Permutations Statistics of Indexed and Poset Permutations

by

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Abstract

The definitions of *descents* and *excedances* in the elements of the symmetric group $\mathcal{S}_d$ are generalized in two different directions.

First, descents and excedances are defined for *indexed permutations*, i.e. the elements of the group $\mathcal{S}_d^n = \mathbf{Z}_n \wr \mathcal{S}_d$, where $\wr$ is wreath product with respect to the usual action of $\mathcal{S}_d$ by permutation of $[d]$. It is shown, bijectively, that excedances and descents are *equidistributed*, and the corresponding *descent polynomials*, analogous to the *Eulerian polynomials*, are computed as the *f-eulerian polynomials* of simple polynomials. The descent polynomial is shown to equal the *h-polynomial* (essentially the *h-vector*) of a certain triangulation of the unit d-cube. This is proved by a bijection which exploits the fact that the h-vector of the triangulation in question can be computed via a *shelling* of the simplicial complex arising from the triangulation. The h-vector, in turn, is computed via the *Ehrhart polynomials* of dilations of the unit d-cube. The famous formula $\sum_{d\geq 0} E_d \frac{x^d}{d!} = \sec x + \tan x$, where E_d is the number of *alternating* permutations in $\mathcal{S}_d$, is generalized in two different ways, one relating to recent work of V.I. Arnold on Morse theory. The resulting formulas are then used to find, in two special cases, a relation between the number of alternating indexed permutations and the value of the corresponding descent polynomial at -1. The definitions of *major index* and *inversion index* are also generalized and their equidistribution is proved.

Secondly, descents and excedances are generalized to all finite posets, the classical case corresponding to the poset $\{1, 2, \ldots, d\}$ of natural numbers in their usual ordering. Again, descents and excedances are equidistributed, which is proved bijectively. This bijection, which is not a generalization of the one described by Foata and Schützenberger in the classical case, has the virtue of translating descents "verbatim" into excedances. Using this, bijective proofs are given of two results concerning the chromatic polynomial of the *incomparability graph* of a poset P.

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Contents

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1 Introduction

There is a wealth of literature on various statistics of the elements of the symmetric group $\mathcal{S}_d$ (see for example [10] and [11] for a bibliography) and some of this has recently been generalized to the hyperoctahedral group B_d (see [16]). In this thesis we generalize the definitions of *descents* and *excedances* in two different directions.

In the classical case of the symmetric group $\mathcal{S}_d$, whose elements we view as permutations of the set $[d] = \{1, 2, \ldots, d\}$, represented as words, a descent in $\pi = a_1 a_2 \ldots a_d \in \mathcal{S}_d$ is an i in $[d]$ such that $a_i > a_{i+1}$, i.e. where a letter in the word π is larger than its successor. The *descent set* $D(\pi)$ of π is the set of those $i \in [d]$ for which $a_i > a_{i+1}$. An excedance in π is an i in $[d]$ such that $a_i > i$. For example, the permutation 34521 has descents at 3 and 4 and excedances at 1, 2 and 3. We construct the *descent polynomial* $D_d(t)$ of $\mathcal{S}_d$ by defining its k-th coefficient to be the number of permutations in $\mathcal{S}_d$ with k descents and the *excedance polynomial* $E_d(t)$ of $\mathcal{S}_d$ in an analogous way. It is well known that $D_d(t) = E_d(t)$, i.e. descents and excedances are *equidistributed* over $\mathcal{S}_d$. Moreover, $D_d(t)$ equals, up to a factor of t, the d-th *Eulerian polynomial* $A_d(t)$. The Eulerian polynomials have been extensively studied in various different contexts.

Other statistics which have been much studied are the *major index* and the *inversion index* of a permutation. The major index $\mathrm{maj}(\pi)$ of $\pi = a_1 a_2 \ldots a_d$ is the sum of all i in the descent set of π. An inversion in π is a pair (i, j) such that $i < j$ and $a_i > a_j$. The inversion index of a permutation π is the number of inversions in π and is denoted $\mathrm{inv}(\pi)$. It is known that inv and maj are equidistributed, i.e. $\sum_{\pi \in \mathcal{S}_d} t^{inv(\pi)} = \sum_{\pi \in \mathcal{S}_d} t^{maj(\pi)}$.

Here, we first generalize the definitions of descents and excedances to the group $S_d^n = \mathbf{Z}_n \wr \mathcal{S}_d$ (where $\wr$ is wreath product with respect to the usual action of $\mathcal{S}_d$ by permutations of $[d]$). We show, bijectively, that excedances and descents are still equidistributed, and we compute the corresponding descent polynomials $D_d^n(t)$ as the *f-eulerian polynomial* of a simple polynomial. We also show that the descent polynomial equals the *h-polynomial* (essentially the *h-vector*) of a certain triangulation of the unit d-cube. This we prove by a bijection which exploits the fact that the h-vector of the triangulation in question can be computed via a *shelling* of the simplicial complex arising from the triangulation. The h-vector, in turn, we compute via the *Ehrhart polynomials* of dilations of the unit d-cube, using a theorem of Stanley and of Betke and McMullen.

Using the work of Brenti [7], we show that the descent polynomials $D_d^n(t)$ have only real roots, which implies that they are *unimodal.*

We also generalize the famous formula $\sum_{d \geq 0} E_d \frac{x^d}{d!} = \sec x + \tan x$, where E_d is the number of *alternating* permutations in $\mathcal{S}_d$, in two different ways, one of which relates to recent work of Arnold [2] on Morse theory. In each case, the resulting formula is then used to find a relation between the number of alternating (respectivley *weakly alternating*) indexed permutations and the value of the corresponding descent polynomial at -1.

We define the *length* of an indexed permutation, in analogy with the corresponding definition for *Coxeter groups*, and we also generalize the definitions of *major index* and the *number of inversions* and show that these are equidistributed as in the classical case.

Secondly, we generalize descents and excedances to all finite posets, the classical case corresponding to the poset defined by the chain $[d]$, i.e. the set $\{1, 2, \ldots, d\}$ of natural numbers in their usual ordering. Again, descents and excedances are equidistributed, which we prove bijectively. Our bijection is not a generalization of the one described by Foata and Schützenberger in the classical case, but it has the virtue of translating descents "verbatim" into excedances. Using this, we give bijective proofs of two results concerning the chromatic polynomial of the *incomparability graph* of a poset P. One of these results was independently obtained earlier by Buhler, Eisenbud, Graham and Wright [8].

2 Preliminaries

We review here some notation which will be adhered to throughout.

We denote by $[n]$ the set $\{1, 2, \ldots n\}$ which, when relevant, is assumed endowed with its usual linear order.

The quotient $\mathbf{Z}/n\mathbf{Z}$ where $\mathbf{Z}$ is the infinite cyclic group of integers and $n \in \mathbf{Z}$ will be denoted $\mathbf{Z}_n$. We always represent the elements of $\mathbf{Z}_n$ by the elements of $[n]$, and when we refer to an ordering of the elements of $\mathbf{Z}_n$ it is the ordering induced by $[n]$.

By id we mean the identity element in a group.

We will denote by $\mathcal{S}_d$ the symmetric group of permutations of $[d]$. An element $\pi \in \mathcal{S}_d$ will usually be represented as a word $\pi = a_1 a_2 \ldots a_d$, where $a_i = \pi(i)$.

We use the boldface letters $\mathbf{z}, \mathbf{w}, \mathbf{x}$ to denote vectors, for example $\mathbf{z} = (z_1, z_2, \ldots, z_d)$. In particular, $\mathbf{0} = (0, 0, \ldots, 0)$.

If x is a real number then $\lfloor x \rfloor$ refers to the largest integer smaller than or equal to x.

We shall be concerned with the elements of the *wreath product* $\mathbf{Z}_n \wr \mathcal{S}_d$. Our definition, which is taken almost verbatim from [13], is not the most general one, but one which is better suited to our purpose (see [14] for the more general definitions).

Let G be any group and let H be a subgroup of the symmetric group $\mathcal{S}_d$. Let $G^d := \{f | f : [d] \to G\}$ be the set of all maps of $[d]$ to G. If $f \in G^d$ and $\pi \in H$, define $f_\pi \in G^d$ by $f_\pi := f \circ \pi^{-1}$. Then, if also $\pi' \in H$, we have $(f_\pi)_{\pi'} = f_{\pi'\pi}$. Define a product on G^d in the obvious way, i.e. if $f, f' \in G^d$ and $i \in [d]$ then $(ff')(i) := f(i) \cdot f'(i)$, where $\cdot$ is the product in G.

The *wreath product of G by H*, denoted $G \wr H$, is the group consisting of the set

$$H \times G^d = \{(\pi; f) | \pi \in H, f \in G^d\}$$

endowed with the product

$$(\pi; f) \cdot (\pi'; f') := (\pi\pi'; ff'_\pi).$$

The identity in $G \wr H$ is $e_{G \wr H} = (e_H; e_{G^d})$, where e_{G^d} is the identity in G^d, i.e. the function satisfying $e_{G^d}(i) = e_G$ for all $i \in [d]$. The inverse of an element is given by

$$(\pi; f)^{-1} = (\pi^{-1}; f^{-1}_{\pi^{-1}})$$

where $f^{-1}(i) := f(i)^{-1}$.

A particular subgroup which we will be concerned with is $H' := \{(\pi; f) | f = e_{G^d}\}$. Clearly H' is isomorphic to H.

Note that we can represent $f : [d] \to G$ in a canonical way as a d-tuple of elements of G, i.e. $f = (f(1), f(2), \ldots, f(d))$.

3 Indexed Permutations

3.1 Definitions and some basic results

Definition 3.1 *An* indexed permutation *is an element of the group* $S_d^n := \mathbf{Z}_n \wr S_d$ *(where* $\wr$ *is wreath product with respect to the usual action of* S_d *by permutation of* $[d]$*). We represent an indexed permutation as the product* $\pi \times \mathbf{z}$ *of a permutation word* $\pi = a_1 a_2 \ldots a_d \in S_d$ *and a d-tuple* $\mathbf{z} = (z_1, z_2, \ldots, z_d)$ *of integers* $z_i \in \mathbf{Z}_n$*. As a convention, we set* $a_{d+1} = d+1$ *and* $z_{d+1} = 0$*.*

Definition 3.2 *A* descent *in* $p = \pi \times z \in S_d^n$ *is an integer* $i \in [d]$ *such that*

1) $z_i > z_{i+1}$ *OR*

2) $z_i = z_{i+1}$ *and* $a_i > a_{i+1}$.

In particular, d is a descent *iff* $z_d > 0$.

Definition 3.3 *An* excedance *in* p *is an integer* $i \in [d]$ *such that*

1) $a_i > i$ *OR*

2) $a_i = i$ *and* $z_i > 0$.

As an example, let $p = 321465 \times (0, 0, 3, 2, 2, 1)$. Then p has descents at 1, 3, 5 and 6 and excedances at 1, 4 and 5.

It is convenient to think of an element of S_d^n as a permutation word in which every letter has a subscript. For example, $p = 321465 \times (1, 0, 3, 2, 2, 1) \in S_5^4$ can be represented by $3_1 2_0 1_3 4_2 6_2 5_1$. We call the subscripts *indices.* Using this, there is an alternative definition of descent. Namely, define an ordering $<_\ell$ on the alphabet $\{i_z | i \in [d], z \in \mathbf{Z}_n$ by setting $i_z <_\ell j_w$ if

i) $z < w$ OR

ii) $z = w$ and $i < j$.

Then a descent in $p = a_{1_{z_1}} a_{2_{z_2}} \ldots a_{d_{z_d}}$ [1] is an i such that $a_{i+1_{z_{i+1}}} <_\ell a_{i_{z_i}}$. This ordering of the letters induces a lexicographic ordering of the indexed permutations in S_d^n.

[1]To make the notation a little less awkward, we write $a_{i_{z_i}}$ instead of $(a_i)_{z_i}$, although z_i is a subscript to a_i rather than to just the i in a_i.

Definition 3.4 *Define an ordering* $<_L$ *of the elements of* S_d^n *by setting* $p = a_{1_{z_1}} a_{2_{z_2}} \ldots a_{d_{z_d}} <_L q = b_{1_{w_1}} b_{2_{w_2}} \ldots b_{d_{w_d}}$ *if* $a_{i_{z_i}} <_\ell b_{i_{w_i}}$ *for the first* i *at which* p *and* q *differ.*

Definition 3.5 *Let* p *be an element of* S_d^n. *Let* $e(p) = \#\{i | i \text{ is an excedance in } p\}$ *and let* $d(p) = \#\{i | i \text{ is a descent in } p\}$. *Then* $E_d^n(t) = \sum_{p \in S_d^n} t^{e(p)}$ *is the* excedance polynomial *of* S_d^n *and* $D_d^n(t) = \sum_{p \in S_d^n} t^{d(p)}$ *is the* descent polynomial *of* S_d^n. *Moreover, let* $E(d,n,k) = \#\{p \in S_d^n | p \text{ has } k \text{ excedances}\}$ *and let* $D(d,n,k) = \#\{p \in S_d^n | p \text{ has } k \text{ descents}\}$, *so that* $E_d^n(t) = \sum_{k=0}^d E(d,n,k)t^k$ *and* $D_d^n(t) = \sum_{k=0}^d D(d,n,k)t^k$.

As a convention, if $n \geq 0$, *we define* S_0^n *to consist of one (empty) indexed permutation and hence we have* $E(0,n,0) = D(0,n,0) = 1$.

Note that when $n = 1, S_d^n$ is essentially $\mathcal{S}_d$ and the definitions of descent and excedance coincide with the classical definitions (see, for example, [21]).

Definition 3.6 *Let* $p \in S_d^n$ *and let* $D(p) = \{i \in [d] | i \text{ is a descent in } p\}$. *Then* $D(p)$ *is the* descent set *of* p.

We will now construct a bijection $S_d^n \to S_d^n$ which takes an indexed permutation with k descents to one with k excedances. First a definition which we will frequently refer to in what follows.

Definition 3.7 *Let* $S_{\mathbf{z}}$ *be the set of permutation words on the letters* $1_{z_1}, 2_{z_2}, \ldots, d_{z_d}$. *That is,* $S_{\mathbf{z}} = \{p = \pi \times \pi(z) \mid \pi \in \mathcal{S}_d\} = \{\pi(1_{z_1} 2_{z_2} \ldots d_{z_d}) \mid \pi \in \mathcal{S}_d\}$.

Note that $S_{\mathbf{0}}$ is the subgroup $\{\pi \times \mathbf{0} | \pi \in \mathcal{S}_d\} \subset S_d^n$ and $S_{\mathbf{z}}$ is the *left coset* $(\pi \times \mathbf{z}) S_{\mathbf{0}}$ for any $\pi \in \mathcal{S}_d$.

Let $\mathbf{Z}_n^d$ be the direct product of d copies of $\mathbf{Z}_n$. Clearly, S_d^n is the disjoint union of the $S_{\mathbf{z}}$'s for all $\mathbf{z} \in \mathbf{Z}_n^d$. The bijection we are about to construct will actually map $S_{\mathbf{z}}$ to itself for each $\mathbf{z} \in \mathbf{Z}_n^d$. However, we need to do this in three steps.

Lemma 3.8 *Suppose* $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbf{Z}_n^d$ *and* $\mathbf{w} = (w_1, \ldots, w_d) \in \mathbf{Z}_n^d$ *have the same number of positive coordinates. Then there is a bijection* $\Theta : S_{\mathbf{z}} \to S_{\mathbf{w}}$ *which preserves the number of descents in each* $p \in S_{\mathbf{z}}$. *In fact,* Θ *preserves the descent set of* p.

Proof: The ordering $<_\ell$ used in Definition 3.4 is a linear ordering of the letters $1_{z_1}, 2_{z_2}, \ldots, d_{z_d}$, respectively of the letters $1_{w_1}, 2_{w_2}, \ldots, d_{w_d}$. Hence there is a unique bijection $\theta : \{i_{w_i} | i \in [d]\} \to \{i_{z_i} | i \in [d]\}$ such that $\theta(i_{z_i}) <_\ell \theta(j_{z_j})$ if and only if $i_{z_i} <_\ell j_{z_j}$. In particular, since $\mathbf{z} = (z_1, \ldots, z_d)$ and $\mathbf{w} = (w_1, \ldots, w_d)$ have the same number of positive coordinates, $z_i > 0$ if and only if $w_j > 0$ where $j_{w_j} = \theta(i_{z_i})$. Now, given $p \in S_{\mathbf{z}}$, define $\Theta : S_{\mathbf{z}} \rightarrow S_{\mathbf{w}}$ by $\Theta(p) = \Theta(a_{1z_1}a_{2z_2}\ldots a_{dz_d}) := \theta(a_{1z_1})\theta(a_{2z_2})\ldots\theta(a_{dz_d})$. Then, by definition of θ, i is a descent in p if and only if i is a descent in $\Theta(p)$. In particular, since $\mathbf{z}$ and $\mathbf{w}$ have the same number of positive coordinates, d is a descent in p if and only if d is a descent in $\Theta(p)$. Hence, Θ preserves not only the number of descents in p but actually the descent set $D(p)$ of p. $\square$

Example 3.9 *Let* $\mathbf{z} = (1, 0, 2, 1)$. *Then* $<_\ell$ *induces the following ordering of the letters* $1_1, 2_0, 3_2, 4_1$*:* $2_0 <_\ell 1_1 <_\ell 4_1 <_\ell 3_2$. *Hence, if, as an example, we let* $p = 3_2 2_0 4_1 1_1$ *and* $\mathbf{w} = (0, 1, 1, 1)$ *, we have* $\Theta(p) = 4_1 1_0 3_1 2_1$.

