ldots, T\}$. This leads to what is referred to in financial markets as a Bermudan Option. By increasing the refinement of a period, this approximation can be made arbitrarily accurate. Further, wlog, we can also assume that, $\tau_T^*(K) < T \Rightarrow K > S_{\tau_T^*(K)}$. This allows us to exclude from consideration possible degenerate policies without any loss of optimality. We can further restrict the set of exercise policies that we need to consider. Let $\mathcal{C}$ refer to the class of exercise policies, that can lead to exercise only at times in $\{0, 1, 2, \ldots, T\}$, satisfy $\tau_T^*(K) < T \Rightarrow K > S_{\tau_T^*(K)}$ and are non-randomized. A policy τ is non-randomized if the event $\tau \leq t$ is deterministic given $S_1, S_2, \ldots, S_t$, or in other words, at any time t one either strikes the option or does not strike, there is no mixing. Proposition A.1 in Appendix A shows that an optimal τ belonging to $\mathcal{C}$ always exists. Thus $\{\tau_T^*(K)\}$ is non-empty $\forall$ K in our setting. We will henceforth consider only policies belonging to class $\mathcal{C}$. Although, an optimal exercise policy within $\mathcal{C}$ exists, it may not necessarily be unique. We will use $\tau_T^*(K)$ to refer to any optimal exercise policy rather than a specific optimal policy. Also for any optimal exercise policy $\tau_T^*(K)$, it follows that $A_T(K) = A_T(K, \tau_T^*(K))$.

2.1 Problem Definition

The problem, that we seek to solve in this thesis is defined as follows. We are interested in bounding the price of an American Put Option with strike K and maturity T. Suppose we have been given the prices $A_1, A_2, \dots, A_N$ of N other American Put Options with the same maturity T but different strikes $K_1, K_2, \dots, K_N$, and the current asset price S_0. We seek to find the minimum and maximum values of $A_T(K)$, for which there exists a risk neutral measure $\mathbb{Q}$, that will price all the securities correctly. In other words, let $\mathcal{Q}$ be the space of admissible measures on the discrete process $S : [0, T] \to \Re^+$. Then we have the following optimization problems

P$^-$:

$$\begin{aligned} A^- &= \inf_{\mathcal{Q}} \max_{\tau \in \mathcal{C}} \mathbb{E}^{\mathbb{Q}}[\beta^\tau (K - S_\tau)^+] \\ \text{s.t. } \max_{\tau \in \mathcal{C}} \mathbb{E}^{\mathbb{Q}}[\beta^\tau (K_i - S_\tau)^+] &= A_i \quad , \quad i = 1, \dots, N \; ; \\ \mathbb{E}^{\mathbb{Q}}[\beta S_t | S_{t-1}, \dots, S_1] &= S_{t-1} \; , \quad t = 1, \dots, T \; . \end{aligned} \tag{2.1}$$

and

P$^+$:

$$\begin{aligned} A^+ &= \sup_{\mathcal{Q}} \max_{\tau \in \mathcal{C}} \mathbb{E}^{\mathbb{Q}}[\beta^\tau (K - S_\tau)^+] \\ \text{s.t. } \max_{\tau \in \mathcal{C}} \mathbb{E}^{\mathbb{Q}}[\beta^\tau (K_i - S_\tau)^+] &= A_i \quad , \quad i = 1, \dots, N \; ; \\ \mathbb{E}^{\mathbb{Q}}[\beta S_t | S_{t-1}, \dots, S_1] &= S_{t-1} \; , \quad t = 1, \dots, T \; . \end{aligned} \tag{2.2}$$

The conditions (2.1) and (2.2) arise because the discounted stock prices must be a martingale under $\mathbb{Q}$. Note that, the fact that τ can be defined independently of the measure $\mathbb{Q}$, allows us to formulate the problem in the stated form.

Problems $\mathbf{P}^-$ and $\mathbf{P}^+$ are not explicit optimization problems but encompass sub-optimization problems that are known to be not amenable to analytical solutions or fast computational procedures. Hence, we approach these problems in an indirect way, by finding structural properties of the price envelope $A_T(\cdot)$. Specifically, we seek to find consistency conditions on the prices, i.e., conditions that are necessary and sufficient for the existence of a no-arbitrage measure $\mathbb{Q}$.

Chapter 3

Necessary Conditions on the Price Curve

We denote by $A_T(\cdot)$, the price function, i.e., the price of the American Put Option as a function of the strike, with maturity T held constant. In this chapter, we will derive a set of necessary conditions that the price function must satisfy. For this we will it find useful to translate back and forth between properties of optimal exercise policies and their implications for $A_T(\cdot)$. We begin with some basic straightforward conditions, that $A_T(\cdot)$ must satisfy to avoid arbitrage opportunities.

3.1 Elementary Necessary Conditions

The following conditions on option prices and optimal exercise policies are immediate.

(N1) Option Price is greater than its intrinsic value:

$$A_T(K) \geq (K - S_0)^+. \tag{3.1}$$

(N2) $A_T(K)$ is increasing in K. This follows trivially as for $K_2 \geq K_1$,

$$\begin{aligned} A_T(K_2) &\geq A_T(K_2, \tau_T^*(K_1)) \\ &\geq A_T(K_1, \tau_T^*(K_1)) = A_T(K_1). \end{aligned}$$

(N3) $A_T(K)$ is convex in K. Let $K = \lambda K_1 + (1-\lambda)K_2$. Then,

$$
\begin{aligned}
(K-S)^+ &\leq \lambda(K_1-S)^+ + (1-\lambda)(K_2-S)^+. \\
\Rightarrow \mathbb{E}[\beta^{\tau_T^*(K)}(K-S_{\tau_T^*(K)})^+] &\leq \lambda\mathbb{E}[\beta^{\tau_T^*(K)}(K_1-S_{\tau_T^*(K)})^+] \\
&\quad +(1-\lambda)\mathbb{E}[\beta^{\tau_T^*(K)}(K_2-S_{\tau_T^*(K)})^+]; \\
\Rightarrow A_T(K) &\leq \lambda A_T(K_1) + (1-\lambda)A_T(K_2).
\end{aligned}
$$

(N4)

Lemma 3.1. *If for a strike K_2, $\exists\ \tau_T^*(K_2) \geq t$, then $A_T(K_2) - A_T(K_1) \leq \beta^t(K_2-K_1)\ \forall K_1 < K_2$.*

Proof. We have,

$$
\begin{aligned}
(K_2-S)^+ &\leq (K_2-K_1) + (K_1-S)^+. \\
\Rightarrow A_T(K_2) &\leq \mathbb{E}^{\mathbb{Q}}[\beta^{\tau_T^*(K_2)}(K_2-K_1)] + \mathbb{E}^{\mathbb{Q}}[\beta^{\tau_T^*(K_2)}(K_1-S_{\tau_T^*(K_2)})^+] \\
&\leq \beta^t(K_2-K_1) + \mathbb{E}^{\mathbb{Q}}[\beta^{\tau_T^*(K_1)}(K_1-S_{\tau_T^*(K_1)})^+] \\
&= \beta^t(K_2-K_1) + A_T(K_1).
\end{aligned}
$$

□

Some special implications of the above property are useful.

(N4.1) Since $\tau_T^*(K_2) \geq 0$,

$$
A_T(K_2) - A_T(K_1) \leq K_2 - K_1\ ,\quad \forall K_1 < K_2. \tag{3.2}
$$

Thus, for a fixed T, $A_T(K)$ is an increasing convex function, whose sub-gradient is bounded by 1.

