A Finite State Machine Framework for Robust Analysis and Control of Hybrid Systems

by

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Abstract

Hybrid systems, describing interactions between analog and discrete dynamics, are pervasive in engineered systems and pose unique, challenging performance verification and control synthesis problems. Existing approaches either lead to computationally intensive and sometimes undecidable problems, or make use of highly specialized discrete abstractions with questionable robustness properties.

The thesis addresses some of these challenges by developing a systematic, computationally tractable approach for design and certification of systems with discrete, finite-valued actuation and sensing. This approach is inspired by classical robust control, and is based on the use of finite state machines as nominal models of the hybrid systems. The development does not assume a particular algebraic or topological structure on the signal sets.

The thesis adopts an input/output view of systems, proposes specific classes of inequality constraints to describe performance objectives, and presents corresponding 'small gain' type arguments for robust performance verification. A notion of approximation that is compatible with the goal of controller synthesis is defined. An approximation architecture that is capable of handling unstable systems is also proposed. Constructive algorithms for generating finite state machine approximations of the hybrid systems of interest, and for efficiently computing a-posteriori bounds on the approximation error are presented. Analysis of finite state machine models, which reduces to searching for an appropriate storage function, is also shown to be related to the problem of checking for the existence of negative cost cycles in a network, thus allowing for a verification algorithm with polynomial worst-case complexity. Synthesis of robust control laws is shown to reduce to solving a discrete, infinite horizon min-max problem. The resulting controllers consist of a finite state machine state observer for the hybrid system and a memoryless full state feedback switching control law.

The use of this framework is demonstrated through a simple benchmark example, the problem of stabilizing a double integrator using switched gain feedback and binary sensing. Finally, some extensions to incremental performance objectives and robustness measures are presented.

Thesis Supervisor: Munther A. Dahleh
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To my Parents...

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Chapter 1

Introduction

1.1 Background and Motivation

Hybrid systems are dynamical systems involving interacting analog and discrete dynamics; that is, dynamics that evolve in continuous and discrete state-spaces. Hybrid systems are ubiquitously present in engineered systems, including automotive systems [8, 56, 67], high performance aircrafts [59], helicopters [55], unmanned aerial vehicles [26, 28], chemical processes [25] and manufacturing plants [60]. Hybrid dynamics also arise in distributed control systems, such as highway [38, 49], air traffic [10, 46, 77] and internet congestion control problems [37, 68, 69], and in biological systems, such as genetic regulatory networks [22, 45].

Hybrid dynamics give rise to unique and challenging simulation, performance verification and control synthesis problems. Given their practical relevance and the multitude of challenges they pose, hybrid systems have received much attention since the early 1990's. Their study has brought together researchers and ideas from two traditionally distant fields, control theory and computer science [5, 36], in an attempt to tackle the inherently multidisciplinary nature of the problems.

Many modeling frameworks have been developed for describing hybrid systems, together with corresponding verification and synthesis tools. The various frameworks were motivated by different problems and address a range of needs. They can be classified into three broad categories, depending on whether the models used are hybrid, continuous or discrete.

Modeling frameworks involving hybrid models, in which both the continuous and discrete evolutions of the hybrid system are explicitly described [33, 51, 58], have been widely researched. Stochastic versions of hybrid automata models have also been proposed [39]. Problems of analysis and safety verification generally reduce to reachability analysis [3]. Approximate solutions to the reachability problems are typically sought, except for special classes of systems where exact solutions are possible [42]. Problems of control synthesis generally reduce to solving an appropriate Hamilton-Jacobi-Bellman equation [11, 48, 50]. The strengths of these modeling frameworks are their generality and their ability to model a wide range of practical problems. Their potential limitation is the computational complexity of their associated analysis and synthesis approaches, that generally scales exponentially with the dimension of the continuous state-space, and the undecidability of certain problems [35]. Thus, much of the current research focuses on developing more efficient computational methods [40, 62, 76] and on identification of decidable (or undecidable) problems [18].

Other approaches exist in which hybrid systems are lifted to a continuous description [17, 21]. Such approaches are possible due to the existence of differential equation models that simulate finite automata and Turing machines [17, 20]. The strengths of these methods is that they allow for a unified approach to hybrid systems analysis and synthesis using standard tools from control theory. However it is not clear that, in doing so, they offer any reduction in complexity [16].

A third set of approaches consists of abstracting a hybrid system to a purely discrete description [4] constructed so as to preserve, typically in a simulation or bisimulation sense, some properties of interest [32]. While these approaches are generally used when the underlying goal is system analysis or verification, problems of controller synthesis based on discrete abstractions have also been considered [15]. Although some classes of hybrid systems are amenable to a finite discrete representation, including rectangular [34] and o-minimal hybrid systems [19, 41], this approach is restrictive. In an attempt to ease these restrictions, current research focuses on approximating hybrid systems so as to guarantee simulation or bisimulation properties asymptotically [30]. However, the relation between the quality of approximation in the proposed metrics and the actual performance of the system, particularly in closed loop, is not well quantified.