Lemma 3.10 *Let* $\mathbf{w} = (w_1, \ldots, w_d) \in \mathbf{Z}_n^d$. *Suppose there is a* $k \in [d]$ *such that* $w_i = 0$ *for all* i *in* $\{1, \ldots, k-1\}$ *and* $w_i = 1$ *for all* i *in* $\{k, \ldots, d\}$. *Then there is a bijection* $\Psi : S_{\mathbf{w}} \rightarrow S_{\mathbf{w}}$ *such that* $e(\Psi(p)) = d(p)$.

Proof: Given $p = a_{1w_{a_1}}a_{2w_{a_2}}\ldots a_{dw_{a_d}} \in S_{\mathbf{w}}$, map p to $\pi = a'_1a'_2\ldots a'_{d+1} \in \mathcal{S}_{d+1}$ where $a'_{d+1} = k$ and $a'_i = a_i$ if $a_i < k$, $a'_i = a_i + 1$ if $a_i \geq k$. Then i is a descent in π if and only if i is a descent in p. Now apply the bijection ϕ in Remark 4.7 to π to obtain $\tau = \phi(\pi)$, where $\tau = b_1b_2\ldots b_{d+1}$ has an excedance $b_i > i$ if and only if $\ldots b_i i \ldots$ appears as a descent in π. Let m be such that $b_m = k$, and observe that, by the definition of ϕ, $m \geq k$, so that m is not an excedance in τ. Let $i' = i$ if $i < m$ and $i' = i + 1$ if $i \geq m$. Now map τ to $q = c_{1w_{c_1}}c_{2w_{c_2}}\ldots c_{dw_{c_d}} \in S_{\mathbf{w}}$ by setting $c_i = b_{i'}$ and $w_{c_i} = 0$ if $b_{i'} < k$, $w_{c_i} = 1$ if $b_{i'} > k$. Thus, k is deleted from τ and each remaining letter of τ is mapped back to what it was in p, that is, b_i in τ is replaced by $(b_i - 1)_1$ if $b_i > k$, but otherwise b_i is left alone. Also, some of the "place numbers" (i.e. the indices) have to be reduced, so that a letter which was in place i with $i > m$ is in place $i - 1$ in q. We claim that i is an excedance in τ if and only if i' is an excedance in q, so that τ and q have the same number of excedances, since m was not an excedance in τ. If $i < m$ then in q we either have $(b_i)_0$ or $(b_i - 1)_1$ in place i. In either case, i is an excedance in q if and only if i is an excedance in τ. If $i > m$ then in place $i - 1$ in q we again have either $(b_i)_0$ or $(b_i - 1)_1$. If $b_i \neq i$ then $i - 1$ is an excedance in q if and only if i is an excedance in τ. Suppose, then, that $b_i = i$. Then, by Corollary 4.9, $b_i < k$, since k is the last letter in π. Hence, we must have $i < k \leq m$, contrary to assumption, so $b_i \neq i$ and we are done. $\square$

Example 3.11 *Let* $p = 4_1 1_0 3_1 2_1$. *Then* $p \longmapsto 51432 \overset{\phi}{\longmapsto} 53421 \longmapsto 4_1 2_1 3_1 1_0$, *so* $\Psi(p) = 4_1 2_1 3_1 1_0$.

Lemma 3.12 *Suppose* $\mathbf{z} = (z_1, z_2, \ldots, z_d)$ *has* $z_k > 0$ *for some* k *and* $z_j = 0$ *for some* j *and that* $\mathbf{w} = (w_1, w_2, \ldots, w_d)$ *satisfies* $w_k = 0$, $w_j > 0$ *and* $w_i = z_i$ *for* $i \notin \{k, j\}$. *Then there is a bijection* $\Phi' : S_{\mathbf{z}} \to S_{\mathbf{w}}$ *such that* $e(\Phi'(p)) = e(p)$.

Proof: A positive coordinate z_i of $\mathbf{z}$ affects excedances in $p = \pi \times \pi(\mathbf{z})$ in a way which is independent of whether $z_i = 1$ or $z_i > 1$. Hence we may assume, without loss of generality, that $z_i \in \{0, 1\}$ for all i. Then, $w_j = z_k = 1$ and $w_k = z_j = 0$. That is, $\mathbf{w}$ is obtained from $\mathbf{z}$ by transposing z_k and z_j. Let $p = \pi \times \pi(\mathbf{z})$ where $\pi = a_1 a_2 \ldots a_d$. We define $\Phi' : S_{\mathbf{z}} \to S_{\mathbf{w}}$ by defining a certain bijection $\phi' : \mathcal{S}_d \to \mathcal{S}_d$ and setting $\Phi'(\pi \times \pi(\mathbf{z})) = \phi'(\pi) \times \phi'(\pi)(\mathbf{w})$. $\phi'(\pi)$ is defined by the following trichotomy.

(1) For all $\pi \in \mathcal{S}_d$ such that π either fixes both j and k or neither, i.e. either $a_j = j$ and $a_k = k$ or $a_j \neq j$ and $a_k \neq k$, we let $\phi'(\pi) = \pi$. Hence, for such p, $q = \Phi'(p)$ is obtained from p simply by interchanging the indices of k and j in p, i.e. k gets the index z_j and j the index z_k. Consequently, the number of excedances is preserved, for in the first case we are moving an excedance from k to j and in the latter case no excedances will be affected since $a_j \neq j$ and $a_k \neq k$. As an example, if $k = 2$ and $j = 5$, we have $\phi'(3_0 2_1 4_1 1_0 5_0) = 3_0 2_0 4_1 1_0 5_1$ and $\phi'(5_1 4_0 2_1 1_0 3_0) = 5_1 4_0 2_0 1_0 3_0$. Clearly, this is injective, for $\phi'(\pi) = \phi'(\tau)$ if and only if $\pi = \tau$.

(2) Suppose $a_k = k$ and $a_j \neq j$. We then define $\phi'(\pi) = \tau = b_1 b_2 \ldots b_d$ in the following way. Let $b_j = j$. Let F be the set of fixed points of π, i.e. $F = \{i \in [d] | a_i = i\}$. In particular, $k \in F$ and $j \notin F$. Given a set S, let S_i denote $S \setminus \{i\}$ and let S^i denote $S \cup \{i\}$. Let $D = [d] \setminus F$. Set $b_j = j$ and set $b_i = i$ for all $i \in F_k$. By definition, the restriction of π to D is a *derangement* of D, i.e. $a_i \neq i$ for all $i \in D$. We have already defined b_i for all $i \in F_k^j$ by declaring such i to be fixed points of τ. Hence, for all $i \in F_k$, i is an excedance in $\Phi'(p)$ if and only if i is an excedance in p, because $a_i = b_i$ and $z_i = w_i$. Moreover, k is an excedance in p and j is an excedance in $\Phi'(p)$. Thus, so far, we have the same number of excedances in p and $\Phi'(p)$.

What remains to be defined is how τ permutes the elements of D_j^k.

There is a unique order preserving bijection $\theta : D \to D_j^k$, i.e. θ maps the smallest element of D to the smallest element of D_j^k, the next smallest element of D to the next smallest element of D_j^k and so on. In other words, $\theta(i) > \theta(m)$ if and only if $i > m$. Now, if $i \in D_j^k$, we set $b_i = \theta(a_{\theta^{-1}(i)})$. Note that this defines a bijection $\tau|_{D_j^k} : D_j^k \to D_j^k$, as required. This further guarantees that $b_i \neq i$ for all $i \in D_j^k$, in particular $b_k \neq k$, and, moreover, that $b_i > i$ precisely when $a_{\theta^{-1}(i)} > \theta^{-1}(i)$. Note also that whether $i \in D_j^k$ is an excedance in $\Phi'(p)$ is not dependent on z_{b_i} since $b_i \neq i$. The same is true of $\theta^{-1}(i)$ and p (and $w_{\theta^{-1}(i)}$), so i is an excedance in $\Phi'(p)$ if and only if $\theta^{-1}(i)$ is an excedance in p.

Let us illustrate this by an example. Let $k = 2, j = 5$ and $q = 3_1 2_1 1_0 4_1 6_0 5_0 7_0$ so that $\pi = 3214657$. Then $F = \{2, 4, 7\}$ and $D = \{1, 3, 5, 6\}$. Hence, τ fixes 4, 5 and 7.

θ maps $\{1,3,5,6\}$ to $\{1,2,3,6\}$ by sending 1 to 1, 3 to 2, 5 to 3 and 6 to 6. Hence, $\tau = 2164537$, so $\Phi'(p) = 2_0 1_0 6_0 4_1 5_1 3_1 7_0$.

Again, this is injective because if $\phi'(\pi) = \phi'(\tau)$ then π and τ have the same fixed points, and so do $\phi'(\pi)$ and $\phi'(\tau)$, and consequently π and τ must be identical on the remaining elements of $[d]$, because the bijection θ was unique.

(3) The case when $a_k \neq k$ and $a_j = j$ is similar to 2). As a matter of fact, it turns out that the similar argument results in this: If $p \in \{q = \pi \times \pi(\mathbf{z}) \in S_{\mathbf{z}} | a_k \neq k$ and $a_j = j\}$ then $\Phi'(p) = (\phi')^{-1}(\pi) \times (\phi')^{-1}(\pi)(\mathbf{w})$, which is well defined, because $(\phi')^{-1}(\pi)$ is (implicitly) defined in 2).

As an example, since we had $\Phi'(3_1 2_1 1_0 4_1 6_0 5_0 7_0) = 2_0 1_0 6_0 4_1 5_1 3_1 7_0$, we have $\Phi'(2_1 1_0 6_0 4_1 5_0 3_1 7_0) = 3_1 2_0 1_0 4_1 6_0 5_1 7_0$.

It is obvious that $S_{\mathbf{z}}$ is the disjoint union of the domains described in (1), (2) and (3) and that $S_{\mathbf{w}}$ is the disjoint union of the images in (1), (2), and (3). $\square$

Example 3.13 *Let* $p = 4_1 2_1 3_1 1_0$ *and let* $\mathbf{w} = (1,0,2,1)$ *(so* $\mathbf{w} = \mathbf{z}$ *in Example 3.9). Then* $\Psi(p) = 1_1 4_1 3_2 2_0$.

Lemma 3.14 *Suppose* $\mathbf{z} = (z_1, , z_2, \ldots, z_d)$ *and* $\mathbf{w} = (w_1, w_2, \ldots, w_d)$ *have the same number of positive coordinates. Then there is a bijection* $\Phi : S_{\mathbf{z}} \to S_{\mathbf{w}}$ *such that* $e(\Phi(p)) = e(p)$.

Proof: Suppose that $\mathbf{z}$ has m positive coordinates z_k such that $w_k = 0$. Label these coordinates $k_1, k_2, \ldots, k_m$ and set $K = \{k_1, k_2, \ldots, k_m\}$. Then $\mathbf{w}$ has m positive coordinates w_j such that $z_j = 0$. Label these coordinates $j_1, j_2, \ldots, j_m$ and set $J = \{j_1, j_2, \ldots, j_m\}$. We can clearly apply Φ' in Lemma 3.12 repeatedly to interchange z_{k_1} with z_{j_1}, then z_{k_2} with z_{j_2}, and so on, until we have "moved" each z_{k_i} to the corresponding z_{j_i}. Once we have done that, we have $z_i > 0$ if and only if $w_i > 0$, so we can simply replace each z_i by w_i since that doesn't affect excedances. However, to simplify the description of Φ, observe that we can perform the procedure described in Lemma 3.12 simultaneously for all the k_i's. Namely, if both $a_{k_i} = k_i$ and $a_{j_i} = j_i$ or neither, then we do nothing. If $a_{k_i} = k_i$ and $a_{j_i} \neq j_i$ then we let $b_{j_i} = j_i$, and if $a_{k_i} \neq k_i$ and $a_{j_i} = j_i$ then we let $b_{k_i} = k_i$. Now, define F and D as before and let $\hat{D} = D \cup \{k \in K | a_k = k\} \setminus \{j \in J | a_j \neq j\}$. Let $\hat{F} = [d] \setminus \hat{D}$ and set $b_i = i$ for all $i \in \hat{F}$. Define the bijection $\theta : D \to \hat{D}$ as in lemma 3.12 and, similarly, let $b_i = \theta(a_{\theta^{-1}(i)})$ for all $i \in \hat{D}$. The reasoning in Lemma 3.12 now goes through without change. $\square$

We now use these lemmas to construc' a bijection $S_{\mathbf{z}} \to S_{\mathbf{z}}$ which takes an indexed permutation with k descents to one with k excedances. Suppose $\mathbf{z}$ has exactly m

positive coordinates. Let $\mathbf{w}$ be defined by $w_i = 0$ if $i \leq d-m$ and $w_i = 1$ if $i > d-m$. Then the composition

$$S_{\mathbf{z}} \xrightarrow{\Theta} S_{\mathbf{w}} \xrightarrow{\Psi} S_{\mathbf{w}} \xrightarrow{\Phi} S_{\mathbf{z}}$$

is a bijection which takes a $p \in S_{\mathbf{z}}$ with k descents to a $q \in S_{\mathbf{z}}$ with k excedances. There follows

Theorem 3.15 *For all $n \geq 1$ and for all $d \geq 0$, $E_d^n(t) = D_d^n(t)$.* □

Let $A_d(t) = tD_d^1(t)$. It has long been known that $A_d(t)$ satisfies $\frac{A_d(t)}{(1-t)^{d+1}} = \sum_{k\geq 1} k^d t^k$ and the polynomial $A_d(t)$ is called the d-th *Eulerian polynomial.* Theorem 3.17 generalizes this relation to our descent polynomials $D_d^n(t)$.

Lemma 3.16 *The coefficients of $E_d^n(t)$ (and hence those of $D_d^n(t)$) satisfy*

$$E(d,n,k) \;=\; (nk+1)E(d-1,n,k) \;+\; (n(d-k)+(n-1))E(d-1,n,k-1).$$

Proof: We can produce any indexed permutation in S_d^n by inserting d_m (for the approppriate $m \in \mathbf{Z_n}$) in an indexed permutation in S_{d-1}^n. Also, each $p \in S_d^n$ arises only once in this way. Let $E_d^k = \{p \in S_d^n | p \text{ has k excedances}\}$. In order to obtain a $p \in E_d^k$ from one in S_{d-1}^n, we can do exactly one of two things:

(1) Given $p' \in E_{d-1}^k$ let i be one of the k excedances of p'. We can pick any $m \in \mathbf{Z_n}$, replace the i-th indexed letter of p' by d_m and append the indexed letter we removed to the end of the resulting indexed permutation to obtain a $p \in E_d^k$. This gives rise to $nkE(d-1,n,k)$ indexed permutations in E_d^k. We can also append d_0 to the end of any $p' \in E_{d-1}^k$ to obtain a $p \in E_d^k$. Hence, E_{d-1}^k gives rise to $(nk+1)E(d-1,n,k)$ indexed permutations in E_d^k.

(2) Given $p' \in E_{d-1}^{k-1}$, let i be one of the $d-k$ non-excedances of p'. We can pick any $m \in \mathbf{Z_n}$, replace the i-th indexed letter of p' by d_m and append the indexed letter we removed to the end of the resulting indexed permutation to obtain a $p \in E_d^k$, since we are adding an excedance at i. This gives rise to $n(d-k)E(d-1,n,k-1)$ indexed permutations in E_d^k. We can also, given any $m \in \{1,2,\ldots,n-1\}$, append d_m to the end of any $p' \in E_{d-1}^{k-1}$ to obtain a $p \in E_d^k$, since we are adding an excedance at d. Hence, E_{d-1}^{k-1} gives rise to $(n(d-k)+(n-1))E(d-1,n,k-1)$ indexed permutations in E_d^k, which, together with 1), proves the lemma. □

As a generalization of the Eulerian polynomials, a polynomial $P(t)$ which satisfies $\frac{P(t)}{(1-t)^{d+1}} = \sum_{k\geq 0} f(k)t^k$, where f is a polynomial of degree d, is called the *f-eulerian polynomial.*

Theorem 3.17

$$\frac{E_d^n(t)}{(1-t)^{d+1}} = \sum_{i\geq 0} (ni+1)^d t^i,$$

i.e. $E_d^n(t)$ is the f-eulerian polynomial where $f(i) = (ni+1)^d$.