(N4.2) If $A_T(K_2) > K_2 - S_0$, then we must have $\tau_T^*(K_2) \geq 1$. Using

Lemma 3.1, then we conclude

$$A_T(K_2) > K_2 - S_0 \Rightarrow A_T(K_2) - A_T(K_1) \leq \beta(K_2 - K_1), \ \forall K_1 < K_2. \tag{3.3}$$

(N4.3) We have either $A_T(K) = K - S_0$ or $A_T(K) > K - S_0$. Suppose the latter is true. Then by (3.3),

$$\begin{aligned} A_T(K) - A_T(0) &\leq \beta K. \\ \Rightarrow K - S_0 &< \beta K, \\ \text{i.e., } K &< \frac{S_0}{1-\beta}. \end{aligned}$$

Hence, we must have

$$A_T(K) = K - S_0, \ \forall K \geq \frac{S_0}{1-\beta}. \tag{3.4}$$

(N5)

Lemma 3.2. *Suppose $K' \geq K$, then $\exists$ an optimal policy $\tau_T^*(K')$ s.t. $\tau_T^*(K') \leq \tau_T^*(K)$ for all $\tau_T^*(K)$ i.e., if in a given state exercising a put option with strike K is optimal, it must also be optimal to exercise any option that has strike $K' > K$. This holds irrespective of the price dynamics process S_t.*

Proof. We prove this on a path by path basis. Let ω refer to a sample path in the evolution process. Suppose for some measure $\mathbb{Q}$, a path $\omega = (S_1, S_2, \ldots, S_t)$ is such that $\tau_T^*(K, \omega) = t$. If $\tau_T^*(K', \omega) \leq t-1$ or $t = T$, then we are done. Else, let $A_T(K, \omega, t)$ and $A_T(K', \omega, t)$, be the time t prices of the two options for the sample path ω. Since it is optimal to exercise the option with strike K at t on ω, $A_T(K', \omega, t) = K - S_t$ and $K \geq S_t$. Then by (3.2) applied to the American Put Option prices at time t on path ω,

$$\begin{aligned} A_T(K',\omega,t) &\leq A_T(K,\omega,t) + (K' - K) \\ &\leq K - S_t + K' - K \\ &\leq K' - S_t. \end{aligned}$$

But this is precisely, the value yield under the policy of exercising the option with strike K' immediately. Hence, $A_T(K',\omega,t) = K' - S_t$ and $\tau_T^*(K',\omega) = t$ is an optimal exercise policy. This concludes the proof. □

(N6) Define,

$$K^* \triangleq \inf\{K \geq 0 : A_T(K) = K - S_0\} \ . \tag{3.5}$$

(Note K^* is well defined in light of (3.4) and $K^* \leq \frac{S_0}{1-\beta}$.) Then, as $A_T(K)$ is convex, and therefore continuous, $A_T(K^*) = K^* - S_0$. Thus $\tau = 0$ is optimal at K^*. Then using Lemma 3.2,

$$A_T(K') = A_T(K',0) = K' - S_0 \ , \ \ \forall K' \geq K^*. \tag{3.6}$$

Condition (3.6), in fact, can also be seen to follow directly from (3.1) and (3.2).

3.2 Strict Necessary Conditions

Our goal is to find a complete set of necessary conditions or a set of conditions which is also sufficient. It would be useful to anticipate their form. Clearly, all the conditions derived in Section 3.1 must somehow be incorporated in any set of Sufficient Conditions that we enlist. Moreover, apriori, since only Option Prices and not Exercise Policies (in fact one can observe but one sample path of an Exercise Policy) are observable, we must state all the conditions solely in terms of the prices.

As we shall see later in Chapter 4, the simple conditions that we have derived so far, with just a little strengthening, also turn out to be sufficient.

The strengthening that is needed is that (3.3) holds for $K_2 = K^*$ (as defined in (3.5)) with a strict inequality, or equivalently at K^*, $A_T(\cdot)$ admits a sub- gradient that is strictly less than β. Interestingly, this condition of strict inequality also turns out to be a necessary condition. This result follows from the following two observations.

One, we know that $\tau = 0$ is not an optimal exercise policy for strikes $K < K^*$ (hence for these strikes, we must have optimal policies $\tau_T * (K) \geq 1$.), while it is an optimal exercise policy for strikes $K > K^*$. Continuity arguments then imply that at the transition point K^*, there exist at least two distinct optimal exercise policies, one of which, say $\bar{\tau}$, does not lead to immediate exercise. Then, from Lemma 3.1, we can conclude readily, that a left sub-gradient l to the price curve $A_T(\cdot)$ at K^* satisfies $l \leq \beta$.

Two, suppose we take a strike K' just less than K^*, and exercise the American Option with strike K' using the non-immediate optimal exercise policy for K^*, i.e., $\bar{\tau}$. Then, along any sample path our payoffs from the two options will differ by at most $\beta(K^* - K')$. Now, the expected difference between the payoffs can be $\beta(K^* - K')$, only if under $\bar{\tau}$, option with strike K^* is always exercised with a positive payoff. However, the event that the option with strike K^* is not exercised with a strictly positive must happen with a non-zero probability. Because if the option is always exercised with a positive payoff, then as the option exercise policy is a stopping time, on an average the discounted stock price faced upon exercise must be S_0, which would then make the option value $A_T(K^*) \leq \beta K^* - S_0 < K^* - S_0$. This then immediately yields the condition that the left sub-gradient at K^* must be strictly less than β. We now make this informal discussion formal.

Strict Necessary Conditions

We shall first prove the following useful lemma.

Lemma 3.3. *(a) The set* $A \triangleq \{K : \{\tau_T^*(K)\} = \{0\}\}$ *is an open interval, i.e., the set of all* K *for which the only optimal policy is to strike immediately is open.*

(b) *The set* $B \triangleq \{K : \{0 \notin \tau_T^*(K)\}\}$ *is an open interval or empty, i.e., the set of all* K *for which exercising immediately is sub-optimal is open (or empty).*

Proof. (a) Suppose $K \in A$, i.e., $\{\tau_T^*(K)\} = \{0\}$. Consider the problem

$$A_T^0(K) = \sup_{\tau \geq 1} \mathbb{E}^{\mathbb{Q}}[\beta^\tau (K - S_\tau^+)].$$

We can show, using techniques similar to those of Proposition A.1 in Appendix A, that an optimal solution to the above problem exists. (We omit the proof, here.) Let $\tilde{\tau}$ be the optimal solution to the above problem. Then, by assumption, $\Delta \triangleq A_T(K) - A_T(K, \tilde{\tau}) = K - S_0 - A_T(K, \tilde{\tau}) > 0$.

Now consider the option with strike $K' = K - \frac{\Delta}{2}$. For any policy $\tau \geq 1$,

$$\begin{aligned} A_T(K', \tau) &\leq A_T(K, \tau) \\ &\leq A_T(K, \tilde{\tau}) \\ &= A_T(K) - \Delta < A_T(K', 0). \end{aligned}$$

Thus, it follows that $\{\tau_T^*(K')\} = \{0\}$. We already know that for any $\tau \geq 1$, and $K' > K$, $A_T(K', \tau) \leq A_T(K, \tau) + \beta(K' - K) < A_T(K) + K' - K$. Thus for all $K' > K$, $\{\tau_T^*(K)\} = \{0\}$. This shows that $A = \{K : \{\tau_T^*(K)\} = \{0\}\}$ is either empty or open. Finally, as for $K > \frac{S_0}{1-\beta}$, $\tau = 0$ is the only optimal policy, we know that the set is not empty, and hence must be an open interval.

(b) Suppose $K \in B$, i.e., $\tau = 0$ is not an optimal exercise policy at K. It then follows directly from Condition (3.6) in Section 3.1, that for any $K' < K$, $0 \notin \{\tau_T^*(K)\}$, i.e., $K' \in B$. By the sub-optimality of $\tau = 0$ at K, $\Delta \triangleq A_T(K) - (K - S_0)^+ > 0$. Then consider the option with strike $K' = K + \frac{\Delta}{2}$. It follows

$$\begin{aligned} A_T(K', 0) &= A_T(K, 0) + \frac{\Delta}{2} \\ &< A_T(K) \\ &\leq A_T(K'). \end{aligned}$$

Thus the set $B = \{K : \{0 \notin \tau_T^*(K)\}\}$ is an open interval or empty[1].

□

Corollary 3.1. *The set $\{K : \{0\} \subset \{\tau_T^*(K)\}\} \neq \emptyset$ and in particular, at K^*, there $\exists$ an optimal exercise policy $\tau_T^*(K)$ satisfying $\tau_T^*(K) \geq 1$. Also, $0 \in \{\tau_T^*(K^*)\}$.*

Proof. From Lemma 3.3, it is clear that the sets A and B defined there are such that $A^c \cap B^c \neq \emptyset$. On the other hand, an optimal exercise policy must exist for every $K > 0$. This means that $\exists$ K which have at least two optimal exercise policies, one of which satisfies $\tau = 0$, and another $\tau \geq 1$. Further, this is true of K^* as by definition (3.5), $K^* \notin A$ and $K^* \notin B$, which are open sets. □

Corollary 3.1 when combined with the following lemma, yields the necessary strengthening of the conditions derived so far.

Lemma 3.4. *Suppose $\beta < 1$. Then, if $\tau_T^*(K) \geq 1$, $\mathbb{P}(K < S_{\tau_T^*(K)}) > 0$.*

Proof. Suppose the lemma is not true and $\mathbb{P}(K < S_{\tau_T^*(K)}) = 0 \Rightarrow K \geq S_{\tau_T^*(K)}$ with probability 1. Then,

$$\begin{aligned} A_T(K) &= \mathbb{E}^{\mathbb{Q}}[\beta^{\tau_T^*(K)}(K - S_{\tau_T^*(K)})^+] \\ &= \mathbb{E}^{\mathbb{Q}}[\beta^{\tau_T^*(K)}(K - S_{\tau_T^*(K)})] \\ &\leq \beta K - S_0 \\ &< K - S_0. \end{aligned}$$

which is of course a contradiction. We made use of the fact that $\mathbb{E}[\beta^\tau S_\tau] = S_0$, if $\beta^t S_t$ is a martingale and if τ is a bounded stopping time (sometimes also referred to as the Optional Stopping Theorem). □

Corollary 3.2. *$\exists$ a sub-gradient $\nu < \beta$ to $A_T(\cdot)$ at K^*.*

[1]We could also have proved this property by showing that the complement of the set of interest must be closed as $A_T(K)$ is a convex and hence a continuous function. Note that this $\Rightarrow 0 \in \{\tau^*(K^*, T)\}$.

