CAPITAL INVESTMENT PLANNING: EXPANSION AND REPLACEMENT

by

ROBERT RONALD TRIPPI

B.A., Queens College of the City University of New York (1966)
S.M., Massachusetts Institute of Technology (1968)

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF
PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF
TECHNOLOGY

January, 1972

Signature of Author...
Alfred P. Sloan School of Management, October 8, 1972

Certified by...
Thesis Supervisor

Accepted by...
Chairman, Departmental Committee on Graduate Students

CAPITAL INVESTMENT PLANNING: EXPANSION AND REPLACEMENT

by

Robert Ronald Trippi

Submitted to the Alfred P. Sloan School of Management on October 8, 1971 in partial fulfillment of the requirements for the degree of Doctor of Philosophy

ABSTRACT

The central topic of this thesis is the problem of gross investment in production facilities at the level of the firm or centrally controlled industry. This subject has particular relevance for managers charged with the responsibility of planning for future additions and deletions to plant or other operations facilities and may also be of interest to the economist, relating more generally to capital budgeting and the micro-economic theory of the firm. A normative approach is taken, focusing on the problem of developing plans which are in some sense either "good" or "optimal". This is one of the few subjects for which a significant body of literature comes from economics, engineering, and business sources.

Many factors must normally be taken into account in the pre-investment planning process. For example, product demand relations and their behavior over time are key input variables, in addition to the technological relationships which determine production costs. Investment costs, cost of capital, and depreciation schemes are other important inputs as, of course, is information about how costs of all types are expected to change with time or facility use. Obviously, expansion and replacement decisions will also be highly dependent on the economic characteristics of production facilities existing at the beginning of the planning interval. Usually a single figure of merit is chosen to evaluate investment plans, such as net present discounted value.

In this thesis several situations are modeled, for which possible solution techniques are suggested. Problems may have elements of aging, represented by upward movement of operating costs through time, encouraging replacement of old producing units. Most problem formulations are nonconvex programming problems and hence are not trivial to solve. Dynamic programming may be used to solve some of these problems, given that certain simplifications are made in the interests of computation. The case of fixed-charge linear investment cost is shown to allow greater computational efficiency using dynamic programming where aging is not present, and an algorithm based upon enumeration of points satisfying the Kuhn-Tucker necessary conditions for an optimum is an alternative to

dynamic programming when retirement of old facilities either does not take place or is pre-specified in time.

Periodic replacement of production units under conditions of static demand is of interest primarily because the model results, if investment costs are fixed-charge linear, in a pure integer program which lends itself readily to solution by a branch-and-bound procedure. Computational experience with the dynamic programming models is described and results of sensitivity analysis presented. More complex problem formulations are likely to be beyond the practical limits of computability for optimal solutions, as will be the case also with serially correlated stochastic demand, so there appears to be much room for future development of procedures which will provide good, although not necessarily optimal, solutions for more realistic models.

Thesis Supervisor: Wallace B. S. Crowston
Title: Associate Professor of Management

ACKNOWLEDGEMENTS

I wish to thank my dissertation committee, Professor Paul R. Kleindorfer, Professor Jeremy F. Shapiro, and especially my chairman, Professor Wallace B.S. Crowston for their help on this project. Professor Crowston was always available with suggestions and welcome encouragement throughout the process. I am grateful also for Professor Warren H. Hausman's efforts in critically reading the near-final draft of this dissertation.

I am indebted to Professor Murat R. Sertel, Donald Lewin, Michael Wagner, and Robert White for their valuable comments and suggestions in preparing those various portions of the dissertation for which they have an especial interest, and to Timothy Warner for his insights into the idiosyncracies of Multics. Thanks must also go to Mrs. Yvonne Wong, Mrs. Martha Siporin, and Mrs. M. Bowman for their patience and care in preparing the typed manuscript.

Computational expenses were very generously supported by the Sloan School of Management.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

EXPANSION DECISIONS IN PERSPECTIVE

The subject matter of this thesis belongs to the normative theory of investment in gross production facilities at the level of the firm or cooperative industry. Descriptive theories of capital investment behavior, for which a large body of economics literature exists, will not be directly addressed. Although the term "capacity" may be loosely used herein, it will simply be as a convenient substitute for "existing production facilities" or "gross quantity of capital plant and equipment of appropriate types," since capacity in the sense of capability to produce at a certain maximal rate may have nebulous meaning in many instances.

For certain chemical and similar processes in near-continuous production, physical capacity may be quite meaningful. However, for many manufacturing and service processes, production can be increased for a given set of facilities by increasing work force, going on overtime production or additional shifts, leasing of space and equipment, changes in purchasing, quality or service policies, subcontracting, or some combination thereof. In such cases, a much more useful concept is that of an output-cost relation of a set of facilities. Although some writers have attempted to define capacity in terms of output-cost relations (DeLeeuw [25]), nothing useful is added for our purposes by the artificial specification of "capacity" levels.

The approaches suggested in this thesis are especially applicable to determining the size and time-phasing of independent production units to be added to existing facilities. Such additions may be complete parallel plant facilities or simultaneous proportional increases in all capital, material, and labor inputs. Under such circumstances the new cost-output behavior can often be readily deduced from the cost-output relations of each of the production units.

Embedded within any capital investment plan are implicit assumptions about a host of operating problems. The relatively uncomplicated operating problem of producing to maximize one-period profit will be considered in the models presented in chapters three four, and five. Within this production plan are lower-level problems involving disaggregation of the period production plan within a fixed plant configuration, such as work force determination, procurement, inventory control, and production scheduling. The optimal solution to such problems is assumed to be summarized by an approximate cost-output relation for the firm.

Many other factors are relevant in determining an optimal expansion plan. Production and investment costs obviously must be known, as must the form and parameters of the demand relation. If those parameters are stochastic, information about their distributions will be useful. Solutions will be highly sensitive to the time-value of money adopted, as it is primarily through this mechanism that multi-step expansion will take place, and also in investment funds available. In addition, aging of facilities may affect costs in a predictable fashion, as may technological progress. Finally, the tax structure and depreciation rate for capital investments must be known.

Chapter two reviews the currently available literature in this field in a non-exhaustive fashion. Chapters three, four and five present several models for expansion and replacement problems along with suggested methods for solution. Chapter six contains computational results for a simplified expansion-replacement situation, and chapter seven discusses present limitations on the structure of problems for which solution to optimality is practical and suggests most promising areas for further research.

Chapter II

SURVEY OF PERTINENT LITERATURE

A. Preliminaries

The literature relating directly to problems of optimal facility expansion is relatively dispersed and disorganized. This chapter will describe models and solution techniques which have been proposed by writers for problems of gross investment in production and operation facilities, as opposed to the timing and selection of individual machine purchases. Also to be excluded from these discussions are the works of investigators which relate primarily to rent-or-buy decisions or warehouse capacity scheduling, as these are rather distinct problems from those of plant expansion. For the reader interested in such topics, strongly suggested are the papers of Veinott and Wagner [100],Fetter [34], and Weeks *et al* [105].

The facility expansion problem has been variously defined by its principal investigators. We will consider a facility expansion problem to be one which includes most or all of the following elements:

1) facility investment costs, where facilities are usually considered to be plants or logistics system elements, but can include the basic producing entity of service industries as well

2) facility operating costs

3) time-dependent demand, where quantity demanded (or sales rate) may be either dependent or independent of other actions of the firm (such as price-setting)

4) essential constraints such as output limitations or financial conditions to be met

5) an objective function or measure of merit of the investment plan.

The goal is to find a plan of action including:

1) the points in time at which investments are to take place (or alternatively the plant configuration which should exist at each point in time)

2) information about the fashion in which the facilities are to be operated in each time period

which will optimize the objective function.

The basis for most of the literature in the facility expansion field is the classical present-value analysis. All costs and all revenues are referred to a common point in time allowing direct comparison of alternative courses of action. Although there are very significant conceptual problems remaining with this analysis (see Baumol and Quandt [5], Lorie and Savage [57], Solomon [90] or Weingartner [107]) for a firm with either limited sources of capital or multiple sources of capital and uncertainty about the future, these have been largely ignored by the investigators in this field. Either an appropriate rate of discount is assumed to exist and be known to the decision-maker for net-present-value (NPV) analysis, or internal rate of return is assumed to be an appropriate measure of merit for the investment plan.

A general model using the criterion of net-present-value for evaluating investment policies has been presented by Riesman and Buffa [80]. The most general situation that they describe is that involving replacement (C), operating expenditures (E), revenues (R), purchase price (B), and salvage value (S). For this "CERBS" case the worth at time zero of the investment plan is P = B - S + E - R, or

$$P = \sum_{j=0}^{n} [B_j e^{-r \sum_{i=1}^{j} T_i}] - \sum_{j=0}^{n} [S_j(T_{j+1}) e^{-r \sum_{i=0}^{j} (T_{i+1})}]$$

$$+ \sum_{j=0}^{n} [e^{-r \sum_{i=0}^{j} T_i} \int_0^{T_{j+1}} E_j(t) e^{-rt} dt]$$

$$- \sum_{j=0}^{n} [e^{-r \sum_{i=0}^{j} T_i} \int_0^{T_{j+1}} R_j(t) e^{-rt} dt] ,$$

where r is the rate of interest and n is the number of replacements being considered. Each item in a succession of replacements, j, may have its own characteristic purchase price B_j, salvage value S_j, revenue and expense functions $R_j(t)$ and $E_j(t)$, and economic life T_j. Other investment models, drawn predominantly from the area of machine replacement policy, are shown to be special cases of this model. For example, the Terborgh [97] model including an "operating inferiority gradient" reduces to the "EB" subcase, in Riesman and Buffa's terminology, while the Dean [23] model is the "ERBS" subcase. Most of the plant expansion problems in this section will fall into the "ERBS" or "CERBS" subclasses and may, additionally, have elements of uncertainty. It should be noted that, although the Riesman-Buffa model can be utilized to evaluate any deterministic plant expansion plan, it does not provide a means of selecting an optimal one; normally there will be a large, often infinite,number of alternative investment plans to consider. This basic framework has also been adopted by Morris [76] in his discussion of problems of "capacity maintenance," actually equipment replacement policy.

As will become evident, most of the investigators in the facility expansion area have directed their efforts to providing solutions to this problem of optimal planning and selection of an optimal investment strategy from the many available. For the most part, the operating problems considered have been quite simple, often merely to provide at least the number of units required in each time period. Forecasts of sales are hence prime input to such models. Although Corrigan and Dean [20] and others have cautioned that the size of the plant should be based on meticulous market research on static price-volume relationship, rate of growth of the product class, and the rate of penetration of the firm's product, many of the analyses have ignored such sources of information. As will be noted, several more ambitious researchers have attempted to include more complex operating problems in their models, such as those involving transportation and backorder decisions.

B. Expansion as an Economic Problem

Much of the early literature in the area of micro-economic theory is concerned with production by the firm. With the usual objective of each firm to maximize profits, the equilibrium conditions in the market have been examined for a variety of pathological cases. This static analysis most often presumes that but one production technology is available to the firm; hence the short-run cost-quantity relation differs from the long-run relation only because of limitations on quantities of factors available, but not due to types of factors (i.e. plant configurations). The dynamic case of production to meet time-dependent demands and appropriate choice of production technologies for the individual producer have been largely neglected by the early writers.

Although the consequences of any investment policy can be evaluated on a period-by-period basis through use of such theory, little guidance is provided for selection of plant size, processes, and time phasing in the classical literature. It has been relatively recently that economists have addressed such questions, motivated to a great extent by the modern-day development of input-output models by Leontief.

Consideration of expansion decisions has sometimes been included in economic theories relating to supply. Lucas [58] has examined present-value optimizing conditions for firms in a competitive industry. Assumptions include output a linear homogeneous function of labor, capital, and investment goods purchases, $Q(t) = F(L(t), K(t), I(t))$, in order to introduce the "fixity" of capital explicitly into the formulation, thereby distinguishing between the short-run and long-run supply behaviors. Hence a transitional period is required for the firm to arrive at its new long-run equilibrium following a change in external market conditions. Physical capital depreciation by exponential decay is assumed.

The usual marginal conditions are obtained, providing the interesting result that for constant prices, net capital stock will grow at a constant rate. Oddly enough, the slope of the short-run firm supply curve may have either sign. Due to the adjustment lag equations, supply price (horizontal long-run supply curve in a competitive industry) increases with the growth rate of industry demand, the demand growth mechanism operating proportionately in the quantity dimension.

The model indirectly provides firm and thus industry demand relations for capital investment goods. From a practical standpoint, however, such information may be of little value to an actual firm facing horizontal supply of capital or purchasing specialized equipment for which supply may even be downward-sloping but independent of other firm's purchases. Homogeneity of capital and lack of purchase economies in capacity are implicit.

Perhaps the best-known application-oriented economics treatment of expansion investment is that of Alan S. Manne [65]. He has examined a succession of models in the area of optimal time-phasing of production facility investments, and he has applied his results to several industries. The simplest model described by Manne is that for a linearly growing deterministic sales rate with plants of infinite life. The object is to always have at least sufficient productive capacity to meet the sales rate, while adding plants of a size which will minimize the present value of costs over an infinite horizon. Excess capacity, when plotted, then displays a sawtooth pattern similar to that of the Wilson-type inventory model (Figure 2-1).

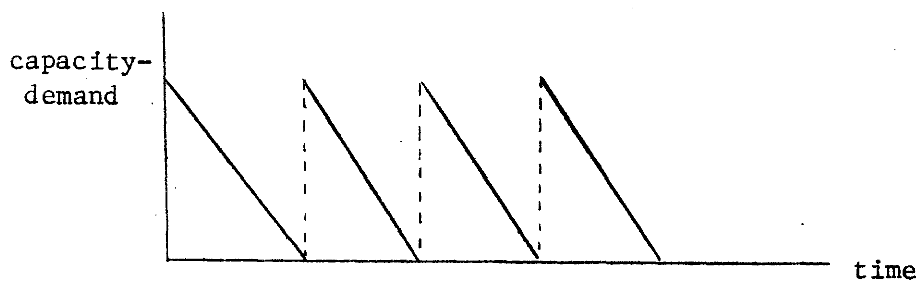


Figure 2-1

The installation costs that result from a single capacity increment capable of producing x units are assumed to be given by a power function relation: kx^a, $k>0$, $0<a<1$ where the physical unit capacity is taken for convenience to be the annual increment in sales. Hence, if $C(x)$ is the sum of all future costs discounted by factor r, looking forward to an infinite horizon, we may write down the following recursive equation: $C(x) = kx^a + e^{-rx}C(x)$.

It follows that $C(x) = \frac{kx^a}{1-e^{-rx}}$. We find the value of plant capacity x which minimizes the stream of costs C by differentiating with respect to x and setting the result equal to zero, obtaining $a = \frac{rx}{e^{rx}-1}$.

For probabilistic sales increments it has been shown that the above formula is modified only by replacing r by a constant factor $-\lambda$ which depends on the degree of uncertainty.[1] It has been further shown by Srinivasan [91] that for exponentially growing demand plant additions should take place at times $t_n = 0, t, 2t, 3t, \ldots, nt, \ldots, T$. Cases involving backlogging, multiple producing areas, and other complications to the basic model have also been worked out by Manne and Erlenkotter [67], and have been applied to data from metals, cement, and fertilizer industries of India. Wein and Sreedharan [104] have applied a quite similar analysis to the Venezuelan steel industry.

The operating problems considered in such models are quite simple: either keeping capacity always above demand or determining

---

[1] The increase in average surplus capacity over time resulting from growing demand is consistent with a proof due to Smith [ ] that an increase in the variance of demand in the static case will result in an increase in unutilized capital stock of the firm for production functions with inelastic substitution of other factors for capital.

how much of demand to meet in the case of penalty costs for not meeting demand (imports create a balance of payments problem; hence an import penalty cost). Marginal operating costs are assumed either zero or constant up to some capacity level of output, at which point they become infinite (Figure 2-2). Furthermore, as demand-price relations are not explicitly considered as a determinant of output, revenues do not appear in these analyses. The objective is always to minimize the present value of costs.

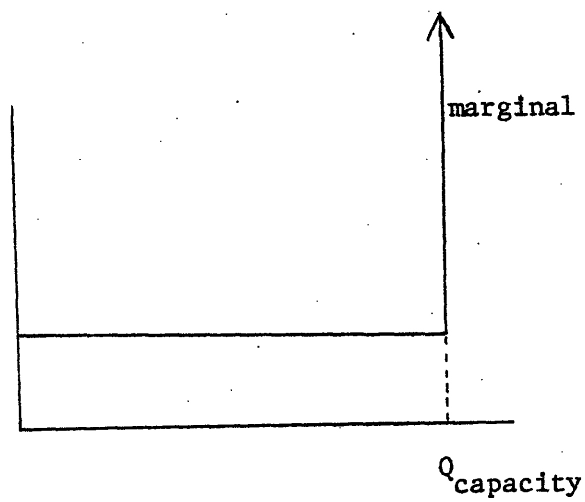


Figure 2-2

Another interesting model has been proposed by Kendrick [51] for programming investment in the Brazilian steel industry. Basically, demand for final product which must always be met is assumed to grow over time, with a transportation-type linear program to be solved in each time period for the optimal routing of intermediate products between plants. Integer variables are used to represent the presence or absence of new plants in each time period. Hence, a rather difficult-to-solve mixed integer program results for the finite horizon case, and a relatively complex operating problem is considered for each period. Algorithms and heuristics for solving such fixed-charge transportation problems have been developed by Sa [82] and others, but only relatively small problems can be solved at this time. As in the Manne-type models, only additions of independent producing units are considered, and a single basic product supplied.

Although such models are often useful for prescribing the optimal growth path of large homogeneous industries, lending themselves well to theories of gross investment behavior, their value to the individual firm for determining its best expansion strategy is questionable. The many assumptions about sales rates, costs, and demand structure are unrealistic reflections of the environment of the individual firm, and the models contain insufficient detail to make use of all of the information that may be available to the manager. Many of these deficiencies, from the point of view of the individual business, have been ameliorated by models proposed by researchers in the process engineering field.

C. Engineering Approaches

The plant expansion investment problem has been treated in some depth in the process engineering literature. Mathematical approaches to the subject may have been encouraged by the relatively reliable relationships among inputs, costs, and outputs, particularly in the chemical industries, and by the analytic training of the management personnel in such industries. The models developed, however, often have more general applicability than to one particular technology.

In many of these analyses, the relation between initial investment or fixed operating costs, K, and capacity, C (as an upper bound on output) is of the form

$$K = b\left(\frac{C}{C_o}\right)^{\theta},$$

where C_o, b and θ are the values for some known investment K_o. Hess and Weaver [43] have utilized this empirically determined relation in determining optimal plant size for uncertain static demand. For the criterion of maximum rate of return they show the optimal capacity C^* to be the solution of

$$\text{prob. (demand} \geqslant C^*) = \frac{\theta K_o C^{*1-\theta}}{C_o^{\theta}}.$$

Using the power function investment cost relation, Salatan and Caselli [83] have examined the optimal design of a multi-stage plant for the case of a static sales rate but uncertain capacity. When sequential stages of production each have stochastic capacities with mean u and variance s^2, the plant capacity will also be a probability distribution, but with mean $u' < u$ and variance $s^{2'} < s^2$. This is known as the concatenation effect (Figure 2-3). All investments are evaluated according to their level of "present cash equivalent" or NPV.

It is assumed in the Salatan-Caselli model that capital costs vary as an exponential power of the <u>mean</u> expected plant capacity of x_o units and that unit average operating costs can be expressed by an equation of the form: $AC = r + fC_o/v$,

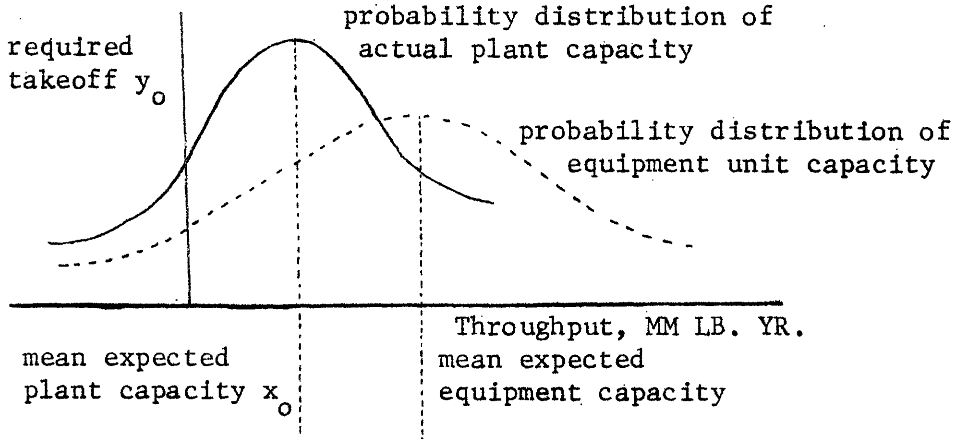
where f = a proportionality factor

r = marginal cost

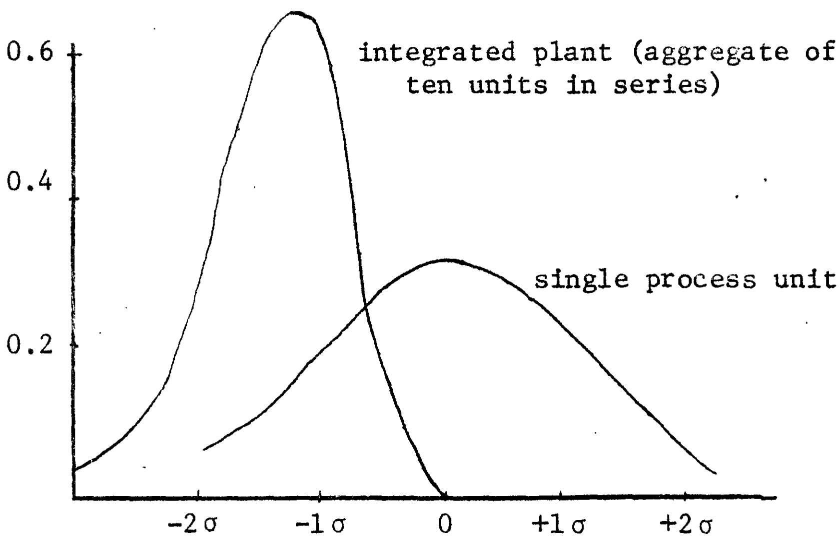
C_o = mean expected capacity, and

v = actual throughput.

Hence, a linear total cost function with positive intercept is required.



Static Case with Uncertainty in Plant Performance



Concatenation Effect on Probability Distribution of Capacity

Figure 2-3

For constant demand rate and uncertain, normally distributed design capacity, the marginal conditions for the optimal plant size C_o with expected throughput x_o have been obtained with the use of the calculus. For increasing sales at an uncertain but constant rate optimization leads to an integral equation which has been solved numerically, under the assumption of deterministic design capacity. The interactions of multiple stochastic elements in capacity, demand, and rate of growth of demand have not been worked out, however. Plant expansion in more than one step is not considered in this analysis. As in the previous models, the marginal cost function is constant up to stochastic capacity output, at which point marginal cost becomes infinite. The mathematical precision of the cost functions, as well as the requirement of constant rate of growth in sales are further limitations of this method, although for products with stable growth and well-defined processes, as are often found in the chemical industries, such assumptions may not be unreasonable.

A quite similar model has been proposed by Coleman and York [17]. The chief innovation of their presentation is the treatment of sales growth uncertainty. Rather than consider sales growth at a constant but stochastic rate, sales are assumed to grow at a constant, known rate until a cutoff date, T_o, at which a leveling off takes place (Figure 2-4).

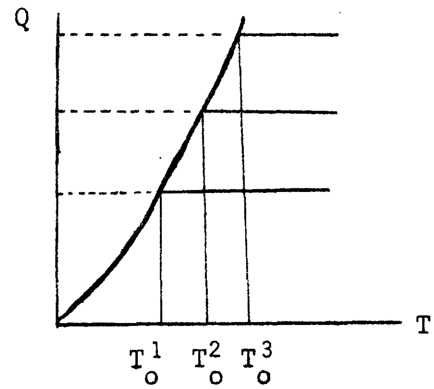


Figure 2-4

Uncertainty enters the model in the form of prior probabilities for several estimates of T_o.

Plant expansion policies can thus be evaluated either according to expected NPV or by following a minimax principle (which is more appropriate for decision-making in the face of a conscious opponent or adverse nature). In the latter case the authors suggest designing the optimal plant for each of the estimates of T_o, and choosing the one which minimizes the maximum loss. By sacrificing some of the economies of scale by expanding in small increments (regularly spaced as in Manne _et al_), the firm is in this case able to hedge against an unfavorable demand outcome and at the same time assure a reasonably good position with respect to the most favorable outcome.

Another problem of interest in the chemical-engineering-economics field is the expansion of multi-stage facilities. Each of N sequential stages may be expanded independently, but the consequences of expanding any stage will depend upon the new state of its following stage. Generoso and Hitchcock [36] have examined the expansion in one step of such multi-stage facilities, based upon an earlier model of Mitten and Nemhauser [73]. They assume that the return from each stage depends only on its own state and the state of the following stage. Three possible decisions θ_j are allowed for each production stage:

1) replace the stage with one of higher capacity
2) add a new unit to the existing stage
3) use the existing stage at a greater throughput.

The optimality criterion is taken to be "venture profit," the incremental return over the minimum acceptable return (defined to be the interest rate times the increase in fixed and working capital). A recursion relation is developed at each stage n of the form

$$f_n(x_{n-1}) = \max_{j=1,2,3} \{V(x_{n-1}, \theta_n^j) + f_{n+1}(x_n)\} ,$$

where V is the venture profit for the stage and x_n is the state resulting from decision θ_n. Computer solution time for a six-stage, three-state-per-stage dynamic program to solve the above is given as one minute (IBM 7044) including calculation of all input parameters.

Although the solution method optimizes expansion of the entire production chain in one step only, the authors suggest (Case II) that multi-step expansion can be treated for a finite horizon if all possible expansion paths of capacity by equally sized increments have each expansion step optimized by use of the single-step procedure. Each expansion route then has embedded within it several single-step problems, and there are likely to be many such routes to consider (Figure 2-5).

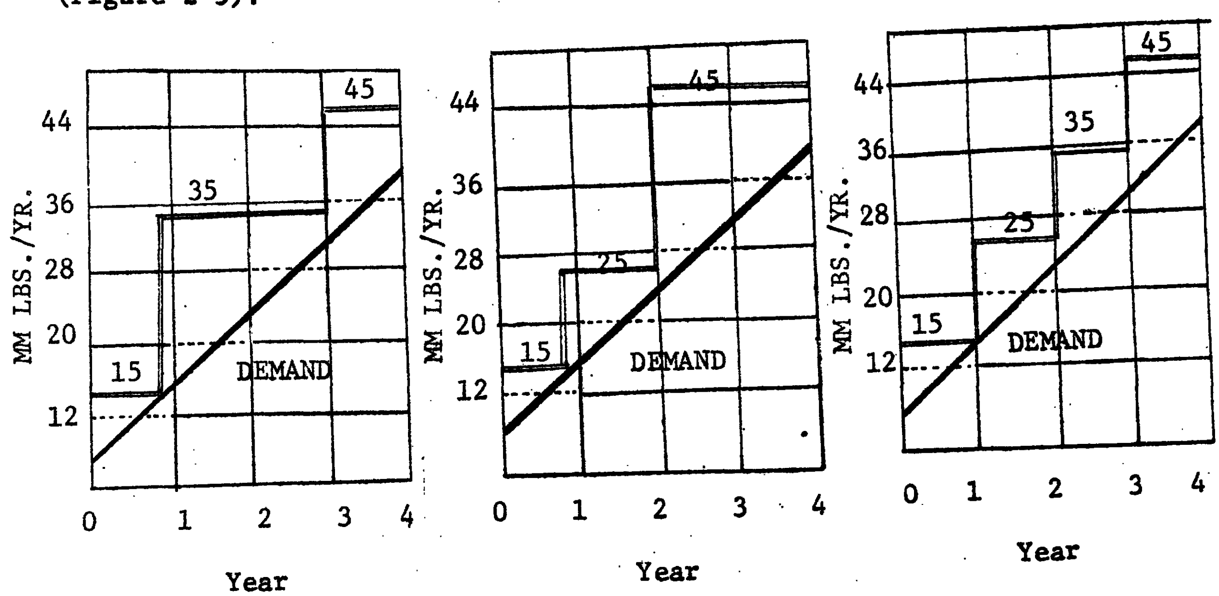


Figure 2-5

D. Plant Expansion in the Management Science and Related Literatures

The management science literature in the area of plant and facilities expansion draws heavily upon the economic and engineering approaches to the subject. Those differences that exist are likely to be ones of emphasis in problem formulation arising from differences in goals, information base, and the degree of abstraction believed to be justifiable. Frequently the scope of the expansion problem considered is somewhere between the macro industry viewpoint of the economist and the viewpoint of the process engineer often concerned with an individual facility producing a particular homogeneous chemical as part of a much larger production complex.