This thesis was motivated by the desire to develop a systematic, semi-automated and computationally tractable paradigm that would allow for analysis and for synthesis of certified controllers for a class of hybrid systems. In particular, the thesis focuses on systems with discrete, finite-valued actuation and sensing. That is, systems where the actuation effectively takes the form of a multi-level switch, and where sensing is coarse or finitely quantized. In addition to its academic relevance, this class of hybrid systems is practically relevant. Generally, there is a trade off between the complexity of engineered systems, their cost and their potential for failures. One way of designing cost-effective systems that exhibit desirable complex behaviors while ensuring low risk of failure is by cleverly switching between several simple subsystems. Such switched systems have been successfully demonstrated in practical applications, such as variable capacity compressors [71], and the trend is expected to continue. On the other hand, coarse sensing is a limitation we often have to live with, either due to cost or power limitations on the sensors themselves or due to the effects of quantization and bit rate constraints when control is done over a network.

1.2 Overview of the Framework

The hybrid design framework proposed in this thesis centers around the use of finite state machines as nominal models of the systems of interest for the purpose of control design. Inspired by classical robust control, the framework provides an approach for quantifying the approximation error resulting from reducing hybrid systems to discrete models, and for designing controllers that are robust to this modeling uncertainty, and that are thus certified by design.

The classical robust control framework [23, 86] and generalizations of it provide paradigms and efficient computational tools for system analysis and optimal controller synthesis in the face of uncertainty (due to modeling errors, linearization, model order reduction, external disturbances, etc...). The idea there is to approximate a given system by a nominal LTI model and to establish a quantitative measure for the degree of fidelity of the nominal model to the original system. This measure is typically an induced gain bound, or more generally an integral quadratic constraint [53], for the system representing uncertainty. An LTI controller is then designed to stabilize the nominal model and to meet other performance objectives, also typically described by induced gain bounds or integral quadratic constraints, in the presence of admissible uncertainty. Robust performance of the closed loop system is verified using a small gain argument or an S-procedure [54, 83].

These 'classical' approaches are not directly applicable when the systems in question are hybrid, as is the case in this thesis, for several reasons. First, the $\mathcal{H}_\infty$ design problem, traditionally solved when synthesizing robust LTI controllers, does not allow for restrictions on the structure of the controller. In particular, when the control signal (the output of the controller) is restricted to take its values in a finite set, as in the case here, $\mathcal{H}_\infty$ control design becomes an unsolved (and likely unsolvable) problem. Second, a particular algebraic structure is assumed and utilized in the classical framework. Namely, the signal spaces are vector spaces, the nominal models and the controllers are linear, and the performance objectives are quadratic. This structure is non-existent in the class of problems of interest. In general, the control input and sensor output signals are strings over arbitrary symbol sets. Thus, quadratic cost functions are not meaningful in this setting. Moreover, even when each of the switched systems has linear internal dynamics, switching and finite output quantization result in highly nonlinear dynamics that are unlikely to be well approximated by LTI models.

Nevertheless, the fundamental ideas of classical robust control can be abstracted and put to use in a different setting. Systems can be intuitively thought of as infinite state machines, with finite state machines as their obvious approximations. Hence, the natural direction is towards a finite state machine based robust control framework that would provide a systematic, tractable and computer-aided approach to tackle the class of problems of interest. This direction was first proposed in [52], and was adopted and further developed in this thesis. The development is in three complementary directions: (i) approaches for generating approximate finite state machine models of hybrid systems, with useful guarantees on the quality of approximation, (ii) a set of tools for robust performance analysis and (iii) methods for synthesizing robust controllers for finite state machine nominal models.

The resulting hybrid control design paradigm is as follows (Figure 1-1). Given a hybrid system P, with discrete, finite-valued actuation and sensing, and given a physical performance objective. The system is first represented as the feedback interconnection of a finite state machine M and an uncertainty block Δ, describing approximation uncertainty. The uncertainty is quantified in terms of some useful constraints on its input and output signals. A 'small gain' type argument is used to determine the desired constraints on the input and output signals of the closed loop M and the switching law φ to be designed, so that the actual system meets its performance objectives. This constraint thus defines a design objective, and the final step is designing a switching law φ so that the interconnected system (M, φ) meets this design objective. The resulting controller K, consisting of a finite state