Proof. First note that, for $K \geq K^*$,

$$\begin{aligned} A_T(K) - A_T(K^*) &= K - K^* \\ &> \nu(K - K^*). \end{aligned}$$

for any $\nu < 1$ and in particular any $\nu < \beta$. Thus all we need to show in order to prove the result is that $\exists \nu < \beta$ s.t. $A_T(K^*) - A_T(K) \leq \nu(K^* - K)$, $\forall K < K^*$. From Corollary 3.1, we know $\exists\ \tau^* \geq 1$ which is optimal for the option with strike K^*. From Lemma 3.4, then $q \triangleq \mathbb{P}[(K - S_{\tau^*})^+ = 0] > 0$. Then, for $K < K^*$,

$$\begin{aligned} A_T(K^*) &= \mathbb{E}^{\mathbb{Q}}[\beta^{\tau^*}(K^* - S_{\tau^*})^+] \\ &= \Pr[K^* > S_{\tau^*}]\mathbb{E}^{\mathbb{Q}}[\beta^{\tau^*}(K^* - S_{\tau^*})|K^* > S_{\tau^*}] \\ &\leq (1-q)\beta(K^* - K) + \Pr[K^* > S_{\tau^*}]\mathbb{E}^{\mathbb{Q}}[\beta^{\tau^*}(K - S_{\tau^*})|K^* > S_{\tau^*}] \\ &\leq (1-q)\beta(K^* - K) + \Pr[K > S_{\tau^*}]\mathbb{E}^{\mathbb{Q}}[\beta^{\tau^*}(K - S_{\tau^*})|K > S_{\tau^*}] \\ &= (1-q)\beta(K^* - K) + \mathbb{E}^{\mathbb{Q}}[\beta^{\tau^*}(K - S_{\tau^*})^+] \\ &\leq (1-q)\beta(K^* - K) + A_T(K). \end{aligned}$$

As $q > 0$, we must have $A_T(K^*) - A_T(K) \leq (1-q)\beta(K - K^*)$, $\forall\ K < K^*$. Thus $\exists$ a sub-gradient $\nu = (1-q)\beta < \beta$, to $A_T(\cdot)$ at K^*. Using this condition for $K = 0$, we get $K^* < \frac{S_0}{1-\beta}$. □

In the next chapter, we show that Corollary 3.2, together with the conditions derived in Section 3.1, are not only necessary but also sufficient.

Chapter 4

Sufficient Conditions

In this chapter, we formally show that the conditions (N1:N6) listed in Section 3.1 and Corollary 3.2 are sufficient as well. Since these conditions were not all independent, we will also reduce them to a convenient minimal set. This chapter summarizes the key results derived in this thesis in the form of Theorem 4.2.

4.1 Sufficiency Conditions for European Call Options

To prove Theorem 4.2, we will use the following theorem that was proved in [15]. This theorem gives (necessary and) sufficient conditions for a function to be a price curve of European Call Options with time to maturity 1 period. For easy reference, we state the theorem here in a form applicable to our setting, without proof.

Theorem 4.1. *Let $C : \Re^+ \rightarrow \Re^+$ be a function such that*

1. *$C(\cdot)$ is non-increasing;*

2. *$C(\cdot)$ is convex;*

3. *$\lim_{K\rightarrow\infty} C(K) = 0$;*

4. *$C(K) \geq C(0) - \beta K$.*

Then, $\exists$ *a random variable* $S \geq 0$ *s.t.* $C(K) = \mathbb{E}[(S - \beta K)^+]$.

4.2 Proof of Sufficiency

Theorem 4.2. *Let* $A_T(K)$ *denote the price of an American Put Option with maturity* T *as a function of its strike* K. *Let* S_0, *be the initial (time* 0*) asset price. Then there exists a martingale measure consistent with the pricing of the American Put Options iff*

(C1) $A_T(0) = 0$.

(C2) $A_T(K) \geq (K - S_0)^+$.

(C3) $A_T(K)$ *is increasing in* K.

(C4) $A_T(K)$ *is convex in* K.

(C5) *Let* $K^* = \inf\{K : A(K,T) = K - S_0\}$. *Then such a* K^* *exists and moreover* $K^* < \frac{S_0}{1-\beta}$. *Also,*

$$\begin{aligned} A_T(K) &= K - S_0 \ , \quad K \geq K^*; \\ A_T(K^*) - A_T(K) &\leq \nu(K^* - K) \ , \quad K < K^*; \end{aligned} \tag{4.1}$$

for some $\nu < \beta$, *i.e.,* $\exists$ *a sub-gradient* ν, *to* $A_T(K)$ *at* K^* *s.t.* $\nu < \beta$.

Proof. Before, we proceed to give a formal proof, we discuss the basic idea underlying our approach. We showed the necessity of all the conditions listed above in Chapter 3. Hence, all we need to show is that these conditions are also sufficient. Using Theorem 4.1, we will argue that upto the point K^*, conditions that apply to $A_T(\cdot)$ are consistent with those needed to construct a European Put Option of Maturity 1. Also, for all strikes beyond K^*, from (4.1), $A_T(\cdot)$ becomes the same as option's intrinsic value. This means that volatility should not play a role for these higher strikes. We will then try to construct an essentially one-change price process such that the European

Put Option Prices (with maturity 1) are the same as the given American Put Option Prices in the region $K \leq K^*$ but are less than them for $K > K^*$. If the discounted prices remained constant after period 1, then it is clear that this price process would explain $A_T(\cdot)$. Such a construction is possible only if $F(K^*) < 1$, where $F(\cdot)$, is the distribution of underlying's price at $t = 1$ implied by the European Put Option Prices with maturity 1. Now the left-gradient to the European Put option price curve at any strike K is simply $\beta F(K)$. Since, the European Put Option prices are to match $A_T(\cdot)$ for $K \leq K^*$, the condition $F(K^*) < 1$, is the same as Condition (C5). Fortunately, this strictness condition, as we know, is also a necessary condition. Figure 4-1 gives a graphical view of the construction.

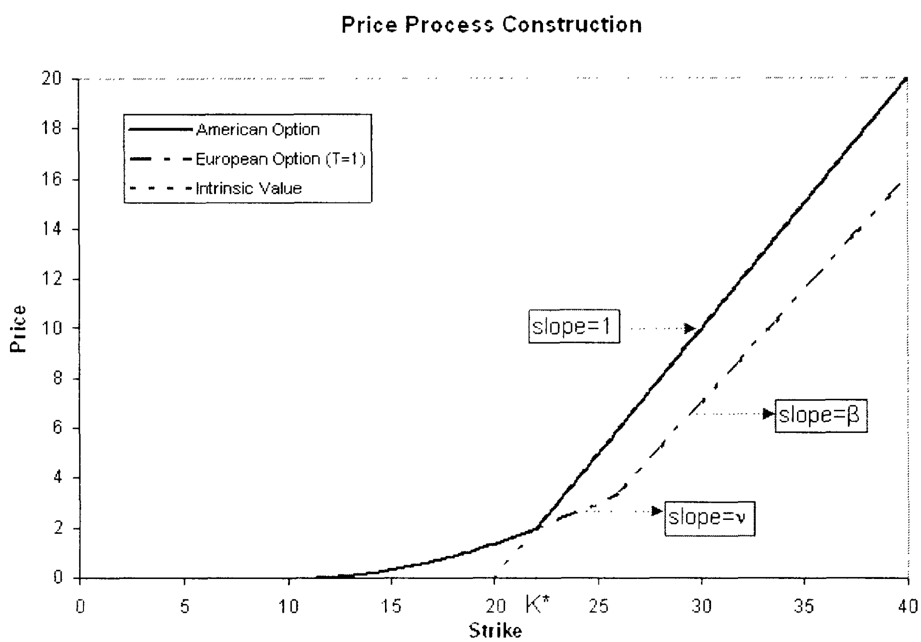

Figure 4-1: Price Process Construction

We will now formally construct a martingale measure consistent with $A_T(\cdot)$. As indicated before, we will construct one for which $\tau_T^*(K) \leq 1 \ \forall \ K$. Let ν be as defined in the statement of the theorem. Define $C(\cdot)$ as follows.

$$C(K) = \begin{cases} A_T(K) - (\beta K - S_0) & \text{if } (0 \le K \le K^*); \\ (C(K^*) - (\beta - \nu)(K - K^*))^+ & \\ = (-(\beta - \nu)K + (1 - \nu)K^*)^+ & \text{if } (K > K^*). \end{cases} \tag{4.2}$$

Then, we claim that $C(\cdot)$ satisfies all the conditions in Theorem 4.1. First note that $C(\cdot)$ is a continuous function by construction.