Before proceeding to the dynamic case of plant expansion to meet changing demand, a discussion of optimal plant size or type for a static environment may be useful. Usually choice of optimal production technology under such conditions requires a tradeoff between several cost categories. One example of such a tradeoff is that between marginal and capital or other fixed costs of the firm. A technology requiring great investment in facilities and equipment usually has lower marginal (predominantly labor and materials) costs than a less-capital-intensive operation, for production of the same product. Yet we see few industries that are either totally capital intensive or totally labor intensive. Thus we suspect that some intermediate mix of the two general factors is likely to be optimal in such industries (Figure 2-6). Similarly, tradeoffs usually exist between the capital costs of specialized machinery and defect costs (perhaps due to uniformity or quality of the product), and between general production costs and transportation costs.

Bowman [9] has considered the problem of warehouse sizing (also applicable to plant sizing) to be a tradeoff between operations and transportation costs. Unit cost is assumed to be a function of both scale of operations in terms of dollars of product supplied (v) and the area served by the facility (A):

$$C = a + b/v + cA^{1/2} ,$$

The parameters a, b, and c are obtained from a cross-sectional regression analysis of existing facilities in each district. As c is an empirically determined constant, the demand environment is assumed to be static. Investment costs are ignored in this analysis. The optimal scale of operations is found through use of the calculus for each existing facility.

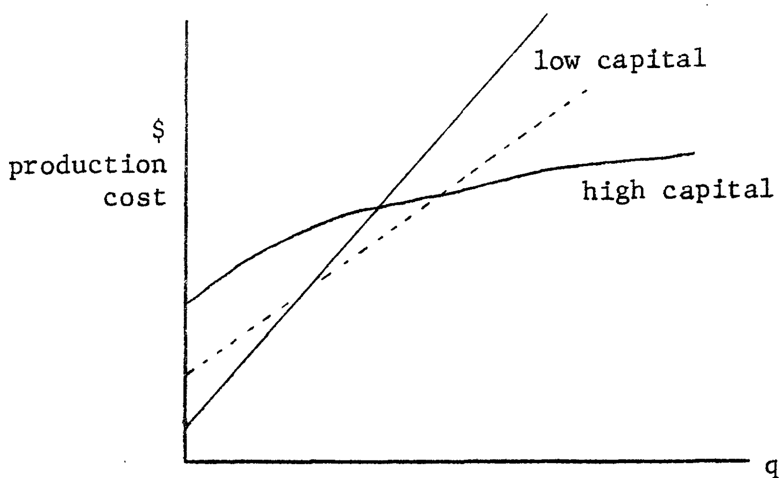


Figure 2-6

A more complex problem of optimizing plant location and sizing, in which only a single-step expansion is allowed, has been formulated by Klein and Klimpel [53]. Total production cost for each of several potential production sites i is represented by a power function of sales plus a fixed charge. Transportation costs are linear functions of the quantity to be shipped annually from each facility to each demand point. For multiple demand points j and either finite or infinite horizon, the nonlinear program with minimization of the present value of all costs as the objective function results:

$$\min: \sum_i \text{NPV of production, investment (fixed), and shipping costs}$$

$$\text{st. } \sum_i P_{ijk} = M_{jk}$$

$$S_i \geq 0, \; P_{ijk} \geq 0 \; ,$$

where P_{ijk} is the number of units shipped from i to j in period k, M_{jk} is the demand at j in period k, and S_i is the plant size selected for site i. It is assumed that the single step establishment of plants will take place simultaneously at all potential sites. Rosen's gradient projection method is used to solve the above nonlinear program for several small problems.

As mathematical programming may be utilized to solve certain other complex single-period operating problems, a possible method of identifying the best multi-step expansion plan is by enumeration of alternative plans for facility expansion, each solved for optimal period operations, selecting the one with the greatest discounted value of all revenues less costs. Rappoport and Drews [79] have adopted this approach in a study of petroleum facilities expansion. A linear program is solved for each period and possible facility configuration to satisfy all demands for petroleum products at minimum total operating cost. The present value of all operation and net investment costs are then compared for each of the alternative facility expansion plans examined. This procedure is obviously useful only when the number of feasible investment plans is relatively small.

Other writers in the field have considered far less complex operating problems, however. Lawless and Haas [55] approach the problem of what size plant to build by considering a set of alternative courses of action over a relatively short horizon. Four possible expansion plans are given in their example:

1) Build to match the six-year sales forecast

2) Build to match the three-year sales forecast and add one increment of expansion during the third year to meet the sixth-year requirement if needed

3) Build to match the two-year sales forecast and add two increments to match the fourth and sixth year forecasts if needed

4) Build the minimum-size plant required for the first year forecast and add an increment of expansion each year for five years if needed.

Thus, only equally spaced expansion increments are considered. Investment costs of plant depend upon output capacity according to a power function relation:

$$\frac{\text{cost of plant a}}{\text{cost of plant b}} = \left(\frac{\text{capacity of a}}{\text{capacity of b}}\right)^{\theta} .$$

Six conditions, corresponding to different patterns of deviation of actual sales from the forecast are examined, and the NPV of each expansion plan is calculated for each condition. The rather detailed NPV calculations have been transformed to a set of easy-to-use nomographs. A feature of this model is that finite construction times for plant and additions can be easily taken into account. Operating costs of the plant configurations are ignored.

White [108] has also examined the problem of developing an investment plan for expansion to meet increasing demands for several products.

The cost function that he uses is linear for each product:

$$E = f + gD_t \quad ,$$

where E is the total annual cost and D_t is the average annual demand during the t^{th} year. New parameters f and g result from each facility expansion investment, assumed to be in increments which cost $10,000 each. Thus the Riesman-Buffa model could be easily utilized to determine the optimum expansion path in the absence of external constraints. However, in this model the firm is assumed to have a limited amount of capital, Z. Dynamic programming is used to determine the optimum allocation of funds to expansion of facilities for each of the products. The basic recursion relation is

$$f_n(z) = \max_{0 \leqslant x_n \leqslant z/v_n} \{g_n(x_n v_n) + f_{n-1}(z - x_n v_n)\},$$

where n designates the product number, x_n is the number of increments of additional facilities for the n^{th} product, v_n is the cost of an additional increment, and z is the unallocated capital at stage n. A maximum of two increments in capacity for each of the products is allowable within the finite horizon. Furthermore, no additional funds are expected to be available in the future in this model. Rather laborious calculations are presented for a four-product, six-year horizon example.

Only rarely in the literature have the prices of factors of production been explicitly considered in seeking optimum expansion policy. Horowitz [46] has considered the problem of optimizing plant size for a dynamic price-quantity relation for a product which requires conversion of raw materials and labor into a more-or-less homogeneous product. He describes his article as "an exercise in algebra of discounting which results in the presentation of (these) answers in a form that is readily understood by management."

The net profit in a given year i from a plant constructed by the conversion of a raw material m into a final product will be equal to

$$\pi_i = (p_f q_f - p_m q_m - W - F - V - D)(1 - \text{tax rate}) ,$$

where q_f is the average quantity sold during the year, p_m is the average price paid for the raw material, q_m is the quantity of raw material used during the year, W is the wage bill, F is fixed cost other than depreciation, D is depreciation, and V is other variable costs. Horowitz then assumes functional forms for q, p, W, V, and D, and finds optimality conditions for the present value of net profits. Multi-step expansion is not considered. Horowitz's analysis differs from most economics approaches in that although the price-quantity relations for the final product change over time, the sales price of the product, once established, is constrained to remain constant over the remainder of the planning period. Horowitz has also examined a simple one-step plant expansion problem in which the price-quantity function for the good is stochastic, taking expected present value as the evaluation criterion.

Another investigator who has explicitly considered the price-quantity relation is Lesso [56]. He has developed a model for the addition of independent producing units for a single product. For a given number of producing units and price-demand relation for each period, an allocation of production to each of the units may be found which will maximize after-tax earnings. Each production unit is assume to have a linear or convex total cost function. An inconsistency in this sub-problem exists, however, for total demand D_t as a forecasted constant appears in constraints of the form

$$\text{total output of existing producing units in period } t \leq D_t$$

although optimizing after-tax earnings will generate prices and total demand quantities which may not correspond to D_t.

A main problem is then formulated assuming that a solution to the sub-problem has been found for each sub-period. A set of integer decision variables represent the point at which each of the producing units is brought into operation, and an integer program to maximize the net present value of all after-tax earnings subject to constraints on the maximum allowable debt-equity ratio of the firm results. A branch-bound algorithm is presented to solve the complete problem. Although the model is a deterministic one and cannot handle arbitrary expansion of existing facilities, it does allow for fairly complex treatment of financial variables and taxes, including depletion and similar allowances.

A simple model which does allow for arbitrary expansion of existing facilities has been presented by Gavett [35]. Given an economy of scale in capital costs and forecasted demands which must be met the problem involves a trade-off between the economy of scale and the capital cost of unutilized capacity. Operating costs are not considered in this model. If we define $K(t,\omega)$ to be the capital cost of expanding in period ω to meet period t's demand and α^{ω} to be the present value of a dollar spent in ω, the functional equation can be written for a finite horizon T:

$$f_t = \min_{0 \leq \omega \leq t} (\alpha^{\omega} K(t,\omega) + f_{\omega-1}), \quad 0 \leq t \leq T \ ,$$

A simple example is presented utilizing this relation. Luenberger[59] has utilized a similar capacity model in illustrating a cyclic dynamic programming procedure using Lagrange multipliers. Here, however, a rather simple aging process is assumed: capacity disappears from the system after a fixed delayof L years regardless of the size of the original capacity increment or date of installation. Unfortunately, his algorithm fails miserably in an example using concave investment costs.

Practically all analyses of expansion have assumed that the investment cost of specific facility alternatives are invariant to when the expansions take place. Hinomoto [45], however, has investigated a problem of expansion in which investment cost W of a facility of size x may either rise or fall as an exponential function of the date of the period, t_a, in which purchase takes place:

$$W = K(z)e^{-kt_a} \ .$$

Similarly, the average operating cost curve of such a facility is assumed to decline exponentially with t_a due to technological progress.

Optimality conditions are worked out for optimum plant size z of each facility to be added to the system and output and price in each period for time-dependent price-demand relation. This type of analysis is more of a contribution to the state of micro-economic theory than an aid to actual decision-making, as the system of equations are likely to be impossible to solve for expansion in more than one step. It is mentioned in this section, however, as it appears in a publication oriented towards management scientists rather than economists.

Operating and planning decisions may require information not only about investment and production costs, but about other costs as well. Erlenkotter [30] has examined multi-step expansion for several producing locations. His model seeks to minimize total discounted shipment and production costs which are directly proportional to quantity plus incurred investment costs over a finite horizon, while meeting projected demand quantities. Revenues are not explicitly considered. Dynamic programming is used with n-dimensional state and decision variables, where n is the number of potential production sites. The operating problem employs a simplex-like algorithm to minimize total transportation and production costs for each state. Computational results are presented for problems involving at most three producing locations.

Other researchers have, while employing relatively simple models, attempted to investigate the relationship among other managerial variables. Chang _et al_ [13] have utilized the basic Manne equation for optimal investment intervals previously described to examine key managerial measures. Principle findings include unit capital costs as a decreasing convex function of the growth rate of demand, risk of idle capacity (taken to be mean absolute deviation of excess capacity as a percentage of average capacity between expansions) relatively insensitive to the discount rate in the short run, and risk of idle capacity as an increasing concave function of demand growth rate. An analysis of the paper industry indicates that the larger the firm, the lower the apparent discount rate that has been applied to capital budgeting (implying a lower risk premium and a greater aggressiveness for such firms). Discount rates are imputed from expansion according to the Manne model, given a capacity scale economy factor of .8 and historic observations of expansion intervals. It is found that market share has generally increased with increasing risk premiums, suggesting that hitherto unexamined operating diseconomies may exist for larger firms.

Along similar lines, Chang and Henderson [12] have noted that, for capacity additions as predicted by such models, industries with linearly growing demand will have a floor beneath which unit capacity costs can never fall, while such is not the case for geometric demand growth. In addition, smaller size firms will in either case exhibit greater changes in unit capacity costs (assuming constant relative market shares over time), presumably contributing to their greater profit volatility.

Although not necessarily associated with a particular model or solution method, corporate simulation techniques have been employed as an aid to business planning in which capital investment is a major factor. Using Industrial Dynamics, Swanson [95] has analyzed the problem of developing effective management decision rules for the firm in a competitive environment. The firm is assumed to control the flow of resources (possibly including physical and working capital, production and engineering personnel, and marketing effort) which determine the firm's competitive position (delivery delays, product performance, reliability, and price, etc.) in the market.

Information gained from observations of the firm's present and past performance is then employed, in part, in making resource control decisions (Figure 2-7). The merit of capital and other resource policies may lie not only in the pattern of future cash flows, but also in the robustness, rapidity, and longevity of uninterrupted sales growth, or other non-monetary measures. Although the projected performance of relatively complex nonlinear feedback systems including the essentials of several functional areas can be

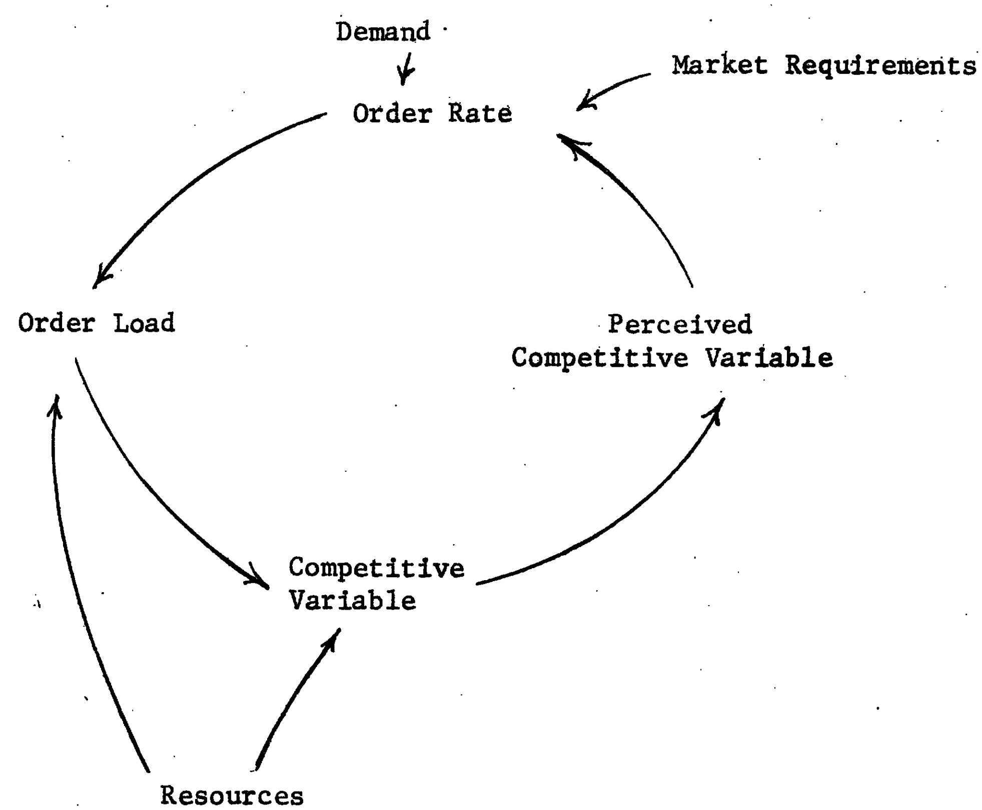


Figure 2-7

Linkage Between Resources and the Market (Swanson) observed through simulation for a variety of different policies, the optimal policy can rarely be found. A good, but suboptimal solution to a more realistic model including capital expenditure may, however, prove to be of greater value than an optimal solution to a simpler model.

E. Comments and Conclusions

The literature in the field of plant expansion appears to be somewhat chaotic. There does not exist two or three basic problem formulations for which investigators have suggested solution techniques, as there is in the facilities location literature, for example. Nearly every investigator has set forth a different problem within the general area of plant and facilities expansion, with a unique set of givens, constraints, and objective function. Hence we can point to no work or group of works that represent the "state of the art" today.

There are many areas for improvement and extension in the treatment of capacity expansion investment decisions. Of course, one could mention that a synthesis of the distinct features of the specialized models would be a significant step to take. For example, including elements of uncertainty, technological improvement, capital rationing, and multi-products, or combinations thereof in a single multi-step model would be an advance. Including the price-quantity relation in those models that ignore revenues would likewise produce useful models, although perhaps ones quite difficult to solve. It is unlikely that solutions including complex operating problems relating to transportation for multi-location expansion will be satisfactorily obtained in the near future for large problems, for even the static plant location problem has by no means yet been completely conquered. The incorporation of seasonal sales fluctuations in forecasts for a model including price-quantity relations should not be too difficult, however, and may provide more accurate estimates of points in time to phase in capacity.

Other significant questions have not been considered at all in the literature to date. For example, deterioration in facility efficiency as a result of age and obsolescence is one important factor influencing actual facility expansion decisions. In addition, all approaches described in this paper for handling uncertainty ignore the learning that may take place when demand either misses or exceeds the forecasted levels at intermediate points of time in the forecasted planning period. It may be possible to apply decision theory in the solution of such a problem.[1]

Another phenomenon worth investigating is the interaction of the expansion plan and long-range pricing policies. Expansion with explicit price-demand relation will, in general, determine the optimal price to prevail at each point of time in the planning horizon. However, management may desire constraints on the price that they will charge in this period, or simply require that significant fluctuations in price (which may become optimal when the design capacity for the facility is approached) be avoided. Evaluation of the effects of these and other constraints on the optimal expansion plan would then be of value to the decision-maker.

---

[1]Sobel [89] has examined a short-range stochastic problem involving the joint regulation of production and capacity which can be acquired and disposed of at constant per-unit cost. The resulting analysis of optimal policies appears to heavily depend upon the convexity of such costs.

CHAPTER III

A MODEL OF EXPANSION

The selection of an optimal expansion plan is essentially a problem in production system design. For each of the possible combinations of facility configurations that may exist at points in time within the planning period, a figure of merit, cumulated net discounted profit, may be obtained. The calculation of net operating profit in each time period assumes optimal operation of the available facilities. In the models to be examined in this chapter, optimal operation simply requires determination of the best output quantity of the product at each point in time for the design-determined operating cost characteristic of the firm. Although a single product is assumed in the analysis to follow, the principles are readily generalizable to the multi-product firm, requiring only operating costs a function of the output of each of the products and a price-demand relation for each product. This section will treat expansion to meet non-decreasing demand in which the production system design for each point in time can be completely described by but one or two parameters in the operating cost function, and in which the set of feasible designs is limited to those for which retirement of production units is either prespecified or nonexistent.

Unfortunately, from the standpoint of efficient computation of optimal investment policies, the objective functions in such models are generally neither convex nor concave functions of all the decision variables. Hence, local-optimum-seeking methods may be of limited use for such problems. Variables will reflect physical changes made to the production system, such as capital additions to take place at points in time. Capital investment will not normally be a continuous function of time, in contrast to models advanced by Lucas [58] and others for homogeneous capital without economies of acquisition or process technology.

A. Revenues

Two examples of concave revenue functions are those associated with linear and constant-elastic demands. The linear demand model is

$$p = \frac{R}{q} = D(\tau) - Cq, \quad C, D > 0 \; ^{1},$$

where τ denotes time.
Revenues are clearly concave in q for this case:

$$R = D(\tau)q - Cq^2$$

$$\frac{\partial^2 R}{\partial q^2} = -2C < 0 \quad .$$

[1] For simplicity of exposition and because of possible practical difficulties in estimating future demand parameters, we assume that only one of these is time-dependent. One could, however, assume that C also changes with time.

For constant-elastic demand,

$$p = \frac{R}{q} = D(\tau)q^{1/e} \qquad D>0,\ e<-1, \text{ where } |e| \text{ is a constant demand elasticity.}$$

Revenues are also concave in this case:

$$R = D(\tau)q^{1/e + 1}$$

$$\frac{\partial^2 R}{\partial q^2} = D(\tau)(\frac{1}{e} + 1)\frac{1}{e}q^{1/e - 1} < 0 .$$

Revenues depend on time-dependent demand parameter(s) $D(\tau)$ and thus $R = R(D(\tau),q)$. For non-stochastic demand parameters we may simply consider $R = R(\tau,q)$. In any case, it will sometimes be assumed that $\frac{dD}{d\tau} > 0$ and that an increase in D will result in the new demand curve being everywhere above the old one.

B. Operating Costs

One useful function that may approximate the actual operating cost behavior of a size S plant is

$$TC(S,q) = FC(S) + \sum_{j=1}^{J} \frac{a_j q^j}{S^{i_j}} \quad , a_j \geq 0, S>0. \qquad 3.2.1$$

Fixed operating costs are represented by FC, while quantity-variable operating cost is a weighted J^{th}-order polynomial of output. If $i_j = j-1$ and if fixed operating cost is proportional to S,

$$FC = a_0 S \ ,$$

then the minimum-average-cost point in q for this cost function will be independent of S. This is easily demonstrated.

Average cost, AC, in this case is

$$\frac{a_0 S}{q} + \sum_{j=1}^{J} \frac{a_j q^{j-1}}{S^{j-1}} \quad .$$

Let AC^* be minimum average cost. At AC^* necessary conditions for a minimum include

$$\frac{\partial AC}{\partial q} = 0 = - \frac{a_0 S}{q^2} + \sum_{j=1}^{J} \frac{a_j (j-1) q^{j-2}}{S^{j-1}} \quad . \qquad 3.2.2$$

Multiplying 3.2.2 by q, we obtain

$$0 = - \frac{a_0}{Q} + \sum_{j=1}^{J} a_j (j-1) Q^{j-1} \ , \qquad 3.2.3$$

where $Q = q/S$. Q^*, the solution to 3.2.3, will be a constant for fixed parameters a_j. Then

$$AC^* = \frac{a_0}{Q^*} + \sum_{j=1}^{J} a_j (Q^*)^{j-1} ,$$

minimum average cost, will be the same for any plant size, so there are no long-run economies of scale (figure 3-1).

A more realistic assumption is that such scale economies are present, and that minimum average cost declines with plant size (figure 3-2). With fixed cost proportional to S this will occur for $i_j > j-1$ in (3.2.1). Another mechanism by which this may take place is by fixed production costs (including overhead) being less than proportional to plant size. For example, take

$$FC = (a_0 S)^{\beta} \quad , \quad 0<\beta<1.$$

With $i_j \geq 0$, the terms $\frac{a_j q^j}{S^{i_j}}$ in (3.2.1) are convex functions of S. Thus for $FC = a_0 S$, TC is convex as the sum of convex functions. Where convexity of TC is required with fixed cost scale economies, the condition

$$\frac{\partial^2 TC}{\partial S^2} = a_0 \beta (\beta-1) S^{\beta-2} + \frac{a_j (i_j+1) i_j q^j}{S^{i_j+2}} \leq 0$$

must obtain for the plant sizes and outputs in question.

Such an operating cost function, with or without the requirement that $i_j = j-1$, is a relatively rich one, as it can easily approximate a wide variety of actual output-cost relationships if J is made sufficiently large. Hence it has been employed in the computational work described in Chapter VI.

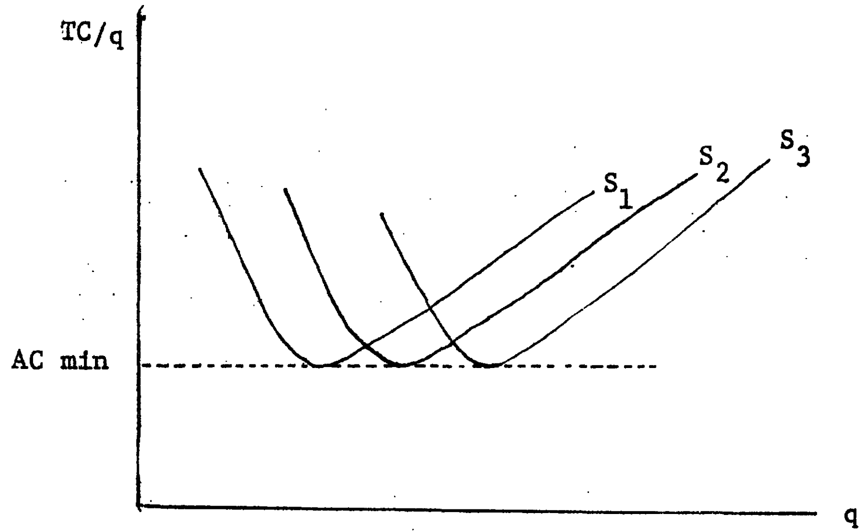


Figure 3-1

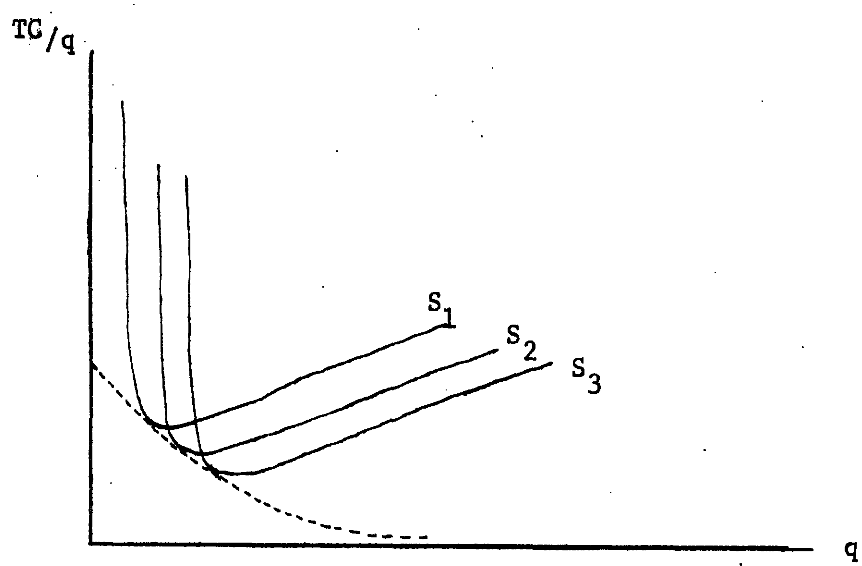


Figure 3-2

In order for an aggegregate size parameter, $S = \sum_k S_k$, where S_k represents the size of the k^{th} past capital increment, to be sufficient to accurately describe the operating cost function in the case of perfectly independently operating production units certain properties of marginal cost (MC(S,q)) are desired. TC(S,q) in this case must implicitly provide an allocation of production to existing units in an optimal fashion - on the basis of equal marginal costs. For this to be so with knowledge only of size parameter S we require that the actual inverse marginal cost function q = q(MC,S) have the following property:

$$q(MC,S) = \sum_k q(MC,S_k). \qquad 3.2.4$$

Consequently, the marginal cost function of size S plant with optimum allocation is the horizontal summation of the marginal cost functions of each production unit. This is illustrated in figures 3-3a,b,and c.

If marginal cost of one or more production units is decreasing for some values of q, actual production system marginal cost can be kinked, as in figure 3-4c. Equation (3.2.4) is not applicable, as the single-valued inverse does not exist in this case. One would then require knowledge of each of the S_k. If each production unit has total operating cost given by (3.2.1) with i_j=j-1 and identical a_j,j=1..J, then (3.2.4) will be valid and an aggregate size parameter will suffice to describe the system.