Proof: By the preceding lemma, it suffices to show that the coefficients of $F_d^n(t) = (1-t)^{d+1}\sum_{i\geq 0}(ni+1)^d t^i$ satisfy the recurrence relation given in the lemma and the same initial conditions as the numbers $E(d,n,k)$. For any $d \geq 0$ and $n \geq 1$, the only indexed permutation with no excedances is $1_0 2_0 \ldots d_0$, so $E(d,n,0) = 1$, and $E(d,n,k) = 0$ if $k > d$. Let $F(d,n,k)$ be the k-th coefficient of $F_d^n(t)$. It is straightforward to check (from the expression below for $F_d^n(t)$) that $F(d,n,0) = 1$ and that $F(d,n,k) = 0$ if $k > d$. Hence, it remains only to be shown that the coefficients $F(d,n,k)$ satisfy the recurrence relation in the lemma. We have

$$F_d^n(t) = (1-t)^{d+1}\sum_{i\geq 0}(ni+1)^d t^i = \sum_{i=0}^{d+1}(-1)^i\binom{d+1}{i}t^i\sum_{i\geq 0}(ni+1)^d t^i =$$

$$\sum_{k=0}^{d}\sum_{i=0}^{k}(-1)^{k-i}\binom{d+1}{k-i}(ni+1)^d t^k,$$

since $F_d^n(t)$ has degree at most d (see, for example, Corollary 4.3.1 in [21]). Hence, $F(d,n,k) = \sum_{i=0}^{k}(-1)^{k-i}\binom{d+1}{k-i}(ni+1)^d$, so we need to show

$$\sum_{i=0}^{k}(-1)^{k-i}\binom{d+1}{k-i}(ni+1)^d =$$

$$(nk+1)\sum_{i=0}^{k}(-1)^{k-i}\binom{d}{k-i}(ni+1)^{d-1} \quad +$$

$$(n(d-k)+(n-1))\sum_{i=0}^{k-1}(-1)^{k-1-i}\binom{d}{k-1-i}(ni+1)^{d-1}.$$

Since $\binom{d}{m} = 0$ if $d \geq 0$ and $m < 0$, we can change the upper bound in the last sum to k, so the equality is equivalent to

$$\sum_{i=0}^{k}(-1)^{k-i}(ni+1)^{d-1}A(i) = 0$$

where

$$A(i) = \binom{d+1}{k-i}(ni+1) - (nk+1)\binom{d}{k-i} + (n(d-k)+(n-1))\binom{d}{k-1-i}.$$

To prove the theorem it thus suffices to show that $A(i) = 0$:

$$\binom{d+1}{k-i}(ni+1) - (nk+1)\binom{d}{k-i} + (n(d-k)+(n-1))\binom{d}{k-1-i} =$$

$$ni\binom{d+1}{k-i} - nk\left[\binom{d}{k-i} + \binom{d}{k-1-i}\right] + n(d+1)\binom{d}{k-1-i} =$$

$$n\left[(d+1)\binom{d}{k-1-i} - (k-i)\binom{d+1}{k-i}\right] = 0,$$

by the straightforward binomial identity $(d+1)\binom{d}{k-1-i} = (k-i)\binom{d+1}{k-i}$. □

There is a way of proving the preceding theorem *combinatorially* when $E_d^n(t)$ is replaced by $D_d^n(t)$. Actually, we can derive the theorem from a finer computation of $D_d^n(t)$. Namely, given $\mathbf{z} \in \mathbf{Z}_n^d$, we compute the descent polynomial $D_{\mathbf{z}}(t) := \sum_{p \in S_{\mathbf{z}}} t^{d(p)}$. The proof of the following theorem is a modification of the proof of Lemma 4.5.1 and of the proof of Theorem 4.5.14 in [21] in the special case where the poset P is an antichain.

Theorem 3.18 *Suppose $\mathbf{z} \in \mathbf{Z}_n^d$ has exactly m positive coordinates and let $D_{\mathbf{z}}(t) := \sum_{p \in S_{\mathbf{z}}} t^{d(p)}$. Then*

$$\sum_{k \geq 0} (k+1)^{d-m} k^m t^k = \frac{D_{\mathbf{z}}(t)}{(1-t)^{d+1}}.$$

Proof: By Lemma 3.8, since we will only be concerned with descent sets of indexed permutations, we may, without loss of generality, assume that $z_i = 0$ for $i \leq d-m$ and $z_i = 1$ for $i > d-m$. Hence, i is a descent in $p = a_{1 z_{a_1}} a_{2 z_{a_2}} \ldots a_{d z_{a_d}} \in S_{\mathbf{z}}$ if $a_i > a_{i+1}$, and d is a descent if and only if $a_d > d-m$. Let $f : [d] \to [k]$ be a function which satisfies $f(i) \geq 2$ if $i > d-m$. Then there is a unique indexed permutation $p = a_{1 z_{a_1}} a_{2 z_{a_2}} \ldots a_{d z_{a_d}} \in S_{\mathbf{z}}$ which satisfies

i. $f(a_1) \geq f(a_2) \geq \cdots \geq f(a_d)$, and

ii. $f(a_i) > f(a_{i+1})$ if $a_i > a_{i+1}$.

Namely, there is a unique ordered partition $< B_1, B_2, \ldots, B_k >$ of $[d]$ such that f is constant on each B_i and $f(B_1) > f(B_2) > \cdots > f(B_k)$. Let $\pi \in \mathcal{S}_d$ be the permutation obtained by ordering the elements of B_1 in increasing order, then the elements of B_2 in increasing order and so on. Then $p = \pi \times \pi(\mathbf{z})$, i.e. p is obtained from $\pi = a_1 a_2 \ldots a_d$ by attaching the index 0 to a_i if $a_i \leq d - m$ and 1 if $a_i > d - m$. We say that p is *compatible* with f. The reason for requiring $f(i) \geq 2$ for $i > d - m$ is that we need to have the possible descent at d force a drop in the value of f as happens with other descents, according to **ii** above. Alternatively, we can think of f as being also defined on $a_{d+1} = d + 1$, with $f(a_{d+1}) = 1$.

Now, a map $f : [d] \to [k]$ satisfies $f(i) \geq 2$ for $i > d - m$ and is compatible with $p = a_{1_{z_{a_1}}} a_{2_{z_{a_2}}} \ldots a_{d_{z_{a_d}}} \in S_{\mathbf{z}}$ if and only if

$$k - d(p) \geq f(a_1) - d_1 \geq f(a_2) - d_2 \geq \cdots \geq f(a_d) - d_d \geq 1, \tag{1}$$

where $d_i = \#\{j \,|\, j \geq i, j \text{ is a descent in } p\}$. Note that (1) forces $f(a_i) \geq 2$ if $a_i > d - m$ because in that case $d_i \geq 1$. Let $\Omega_p(k)$ be the number of maps satisfying (1). Then $\Omega_p(k) = \left(\binom{k-d(p)}{d}\right) = \binom{k-d(p)+d-1}{d}$ and

$$\sum_{k \geq 0} \Omega_p(k) t^k = \frac{t^{d(p)+1}}{(1-t)^{d+1}}.$$

Hence, if we let $\Omega(k) = \sum_{p \in S_{\mathbf{Z}}} \Omega_p(k)$, we get

$$\sum_{k \geq 0} \Omega(k) t^k \quad = \quad \frac{\sum_{p \in S_{\mathbf{Z}}} t^{d(p)+1}}{(1-t)^{d+1}} \quad = \quad \frac{t D_{\mathbf{z}}(t)}{(1-t)^{d+1}}. \tag{2}$$

Now, $\Omega(k)$ is the number of maps $f : [d] \to [k]$ such that $f(i) \geq 2$ if $i > d - m$, so clearly $\Omega(k) = k^{d-m}(k-1)^m$. Dividing both sides of (2) by t yields the theorem. $\square$

If we now sum over all $\mathbf{z} \in \mathbf{Z}_n^d$, then, since there are exactly $\binom{d}{m}(n-1)^m$ $\mathbf{z}$'s with exactly m positive coordinates, we get

$$\frac{D_d^n(t)}{(1-t)^{d+1}} \quad = \quad \frac{\sum_{\mathbf{z} \in \mathbf{Z}_n^d} D_{\mathbf{z}}(t)}{(1-t)^{d+1}} \quad = \quad \sum_{m=0}^{d} \binom{d}{m} (n-1)^m \sum_{k \geq 0} (k+1)^{d-m} k^m t^k \quad =$$

$$\sum_{k \geq 0} \left(\sum_{m=0}^{d} \binom{d}{m} (n-1)^m (k+1)^{d-m} k^m \right) t^k \quad =$$

$$\sum_{k\geq 0}((n-1)k+(k+1))^d t^k \quad = \quad \sum_{k\geq 0}(nk+1)^d t^k,$$

as in Theorem 3.17.

From these expressions for $D_{\mathbf{z}}(t)$ and $D_d^n(t)$, we get some further interesting results about these polynomials. In [7], Brenti shows that if a polynomial $f(n)$ has all its roots in the interval $[-1,0]$, then its f-eulerian polynomial $W(t) = w_0+w_1t+\cdots+w_dt^d$ (defined on page 13 here) has only real zeros (see Theorems 4.4.4 and 2.3.3 in [7]). That, in turn, implies that the sequence $w_0, w_1, \ldots, w_d$ of coefficients of $W(t)$ is *unimodal*, i.e. $w_0 \leq w_1 \leq \cdots \leq w_k \geq w_{k+1} \geq \cdots \geq w_d$ for some k with $0 \leq k \leq d$. Thus, the following theorem is an obvious consequence of Theorems 3.17 and 3.18.

Theorem 3.19 *For any d and n, the polynomial $D_d^n(t)$ has only real zeros. In particular, there is a $k \in \{0,1,\ldots,d\}$ such that*

$$D(d,n,0) \leq D(d,n,1) \leq \cdots \leq D(d,n,k) \geq D(d,n,k+1) \geq \cdots \geq D(d,n,d).$$

The same is true of the polynomial $D_{\mathbf{z}}(t)$ for any $\mathbf{z} \in \mathbf{Z}_n^d$. □

We will expand on Theorem 3.17 in Section 3.5. Before that we will give yet another definition of descent (which explains why we choose to define indexed permutations the way we do), compute the exponential generating functions of the descent polynomials $D_d^n(t)$ and make some comments on S_d^n in the special case when $n = 2$.

3.2 The length function

The symmetric group S_d can be defined as the group generated by the set $S = \{s_1, s_2, \ldots, s_{d-1}\} = \{(12), (23), \ldots, ((d-1)d)\}$ where, for example, (12) denotes the permutation which transposes the first two letters in a permutation word. The relations among the generators are given by $(s_is_{i+1})^3 = (s_is_j)^2 = (s_i)^2 = id$ for $1 \leq i,j \leq d$ such that i and j differ by at least 2. Given this, one defines the *length* $\ell(\pi)$ of a permutation $\pi \in S_d$ to be the least number r such that $\pi = s_{i_1}s_{i_2}\ldots s_{i_r}$ for some multiset $\{s_{i_1}, s_{i_2}, \ldots, s_{i_r}\}$ on S. It is well known that the descent set of π can be defined as the set of those generators s_i for which $\ell(\pi s_i) < \ell(\pi)$.

Similarly, the hyperoctahedral group, which coincides with our S_d^2, is generated by the set $H = \{s_1, s_2, \ldots, s_{d-1}, s_d\} = \{(12), (23), \ldots, ((d-1)d), (d_1)\}$, where $s_d = (d_1)$ is the indexed permutation which adds 1 (mod 2) to the index of the last letter of an indexed permutation word. As an example (where we use mixed notation for the indexed permutations), $3_01_12_0(3_1) = 3_01_12_1$. Here, however, we need to modify our definition of descent.

Definition 3.20 *A* g-descent *in $p = a_{1z_1}a_{2z_2}\dots a_{dz_d} \in S_d^n$ is an $i \in [d]$ such that one of the following holds:*

i. $z_i = z_{i+1} = 0$ and $a_i > a_{i+1}$,

ii. $z_i, z_{i+1} > 0$ and $a_i < a_{i+1}$,

iii. $z_i > 0$ and $z_{i+1} = 0$.

It is easy to see that g-descents are equidistributed with descents. Namely, Definition 3.20 is tantamount to replacing the ordering $<_\ell$ by a new ordering of the letters $\{a_{iz_i}\}$, so a straightforward modification of the bijection Θ in Lemma 3.8 will send an indexed permutation with k descents to one with k g-descents.

Under this definition, a g-descent in $p \in S_d^2$ coincides with a generator s_i such that $\ell(ps_i) < \ell(p)$. Because (s_d) has order 2 when $n = 2$, it is clear that both the symmetric groups and the octahedral groups are *Coxeter groups* (for a definition and further information see [12]).

For $n > 2$, S_d^n is generated by $H = \{s_1, s_2, \dots, s_{d-1}, s_d\}$, where now $s_d = (d_1)$ is the indexed permutation which adds 1 (mod n) to the index of the last letter of an indexed permutation word. Although the groups S_d^n are not (at least not in any obvious way) Coxeter groups when $n > 3$, they are what is called *unitary groups generated by reflections* (see [17]). If we define length as before, then a g-descent in $p \in S_d^n$ coincides either with a generator s_i with $i < d$ such that $\ell(ps_i) < \ell(p)$ or with a power of s_d such that $\ell(p(s_d)^k) < \ell(p)$. We will sketch a proof of this, following §5 in [20].

We claim that if $p = \pi \times \mathbf{z} = a_1a_2\dots a_d \times (z_1, z_2, \dots, z_d) \in S_d^n$, then

$$\ell(p) = \ell(\pi) + \sum_i (2d_i + z_i)$$

where the sum is over all i such that $z_i > 0$ and d_i is the number of a_j's to the right of a_i which are larger than a_i. This is so, because π has to appear as the product of some subset of the generators in any word $s_{i_1}s_{i_2}\dots s_{i_r} = p$, and to equip each a_i with the appropriate index, if $z_i > 0$, a_i must be moved to the end of the permutation and then back to the i-th place. To do that, a_i must be moved at least past all a_j's to the right of a_i that are larger than a_i and then back, and clearly p can always be obtained by this algorithm. That is, we build up π by the minimum number of generators required, and interrupt that process only to move each a_i with $z_i > 0$ to the end and back (giving a_i the right index while it sits at the end) *precisely when a_i is as close to the end as it does get*, at which point a_i is separated from the end by those a_j's which are to the right of a_i in π and which are larger than a_i.

With this way of writing p with a minimal number of generators, which we call the *canonical word for* p, a g-descent at $i < d$ in p is easily seen to correspond to a generator s_i with $i < d$ such that $\ell(ps_i) < \ell(p)$. Namely, s_i acts on π by transposing a_i and a_{i+1}, so we know from the classical case that the length of π will either be increased by 1 or decreased by 1. A straightforward case analysis concernig the indices of a_i and a_{i+1} shows that the length of p will be reduced by 1 precisely when i is a g-descent in p and increased by 1 when i is not a g-descent in p. A g-descent at d entails that in the canonical word for p there is no occurrence of s_{d-1} after the last occurrence of s_d, so, since s_d commutes with s_i for any $i < d-1$, appending the appropriate power of s_d to the canonical word for p results in reducing the length of p by z_d.