- *Non-negativity:*

  $C(\cdot)$ is non-negative for $K > K^*$ by definition. For $K \le K^*$, note that

$$\begin{aligned} C(K) &= A_T(K) - (\beta K - S_0) \\ &\ge A_T(K) - (K - S_0)^+ \\ &\ge 0. \end{aligned}$$

  by Condition (C2).

- *Non increasingness:*

  Again as $\nu < \beta$, non-increasingness in (K^*, ∞) follows from the definition of $C(\cdot)$. Consider $K_1, K_2 : K_1 \le K_2 \le K^*$. Since $\nu < \beta$ is a sub-gradient at K^* to $A_T(\cdot)$, and as $A_T(\cdot)$ is convex by Condition (C4), $\exists$ a sub-gradient $\nu_1 \le \nu < \beta$ to $A_T(\cdot)$ at K_1. Then

$$\begin{aligned} A_T(K_2) - A_T(K_1) &\le \nu_1(K_2 - K_1) \\ &\le \beta(K_2 - K_1). \\ \Rightarrow A_T(K_2) - (\beta K_2 - S_0) &\le A_T(K_1) - (\beta K_1 - S_0). \\ \Rightarrow C(K_2) &\le C(K_1). \end{aligned}$$

  Thus $C(\cdot)$ is also decreasing in $[0, K^*]$. As $C(\cdot)$ is continuous at K^*, it follows that it is a non-increasing function.

- *Convexity:*

  $C(\cdot)$ is convex in the domain $[0, K^*]$ as $A_T(\cdot)$ is a convex function. It is also convex in (K^*, ∞) by construction. Further, we note that, $C(\cdot)$ is continuous and has a left sub-gradient ν at K^*, which is less than a right sub-gradient β at K^*. Hence $C(\cdot)$ is a convex function.

- $\lim_{K\to\infty} C(K) = 0$:

  By construction, $C(K) = 0$ for $K > \frac{1-\nu}{\beta-\nu}K^*$. Hence $\lim_{K\to\infty} C(K) = 0$.

- $C(K) \geq C(0) - \beta K$:

  Suppose $K \leq K^*$. Then,

$$\begin{aligned} C(K) - C(0) &= A_T(K) - \beta K \\ &\geq -\beta K. \end{aligned}$$

  as $A_T(K) \geq 0$. Now for $K > K^*$, note that

$$\begin{aligned} C(K) &\geq C(K^*) - (\beta - \nu)(K - K^*) \\ &\geq C(0) - \beta K^* - \beta(K - K^*) \\ &= C(0) - \beta K. \end{aligned}$$

Using Theorem 4.1, then there exists a random variable $X \geq 0$ s.t. $\mathbb{E}[(X - \beta K)^+] = C(K)$. Now we define a risk neutral measure on S_t, for which $A_T(\cdot)$ will be an American Put Option price curve. This measure can be described in terms of the dynamics of S_t under the same, which are given as follows:

$$\begin{aligned} S_1 &\stackrel{d}{=} \frac{X}{\beta}; \\ S_t &= \frac{S_1}{\beta^{t-1}}, \text{ if } \ldots\ 1 < t \leq T. \end{aligned}$$

First, we verify that this process is a martingale. We have

$$\begin{aligned}
\mathbb{E}[\beta^t S_t | S_{t-1}, \ldots, S_1] &= \beta S_1 = \beta^{t-1} S_{t-1} \;\text{ if } t \geq 1, \text{ and} \\
\mathbb{E}[\beta S_1] &= \mathbb{E}[X] \\
&= C(0) = S_0 \;, \quad \text{by construction.}
\end{aligned}$$

Let $\bar{A}_T(\cdot)$ denote the American Put Option Price Curve for this process. We claim that an optimal exercise policy $\tau^* \leq 1$ exists for all strikes. This is because, for $\tau > 1$

$$\begin{aligned}
\beta^\tau (K - S_\tau)^+ &= \beta^\tau (K - S_1 \frac{1}{\beta^{\tau-1}})^+ \\
&= \beta(\beta^{\tau-1} K - S_1)^+ \\
&\leq \beta(K - S_1)^+.
\end{aligned}$$

Thus for $\beta < 1$, if is optimal to exercise at $\tau > 1$ for some sample path ω, it must also be optimal to exercise at $\tau = 1$ for that sample path. Then, wlog, we can take $\{\tau_T^*(K)\} \subseteq \{0, 1\}$. Note that $\bar{A}_T(K, 0) = (K - S_0)^+$ and $\bar{A}_T(K, 1) = \mathbb{E}[\beta(K - S_1)^+]$. By the Put-Call parity,

$$\begin{aligned}
\mathbb{E}[\beta(K - S_1)^+] &= \mathbb{E}[(\beta S_1 - \beta K)^+] + (\beta K - S_0) \\
&= \mathbb{E}[(X - \beta K)^+] + (\beta K - S_0) \\
&= C(K) + (\beta K - S_0). \\
\text{Hence, } \bar{A}_T(K, 1) &= \begin{cases} A_T(K) & \text{if } K \leq K^*; \\ (-(\beta - \nu)K + (1 - \nu)K^*)^+ + \beta K - S_0 & \text{if } K > K^*. \end{cases}
\end{aligned}$$

As $A_T(K) > K - S_0$ for $K \leq K^*$, it follows that $\bar{A}_T(K, 1) \geq \bar{A}_T(K, 0) \Rightarrow \bar{A}_T(K) = A_T(K)$ for $K \leq K^*$. For $K > K^*$, we have

$$\begin{aligned}
\bar{A}_T(K, 1) &= (\nu(K - K^*) - \beta K + K^*)^+ + \beta K - S_0 \\
&\leq ((K - K^*) - \beta K + K^*)^+ + \beta K - S_0 \\
&= K - S_0.
\end{aligned}$$

Hence for $K \geq K^*$, $\bar{A}_T(K,1) \leq \bar{A}_T(K,0) \Rightarrow \bar{A}_T(K) = K - S_0 = A_T(K)$. This means that $\bar{A}_T(K) = A_T(K)\ \forall K$. □

If we compare the necessary and sufficient conditions from Theorem 4.2 to those for European Put options with maturity T, then Conditions (C1), (C3) (increasingness) and (C4) (convexity) apply to the latter as well. We, in fact, also have counterparts to conditions (C2) and (C5). If we denote by $E_T(K)$, the price of a European Put option of maturity T and strike K, then we will have $E_T(K) \geq (\beta^T K - S_0)^+$. Also, even for European Put Options, if for some K_E, $E_T(K_E) = \beta^T K_E - S_0$, then $E_T(K) = \beta^T K - S_0$, $\forall\ K \geq K_E$. However such a K_E need not exist for a European Put Option, unlike the American Put Option case[1]. We observe that the conditions for $A_T(\cdot)$ are closer to those for $E_1(\cdot)$ (European Put Options with maturity 1) rather than the ones for $E_T(\cdot)$. Indeed, an interesting observation from Theorem 4.2 is that the time to maturity T, in fact, does not figure in the set of necessary and sufficient conditions.

As a final remark, in the proof of Theorem 4.2, we constructed one measure that was consistent with an American put option price curve $A_T(\cdot)$ satisfying the conditions listed therein. However, this measure need not be the only one consistent with $A_T(\cdot)$.

[1]However, we do require $\lim_{K\to\infty} E_T(K) - (\beta^T K - S_0) = 0$.

Chapter 5

Algorithm to find bounds on the Price of an American Put

In this chapter, we translate the conditions that we derived in terms of the entire price curve $A_T(\cdot)$ to their implications on discrete samples from this curve, i.e., strike and price point pairs. This translation is loss-less in the sense that we have equivalent necessary and sufficient no-arbitrage conditions that can be expressed in terms of discrete points on the $A_T(\cdot)$ curve. We then extend the results derived in Chapters 3 and 4 to derive tight no-arbitrage bounds on an American Put option with a given maturity and strike, based on prices of other put options with the same maturity and different strikes $K_1, K_2, \ldots, K_N$ and the current underlying price S_0, i.e., we achieve our goal of solving problems $\mathbf{P}^-$ and $\mathbf{P}^+$ defined in Chapter 2.