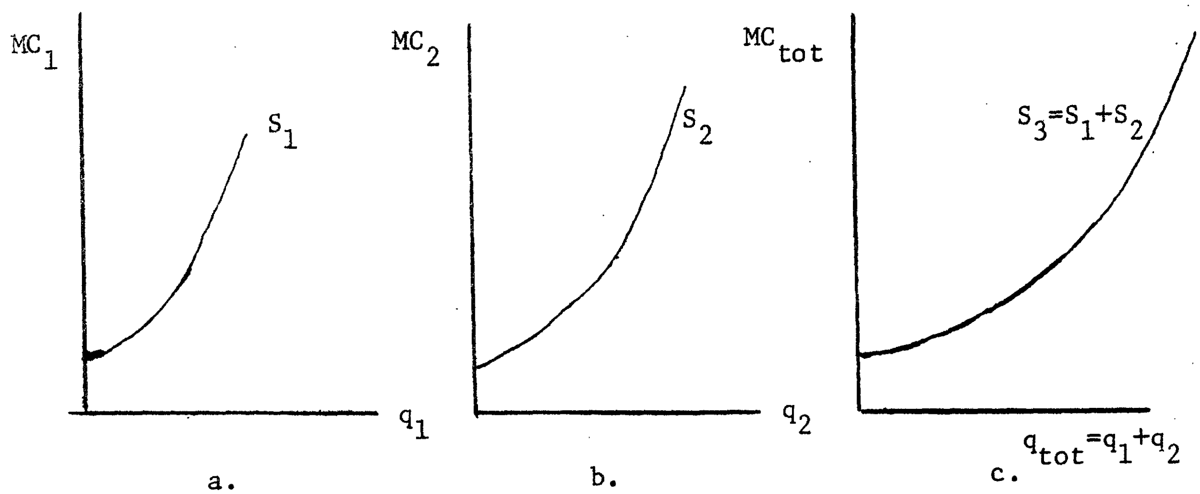


Figure 3-3

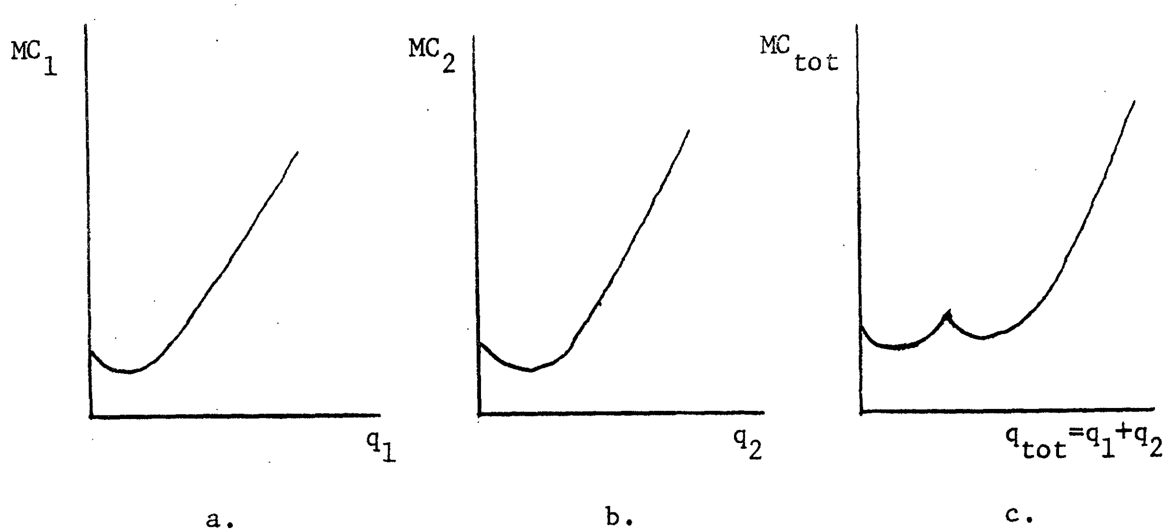


Figure 3-4

In addition to size, the plant age may be an important determinant of operating costs. A parameter H which increases with age may then be included in the production cost function. For example, we may replace some or all of the a_j by Ha_j or some more complex function of H. In any case, the essential idea is to include a simple means by which aging can be reflected in production costs.

C. Operating Profits

Profits are assumed to depend on demand parameter which changes with time $(D(\tau))$, production system state (θ), and output (q) only. operating profits are defined as revenues (R) less operating costs (TC) adjusted for taxes[1]:

$$\Pi(D(\tau),\theta,q) = [R(D(\tau),q) - TC(\theta,q)]\ [1 - \text{tax rate}] .$$

System state θ may be size (S), size and average age (scalars S,H), sizes and ages of individual units (vectors S,H), etc. The optimal profit function π is defined to be the maximum profit obtainable at time τ from facility of configuration θ:

$$\pi(D(\tau),\theta) = \max_q \ \{\Pi(D(\tau),\theta,q)\} \quad . \qquad 3.3.1$$

At optimal output q^* satisfying (3.3.1) marginal cost will be equated with marginal revenue:

$$\left.\frac{\partial TC}{\partial q}\right|_{q^*} = \left.\frac{\partial R}{\partial q}\right|_{q^*} \quad . \qquad 3.3.2$$

[1] This is not the accountant's "operating profit," usually defined to be sales revenue less all production and operating expenses, since depreciation is not included as an operating expense.

Should cost be a convex function of q and S and revenues be concave in q, then Π will be a concave function of both q and S and π will be a concave function of S as the maximum over q of a concave function.

For discrete-time formulations the average n^{th}-period demand parameter, D_n, may be substituted for $D(\tau)$ in the profit function π .

With single size parameter S the operating profit function is likely to have the general shape illustrated in figure 3-5.

With inclusion of a single age parameter H in the operating cost function, the isoprofit lines will be everywhere further to the right (left) in the diagram as H is increased (decreased).

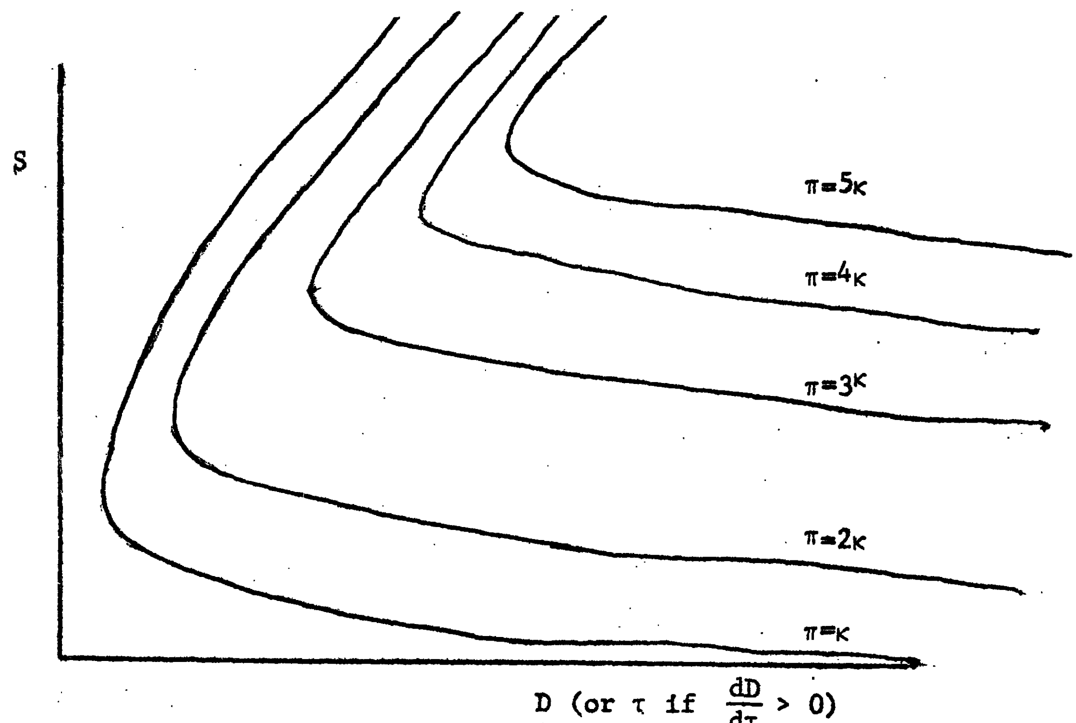


Figure 3-5

D. Investment Costs

Investment costs are defined as those net costs to the firm after subtraction of discounted future tax savings through depreciation. For example, with accelerated double-declining balance depreciation factor $0<d<1$, the one-period depreciation allowance n years after capital increment of size s costing $\mathcal{I}(s)$ is $2\mathcal{I}d^n(1-d)$. Hence, the increment in after-tax profit in year h is $2(\text{taxrate})\mathcal{I}^n d^n(1-d)$, and the present value discounted by factor $\alpha = \frac{1}{1+r}$ of all such potential net profit increments at the time of investment is

$$2(\text{taxrate})\mathcal{I}(s)(1-d)\sum_{h=0}^{\infty}\alpha^n d^h .$$

So net investment cost is

$$I(s) = \mathcal{I}(s)\left[1 - \frac{2(\text{taxrate})(1-d)}{1-\alpha d.}\right] .$$

It should be noted that the double-declining balance method of depreciation assumed here is quite common and requires that the depreciation for each year be found by applying a rate to the book value of the asset at the beginning of that year rather than to the original cost of the asset. Book value is cost less total depreciation accumulated up to that time. If the declining-balance method is used, the tax law permits the firm to take double the rate allowed under the straight-line method. For the purposes of this model, the depreciation rate d chosen is assumed to reflect the average life of similar plants (probably in the neighborhood of 25 years), chosen solely to satisfy Internal Revenue Service regulations. A typical value of d might be .9. In actuality, the rate of obsolescence and deterioration of the plant may be treated quite independently of the depreciation structure in this model, being perhaps reflected by rising operating cost curves for the aging facility.

This analysis assumes, further, that significant operating profits will be obtained in each subsequent year, so that the anticipated tax savings will be realized. For a well-established firm contemplating major capital expenditures such an assumption is not unreasonable, and this treatment constitutes also a good approximation to costs when losses occurring in anomalous years can be carried forward for tax purposes.

Facility investment costs are often characterized by economies of size. Thus, the per-unit cost of the production unit decreases with increasing size. Approximation to such costs may be through a variety of functional forms.[1] For example, the fixed-charge linear investment cost function

$$I(s) = k_1 + k_2 s, \quad s>0 \quad k_1, k_2>0$$
$$= 0 \qquad s=0$$

has such economies, as does the power-function relation

$$I(s) = k_1 s^{k_2} \quad 0<k_2<1, \quad k_1>0 \qquad 3.4.1$$

The latter has been observed to hold for certain industry groups (Chilton [16]) with $.5<k_2<.9$. In addition, such investment costs may depend on the time in which such investment takes place, and thus have time-dependent parameters.

E. Expansion in k Steps

A k-step expansion policy is defined as the set of expansion time-action pairs $(\tau_i;\Theta_i)$, $i = 1...k$, where Θ_i represents the parameter(s) which completely describe the operating cost function of the firm after

[1]One can think of S and s as being measured in "natural" units of capital, defined solely by the way in which operating costs are affected. If different production technologies are available at the same point in time, the investment cost function would then be the lower envelope of the investment cost functions of all such technological alternatives.

the i^{th} action. For the simplest model of expansion a single parameter S_i, representing cumulated capital investments, will be employed; hence $\Theta_i = S_i = S_0 + \sum_{j=1}^{i} s_j$. The objective function to be maximized is the present value of profits less investment costs, where all prices and costs are relative to the price of capital goods (a numeraire):

$$f_o = \sum_{i=o}^{k} \int_{\tau_i}^{\tau_{i+1}} \pi(D(t),\Theta_i)e^{-rt}dt - \sum_{i=1}^{k} I(s_i)e^{-r\tau_i} \qquad 3.5.1$$

$$\bar{\tau}_o \leqslant \tau_1 < \tau_2 \ldots < \tau_k \leqslant \bar{\tau}_{k+1} .$$

The fixed planning period in this case is $(\bar{\tau}_o, \bar{\tau}_{k+1})$. As operating profits are taken to be continuously twice differentiable in τ, f_o will be quasi-concave in each of the τ_i if at every stationary point, f_o is locally concave in τ_i.[1]

$$\frac{\partial f_o}{\partial \tau_i} = 0 = -\pi(D(\tau_i),\Theta_i)e^{-r\tau_i} + rI(s_i)e^{-r\tau_i} + \pi(D(\tau_i),\Theta_{i-1})e^{-r\tau_i} \qquad 3.5.2$$

---

[1] For fixed delay L_i in operationality of the i^{th} investment, the limits of integration in (3.5.1) may have the constant L_i added, and the quasi-concavity property will remain.

$$\frac{\partial^2 f_o}{\partial \tau_i^2} = + r\pi(D(\tau_i),\Theta_i)e^{-r\tau_i} - r^2 I(s_i)e^{-r\tau_i} - r\pi(D(\tau_i),\Theta_{i-1})e^{-r\tau_i}$$

$$- \frac{\partial \pi(D(\tau_i),\Theta_i)}{\partial \tau_i} e^{-r\tau_i} + \frac{\partial \pi(D(\tau_i),\Theta_{i-1})}{\partial \tau_i} e^{-r\tau_i} \quad . \qquad 3.5.3.$$

Combining Equations (3.5.2) and (3.5.3)

$$\frac{\partial^2 f_o}{\partial \tau_i^2} = \left[- \frac{\partial \pi(D(\tau_i),\Theta_i}{\partial \tau_i} + \frac{\partial \tau(D(\tau_i),\Theta_{i-1})}{\partial \tau_i} \right] e^{-r\tau_i}$$

Thus, $\frac{\partial^2 f_o}{\partial \tau_i^2} \leqslant 0$ if

$$- \frac{\partial \pi(D(\tau_i),\Theta_i)}{\partial \tau_i} + \frac{\partial \pi(D(\tau_i),\Theta_{i-1})}{\partial \tau_i} \leqslant 0 \quad . \qquad 3.5.4$$

For the simple expansion model, a sufficient condition for (3.5.4) to hold is

$$\frac{\partial^2 \pi}{\partial S \partial \tau} \geqslant 0 \quad . \qquad 3.5.5$$

Although it is not immediately apparent, if optimal output is non-decreasing with time ($\frac{dq^*}{d\tau} \geqslant 0$), as will always be the case with the cost and revenue functions considered herein, (3-11) will hold for a remarkably large class of π-functions. As a simple example, consider

$$\pi(\tau,S) = R(q(t)) - TC(q(\tau),S) = R(q(t)) - FC(S)$$

$$- \sum_{j=1}^{J} \frac{a_j q(\tau)^j}{S^{j-1}} \quad a>0, \qquad q(\tau) = D(\tau)$$

the case of infinitely inelastic demand and a cost function with variable operating costs a J^{th} order weighted positive polynomial of output.

Then

$$\frac{\partial\pi}{\partial\tau} = \frac{dR}{dq} \cdot \frac{dq}{dt} - \left[\sum_{j=1}^{J} \frac{j a_j q^{j-1}}{S^{j-1}}\right] \frac{dq}{d\tau}$$

$$\frac{\partial^2\pi}{\partial S\partial\tau} = \left(\sum_{j=1}^{J} \frac{j(j-1)a_j q^{j-1}}{S^j}\right) \frac{dq}{d\tau} \geqslant 0 \quad .$$

More generally,

$$-\frac{\partial \pi(D(\tau),\theta_j)}{\partial\tau} + \frac{\partial\pi(D(\tau),\theta_{j-1})}{\partial\tau} = \left(-\frac{\partial TC(q,\theta_j)}{\partial q} + \frac{\partial TC(q,\theta_{j-1})}{\partial q}\right) \frac{\partial q^*}{\partial R} \frac{\partial R}{\partial D} \frac{d(D)}{d\tau}$$

can never be positive with revenues nondecreasing with time, positive marginal revenue at optimal output q^*, and marginal cost no greater after the investment than before, conditions all of which hold for the functions previously described and are likely to exist for most problems of this sort.

From (3.5.2) it may be noted that $\frac{\partial f_0}{\partial \tau_i} = g(\tau_i, \Theta)$, a function of τ_i only for fixed expansion sizes. Hence, f_0 is separable in τ: $f_0 = \sum_i f_i(\tau_i, \theta)$. However this does not ensure that f_0 is a quasi-concave function of τ; indeed, it can readily be shown that the stationary point in τ of $f_0(\tau_1, \tau_2, \Theta)$ may be locally convex for $\tau_2 = \lambda_1\tau_1 + \lambda_2$, $0 < \lambda_1 < 1$.

F. A Solution Procedure

Gross expansion planning problems typically involve a relatively few expansion steps (perhaps $k \leq K = 5$), over realistic planning horizons of perhaps 10-25 years, due to the compound effects of discounting and uncertainty in long-range forecasts. Thus, if optimal solutions for $k = 1, \ldots, K$ have been obtained, one might with reasonable certainty assume that the optimal expansion policy has not been overlooked.

If the number of non-zero potential expansion sizes s is m, the number of potential expansion size combinations for a k expansion policy is m^k. For m fairly small the continuous-time formulation admits to ready solution. For a given sequence of facility configurations f_0 may be maximized by maximizing each of the f_i over the planning period $(\bar{\tau}_0, \bar{\tau}_{k+1})$ individually with respect to τ_i due to separability. Golden section, bisection or similar search techniques may be utilized for these quasiconcave maximizations. If the solution τ_i*, i=1,...k, obtained satisfies the feasibility constraints $\bar{\tau}_0 \leq \tau_1 < \tau_2 \ldots \tau_k \leq \bar{\tau}_{k+1}$ expansion policy (τ^*, Θ) is retained. If not, the expansion set corresponding to Θ is rejected. This process is repeated for each of the m^k expansion sets, and of those whose expansion timings are feasible, the optimal one is that which yields the greatest value of f_0. Note that when infeasible timings $\tau_i^* \geq \tau_{i+1}^*$ result from the unconstrained optimization,[1] no solution enforcing feasibility $\tau_i < \tau_{i+1}$ for the expansion sizes need be considered, for a solution of no lower value of f_0 must exist with $\tau_i = \tau_{i+1}$, due to quasiconcavity, and this possibility has already been covered by the k-1 expansion optimization problem.

[1] This can occur only at the end-points $\bar{\tau}_o$ or $\bar{\tau}_{k+1}$ for $\theta_i = S_i$ if (3.5.5) holds.

G. Properties of the Optimal Solution

From the optimality conditions (3.5.2) and (3.5.4), several intuitively plausible observations may be made about the behavior of this solution. First, an increase in investment cost for a given expansion size s_i will have the effect of delaying only the i^{th} expansion, while an increase in the i^{th} expansion size itself will always result in a delay in the i^{th} and all following investments, in each case possibly requiring a fewer number of expansions to become optimal.

In addition, by use of (3.5.2) it is found that

$$\frac{\partial \tau_i^*}{\partial r} = \frac{I(s_i)}{-\frac{\partial \pi(D(\tau_i),\Theta_i)}{\partial \tau_i} + \frac{\partial \pi(D(\tau_i)\Theta_{i-1})}{\partial \tau_i}} \quad < \quad 0 \ ,$$

so a marginal increase in the discount rate will have the effect of advancing all expansions, possibly allowing a greater number of expansions to become optimal. For the case of convex or linear investment costs (including fixed charge with the number of investments given) the objective function (3.5.1) for simple expansions will be a concave function of s for fixed values of the τ_i, as a concave function of linear functions.

H. Forecast Uncertainty

For a maximum expected value solution to the simple expansion problem with known and independently distributed investment costs and demands, one need only substitute expectations $E(I)$ and $E(\pi)$ for the quantities I and π appearing in the preceding sections and solve as a deterministic problem. However, stochastic demand parameters $D(\tau)$ more typically will be correlated in some fashion. For example, one might postulate behavior of demand parameter $D(\tau+d\tau)$ given $D(\tau)$ according to a continuous diffusion process with trend $\delta(\tau)$ and independently distributed uncertainty terms $\tilde{\xi}(\tau)d\tau$; $E(\tilde{\xi}(\tau)) = 0$. Hence $\tilde{D}(\tau+d\tau) = \tilde{D}(\tau) + [\delta(\tau) + \tilde{\xi}(\tau)]d\tau$.[1]

Nevertheless, it is interesting to note that solution as a deterministic problem with approximation of $E(\pi(\tilde{D}(\tau), \theta_i))$ by $\pi(E(\tilde{D}(\tau)),\theta_i)$ will normally result in planned expansion timings being delayed from those projected to be optimal at time τ_o for a given set of expansion sizes in this naive case.[2] For the process above expectations $E(\tilde{D}(\tau))$ at τ_o can be readily obtained:

$$E_{\tau_o}(\tilde{D}(\tau)) = D(\tau_o) + \int_{\tau_o}^{\tau} \delta_{\tau_o}(t)dt. \quad \text{With}$$

---

[1] Under some circumstances it may be more realistic to take rates of change of demand parameter, $d(D)/d\tau$, to be governed by a similar stochastic process.

[2] Of course, with constant review of actual demands and other market information distributions for future demand can be updated over the planning period, resulting in a strategy contingent on realized demands, assuming that the firm makes efficient use of all available information.

marginal revenue increasing with D, π will be convex in D;

$$\frac{\partial^2 \pi}{\partial D^2} = \frac{\partial(\frac{\partial R}{\partial q})}{\partial D} \cdot \frac{\partial q^*(D,\Theta)}{\partial D} \geqslant 0 \qquad 3.8.1$$

since optimal output q* will never decrease with upward shifts in demand, $\frac{\partial q^*}{\partial D} \geqslant 0$. 3.8.2

In addition, with $\frac{\partial q^*}{\partial D} = \frac{\frac{\partial^2 R}{\partial q \partial D}}{\frac{\partial^2 R}{\partial q^2} - \frac{\partial^2 TC}{\partial q^2}}$

from differentiation of the optimal output relation implied by (3.3.2) and

$$\frac{\partial^2 TC(q,\Theta_{i-1})}{\partial q^2} \geqslant \frac{\partial^2 TC(q,\Theta_i)}{\partial q^2} \qquad 3.8.3$$ [1]

it follows that

$$\frac{\partial q^*(D,\Theta_{i-1})}{\partial D} \leqslant \frac{\partial q^*(D,\Theta_i)}{\partial D} , \qquad 3.8.4$$

1. For $\Theta_i = S_o + \sum_{j=1}^{i} s_j$ this inequality will prevail for any non-decreasing marginal cost function satisfying the horizontal additivity requirement (3.2.4), as well as for others.

so π is everywhere no less convex in D for Θ_i than for Θ_{i-1} from (3.8.1) and (3.8.4). The following inequality will hold for these unequally convex functions of random variables:

$$E(\pi(\tilde{D},\Theta_{i-1}))-E(\pi(\tilde{D},\Theta_i)) \leqslant \pi(E(\tilde{D}),\Theta_{i-1})- \pi(E(\tilde{D}),\Theta_i) \quad .$$

It is obvious then that the expansion timings determined through use of $\pi(E(\tilde{D}),\Theta)$ as a substitute for $\pi(D,\Theta)$ to satisfy equation (3.5.3) will be no earlier than the maximal expected value timings employing $E(\pi(\tilde{D},\Theta))$ in place of $\pi(D,\Theta)$ in this equation.

I. Discrete-Time Formulation

The simple expansion investment problem can be formulated as a discrete-time dynamic program. Period profits $\pi_n(D_n,S_n)$ represent the total maximal operating profit in period n from capacity S_n, based upon the average demand characteristics for that period. For finite horizon N problems and known demand relations, the basic recursion relation is

$$f_n(S_n) = \pi_n(D_n,S_n) + \max_{s_n \geq 0} \{-I_n(s_n) + \alpha f_{n+1}(S_n + s_n)\}$$

$$n = 1,\ldots,N-1$$

with $f_N(S_N) = \pi_N(D_N,S_N)$, where s_n is the expansion size in period n, $\alpha = \frac{1}{1+r}$ is a discount factor, and the planning interval [1,N] is entered with some initial plant size $\bar{S}_1$.[1] With demand parameter D_n for each period known with certainty, one-period profits may be represented by $\pi_n(S_n)$ instead of $\pi_n(D_n,S_n)$. For stochastic demand parameters and investment costs with initially known independent distributions for each period a maximal expected-value plan given available information in period 1 can be obtained by using the expected one-stage returns $E_1(\pi(D,S))$ and $E_1(I(s))$ in place of $\pi(D,S)$ and $I(s)$ above. More realistically, however, distributions of future demand parameters will depend on their past values. If demand distributions are (discrete) markovian in nature (depending only on prior period demand) with transition probabilities $P_n(D_{n+1}/D_n)$ and realized demands observable (as will usually be the case), then D_n can be treated as an additional state variable, and the recursion relation rewritten as

[1]Periods may be of unequal length, suitably reflected in the values for π_n and α_n. For computational purposes it may be desirable to use periods of longer lengths towards the end of the planning period, as one might expect early decisions to be least sensitive to errors due to time discretization later in the planning period.

$$f_n(D_n,S_n) = \pi_n(D_n,S_n) + \max_{s_n \geq 0} \{-I_n(s_n) + \alpha E(f_{n+1}(D_{n+1}, S_n+s_n))\}$$

where

$$E(f_{n+1}(D_{n+1},S_n+s_n)) = \sum_{D_{n+1}} f_{n+1}(D_{n+1},S_n+s_n)P_n(D_{n+1}/D_n).^1$$

The above procedure will provide an optimal solution to the problem as stated. As a large number of other inputs will usually also not be known with precision initially, it would be heuristically desirable to update these as new information becomes available, and to resolve the problem periodically (a "rolling strategy").

J. Fixed-Charge Linear Investment Costs

The search procedure for identifying optimal values of expansion size (s) for each capital level (S) can be simplified considerably for the case of fixed-charge linear investment costs.

---

[1] Similar treatment of markovian demand is possible for the more complex dynamic programming models in later sections.

If we let $F_n(S_n, s_n) = -I_n(s_n) + \alpha f_{n+1}(S_n + s_n)$, then $f_n(S_n) = \pi_n(S_n) + \max_{s_n \geq 0} F_n(S_n, s_n)$. For clarity we temporarily drop the subscript n on s and S. Let $s^*(S)$ be the value of s which maximizes $F_n(S,s)$. For the fixed-charge linear investment cost function,

$$I_n(S) = a_n + I_n s, \quad s>0 \quad I_n, a_n > 0,$$
$$= 0 \qquad s = 0\ ,$$

we have

$$f_n(S) = \pi_n(S) + \max\ \{-a_n + \max_{s \geq 0}\{F_n^+(S,s)\},\ F_n^0(S)\}$$

where $F_n^+(S,s) = -I_n s + \alpha f_{n+1}(S + s)$ and $F_n^0(S,s) = \alpha f_{n+1}(S)$. Let s^{**} maximize $F_n^+(S,s)$. Then $s^*(S)$, the optimal expansion size for capital level S, is

$$\begin{Bmatrix} s^{**} \\ 0 \end{Bmatrix} \text{ according as } -a_n + F_n^+(S, s^{**}) \begin{Bmatrix} > \\ \leq \end{Bmatrix} F_n^0(S).$$

For the remainder of this section subscripts n, n+1 will be omitted entirely, and we define $s_1 \equiv s^{**}(S_1)$.

Lemma 1: If s, δ, $s_1 - \delta$ are ≥ 0, then $s_1 - \delta = s^{**}(S_1 + \delta)$.

Proof: Since s_1 maximizes $F^+(S_1, s)$ s.t. $s \geq 0$, we have

$$F^+(S_1, s_1) \geq F^+(S_1, s) \qquad \forall s \geq 0, \text{ or}$$

$$-Is_1 + \alpha f(S_1 + s_1) \geq -Is + \alpha f(S_1 + s).$$

Letting $s' = s - \delta$, so that $s = s' + \delta$, we substitute into the above inequality, obtaining

$$-Is_1 + \alpha f(S_1 + s_1) \geq -I[s' + \delta] + \alpha f(S_1 + s' + \delta), \quad \forall s' + \delta \geq 0$$

$$-I[s_1 - \delta] + \alpha f((S_1 + \delta) + (s_1 - \delta)) \geq -Is' + \alpha f((S_1 + \delta) + s') \forall s' \geq -\delta \ ,$$

from which it follows that

$$F^+((S_1 + \delta), (s_1 - \delta)) \geq F^+((S_1 + \delta), s') \ \forall s' \geq -\delta; \text{ hence also } \forall s' \geq 0.$$

Therefore it must be that $s_1 - \delta = s^{**}(S_1 + \delta) \geq 0$.

QED

<u>Corollary to Lemma 1</u>: If δ, s_1 are ≥ 0, then <u>only</u> if $s_1 - \delta < 0$ can $s^{**}(S_1 + \delta)$ be $> s_1$.