3.3 Generating functions for the descent polynomials

It was known already to Euler that the polynomials $A_d(t) = tD_d^1(t)$ satisfied

$$t^{-1}\sum_{d\geq 0} A_d(t)\frac{t^d}{d!} \quad = \frac{(1-t)e^{x(1-t)}}{1-te^{x(1-t)}}.$$

This can be derived in a way which trivially generalizes to the derivation for $D_d^n(t)$, any n (the author is grateful to Victor Reiner for pointing out this derivation):

$$\sum_{d\geq 0}\frac{D_d^n(t)}{(1-t)^{d+1}}\frac{x^d}{d!} = \sum_{d\geq 0}\left(\sum_{k\geq 0}(nk+1)^d t^k\right)\frac{x^d}{d!} = \sum_{k\geq 0} t^k \sum_{d\geq 0}\frac{(nk+1)^d x^d}{d!} = \sum_{k\geq 0} t^k e^{(nk+1)x}.$$

Now, multiply both sides by $(1-t)$ and replace k by d in the RHS to get

$$\sum_{d\geq 0}\frac{D_d^n(t)}{(1-t)^d}\frac{x^d}{d!} \quad = \quad (1-t)\sum_{d\geq 0} t^d e^{(nd+1)x} \quad = \quad (1-t)e^x\frac{1}{1-te^{nx}}.$$

Finally, replace x by $x(1-t)$ to obtain

Theorem 3.21

$$\sum_{d\geq 0} D_d^n(t)\frac{x^d}{d!} \quad = \quad \frac{(1-t)e^{x(1-t)}}{1-te^{nx(1-t)}}. \qquad \square$$

3.4 The hyperoctahedral group

The hyperoctahedral group is the group of *permutations and sign changes* of the coordinates in $\mathbf{R}^d$, i.e. the group of automorphisms $\phi : \mathbf{R}^d \to \mathbf{R}^d$ which send each standard basis vector $\mathbf{e}_i$ to $\pm\mathbf{e}_j$ for some $\mathbf{e}_j$. Requiring that ϕ be an autmorphism ensures that either $\mathbf{e}_j$ or $-\mathbf{e}_j$ is the image of exactly one of the $\mathbf{e}_i$'s, so an element of the group can be represented by a word $a_{1\epsilon_1}a_{2\epsilon_2}\dots a_{d\epsilon_d}$, where $\epsilon_i = \pm 1$, indicating that the standard basis vector $\mathbf{e}_i$ goes to $\epsilon_i \mathbf{e}_{a_i}$. The hyperoctahedral group can also be described as the group of symmetries of the d-hyperoctahedron, whence the name of the group. This group is isomorphic to our S_d^2. Descents have been defined previously for this group (but not excedances), in a way which is equidistributed with ours (see, for example, [16]), and it is known that the polynomial $D_d^2(t)$ is *symmetric*, i.e. the coefficients $D(d,2,k)$ satisfy $D(d,2,k) = D(d,2,d\text{-}k)$. This is also true in the case of the symmetric group S_d, where we have $D(d,1,k) = D(d,1,d\text{-}1\text{-}k)$. It is also known that $D_d^2(t)$ equals the *h-polynomial* $h(\hat{O}_d,t)$ of the first barycentric subdivision of the *d-hyperoctahedron* O_d. We will generalize this relationship in section 3.6, although we have to replace the d-hyperoctahedron by a certain triangulation of the unit d-cube.

The symmetry of $D_d^2(t) = h(\hat{O}_d,t)$ is a consequence of the fact that the hyperoctahedron is a simplicial polytope, but it is easy to show directly that the descent polynomial $D_d^2(t)$ is symmetric. Namely, if $p = a_{1z_1}a_{2z_2}\dots a_{dz_d}$ has k descents, then $p' = b_{1w_1}b_{2w_2}\dots b_{dw_d}$ defined by $b_i = d+1-a_i$ and $w_i = 1-z_i$ has $d-k$ descents, because i is a descent in p' if and only if i is not a descent in p.

For $n \geq 3$, $D_d^n(t)$ is not symmetric. For example, there is always only one indexed permutation with no excedances, namely $1_0 2_0 \dots d_0$, but there are $(n-1)^d$ with d excedances, namely $1_{z_1}2_{z_2}\dots d_{z_d}$ for any $(z_1,z_2,\dots,z_d)$ with all $z_i > 0$, of which there are $(n-1)^d$.

3.5 The connection to Ehrhart polynomials

What is perhaps most interesting about Theorem 3.17 is that it suggests a connection between our descent polynomials and the *Ehrhart polynomials* of certain integral polytopes. In order to develop that we need to make a digression here and review some basic facts about Ehrhart polynomials.

Let $\mathcal{P}$ be a *lattice* d-polytope, i.e. a convex d-polytope in $\mathbf{R}^m$ with integral (or lattice) vertices, i.e. $v_i \in \mathbf{Z}^m$ for all vertices v_i of $\mathcal{P}$. For $n \in \mathbf{N}$ let $n\mathcal{P} = \{nx \,|\, x \in \mathcal{P}\}$, i.e. $n\mathcal{P}$ is the (lattice) polytope obtained by dilating $\mathcal{P}$ by a factor of n.

For $n \in \mathbf{N}$ define the function

$$i(\mathcal{P},n) = \#\{x \in \mathbf{R}^m \,|\, x \in n\mathcal{P} \cap \mathbf{Z}^m\}.$$

Thus, $i(\mathcal{P},n)$ is the number of lattice points contained in $n\mathcal{P}$. By Cor. 4.6.28 in [21], $i(\mathcal{P},n)$ is a polynomial in n of degree d, called the *Ehrhart polynomial* of $\mathcal{P}$. Now define the generating function

$$E(\mathcal{P},\lambda)=\sum_{n\geq 0} i(\mathcal{P},n)\lambda^n.$$

By Thm. 2.1 in [19], we have

$$E(\mathcal{P},\lambda)=\frac{h^*(\mathcal{P},\lambda)}{(1-\lambda)^{d+1}}$$

where $h^*(\mathcal{P},\lambda)$ is a polynomial of degree at most d with non-negative integer coefficients, called the *Ehrhart h^*-polynomial of $\mathcal{P}$.*

Now, it is easy to compute the Ehrhart polynomial $i(C^d,k)$ of a d-dimensional unit cube C^d. Namely, as the lattice points in C^d consist of all d-tuples $(z_1,z_2,\ldots,z_d)$ with each $z_i\in\{0,1\}$, the cube contains 2^d lattice points, $2C^d$ contains 3^d lattice points, and so on. In general, the dilation of C^d by k contains $(k+1)^d$ lattice points, so the Ehrhart polynomial of C^d is $i(C^d,k)=(k+1)^d$. Likewise, if we start with the dilation nC^d of C^d by n, we see that its Ehrhart polynomial is $i(nC^d,k)=(nk+1)^d$. But this is the same polynomial as the one appearing in Theorem 3.17, which leads us to the following observation:

Theorem 3.22 *$D_d^n(t)=h^*(nC^d,t)$, where nC^d is the dilation of the unit d-cube by n.* □

This, of course, leads one to speculate whether it might be possible to find some sort of a "natural" connection between descents in indexed permutations and the structure of the dilated cubes in question, i.e. a bijective proof of Theorem 3.22. Unfortunately, although there is a nice geometric interpretation of the coefficients of $h^*(\mathcal{P},t)$ when $\mathcal{P}$ is a lattice simplex (see [3]), no such interpretation is known for polytopes in general. However, there is an alternative approach...

First some definitions. A simplicial complex K is *pure* if all its maximal faces have the same dimension $d=\dim(K)$. If K is a pure simplicial complex of dimension d, then a *facet* of K is a d-face, i.e. a d-dimensional face, of K. The *h-vector* $h(K)=(h_0,h_1,\ldots,h_d)$ of a simplicial complex K of dimension d-1 is defined as follows: Let $f_i=f_i(K)$ be the number of i-dimensional faces in K, where we set $f_{-1}=1$ (corresponding to the empty set), and define $h(K)=(h_0,h_1,\ldots,h_d)$ by setting

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i}=\sum_{i=0}^{d} h_i x^{d-i}.$$

We define the *h-polynomial* $h(K,t)$ of K by $h(K,t)=h_0+h_1t+\cdots+h_dt^d$. For further information about h-vectors, see [22].

Let K be a pure simplicial lattice complex of dimension d. If all facets of K have volume $1/d!$ (which is the least volume a lattice d-simplex can have) then we say that K is *primitively triangulated*. (What is primitively triangulated is of course the geometric realization $|K|$ of K, but we will allow ourselves this abuse of notation). The following theorem is essentially a consequence of Cor. 2.5 in [19], whose conclusion is expressed in greater generality in Thm. 2 in [3].

Theorem 3.23 *Suppose K is a primitively triangulated simplicial lattice complex. Then $h^*(K,t) = h(K,t)$.* □

It is easy to see that if nC^d is triangulated in such a way that all the maximal simplices have volume $1/d!$ then the triangulation must consist of $d! \cdot n^d$ maximal simplices. But $d! \cdot n^d$ is also the cardinality of S_d^n, hence the sum of the coefficients of $D_d^n(t)$. Moreover, it is known that for certain pure simplicial complexes K the coefficients of $h(K,t)$ can be interpreted in a way that partitions the facets of K according to how they intersect other facets. We will briefly review this now. For further information see [4] and [5].

Definition 3.24 *Let K be a finite pure simplicial complex of dimension d. If F is a face of K, let $\bar{F}$ be the complex consisting of F and all its faces. An ordering $F_1, F_2, \ldots, F_n$ of the facets of K is called a* shelling *if, for all k with $1 < k \leq n$, $\bar{F}_k \cap \bigcup_{i=1}^{k-1} \bar{F}_i$ is a pure complex of dimension (d-1). A complex K is said to be* shellable *if there exists a shelling of K.*

That is, a complex is shellable if it can be built up by adding one facet at a time in such a way that, for $k > 1$, the intersection of each $\bar{F}_k$ with the complex generated by the previous $\bar{F}_i$'s is a nonempty union of (d-1)-faces of $\bar{F}_k$.

If $F_1, F_2, \ldots, F_n$ is a shelling of K, then for each F_k there is a unique minimal face G_k of $\bar{F}_k$ which is not contained in $\bar{F}_k \cap \bigcup_{i=1}^{k-1} \bar{F}_i$, and the cardinality of G_i ($= \dim(G_i) + 1$) equals the number of $(d-1)$-faces in $\bar{F}_k \cap \bigcup_{i=1}^{k-1} \bar{F}_i$. For example, if $\bar{F}_k \cap \bigcup_{i=1}^{k-1} \bar{F}_i$ consists of a single $(d-1)$-face of $\bar{F}_k$, then there is a unique vertex v of $\bar{F}_k$ which is not contained in that $(d-1)$-face and thus not contained in $\bigcup_{i=1}^{k-1} \bar{F}_i$.

As it turns out, the h-vector of a shellable complex can be computed from the shelling. The following theorem is essentially due to McMullen [15].

Theorem 3.25 *Let $F_1, F_2, \ldots, F_n$ be a shelling of K and let, for k with $1 \leq k \leq n$, G_k be the minmal face of $\bar{F}_k$ which is not contained in $\bigcup_{i=1}^{k-1} \bar{F}_i$. Let $c(k)$ be the cardinality of G_k. Then we have the following formula for the h-polynomial of K:*

$$h(K,t) = \sum_{i=1}^{n} t^{c(i)}. \qquad \square$$

Thus, given a shelling $F_1, F_2, \ldots, F_n$ of a simplicial complex K, we can compute the h-polynomial $h(K,t)$ of K via Theorem 3.25. In doing that, we say that a facet F_i of K *contributes to the k-th coefficient of* $h(K,t)$ if $c(i) = k$.

So, if we could find a shellable primitive triangulation T of nC^d, we would have, via Theorem 3.23, a topological interpretation of the coefficients of $h^*(T,t)$. On the other hand, we know how to interpret these coefficients in terms of indexed permutations, partitioned by number of descents, since $h^*(T,t) = D_d^n(t)$. We might therefore hope to be able to construct a bijection between the facets of T and the elements of S_d^n that sent an indexed permutation with k descents to a facet of T which (in an appropriate shelling) contributed to the k-th coordinate of $h(K)$. The problem is, of course, to find the right triangulation of nC^d and then to find the right shelling of that triangulation.

It is not clear, a priori, whether it is easier to find the bijection desired by looking at descents or excedances. There is, however, a reason to believe that descents are the way to go. That reason can be be explained by Lemma 3.29.

Let σ be a simplex. In what follows we will, by abuse of notation, also let σ denote the complex consisting of σ and all its faces and, in case σ has a *geometric realization* in the euclidean space $\mathbf{R}^d$, the subspace of $\mathbf{R}^d$ realizing σ.

Definition 3.26 *Let C^d be the standard unit d-cube. For each permutation word $\pi = a_1a_2\ldots a_d$ in S_d, let $\sigma_\pi = \{\mathbf{x} = (x_1, x_2, \ldots, x_d) \in C^d | 1 \geq x_{a_1} \geq x_{a_2} \geq \cdots \geq x_{a_d} \geq 0\}$. We call σ_π the* path simplex *defined by* π.

The reason for calling them *path*-simplices is that if $\pi = a_1a_2\ldots a_d$ then σ_π can be defined as the convex hull of the path traveling through vertices $\mathbf{0}$, $\mathbf{e}_{a_1}$, $\mathbf{e}_{a_1} + \mathbf{e}_{a_2}$, $\ldots$, $\mathbf{e}_{a_1} + \mathbf{e}_{a_2} + \cdots + \mathbf{e}_{a_d}$, where $\mathbf{e}_i$ is the i-th standard basis vector in $\mathbf{R}^d$.

The collection $\{\sigma_\pi | \pi \in S_d\}$ of path-simplices induces a simplicial subdivision of the unit d-cube C^d. Namely, their union covers C^d and the intersection of any two of the path-simplices is a face of each one, as we point out now.

Remark 3.27 *Let $\pi = a_1a_2\ldots a_d$, so that $\sigma_\pi = \{\mathbf{x} = (x_1, x_2, \ldots, x_d) \in C^d | 1 \geq x_{a_1} \geq x_{a_2} \geq \cdots \geq x_{a_d} \geq 0\}$ is a path-simplex. A k-dimensional face of σ_π is defined by replacing $d-k$ of the $\geq$'s by $=$'s, i.e. by replacing $d-k$ of the linear inequalities defining σ_π by their boundary equalities.*

For example, the 2-faces of $\sigma_{213} = \{\mathbf{x} = (x_1, x_2, x_3) \in C^3 | 1 \geq x_2 \geq x_1 \geq x_3 \geq 0\}$ are defined by

$$\{\mathbf{x} \in C^3 \mid 1 = x_2 \geq x_1 \geq x_3 \geq 0\},$$
$$\{\mathbf{x} \in C^3 \mid 1 \geq x_2 = x_1 \geq x_3 \geq 0\},$$
$$\{\mathbf{x} \in C^3 \mid 1 \geq x_2 \geq x_1 = x_3 \geq 0\},$$
$$\{\mathbf{x} \in C^3 \mid 1 \geq x_2 \geq x_1 \geq x_3 = 0\}.$$

The following lemma is a straightforward consequence of Remark 3.27.

Lemma 3.28 *Let $\pi = a_1a_2\ldots a_d$ and $\tau = b_1b_2\ldots b_d$. The intersection $\sigma_\pi \cap \sigma_\tau$ of the path simplices σ_π and σ_τ can be described as follows: Let i_1 be the least positive integer such that $a_{i_1} \neq b_{i_1}$. Let i_1' be the least integer greater than i_1 such that $T_1 = \{a_i, a_{i+1}, \ldots, a_{i'}\} = \{b_i, b_{i+1}, \ldots, b_{i'}\}$. Let i_2 be the least integer greater than i_1' such that $a_{i_2} \neq b_{i_2}$ and define T_2 similarly. Continue this way until we have come to the end of π and τ and have a collection of sets $T_1, T_2, \ldots, T_k$ obtained in the process. Then $\sigma_\pi \cap \sigma_\tau = \sigma_\pi \cap X$ where $X = \{\mathbf{x} | x_{a_i} = x_{a_j}$ if $a_i, a_j \in T_m$ for some $m\}$. In particular, two path-simplices intersect maximally if and only if their corresponding permutations differ by a single transposition $\ldots a_ia_{i+1}\ldots \rightarrow \ldots a_{i+1}a_i\ldots$ of adjacent letters.* □

In other words, $\sigma_\pi \cap \sigma_\tau$ can be obtained by converting to $=$ those $\geq$'s in $\{\mathbf{x} = (x_1, x_2, \ldots, x_d) \in C^d | 1 \geq x_{a_1} \geq x_{a_2} \geq \cdots \geq x_{a_d} \geq 0\}$ that stand between x_{a_i} and x_{a_j} where a_i and a_j belong to the same T_m. For example, let $\pi = 24375168$ and $\tau = 24537186$. Then $\sigma_\pi \cap \sigma_\tau = \{\mathbf{x} \in C^8 | 1 \geq x_2 \geq x_4 \geq x_3 = x_7 = x_5 \geq x_1 \geq x_6 = x_8 \geq 0\}$.

Lemma 3.29 *Let K_d be the collection $\{\sigma_\pi | \pi \in S_d\}$ of path-simplices which triangulate the unit d-cube. Order the simplices in K_d by the lexicographic ordering of their corresponding permutation words. This ordering is a shelling of the unit d-cube.*

Proof: Let B_d be the Boolean algebra on d elements. Then K_d is the *order complex* of B_d and the lemma is just a special case of *lexicographic shellability* (see [4]). However, for the sake of completeness, we will prove the lemma directly.