5.1 Conditions for Strike-Price Pairs

Proposition 5.1. *Given $N+1$ American options with distinct strikes $K_0 = 0, K_1, K_2, \ldots, K_N$, s.t. $0 = K_0 < K_1 < K_2 < \ldots < K_N$ and prices $A_0 = 0, A_1, A_2, \ldots, A_N$ respectively, an initial underlying price S_0; there exists a martingale measure consistent with the option prices iff the conditions listed below are satisfied. Wlog. assume that the strike-price pair $(\bar{K} = \frac{S_0}{1-\beta}, \bar{K} - S_0)$ exists as a data point*[1].

[1] If this pair cannot be accommodated, then there is an arbitrage opportunity.

1. $A_i \geq (K_i - S_0)^+ \ , \quad i = 0, \ldots, N.$

2. Define $l_i \triangleq \frac{A_i - A_{i-1}}{K_i - K_{i-1}}$. *Then* $0 \leq l_{i-1} \leq l_i \ , \quad i = 1, \ldots, N.$

3. Let $i^* \triangleq \min_{1 \leq i \leq N}\{i : A_i = K_i - S_0\}$. *Then,*

$$\begin{aligned} A_i &= K_i - S_0 \ , \quad \forall i \geq i^* \ ; \\ l_{i^*} &< 1 \ ; \\ l_{i^*-1} &< \beta \ . \end{aligned}$$

Further, if $K_{i^*} = \bar{K}$, *then* $l_{i^*-1} < l_{i^*}$.

Proof. Suppose there exists a martingale measure $\mathbb{Q}$. The necessity of Conditions 1, 2, 3 follows immediately from Theorem 4.2. We only remark about the qualification for Condition 3.

Suppose $A_T(K)$ is the American Put Price function for $\mathbb{Q}$, then $K_{i^*-1} < K^* \leq K_{i^*}$ if $K_{i^*} < \bar{K}$ and $K_{i^*-1} < K^* < K_{i^*}$ if $K_{i^*} = \bar{K}$. Suppose, $K_{i^*} = \bar{K}$. Then

$$\begin{aligned} l_{i^*} &= \frac{A_{i^*} - A_{i^*-1}}{K_{i^*} - K_{i^*-1}} \\ &= \frac{K^* - K_{i^*-1}}{K_{i^*} - K_{i^*-1}} \frac{A_T(K^*) - A_T(K_{i^*-1})}{K^* - K_{i^*-1}} + \frac{K_{i^*} - K^*}{K_{i^*} - K_{i^*-1}} \frac{A_T(K_{i^*}) - A_T(K^*)}{K_{i^*} - K^*} \\ &\geq \frac{K^* - K_{i^*-1}}{K_{i^*} - K_{i^*-1}} l_{i^*-1} + \frac{K_{i^*} - K^*}{K_{i^*} - K_{i^*-1}} \\ &> l_{i^*-1}. \end{aligned}$$

To prove sufficiency, let $\lambda = \frac{1 - l_{i^*}}{1 - l_{i^*-1}}$. Then, $0 < \lambda \leq 1$, if $K_{i^*} < \bar{K}$ and $0 < \lambda < 1$, if $K_{i^*} < \bar{K}$. Add, if necessary, the point $(\bar{K}^*, A^*) = (K_{i^*-1} + \lambda(K_{i^*} - K_{i^*-1}), \bar{K}^* - S_0)$ to data (K_i, A_i) and consider the piecewise linear function passing through these points. The left derivative of this function at $\bar{K}^*$ is

$$\begin{aligned} m_- &= \frac{A^* - A_{i*-1}}{\bar{K}^* - K_{i^*-1}} \\ &= \frac{\bar{K}^* - S_0 - (A_{i^*} - l_{i^*}(K^* - K_{i^*-1}))}{\lambda(K^* - K_{i^*-1})} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda + l_{i^*} - 1}{\lambda} \\
&= l_{i^*-1}.
\end{aligned}
$$

If $\lambda < 1$, the right derivative is

$$
\begin{aligned}
m_+ &= \frac{A_{i^*} - A^*}{K_{i^*} - \bar{K}^*} \\
&= 1.
\end{aligned}
$$

Also,if $\lambda = 1$, then $m^+ = 1$ trivially. Thus the resulting function is convex. Moreover as $l_{i^*-1} < \beta$ and $\lambda < 1$ when $K_{i^*} = \bar{K} \Rightarrow \bar{K}^* < \bar{K}$, this function satisfies all the properties to be a valid American Put Price function as listed in Theorem 4.2, with $K^* = \bar{K}^*$. Thus a price process fitting the given option prices exists. □

5.2 Algorithm to find Bounds

Now we are ready, to solve the problems $\mathbf{P}^-$ and $\mathbf{P}^+$ defined in Chapter 2. We assume that we have been given data in the augmented form described in Proposition 5.1 and the conditions listed therein are satisfied (else there must exist an arbitrage opportunity). We are interested in finding the upper bound A^+ and the lower bound A^- on the price of an option that has strike K. Consider the following exhaustive cases:

1. $\exists\, i$ s.t. $K < K_i$ and $A_i = 0$. As $A_T(K)$ is non-negative and increasing, it then follows trivially that $A^+ = A^- = 0$ in this case.

2. $K > K_{i^*}$. This $\Rightarrow K > K^*$, which in turn $\Rightarrow A^+ = A^- = K - S_0$.

3. $K < K_{i^*-1}$. Then $K < K^*$. In this case, let $j > 1$ be such that $K_{j-1} < K < K_j$. Then the only conditions that the price A of the option must satisfy are

$$
\begin{aligned}
A &\geq A_{j-1} + l_{j-1}(K - K_{j-1}), \\
A &\geq A_j - l_{j+1}(K_j - K),
\end{aligned}
$$

$$\begin{aligned} A &> A_j - \beta(K_j - K), \\ A &\leq A_{j-1} + l_j(K - K_{j-1}). \end{aligned}$$

These are simple linear constraints and the problems $\mathbf{P}^-$ and $\mathbf{P}^+$ are almost standard linear programming problems but for the strict inequality in the third constraint. The problems are feasible as $l_{j+1} \geq l_j \geq l_{j-1}$ and $\beta > l_j \geq l_{j-1}$. For the problem $\mathbf{P}^+$, only the last constraint in fact will be active. We have

$$\begin{aligned} A^- &= \max(A_j + l_{j-1}(K - K_{j-1}), A_j - l_{j+1}(K_j - K), A_j - \beta(K_j - K)) \; ; \\ A^+ &= A_j + l_j(K - K_{j-1}). \end{aligned}$$

Also, note that if $A^- = A_j - \beta(K_j - K)$, then although the bound A^- is tight, there doesn't exist an optimal solution, i.e., $\nexists\, \mathbb{Q}$ under which the optimal price is realized.

4. $K_{i^*-1} < K < K_{i^*}$ and $K_{i^*} < \bar{K}$. Then the conditions that the price A for the option must satisfy are

$$\begin{aligned} A &\geq A_{i^*-1} + l_{i^*-1}(K - K_{i^*-1}), \\ A &\geq A_{i^*} - (K_{i^*} - K), \\ A &\leq A_{i^*-1} + l_{i^*}(K - K_{i^*-1}), \\ A &< A_{i^*-1} + \beta(K - K_{i^*-1}). \end{aligned}$$

Again the problems $\mathbf{P}^-$ and $\mathbf{P}^+$ are almost standard linear but for the strict inequality in the last constraint. These problems are also feasible. And

$$\begin{aligned} A^- &= \max\{A_{i^*-1} + l_{i^*-1}(K - K_{i^*-1}), A_{i^*} - (K_{i^*} - K)\}; \\ A^+ &= A_{i^*-1} + (K - K_{i^*-1}) \min(l_{i^*}, \beta). \end{aligned}$$

Again, if $\beta < l_{i^*}$, then the bound A^+ is tight but there is no optimal solution.

5. $K_{i^*-1} < K < K_{i^*}$ and $K_{i^*} = \bar{K}$. Then $K^* < K_{i^*}$. Then the conditions that the price A for the option must satisfy are

$$\begin{aligned} A &\geq A_{i^*-1} + l_{i^*-1}(K - K_{i^*-1}), \\ A &\geq A_{i^*} - (K_{i^*} - K), \\ A &< A_{i^*-1} + l_{i^*}(K - K_{i^*-1}), \\ A &< A_{i^*-1} + \beta(K - K_{i^*-1}). \end{aligned}$$

We have

$$\begin{aligned} A^- &= \max\{A_{i^*-1} + l_{i^*-1}(K - K_{i^*-1}), A_{i^*} - (K_{i^*} - K)\}; \\ A^+ &= A_{i^*-1} + (K - K_{i^*-1}) \min(l_{i^*}, \beta). \end{aligned}$$

Again, the bound A^+ is tight but there is no optimal solution.

Thus, the problems $\mathbf{P}^-$ and $\mathbf{P}^+$ that we defined in Chapter 2, can, in fact, be solved very efficiently using a rather simple algorithm and in effect one has closed form expressions for the bounds.