<u>Proof</u>: None is necessary. The case of $s_1 - \delta < 0$ is not covered by Lemma 1.

The import of Lemma 1 and corollary is that as S is increased, $s^{**}(S)$ will decrease by equal amounts (if originally non-zero), until some point S_b at which $s^{**}(S_b) = 0$. As S is further increased beyond S_b, $s^{**}(S)$ may again at some point take on a positive value, then decrease again in the same fashion. A sawtooth-like pattern will then result (Figure 3-6). Lemma 2 will aid in computing the positive value which may possibly be taken on by $s^{**}(S)$ as S is increased beyond such points as S_b.

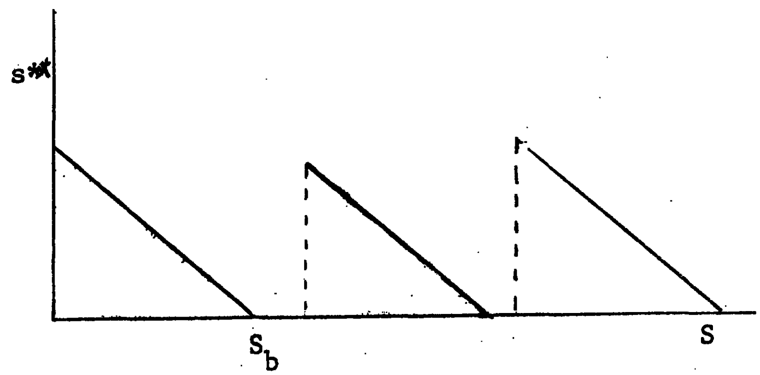


Figure 3-6

Lemma 2: Take S_0 to be the smallest feasible value of S and $s^i>0$, $i = 1,\ldots,m$, the m positive values (if any exist) of s' which locally maximize $F^+(S_0,s')$ s.t. $s'\geq 0$. Let $G(S) = \{s^i\}/s^i - S + S_0>0$. If $S_1>S_0$ and $s_1>0$, then $s_1 = s^j - S_1 + S_0$, where $s^j\varepsilon G(S_1)$ and $F^+(S_0,s^j)\geq F^+(S_0,s^i)\forall s^i\varepsilon G(S_1)$.

Proof: The proof is in two parts. First, we will prove that $s_1 = s^j - S_1 + S_0$, where $s^j\varepsilon G(S_1)$. Then it will be proven by contradiction that $F^+(S_0,s^j)\geq F^+(S_0,s^i)\forall s^i\varepsilon G(S_1)$

Part 1: Consider any state, optimal-decision pair, S_1, s_1, with $S_1>S_0$ and $s_1>0$. Since s_1 maximizes $F^+(S_0,s)$ s.t. $s\geq 0$, we have

$$F^+(S_1,s_1) \geq F^+(S_1,s) \quad \forall s \geq 0, \text{ or}$$

$$-Is_1 + \alpha f(S_1 + s_1) \geq -Is + \alpha f(S_1,s) \quad \forall s \geq 0. \qquad (3.10.1)$$

Let $s^j = s_1 + S_1 - S_0$ and $s' = s + S_1 - S_0$, so that $s_1 = s^j - S_1 + S_0$ and $s = s' - S_1 + S_0$. Substituting into (3.10.1) we obtain

$$-I[s^j - S_1 + S_0] + \alpha f(S_1 + s^j - S_1 + S_0) \geq -I[s' - S_1 + S_0]$$

$$+ \alpha f(S_1 + s' - S_1 + S_0) \quad \forall\ s' - S_1 + S_0 \geq 0 \text{ ; hence}$$

$$-Is^j + \alpha f(S_0 + s^j) \geq -Is' + \alpha f(S_0 + s') \forall s' \geq S_1 - S_0 > 0$$

or

$$F^+(S_0,s^j) \geq F^+(S_0,s') \qquad \forall s' \geq S_1 - S_0 > 0.$$

Therefore s^j maximizes $F^+(S_0,s')$ s.t. $s' \geq S_1 - S_0 > 0$. Note that since $s^j = s_1 + S_1 - S_0$ and $s_1 > 0$, $s^j > S_1 - S_0$. It follows that s^j at least locally maximizes $F^+(S_0,s')$ s.t. $s' \geq 0$. In addition, $s^j - S + S_0 = s_1 > 0$. Hence $s^j \varepsilon G(S_1)$.

Part 2: Proof is by contradiction. It has been proven in Part 1 that $s_1 = s^j - S_1 + S_0$ where $s^j \varepsilon G(S_1)$. Suppose that it is not the case that $F^+(S_0,s^j) \geq F^+(S_0,s^i)\ \forall s^i \varepsilon G(S_1)$. Then there exists $s^k\ \varepsilon G(S_1)$ such that $F^+(S_0,s^j) < F^+(S_0,s^k)$. Thus

$$-Is^j + \alpha f(S_0 + s^j) < -Is^k + \alpha f(S_0 + s^k).$$

We have $s_1 = s^j - S_1 + S_0$, so $s^j = s_1 + S_1 - S_0$. We may also let $s_2 = s^k - S_1 + S_0$, so that $s^k = s_2 + S_1 - S_0$, and substitute into the above inequality, obtaining

$$-I[s_1 + S_1 - S_0] + \alpha f(S_0 + s_1 + S_1 - S_0) < -I[s_2 + S_1 - S_0]$$
$$+ \alpha f(S_0 + s_2 + S_1 - S_0)$$
$$- Is_1 + \alpha f(S_1 + s_1) < -Is_2 + \alpha f(S_1 + s_2). \qquad 3.10.2$$

Since $s^k \varepsilon G(S_1)$, $s_2 = s^k - S_1 + S_0 > 0$, resulting with 3.10.2 in a contradiction to (3.10.1). Therefore, it must be that $F^+(S_0, S^j) \geq F^+(S_0, s^i)$ $\forall$ $s^i \varepsilon G(S_1)$.

QED

Lemma 2 tells us that we need only consider as a potential non-zero level of s_1 the value $s^j - S_1 + s_0$, where s^j provides the greatest non-zero local maximum to $F^+(S_0, s')$ s.t. $s' \geq 0$ for which $s_1 = s^j - s_1 + S_0 > 0$. The values $s^i \varepsilon G(S_0)$ may be initially arranged in decreasing order of $F^+(S_0, s^i)$ and thus only the first element in the list need be examined for any value of S. As S is increased, once a value s^i becomes less than $S - S_0$, it may be permanently (for this stage) removed from G(S). It is of computational value to note also that if S_e is any point at which $G(S_e)$ becomes empty, $s^{**}(S) = 0$ for $S \geq S_e$.[1]

---

[1] At the expense of considerably more notational complexity the same results would have been obtained for every fixed value of D_n in the markovian demand formulation of section III-I.

K. Sufficiency of Unimodal Search

If $I_n(s_n)$ is convex and $\pi_n(S_n)$ is concave for all n, then one-stage returns are concave. As a result of concave one-stage returns, n^{th}-stage cumulated returns $f_n(S_n)$ will be concave.[1] $F_n(S_n, s_n)$ will then obviously be cancave. A less-restrictive sufficient condition for concavity of F_n exists, however.

Suppose that $F_{n+1}(S_{n+1}, s_{n+1})$ is concave. Then $\max_{s_{n+1} \geq 0} \{F_{n+1}(S_{n+1}, s_{n+1})\}$ must be concave.[2] But $F_n(S_n, s_n) = [-I_n(s_n) + \alpha\pi_{n+1}(S_{n+1})] + \max_{s_{n+1} \geq 0} \{F_{n+1}(S_{n+1}, s_{n+1})\}$, where $S_{n+1} = S_n + s_n$. Hence, a sufficient condition for $F_n(S_n, s_n)$ to be concave is that the expression in squared brackets be concave, or that the concavity of I_n be less than that of $\alpha\pi_{n+1}$ for admissible values of s_n and S_n. Since last-stage returns π_N are concave, these conditions are sufficient for $F_n(S_n, s_n)$ to be concave for all n (by induction). Under such circumstances, golden section, bisection or similar unimodal search techniques may be employed to determine optimal expansion sizes $s_n^*(S_n)$ at each stage.

---

[1] A discussion of this well-known property appears in Hadley [40], p. 375.

[2] The maximum within a convex set of a concave function in any subset of its arguments is concave.

L. Age-Dependent Production Costs

Aging may affect the producing facilities of the firm in a variety of ways. Replacement of production units may become mandatory at points in time due to their physical exhaustion alone, or the effects of aging may show up in the output-cost relation of the firm. In the latter case, it will be assumed here that a single state variable, H_n, as a parameter in the operating cost function will be sufficient to completely describe all aging effects. Implicit in this assumption is the property that if at any point in time additional capital, s_n, is added to the production system of size S_n the new age variable H_{n+1} will depend only on the variables S_n, s_n and H_n. Thus no "memory" of each specific past addition size is necessary.

For example, if no action is taken between periods n and n+1, we may adopt a growth pattern for H: $H_{n+1} = H_n\gamma_n$, $\gamma_n > 1$. For an addition of size s in n, the new value H_{n+1} might be weighted by old and new capacities, where a new unit begins with new-unit parameter h_n, adjusted by γ_n.[1,2] Anticipated technological improvements may allow h_n to decrease with increasing n. Hence,

---

1 $H_n - h_n$ corresponds to Terborgh's cumulative inferiority gradient for the entire production system.

2. The correct age parameter must always result after several additions by this process since weighted average ω has the property that $\omega(a,b,c) = \omega(\omega(a,b), c)$.

$$H_{n+1} = \left(\frac{S_n H_n + s_n h_n}{S_n + s_n}\right) \gamma_n \qquad \text{or}$$

$$(S_n + s_n) H_{n+1} = S_n H_n \gamma_n + s_n h_n \gamma_n .$$

This is probably the simplest fashion in which H_{n+1} can be a function of S_n, s_n, and H_n which makes reasonable economic sense. Consider H_n affecting only fixed operating costs which are a power function of plant size (as suggested in section 3B) in the following manner:

$$FC_n(S_n, H_n) = (H_n a_o S_n)^{\beta} \quad 0<\beta<1$$

$$FC_{n+1}(S_n + s_n, H_{n+1}) = FC_{n+1}(S_n, s_n, H_n) = [S_n H_n a_o \gamma_n + s_n h_n a_o \gamma_n]^{\beta}$$

Note that for $\beta = 1$ (fixed costs directly proportional to facility size), the fixed costs of old and newly added units are simply summed to obtain the total fixed cost for the period, which is precisely what one would expect in the absence of fixed-cost economies. Of course, the age factor H may appear in the quantity-variable terms of the operating cost function as well, giving rise to the effect of increasing marginal cost with age.

The dynamic programming recursion relation for this situation of aging with expansion but inadmissible replacement of old units is

$$f_n(S_n,H_n) = \pi_n(S_n,H_n) + \max_{s_n \geq 0} \{ -I_n (s_n) + \alpha f_{n+1}(S_n + s_n,H_{n+1}) \}, n=1,\ldots N-1$$

with

$$f_N(S_N,H_N) = \pi_N(S_N,H_N).$$

As there are two-state variables and but one decision variable in this formulation, computational feasibility is likely for reasonably fine discrete finite grids imposed on s_n, S_n, and H_n, and interpolation of values of f_{n+1} over H_{n+1} should be satisfactory due to continuity of π and thus f in H. Assuming that a realistic upper bound $\bar{s}_n$ can be placed on expansion size, H_{n+1} can be also be bounded:

$$\gamma_n \left[\frac{S_n H_n + \bar{s}_n h_n}{S_n + \bar{s}_n} \right] \leq H_{n+1} \leq H_n \gamma_n .$$

M. Other Approaches

Dynamic programming as a solution method has several disadvantages, including a significant increase in computational effort required for finer finite grids imposed upon variables, the difficulty of tightly bounding potentially optimal variable values, and the non-availability of good solutions short of the final iterations. In addition, one cannot easily constrain the number of periods between investments or number of expansions for the purpose of sensitivity analysis, to avoid implicit consideration of policies which are believed to be non-optimal, or because of practical restraints related to cash flow, debt levels, etc. We will discuss another approach which avoids many of these difficulties while introducing a few of its own.

Let us examine the programming formulation of the investment-aging problem, which requires maximization of all discounted operating profits less investment costs subject to state-stage transition constraints:

$$\text{Max:} \qquad fo = \sum_{n=1}^{N} \alpha^n \pi_n (S_n, H_n) - \sum_{n=1}^{N-1} \alpha^n I_n(s_n)$$

$$\text{s.t.} \qquad S_1 - \bar{S}_1 = 0 \qquad (1a)$$

$$S_{n+1} - S_n - s_n = 0 \qquad n=1,\ldots..N-1 \qquad (1b)$$

$$H_1 - \bar{H}_1 = 0 \qquad (2a)$$

$$H_{n+1} - \gamma\left[\frac{S_n H_n + s_n h_n}{S_n + s_n}\right] = 0 \qquad n=1,\ldots..N-1 \qquad (2b)$$

$$s_n \geq 0 \qquad n=1,\ldots..N-1 \qquad (3)$$

The firm is assumed to enter the planning interval with given size and age, $\bar{S}_1$ and $\bar{H}_1$. Consider next the Kuhn-Tucker (K-T) necessary conditions for an optimal solution, assuming differentiability of all π_n and I_n (the constraint qualification is trivially satisfied). With multipliers λ, μ, and ν associated with constraint sets (1), (2), and (3) above, respectively, these are

$$\alpha^n \frac{\partial \pi_n}{\partial S_n} + \lambda_{n-1} - \lambda_n = 0 \qquad n=1,\ldots N-1 \qquad (4)$$

$$\alpha^N \frac{\partial \pi_N}{\alpha S_N} + \lambda_{N-1} = 0 \qquad (5)$$

$$\alpha^n \frac{\partial \pi_n}{\partial H_n} + \mu_{n-1} - \zeta_n = 0 \qquad n=1,\ldots N-1 \qquad (6)$$

$$\alpha^N \frac{\partial \pi_N}{\partial H_N} + \mu_{N-1} = 0 \qquad (7)$$

$$-\alpha^n \frac{dI_n}{ds_n} - \lambda_n + \frac{\zeta_n(H_n - h_n)}{S_n + s_n} + \nu_n = 0 \qquad n=1,\ldots N-1 \qquad (8)$$

$$\nu_n s_n = 0 \qquad n=1,\ldots N-1 \qquad (9)$$

$$\nu_n \geqslant 0 \qquad n=1,\ldots N-1 \qquad (10)$$

where

$$\zeta_n = \mu_n \frac{\gamma S_n}{S_n + s_n} \qquad n=1,\ldots N-1 \qquad (11)$$

Suppose that we are given values for λ_o (associated with 1a) and μ_o (associated with 2a). Assume for the moment that these are optimal values for λ_o and μ_o. Then it will be possible to obtain K-T satisfying points by solving (1)-(11) sequentially rather than simultaneously.

The procedure is straightforward. Suppose that at any intermediate stage we have values for S_n, H_n, λ_{n-1}, and μ_{n-1}. Then (4) may be solved for λ_n and (6) solved for ζ_n. If it is possible for s_n to be positive, solving (8) with $\nu_n = 0$ will provide the appropriate value. One must consider also the possibility at each stage that $s_n=0$, for this will always satisfy K-T. Equations (11) will provide a value for μ_n given values for S_n , s_n, and ζ_n, and (1b) and (2b) will give values for S_{n+1} and H_{n+1}, respectively. The process may now be repeated for stage n+1.

Several observations may be made about this procedure:

(1) with π concave in S and concave or convex in H there will be no ambiguity in determining values for λ_n and ζ_n from (4) and (6). With I concave in s there will be no ambiguity about a positive value for s_n.

(2) At most 2^{N-1} combinations need be examined (for $s_n=0$ or $s_n>0$). Preliminary computational experience suggests, however, that the actual number will be nearer to 2^m, where m is the number of times that investment takes place in the optimal solution, (this may be a small number) since (8) and (9) cannot always be satisfied with $s_n>0$.[1]

(3) The best of the solutions so generated will, if λ_o and μ_o were chosen correctly, be the optimal and thus (5) and (7) will be automatically satisfied.

---

[1] As partial solutions may be discarded when their upper bounds (some of which are obvious) exceed the objective function value for the best solution at hand, this number may be reduced further.

The optimal values of λ_o and μ_o are usually not known beforehand, however. If the initial quantity of capital ($\bar{S}_1$) were optimal with respect to the planning interval employed, optimal λ_o would equal zero (this may be a poor state to arrive at with respect to the firm's present value of cash flow at earlier points in time, though). A smaller value for $\bar{S}_1$ would imply optimal λ_o positive while a larger value for $\bar{S}_1$ would imply optimal λ_o negative. The optimal values of all μ_n must be negative with operating costs increasing in age parameter. Some search over λ_o and μ_o will generally be required.[1] Investment periods may be fixed at any point in the search, avoiding the combinational problem. Terminal conditions (5) and (7) will come close to being satisfied with good solutions for a sufficiently long horizon (as α^N, λ_{N-1}, and μ_{N-1} all approach zero), although with expansion periods fixed λ_o and μ_o may be perturbed in an attempt to secure exact fulfillment of these conditions (due to resulting continuity of the terms in (5) and (7) in λ_o and μ_o).[2]

---

[1] A similar procedure applied to the simple problem of expansion without aging would require only the one-dimensional search over λ_o and may thus be more competitive with the dynamic programming approach.

[2] It is worth noting that the discrete maximum principle is not applicable to this problem. Since $H_n > h_n$ and $\mu_n < 0$, the Hamiltonian is found to be always maximized either with $s_n = 0$ or $s_n = \infty$, values which will usually result in a poor solution. The weak form of the discrete maximum principle provides only necessary conditions for an optimum and does not avoid the combinational problem since the stationary point of the Hamiltonian is not always the maximizing point (Converse[19], section 5.8). Many of the classical iterative procedures based on this principle (for example, Fan and Wang [32] p. 17 ff.) will be highly dependent on starting values for λ_o, μ_o, and each of the s_n in this problem and thus appear to be most useful for perturbing a solution in order to get conditions (5) and (7) to hold, for which they are unlikely to be worth the effort.

N. Stochastic Independently Distributed Demand Parameters and Expansion Costs

In the case of uncertain forecast of expansion costs and demand parameters, the dynamic programming structure may, under some circumstances, be appropriate for obtaining efficient expected value - variance solution pairs. For the case of stochastic expansion costs and demand parameters with known and independent distributions, use of multipliers $\lambda, \bar{\lambda} \in [0,1]$, $\bar{\lambda} = 1 - \lambda$ associated with total return expected value and variance, respectively, may be possible without increasing the state dimensionality of the problem to be solved for each value of λ.[1] Since the objective function will take the form of a weighted sum of single-stage returns and variances, the following recursion relation for the n^{th} stage is found to result:

$$g_n(\Theta_n) = \max_{s \geq 0} \{\lambda[E(\pi(\tilde{D}_n,\Theta_n) - E(\tilde{I}(s))] -$$

$$\bar{\lambda}\alpha^n \left[\left(\frac{\partial\pi(D,\Theta_n)}{\partial D} \Bigg|_{E(\tilde{D}_n)} \right)^2 V(\tilde{D}_n) + V(\tilde{I}(s)) \right]$$

$$+ \alpha g_{n+1}(\Theta_{n+1}) \},$$

---

[1]Again we seek an open-loop solution without considering feedback from observing realized values of stochastic variables over the planning interval. This is, admittedly, an unrealistic situation, but it may provide an approximate solution to problems involving serially correlated demand.

a linear approximation to π at $E(\tilde{D}_n)$ being taken to estimate $V(\pi_n)$. Unfortunately, multipliers may not exist for all efficient pairs, as the expected total return is not generally a concave function of the decision variables.

To arrive at efficient solutions explicitly considering positive covariances between stochastic quantities between periods would necessitate the addition of at least another state variable, resulting in a severe increase in computational difficulty. Note, however, that reducing any expansion size s_n will always lower the total solution variance (as $0 < \frac{\partial\pi(D,\Theta_{i-1})}{\partial D} \leq \frac{\partial\pi(D,\Theta_i)}{\partial D}$, $\frac{\partial D}{\partial\tau} \geqslant 0$) ,

so that an unimaginative heuristic procedure for reducing solution variance beginning with a maximal expected-value solution might involve successive decrements of expansion sizes by amounts proportional to marginal contribution to variance,[1] followed by re-optimization of timings, repeated until a tolerable value of variance is obtained.

1/ Profit covariances can be approximated as

$$\mathrm{cov}_{\pi_{n,k}} \simeq \left.\frac{\partial\pi(D,\Theta_n)}{\partial D}\right|_{E(\tilde{D}_n)} \left.\frac{\partial\pi(D,\Theta_k)}{\partial D}\right|_{E(\tilde{D}_k)} \mathrm{cov}(D_n,D_k)$$

O. Retirement of Production Units

The major results of this chapter can easily be extended to encompass the case of new investment with retirement at prespecified times of production units in existence at the beginning of the planning interval. All that is required is to subtract from S at any point in time τ or period n the sum of sizes of production units already retired. Age parameters can similarly be computed relative to existing old production units (for example using weighted averages) at any point in time. The solution method suggested in section IIIM additionally allows the possibility of retiring production units introduced within the planning interval, as partial solutions are available at every point in the procedure. Without retirement times being prespecified, however, the combinatorial aspects of this problem would obviously be greatly increased.

The retirement decision, particularly when existing capital is large relative to future increments, is often as important economically as the investment decision and difficult to separate from the latter. In many cases investments will simply coincide in time with retirements (due in part to capital acquisition economies). Retirement time will be treated as a decision variable in the remaining chapters.

Chapter IV

THE STATIC REPLACEMENT PROBLEM

A. Nature of the Problem

For the static replacement problem, it is assumed that the output of each of the several existing production units is invariant with time. Such will be the case under either of the following conditions:

1) marginal costs unaffected by age of the production unit

2) outputs of each production unit are not interchangeable (for example, different product lines or sequential production), and demands are infinitely inelastic

if the price-demand function is stationary with respect to time from the beginning of the planning period and known with certainty.

As production units age, then, their production costs rise in some assumedly predictable fashion. Total production costs will rise in the absence of replacement of production units.

Replacement units may or may not be more efficient than the units they replace, to be reflected in the production cost-age function. Investment costs will, in general, depend on the number, sizes and ages of the production units replaced at any point in time, and may usually be concave in the total capacity replaced. Both initial and replacement production units are assumed indivisible.

Clearly, if demand were stationary over all time, past and future, optimal expansion policy with economies of capacity acquisition would likely require but a single production unit, with periodic replacement of same, and indeed over an infinite horizon such a policy may become optimal beyond some point in time. However, the firm begins the planning period with a set of Φ production units of possibly differing ages and sizes, presumably due to demand having been growing during some prior time interval. Revenues may be neglected as they remain constant in this static case.

B. Fixed-Charge Linear Investment Costs

For the case in which investment costs are fixed-charge linear functions of replacement size, the static replacement problem can be formulated as an integer program. Let $H_{kni} = H_{kni}(s_i)$ be the production cost in period n associated with unit i introduced in period k.

$H_{kni} = \infty, k>n$. Let $D_{ki} = d_k s_i$ be the linear portion of net investment cost associated with replacing a unit of size s_i in period k. F_k will represent the fixed charge incurred in period k from <u>any</u> new investment. Salvage values are assumed independent of age and may be netted from replacement costs.

The variables are:

X_{kni} = 1 if production unit i introduced in k is in use in period n
= 0 otherwise

Y_{ki} = 1 if production unit i is replaced in period k
= 0 otherwise

Z_k = 1 if any production unit is replaced in k
= 0 otherwise

As the planning period is entered with a set of Φ production units, $Y_{oi} = 1$, $i = 1, \ldots \Phi$. The object is to find a minimal discounted cost replacement policy over the finite discrete time interval [1,N] while maintaining constant total capacity $\sum_i s_i$. The mathematical program is

MPI

(1) $\min \sum_i \sum_k \sum_n X_{kni} H_{kni} \alpha^n + \sum_i \sum_k D_{ki} \alpha^k + \sum_k F_k Z_k \alpha^k$

subject to

(2) $\sum_{0 \leqslant k \leqslant n} X_{kni} = 1 \qquad i = 1,\ldots,\Phi, \quad n = 1,\ldots,N$

(3) $X_{kni} \leqslant Y_{ki} \qquad i = 1,\ldots,\Phi, \quad n = 1,\ldots,N$
$0 \leqslant k \leqslant n$

(4) $Y_{ki} \leqslant Z_k \qquad i = 1,\ldots,\Phi, \quad k = 1,\ldots,N$

(5) $0 \leqslant X, Y, Z \leqslant 1$

(6) Y integer

(7) $Y_{ik} \geqslant Y_{jk} Y_{j\ell} Y_{i\ell} \qquad i = 1,\ldots,\Phi, \quad j = 1,\ldots,\Phi, \quad \ell = 1,\ldots,k-1$

Constraint set (2) ensures that total output remains constant over the planning interval. Constraint sets (3) and (4) assure that the linear and fixed-charge, respectively, portions of period k investment cost are incurred if a production unit of a given vintage n-k is employed in n. Constraint set (7), perhaps difficult to interpret, ensures only that whole production units are replaced.

For any feasible solution in Y to MPI, clearly an integer solution in Z is optimal; Z_k is fixed at zero unless constraint set (4) forces it to be fixed at one. In addition, X_{kni} is set to one for the greatest $k \leq n$ for which $Y_{ik} = 1$; to zero for all other k. Thus, only the newest production units in existence are used, as these have the lowest operating costs H_{kni}.

C. A Branch-Bound Algorithm

One technique that may be employed to solve this problem is that of branch-and-bound.

Lower bounds on the optimal solution to MPI can be obtained by solving a series of less-constrained problems. Define the surrogate constraints:

$$(3a) \quad \sum_{n>k} X_{kni} \leq Y_{ki}(N - k)$$

$$(4a) \quad \sum_i Y_{ki} \leq Z_k(\Phi)$$

MP2 is the less-constrained problem (1), (2), (3a), (4a), (5). Branching takes place by successively fixing variables Y_{ki} to their extreme values, zero and one. If at any stage of the branching process, U is the set of indices ik of Y variables fixed at 0, V is the set of indices of Y variables fixed at 1, and W contains free variable indices, then the solution to MP2 will provide a lower bound to the solution to MPI for Y_{ik}, $ik \in U \cup V$, fixed at integer values. Search is terminated beyond any node (representing a set $U \cup V$) for which the lower bound obtained from MP2 exceeds the value of the least-cost feasible solution to MPI obtained thus far (the incumbent).

Upon inspection of the dual to the linear program MP2, an optimal solution is found to be one in which

$X^*_{kni} = 1$ for all k which provide the minimum value to

$$\left[\{ \min_{\substack{ik \in V \\ k \leq n}} H_{kni}\alpha^n \}, \{ \min_{\substack{ik \in W \\ k \leq n}} (H_{kni}\alpha^n + \frac{D_{ki}}{N-k}\alpha^k + \frac{F_k\alpha^k}{(N-k)r}) \} \right]$$

$X^*_{kni} = 0$ for all other k

$$Y^*_{ki} = \frac{\sum_{n \geq k} X_{kni}}{N-k} \qquad ki \in W$$

$$Z^*_k = \frac{\sum_i Y_{ki}}{\Phi} \qquad \text{all } ki \in W$$

and thus a lower bound to MPI is obtained. In addition, constraint set (7) involving only Y variables can be utilized to prevent branching to infeasible replacement policies. One may scan fixed values of Y_{ik} for the most recent (if any) period ℓ in which replacement of i took place ($Y_{i\ell} = 1$). If any other facility j was replaced simultaneously with i in ℓ ($Y_{j\ell} = 1$), and if facility j has been replaced in k($Y_{jk} = 1$), then a branch to $Y_{ik} = 0$ cannot be made. Similarly, if facility j has been denied replacement in k ($Y_{jk} = 0$), then a branch to $Y_{ik} = 1$ cannot be made.