The case when $d = 1$ is trivial, since then there is only one facet. So we may assume $d \geq 2$. Suppose π and τ are two permutation words with $\pi > \tau$ in the lexicographic ordering. Let i be the first place in which π and τ differ and set $\pi = a_1a_2\ldots a_d$ and $\tau = a_1a_2\ldots a_{i-1}b_i\ldots b_d$. Since $\pi > \tau$ we must have $a_i > b_i$. Let j be the first place after i such that $\{a_i, a_{i+1}, \ldots, a_j\} = \{b_i, b_{i+1}, \ldots, b_j\}$. Then, by Lemma 3.28, we know that $x_{a_i} = x_{a_{i+1}} = \cdots = x_{a_j}$ for all $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \sigma_\pi \cap \sigma_\tau$. Let k be the least integer in $\{i, i+1, \ldots, j-1\}$ such that k is a descent in π. Such a k must exist, for otherwise we would have $a_i < a_{i+1} < \cdots < a_j$ and we couldn't have

$\pi > \tau$. Define π' by $\pi' = a_1a_2\ldots a_{k-1}a_{k+1}a_ka_{k+2}\ldots a_d$. Then $\pi' < \pi$ and we claim that $\sigma_\pi \cap \sigma_\tau \subset \sigma_\pi \cap \sigma_{\pi'}$. To see that, observe that $\sigma_\pi \cap \sigma_{\pi'}$ is the $(d\text{-}1)$-face of π defined by setting $x_{a_k} = x_{a_{k+1}}$ and clearly $\sigma_\pi \cap \sigma_\tau$ is contained in this face of π, since $x_{a_k} = x_{a_{k+1}}$ for all $x \in \sigma_\pi \cap \sigma_\tau$. Thus, for any $\tau < \pi$, $\sigma_\pi \cap \sigma_\tau$ is contained in some $(d\text{-}1)$-face of $\sigma_\pi \cap \bigcup_{\pi'<\pi} \sigma_{\pi'}$, which completes the proof. □

3.6 The triangulation and shelling of nC^d

We will now construct a triangulation $\widehat{nC^d}$ of nC^d and then shell that triangulation. The shelling will give rise to a bijection associating an indexed permutation in S_d^n with k descents to a facet of $\widehat{nC^d}$ that contributes to the k-th coordinate of $h(nC^d)$ when the h-vector is computed from the shelling.

Embed nC^d in $\mathbf{R}^d$ so that the coordinates of its vertices are all d-tuples which consist of only 0's and n's. That is, nC^d is the image of the standard unit d-cube under the map $f : \mathbf{R}^d \to \mathbf{R}^d$ defined by $f(x) = nx$. Subdivide nC^d into n^d cubes of volume 1 in the obvious way, i.e. given any vector $\mathbf{v} = (v_1, v_2, \ldots, v_d)$ such that $v_i \in \{0, 1, \ldots, n-1\}$, we obtain a unique d-cube contained in nC^d by translating the standard unit d-cube by this vector. We label each of these cubes with the corresponding vector, so that the standard unit cube is $c_\mathbf{0}$ and $c_\mathbf{v} = c_\mathbf{0} + \mathbf{v}$. Subdivide $c_\mathbf{0}$ into the path-simplices defined in 3.26. This induces a simplicial subdivision of $c_\mathbf{0}$. The other cubes are subdivided in an analogous way, so that a triangulation of a cube labeled with $\mathbf{v}$ coincides with the translation by $\mathbf{v}$ of the triangulated standard unit cube. This induces a simplicial subdivision of nC^d which we call $\widehat{nC^d}$.

To order the simplices of $\widehat{nC^d}$ we proceed as follows: A facet σ of the cube $c_\mathbf{0}$ is labeled by $\pi \times \mathbf{0}$ where π is the permutation defining σ (cf. 3.26). For $\mathbf{z} \neq \mathbf{0}$, if σ is a facet in the cube $c_\mathbf{z}$ and $\sigma = \sigma_{\pi\times\mathbf{0}} + \mathbf{z}$ (i.e. σ is the translation by $\mathbf{z}$ of the path-simplex defined by π), then σ is labeled by $\pi \times \pi(\mathbf{z})$. Note that by permuting the coordinates of $\mathbf{z}$ in this way, so that the i-th coordinate of $\mathbf{z}$ follows i, we are actually labeling the facets of the cube $c_\mathbf{z}$ by all the permutation words on the letters $1_{z_1}, 2_{z_2}, \ldots, d_{z_d}$, that is, by the elements of $S_\mathbf{z}$ defined in 3.7.

Let $<$ denote the lexicographic ordering of vectors of the same length. That is, if $\mathbf{z} = (z_1, z_2, \ldots, z_d)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_d)$, then $\mathbf{z} < \mathbf{w}$ if and only if $z_i < w_i$ for the first i at which $\mathbf{z}$ and $\mathbf{w}$ differ. We now order the facets of $\widehat{nC^d}$ in the following way:

Definition 3.30 *Let $\mathcal{O}$ be the following ordering $<_\mathcal{O}$ of the facets of $\widehat{nC^d}$:*

$\mathcal{O}1)$ If $\mathbf{z} < \mathbf{w}$ then $\sigma_{\pi\times\pi(\mathbf{z})} <_\mathcal{O} \sigma_{\tau\times\tau(\mathbf{w})}$ for all π and τ.

$\mathcal{O}2)$ If $\pi \times \pi(\mathbf{z}) <_L \tau \times \tau(\mathbf{z})$ then $\sigma_{\pi\times\pi(\mathbf{z})} <_\mathcal{O} \sigma_{\tau\times\tau(\mathbf{z})}$, where $<_L$ is as in 3.4.

Thus, a facet in $c_{\mathbf{z}}$ comes before any facet in $c_{\mathbf{w}}$ if $\mathbf{z} < \mathbf{w}$. The ordering of the facets in a single cube $c_{\mathbf{z}}$ is a permutation of the shelling order described in 3.29. Moreover, it is induced by permuting the coordinate axes in $\mathbf{R}^d$. Let us illustrate that by an example. Let $d = 4$, let $\mathbf{z} = (2,0,1,0)$. Then the first few indexed permutations in $S_{\mathbf{z}}$ (in the ordering $<_L$) are $2_0 4_0 3_1 1_2$, $2_0 4_0 1_2 3_1$, $2_0 3_1 4_0 1_2$, $2_0 3_1 1_2 4_0$, $2_0 1_2 4_0 3_1 \ldots$. Let $\pi = 2431$.

Then the first one of these indexed permutations is $\pi \times \pi(\mathbf{z}) = \pi(id) \times \pi(\mathbf{z})$. The second one is $\pi(1243) \times \pi(0,0,2,1)$. In general, the n-th one of these indexed permutations is simply π applied to the n-th permutation in S_4 crossed with an appropriate permutation of $(2,0,1,0)$ (namely the permutation of $(2,0,1,0)$ by $\pi(\pi_n)$, where π_n is the n-th permutation in S_4).

Now, to simplify the notation, let us translate the cube $c_{\mathbf{z}}$ by -$\mathbf{z}$, so that it is embedded in $\mathbf{R}^4$ as the standard unit cube. Then the first facets in our ordering would be defined thus:

$$
\begin{aligned}
&\{\mathbf{x} \in C^4 \mid 1 \geq x_2 \geq x_4 \geq x_3 \geq x_1\},\\
&\{\mathbf{x} \in C^4 \mid 1 \geq x_2 \geq x_4 \geq x_1 \geq x_3\},\\
&\{\mathbf{x} \in C^4 \mid 1 \geq x_2 \geq x_3 \geq x_4 \geq x_1\},\\
&\{\mathbf{x} \in C^4 \mid 1 \geq x_2 \geq x_3 \geq x_1 \geq x_4\},\\
&\{\mathbf{x} \in C^4 \mid 1 \geq x_2 \geq x_1 \geq x_4 \geq x_3\}.
\end{aligned}
$$

If we now map $\mathbf{R}^4 \to \mathbf{R}^4$ by sending the ordered basis $< \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 >$ to the ordered basis $< \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_3, \mathbf{e}_1 >$, then the above list becomes

$$
\begin{aligned}
&\{\mathbf{x} \in C^4 \mid 1 \geq x_1 \geq x_2 \geq x_3 \geq x_4\},\\
&\{\mathbf{x} \in C^4 \mid 1 \geq x_1 \geq x_2 \geq x_4 \geq x_3\},\\
&\{\mathbf{x} \in C^4 \mid 1 \geq x_1 \geq x_3 \geq x_2 \geq x_4\},\\
&\{\mathbf{x} \in C^4 \mid 1 \geq x_1 \geq x_3 \geq x_4 \geq x_2\},\\
&\{\mathbf{x} \in C^4 \mid 1 \geq x_1 \geq x_4 \geq x_2 \geq x_3\},
\end{aligned}
$$

which corresponds to the permutations 1234, 1243, 1324, 1243, 1324, which are just the first five permutations in S_4 in the lexicographic ordering. In general, under this mapping, the triangulation of C^4 by path-simplices is mapped to itself in such a way that a path-simplex which is number n in the shelling order originally defined in Definition 3.29 is sent to a path-simplex which is number n in our current order. That is, by *permuting the coordinate axes* in $\mathbf{R}^4$, we turn the original ordering of the facets into our current ordering. The permutation which does this is, of course, determined solely by $\mathbf{z}$. Since $\mathbf{z} = (2,0,1,0)$, the permutation must be 2431, i.e. the permutation which takes $1_2 2_0 3_1 4_0$ to the lexicographically least indexed permutation in $S_{\mathbf{z}}$. In short, if we *relabel* the coordinate axes in $\mathbf{R}^4$, we turn the original ordering of the facets of C^4 into the ordering of the facets of $c_{\mathbf{z}}$. Hence, this ordering must also be a shelling of the cube in question, because the shelling in Definition 3.29 is clearly independent of how the coordinate axes are labeled. There was, naturally, nothing special about $d = 4$; the argument generalizes in an obvious way and we have proved:

Lemma 3.31 *The restriction of the ordering $\mathcal{O}$ to the facets of a cube $c_{\mathbf{z}}$ in $\widehat{nC^d}$ is a shelling of that cube.* $\square$

For the next lemma, we need the following remark.

Remark 3.32 *Let $\pi = a_1 a_2 \dots a_d$ and let $\mathbf{z} = (z_1, z_2, \dots, z_d)$. The translation $\sigma_\pi + \mathbf{z}$ of σ_π by $\mathbf{z}$ satisfies*

$$\sigma_\pi + \mathbf{z} = \{\mathbf{x} \in \mathbf{R}^d | 1 \geq x_{a_1} - z_{a_1} \geq x_{a_2} - z_{a_2} \geq \cdots \geq x_{a_d} - z_{a_d} \geq 0\}.$$

Lemma 3.33 *Let σ_p be a facet of $c_{\mathbf{z}}$ in $\widehat{nC^d}$ and let $p = \pi \times \pi(\mathbf{z})$ where $\pi = a_1 a_2 \dots a_d$. Then σ_p has two (d-1)-faces which lie on the boundary of $c_{\mathbf{z}}$. These faces are defined by $\sigma_p^0 := \{\mathbf{x} \in C^d | 1 \geq x_{a_1} \geq x_{a_2} \geq \cdots \geq x_{a_d} = 0\} + \mathbf{z}$ and $\sigma_p^1 := \{\mathbf{x} \in C^d | 1 = x_{a_1} \geq x_{a_2} \geq \cdots \geq x_{a_d} \geq 0\} + \mathbf{z}$, respectively. If $z_{a_d} \geq 1$ then σ_p^0 is a (d-1)-face of a facet of the cube $c_{\mathbf{z}-\mathbf{e}_{a_d}} = c_{\mathbf{z}} - \mathbf{e}_{a_d}$. If $z_{a_1} \leq n-2$ then σ_p^1 is a (d-1)-face of a facet of the cube $c_{\mathbf{z}+\mathbf{e}_{a_1}} = c_{\mathbf{z}} + \mathbf{e}_{a_1}$.*

Moreover, the intersection of σ_p with any cube $c_{\mathbf{w}} \neq c_{\mathbf{z}}$ is contained in the union of σ_p^0 and σ_p^1. More specifically, if $\mathbf{w} < \mathbf{z}$ then $\sigma_p \cap c_{\mathbf{w}} \subset \sigma_p^0$ and if $\mathbf{w} > \mathbf{z}$ then $\sigma_p \cap c_{\mathbf{w}} \subset \sigma_p^1$

Proof: Clearly, σ_p^0 and σ_p^1 are (d-1)-faces of σ_p. Since each lies in a hyperplane supporting the cube $c_{\mathbf{z}}$, they must lie on the boundary of $c_{\mathbf{z}}$. Now, if $z_{a_d} \geq 1$ then

$$\sigma_p^0 = \{\mathbf{x} \in \mathbf{R}^d | 1 \geq x_{a_1} - z_{a_1} \geq x_{a_2} - z_{a_2} \geq \cdots \geq x_{a_d} - z_{a_d} = 0\} =$$

$$\{\mathbf{x} \in \mathbf{R}^d | 1 = x_{a_d} - z_{a_d} + 1 \geq x_{a_1} - z_{a_1} \geq x_{a_2} - z_{a_2} \geq \cdots \geq x_{a_{d-1}} - z_{a_{d-1}} \geq 0\} = \sigma_{\pi' \times \pi'(\mathbf{z}')}$$

where $\pi' = a_d a_1 a_2 \dots a_{d-1}$ and $\mathbf{z}' = \mathbf{z} - \mathbf{e}_{a_d}$, so $\sigma_{\pi' \times \pi'(\mathbf{z}')} \subset c_{\mathbf{z}'}$. Similar reasoning shows that if $z_{a_1} \leq n-2$ then $\sigma_p^1 = \sigma_r^0$ where $r = \pi'' \times \pi''(\mathbf{z} + \mathbf{e}_{a_1})$ and $\pi'' = a_2 a_3 \dots a_d a_1$. To show that $\sigma_p \cap c_{\mathbf{w}} \subset \sigma_p^0 \cup \sigma_p^1$ for any $\mathbf{w} \neq \mathbf{z}$, observe that a point $\mathbf{x}_0 = (x_1, x_2, \dots, x_d) \in \sigma_p \cap c_{\mathbf{w}}$ must lie on the boundary of $c_{\mathbf{z}}$ and must have $x_i = z_i$ (if $\mathbf{w} < \mathbf{z}$) or $x_i = z_i + 1$ (if $\mathbf{w} > \mathbf{z}$), where i is the first coordinate in which $\mathbf{w}$ and $\mathbf{z}$ differ. Suppose $a_j = i$. Then, if $\mathbf{w} < \mathbf{z}$, $\mathbf{x}_0$ must belong to the set

$$\{\mathbf{x} \in \mathbf{R}^d | 1 \geq x_{a_1} - z_{a_1} \geq x_{a_2} - z_{a_2} \geq \cdots \geq x_{a_j} - z_{a_j} = x_{a_{j+1}} - z_{a_{j+1}} = \cdots = x_{a_d} - z_{a_d} = 0\} \subset \sigma_p^0$$

and, if $\mathbf{w} > \mathbf{z}$, $\mathbf{x_0}$ must belong to the set

$$\{\mathbf{x} \in \mathbf{R}^d | 1 = x_{a_1} - z_{a_1} = x_{a_2} - z_{a_2} = \cdots = x_{a_j} - z_{a_j} \geq x_{a_{j+1}} - z_{a_{j+1}} \geq \cdots \geq x_{a_d} - z_{a_d} \geq 0\} \subset \sigma_p^1,$$

as claimed. □

Proposition 3.34 *The ordering $\mathcal{O}$ defines a shelling of $\widehat{nC^d}$.*

Proof: Let σ_p be a facet of the cube $c_{\mathbf{z}}$ in $\widehat{nC^d}$ with $p = \pi \times \pi(\mathbf{z}) = a_1 a_2 \ldots a_d \times (z_{a_1}, z_{a_2}, \ldots, z_{a_d})$. If $\mathbf{z} = \mathbf{0}$ then we are done, by Lemma 3.29. So assume $\mathbf{z} \neq \mathbf{0}$.

We need to show that $I_p := \sigma_p \cap \bigcup_{q<p} \sigma_q$ is a nonempty union of (d-1)-faces of σ_p (where, by abuse of notation, $q < p$ means $q <_{\mathcal{O}} p$). By the preceding lemma, since the restriction of $\mathcal{O}$ to the cube $c_{\mathbf{z}}$ is a shelling of $c_{\mathbf{z}}$, the intersection $I_{\mathbf{z}}$ of σ_p with those facets in $c_{\mathbf{z}}$ which are prior to σ_p must be a union (possibly empty) of (d-1)-faces of σ_p. If this union is empty, p must be the least indexed permutation in $S_{\mathbf{z}}$, so $z_{a_d} > 0$ since $\mathbf{z} \neq \mathbf{0}$. Hence, σ_p^0 belongs to a facet of the cube $c_{\mathbf{z}'} = c_{\mathbf{z}-\mathbf{e}_{a_d}}$, so $I_p = \sigma_p^0$, a (d-1)-face of σ_p as desired.