Chapter 6

Numerical Illustrations

In this chapter, we provide numerical examples of bounds computed on select option prices using actual market data. Our purpose is three-fold:

- To explore the quality of bounds that we have derived and their usefulness in a real-life setting.
- To see how efficient the markets are and if arbitrage opportunities exist.
- To see how deep the markets are. We can get an idea of the deepness or liquidity in the markets by comparing the spread of bid-ask quotes with the spread of no-arbitrage bounds that we compute.

We, however, will first need to somewhat adapt the algorithm discussed in Section 5.2 to apply it to market data.

6.1 Modifications

Markets of the real world do not directly fit our framework that we described in Chapter 2. There are two important differences between the markets and our model. First, note that our model is based on a discrete time approximation, i.e., the bounds were derived for a Bermudan approximation to the American Put Option rather than the American Put option itself. The quality of the approximation improves as the discretization intervals become smaller. A fallout of the tightening of this approximation is that the value of β approaches 1. As this happens, we have in effect a relaxation of Condition 3 in Proposition 5.1. Interestingly this leads to the conclusion that the risk-free interest rate, which makes the pricing of an American Put Option a complicated exercise doesn't feature directly in the computation of bounds that are based on the prices of other Put Options of the same maturity. The effect of the interest rates on the price of the option is an indirect one and is conveyed only through the prices of other put options and the constraints they impose on the price of the option of interest. With this modification, the primary constraint that we must test for no-arbitrage in American Put option prices is that of convexity. Note that this is also a sufficient condition. Another implication of this refinement is that to find bounds on the price of an option with strike K, we need only 'local' price information, or more precisely, only the prices of options with two nearest strikes above and below K matter. This is of course assuming that the prices of these options are known accurately. This, however, is the second significant point of departure of the model from the markets. Markets have frictions and can indicate only a range for the price (bid-ask spreads) and not the price itself accurately. There are two approaches that we may use to address this issue, and we will use both to compute bounds. The first and the simpler approach is to take a point estimate of the fair price using the bid-ask quotes. For example, we can use their average as the 'true' price. We shall refer to the bounds computed using this approach as 'nominal' bounds. The other, stricter approach is to interpret the bid and ask quotes as lower and upper envelopes respectively on the option prices. For no arbitrage to hold, then these envelopes must be such that they allow a convex function to 'pass through' them. See Figure 6.1 for an illustration. This concept can be made formal and bounds can be computed based on the bid ask quotes by solving a simple linear programming problem, which is given in Appendix B. We will refer to the bounds computed using this approach as the 'true' no-arbitrage bounds.

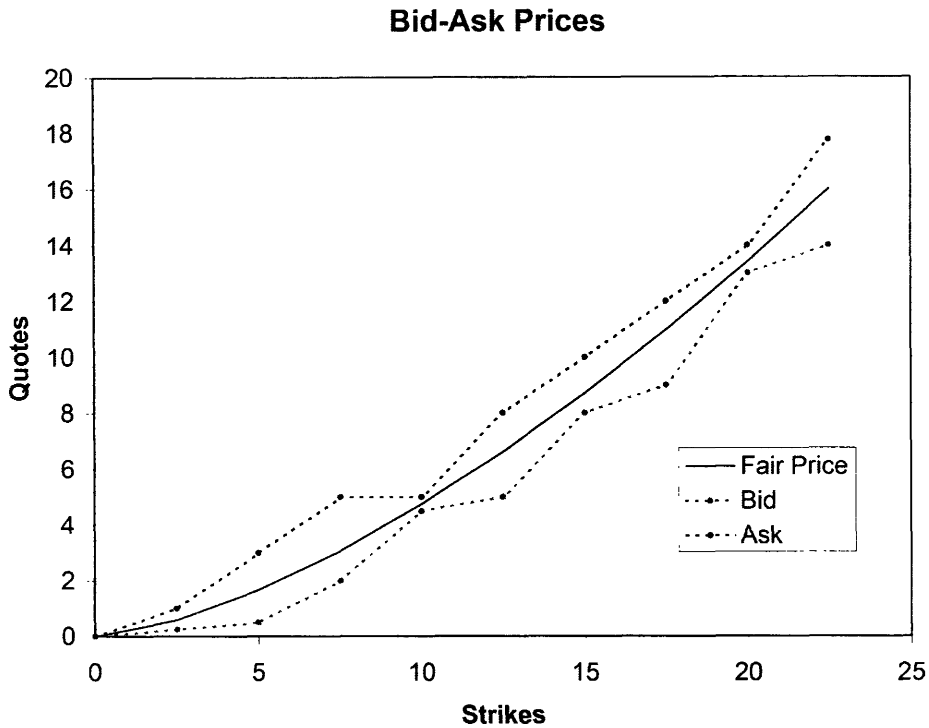

Figure 6-1: Bid-Ask Price enforced constraints

6.2 Data

We used data as on close of Feb 14, '06 on 3 types of Options - the American Put Options on the S&P500 Index maturing on Dec. 15, '06. traded on the CBOE (Chicago Board Options Exchange), which have a relatively deep market, the American Put Options on General Electric (GE) Stocks maturing on Mar 17, '06 - an option that would have a rather low volatility and American Put Options on Intel Corp. Stock maturing on Jan 17, '08 - which would be likely to be more volatile. The latter two options are relatively much less liquid and are traded on the American Exchange (AMEX). We obtained all our quotes from Yahoo! Finance [18].

6.3 Quality of Bounds

We first compute the true no-arbitrage implied bounds on the S&P-500 options based on the bid and ask quotes. Figure 6.3 plots the bid and ask prices against the strikes. For each strike, we take the bid-ask quotes of all other strikes and solve the program in Appendix B to compute the true no-arbitrage upper and lower bounds on its price. A comparison of the true no-arbitrage bounds with the bid-ask quotes for that strike allows us to check if any arbitrage opportunities existed.

The results of our computations are tabulated in Table 6.1. The index (underlying) had closed at 126.41.

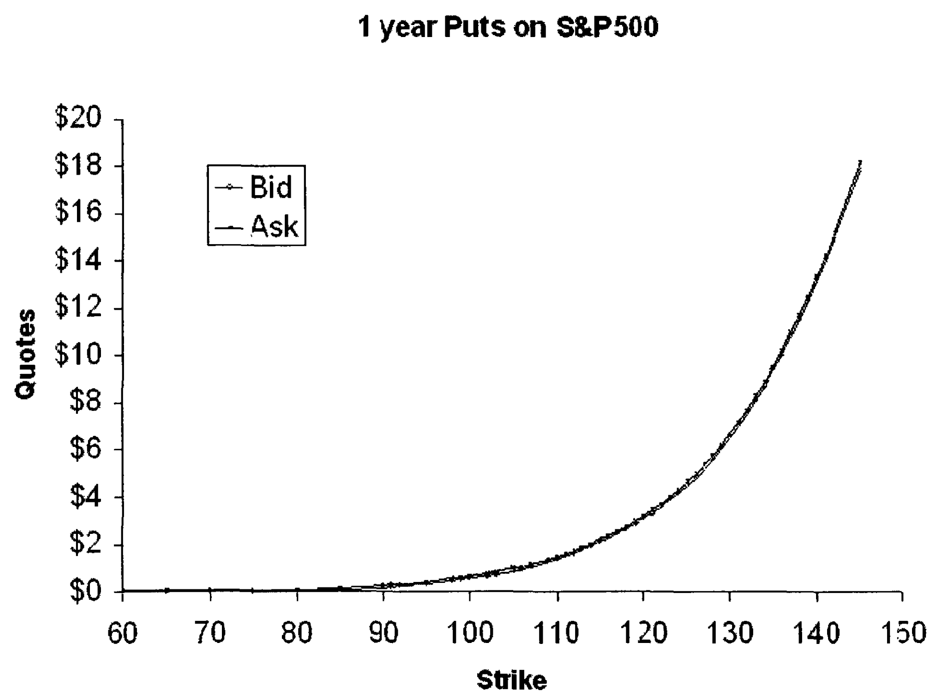

Figure 6-2: Price v/s Strike for Dec. 15, '06 Put Options on S&P500 (as on Feb. 14, '06)