D. Piecewise-Linear Concave Investment Costs

At the expense of considerably more computational difficulty, the static investment-replacement model may be generalized to encompass the case of piece-wise-linear fixed-charge concave investment costs. As is well-known, a piecewise linear concave fixed-charge function g(s) may be represented in the following fashion:

$$g(s) = \min_{s_r}[\sum_r f_r(s_r)], \qquad \sum_r s_r = s \qquad s_r, s \geqslant 0$$

$$= \min_r[f_r(s) + \sum_{m \neq r} f_m(0)] = \min_r[f_r(s)]$$

where f_r are each fixed-charge linear functions. Therefore, it is only necessary to allow any one of a number of fixed-charge investment cost functions to be utilized in each time period to represent any piecewise linear concave investment function. All linear portions of investment costs must remain proportionate across facilities: $D_{kir} = s_i d_k \beta_r$, where β_r is the proportionality factor associated with investment cost function r. Fixed charges F_{kr} now also depend on the investment cost function selection, r. Bounds are developed in the same way as before, although new branching restrictions are that Y_{kir} cannot be fixed at one if Y_{kim}, $m \neq r$, has already been fixed at one. It is doubtful whether large problems can be solved in this fashion, although it may be possible to solve small problems involving two- or three-segmented investment functions.

CHAPTER V

EXPANSION WITH REPLACEMENT AND OTHER EXTENSIONS

A. General Model

The case in which new investment in production facilities may take place in response to both nonstationary product demand and the aging of existing facilities, requiring replacement, is the subject of this chapter. In addition, technological improvements may reduce the operating costs of a potential new production unit in a specific fashion, further encouraging replacement of existing production facilities. This situation may be most compactly described using recursion relations, for which dynamic programming is, in principle, a solution technique. It may be noted at the outset that all of the dynamic programming approaches to solution of problems allowing retirement of production units are especially suited to cases in which demand is expected to decline beyond some point in time, as the computational method does not require nondecreasing demand, and as it is admissible for retired production units to be replaced with ones of smaller or zero size.

For each of the N discrete time periods within the planning interval, two state vectors, describing the sizes and ages of potentially existing production units may be defined:

$S = (S_1, \ldots S_{r+n-1})^T$, production unit sizes representing initial plus subsequent capital additions

$H = (H_1, \ldots H_{r+n-1})^T$, production unit age parameters.

If the planning interval is entered with r production units, the state vectors for the n^{th} stage will have r + n - 1 dimensions. Decisions are s, the size of the addition to be made in period n, if any, and a replacement vector X, where $X_j = 1$ if production unit j is to be retired, $X_j = 0$ otherwise.

Recursion relations for this problem are

$$f_n(S,H) = \max_{\substack{s \geq 0 \\ X}} \{\pi_n(S,H) - I_n(s) + R_n(SX^T, HX^T) + \alpha f_{n+1}(S',H')\}$$

$$n = 1 \ldots N-1$$

with $f_N(S,H) = \pi_N(S,H)$,

where

$$S' = \begin{pmatrix} S(\bar{1}-X)^T \\ s \end{pmatrix}, \quad H' = \gamma_n \begin{pmatrix} H(\bar{1}-X)^T \\ h_n \end{pmatrix}.$$

Period profits π_n depend on the sizes (S) and ages (H) of existing production units.

R is some salvage value of retired units, γ_n defines the growth of each age parameter H_j from one period to the next, and h_n is the age parameter of a new unit introduced in period n. If m addition sizes are possible in each period, then there will be $2^r(n-1)^m$ states and as many as $m2^{r+n-1}$ decisions for each possible state even for this highly structured model. Thus, computation is likely to be impractical for problems of realistic size using this approach.

B. <u>Suboptimal Solutions</u>

For a given sequence of expansion-retirement actions, optimal timings can in many cases be obtained by solving the set of quasi-concave optimization problems described in Chapter 3. For consistency, assume again the timing for the i^{th} action can be represented by the continuous variable τ_i. From (3.5.4) the only condition required for quasi-concavity of total discounted net profits in each of the τ_i is

$$-\frac{\partial\pi(D(\tau),H',S')}{\partial\tau}+\frac{\partial\pi(D(\tau),H,S)}{\partial\tau} \leqslant 0, \quad i = 1\ldots k.$$

It is readily seen that these conditions will obtain under non-decreasing demand, positive marginal revenues at optimality, and marginal cost non-increasing in each of the S_j and non-decreasing in each of the H_j if replacements are always with units of at least equal size S_j. As in the expansion-only situation of Chapter 3, demand uncertainty will again result in advances in optimal timings for the initial plan for independently distributed demand parameters.

For k given expansion timings $\bar{\tau}_0 \leq \tau_i \leq \bar{\tau}_{k+1}$, dynamic programming might be employed to solve realistically sized action-only problems. The recursion relation at each stage i is

$$f_i(S,H) = \max_{\substack{s \geq 0 \\ X}} \left\{ \int_{\tau_i}^{\tau_{i+1}} \pi(D(t), S, H)\, e^{-rt} dt \right.$$

$$\left. + \, e^{-r(\tau_{i+1}-\tau_i)} \left[-I(s) + R(SX^T, HX^T) + f_{i+1}(S', H') \right] \right\}$$

$$i = 0 \ldots k-1$$

with $f_k(S,H) = \int_{\tau_k}^{\tau_{k+1}} \pi(D(t), S, H)\, e^{-rt} dt$.

The number of possible states at stage i in this case is $2^r(m)^{i-1}$, while the number of possible decisions is at most $m2^{r+i-1}$. Typically, k will be a fairly small number, and thus the total number of states and decisions could easily be manageable.

Unfortunately, there appears to be no obvious way of simultaneously determining both optimal actions and timings. Although it may be possible to alternately solve to optimality the pure action and pure timing problems, holding the complementary set of variables constant in each case until no further improvement results, a globally optimal set of action-timing pairs will not necessarily be forthcoming as the objective function, as in the case of simple expansion, is not generally a unimodal function of all its variables, although it may be everywhere differentiable for continuous time. Furthermore, if at some point in this process the optimal solution to the timing problem involves a reduction from k to k - 1 actions, such a method would not allow for one to subsequently consider actions at k points in time, which may be the optimal number for the action-timing problem.

C. Restricted Replacement Policy

If candidates for replacement are limited to those production units in operation at the beginning of the planning period, the number of states may be reduced considerably in a dynamic programming formulation of the problem. Let S represent total capital installed between periods 1 and n and s represent the addition size in period n. Define vectors X,Y such that $X_j = 1$ if production unit j (of size S_j and age factor H_{jn}) is replaced, $X_j = 0$ otherwise and $Y_j = 1$ if production unit i has not yet been replaced , $Y_j = 0$ otherwise. H is the summary age factor for all units added within the planning period, as discussed in section 3 . The recursion relations are

$$f_n(S,H,Y) = \max_{\substack{s\geq 0 \\ X/X\leq Y}} \{\pi_n(S,HY) - I_n(s) + R_n(X)$$

$$+ \alpha f_{n+1}(S,H'',(Y-X)) \qquad n = 1\ldots N-1$$

with $f_N(S,H,Y) \quad = \pi_N(S,H,Y)$,

where

$$H'(H,S,s) = \left(\frac{SH + sh_n}{S+s}\right)\gamma_n \quad .$$

Period profits π_n depend only on new capital and its current age, S,H, and existing old production units of given sizes and age parameters H_{jn} which may increase in a similar fashion:

$$H_{jn+1} = H_{jn}\gamma_n.$$ [1]

It is likely that in many situations solution of a problem allowing retirement of facilities added within the planning period will result in the solution not requiring such retirements. Furthermore, retirement of facilities not yet in existence, if optimal, will probably occur quite late in the planning period, and thus have minimal impact on the early investment decisions, which are of most immediate concern to the planner. If these models are to be used periodically as new information and revised predictions become available, all production units will obviously eventually become candidates for replacement.

D. Treatment of Horizon

In many cases the use of a finite planning horizon determined by the ability to provide usable forecasts may be inappropriate for an enterprise contemplating major capital expansion. Except possibly for the case of a firm owned by the entrepreneur who can predict his end with certainty and will leave no survivors, the assumption of the firm going out of business after N years may be unrealistic. As with many non-stationary sequential decision problems, the alternatives are limited to

[1]For consistency, whenever an age parameter appears in the firm's operating cost function, one should use the weighted average of $\mathcal{H}$ and each of the H_j's as a single age parameter.

a) fixing a terminal production system configuration. In this case the solution obtained may coincide with an infinite horizon solution. The difficulty is that there is usually no way of identifying the optimal terminal state from the large, perhaps infinite, number of possibilities.

b) arbitrarily assuming that the problem at some point becomes stationary. Aging ceases and demands and factor prices become proportional and remain so. In this case it may be possible to evaluate infinite horizon returns directly once a state is observed to be repeated.

c) allowing either variable or fixed terminal system state with a finite horizon larger than the desired planning period would require. The solutions obtained with this approach are likely to have the early investment decisions closely approximating those of the theoretical infinite-horizon solutions, since these early actions should be relatively insensitive to the decisions for the later portion of the planning period due to the effects of discounting. It is also straightforward to implement.

E. Integration of Investment, Pricing and Financing Policy

For the expansion-replacement problems that have been discussed, the optimal product prices generated when plotted against time will typically display a sawtooth-like pattern such as that in Figure 5-1.

Figure 5-1

Each major discontinuity will correspond to a point in time at which a change in production system configuration is to take place, causing a reduction in marginal cost and allowing a greater output and thus lower price to become optimal. Between consecutive investments optimal price will increase as a result of a rising demand function and/or marginal cost (if aging is reflected in this way).

Although such manipulation of product prices from one period to the next may be neither practical nor desirable, the long-term trend obtained could be of some interest to management. An exponential or other function might be fitted to the resultant price-time curve, or the investment problem solved for various fixed rates of increase or decrease in price. The effects of simply imposing constraints on the number or spacing of price changes within the planning period (resulting in price a step-function of time) might also be investigated.

Other aspects of pricing policy include those associated with competitors' responses to price changes. For example, one elementary static economic model (Sweezy [96]) suggests that demand elasticities of firms in oligopolistic markets are greater for increases than for decreases in price. To describe this situation fully would require that demand shift leftward for any production system change resulting in a lower optimal price. This case could be represented in the dynamic programming format by allowing the demand parameter(s) to constitute an additional state descriptor to be modified by potential investment decisions. If competitors' responses are stochastic and immediate, a sequential markovian decision problem will result.

Should the investment problem have a sufficiently limited number of production system states, additional variables might also be employed to represent financial and other conditions relevant to the investment decision. Existing debt may, as suggested by Lesso [56], be an important determinant of investment feasibility. For example, an upper limit $0<\eta<1$ could be placed on the allowable debt-equity ratio of the firm;

$$\frac{\text{DEBT}_n}{\text{EQUITY}_n} \leq \eta \quad .$$

If earnings are applied to dividends and debt retirement (DR) only, depending on the level of operating profits, we have

$$\text{DEBT}_{n+1} = \text{DEBT}_n + I_n(s) - \text{DR}(\pi_n(\theta))^1 \qquad 5.5.1$$

and

$$\text{EQUITY}_{n+1} = \text{DEBT}_{n+1} + \text{OWNEREQUITY}_{n+1} \quad , \qquad 5.5.2$$

$$\text{OWNEREQUITY}_{n+1} = \text{OWNEREQUITY}_n + \text{DR}(\pi_n(\theta)) + \text{NEWOWNEREQUITY}_n .$$

The choice of expansions in each period n would then be limited by constraints of the form

---

[1] Salvage value from sale of retired equipment may be included here as well.

$$I_n(s) \leq \eta\,[\text{OWNEREQUITY}_n]/(1-\eta) - \text{DEBT}_n \qquad 5.5.3$$

and the impact of differing long-term financing plans upon average and cumulated discounted earnings per share or other performance measures could be investigated.[1]

One might alternatively employ financial constraints more consistent with the contemporary "debt capacity" approach to investment financing decisions.[2] In this case the additional debt and hence indirectly the investment size is constrained by the debt capacity of the firm rather than by debt-equity level. Debt capacity, DC_n, is assumed to depend upon the temporal probability distributions of cash flows, which are further assumed (as a first approximation) to be directly related to the physical assets of the firm, θ, due to the possibility of cash inadequacy.[3]

---

[1] This formulation allows part or all of actual accounting net profits less dividends to remain in the form of liquid assets, without immediate debit to long-term liabilities in periods in which debt/equity does not constrain the investment decision. In such a case, (5.5.1) and (5.5.2) define potential debt and equity levels, respectively. However, it is assumed that in periods when (5.5.3) is binding the cumulated past additions to liquid assets are used to retire debts, relaxing this constraint to the fullest.

[2] For a linear programming approach to simultaneous determination of investment projects and financing methods using this approach see Myers[77].

[3] DC_n obviously depends, in a complex fashion, on future decisions as well.

Hence investment constraints for each period (assuming but one class of debt) would have the form

$$I_n(s) \leq DC_n(\theta_n) - DEBT_n \quad . ^{1} \qquad 5.5.4$$

In either case, wherever period profits π appear terms $TX(DEBT_n)$ and $-FIN(I_n(s))$ may be added to reflect tax savings due to the interest expense and net cost of financing without introducing additional complication. Both of these approaches, although crude, would appear at least to preclude arriving at solutions requiring clearly impossible financing actions.

[1] Again assuming that investments are financed by debt.

CHAPTER VI
COMPUTATIONAL RESULTS

In this chapter we will discuss some computational experience acquired using the deterministic dynamic programming models of sections III-L and V-C for expansion with and without replacement of production units available at the beginning of the planning interval. An algorithm which will solve both of these problems has been programmed for use on MIT's Multics (Multiplexed Information and Computing Service, currently implemented on the GE-645) time-sharing computer system. Multics has proven to be a useful tool for developing and debugging the experimental programs and for performing model sensitivity analyses.

A. Model Description

The investment cost function employed is the power-function (3.4.1), while linear price-demand function is assumed for each of the discrete time periods. Operating costs are given by (3.2.1), with fixed operating costs proportional to $S^{.7}$ and variable operating costs a third-degree weighted polynomial of output, q (in (3.2.1) J=3 and i_j=j-1). The square of age parameter (or weighted average of age parameters of old and new production units in the case of retirable production units) is applied to both fixed and variable production costs. All revenues and costs are assumed to be less taxes.

S and H have been discretized into eleven and seven values, respectively, for most but not all of the computations, with linear interpolation used to evaluate returns for intermediate values (since one-stage returns are continuous in S and H, so are n^{th}-stage returns). With retirement, integer variables indicating the presence or absence of production units are employed. Expansion size s is discretized into at least six values in evaluating n^{th}-stage returns for each of the possible states, and into at least twenty-one values in recovering the optimal solution.[1] Neither of the two increment sizes for s need be multiples of those for S, and will not necessarily correspond with the latter in the resulting solution even using linear interpolation. Program listings and more detailed descriptions of usage may be found in the Appendix.

B. Computation Time

Computation time (CPU) can be divided into two parts: a fixed component and a variable component. Fixed computation time includes that required for locating and reading files, establishing linkages to subroutines, and performing calculations of a set-up nature. A typical value would be 20 seconds. After an initial problem has been solved, however, and with changes in inputs made, the fixed computation time is

[1] Optimal decisions are never actually tabulated for each state and stage; it is pointed out in Hadley [40], pp.370-72, that the only method which guarantees that continuous decisions are determined accurately when discretizing continuous state variables is the one in which direct computation is used to recover decisions. However, all state-stage returns f(θ) need to be tabulated when using this method, so storage requirements are not greatly reduced.

reduced to about 14 seconds for subsequent problem runs. Variable computation time is about 5ms/iteration, with the number of iterations approximately equal to

$$\underline{S}\underline{s}\underline{H}N3^{r} ,$$

where N is the number of time periods, r is the number of replaceable production units, and $\underline{S}$,$\underline{s}$, and $\underline{H}$ are the number of discrete values for which intermediate returns are evaluated. Recovery of the optimal solution even with s finely discretized requires only a relatively small (on the order of a few percent of the total) amount of CPU time. As an example then, a problem with r=2, N=22, $\underline{S}$=11, $\underline{s}$=6, and $\underline{H}$=7 would have a variable computation time of about 500 seconds using this formula.[1] Increasing the complexity of operating cost or revenue functions would in most cases increase variable computation time further; use of a subroutine allowing operating costs to be a fourth degree polynomial in q was found to triple variable computation time.

---

[1]We are assuming the best of conditions. Speed of computation becomes significantly degraded when many users are on Multics.

C. Sensitivity Analyses

The sensitivity analyses discussed in the remainder of this chapter involve one-at-a-time changes in inputs, since consideration of all combinations of input parameters would result in too large a number to be practical for the purposes of this thesis. Results must therefore be cautiously interpreted, being rather more illustrative than definitive. The solutions are presented on the diagrams in the following manner: prices and age parameters are plotted on the same scale, with expansion size in a square box (□) immediately above the graph and aligned with the appropriate time period. The lower solid curve is age parameter, while the upper one is price. In the case of retirement of production units, the weighted average of old and new production unit age parameters (which directly enters the operating cost function) is represented separately by a broken line, and a period during which a retirement takes place is indicated by a diamond (◇) containing the number of the retired unit.

Problem 1: Base Case I; Expansion Without Replacement

This is a typical problem of expansion planning without replacement. Inputs are as follows: horizon N=22, discount factor α =.85, investment cost parameters k_1=2.0,k_2=.7, initial capital $\overline{S}_1$= 1.5, initial age parameter $\overline{H}_1$ = 1.0, new capital age parameters h_n

decrease at the rate techrate=.05 per period. Cost function parameters are $a_o = .3$, $a_1 = .5$, $a_2 = 0$, $a_3 = 1.0$. Demand slope $C = .25$, vertical intercept D(n) in periods 1-18 according to a quadratic function which begins with D(1) = 2.0 and rises to its maximum in period 18 with D(18) = 8.0. D(n) = 8.0 for n ≥ 18, and age parameter growth factor $\gamma = 1.06$ (six percent growth per period). Discretization numbers are $\underline{S} = 11$, $\underline{H} = 7$, and $\underline{s} = 5$.

In this case demand parameter D increases smoothly, but at decreasing rate, and becomes stationary after reaching its maximum value of 8.0 in period 18. The trend is for price to rise, but at a decreasing rate, and, as seen on the diagram, investments of sizes s=11,15, and 9 take place in periods n=5,10, and 16 in the optimal solution.

Problem 1: Base Case I (see text)

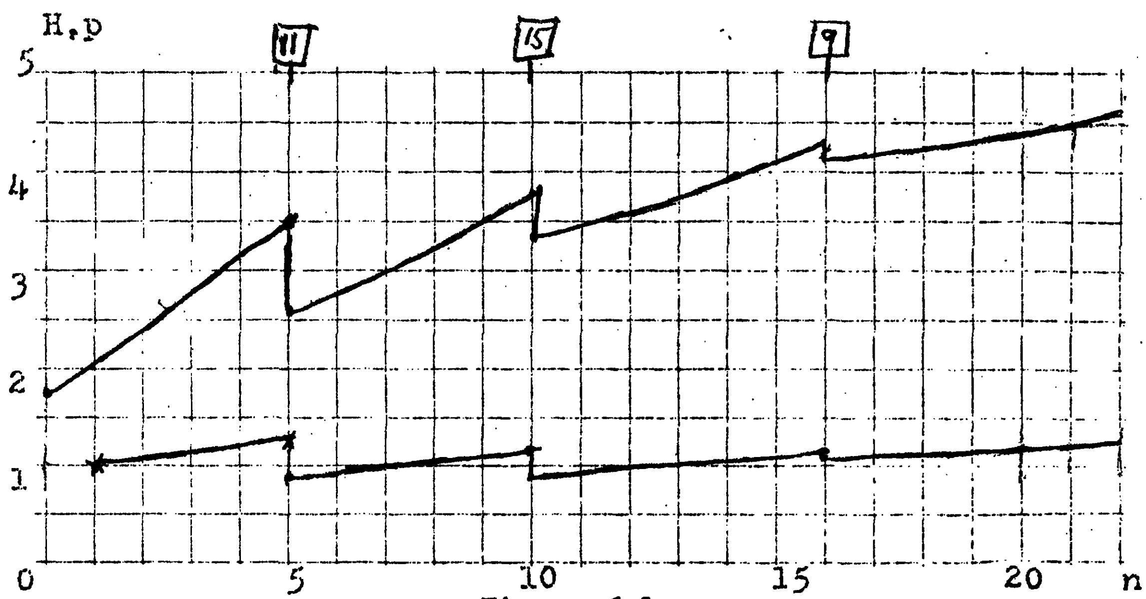


Figure 6-1

Problems 2-5: Changes in Computational Parameters (not illustrated)

In order to test the effects of discretization, $\underline{S}$, $\underline{H}$, and $\underline{s}$ were each doubled to 22, 14, and 10, respectively (problem 2). No change in the solution resulted, although computation time increased dramatically as expected. Although finer discretizations may increase accuracy of intermediate returns, $f(\theta)$, the effects on the optimal decisions themselves appear to be minimal.

In order to test for effects of increasing length of horizon Base Case I was run with horizon N = 30 (Problem 3) and N = 50 (problem 4). No change in the solution resulted in the first case, while in the second the size of the third capital increment was reduced from 9 to 8, with an additional investment of size s = 8 taking place at n = 29. Horizon length thus appears not to be critical.

In problem 5 the investment scheduled for period 10 in Base Case I was prevented by temporarily placing a great cost on all investments for this period ($k_1(10) = 10000$). The new solution has investments of size 11, 16, and 8 at periods 5, 11, and 16, but the objective function value is increased by only .94 % from that of Base Case I, indicating a relative insensitivity of the problem to investment timings, due primarily to the possibility of compensation by changing investment sizes and prices.

Problems 6-17 use the data of Base Case I with the exception of the input parameter(s) indicated on the corresponding diagram. We observe that inflation appears to be a way of life for the firm that cannot divest itself of deteriorating capital.

Problems 6-8: Investment Economies

As economies of capital acquisition are reduced, investments predictably become more frequent and of smaller size in these problems, leading ultimately, one would expect, to investment becoming a continuous function of continuous time, as in the Lucas[58] model.

Problem 6
k_2=.95

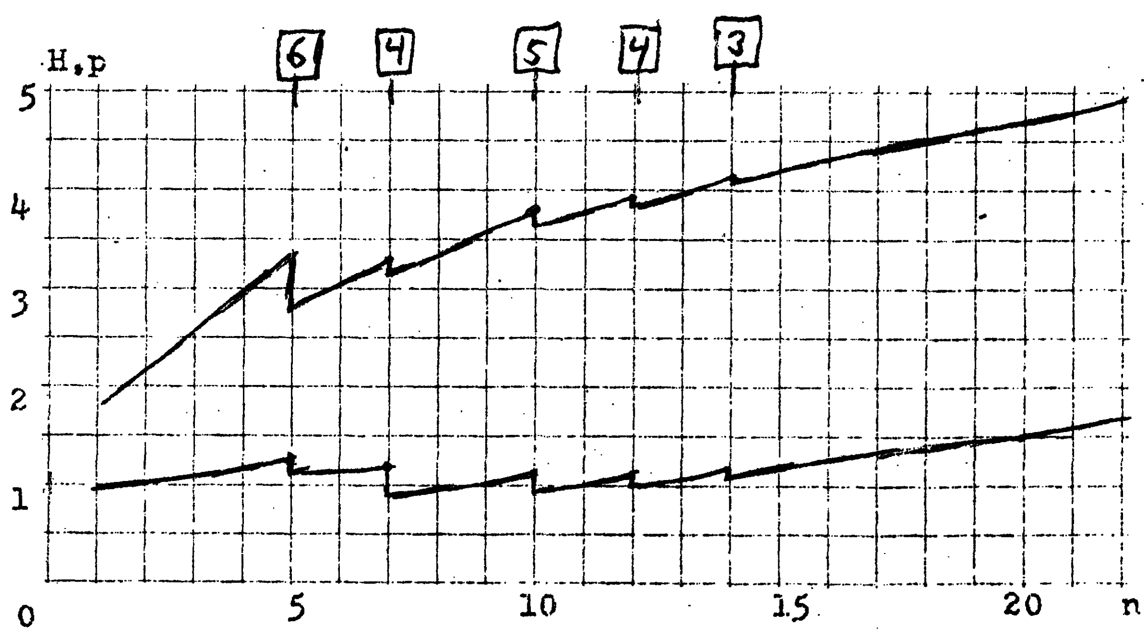
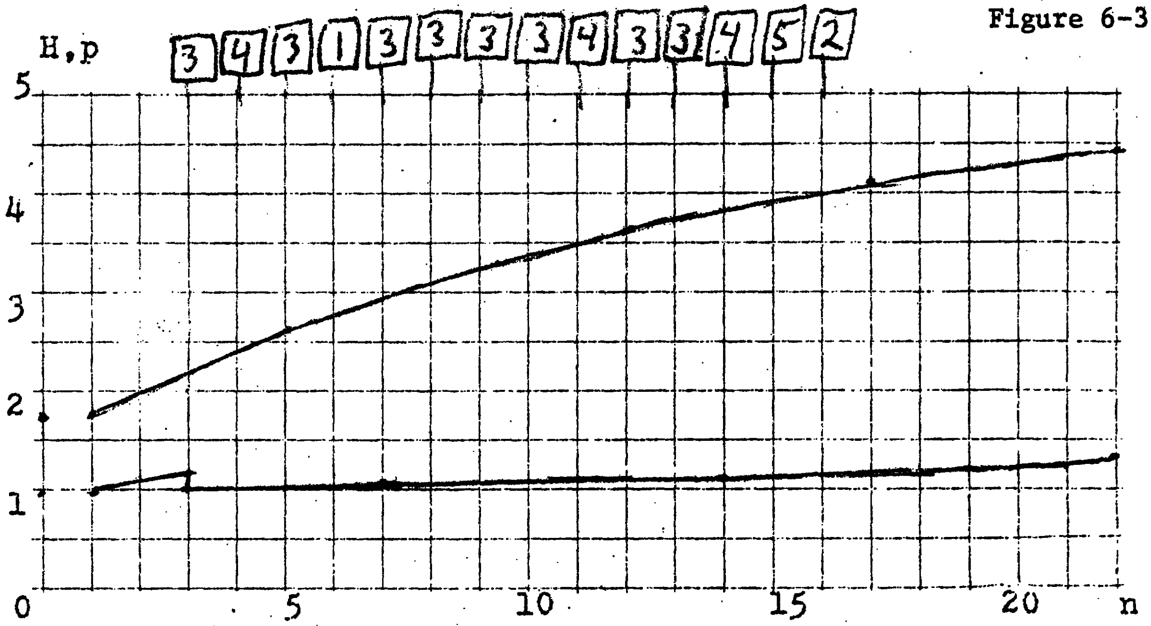


Figure 6-2

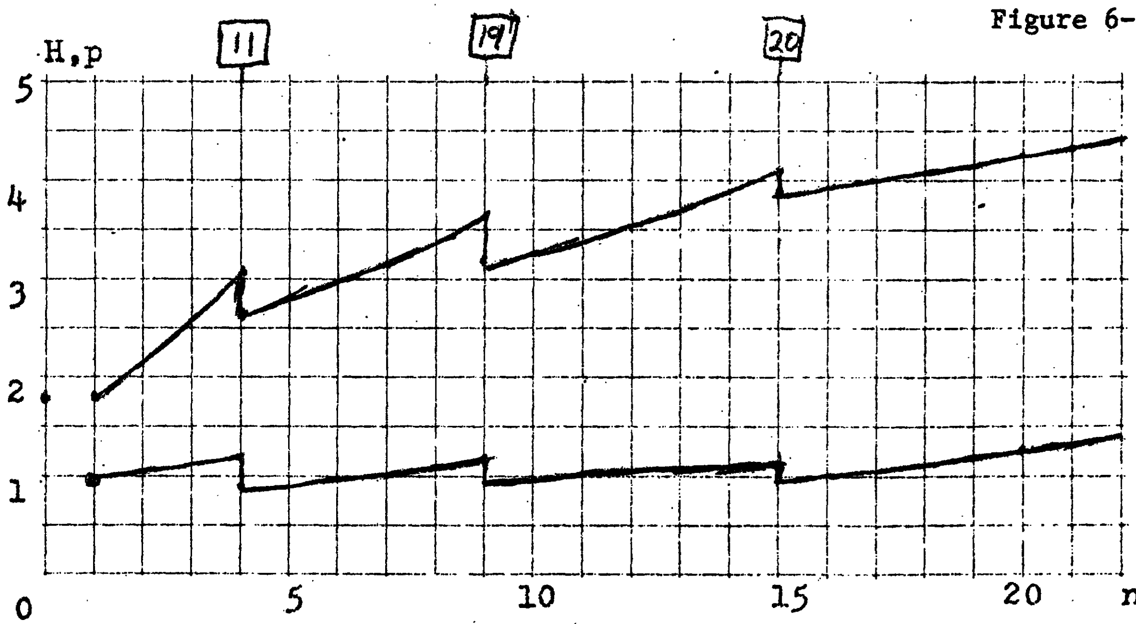
Problem 7
k_2=1.0

Figure 6-3



Problem 8
k_2=.55

Figure 6-4



Problems 9,10: Discount Factor

Choice of discount factor for the data employed is non-critical with respect to the first investment; for $\alpha = .75, .85$, and $.95$ the first investment will be of size s = 10 or 11 in period 5. It is interesting to note that this behavior is in contradistinction to that of the classical "capacity" models, in which investment sizes rapidly increase with α (in the limit becoming infinite as $\alpha \to 1.0$ for strictly increasing demand) due to acquisition economies. This does not occur in our case due both to the presence of aging (the firm would otherwise be left with a large, old, and thus expensive to operate plant later on in the planning interval) and to fixed production costs which can increase with plant size as fast as, or faster than, savings from acquisition economies.