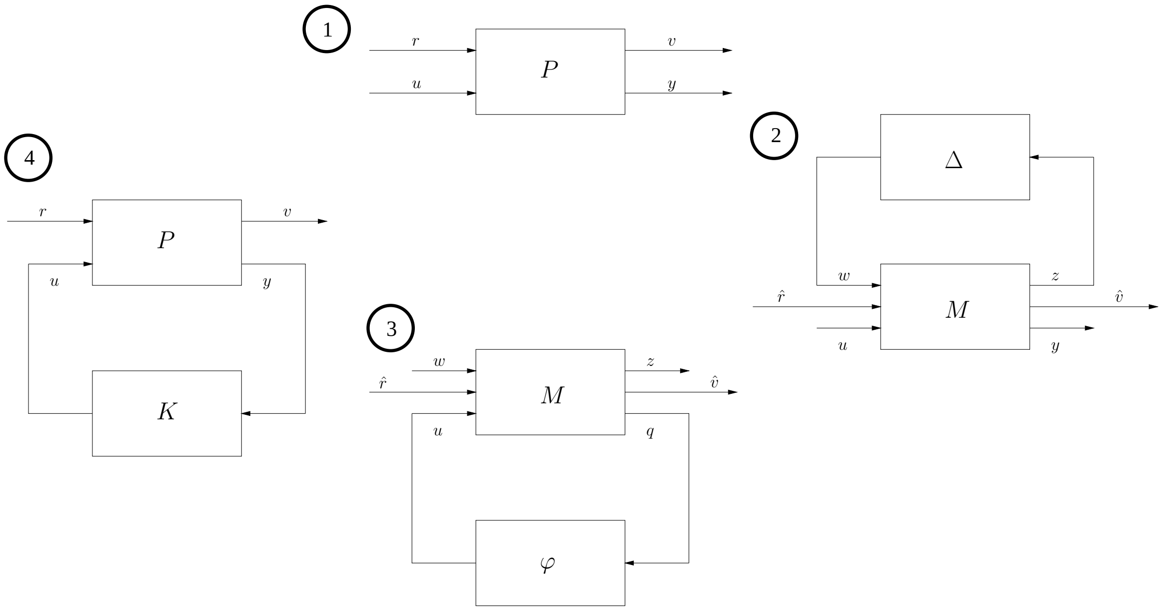

Figure 1-1: Hybrid control design paradigm

machine observer and the switching law φ, can then be implemented in feedback with the original hybrid system, and the physical performance objectives are guaranteed to be met by design.

1.3 Contributions of the Thesis

The foundations of the hybrid design framework and its various steps are discussed in detail in Chapters 2-5. An illustrative benchmark example is presented in Chapter 6. Extensions of this approach to incremental descriptions of performance objectives and quality of approximation are discussed in Chapter 7. The conclusions and recommendations for future work are presented in Chapter 8.

The contributions of the thesis are:

- A unified input/output view of systems, performance and robustness that is appropriate for the hybrid systems of interest in that it does not assume a particular algebraic or topological structure. Systems are understood to be sets of signals, performance objectives are described in terms of specific classes of constraints on the signals, and a corresponding new formulation of a 'Small Gain' theorem is derived (Chapter 2).

- An approach for approximating systems with finite actuation and sensing by finite state machines that is compatible with the objectives of control design. A suitable notion of approximation is introduced and an observer-like structure is proposed for the approximation error, thus allowing for approximation of systems that are externally unstable. Two constructive algorithms for generating nominal finite state machine models are presented, together with algorithms for computing a-posteriori gain bounds for the resulting approximation error (Chapter 3).

- A set of analytic and algorithmic tools for verifying stability and computing gains of deterministic finite state machine models. Existence of appropriate storage functions are shown to be necessary and sufficient conditions for stability and gain verification. A connection to discrete shortest path problems is established, allowing for strongly polynomial algorithms for stability and gain verification, with $\mathcal{O}(n^2)$ worst case computational complexity for an n state machine, under appropriate assumptions (Chapter 4).

- A characterization of the robust control design problem as a full state feedback, infinite horizon min-max problem parametrized by a scaling factor, whose solution satisfies a (scale dependent) Bellman inequality (Chapter 5).

- A demonstration of the hybrid design paradigm using a simple academic benchmark example: the problem of exponentially stabilizing a double integrator using switched gain feedback based on binary position measurements (Chapter 6).

- An extension of the analysis tools derived in the thesis to instances where performance objectives, system characteristics and robustness measures are described in terms of incremental constraints on the signals. A new formulation of an 'Incremental Small Gain' theorem is derived, as well as necessary and sufficient conditions for incremental stability and gain verification for finite state machine models (Chapter 7).

Earlier versions of these results as well as complementary results can be found in [72, 73, 74, 75].

1.4 Notation

The following notation is used throughout the thesis: given a set $\mathcal{A}$, $card(\mathcal{A})$ denotes its cardinality. $\mathcal{P}(\mathcal{A})$ denotes the power set of $\mathcal{A}$, that is the set of all its subsets, while $\emptyset$ denotes the empty set. Given two sets $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}\times\mathcal{B}$ denotes their Cartesian product while $\mathcal{A}\setminus\mathcal{B}$ denotes their difference, that is the set of all $x\in\mathcal{A}$ such that $x\notin\mathcal{B}$. For $a\in\mathcal{A}^p$, a_i denotes the i^{th} component of a. $\mathbf{Z}_+$ and $\mathbf{R}_+$ denote the set of non-negative integers and the set of non-negative reals, respectively. For every $k\in\mathbf{Z}_+$, ordered set $\mathbf{Z}_k=\{i\in\mathbf{Z}_+|i\leq k\}$. $\mathcal{A}^{\mathbf{Z}_+}$ is the set of all infinite sequences over set $\mathcal{A}$: that is, $\mathcal{A}^{\mathbf{Z}_+}=\{h:\mathbf{Z}_+\to\mathcal{A}\}$. An element of $\mathcal{A}$ is denoted by a while an element of $\mathcal{A}^{\mathbf{Z}_+}$ is denoted by $\mathbf{a}$ or $\{a(t)\}_{t=0}^{\infty}$. For $\mathbf{a}\in\mathcal{A}^{\mathbf{Z}_k}$ and for $T<k$, $\mathcal{P}_T^+(\mathbf{a})$ denotes the subsequence $\{a(t)\}_{t=T+1}^{k}$, while $\mathcal{P}_T^-(\mathbf{a})$ denotes the subsequence $\{a(t)\}_{t=0}^{T}$ for $T\leq k$. The notation $\subseteq$ and $\subset$ denotes inclusion and proper inclusion, respectively. For $\mathcal{A}_o\subset\mathcal{A}$, $\mathbf{I}_{\mathcal{A}_o}$ denotes the indicator function of set $\mathcal{A}_o$, that is the function $\mathbf{I}_{\mathcal{A}_o}:\mathcal{A}\to\{0,1\}$ defined by $\mathbf{I}_{\mathcal{A}_o}(a)=1$ for $a\in\mathcal{A}_o$ and $\mathbf{I}_{\mathcal{A}_o}(a)=0$ otherwise. For $f:D_1\to R_1$ and $g:D_2\to R_2$ with $R_1\subseteq D_2$, $g\circ f$ denotes the composition of f and g, that is the map $g\circ f:D_1\to R_2$ defined by $g\circ f(x)=g(f(x))$.