Table 6.1: True no-arbitrage Bounds on S&P 500 Index Put Options

Strike	Volume	Bid	Ask	Lower Bound	Upper Bound
60	0	0.00	0.05	0.00	0.04
65	0	0.00	0.05	0.00	0.04
70	0	0.00	0.05	0.00	0.05
75	0	0.00	0.05	0.00	0.08
80	1000	0.00	0.10	0.00	0.10
85	0	0.05	0.15	0.00	0.18
90	6	0.15	0.25	0.15	0.26
91	10	0.20	0.30	0.20	0.28
92	40	0.25	0.30	0.21	0.31
95	50	0.30	0.40	0.29	0.43
98	300	0.45	0.55	0.43	0.55
99	10	0.50	0.60	0.48	0.60
100	10	0.55	0.65	0.53	0.65
102	2	0.65	0.75	0.61	0.75
103	10	0.70	0.80	0.69	0.83
105	3	0.85	1.00	0.83	0.97
106	8	0.95	1.05	0.92	1.06
107	10	1.05	1.15	1.00	1.15
109	100	1.25	1.35	1.19	1.35
110	2135	1.35	1.45	1.33	1.47
111	1223	1.50	1.60	1.43	1.58
112	10	1.60	1.70	1.59	1.72
113	120	1.75	1.85	1.69	1.85
114	500	1.90	2.00	1.89	2.02
115	8	2.10	2.20	2.02	2.18
116	1	2.25	2.35	2.23	2.37

Continued on next page

Table 6.1 – continued from previous page

Strike	Volume	Bid	Ask	Lower Bound	Upper Bound
117	120	2.45	2.55	2.39	2.55
118	4	2.65	2.75	2.60	2.75
119	5	2.85	2.95	2.81	2.98
120	1	3.10	3.20	3.02	3.20
121	1	3.30	3.50	3.29	3.45
122	13	3.60	3.70	3.53	3.73
123	1	3.90	4.00	3.82	4.00
124	11	4.20	4.30	4.14	4.33
125	132	4.50	4.70	4.45	4.65
126	3	4.80	5.00	4.77	5.03
127	1	5.20	5.40	5.08	5.40
128	1	5.60	5.80	5.57	5.80
129	2	6.10	6.20	5.93	6.25
130	167	6.50	6.70	6.47	6.70
131	68	7.00	7.20	6.88	7.20
132	484	7.50	7.70	7.40	7.75
133	1	8.10	8.30	7.95	8.30
134	20	8.70	8.90	8.63	8.90
135	62	9.40	9.50	9.20	9.55
136	2500	10.00	10.20	9.97	10.23
137	1	10.80	11.00	10.58	10.95
138	1	11.50	11.70	11.45	11.73
139	1522	12.30	12.50	12.20	12.55
140	390	13.20	13.40	13.10	13.40
141	1969	14.10	14.30	14.00	14.30
142	250	15.00	15.20	14.90	15.28
145	100	17.90	18.20	17.50	18.20

We found that while there were no arbitrage opportunities in the market, the bounds we derived are close to the bid-ask quotes in many cases. This is especially true of the upper bounds which are in some cases even lower than the quoted Ask price, indicating that the markets for these options are unusually wide and illiquid and that for options, the demand forces are in general stronger than the supply forces.

6.4 Impact of Information

Next, we examine the efficacy of our methods for market-makers. There are usually a few options that are traded more frequently than others and whose prices may be considered reliable. Market-makers use the prices of these options to arrive at the prices of other options. To mimic this procedure we take the data in Table 6.1, but instead of using the bid and ask prices directly, we compute the nominal price of an option as a simple average of the quoted bid and ask prices. This is because for highly liquid options, the bid-ask gap is usually small and the mid-point a good indicator of the fair-price of the option. To investigate the strengths of the bounds and their dependency on data and possible noise effects, we group the options in three classes based on their liquidity or traded volumes (Table 6.1 also gives the traded volume in number of contracts traded for different options.):

- Class A contains options whose volume exceeded 100 contracts,
- Class B contains those whose volume is between 10 and 100 contracts, and
- Class C contains the remaining options with volume less than 10 contracts.

We than take a few strikes in Class C, and derive the bounds on their prices in the following 3 settings,

1. All other option prices are considered to be reliable,
2. Only class A and class B options are considered reliable
3. Only class A options are reliable signals.

Option Data				All Options		Class A,B Options		Class A Options	
Strike	Volume	Bid	Ask	Lower	Upper	Lower	Upper	Lower	Upper
75	0	0.00	0.05	0.03	0.04	0.00	0.05	0.00	0.05
90	6	0.15	0.25	0.23	0.23	0.23	0.23	0.06	0.30
105	3	0.85	1.00	0.90	0.92	0.90	0.93	0.90	1.01
115	8	2.10	2.20	2.10	2.13	2.10	2.13	2.10	2.13
118	4	2.65	2.75	2.70	2.70	2.68	2.73	2.68	2.76
120	1	3.10	3.20	3.15	3.15	3.05	3.19	3.05	3.29
123	1	3.90	4.00	3.90	3.95	3.90	3.95	3.80	4.08
127	1	5.20	5.40	5.25	5.30	5.30	5.40	5.13	5.40
129	2	6.10	6.20	6.10	6.15	6.10	6.20	6.10	6.20
137	1	10.80	11.00	10.80	10.85	10.75	10.87	10.73	10.87

Table 6.2: Bounds with different levels of information for S&P 500 Dec-06 Options

In all the three settings, we use the nominal prices and not the bid-ask quotes directly as before. The computed 'nominal' bounds are shown in the Table 6.4. We see that in most cases, information conveyed through Class C option prices doesn't refine the bounds much; although ignoring Class B option prices can result in a significant weakening of the bounds. The results seem to indicate that in general the derived bounds are quite robust to sub-sampling and one can ignore the noisy information in the quotes of illiquid options without a significant loss in efficacy.

6.5 Comparison for different Maturities

In Table 6.3, we compute the true no-arbitrage bounds (using the bid-ask quotes directly) and nominal bounds (taking the mid-point of the bid-ask quotes as the nominal 'price') for put options on the GE stock maturing in roughly 1 - month, using the same methods as described for the of S&P 500 Options. These options are characterized by low volatilities because of their imminent expiry. We note that the bounds given are very narrow for most strikes. We also note an inversion for some strikes in the bounds computed using 'nominal' prices i.e., the upper bound implied is in fact less than the lower bound. If the nominal prices can be taken as fair-values then this implies a mis-pricing in the options. This mis-pricing however has to be large enough to allow arbitrage opportunities given the wide bid-ask spreads. Also, the fair value is likely to be different from 'nominal' prices for options that are not liquid. This inference is more plausible given the fact that the true no-arbitrage bounds do not show any 'inversion'.

Option Data				True no-arbitrage bounds		'Nominal' bounds	
Strike	Volume	Bid	Ask	Lower	Upper	Lower	Upper
22.5	0	0.00	0.05	0.00	0.04	0.03	0.02
25.0	0	0.00	0.05	0.00	0.05	0.03	0.03
27.5	102	0.00	0.05	0.00	0.08	0.03	0.05
30.0	4	0.05	0.10	0.00	0.20	0.03	0.18
32.5	7339	0.30	0.35	0.05	0.98	0.13	0.94
35.0	642	1.75	1.85	1.70	2.38	1.85	2.31
37.5	231	4.20	4.40	4.20	4.33	4.25	4.28
40.0	10	6.70	6.80	6.70	6.82	6.80	6.78
42.5	146	9.20	9.30	9.20	9.30	9.25	9.25
45.0	51	11.60	11.90	11.70	11.80	11.75	11.75
47.5	901	14.20	14.30	14.10	14.30	14.25	14.25

Table 6.3: True no-arbitrage and nominal price based bounds for Mar-06 Options on General Electric(GE). Stock closed at \$33.25.

In Table 6.4, we list the true no-arbitrage bounds and nominal price based bounds for put options on the Intel stock maturing in roughly about 2 years. We would expect these options to have relatively high volatility effects. We note that the bounds are somewhat wide for low strikes but strengthen as the strike increases. The bounds are however not as strong as in the previous cases.

Summary

To summarize our observations from the numerical experiments,

- We did not observe any actionable arbitrage opportunities in the markets, i.e., all put options were consistently priced.

- The upper and lower bounds that we obtain are in general, of the same order as the quoted bid-ask spreads. For some strikes, the quoted bid-ask spreads are in fact wider than the computed no-arbitrage bounds, indicating that the market is not deep in those options.

Option Data				True no-arbitrage bounds		'Nominal' bounds	
Strike	Volume	Bid	Ask	Lower	Upper	Lower	Upper
15.0	2	0.60	0.70	0.20	1.03	0.35	0.99
17.5	26	1.10	1.20	0.70	1.35	0.80	1.30
20.0	25	1.90	2.00	1.50	2.20	1.65	2.13
22.5	46	3.00	3.20	2.60	3.35	2.75	3.28
25.0	360	4.50	4.70	4.00	4.85	4.25	4.78
27.5	15	6.40	6.50	6.00	6.75	6.20	6.63
30.0	47	8.50	8.80	8.50	8.85	8.60	8.78
32.5	25	11.00	11.20	11.00	11.25	11.10	11.13
35.0	12	13.50	13.70	13.50	13.70	13.55	13.60
45.0	100	23.50	23.70	22.90	23.70	23.60	23.60

Table 6.4: Absolute no-arbitrage and nominal price based bounds for Jan-08 Options on Intel Corp(INTC). Stock closed at $21.37.