Problem 9
$\alpha = .75$

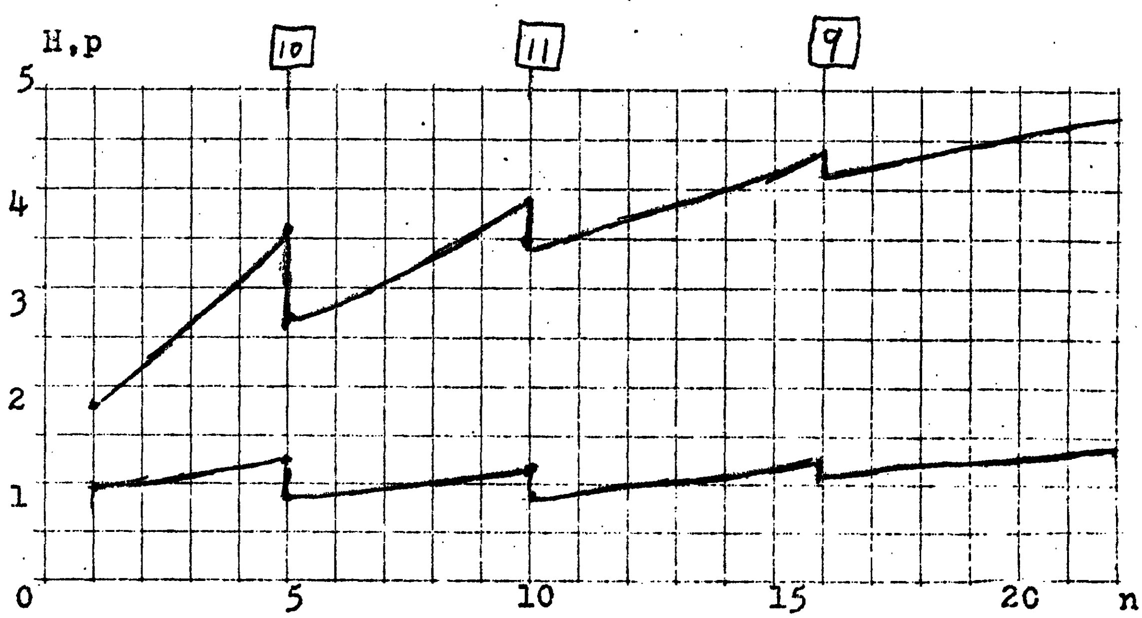


Figure 6-5

Problem 10
$\alpha = .95$

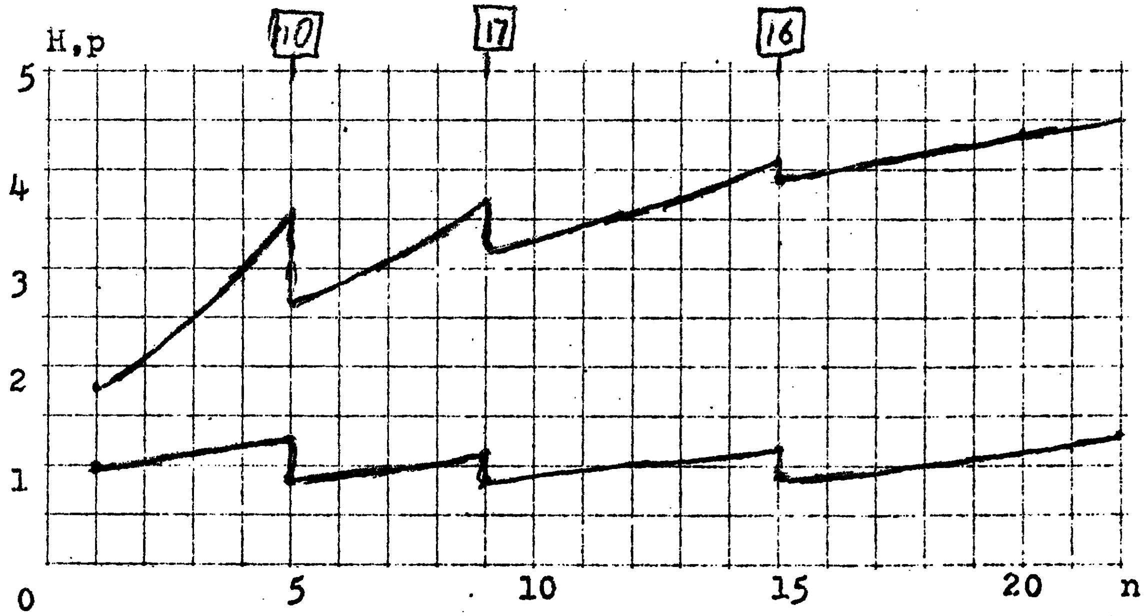


Figure 6-6

Problems 11,12: Aging

The proportional increase in age parameter per period (γ) affects price primarily early in the planning interval in these problems. Towards the end prices become nearly the same.

Problem 11
$\gamma=1.0$

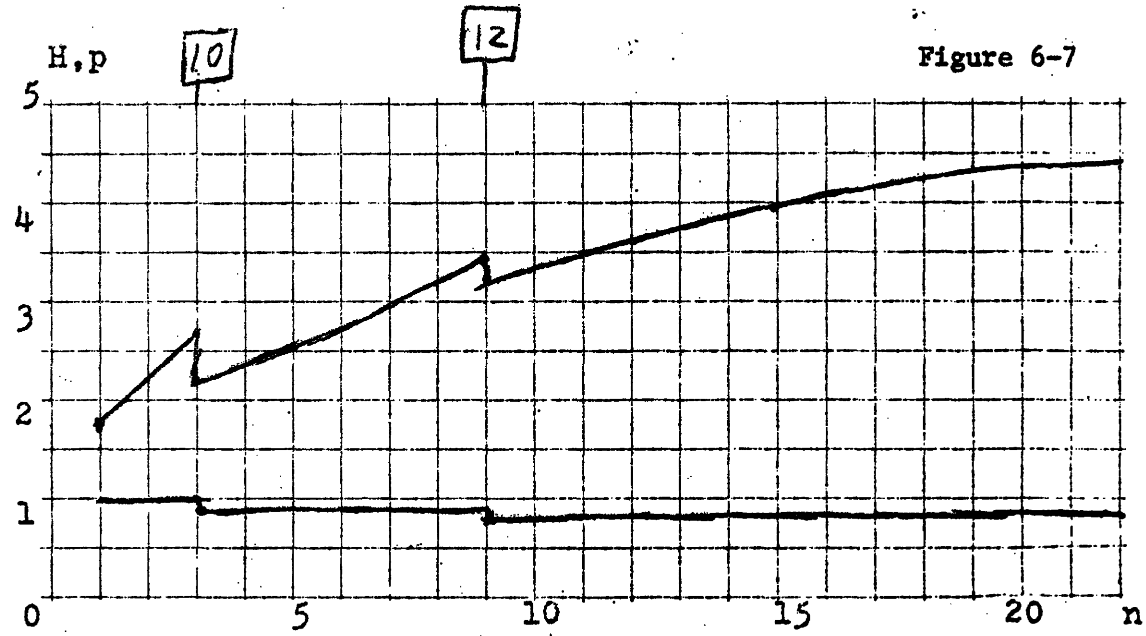


Figure 6-7

Problem 12
$\gamma=1.1$

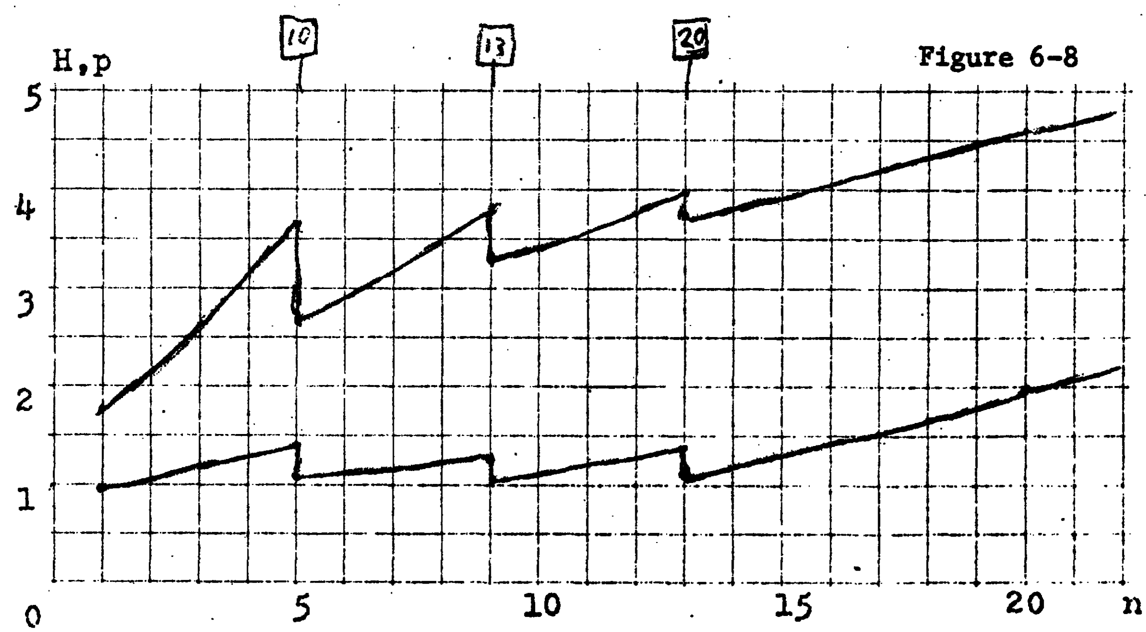


Figure 6-8

Problems 13-15: Fixed Operating Costs

As fixed operating costs rise relative to variable costs (the greater a_o) the first investment is delayed, but subsequent investments may be delayed or advanced. As a point of reference, fixed operating costs average about one-half of total operating costs over the planning interval in Base Case I (a_o=.3).

Problem 13
a_o=.6

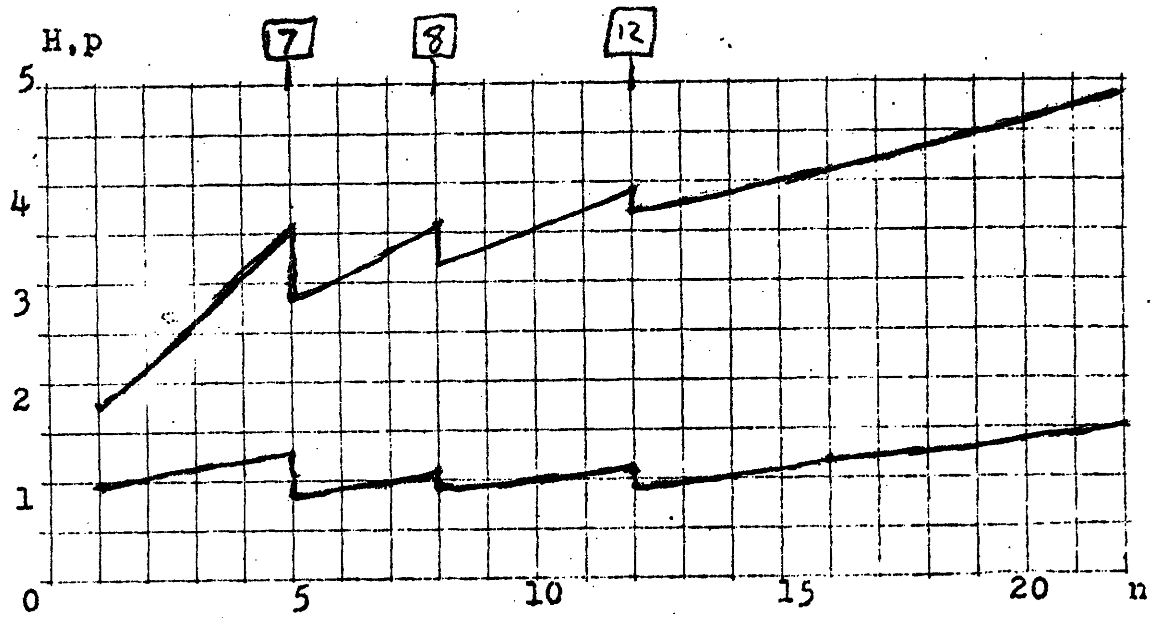
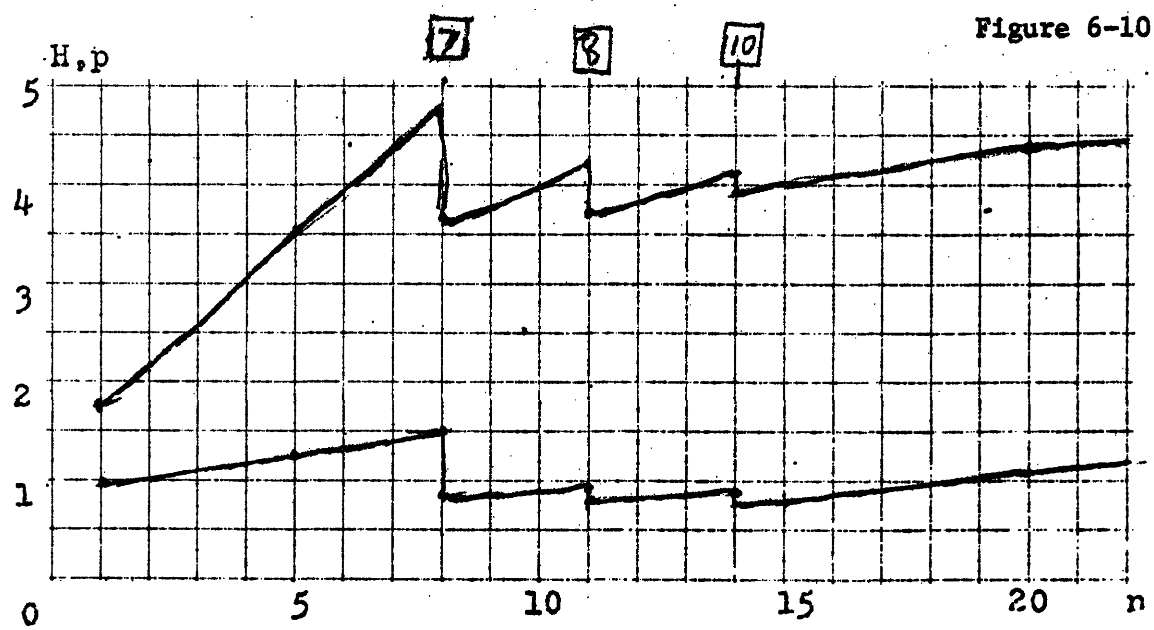


Figure 6-9

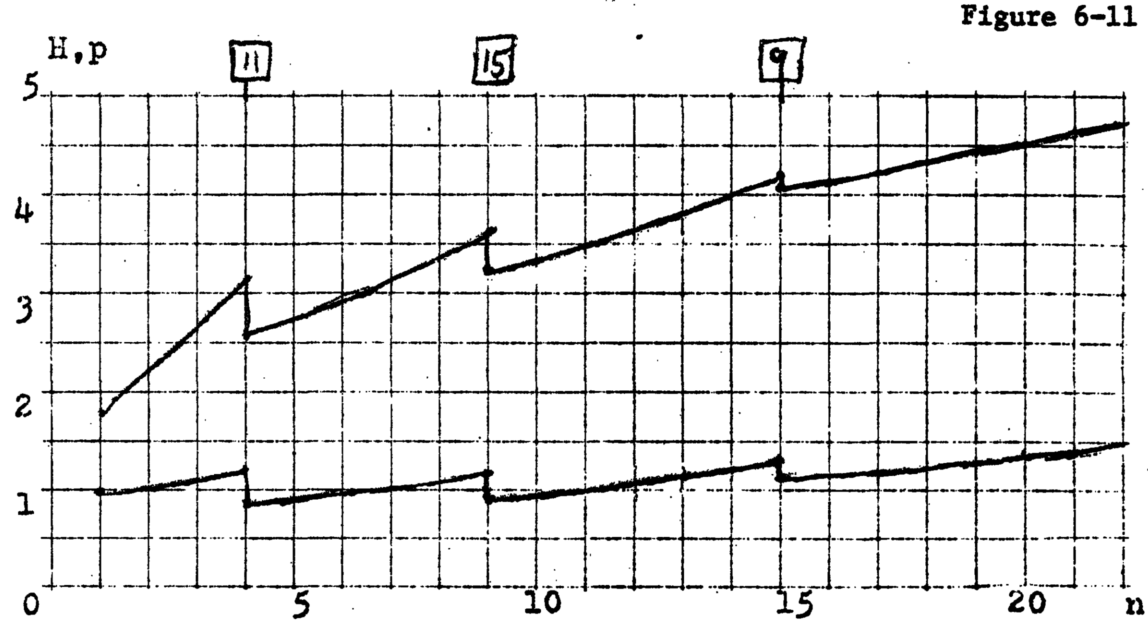
Problem 14
a_0=1.5

Figure 6-10



Problem 15
a_0=.15

Figure 6-11



Problems 16,17: Technological Change

The greater the (embodied) technological improvement rate (techrate), the more moderate increases in optimal price becomes in these problems, a very reasonable result.

Problem 16
techrate=0.0

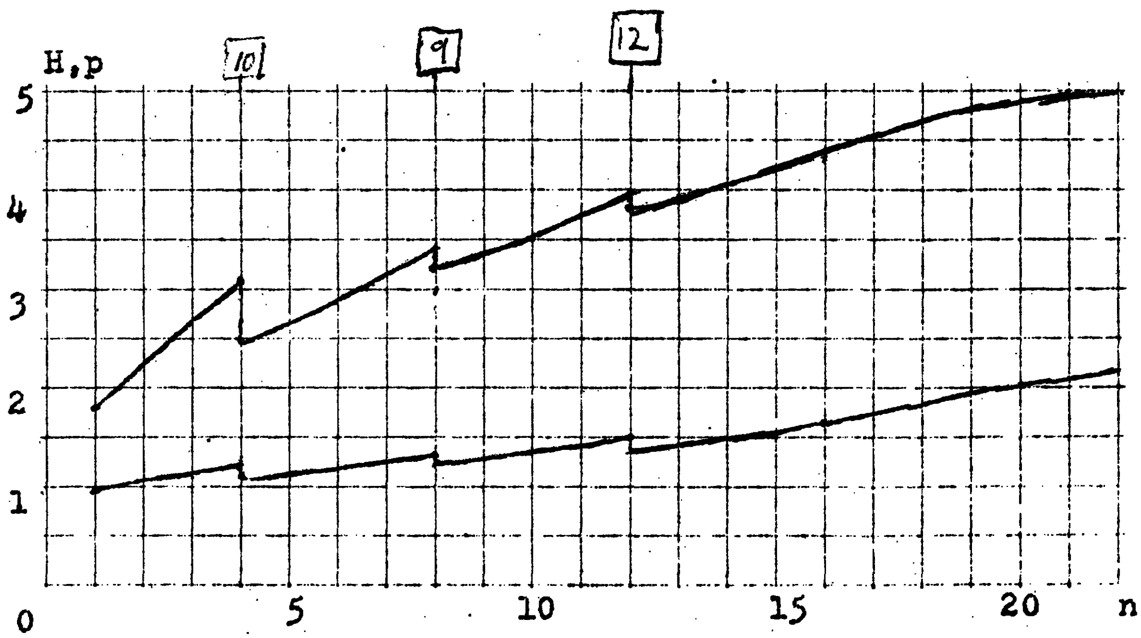


Figure 6-12

Problem 17
techrate=.08

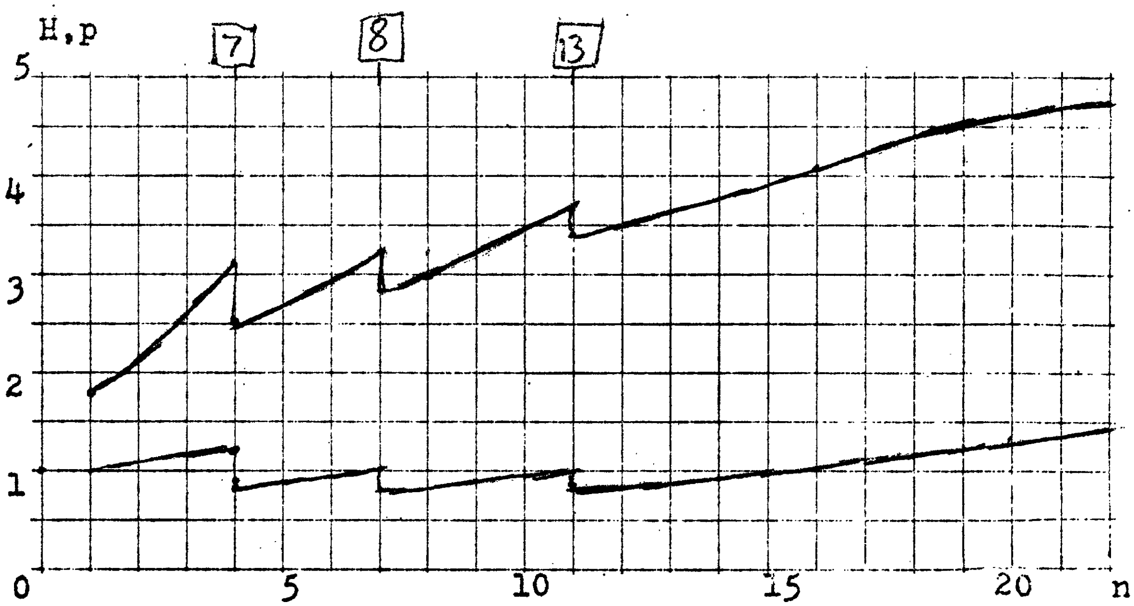


Figure 6-13

Problem 18 (Base Case II): Expansion with Replacement

This case has identical input data as Base Case I (Problem 1) except that two "old" replaceable production units of sizes S1, S2 and ages H1, H2 are present in the system initially, and that demand parameter growth begins with D(1) = 4.0 instead of 2.0 as in Base Case I. Salvage values are zero.

Problem 18: Base Case II (see text)
S1=8 S2=4
H1=1.5 H2=1.1

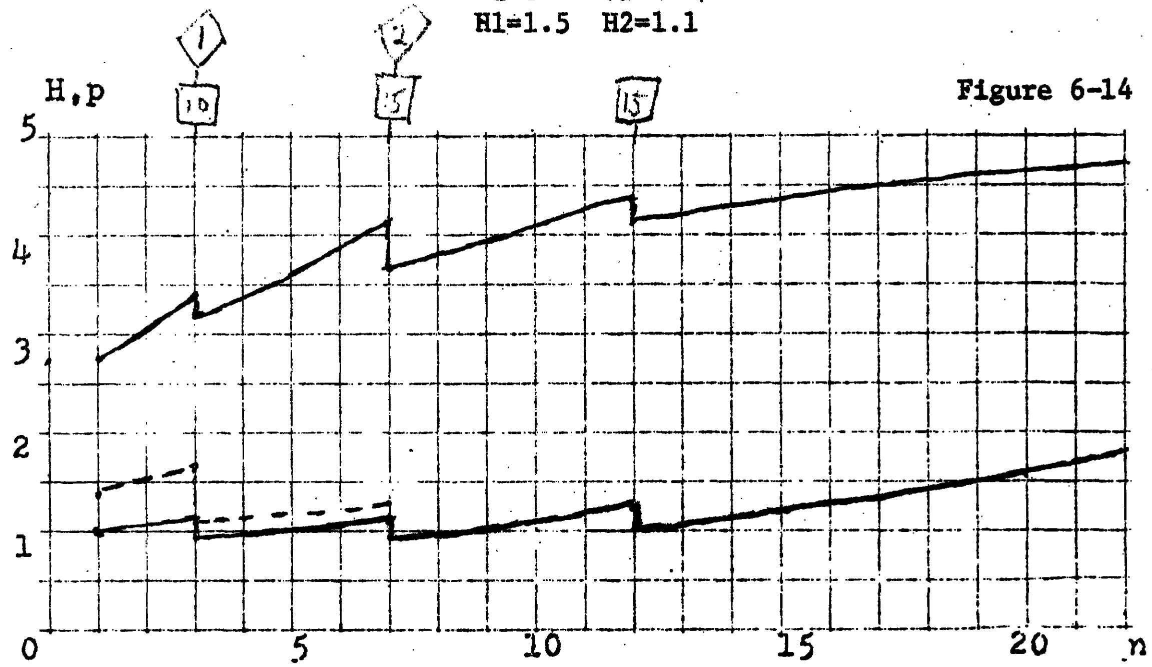


Figure 6-14

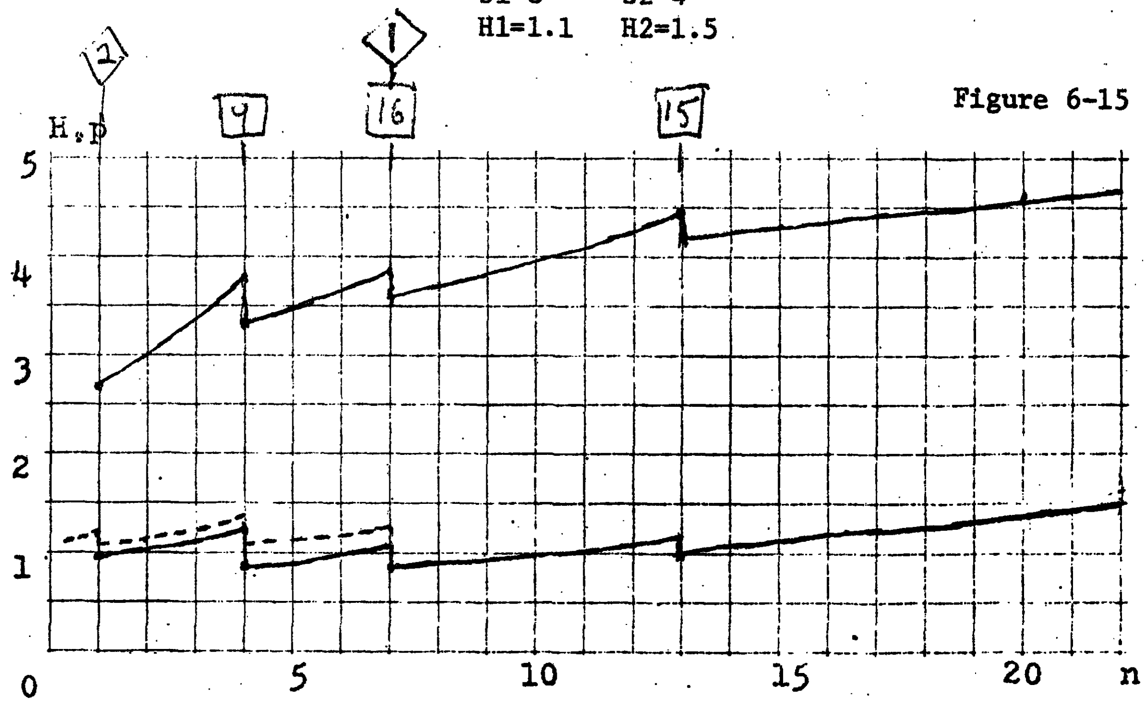
Problems 19-28 use the data of Base Case II except for changes noted in the text and on the diagrams. Note that Problems 20 and 21 have identical output and so employ a common graph.