Chapter 2

A Framework for Systems, Performance and Robustness

2.1 Introduction

This chapter lays the foundations for the thesis by presenting a unified view of (discrete-time) systems, interconnections, performance objectives and robustness. This view integrates ideas from classical robust control theory [23, 86], behavioral systems theory [81], Integral Quadratic Constraint (IQC) analysis [53] and the theory of dissipative systems [79, 80], and develops them in new directions compatible with the hybrid problems of interest.

The philosophy is to view systems as sets of input and output signals, thus clearly differentiating between systems (signal sets) and models of systems (mathematical descriptions of the processes that generate the signal sets). Performance requirements for systems are described in terms of specific classes of inequality constraints on the elements of their signal sets, while interconnections are described in terms of intersections of signal sets. Performance of an interconnected system is described in terms of the performance of its components. Deterministic finite state machine models, used as the central building blocks of the hybrid systems design paradigm, are also introduced in this chapter. Simple illustrative examples of the definitions and results of this chapter are provided in Section 2.5.

2.2 Signals, Systems and Automata Models

2.2.1 An Input/Output View of Systems

A discrete-time signal is understood to be an infinite sequence over some prescribed set, which is referred to as an *alphabet set*.

Definition 2.1. *A discrete-time* ***system*** *S is a set of pairs of signals, $S \subset \mathcal{U}^{\mathbf{Z}_+} \times \mathcal{Y}^{\mathbf{Z}_+}$.*

A system is thus a process characterized by its *feasible signals set*, which is simply a list of ordered pairs of all the signals (sequences over input alphabet set $\mathcal{U}$) that can be applied as an input to this process, and all the output signals (sequences over output alphabet set $\mathcal{Y}$) that can be potentially exhibited by the process in response to each of the input signals. The notation S will be used interchangeably throughout the thesis to denote the system and its feasible signals set.

Typically, the input signals consist of control inputs (signals that are chosen by the controller), disturbance inputs (signals that negatively affect the system and that we have no control over) and exogenous inputs (possibly representing a reference signal or the effect of the environment on the system). The output signals typically include measured outputs (sensor measurements made available to the controller), performance outputs (signals that are used to describe system performance) and other outputs (possibly representing the effect of the system on the environment or on another system).

In general, the alphabet sets can be continuous, discrete (countable or uncountable) or both, particularly when the systems in question are hybrid. The particular class of hybrid systems considered in this thesis, referred to as *systems with finite actuation and sensing*, are those in which the control input and measured output are restricted to finite alphabet sets; no other a-priori assumptions are made about the system. The case when all the alphabet sets are finite is also of special interest in this thesis. The systems in question are then said to be *systems over finite alphabets*.

2.2.2 Deterministic Finite State Machine (DFM) Models

A special class of systems over finite alphabets provides a set of nicely tractable models that are used in this thesis to approximate the hybrid systems of interest. The common feature of systems in this class is that a specific process, mathematically modeled by a deterministic finite state machine (DFM) as described in Definition 2.2, generates the feasible signals set.

Definition 2.2. *A **deterministic finite state machine (DFM)** is a mathematical model of a discrete-time system described by two equations, a state transition equation (2.1) and an output equation (2.2):*

$$q(t+1) = f(q(t), u(t)) \tag{2.1}$$

$$y(t) = g(q(t), u(t)) \tag{2.2}$$

where $t \in \mathbf{Z}_+$, $q(t) \in \mathcal{Q}$, $u(t) \in \mathcal{U}$, $y(t) \in \mathcal{Y}$, *and where* $\mathcal{Q}$, $\mathcal{U}$ *and* $\mathcal{Y}$ *are finite sets.*

$\mathcal{U}$ and $\mathcal{Y}$ are finite alphabet sets of possible instantaneous values of the input signal and the output signal, respectively. $\mathcal{Q}$ is a finite set of states of the DFM. $f : \mathcal{Q} \times \mathcal{U} \to \mathcal{Q}$ is the state transition function and $g : \mathcal{Q} \times \mathcal{U} \to \mathcal{Y}$ is the output function. $\mathcal{D}_M$ is the feasible signals set of the system modeled by deterministic finite state machine M:

$$\mathcal{D}_M = \{(\mathbf{u}, \mathbf{y}) \in \mathcal{U}^{\mathbf{Z}_+} \times \mathcal{Y}^{\mathbf{Z}_+} | \ \exists \mathbf{q} \in \mathcal{Q}^{\mathbf{Z}_+} \text{ such that } \mathbf{q}, \mathbf{u}, \mathbf{y} \text{ satisfy (2.1), (2.2), } \forall t \in \mathbf{Z}_+\} \tag{2.3}$$

2.3 Gain Stability

A notion of gain stability is proposed in this section. In line with the general philosophy of the thesis, this notion of stability is an input/output property described in terms of constraints on the feasible signals of a system. It will be used to mathematically describe physical performance objectives of the system, in addition to representing robustness measures.