If $I_{\mathbf{z}} \neq \emptyset$, then, by Lemma 3.31, $I_{\mathbf{z}}$ is a nonempty union of (d-1)-faces of σ_p, so what remains to be taken into account is how σ_p intersects other small cubes than its own. Obviously, we need only check those cubes $c_{\mathbf{w}}$ for which $\mathbf{w} < \mathbf{z}$. By Lemma 3.33, we need only check how σ_p^0 intersects such small cubes. Now, if $z_{a_d} \neq 0$ then, by Lemma 3.33, $\sigma_p^0 = \sigma_q^1$ for some $q < p$, so I_p is a union of $(d-1)$-faces of σ_p, viz. $I_p = I_z \cup \sigma_p^0$.

Suppose, then, that $z_{a_d} = 0$ and that σ_p^0 intersects $c_{\mathbf{w}}$ where $\mathbf{w} < \mathbf{z}$. Then, for each $i \in [d]$, w_i can differ by at most 1 from z_i. Let i be the first coordinate in which $\mathbf{w}$ and $\mathbf{z}$ differ. Then, since $\mathbf{w} < \mathbf{z}$, we must have $w_i = z_i - 1$. Hence, any point $\mathbf{x_0}$ in $\sigma_p^0 \cap c_{\mathbf{w}}$ must belong to the set

$$\{\mathbf{x} \in \mathbf{R}^d | 1 \geq x_{a_1} - z_{a_1} \geq x_{a_2} - z_{a_2} \geq \cdots \geq x_i - z_i = x_{i+1} - z_{i+1} = \cdots = x_{a_d} - z_{a_d} = 0\}.$$

Let j be such that $z_{a_j} > 0$ and $z_{a_k} = 0$ for all $k > j$. Such a j must exist, since $\mathbf{z} \neq 0$ and $z_{a_d} = 0$. Also, $a_j \geq i$, since $z_i = w_i + 1 \geq 1$. But then

$$\{\mathbf{x} \in \mathbf{R}^d | 1 \geq x_{a_1} - z_{a_1} \geq x_{a_2} - z_{a_2} \geq \cdots \geq x_i - z_i = x_{i+1} - z_{i+1} = \cdots = x_{a_d} - z_{a_d} = 0\} \subset$$

$$\{\mathbf{x} \in \mathbf{R}^d | 1 \geq x_{a_1} - z_{a_1} \geq x_{a_2} - z_{a_2} \geq \cdots \geq x_{a_j} - z_{a_j} = x_{a_{j+1}} - z_{a_{j+1}} \geq \cdots \geq x_{a_d} - z_{a_d} = 0\}.$$

This last set is a (d-1)-face of σ_p and of $\sigma_q = \sigma_{\tau \times \tau(\mathbf{z})}$ where $\tau = a_1 a_2 \ldots a_{j+1} a_j \ldots a_d$, so $\tau(\mathbf{z}) = (z_{a_1}, z_{a_2}, \ldots, z_{a_{j+1}}, z_{a_j}, \ldots, z_{a_d})$. Hence, since $z_{a_j} > 0$ and $z_{a_{j+1}} = 0$, q is prior to p in $\mathcal{O}$, so the (d-1)-face $\sigma_p \cap \sigma_q$ of σ_p is contained in I_p and we have shown that any $\mathbf{x}_0 \in \sigma_p^0 \cap c_{\mathbf{w}}$ lies in this face. Hence, I_p is a union of (d-1)-faces of σ_p and the proof is complete. $\square$

Recall that by Theorem 3.25 we can compute the h-vector of a simplicial complex K from a shelling of K. Namely, if $F_1, F_2, \ldots, F_n$ is a shelling of K and $c(i)$ is as in Theorem 3.25, then $h_k = \#\{i | c(i) = k\}$, where h_k is the k-th coordinate of the h-vector of K. That is, h_k equals the number of facets F_i such that $\bar{F}_i$ intersects $\bigcup_{j=1}^{i-1} \bar{F}_j$ in k distinct faces of dimension $(d-1)$. Now, in the shelling of $\widehat{nC^d}$ the facets in a single cube $C_{\mathbf{z}}$ were ordered so that $\sigma_q = \sigma_{\tau \times \tau(\mathbf{z})}$ was prior to $\sigma_p = \sigma_{\pi \times \pi(\mathbf{z})}$ if and only if $q <_L p$ in the lexicographic ordering of indexed permutations. Also, by Lemma 3.28, σ_p intersects σ_q maximally if and only if π and τ (hence p and q) differ by a single transposition. Suppose now that σ_p and σ_q intersect maximally. Then, if $p = a_{1_{z_1}} a_{2_{z_2}} \ldots a_{d_{z_d}}$, we must have $q = a_{1_{z_1}} a_{2_{z_2}} \ldots a_{k+1_{z_{k+1}}} a_{k_{z_k}} \ldots a_{d_{z_d}}$ for some $k \in [d-1]$. If σ_q is prior to σ_p then we must have that $a_{k+1_{z_{k+1}}} <_\ell a_{k_{z_k}}$ and hence that k constituted a descent in p. Conversely, every *internal* descent k (i.e. $k \in [d-1]$) in p corresponds to a facet σ_s in $c_{\mathbf{z}}$ which intersects σ_p maximally and for which $s <_L p$. That is, there is a one-to-one correspondence between internal descents in p and facets in $c_{\mathbf{z}}$ which are prior to σ_p and which intersect σ_p maximally. The only other facets of $\widehat{nC^d}$ which σ_p intersects maximally are those which contain σ_p^0 and σ_p^1. A facet containing σ_p^1 must come after σ_p. A facet containing σ_p^0 must be prior to σ_p and belong to the cube $c_{\mathbf{z}-\mathbf{e}_{a_d}}$, which exists in $\widehat{nC^d}$ *if and only if* $z_{a_d} > 0$, i.e. if and only if d is a descent in p. Hence, the number of descents in p equals the number of facets in $\widehat{nC^d}$ which are prior to σ_p and which intersect σ_p maximally. This number must equal the number of $(d-1)$-faces in $\sigma_p \cap \bigcup_{q<p} \sigma_q$, because nC^d is a manifold with boundary, so a $(d-1)$-face can belong to at most two facets. We have proved (what we already knew):

Theorem 3.35 *For all $d \geq 0$ and for all $n \geq 1$, $D_d^n(t) = h(\widehat{nC^d}, t)$.* $\square$

3.7 Alternating permutations

In the classical case of the symmetric group, a permutation $\pi = a_1 a_2 \ldots a_d \in \mathcal{S}_d$ is said to be *alternating* if it has descent set $D(\pi) = \{1, 3, 5, \ldots, d-1\}$ for d even and $D(\pi) = \{1, 3, 5, \ldots, d-2\}$ for d odd, so that $a_1 > a_2 < a_3 > \ldots$. A permutation is *reverse alternating* if $a_1 < a_2 > a_3 < \ldots$. There is a one-to-one correspondence between alternating and reverse alternating permutations, viz. $a_1 a_2 \ldots a_d \to b_1 b_2 \ldots b_d$ where $b_i = d + 1 - a_i$. The number E_d of alternating permutations in $\mathcal{S}_d$ is called an *Euler number* and there is a remarkable formula, due to André [1], related to these. Namely, we have $\sum_{d \geq 0} E_d \frac{x^d}{d!} = \tan(x) + \sec(x)$.

It seems that to generalize the definition of alternating permutation to our S_d^n, one ought to consider the descent/ascent at d, and we will do this later. However, such a definition isn't altogether satisfying, beause it means that in the case of S_d^1, i.e. essentially the symmetric group $\mathcal{S}_d$, there would be alternating permutations only for even d and reverse alternating only for odd d. Moreover, there is something to be gained from the definition which ignores the descent/ascent at d and thus has the classical case as a specialization.

Definition 3.36 *An indexed permutation $p \in S_d^n$ is* weakly alternating *if, for $i \in [d\text{-}1]$, i is a descent if and only if i is odd.*

Thus, $2_1 3_0 4_2 1_1$ and$2_1 3_0 4_2 1_0$ are both weakly alternating, because we are ignoring the descent/ascent at $d = 4$.

This definition allows us to generalize the mysterious formula of André in a very simple way.

Theorem 3.37 *Let E_d^n be the number of weakly alternating permutations in S_d^n. Then $\sum_{d \geq 0} E_d^n \frac{x^d}{d!} = \tan(nx) + \sec(nx)$.*

Proof: In the classical case, i.e. when $n = 1$, this is well known (see, e.g., page 149 in [21] or page 89 in [10]). When $n > 1$ we proceed as follows. Fix $\mathbf{z} \in \mathbf{Z}_n^d$ and let $S_\mathbf{z}$ be as in Definition 3.7. We claim that the number of weakly alternating permutations in $S_\mathbf{z}$ equals E_d, the number of weakly alternating permutations in $\mathcal{S}_d$ (i.e. alternating in the classical sense). To show that, let $\theta : A_\mathbf{z} = \{1_{z_1}, 2_{z_2}, d_{z_d}\} \to [d]$ be the map which takes the k-th element of $A_\mathbf{z}$ (in the ordering $<_\ell$ used in Definition 3.4) to k. Then, $\theta(a_{i_{z_{a_i}}}) < \theta(a_{j_{z_{a_j}}})$ if and only if $a_{i_{z_{a_i}}} <_\ell a_{j_{z_{a_j}}}$. Define $\Theta : S_\mathbf{z} \to \mathcal{S}_d$ by $\Theta(p) = \Theta(a_{1_{z_1}} a_{2_{z_2}} \dots a_{d_{z_d}}) = \theta(a_{1_{z_1}})\theta(a_{2_{z_2}}) \dots \theta(a_{d_{z_d}})$. Thus i is a descent in p if and only if i is a descent in $\Theta(p)$, and conversely, except when $i = d$. Hence, since we are ignoring the descent/ascent at d, p is weakly alternating if and only if $\Theta(p)$ is weakly alternating, which proves our claim. Now, S_d^n is the disjoint union of $S_\mathbf{z}$'s for all $\mathbf{z} \in \mathbf{Z}_n^d$. There are n^d such $\mathbf{z}$'s, so the number of weakly alternating indexed permutations in S_d^n is $E_d^n = n^d E_d$. Consequently, $\sum_{d \geq 0} E_d^n \frac{x^d}{d!} = \sum_{d \geq 0} n^d E_d \frac{x^d}{d!} = \sum_{d \geq 0} E_d \frac{(nx)^d}{d!} = \tan(nx) + \sec(nx)$. □

In light of André's theorem, the Euler number E_d is called a *tangent number* or a *secant number*, according as d is odd or even. An interesting formula relating Euler numbers to the Eulerian polynomials states that $E_{2d+1} = (-1)^{d+1} A_{2d+1}(-1)$ (where $A_d(t)$ is the d-th Eulerian polynomial) or, in terms of our descent polynomials, $E_{2d+1} = (-1)^d D_{2d+1}^1(-1)$. We can generalize this to the hyperoctahedral group, i.e. the case $n = 2$.

Theorem 3.38 *Let E_d^2 be the number of weakly alternating permutations in S_d^2. Then $E_{2d}^2 = (-1)^d D_{2d}^2(-1)$.*

Proof: By Theorem 3.21, $\sum_{d\geq 0} D_d^2(t)\frac{x^d}{d!} = \frac{(1-t)e^{x(1-t)}}{1-te^{2x(1-t)}}$. Substitute -1 for t to get
$\sum_{d\geq 0} D_d^2(-1)\frac{x^d}{d!} = \frac{2e^{2x}}{1+e^{4x}}$. Hence, if $i=\sqrt{-1}$, we have

$$\sum_{d\geq 0} D_d^2(-1)\frac{(ix)^d}{d!} = \frac{2e^{2ix}}{1+e^{4ix}} = \frac{2}{e^{-2ix}+e^{2ix}} = \frac{1}{\cos(2x)} = \sec(2x).$$

But, since $D_d^2(t)$ is symmetric, with $D(d,2,k) = D(d,2,d\text{-}k)$, we have $D_{2d+1}^2(-1) = 0$, so $\sum_{d\geq 0}(-1)^d D_{2d}^2(-1)\frac{x^{2d}}{(2d)!} = \sec(2x)$. Comparing this with Theorem 3.37 (and the Taylor expansion of $\sec x$ and $\tan x$ at 0) yields the theorem. □

We now turn to a new definition of alternating indexed permutations.

Definition 3.39 *An indexed permutation $p \in S_d^n$ is* alternating *if, for $i \in [d]$, i is a descent if and only if i is even. p is* reverse alternating *if, for $i \in [d]$, i is a descent if and only if i is odd.*

Note that this actually interchanges the definitions from the classical case. The reason for doing so is aesthetic (or else due to the quirkiness of the author). Namely, alternating and reverse alternating indexed permutations are *not* in general equinumerous (although they do satisfy a certain duality property) and the generating function for alternating indexed permutations is simpler than the one for reverse alternating ones. Whence it seems that we are justified in doing this and we might say, if we didn't care to disguise our immodesty, that we are simply correcting an error caused by a historical accident.

In the hyperoctahedral group, alternating and reverse alternating permutations are in a bijective correspondence. Namely, if $a_{1_{z_1}}a_{2_{z_2}}\ldots a_{d_{z_d}} \in S_d^2$ is alternating then $b_{1_{w_1}}b_{2_{w_2}}\ldots b_{d_{w_d}} \in S_d^2$ defined by $b_i = d+1-a_i$ and $z_i = 1-w_i$ is reverse alternating. For alternating permutations in this group, Reiner [16] has computed the generating function $\sum_{d\geq 0} E_d^2\frac{x^d}{d!} = \frac{\cos x + \sin x}{\cos 2x}$. However, he defines the descent set to be $\{d-1, d-3, \ldots\}$. This makes no difference for the hyperoctahedral group, but for $n > 2$ there are aesthetic reasons to prefer our Definition 3.39. Before this will be evident, we need to make a detour in order to understand the distribution of alternating indexed permutations on a finer scale. That is, we will compute the number of alternating indexed permutations in $S_{\mathbf{z}}$ for any $\mathbf{z} \in \mathbf{Z}_n^d$.

Consider the following triangle, defined by setting $a_0^0 = 1$ and, in general, $a_d^k = \sum_{i=k}^{d-1} a_{d-1}^i$ for d even and $a_d^k = \sum_{i=0}^{k-1} a_{d-1}^i$ for d odd. The first line is number 0 and a_d^k is the entry number k from the right in line d, where the rightmost entry in line d is a_d^0.

$$\begin{array}{c}
1 \\
1 \quad 0 \\
0 \quad 1 \quad 1 \\
2 \quad 2 \quad 1 \quad 0 \\
0 \quad 2 \quad 4 \quad 5 \quad 5 \\
16 \quad 16 \quad 14 \quad 10 \quad 5 \quad 0 \\
0 \quad 16 \quad 32 \quad 46 \quad 56 \quad 61 \quad 61 \\
272 \quad 272 \quad 256 \quad 224 \quad 178 \quad 122 \quad 61 \quad 0 \\
\vdots
\end{array}$$

This triangle appears in [2], where it is called the *Bernoulli-Euler triangle.* We will show shortly that the numbers on the diagonal edges of the triangle are the Euler numbers. In [2], Arnold states that each line in the triangle defines *finite mass distributions* and he shows, among other things, that the Euler number E_d is the number of *maximal morsifications* of the function x^{d+1}.

Theorem 3.40 *Suppose* $\mathbf{z} \in \mathbf{Z}_n^d$ *has* $d-k$ *positive coordinates. Then* a_d^k *is the number of alternating indexed permutations in* $S_{\mathbf{z}}$ *and* a_d^{d-k} *is the number of reverse alternating indexed permutations in* $S_{\mathbf{z}}$.

Proof: It is easy to prove, by induction on d and k, that a_d^k is the number of *zig-zag paths* from a_0^0 to a_d^k, i.e. the number of sequences $a_0^0, a_1^{k_1}, a_2^{k_2}, \ldots, a_d^{k_d} = a_d^k$ such that $k_i < k_{i+1}$ if i is even and $k_i \geq k_{i+1}$ if i is odd. We claim that each such path corresponds to a unique weakly alternating permutation $\pi = a_1 a_2 \ldots a_d$ in S_d such that $a_d \geq k+1$ if d is even and $a_d < k+1$ if d is odd. [2] Namely, given such a path, define π recursively by setting $a_d = k_{d-1} + 1$ (so $a_d \geq k+1 = k_d + 1$ if d is even and $a_d < k+1$ if d is odd) and, in general, let a_{i+1} be the $(k_i + 1) - th$ largest element in the set $[d] \setminus \{a_d, a_{d-1}, \ldots, a_{i+1}\}$. As an example, the path $a_0^0, a_1^1, a_2^0, a_3^2, a_4^2$ gives rise to the permutation 2413. What remains is to show that a zig-zag path gives rise to an *alternating* permutation and vice versa. Assume that d is even. Then $k_{d-1} \geq k_{d-2}$ so clearly $a_d > a_{d-1}$, as desired. In general, suppose i is even with $1 \leq i < d$. Then $k_i \neq i$ because that would force $k_{i-1} < k_i$, contrary to assumption. Hence, the $(k_i + 1) - th$ element of $[d] \setminus \{a_d, a_{d-1}, \ldots, a_{i+1}\}$ is not the largest element of the set. Thus, since k_{i-1} is at least k_i, we have $a_d > a_{d+1}$ as claimed, because the $(k_i + 1) - th$

[2]This is actually proved in [2], but in an indirect way.

element of $[d] \setminus \{a_d, a_{d-1}, \ldots, a_{i+1}, a_i\}$ must be larger than the $(k_i+1)-th$ element of $[d] \setminus \{a_d, a_{d-1}, \ldots, a_{i+1}\}$. If i is odd then $k_{i-1} < k_i$, so $a_i < a_{i+1}$ since the $(k_{i-1}+1)-th$ element of $[d] \setminus \{a_d, a_{d-1}, \ldots, a_{i+1}, a_i\}$ must be smaller than the $(k_i+1)-th$ element of $[d] \setminus \{a_d, a_{d-1}, \ldots, a_{i+1}\}$. Reversing the above procedure to obtain a zig-zag path given an alternating permutation is straightforward and will be omitted.