Problems 19-25: Characteristics of Existing Production Units

There is a pronounced tendency for identical initial age parameters to require simultaneous replacement when "old" production units are relatively large or when their age parameters are nearly the same (problems 20-24). However, even with very nearly identical age parameters, if production units are small retirements are more likely to be staggered (problem 25). As expected, retirements and new investment usually take place simultaneously (the only exception occurs in problem 19). In problem 22 price (and by implication marginal cost) rises rather than falls as is usually the case, immediately after the first investment. The decision is nevertheless justified by the decrease in fixed operating costs which result from replacement of old capital.

Problem 19
S1=8 S2=4
H1=1.1 H2=1.5

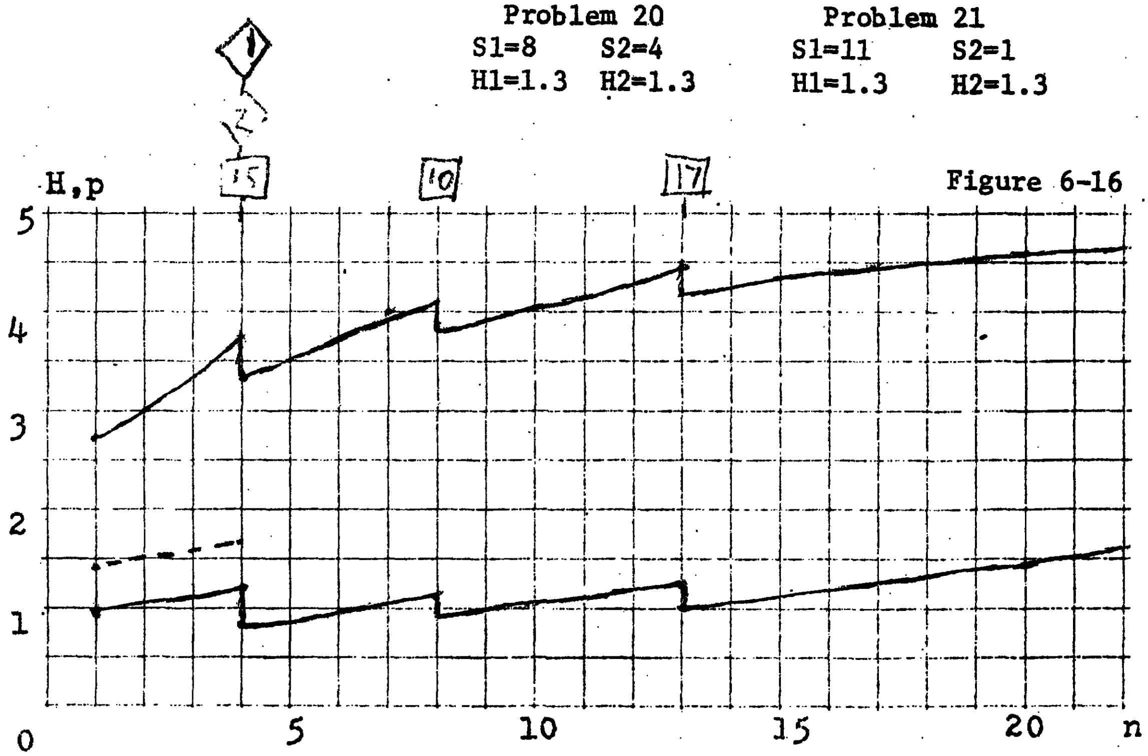
Figure 6-15



Problem 20
S1=8 S2=4
H1=1.3 H2=1.3

Problem 21
S1=11 S2=1
H1=1.3 H2=1.3

Figure 6-16



Problem 22

S1=12 S2=12
H1=1.3 H2=1.3

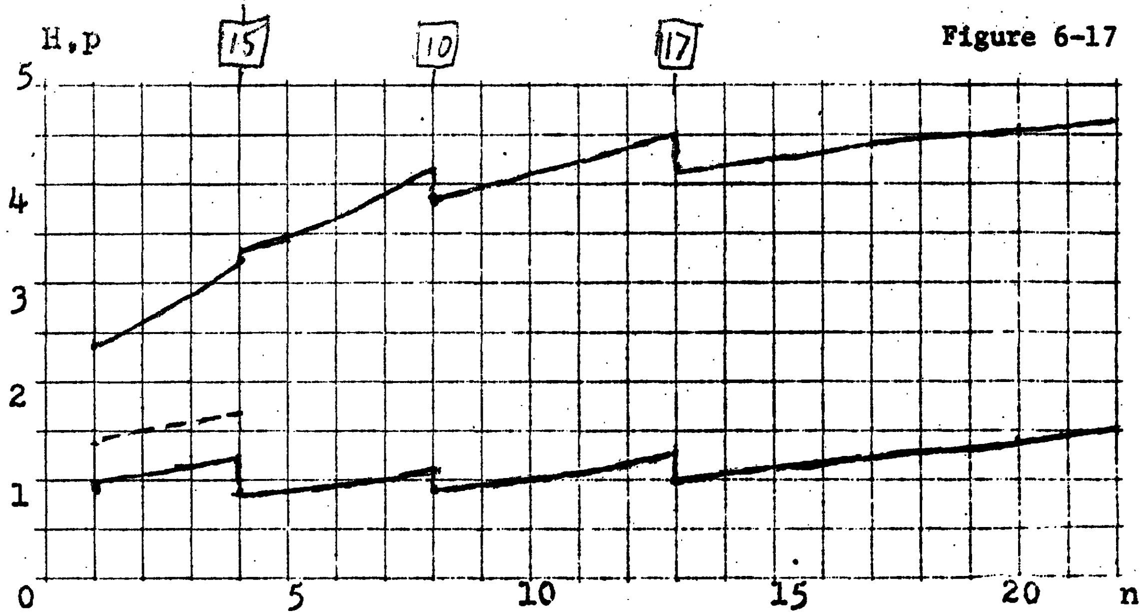


Figure 6-17

Problem 23

S1=8 S2=8
H1=1.1 H2=1.5

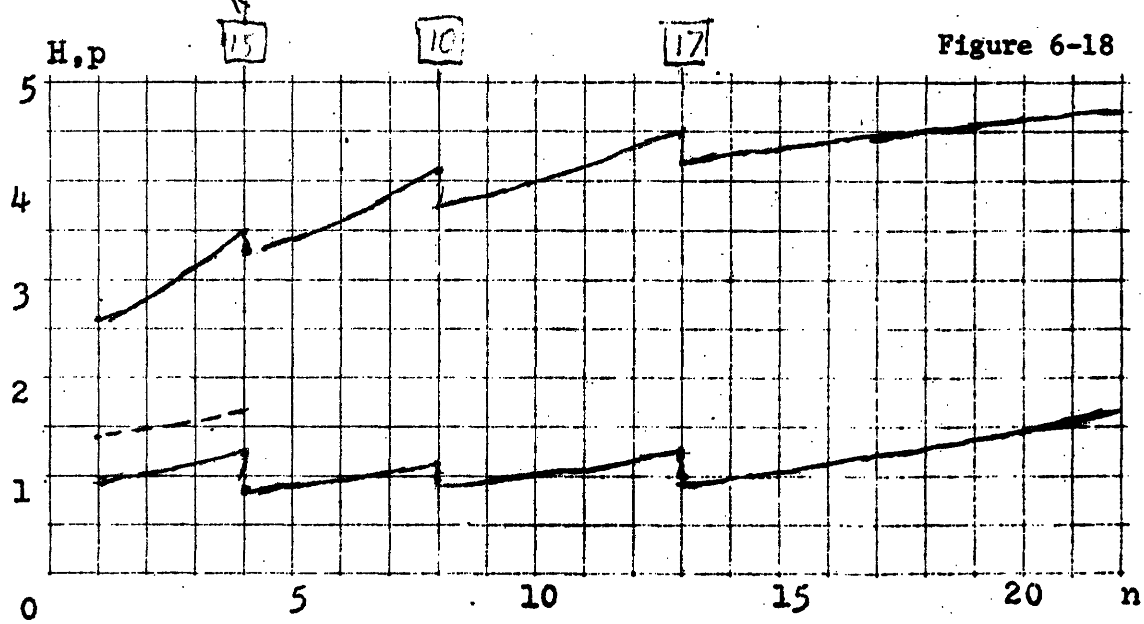
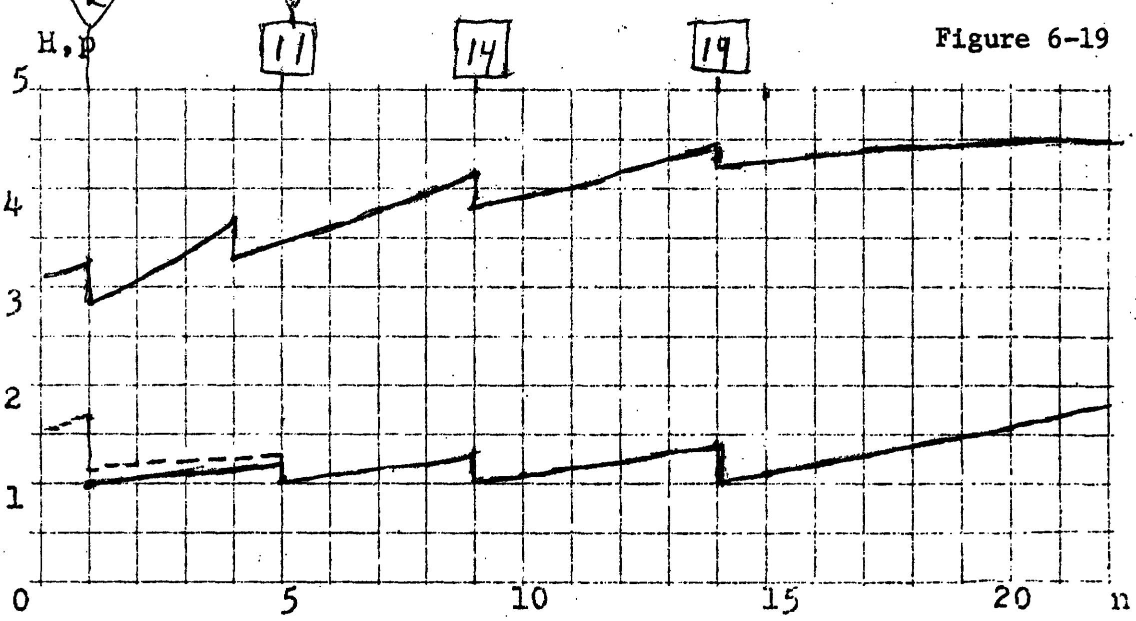


Figure 6-18

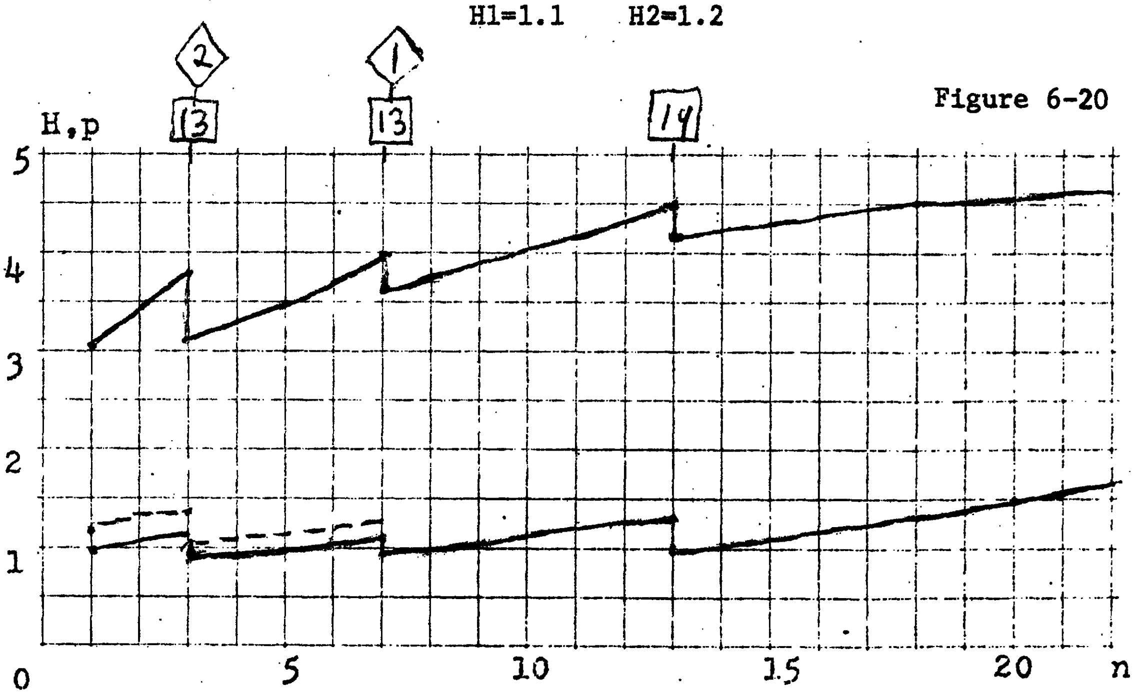
Problem 24
S1=3 S2=3
H1=1.1 H2=2.0

Figure 6-19



Problem 25
S1=3 S2=3
H1=1.1 H2=1.2

Figure 6-20



Problems 26-28: Changes in Demand

In problem 26 the demand slope C is increased to .35 from the value of .25 in Base Case II. Problem 27 is identical to Base Case II except that demand parameter D(n) reaches its maximum value of 8.0 in period 9 rather than in period 18, and thereafter remains constant. In problem 28 D(n) also reaches its maximum in period 9, but then continues to decline (in the same quadratic fashion) to a value of 2.0 in period 18, beyond which it remains constant.

We observe that the first investment is increased in size, but unaltered in time for problems 27 and 28 relative to Base Case II. The first decision is not very different for the two cases although the behavior of demand later in the planning interval is quite different, suggesting for this data a relatively low value for perfect information about demand far into the future.

Problem 26
C=.35

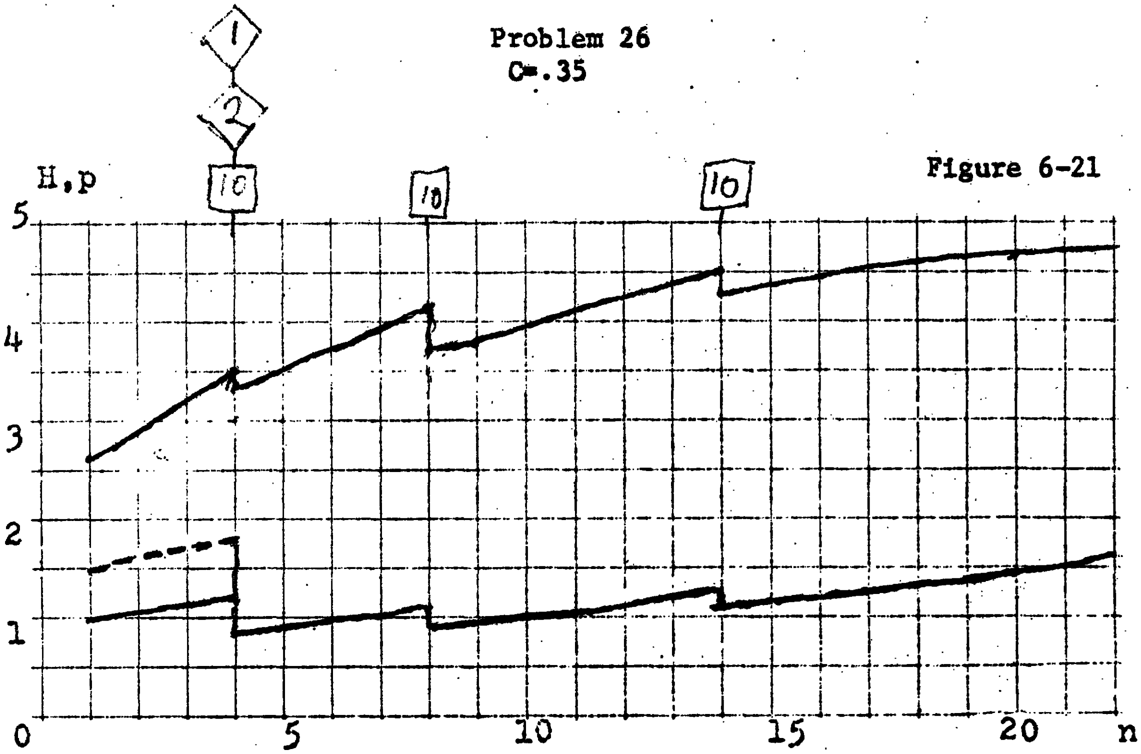


Figure 6-21

Problem 27
(see text)

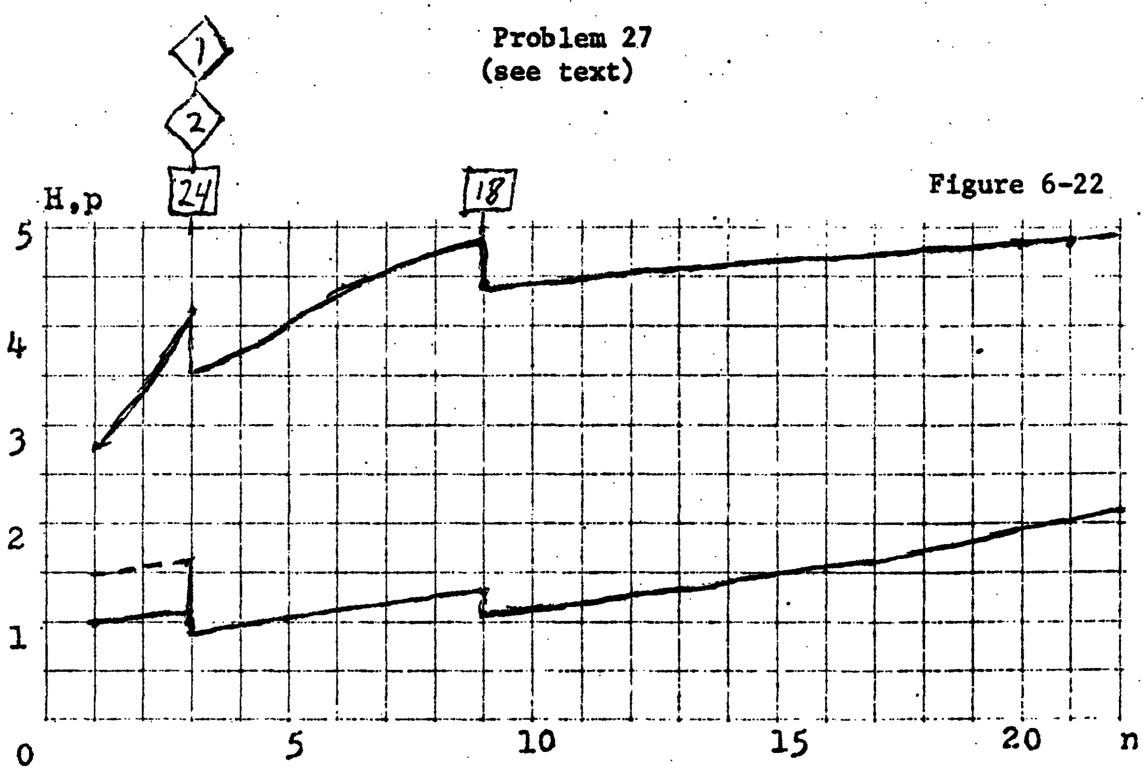


Figure 6-22

Problem 28
(see text)

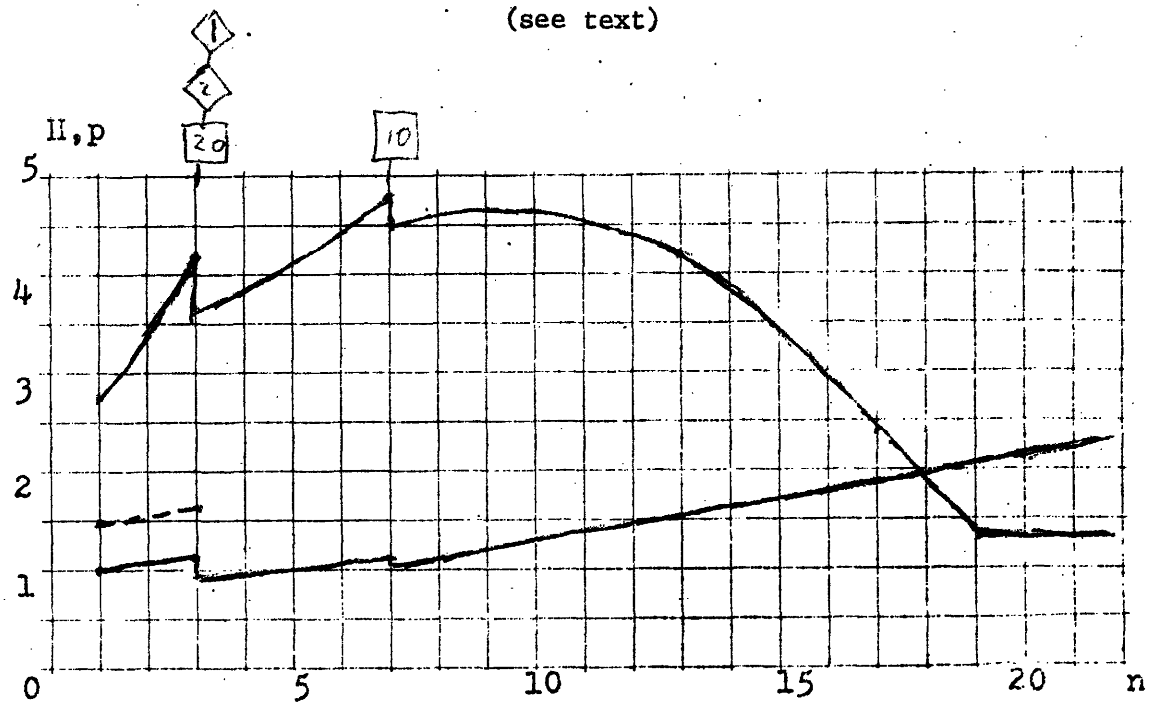


Figure 6-23

It is hoped that these problems are suggestive of the potential utility of such investment-replacement models for managerial decision-making. In particular, since solutions to these fairly rich models require only minutes of computer time, it should be possible to obtain solutions to some of the more complex formulations (such as with markovian demand) with costs of computation which are negligible relative to the anticipated solution payoffs for the larger firm.

CHAPTER VII

CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

Any discussion of capital investment problems is likely to include the subjects of data collection and reliability, solution methods, and objectives. With regard to data required for the kinds of interdependent investment problems examined in this thesis little of novelty will be said. Statistical cost analysis can, of course, be employed in developing cost relations in those industries of long history with homogeneous outputs and stable technology. Reliable point estimates or distributions are unlikely to be easily obtained, however, for such required inputs as technological change and aging coefficients, or for long-term rates of inflation in factor and product markets. Demand parameter estimates beyond a few years may be mere guesses for many firms not providing a utility or engaged in one of the basic industries. Fortunately, though, the objective function of discounted cash flows is very forgiving of errors in forecasts far into the future; immediate optimal actions may be little affected by alternative levels of demand, let us say, twenty years hence.

Development of efficient solution procedures for the more complex problem formulations presents a real challenge for the more mathematically inclined. Inclusion of additional variables to represent financial and other measures or for multiple producing locations will quickly limit the practical size of problems which can be handled by dynamic programming (the "curse of dimensionality" at work). Another difficulty is that this method cannot satisfactorily handle correlated stochastic quantities or frequently recurring accounting losses. In the latter case net investment costs cannot properly be evaluated, since they will depend on the periods (if any) in which losses occur, since tax savings due to the depreciation allowance will not be forthcoming in such periods. The timings and magnitudes of losses will, in turn, depend to some extent on the investment strategy followed, so will usually not be known beforehand. Algorithms capable of economically solving large general nonconvex mixed programming problems appear not yet to be available, so further research might profitably focus on the development of specialized algorithms for the more realistic formulations.

It is likely that the selection of investment strategy using more comprehensive models, particularly with the inclusion of uncertainty, can best be approached with the aid of simulation techniques. However, the investment decision process will, in general, be more involved than the usual use of simple "decision rule" equations. A fairly complex subproblem might be solved at each point in time, for example, to arrive at locally optimal decisions based on an approximation to the marginal conditions which ought to prevail. Moreover, since fixing some variables will usually cause the partial optimization problem in the remaining variables to become much simpler, simulation runs might be made to determine "good" values for one set of variables (such as retirement times) according to one set of criteria, while the others are subsequently chosen using an optimal-seeking method, possibly even according to different criteria.

Although simulation can be employed to determine period returns for more detailed systems (for example including the intermediate-range production planning decisions, aspects of stochastic consumer and competitor response to marketing policy, logistics problems, etc.), evaluation of all state-period profits, even within the restricted state space provided by transition feasibility constraints, is likely to be impractical. One could very well, however, employ an elaborate detail model to examine the response surface in the neighborhood of any solution, in order to provide more accurate information for further computations. Several iterative procedures can be easily envisioned. Furthermore, in this approach the detailed economic responses to every possible decision and system state need not be pre-specified; the manager can possibly supply these as needed.[1,2]

---

[1] Green [37] has shown that heuristically coupling a detailed simulation model of the production environment with an intermediate-range (aggregate production planning) model could yield significant cost benefits. We are suggesting that some sort of coupling between a more detailed intermediate-range and a long-range planning model is likely also to result in better decisions.

[2] For a compendium of recent corporate simulation models see Schreiber [85].

It would be a mistake to interpret the lack of analytic solutions in this thesis to mean that none were sought. The results published for the classical capacity expansion problem could lead one to strongly suspect that investments or inter-investment intervals of constant or systematically increasing or decreasing size may be optimal for certain patterns of demand and profit and investment functions over an infinite horizon.

For the relatively uncomplicated case of a single size parameter describing the production system a variety of such relations which might have some economic justification (including period profits linear or exponential in time, power-function, linear, and exponential in size, and linear homogeneous in size and time; and investment costs fixed-charge linear and power-function of capital increment) were fruitlessly investigated with the objective of maximizing either the sum of discounted net profits or average per-period net profits (in those cases in which discounted net profits over an infinite horizon can become infinite). A difficulty is that the recursive expression for returns at any point in time, if one exists, will probably not be of an obvious or simple form. Crowston and Sjogren [21] have worked out a periodic policy for a similar one-parameter problem using nonlinear production costs, but their solution will generally be a suboptimal one as allocation of production is done in a suboptimal fashion, with all but the most recently added production unit operating at its minimum-average-cost point. Clearly, much remains to be done in this area.

Actual objectives for investment policy may not be adequately represented by the measure of net discounted present value.[1] Such a criterion may be appropriate for making decisions on a smaller scale, such as for individual machine purchases, or for the well-diversified firm, since the effects on the firm's cash flow and capital structure will tend to even out when many investments take place, each of which is of relatively small size. In addition, it is the rare company (or shareholder) which evaluates the performance of its decision-makers solely on the basis of the expected present value of long-term plans. It is probable that the manager's utility function will then be vector-valued, including several other more difficult-to-quantify elements.

Of course, one component of the manager's utility function will include accounting profits, but probably in a nonlinear fashion. Negative profits, in particular, are likely to receive a greater weighting than positive profits.[2] Such factors as rapidity and smoothness of growth in profits and sales through time may be important also, as well as longevity of such growth. Robustness in the face of uncertainty is to be desired as well. For problems of this nature the Industrial Dynamicists

---

[1]Rate of return is even less appropriate for large-scale capital decisions since the pattern of cash flows which occur (a sequence of positive and negative flows, both generally increasing in absolute value through time) is the sort which is likely to result in multiple rates of return.

[2]Another weakness of dynamic programming as a solution technique is that a nonlinear utility function for accounting profits cannot be employed since, unlike present value of cash flows, intertemporal utility is usually not additive.

would probably be correct in giving greater attention to the paths taken by key variables through time than to scalar measures of performance. Growth in market share, size, and consequently importance of the firm and prestige for its officers often will take precedence over return to present equity holders (witness some Japanese textile producers who have proudly claimed that, while greatly increasing sales over the past decade, they have been working with a net margin of 2% or less).

Weingartner[1] has proposed a model for selecting independent investment projects in which the objective is to maximize the last of a finite sequence of nondecreasing dividend payments; such an objective may be appropriate for our problem as well. Nevertheless, incorporating a nonlinear utility function appears to be a worthwhile direction for extending the present work, but one that would probably require a completely new approach to solution than those discussed in this thesis, since the difficulty of obtaining an optimal solution is increased by several orders of magnitude by the "utility cost" of any investment being dependent on other investment and retirement decisions. The operationality of developing more sophisticated criteria for investment, however, is likely to remain questionable for some time to come.