- The bounds obtained are fairly robust, in the sense, that they do not change significantly if we ignore some of the more unreliable price signals, i.e., prices of highly illiquid options.
- The bounds obtained for options with shorter maturities are sharper compared to those obtained for options with longer maturities.

Chapter 7

Conclusions and Final Remarks

We derived in this thesis, the necessary and sufficient conditions that prices of American Put Options on a non-dividend paying stock must satisfy to be consistent i.e., allow no arbitrage. We discover that these conditions are surprisingly few and simple. Using these conditions, we then derive the bounds on the price of an American Put option of a specified strike, given the price of other American Put Options with the same maturity. The problem of finding bounds can be cast as a simple linear programming problem that can be solved directly using the algorithm presented in Section 5.2. We also applied our results to data obtained from real markets. Our methods could be readily modified, with only a little increase in complexity, to accommodate the form in which price information is available in real markets, i.e., a bid-ask interval. We discover that in most cases, especially for options with relatively moderate volatilities, the bounds are quite strong compared to the quoted bid-ask spreads.

We have, however, accounted only for the effect of one particular type of derivative security on the American Put Option Price, that is the prices of other American Put Options having the same maturity. It will be interesting to see how the bounds obtained can be tightened if it were possible to constrain the prices using additional information from other derivative securities linked to the underlying.

Appendix A

Existence of an Optimal Exercise Policy

Proposition A.1. *Given an American Put Option with strike K, maturity T and a risk neutral measure $\mathbb{Q}$ for the underlying stock price process, an optimal exercise policy τ^* satisfying the following properties exists*

1. *τ^* is non-randomized i.e., given a path ω of the stock prices, at any time t, $\tau^*(\omega, t) = 1$ or $\tau^*(\omega, t) = 0$, where $\tau^*(\omega, t)$ is the optimal probability of exercise at time t for the path ω.*

2. *$\tau^*(\omega, t) = 1 \Rightarrow t = T$ or $K > S_t$.*

Proof. Let $\mathcal{F}$ denote the space of exercise policies and $\mathcal{F}^d$ that of non-randomized exercise policies. ω denotes a particular evolution of stock prices i.e., the vector $(S_0, S_1, \ldots, S_T)$. For $t \geq s$, let $\omega^{s:t}$ denote the vector $(S_s, S_{s+1}, \ldots, S_t)$. Let $\tau(\omega, t)$ denote the probability that the option is exercised at time t, for sample path ω, under the policy τ. $\tau \in \mathcal{F} = \Re^{T+1} \rightarrow [0,1]^{T+1}$ is a valid exercise policy (i.e., a stopping time) iff

1. $\mathbf{1}'\tau(\omega, t) = 1$;

2. $\omega_1^{0:t} = \omega_2^{0:t} \Rightarrow \tau^k(\omega_1, t) = \tau^k(\omega_2, t)$.

Further $\tau \in \mathcal{F}^d$ if $\tau(\omega) \in \{0,1\} \ \ \forall\omega, t$. Also,

$$\begin{aligned} A_T(K,\tau) &= \sum_{t=0}^{T} \mathbb{E}^{\mathbb{Q}}[\beta^t (K - S_t)^+ \tau(\omega)]; \\ A_T(K) &= \sup_{\tau\in\mathcal{F}} \sum_{t=0}^{T} \mathbb{E}^{\mathbb{Q}}[\beta^t (K - S_t)^+ \tau(\omega)]. \end{aligned}$$

We first prove the existence of a non-randomized optimal exercise policy. Property 2 would then follow from a straightforward argument. Suppose the support set of the process S_t is finite. Then, clearly $\mathcal{F}^d$, the the set of non-randomized exercise policies is finite and a randomized exercise policy is just a convex combination of non-randomized exercise policies. It then follows that an optimal exercise policy belonging to $\mathcal{F}^d$ exists by simple Linear Programming Theory. Suppose the state space for S_t is not finite. In this case, we can use induction on T and the dynamic programming principle to prove the existence of a non-randomized τ^*. For $T = 1$, this follows almost immediately, as payoffs from all exercise policies can be characterized in terms of a parameter $p : 0 \le p \le 1$ as

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[\beta^\tau (K - S_\tau)^+] &= p(K - S_0)^+ + (1-p)\mathbb{E}^{\mathbb{Q}}[\beta(K - S_1)^+] \\ &\le \max((K - S_0)^+, \mathbb{E}^{\mathbb{Q}}[\beta(K - S_1)^+]). \end{aligned}$$

Thus one of $\tau_0 : \tau_0(\omega, 0) = 1, \tau_0(\omega, 1) = 0 \ \forall\omega$ or $\tau_1 : \tau_1(\omega, 0) = 0, \tau_1(\omega, 1) = 1 \ \forall\omega$ is an optimal policy and hence the hypothesis holds for $T = 1$. Assume it is true for $T = m$ for some m and consider the case when $T = m + 1$. At $t = 1$, then by the induction hypothesis, for all possible values of S_1 a henceforth non-randomized optimal exercise policy, $\tau^*_{m-1}(S_1, \omega^{1:m+1}, t)$ exists. Let $V_{m-1}(s_1)$ denote the value of the corresponding American Put Option that can be exercised anywhere in periods $1, 2, \cdots, m$, for all paths $\omega : S_1 = s_1$. Then set

$$\begin{aligned} \tau^*(\omega, 0) &= 1, \\ \tau^*(\omega, t) &= 0\ , \ \ \forall t \ge 1; \end{aligned}$$

$$
\begin{aligned}
& \text{if } K - S_0 \geq \beta \mathbb{E}^{\mathbb{Q}}[V_{m-1}(S_1)] \text{ and} \\
\tau^*(\omega, 0) &= 0, \\
\tau^*(\omega, t) &= \tau^*_{m-1}(S_1, \omega^{1:m+1}, t-1)\ , \quad \forall t \geq 1; \\
& \text{otherwise.}
\end{aligned}
$$

From the principle of optimality, it follows that τ^* so defined must be optimal. Further, it also non-randomized by construction and the induction hypothesis.

Finally, if τ^* violates property 2 of the proposition, then consider a modification $\bar{\tau^*}$, s.t.,

$$
\begin{aligned}
\bar{\tau^*}(\omega, t) &= \begin{cases} \tau^*(\omega, t) & \text{if } t \leq T-1 \text{ and } \tau^*(\omega, t) = 0 \text{ or } S_t < K; \\ 0 & \text{if } t \leq T-1 \text{ and } \tau^*(\omega, t) = 1 \text{ and } S_t \geq K. \end{cases} \\
\bar{\tau^*}(\omega, T) &= 1 - \sum_{t=1}^{T-1} \bar{\tau^*}(\omega, t).
\end{aligned}
$$

It can be immediately verified that $A_T(\omega, \bar{\tau^*}) \geq A_T(\omega, \tau^*) = A_T(K)$. Thus, we always have a non-randomized optimal policy satisfying property 2 as well.

As τ^* is non-randomized, and the option (by our notational convention) is always struck, we can alternatively describe $\tau^*(\omega)$ unambiguously by the time when the option is exercised on path ω i.e., $\tau^*(\omega) \in \{0, 1, 2, \ldots, T\}$. This is the notation that we use in the thesis. □

Appendix B

Bounds from Bid-Ask Quotes

Suppose, we have been given a set of $N+1$ strikes $K_0, K_1, K_2, \ldots, K_N$ and the corresponding bid $(B_0, B_1, \ldots, B_N)$ and ask quotes $(A_0, A_1, \ldots, A_N)$ for all but one of them, say strike K_J, as well as the initial stock price S_0. We also assume wlog, that $0 = K_0 < K_1 < \ldots < K_N$. Then the linear program given below enforces the convexity and other no arbitrage constraints to give the upper and lower no-arbitrage bounds on the price of the option with strike K_J.

$$\max_{P_0,P_1,\ldots,P_N} / \min_{P_0,P_1,\ldots,P_N} \quad P_J$$

s.t.

$$\begin{aligned}
&P_0 \;=\; 0 \\
&B_i \;\leq\; P_i \;\leq\; A_i && , \; i = 0,\ldots,J-1,J+1,\ldots,N; \\
&P_i \;\geq\; (K_i - S_0)^+ && , \; i = 0,\ldots,N; \\
&\tfrac{1}{K_{i+1}-K_i}(P_{i+1} - P_i) \;\geq\; \tfrac{1}{K_i - K_{i-1}}(P_i - P_{i-1}) && , \; i = 1,\ldots,N-1; \\
&0 \;\leq\; \tfrac{1}{K_i - K_{i-1}}(P_i - P_{i-1}) \;\leq\; 1 && , \; i = 1,\ldots,N.
\end{aligned}$$

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