We show next that if d is even then the number of weakly alternating permutations in $\mathcal{S}_d$ whose last letter is greater than or equal to $k+1$ is equal to the number of alternating indexed permutations in $S_{\mathbf{z}}$ if $\mathbf{z}$ has exactly $d-k$ positive coordinates. Apply the bijection Γ in Lemma 3.8 to map $S_{\mathbf{z}}$ to $S_{\mathbf{w}}$ where $\mathbf{w} = (0, 0, \ldots, 0, 1, 1, \ldots, 1)$ has its last $d-k$ coordinates equal to 1 and the first k equal to 0. Then, if $\pi = a_1 a_2 \ldots a_d \in \mathcal{S}_d$ is alternating and $a_d \geq k$, $p = a_{1_{w_{a_1}}} a_{2_{w_{a_2}}} \ldots a_{d_{w_{a_d}}}$ must be alternating because $w_i \geq w_{i-1}$ for all $i \in [d]$ so $a_{i_{w_{a_i}}} <_\ell a_{i+1_{w_{a_{i+1}}}}$ if and only if $a_i < a_{i+1}$ and, since $a_d \geq k+1$, $w_{a_d} = 1$, so d is a descent in p as required. When d is odd, a similar argument shows that p is alternating if and only if π is.

The case for reverse alternating permutations is similar, but it can also be proved by noting that if $p = a_{1_{w_{a_1}}} a_{2_{w_{a_2}}} \ldots a_{d_{w_{a_d}}} \in S_{\mathbf{w}}$ is alternating with $\mathbf{w}$ as before, then $p = b_{1_{v_{b_1}}} b_{2_{v_{b_2}}} \ldots b_{d_{v_{b_d}}} \in S_{\mathbf{v}}$ defined by $b_i = d+1-a_i$ and $v_i = 1 - w_i$ is reverse alternating and $\mathbf{v}$ has exactly k positive coordinates. □

Porism 3.41 *If d is even then $a_d^0 = E_d$ and if d is odd then $a_d^d = E_d$, where E_d is the d-th Euler number.* □

One can derive several recurrence relations between the entries in the *BE*-triangle, but there is a particular one which we will need. If we cut off the first $d+1$ lines of the triangle and turn this initial segment upside down, then we can express the entries a_d^k in the top line as a polynomial in k. Let us say that we take the first 5 lines and turn them upside down. If we then change the sign of every entry in lines 3 and 4 from the top, we get the following triangle

$$\begin{array}{ccccccccc}
5 & & 5 & & 4 & & 2 & & 0 \\
& 0 & & 1 & & 2 & & 2 & \\
& & -1 & & -1 & & 0 & & \\
& & & 0 & & -1 & & & \\
& & & & 1 & & & &
\end{array}$$

which constitutes a *difference table*, i.e. each entry is the difference between the entries just above it. More precisely, if we have $\begin{smallmatrix} a & & b \\ & c & \end{smallmatrix}$ then $c = b - a$. This yields a formula for the entries a_4^k now sitting in the top line: $a_4^k = 5 + 0\binom{k}{1} - 1\binom{k}{2} - 0\binom{k}{3} + 1\binom{k}{4}$. In general (see, e.g., [21], Proposition 1.4.2), the entries on the far left diagonal constitute the coefficients of a polynomial in k *in the basis* $\{\binom{k}{i} | i \in \mathbb{N}\}$. Making use of the fact that every other entry on this diagonal is 0 we get the following result.

Lemma 3.42

$$a_{2d}^k = \sum_{i=0}^{d} (-1)^i \binom{k}{2i} a_{2d-2i}^0 \qquad \textit{and} \qquad a_{2d+1}^k = \sum_{i=0}^{d} (-1)^i \binom{k}{2i+1} a_{2d-2i}^0 . \qquad \square$$

Note that this expresses a_d^k in terms of the Euler numbers, since $a_{2d}^0 = E_{2d}$ by Porism 3.41.

Theorem 3.43 *Let A_d^n be the number of alternating indexed permutations in S_d^n and R_d^n the number of reverse alternating such. Then*

$$\sum_{d\geq 0} A_d^n \frac{x^d}{d!} = \frac{\cos x + \sin x}{\cos(nx)} \qquad \textit{and} \qquad \sum_{d\geq 0} R_d^n \frac{x^d}{d!} = \frac{\cos((n\text{-}1)x) + \sin((n\text{-}1)x)}{\cos(nx)}.$$

Proof: Because $\frac{1}{\cos(nx)}$ has only terms of even degree, the theorem claims, among other things, that $\sum_{d\geq 0} A_{2d}^n \frac{x^{2d}}{(2d)!} = \frac{\cos x}{\cos(nx)}$. We will prove this. The other three cases are similar.

By the proof of Theorem 3.37 and Porism 3.41, $\sec(nx) = \sum_{d\geq 0} n^{2d} a_{2d}^0 \frac{x^{2d}}{(2d)!}$, so $\frac{\cos x}{\cos(nx)} = \sum_{d\geq 0} \left(\sum_{k=0}^{d} (-1)^{d-k} \binom{2d}{2k} n^{2k} a_{2k}^0 \right) \frac{x^{2d}}{(2d)!}$.

Also, $A_{2d}^n = \sum_{k=0}^{2d} \binom{2d}{k} (n-1)^{2d-k} a_{2d}^k$, because a_{2d}^k is the number of alternating permutations in $S_{\mathbf{z}} \subset S_{2d}^n$ if $\mathbf{z}$ has exactly $2d-k$ positive coordinates, and there are exactly $\binom{2d}{k}(n-1)^{2d-k}$ such $\mathbf{z}$. Hence, we need to show

$$\sum_{k=0}^{2d} \binom{2d}{k} (n-1)^{2d-k} a_{2d}^k \quad = \quad \sum_{k=0}^{d} (-1)^{d-k} \binom{2d}{2k} n^{2k} a_{2k}^0 .$$

Let $m = n-1$ and use Lemma 3.42 to obtain

$$\sum_{k=0}^{2d} \binom{2d}{k} m^{2d-k} \sum_{i=0}^{d} (-1)^i \binom{k}{2i} a_{2d-2i}^0 \quad = \quad \sum_{k=0}^{d} (-1)^{d-k} \binom{2d}{2k} a_{2k}^0 \sum_{i=0}^{2k} \binom{2k}{i} m^i . \qquad (3)$$

Clearly, each side of (3) is a polynomial in m, so it suffices to show that the coefficient to m^j is the same on both sides for each j. Let L_j be the coefficient to m^j in the LHS and let R_j be the coefficient to m^j in the RHS. Then we have

$L_{2d-k} = \binom{2d}{k}\sum_{i=0}^{d}(-1)^i\binom{k}{2i}a^0_{2d-2i}$, so $L_j = \binom{2d}{2d-j}\sum_{i=0}^{d}(-1)^i\binom{2d-j}{2i}a^0_{2d-2i}$. Now, using the identity $\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c}$ we get

$$L_j = \sum_{i=0}^{d}(-1)^i\binom{2d}{2i}\binom{2d-2i}{2d-j-2i}a^0_{2d-2i} = \sum_{i=0}^{d}(-1)^i\binom{2d}{2i}\binom{2d-2i}{j}a^0_{2d-2i}\,. \tag{4}$$

As for the right hand side we have

$$R_j = \sum_{k=0}^{d}(-1)^{d-k}\binom{2d}{2k}\binom{2k}{j}a^0_{2k} = \sum_{k=0}^{d}(-1)^k\binom{2d}{2k}\binom{2d-2k}{j}a^0_{2d-2k}\,,$$

which agrees with (4) as desired. □

Theorem 3.43 yields the following result, akin to Theorem 3.38:

Theorem 3.44 $$(-1)^{\lfloor\frac{d+1}{2}\rfloor}D^3_d(-1) = A^3_d.$$

Proof: By Theorem 3.21, $\sum_{d\geq 0} D^3_d(t)\frac{x^d}{d!} = \frac{(1-t)e^{x(1-t)}}{1-te^{3x(1-t)}}$. Substitute -1 for t and let $i = \sqrt{-1}$. Then we have

$$\sum_{d\geq 0} D^3_d(-1)\frac{(ix)^d}{d!} = \frac{2e^{2ix}}{1+e^{6ix}} = \frac{2e^{-ix}}{e^{-3ix}+e^{3ix}} = \frac{\cos x - i\sin x}{\cos(3x)}\,.$$

By Theorem 3.43,

$$\sum_{d\geq 0} A^3_d\frac{(ix)^d}{d!} = \frac{\cos(ix)+\sin(ix)}{\cos(3ix)},$$

so, since $\cos x$ has only even degree terms and $\sin x$ only odd degree terms, our claim is equivalent to saying that the Taylor series of $\frac{\cos x}{\cos(3x)}$ differs from that of $\frac{\cos(ix)}{\cos(3ix)}$ only by the sign changes afforded by $(-1)^{\lfloor\frac{d+1}{2}\rfloor}$ and that the same is true of the series for $\frac{-i\sin x}{\cos(3x)}$ and $\frac{\sin(ix)}{\cos(3ix)}$.

Since $\frac{\cos x}{\cos(3x)}$ has only even degree terms, we can write $\frac{\cos x}{\cos(3x)} = \sum_{d\geq 0} a_d\frac{x^{2d}}{(2d)!}$, so $\frac{\cos(ix)}{\cos(3ix)} = \sum_{d\geq 0} a_d\frac{(ix)^{2d}}{(2d)!} = \sum_{d\geq 0}(-1)^d a_d\frac{x^{2d}}{(2d)!} = \sum_{d\geq 0}(-1)^{\lfloor\frac{2d+1}{2}\rfloor}a_d\frac{x^{2d}}{(2d)!}$, as claimed. The other case is similar. □

It is clear why the approach employed in the preceding theorem can't yield any similar results for $n \neq 3$. Why no such results are true, or what modifications might yield similar results, remains a mystery, which perhaps could be solved by finding a "bijective" proof of Theorem 3.44.

3.8 Major index and inversions

Apart from descents and excedances, there are two other statistics of the elements of the symmetric group $\mathcal{S}_d$ that have been extensively studied. These are the *inversion index* and the *major index* of $\pi \in \mathcal{S}_d$. An inversion in $\pi = a_1 a_2 \dots a_d$ is a pair (i, j) such that $i < j$ and $a_i > a_j$. The inversion index $\mathrm{inv}(\pi)$ of π is the number of inversions in π. The major index $\mathrm{maj}(\pi)$ of π is the sum of the elements of the descent set $D(\pi)$ of π.

Foata [9] has constructed a bijection $\phi : \mathcal{S}_d \to \mathcal{S}_d$ such that $\mathrm{maj}(\pi) = \mathrm{inv}(\phi(\pi))$, which shows that maj and inv are equidistributed over $\mathcal{S}_d$. A nice description of ϕ can be found in [6].

By definition, Foata's bijection ϕ has the property that if $\pi = a_1 a_2 \dots a_d$ and $\phi(\pi) = b_1 b_2 \dots b_d$, then $a_d = b_d$. Hence the following.

Remark 3.45 *Let $k \in [d]$ and let $A_{d,k} = \{\pi = a_1 a_2 \dots a_d \in \mathcal{S}_d | a_d = k\}$. Then*

$$\sum_{\pi \in A_{d,k}} t^{\mathrm{maj}(\pi)} = \sum_{\pi \in A_{d,k}} t^{\mathrm{inv}(\pi)}.$$

Definition 3.46 *For $p \in S_d^n$, the* major index of p *is* $\mathrm{maj}(p) = \sum_{j \in D(p)} j$.

Definition 3.47 *For $p = a_{1_{z_1}} a_{2_{z_2}} \dots a_{d_{z_d}} \in S_d^n$, an* inversion *in p is a pair (i, j) such that $1 \le i < j \le d+1$ and $a_{j_{z_j}} <_\ell a_{i_{z_i}}$. Let $I(p) = \{(i, j) | (i, j)$ is an inversion in $p\}$. Then* $\mathrm{inv}(p) = \#I(p)$ *is the* inversion index of p.

Note that this differs from the classical definition in that we consider an indexed permutation in S_d^n to have $a_{d+1} = d+1$ so $(i, d+1)$ is an inversion for any i such that $z_i > 0$. For example, $2_0 3_1 1_0$ has three inversions, namely $(1, 3)$, $(2, 3)$, and $(2, 4)$.

In the symmetric group $\mathcal{S}_d$, the number of inversions in $\pi \in \mathcal{S}_d$ equals the length $\ell(\pi)$ as defined in section 3.2. This is not the case in S_d^n for $n \ge 2$, where length is generally larger than inv. Reiner [16] has obtained a certain relation between maj and length in S_d^n (via S_d^2). There is, however, a much more elegant relation between maj and inv.

Theorem 3.48 *For any $\mathbf{z} \in \mathbf{Z}_n^d$,*

$$\sum_{p \in S_{\mathbf{z}}} t^{\mathrm{maj}(p)} = \sum_{p \in S_{\mathbf{z}}} t^{\mathrm{inv}(p)}.$$

Proof: Suppose $\mathbf{z}$ has $z_i = 0$ for exactly $k-1$ values of i. Let $\theta : \{i_{z_i} | i \in [d]\} \to \{1, 2, \ldots, k-1, k+1, \ldots, d+1\}$ be the bijection which takes the i-th element of $\{i_{z_i} | i \in [d]\}$ (in the ordering $<_\ell$) to the i-th element of $\{1, 2, \ldots, k-1, k+1, \ldots, d+1\}$. That is, the i-th element of $\{i_{z_i} | i \in [d]\}$ goes to i if $i < k$ and to $i+1$ if $i \geq k$. Let $A_{d,k}$ be as in Remark 3.45 and define $\Theta : S_{\mathbf{z}} \to A_{d+1,k}$ by $\Theta(a_{1z_1} a_{2z_2} \ldots a_{dz_d}) = \theta(a_{1z_1})\theta(a_{2z_2}) \ldots \theta(a_{dz_d})$. It follows that i is a descent in p *iff* i is a descent in $\Theta(p)$ and that (i, j) is an inversion in p *iff* (i, j) is an inversion in $\Theta(p)$. Hence, $\sum_{p \in S_{\mathbf{z}}} t^{\mathrm{maj}(p)} = \sum_{\pi \in A_{d+1,k}} t^{\mathrm{maj}(\pi)}$ and $\sum_{p \in S_{\mathbf{z}}} t^{\mathrm{inv}(p)} = \sum_{\pi \in A_{d+1,k}} t^{\mathrm{inv}(\pi)}$. By Remark 3.45, this implies the desired result. $\square$

Corollary 3.49 inv *and* maj *are equidistributed over S_d^n, i.e.*

$$\sum_{p \in S_d^n} t^{\mathrm{maj}(p)} = \sum_{p \in S_d^n} t^{\mathrm{inv}(p)}.$$

$\square$

4 Poset permutations

4.1 Descents and excedances in self-bijections of posets

In the classical case of the symmetric group $\mathcal{S}_n$, one refers to the order relations among the positive integers $1, 2, \ldots, n$ when defining excedances and descents. It is natural to ask what happens if we remove some of these relations, that is if we replace the *chain* $1 < 2 < \ldots < n$ with an arbitrary poset (partially ordered set) on n elements.

Definition 4.1 *Let* P *be a finite poset with* n *elements. Label the elements of* P *by the integers in* $[n]$*. As a convention, require that if* $x_i > x_j$ *in* P *then* $i > j$*, i.e. the labeling represents a* linear extension *of* P*. Let* $\phi_\pi : P \to P$ *be a bijection of* P *to itself represented by a permutation word* $\pi \in \mathcal{S}_n$*, i.e.* $\phi_\pi(x_i) = x_{\pi(i)}$*.*

A P-excedance *in* π *is an* $i \in [n]$ *such that* $x_{\pi(i)} > x_i$*.*

A P-descent *in* π *is an* $i \in [n]$ *such that* $x_{\pi(i)} > x_{\pi(i+1)}$*.*

When no confusion can arise as to which poset P we are referring to, we will simply talk about *excedances* and *descents.* Note that when P is a chain, P-descents and P-excedances coincide with the classical definitions of excedances and descents in $\mathcal{S}_n$.