---

[1] "Criteria for Programming Investment Project Selection," pp.201-212, in Weingartner [106].

BIBLIOGRAPHY

1. Alpert, S. B. "Economics of Scale in the Metal Removal Industry," Journal of Industrial Economics 7:175-181, July, 1959.

2. Arnoff, E. L. and J. C. Chambers. "On the Determination of Optimum Reserve Generating Capacity," Ch.19 in Analysis of Industrial Operations, Bowman and Fetter, Richard D. Irwin, Inc.

3. Arrow, K. J. "Optimal Capital Policy, the Cost of Capital, and Myopic Decision Rules," Annals of the Institute of Statistical Mathematics 16:21-30, Tokyo, 1964.

4. Babb, E. M., and T. H. Yu. "Analysis of Factors Affecting Least-Cost Size of Plant," Management Science, Vol. 16, No.10, June 1970.

5. Baumol, W. J. and Quandt, R. E. "Mathematical Programming and the Discount Rate under Capital Rationing," Economic Journal, June 1965, pp. 317-329.

6. Bellman, R. E. Dynamic Programming, Princeton University Press, Princeton, New Jersey, 1957.

7. Bellman, R. E. and Dreyfus, S. E. Applied Dynamic Programming (Princeton, New Jersey: Princeton University Press), 1962.

8. Bowman, E. H. "Scale of Operations - An Empirical Study," Operations Research, 6, 3 (May-June 1958).

9. Bowman, E. H. and Stewart, J. B. "A Model for Scale of Operations," Journal of Marketing, 20, 3 (January 1956).

10. Canada, J. R. and Wadsworth, H. M. "Methods for Quantifying Risk in Economic Analyses of Capital Projects," Journal of Industrial Engineering, XIX, 1 (January 1968).

11. Carlson, S. A Study on the Pure Theory of Production (New York: Kelley and Millman, Inc.), 1956.

12. Chang, Y. S. and Henderson, B. D. "Strategies in Capacity Investment," Working Paper, College of Business Administration, Boston University, 1971.

13. Chang, Y. S.; Woon, Chung; Lloyd, George; Maddison, George and Lockard, Ron. "The Capacity Investment Decision" Working Paper, Department of Operations Management - Quantitative Methods, Boston University, 1971.

14. Chenery, H. B., "Overcapacity and the Deceleration Principle," Econometrics, 20, (January 1952), pp. 1-28.

15. Chenery, H. B. "The Interdependence of Investment Decisions," in M. Abramovitz, ed, The Allocation of Economic Resources, Stanford, California: Stanford University Press, 1959, pp. 82-120.

16. Chilton, C. H. "Six-Tenths Factor Applies to Complete Plant Costs," in C. H. Chilton, Cost Engineering in the Process Industries (New York: McGraw-Hill), 1960.

17. Coleman, J. R. and York, R. "Optimum Plant Design for a Growing Market," Journal of Industrial and Engineering Chemistry, 56, 1 (January 1964), pp. 28-34.

18. Coleman, J. R., Jr., Smidt, S., and York, P. "Optimal Plant Design for Seasonal Production," Management Science, 10, 4.

19. Converse, A. O. Optimization. Holt, Rinehart and Winston, 1970.

20. Corrigan, T. E. and Dean, M. J. "Determining Optimal Plant Size," Chemical Engineering, 74, 17 (14 August 1967), pp.152-156.

21. Crowston, W. B. and Sjogren, J. G. "Capacity Expansion with Non-Proportional Operating Cost Functions," Working Paper 544-71, Sloan School of Management, MIT, May, 1971.

22. Daniels, Everett J. "A Long Range Planning Model for a New Firm," Journal of Industrial Engineering, XV, 2 (March-April 1964), pp. 102-105.

23. Dean, J. Managerial Economics (New York: Prentice-Hall, Inc.), 1951.

24. de Ghellinck, G. "Les Problemes de Decisions Sequentielles," Cahiers du Centre d'Etudes de Recherche Operationelle, 2:161-179, Brussels. 1960.

25. DeLeeuw, F. "The Concept of Capacity," The American Statistical Association, 57, 300.

26. Domar, E. D. "Capital Expansion, Rate of Growth, and Employment," Econometrica, 14, 1946.

27. Drucker, P. F. "Long Range Planning," Management Science, 5, 3 (April 1959), pp.238-249.

28. Erlenkotter, D. "Two Producing Areas - Dynamic Programming Solutions," Chapter 13 in Manne, Investments for Capacity Expansion, Size, Location and Time-Planning, MIT Press, 1967.

29. Erlenkotter, D. "Optimal Plant Size with Time-Phased Imports," in Manne (1967), pp. 157-177.

30. Erlenkotter, D. "Preinvestment Planning for Capacity Expansion: A Multi-Location Dynamic Model," Doctoral Dissertation, Graduate School of Business, Stanford University, 1969.

31. Falquier, A. D. "Evaluating Consequences of Optimal Investment Decisions for Capacity," 15.781 Paper, Sloan School of Management, 1967.

32. Fan, L. and Wang, C. The Discrete Maximum Principle, Wiley, 1964.

33. Faur, P. "Elements for the Selection of Investments in the Refining Industry," Proceedings of the Second International Conference on Operations Research, J. Wiley and Sons.

34. Fetter, R. B. "A Linear Programming Model for Long-Range Capacity Planning," Management Science, 7.

35. Gavett, J. W. "Capacity Decisions," Chapter 11 in Production and Operations Management, Harcourt Brace and World, New York, 1968.

36. Generoso, E. and Hitchcock, L. B. "Optimizing Plant Expansion: Two Cases," Industrial and Engineering Chemistry, 60, 1 (January 1968), pp.12-21.

37. Green, R. S. Heuristic Coupling of Aggregate and Detail Models in Factory Scheduling. Ph.D. Thesis, Sloan School of Management, MIT, January, 1971.

38. Gregory, J. K. "Optimum Plant Size and Location under Uncertainty," unpublished Doctoral Dissertation, Stanford University, 1966.

39. Gupta, S. K. and Rosenhead, J. "Robustness in Sequential Investment Decisions," Management Science, 15, 2 (October 1958).

40. Hadley, George. Nonlinear and Dynamic Programming (Reading, Mass: Addison Wesley), 1964.

41. Happel, J. Chemical Process Economics (New York: Wiley), 1958.

42. Haselbarth, J. E. "Updated Investment Costs for 60 Types of Chemical Plants," Chemical Engineering, (December 1967), pp.214-216.

43. Hess, S. W. and Weaver, J. B. "How Big a Plant?" Industrial Engineering Chemistry, 53, 7 (July 1961).

44. Hickman, B. G. "On a New Method of Capacity Estimation," Journal of the American Statistical Association, 59, 306.

45. Hinomoto, H. "Capacity Expansion with Facilities under Technological Improvement," Management Science, 11, 5 (March 1965).

46. Horowitz, Ira. "Formulation of the Price-Cost Relationship in a Class of Plant Investment Decisions," Journal of Industrial Engineering, XIV, 6, pp. 300-305.

47. Horowitz, Ira. "The Plant Investment Decision Revisited," The Journal of Industrial Engineering, XVII, 8, pp. 416ff.

48. Howard, R. Dynamic Programming and Markov Processes, New York: Wiley, 1960.

49. Jernick, T. L. "Inventory Control and Investment Analysis of Production Facilities by Simulation," M. S. Thesis, Sloan School of Management, 1967.

50. Kaufman, G. M. "Sequential Investment Analysis under Uncertainty," The Journal of Business, XXXVI, 1 (1963).

51. Kendrick, D. A. Programming Investment in the Steel Industry. (Cambridge, Massachusetts: Department of Economics, MIT, Doctoral Thesis), 1965.

52. Kendrick, D. Programming Investment in the Process Industries, Cambridge, Mass.: Technology Press, 1967.

53. Klein, M. and Klimpel, R. R. "Application of Linearly Constrained Nonlinear Optimization to Plant Location and Sizing," Journal of Industrial Engineering, XVIII, 1 (January 1967), pp.90-95.

54. Kurakoshi, K. "The Optimal Expansion of Production Facilities Under Limited Capital," Electrical Communications Laboratory Technical Journal, 13, 4.

55. Lawless, R. M. and Haas, P. R. "How to Determine the Right Size Plant," Harvard Business Review, 40, 3.

56. Lesso, William G. An Algorithm for the Optimal Selection, Sequence and Timing of Interrelated Major Capital Investments, Austin, Texas: unpublished Ph.D. Thesis, University of Texas, May, 1968.

57. Lorie, J. H. and Savage, L. J. "Three Problems in Rationing Capital," Journal of Business, October 1955.

58. Lucas, R. E. Jr. "Adjustment Costs and the Theory of Supply," Journal of Political Economy, Vol. 75, No. 4, August 1967.

59. Luenberger, D. G. "Cyclic Dynamic Programming: A Procedure for Problems with Fixed Delay," Operations Research, Vol. 19, No. 4, July-August 1971.

60. Manne, A. S. Scheduling of Petroleum Refinery Operations, Cambridge Mass.: Harvard University Press, 1956.

61. Manne, A. S. "Linear Programming and Sequential Decisions," Management Science, 6:259-267, April 1970.

62. Manne, A. S. "Capacity Expansion and Probabilistic Growth," Econometrica, 29, 4 (October 1961), pp. 632-649.

63. Manne, A. S. Economic Analysis for Business Decisions, McGraw-Hill, 1961.

64. Manne, A. S. "Plant Location under Economies of Scale," Management Science, November 1964.

65. Manne, A. S. Investments for Capacity Expansion, Size, Location and Time Planning, Cambridge, Mass.: MIT Press, 1967.

66. Manne, A. S. and Markowitz (eds.) Studies in Process Analysis, Wiley, 1963.

67. Manne, A. S. and Erlenkotter, D. "Capacity Expansion for India's Nitrogenous Fertilizer Industry," Management Science, 14, 10 (June 1968), pp. 553-572.

68. Marcus, B. T. and Ungar, A. J. A Multi-period Two-stage Model for Locating Alumina and Aluminum Reduction Capacity, Kaiser Aluminum and Chemical Corporation.

69. Marglin, S. A. Approaches to Dynamic Investment Planning. Amsterdam: North-Holland Publishing Company.

70. Masse, P. and Gibrat, R. "Application of Linear Programming in Investments in the Electric Power Industry," Management Science, January 1957.

71. McDowell, I. "The Economical Planning Period for Engineering Works," Operations Research, 8:533-542, July-August 1960.

72. Meyer, J. R. and Kuh, E. The Investment Decision, An Empirical Study. Cambridge, Mass.: Harvard University Press, 1957.

73. Mitten, L. G. and Nemhauser, G. L. "Multi-stage Optimization," Chemical Engineering Progress, 59, 1963, pp.52-60.

74. Modigliani, F. and Hohn, F. E. "Production Planning over Time and the Nature of the Expectation of Planning Horizon," Econometrica, 23, January 1955, pp. 46-66.

75. Moore, F. T. "Economies of Scale: Some Statistical Evidence," Quarterly Journal of Economics. 73-232-245.

76. Morris, W. T. The Capacity Decision System, Irwin, 1967.

77. Myers, S. C. "Linear Programming for Financial Planning Under Uncertainty." Working Paper 387-69, Sloan School of Management, MIT, May 1969.

78. Nord, O. C. Growth of a New Product - Effects of Capacity Acquisition Policies. Cambridge, Mass.: MIT Press, 1963.

79. Rappaport, L. A. and Drews, W. P. "Mathematical Approach to Long-Range Planning, Harvard Business Review, pp. 75-87.

80. Reisman, A. and Buffa, E. S. "A General Model for Investment Policy," Management Science, 8, 3 (1963)

81. Rosen, J. B. "The Gradient Projection Method for Nonlinear Programming -- Part I, Linear Constraints," Journal of the Society of Applied and Industrial Mathematics, 8, 1 (1960)

82. Sa, Graciano. Concave Programming in Transportation Networks. Doctoral Thesis, Sloan School of Management, MIT, 1968.

83. Salatan, D. I. and Caselli, A. V. "Optimum Design Capacity of New Plants," Chemical Engineering Progress, 59, 5 (May 1963), pp.69-75.

84. Scarpati, J. S. "Review of Some Methods of Planning Capacity Increases to Meet Expanding Demand Capacity Over-time," Term Paper, 15.781 Sloan School of Management, 1967.

85. Schreiber, A. N. Corporate Simulation Models, University of Washington Press, 1970.

86. Sjogren, J. G. Capacity Expansion Models Allowing Use of Non-Proportional Operating Cost Functions, S. M. Thesis, Sloan School of Management, MIT, 1971.

87. Smith, K. R. "Risk and the Optimal Utilization of Capital," Review of Economic Studies, Vol. 37, No. 110, April 1970.

88. Smith, U. L. "Economic Equipment Policies: An Evaluation," Management Science, 4, 1957, pp.20-37.

89. Sobel, M. J. "Employment Smoothing (Capital Accumulation) for Stochastic Demand," Management Science, Vol.16, No.5, January 1971.

90. Solomon, E. The Theory of Financial Management. New York: Columbia University Press, 1963.

91. Srinivasan, T. N. "Geometric Rate of Growth of Demand," Chapter 9 in Manne, Investments for Capacity Expansion....

92. Starr, M. K. "Planning Models," Management Science, 13, 4 (December 1966), pp.b-115ff.

93. Stigler, G. J. "The Economies of Scale," Journal of Law and Economics 1, 1 (1958).

94. Stigler, G. J. "Production and Distribution in the Short-Run," Journal of Political Economy, 47, June 1939, pp. 305-327.

95. Swanson, C. V. Resource Control in Growth Dynamics. Ph.D. Thesis, Sloan School of Management, MIT, June, 1969.

96. Sweezy, P. "Demand under Conditions of Oligopoly," Journal of Political Economy, 1939.

97. Terborgh, G. Dynamic Equipment Policy, New York: McGraw-Hill, 1949.

98. Terborgh, J. Business Investment Policy. Washington, D. C.: Machinery and Allied Products Institute, 1958.

99. Thompson, R. G., and Proctor, M.S. "Optimal Production, Investment, Advertising, and Price Controls for the Dynamic Monopoly Firm," Management Science, Vol. 16, No. 3, November 1969.

100. Veinott, A. F. and Wagner, H. M. "Optimal Capacity Scheduling I, II", Operations Research, July-August, 1962.

101. Veitoritz, T. and Manne, A. S. Chemical Processes, Plant Location and Economies of Scale: Studies in Process Analysis, edited by Manne and Markowitz. New York: John Wiley and Sons, 1963.

102. Waddell, R. M. Capacity Expansion Planning Factors, 1966.

103. Walters, A. A. "Production and Cost Functions: An Econometric Survey," Econometrica, 33:1-66, January 1963.

104. Wein, H. H. and Sreedharan, V. P. The Optimal Staging and Phasing of Multiproduct Capacity. Michigan State University, 1968.

105. Weeks, Chambers and Mullick. "Lease-Buy Planning Decisions," Management Science, 15, 6 (February 1969), pp.293-307.

106. Weingartner, H. M. Mathematical Programming and the Analysis of Capital Budgeting Problems. Chicago: Markhaus Publishing Co., 1967.

107. Weingartner, H. M. "Capital Budgeting of Interrelated Projects: Survey and Synthesis," Management Science, 12, 7 (March 1960), pp.485-576.

108. White, James M. "A Dynamic Model for the Analysis of Expansion," Journal of Industrial Engineering, XVII, 5 (May 1966), pp.275-281.

109. Whiting, I. J. "Planning for an Expanding Electricity Supply," The Operational Research Quarterly, 14, 2.

110. Zabel, E. "On the Meaning and Use of a Capacity Concept," Naval Research Logistics Quarterly, 2, 1955.

111. Zabel, E. "Measures of Industrial Capacity," Naval Research Logistics Quarterly, 3, 1956.

APPENDIX

This appendix contains the PL-1 source text for the programs employed in the computational work of Chapter VI. Complete understanding of the functioning of these programs, particularly with regard to input and output, is best obtained after an understanding of the operation of interactive computational systems, and of Multics, a sophisticated remote access service currently implemented on the GE-645, in particular. Programs are of two general types, those that relieve the user from the laborious task of entering masses of data, and those associated with the algorithm itself. Brief descriptions of the programs are below, followed by the source texts.

QDEMAND: The demand parameter generating program for the case of quadratic growth. Acquires key inputs concerning demand growth over time from the user.

ATANDEMAND: Similar to QDEMAND, but generates demand parameters which grow in an asymptotic fashion (according to the arctangent function). This program was not used in the computational work discussed.

TECH: Generates exponentially declining production unit age parameters to represent technological change.

ANEW: The most recent version of the control program for the backward and forward dynamic programming subroutines, ONE and TWO. Acquires and processes data from files established by the demand program used and TECH, and information from the user, including input constants, problem options, variable bounds and discretization, output format, etc. Allows any number of data elements to be individually changed for recomputation and allows re-execution of TWO alone with certain changes of data.

ONE: Computes state-stage cumulated returns f(.) using one-period profits from subroutine PIFUN.

TWO: Recovers the optimal solution, also using PIFUN.

PIFUN: Computes one-period profits with weighted cubic operating cost function and linear demand.

Further information concerning these programs may be obtained from the author.

QDEMAND

```
qdemand: procedure;

dcl d(50);

put list ("program for quadratic lowbounded demand p
   arameter");
 put list ("input initial d, maximum d, lower bound,
and peaking period");
put skip;

get list (initial_d,dmax,bound,npeak);

comput:do n=1 to 50;

d(n)= dmax - ((n-npeak)**2*(dmax-initial_d))/(1-npea
   k)**2;
if n>npeak & d(n)<bound then d(n)=bound;

end comput;

put file(filedem) list ((d(n) do n=1 to 50),"a");

close file(filedem);end qdemand;
```

ATANDEMAND

```
atandemand:proc;

dcl (d(50),initiald.dmax.b,c)float bin(27) real,(n,
 nmax) fixed bin(17) real;
put list ("program for atan demandparameter. Input
  initial d, dmax, n/d(n)=.9dmax");
get list (initiald,dmax,nmax);

c=(dmax-initiald)*2e0/3.14156e0;

comput:do n=1 to 50;

b=(n-1) *6.314e0/(nmax-1);

d(n)=c*atan(b)+initiald;

end comput;

put file(filedem) list ((d(n) do n=1 to 50),"a");

close file(filedem);

end atandemand;
```

TECH

```
tech:procedure;
dcl h(50);
put list ("input h(1), tech rate");
put skip;
get list (h(1),tech_rate);
comput:do i=2 to 50;
h(i)=h(i-1)*(1-tech_rate);end comput;
put file(filetech) list ((h(i) do i=1 to 50));
put file(filetech) list ("a");
close file(filetech);
end tech;
```

ANEW,ONE,TWO

```
anew:procedure;

dcl (os1,os2,c1,oh1.oh2,ss,hh,avh,b1,b2,a1,a2,a4,pi,qopt)
                        float bin(27) real external static;
dcl dem(50) float bin(27) real;

dcl (n,iy1,iy2) fixed bin(17) real external static;

dcl h(50) float bin(27) real,(ssmax,smax,hhmax,s,alpha,pi
max,ssdel,hhdel,sdel,
w1.w2,w3,w4,upwts,upwth,xlowts,xlowth,cinvfun,turn,hhh,ss
 s,fb.k1.k2,rate,rrate,cinv(0:20,50)) float bin(27) real;
dcl (expan_size,fa) float bin(27)  real,(ix1opt,ix2opt,kk
                              k) fixed bin(17) real;
dcl  f(50,0:10,0:6,0:1,0:1) float bin(27) real initial((1
                                         5400)0) ;
dcl (ii1,nmax,issmax,ihhmax,ismax,n1,ny1,ny2,isss,ihhh,ih
                              h,iss,i,is) fixed bin(17) real,
dcl iter fixed bin(17) real;

put list ("to retire type 1");

put skip;

get list (ii1);

put list ("type nmax,issmax,ssmax,ihhmax,hhmax,ismax,smax
                         ,gamma,alpha,rate,k1,k2");
put skip;

get list(nmax,issmax,ssmax,ihhmax,hhmax.ismax.smax.gamma,
                                    alpha,rate,k1,k2);
put list ("type os1,os2,oh1init,oh2init,b2,a1,a2,a4,c1");

put skip;

get  list (os1,os2,oh1init,oh2init,b2,a1,a2,a4,c1);

put skip;

get file(filedem) list ((dem(n) do n=1 to 50));
```

```
get file(filedem) list ((dem(n) do n=1 to 50));
close file(filedem);
get file(filetech) list ((h(i) do i=1 to 50));
close file(filetech);
repeat:ssdel = ssmax/issmax;hhdel = hhmax/ihhmax;sdel =
                                              smax/ismax;
rrate=1e0;comput:do n=1 to 50;rrate=rate*rrate;do is=0
                                                    to ismax;
cinv(is,n)=rrate*k1*((is*sdel)**k2);
end comput;
call one;call two;
one:procedure;
iter=0;
ndec: do n = nmax to 1 by -1; b1=dem(n); oh1=oh1init*ga
 mma**n;oh2=oh2init*gamma**n;n1 = n+1;state: do ihh = 0
 to ihhmax;
     hh = ihh*hhdel; do iss = 0 to issmax;ss = iss*ssdel;
do iy1 = 0 to ii1;do iy2 = 0 to ii1;
call pifun;
     f(n,iss,ihh,iy1,iy2) = -1e6;
decision:do is=0 to ismax;s=is*sdel;
cinvfun = cinv(is,n);turn = pi-cinvfun;
sss= ss+s; if sss>0 then hhh = gamma* (s*h(n)+ss*hh)/sss;
else hhh=gamma*hh;
if hhh>hhmax then hhh=hhmax;
if sss>ssmax then sss=ssmax;
ihhh = hhh/hhdel;isss = sss/ssdel;
upwts = mod(sss,ssdel)/ssdel;xlowts = 1-upwts;
```

```
upwts = mod(sss,ssdel)/ssdel;xlowts = 1-upwts;

upwth = mod(hhh,hhdel)/hhdel; xlowth = 1-upwth;

w1 = xlowts*xlowth*alpha;w2 = upwts*xlowth*alpha;w3 = x
        lowts*upwth*alpha;w4 = upwts*upwth*alpha;
 wierd:do ny1 = 0 to iy1;do ny2 = 0 to iy2;

     if n<nmax then fb = turn+

     w1*f(n1,isss,ihhh,ny1,ny2)+

     w2*f(n1,isss+1,ihhh,ny1,ny2)+

     w3*f(n1,isss,ihhh+1,ny1,ny2)+

     w4*f(n1,isss+1,ihhh+1,ny1,ny2);

                                        else fb = turn;

    if fb>f(n,iss,ihh,iy1,iy2) then f(n,iss,ihh,iy1,iy2
   ) = fb;
iter=iter+1;

end wierd;

skip0:iter=iter;

end decision;

end state;

put list (n);

if n=1 then gg:do; put data (iter); end gg;

put skip;

end ndec;

end one;

two:procedure;

put list ("type ismax,initial_new_s");
```

```
put list ("type ismax,initial_new_s");
put skip;
get list (ismax,ss);
sdel=smax/ismax;
rrate=1e0;comput:do n=1 to 50;rrate=rate*rrate;do is=0
cinv(is,n)=rrate*k1*((is*sdel)**k2);           to ismax;
end comput;
hh=1;
iy1=ii1;iy2=ii1;
iter=0;
ndec:do n=1 to nmax; n1=n+1;
oh1=oh1init*gamma**n;oh2=oh2init*gamma**n;
b1=dem(n);
call pifun;
     fa = -1e6;
loop:do is=0 to ismax;s=is*sdel;
cinvfun = cinv(is,n);turn = pi-cinvfun;
sss = ss+s; if sss>0 then  hhh = gamma* (s*h(n)+ss*hh)/
                                      sss;
else hhh=gamma*hh;
if hhh>hhmax then hhh=hhmax;
if sss>ssmax then sss=ssmax;
ihhh = hhh/hhdel;isss = sss/ssdel;
if ihhh>ihhmax then ihhh=ihhmax; if isss>issmax then i
                  sss=issmax;
```

```
if ihhh>ihhmax then ihhh=ihhmax; if isss>issmax then iss
upwts = mod(sss,ssdel)/ssdel;xlowts = 1-upwts; s=issmax;
upwth = mod(hhh,hhdel)/hhdel; xlowth = 1-upwth;
w1 = xlowts*xlowth*alpha;w2 = upwts*xlowth*alpha;w3 = xl
                   owts*upwth*alpha;w4 = upwts*upwth*alpha;
wierd:do ny1 = 0 to iy1;do ny2 = 0 to iy2;
     if n<nmax then fb = turn+
     w1*f(n1,isss,ihhh,ny1,ny2)+
     w2*f(n1,isss+1,ihhh,ny1,ny2)+
     w3*f(n1,isss,ihhh+1,ny1,ny2)+
     w4*f(n1,isss+1,ihhh+1,ny1,ny2);
     else fb = turn;
     if fb>fa then eat:do; fa = fb;
quantity=qopt;price=b1-b2*qopt;
pimax=pi;
expan_size=s;ix1opt=iy1-ny1;ix2opt=iy2-ny2;end eat;
iter=iter+1;
end wierd;
skip0:iter=iter;
end loop;
put data (n,fa);
put skip;
put data (iter);
put skip;
```

```
put data (ss,hh,avh,expan_size,quantity,price.pimax.iy
                                1,iy2);
if ix1opt=1 then rr:do; put list ("//////////replace o
                                ne///////////"); end rr;
if ix2opt=1 then ww:do; put list ("//////////replace t
                                wo///////////"); end ww;
put skip;

put skip;

if ss+expan_size>0 then hh=gamma*(hh*ss+h(n)*expan_siz
                                e)/(ss+expan_size);
else hh=gamma*hh;

ss=ss+expan_size;

iy1=iy1-ix1opt;iy2=iy2-ix2opt;

end ndec;

end two;

put data (ii1);

put skip;

put list ("again?"); put skip;get list (kkk); if (kkk)
                                =1 then  sub:do;
put list ("enter changes"); put skip; get data; put sk
                                ip; go to repeat; end sub;
if (kkk)>1 then call two;

end anew;

EOF
```

PIFUN

```
pifun:procedure;

 dcl (os1,os2.c1, oh1.oh2,ss,hh,avh,sstot,b1,b2,a1,a2,
   a4,pi,qopt) float bin(27)  real external static;
dcl (n,iy1,iy2) fixed bin(17) real external static;

sstot=ss+os1*iy1+os2*iy2;

if sstot<.0001e0 then sstot=.0001e0;

avh=(hh*ss+oh1*iy1*os1+oh2*iy2*os2)/sstot;

if avh<0 |b1<a2 then aa:do;

put data (hh,ss,os1,os2,sstot,avh,a1,a2,a4,c1,n,b1,b2);

end aa;

if avh<.0001e0 then avh=.0001e0;

qopt=(sstot**2*(2e0*b2-((2e0*b2)**2+12e0*(b1-a2)*a4*(
     avh**2)/sstot**2)**.5e0))/(-6e0*(avh**2)*a4);
pi=-((avh**2)*(sstot**c1)*a1 +a2*qopt +((avh**2)*a4*q
   opt**3e0)/sstot**2) + b1*qopt-b2*qopt**2;
end pifun;
```

Biographical Note

Robert Ronald Trippi was born on July 10, 1944 in Brooklyn, New York. He entered the Polytechnic Institute of Brooklyn in 1962 as an Engineering major, but as a result of a shift of interests in 1964 he transferred to Queens College of the City University of New York, from which in September, 1966 he graduated with the B.A. degree (Magna Cum Laude) in Economics. He entered MIT in September, 1966, earned the M.S. degree in Management in June, 1968, and completed the Ph.D. degree in January, 1972. Employment while an undergraduate ranged from painting the insides of sewer pipes to the design of ultra-high-stability miniature variable capacitors. While at MIT he was a teaching assistant (September 1967-June 1968), Instructor at Boston University (Spring, 1969), and worked summers as Instructor at Queens College (1967), consultant to the Provost of Lowell Technological Institute (1968), and as Research Associate at Charles River Associates (1969), in addition to being intermittently self-employed. He is a member of Sigma Xi. AAAS, TIMS, and Omicron Delta Epsilon.