Definition 2.3. *Consider a system* $S \subset \mathcal{U}^{\mathbf{Z}_+} \times \mathcal{Y}^{\mathbf{Z}_+}$ *and let* $\rho : \mathcal{U} \to \mathbf{R}$ *and* $\mu : \mathcal{Y} \to \mathbf{R}$ *be given functions.* S *is* $\boldsymbol{\rho/\mu}$ ***gain stable*** *if there exists a finite non-negative constant* γ *such that the following inequality is satisfied for every pair* $(\mathbf{u}, \mathbf{y})$ *in* S*:*

$$\inf_{T \geq 0} \sum_{t=0}^{T} \gamma\rho(u(t)) - \mu(y(t)) > -\infty \tag{2.4}$$

Remark 2.1 The ρ/μ gain stability property, described by the inequality constraint in (2.4) can be equivalently expressed as follows: there exists a $\gamma > 0$ such that for every pair $(\mathbf{u}, \mathbf{y}) \in S$, there exists a constant $C_{\mathbf{u},\mathbf{y}} \in \mathbf{R}_+$ such that the following inequality:

$$\sum_{t=0}^{T} \mu(y(t)) \leq C_{\mathbf{u},\mathbf{y}} + \gamma \sum_{t=0}^{T} \rho(u(t))$$

holds for all $T \geq 0$. $\diamond$

The formulation in Remark 2.1 is reminiscent of induced norm descriptions of stability for LTI systems in the traditional setting. There, the functions ρ and μ are signal norms, typically l_1, l_2 or l_∞ norms. In the framework developed in this thesis, no algebraic or topological structure is assumed a-priori on the signal sets. Thus, no constraints are imposed on the functions ρ and μ. As a result, any system will be ρ/μ gain stable for some choice of ρ and μ. A judicious choice of ρ and μ is needed to ensure that the corresponding stability property provides a useful characterization of system S. The choice of ρ and μ is guided by the specifics of the physical problem. In particular, when ρ and μ are restricted to non-negative values, a notion of gain emerges.

Definition 2.4. *Consider a system $S \subset \mathcal{U}^{\mathbf{Z}_+} \times \mathcal{Y}^{\mathbf{Z}_+}$ and let $\rho : \mathcal{U} \rightarrow \mathbf{R}_+$ and $\mu : \mathcal{Y} \rightarrow \mathbf{R}_+$ be given non-negative functions. The* ***ρ/μ gain*** *of S is the greatest lower bound of γ such that (2.4) is satisfied for every pair $(\mathbf{u}, \mathbf{y})$ in S.*

A further refinement is possible when the alphabet sets $\mathcal{U}$ and $\mathcal{Y}$ are finite, and when ρ and μ are zero on some $\mathcal{U}_o \subset \mathcal{U}$ and $\mathcal{Y}_o \subset \mathcal{Y}$, respectively, and strictly positive elsewhere. In this case, a notion of gain stability that is independent of the particular choice of ρ and μ emerges (see Remark 2.2).

Definition 2.5. *Consider a system $S \subset \mathcal{U}^{\mathbf{Z}_+} \times \mathcal{Y}^{\mathbf{Z}_+}$ where $\mathcal{U}$ and $\mathcal{Y}$ are finite sets and let $\mathcal{U}_o \subset \mathcal{U}$ and $\mathcal{Y}_o \subset \mathcal{Y}$ be given sets. S is* ***gain stable about $(\mathcal{U}_o, \mathcal{Y}_o)$*** *if there exists a $\gamma > 0$ and $\rho : \mathcal{U} \rightarrow \mathbf{R}_+$, $\mu : \mathcal{Y} \rightarrow \mathbf{R}_+$ zero on $\mathcal{U}_o$ and $\mathcal{Y}_o$, respectively and strictly positive elsewhere, such that every pair $(\mathbf{u}, \mathbf{y})$ in S satisfies (2.4).*

Remark 2.2 Let $\rho_1 : \mathcal{U} \rightarrow \mathbf{R}_+$ and $\rho_2 : \mathcal{U} \rightarrow \mathbf{R}_+$ be zero on $\mathcal{U}_o \subset \mathcal{U}$ and strictly positive elsewhere. Let $\mu_1 : \mathcal{Y} \rightarrow \mathbf{R}_+$ and $\mu_2 : \mathcal{Y} \rightarrow \mathbf{R}_+$ be zero on $\mathcal{Y}_o \subset \mathcal{Y}$ and strictly positive elsewhere. Set

$$c_\rho = \max_{u \in \mathcal{U} - \mathcal{U}_o} \frac{\rho_1(u)}{\rho_2(u)}$$

and

$$c_\mu = \min_{y \in \mathcal{Y} - \mathcal{Y}_o} \frac{\mu_1(y)}{\mu_2(y)}$$

The following inequality holds for any non-negative constant γ, and any $T \geq 0$:

$$c_\mu \sum_{t=0}^{T} \frac{\gamma c_\rho}{c_\mu} \rho_2(u(t)) - \mu_2(y(t)) \geq \sum_{t=0}^{T} \gamma \rho_1(u(t)) - \mu_1(y(t))$$