Definition 4.2 *Let* $\pi = a_1 a_2 \ldots a_n$ *be a permutation in* $\mathcal{S}_n$*. The* reverse of π*, denoted* π^{rev} *is defined by* $\pi^{\text{rev}} = a_n a_{n-1} \ldots a_1$*.*

Remark 4.3 *Define a* P-ascent *to be the "complement" of a* P*-descent, i.e. a* P*-ascent in* π *is an* i *such that* i *is not a* P*-descent in* π*. Clearly, if* i *is a* P*-descent in* π *then* i *is a* P*-ascent in* π^{rev}*, so* P*-ascents and* P*-descents are equidistributed for any poset* P*, since* $\pi \to \pi^{\text{rev}}$ *defines a bijection* $\mathcal{S}_n \to \mathcal{S}_n$*.*

Definition 4.4 *Let* P *be a finite poset with* n *elements and let* $\pi \in \mathcal{S}_n$*. Let* $d_P(\pi) = \#\{i | i \text{ is a descent in } \pi\}$ *and let* $e_P(\pi) = \#\{i | i \text{ is an excedance in } \pi\}$*. Then* $D_P(t) = \sum_{\pi \in \mathcal{S}_n} t^{d_P(\pi)}$ *is the* descent polynomial *of* P *and* $E_P(t) = \sum_{\pi \in \mathcal{S}_n} t^{e_P(\pi)}$ *is the* excedance polynomial *of* P*.*

What one might hope to show is that $D_P(t) = E_P(t)$, in other words, that excedances and descents are equidistributed for any finite poset P. In that case, a bijective proof would of course be desirable. In [10], Foata and Schützenberger give a bijective proof of the equidistribution of excedances and ascents in the classical case. Looking at this proof, one quickly sees that it will not work for an arbitrary poset.

An example will explain this. Let $\pi = 641253$ and let $\phi : \mathcal{S}_n \to \mathcal{S}_n$ be Foata and Schützenberger's bijection in Theorem 1.15 in [10]. Then $\phi(\pi) = 632145$ and the excedances in π, namely $6 > 1$ and $4 > 2$, are "translated" into the ascents $1 < 4$ and $4 < 5$ in $\phi(\pi)$. Hence, this cannot work for an arbitrary poset and our definition of P-excedances and P-descents/ascents, because we could have, for example, a poset P in which the only order relations were $x_6 > x_1$ and $x_4 > x_2$, so that π had two P-excedances but $\phi(\pi)$ had no P-ascents. This suggests that for a bijection ϕ to work for an arbitrary poset, it would have to translate excedances "verbatim" into descents. That is, if $a_b > b$ is an excedance in π, then $\ldots ab \ldots$ must appear as a descent in $\phi(\pi)$ and conversely.

We will now describe such a bijection.

Let $\pi = \pi_n = a_1 a_2 \ldots a_n$ be a permutation word in $\mathcal{S}_n$. Remove the largest letter, n, from π_n to obtain π_{n-1}. Continue this process until reaching $\pi_m = 12...m = id$. Set $\tau_m = \pi_m$. If π_{m+1} is obtained from π_m by inserting (m+1) at the end of π_m then insert (m+1) at the end of τ_m to obtain τ_{m+1}. Otherwise, if π_{m+1} is obtained from π_m by inserting (m+1) before k in π_m then put (m+1) in the k-th place in τ_m and move what was in that place to the end of τ_m to obtain τ_{m+1}. Continue this process until reaching $\tau = \tau_n$ and define $\phi : \mathcal{S}_n \to \mathcal{S}_n$ by $\phi(\pi) = \tau$. It's easy to see that the number of excedances in τ_k equals the number of descents in π_k for each k, in particular for $k = n$. It is obvious how to reverse this procedure to go from τ_n to π_n. Hence, this describes a bijection $\phi : \mathcal{S}_n \to \mathcal{S}_n$ such that $e(\phi(\pi)) = d(\pi)$.

Example 4.5 Let $\pi = 35142$. Then we reach τ in the following way:

$$\begin{array}{ccccccc} 35142 & \to & 3142 & \to & 312 & \to & 12 \\ & & & & & & \downarrow \\ 54123 & \leftarrow & 3412 & \leftarrow & 321 & \leftarrow & 12 \end{array}$$

In this case, the descents in π consist of $\ldots 54 \ldots$ and $\ldots 63$, and, consequently, the excedances in τ occur at 3 and 4, with 6 in the third place and 5 in the fourth place.

It is clear from the definition of ϕ that ϕ translates descents "verbatim" into excedances, i.e. if $\ldots ab \ldots$ is a descent in π, so $a > b$, then a is in the b-th place in τ and thus constitutes an excedance in τ. Conversely, an excedance in τ derives from a descent in π. Therefore, if $\ldots ab \ldots$ constitutes a P-descent in π for some poset P, a being in the b-th place in τ constitutes a P-excedance in τ and conversely. Thus, we have $d_P(\pi) = e_P(\phi(\pi))$ for any finite poset P and we have shown:

Theorem 4.6 *For any finite poset P, $D_P(t) = E_P(t)$.* □

If we examine the process by which we built $\tau = \phi(\pi)$ from π, we discover the following alternative description of ϕ:

Remark 4.7 *Let $\pi = a_1 a_2 \ldots a_n$ be a permutation word in S_n and let $\phi : S_n \to S_n$ be as above. Then $\tau = \phi(\pi)$ can be constructed from π as follows: Define $a_0 := 0$ and let $k \in [n]$. If a number less than a_k appears after a_k in π then a_k is in place number a_{k+1} in τ. Otherwise, find the first (rightmost) number in π which is left of a_k and which is less than a_k. If this number is a_i then a_k is in place number a_{i+1} in τ. In particular, this means that if $\ldots ab \ldots$ is a descent in π then a is in the b-th place in τ and hence constitutes an excedance in τ.*

Example 4.8 $\phi(35142) = 54123$. 5 goes to the first place and 4 to the second because 51 and 42 are descents in 35142. To place 1, since no number less than 1 appears after 1 we trace back until we hit 0. The successor of 0 is 3 so 1 goes to the third place. 2; trace back to 1, whose successor is 4 so 2 goes to the fourth place. 3 has smaller numbers to its right so 3 goes to the fifth place, 5 being the successor of 3.

The following is a straightforward consequence of Remark 4.7.

Corollary 4.9 *Let $\pi = a_1 a_2 \ldots a_n$ and let $\tau = \phi(\pi) = b_1 b_2 \ldots b_d$. Suppose that τ fixes i, i.e. $b_i = i$, and let k be such that $a_k = i$. Then $a_m > a_k$ for any $m > k$ and $a_{k-1} < a_k$. In particular, if $a_d = i$ then $a_m \neq m$ for any $m > i$.* □

4.2 P-descents and the incomparability graph of P

Let P be a finite poset with d elements and let $IN(P)$ be the *incomparability graph of P*, i.e. $IN(P)$ is the graph whose vertices are the elements $\{x_1, x_2, \ldots, x_d\}$ of P and whose edges are those pairs (x_i, x_j) which are *incomparable*, i.e. for which neither $x_i \leq x_j$ nor $x_j \leq x_i$ holds. Recall that the *chromatic polynomial* of a graph G is defined by $\chi(G, n) :=$ number of n-colorings of G, where an n-coloring is a map $c : G \to [n]$ such that $c(x_i) \neq c(x_j)$ if (x_i, x_j) is an edge in G.

The following theorem, with a proof similar to ours, was independently obtained earlier by Buhler, Eisenbud, Graham and Wright [8].

Theorem 4.10 *Let P be a finite poset on d elements, let $\chi(P, n)$ be the chromatic polynomial of the incomparability graph $IN(P)$ of P and let $C_P(t)$ be the $\chi-$eulerian polynomial. Let D_k be the k-th coefficient of $D_P(t)$ and let $D_P^{\mathrm{rev}}(t) = D_d + D_{d-1}t + D_{d-2}t^2 + \ldots D_0 t^d$. Then $D_P^{\mathrm{rev}}(t) = C_P(t)$.*

Proof: Throughout the proof, a descent will mean a P-descent.

The conclusion of the theorem is that

$$\sum_{n\geq 0} \chi(P,n)t^n = \frac{D_P^{\mathrm{rev}}(t)}{(1-t)^{d+1}},$$

which is equivalent to $\chi(P,n) = \sum_{k=0}^{k=d} \binom{n+k}{d} D_k$. This is equivalent to saying that the number of ways of coloring $IN(P)$ with n colors equals $\sum_{k=0}^{k=d} \binom{n+k}{d} D_k$ where D_k is the number of permutations in $\mathcal{S}_d$ with k descents.

We give a coloring scheme which associates to each permutation with k descents $\binom{n+k}{d} D_k$ distinct colorings of $IN(P)$ with n colors. We also show that each coloring of $IN(P)$ gives rise to exactly one permutation together with certain markers which show that this is a bijective correspondence.

Given n colors, order them linearly once and for all. For example, call them $a, b, c, \ldots$.

Given a permutation $\pi \in \mathcal{S}_d$ with k descents, pick i of the k descents in π and pick $d-k+i$ of the n colors . This can be done in $\binom{k}{i}\binom{n}{d-k+i}$ ways. The descents picked will be *ignored* when we now assign colors to the letters of π. Associate one of the colors chosen to each of the letters of π as follows: The first letter of π gets the smallest of the colors chosen (a if a was picked), the second gets the next smallest of the colors chosen (b if both a and b were picked) and so on, *except* that whenever there is an unignored descent in π, say $\ldots rs \ldots$ where $x_r > x_s$ in P, r and s get the same color. Hence, if there are k descents and i of them are ignored, we will use exactly $d-k+i$ colors. It is clear that if two vertices x_i and x_j of $IN(P)$ receive the same color, then x_i and x_j are comparable in P and hence are not connected by an edge in $IN(P)$. This describes $\binom{k}{i}\binom{n}{d-k+i}$ distinct colorings of $IN(P)$ if the vertex x_i in $IN(P)$ is colored by the color assigned to the letter i in π. Hence, each permutation in $\mathcal{S}_d$ with k descents gives rise to $\sum_{i=0}^{i=k}\binom{k}{i}\binom{n}{d-k+i}$ distinct colorings of $IN(P)$ so all the permutations in $\mathcal{S}_d$ give rise to $\sum_{k=0}^{d-1}\left[\sum_{i=0}^{i=k}\binom{k}{i}\binom{n}{d-k+i}\right] D_k$ distinct colorings of $IN(P)$.

Example: P is described by the cover relations $x_1 < x_2$, $x_3 < x_4$ and $x_2 < x_5$. Then $\pi = 52431$ has two descents, namely $52\ldots$ and $\ldots 43\ldots$. Given colors a, b, c, d, and e, suppose we pick a, c and d and suppose that we ignore the descent $52\ldots$. Then π gives rise to the coloring sequence $abccd$, so vertex x_1 in $IN(P)$ is colored with d, vertex x_2 with b, x_4 and x_3 with c and x_5 with a.

Conversely, given a coloring of $IN(P)$, that coloring describes a permutation π, which colors have been chosen and which descents in π have been ignored, in the following way: Order the letters $1, \ldots, d$ so that all the letters whose vertices were colored a come first, next come those whose vertices were colored b and so on, and so that the letters corresponding to each set of like-colored vertices are in decreasing order (as integers). This describes a permutation $\pi \in \mathcal{S}_d$. Obviously the coloring indicates which colors have been used, and a descent $\ldots a_i a_{i+1} \ldots$ in π was ignored if and only if x_{a_i} and $x_{a_{i+1}}$ received different colors.

Example: Let P be as in the above example and suppose that the vertices x_1, x_2, x_3, x_4, x_5 of $IN(P)$ were colored with b, d, a, a, c, respectively. Then, since x_3 and x_4 received the least of the colors, $\pi = 43\ldots$. Since x_1 was colored with b, $\pi = 431\ldots$. Finally, x_5 was colored with c and x_2 with d, so $\pi = 43152$ and there is precisely one descent ignored, namely $\ldots 52$.

We have shown that there is a bijectiive correspondence between the set of colorings of $IN(P)$ on one hand and the set of permutations with a choice of descents to ignore and a choice of the colors used on the other hand.

To prove the theorem it remains only to show that $\sum_{i=0}^{i=k} \binom{k}{i}\binom{n}{d-k+i} = \binom{n+k}{d}$. This is done in the following lemma. □

Lemma 4.11 $\sum_{i=0}^{i=k} \binom{k}{i}\binom{n}{d-k+i} = \binom{n+k}{d}$.

Proof: The equality is equivalent to $\sum_{i=0}^{i=k} \binom{k}{i}\binom{n}{n-d+k-i} = \binom{n+k}{n+k-d}$. Let $m = k+n$, so we need to show $\sum_{i=0}^{i=k} \binom{k}{i}\binom{m-k}{m-d-i} = \binom{m}{m-d}$. The right hand side counts the number of ways to choose $m-d$ elements from a set of m elements. The left hand side counts the same, by partitioning the set into two sets, one of size k and the other of size $m-k$ and then counting, for each i, the number of ways of first choosing i of the k elements and then choosing $m-d-i$ of the remaining $m-k$ elements. □

Let $C_P(t)$ be as in Theorem 4.10. It is well known (see, for example, [18]) that the leading coefficient of $C_P(t)$ equals the number of *acyclic orientations* of $IN(P)$. We now give a bijective proof of this special case of Theorem 4.10.

Theorem 4.12 *Let P be a poset on d elements and let D_0 be the number of permutations in $\mathcal{S}_d$ with no P-descents. Then D_0 equals the number of acyclic orientations of $IN(P)$.*

Proof: Every permutation word $\pi \in \mathcal{S}_d$ defines an orientation of $IN(P)$ (actually of any graph on d vertices) by orienting $x_a \to x_b$ if a appears before b in π. Clearly such an orientation of $IN(P)$ is acyclic, although two permutations can give rise to the same orientation (unless P is an antichain). Conversely, an acyclic orientation of $IN(P)$ gives rise to a permutation word π(perhaps more than one) by requiring that a precede b in π if there is an edge between x_a and x_b in $IN(P)$ which is oriented

$x_a \to x_b$. Define an equivalence relation $\sim$ on the set of permutation words in S_d by declaring $\pi \sim \tau$ if and only if π and τ give rise to the same orientation of $IN(P)$.

We claim that each equivalence class contains exactly one permutation with no descents. Suppose $\ldots ab \ldots$ is a descent in π. Then $x_a > x_b$ in P so there is no edge between x_a and x_b in $IN(P)$. Hence, transposing ab to get $\ldots ba \ldots$ induces the same orientation of $IN(P)$ as π does. Thus we can eliminate the descents in π, one by one, without affecting the orientation induced by the resulting permutation. Since transposing a descent results in a permutation which is smaller (in the lexicographic ordering), this process must come to an end, at which point the resulting permutation has no descents. So each equivalence class contains at least one permutation with no descents.

To complete the proof, we need only show that two distinct permutations with no descents cannot induce the same orientation. Suppose $\pi = a_1a_2\ldots a_d$ and $\tau = b_1b_2\ldots b_d$ induce the same orientation of $IN(P)$ and that π and τ have no descents. Assume $a_d \neq b_d$. Then $a_d = b_k$ for some $k < d$. Hence, x_{b_k} is comparable to $x_{b_{k+1}}$ for otherwise π would induce the orientation $x_{b_{k+1}} \to x_{b_k} = x_{a_d}$ while τ would induce the orientation $x_{b_k} \to x_{b_{k+1}}$, contrary to assumption. Similarly, x_{b_k} is comparable to $x_{b_{k+2}}$, so $x_{b_k} < x_{b_{k+2}}$ since otherwise $x_{b_{k+2}} < x_{b_k} < x_{b_{k+1}}$, so $\ldots b_{k+1}b_{k+2}\ldots$ would constitute a descent in τ. Clearly, similar reasoning leads to $x_{b_k} < x_{b_{k+3}}$ and, eventually, $x_{b_k} < x_{b_d}$. Hence, $x_{a_d} < x_{b_d}$. By symmetry, we can also show $x_{b_d} < x_{a_d}$, a contradiction, so we must have $x_{a_d} = x_{b_d}$ and thus $a_d = b_d$. But now we can apply the same reasoning to $\pi' = a_1a_2\ldots a_{d-1}$ and $\tau' = b_1b_2\ldots b_{d-1}$ (and the poset $P' = P \setminus \{x_{a_d}\}$) to show that $a_{d-1} = b_{d-1}$ and so on, so that $\pi = \tau$. $\square$

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