It follows from this inequality and from finiteness of the alphabet sets that if, for some choice of functions ρ_1 and μ_1 zero on $\mathcal{U}_o$ and $\mathcal{Y}_o$ respectively and positive elsewhere, there exists a non-negative constant γ, say $\gamma = \gamma_1$, such that (2.4) holds; then for any other choice of functions ρ_2 and μ_2 zero on $\mathcal{U}_o$ and $\mathcal{Y}_o$ respectively and positive elsewhere, there exists a value $\gamma_2 \geq 0$, in particular $\gamma_2 = \frac{c_\rho \gamma_1}{c_\mu}$, such that (2.4) also holds for all $(\mathbf{u}, \mathbf{y}) \in \mathcal{D}$. Thus, given a system S and a particular choice of $\mathcal{U}_o$ and $\mathcal{Y}_o$, the existence (or non-existence) of a finite γ and functions $\rho : \mathcal{U} \rightarrow \mathbf{R}_+$ and $\mu : \mathcal{Y} \rightarrow \mathbf{R}_+$, zero on $\mathcal{U}_o$ and $\mathcal{Y}_o$ respectively and positive elsewhere, such that (2.4) is satisfied is an intrinsic property of the system. $\diamond$

Remark 2.3 In instances where the alphabet sets have some particular algebraic structure, there is a natural choice for $\mathcal{U}_o$ and $\mathcal{Y}_o$. For example, for an alphabet set with a monoid structure or a field structure, the natural choice is the singleton consisting of the identity element of the monoid and the additive identity element of the field, respectively. The signals $\mathbf{u} \in \mathcal{U}_o^{\mathbf{Z}_+}$ and $\mathbf{y} \in \mathcal{Y}_o^{\mathbf{Z}_+}$ are then 'zero' signals. $\diamond$

2.4 Stability of Interconnections

2.4.1 Feedback Interconnections

Consider two systems $S \subset (\mathcal{U} \times \mathcal{W})^{\mathbf{Z}_+} \times (\mathcal{Y} \times \mathcal{Z})^{\mathbf{Z}_+}$ and $\Delta \subset \mathcal{Z}^{\mathbf{Z}_+} \times \mathcal{W}^{\mathbf{Z}_+}$. Their feedback interconnection, denoted by (S, Δ) and shown in Figure 2-1, is a system $I \subset \mathcal{U}^{\mathbf{Z}_+} \times \mathcal{Y}^{\mathbf{Z}_+}$ with feasible signals set:

$$I = \{(\mathbf{u}, \mathbf{y}) | \; \exists \; \mathbf{w} \in \mathcal{W}^{\mathbf{Z}_+}, \mathbf{z} \in \mathcal{Z}^{\mathbf{Z}_+} \text{ such that } ((\mathbf{u}, \mathbf{w}), (\mathbf{y}, \mathbf{z})) \in S \text{ and } (\mathbf{z}, \mathbf{w}) \in \Delta\}$$

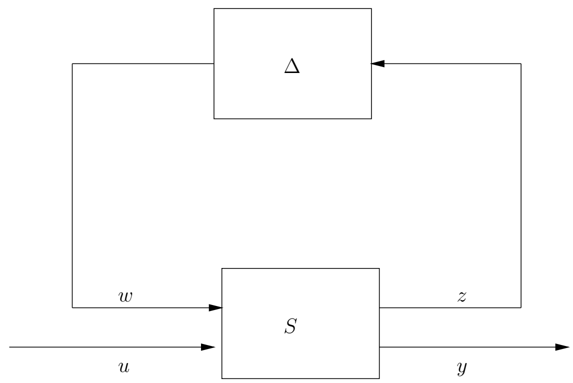

Figure 2-1: Feedback interconnection of S and Δ

Consider the feedback interconnection of S and Δ and suppose that both systems are gain stable. The question is, what can be said about their feedback interconnection, the system I with input u and output y?

2.4.2 A 'Small Gain' Theorem

The following Theorem describes the gain stability properties of an interconnected system in terms of those of its component systems.

Theorem 2.1. *(A 'Small Gain' Theorem) Suppose that system S is ρ_S/μ_S gain stable and satisfies (2.4) with $\gamma = 1$, for some $\rho_S : \mathcal{U} \times \mathcal{W} \to \mathbf{R}$ and $\mu_S : \mathcal{Y} \times \mathcal{Z} \to \mathbf{R}$. Suppose also that system Δ is ρ_Δ/μ_Δ gain stable and satisfies (2.4) with $\gamma = 1$, for some $\rho_\Delta : \mathcal{Z} \to \mathbf{R}$ and $\mu_\Delta : \mathcal{W} \to \mathbf{R}$. The interconnected system (S, Δ) with input u and output y is ρ/μ gain stable for $\rho : \mathcal{U} \to \mathbf{R}$ and $\mu : \mathcal{Y} \to \mathbf{R}$ defined by:*

$$\rho(u) \doteq \sup_{w \in \mathcal{W}} \{\rho_S(u, w) - \mu_\Delta(w)\} \tag{2.5}$$

$$\mu(y) \doteq \inf_{z \in \mathcal{Z}} \{\mu_S(y, z) - \rho_\Delta(z)\} \tag{2.6}$$

and satisfies (2.4) with $\gamma = 1$.

Proof. By assumption, all feasible signals of S satisfy:

$$\inf_{T \geq 0} \sum_{t=0}^{T} \rho_S(u(t), w(t)) - \mu_S(y(t), z(t)) > -\infty \tag{2.7}$$

and those of Δ satisfy:

$$\inf_{T\geq 0}\sum_{t=0}^{T}\rho_\Delta(z(t))-\mu_\Delta(w(t))>-\infty \tag{2.8}$$

For functions ρ and μ defined in (2.5) and (2.6), (2.7) implies that all feasible signals of system S satisfy the following condition:

$$\inf_{T\geq 0}\sum_{t=0}^{T}\rho(u(t))+\mu_\Delta(w(t))-\mu(y(t))-\rho_\Delta(z(t))>-\infty \tag{2.9}$$

Adding (2.8) to (2.9), and noting that the infimum of the sum of two functions is larger than or equal to the sum of the infimums of the functions, we get:

$$\inf_{T\geq 0}\sum_{t=0}^{T}\rho(u(t))-\mu(y(t))>-\infty \tag{2.10}$$

Hence, the interconnected system (S,Δ) is ρ/μ gain stable and satisfies (2.4) with $\gamma=1$. □

Traditionally, the Small Gain Theorem [23] (see [64, 84, 85] for a historical perspective) gives sufficient conditions, expressed in terms of the gains of the component systems, for an interconnection to be stable given that the components are stable. In contrast, this new 'Small Gain' Theorem does not expressly depend on a notion of gain. Anytime two systems are gain stable, their interconnection is also gain stable in some sense. Whether or not this stability property is practically useful depends on the functions ρ and μ. Nevertheless, this result is referred to as a 'Small Gain' Theorem to emphasize its usage, which is analogous to that of the traditional Small Gain Theorem, and the fact that it reduces to the standard Small Gain Theorem under appropriate conditions, as described in the following remark.

Remark 2.4 An interesting special case is when all the alphabet sets are finite subsets of $\mathbf{R}$, and gain stability of systems S and Δ are interpreted as l_2 gain conditions in $\mathbf{R}$. In this case, we have: $\rho_S(u,w)=|u|^2+|w|^2$, $\mu_S(y,z)=|y|^2+|z|^2$, $\mu_\Delta(w)=|w|^2$, $\rho_\Delta(z)=|u|^2$, and consequently $\rho(u)=|u|^2$ and $\mu(y)=|y|^2$. This formulation thus reduces to the standard small gain result: if each of S and Δ are stable with l_2 gain not exceeding 1, then so is their interconnection. ◇

Corollary 2.2. *The interconnection (S,Δ) is ρ/μ gain stable with $\gamma=1$ for any $\rho:\mathcal{U}\to\mathbf{R}$,*

$\mu : \mathcal{Y} \to \mathbf{R}$ *defined by:*

$$\rho(u) \doteq \sup_{w \in \mathcal{W}} \{\rho_S(u, w) - \tau_d \ \mu_\Delta(w)\} \tag{2.11}$$

$$\mu(y) \doteq \inf_{z \in \mathcal{Z}} \{\mu_S(y, z) - \tau_d \ \rho_\Delta(z)\} \tag{2.12}$$

with $\tau_d > 0$.

Proof. If the feasible signals of system Δ satisfy (2.8), they also satisfy

$$\inf_{T \geq 0} \sum_{t=0}^{T} \tau_d \ \rho_\Delta(z(t)) - \tau_d \ \mu_\Delta(w(t)) > -\infty$$

for any positive scaling parameter $\tau_d > 0$. The statement thus follows by simply replacing μ_Δ and ρ_Δ by $\tau_d \mu_\Delta$ and $\tau_d \rho_\Delta$ respectively in (2.5) and (2.6). □

Thus, the 'Small Gain' Theorem (Theorem 2.1) is in fact a statement that under the stated assumptions, the interconnected system (S, Δ) satisfies an *infinite* number of gain stability properties corresponding to an infinite number of (positive) values of scale τ_d. This scaling factor is comparable to the "D-scales" in the classical robust control framework.

The 'Small Gain' Theorem can be used as an analysis tool, as will be illustrated in the last example in Section 2.5. In particular, we may be interested in proving stability of the interconnection (S, Δ) about $(\mathcal{U}_o, \mathcal{Y}_o)$, for some specific choice of $\mathcal{U}_o \subset \mathcal{U}$ and $\mathcal{Y}_o \subset \mathcal{Y}$. Theorem 2.1 allows us to verify this if there exists some scaling factor $\tau_d > 0$ such that ρ and μ defined in (2.5) and (2.6) satisfy the requirement that they're zero on $\mathcal{U}_o$ and $\mathcal{Y}_o$, respectively, and strictly positive elsewhere.

The Small Gain Theorem can also be used as a tool for posing a robust control synthesis problem. This will be discussed in detail in Chapter 5 and illustrated in Chapter 6.

2.5 Examples

The following simple examples practically illustrate each of the definitions proposed in Sections 2.3 and 2.4.

Example 2.1 The system over finite alphabets defined by its feasible signals:

$$\mathcal{D} = \{(\mathbf{u}, \mathbf{y}) \in \{-1, 0, 1\}^{\mathbf{Z}_+} \times \{-K, 0, K\}^{\mathbf{Z}_+} | y(t) = Ku(t), \forall t \in \mathbf{Z}_+\}$$

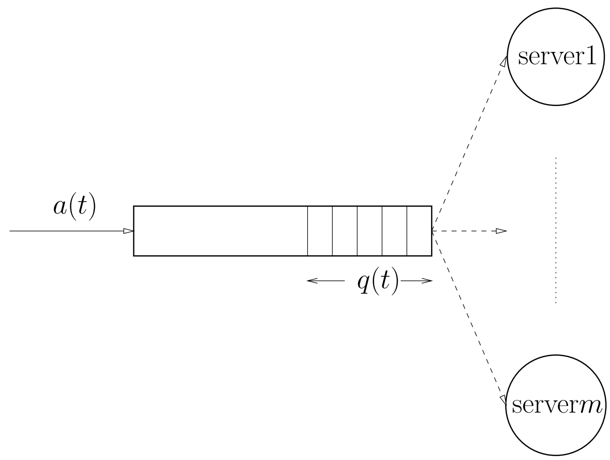

Figure 2-2: A queuing system

is a gain K whose input is restricted to three values: -1, 0 and 1, and whose output is consequently restricted to the three values: $-K$, 0, K. $\triangledown$

Example 2.2 Consider a queuing system consisting of a single buffer and m deterministic servers, of which only one can be used at any given time (Figure 2-2). The i^{th} server operates at a fixed rate π_i and incurs operating cost c_i per unit time, with $\pi_i < \pi_j$ and $c_i < c_j$ for $i < j$. Let $a(t)$ be the number of packets arriving at the buffer at time step t, where $t \in \mathbf{Z}_+$, and suppose that there is a physical limitation to how many packets can arrive at any time step (i.e. an upper bound on $a(t)$). The queuing system is said to be stable if the queue size remains finite at all times.

The system can be modeled as a system over finite alphabets with two inputs, the server rate $\pi(t) \in \mathcal{R} = \{\pi_1, \ldots, \pi_m\}$ (a control input) and the number of arrivals $a(t) \in \mathcal{A} = \{0, 1, \ldots, \beta\}$. The internal state of the system is q, the length of the queue, described by the following update equation:

$$q(t+1) = \max\{0, q(t) + a(t) - \pi(t)\}$$

The performance output is δq, where $\delta q(t) = q(t+1) - q(t)$. The performance objective (queue stability), described by the requirement that $q(T) < \infty$ for all $T \geq 0$, can be equivalently expressed as a gain stability condition:

$$\inf_{T \geq 0} \sum_{t=0}^{T} -\delta q(t) > -\infty \tag{2.13}$$

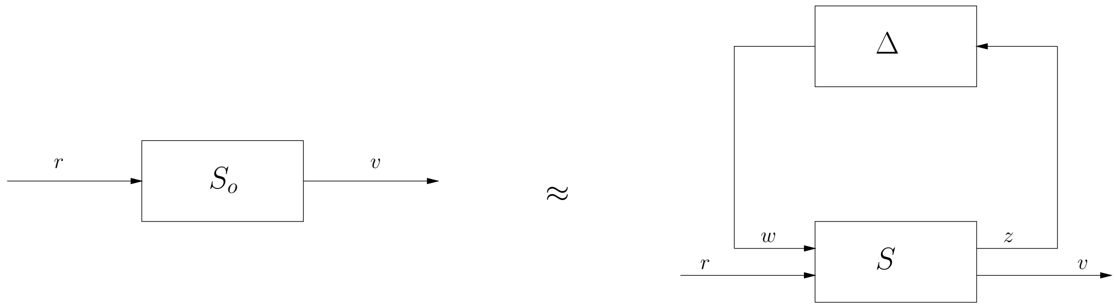

Figure 2-3: Feedback interconnection of a nominal model and a Δ block representing a more complex system

∇

A typical usage of the 'Small Gain' Theorem (Theorem 2.1) as an analysis tool is as follows: given a system S_o with complex or partially unknown dynamics, and a physical performance objective, mathematically described by a gain condition as in equation (2.10), that we wish to verify for the system. The complex system is first represented as the feedback interconnection of a simpler system S, essentially a nominal or lower order model of the original system (possibly a deterministic finite state machine model as described in Section 2.2.2), and a system Δ, representing the modeling uncertainty or the approximation error (Figure 2-3), and characterized in terms of some gain stability property as in equation (2.8). A gain bound for M is computed as in equation (2.7), and the 'Small Gain' Theorem is used to verify the performance of the closed loop system (S, Δ), and hence of the original system S_o. The following simple example illustrates this procedure.

Example 2.3 Consider the queuing system introduced in Example 2.2. A controller for this queuing system is a system that implements a control law which maps the length of the queue in the buffer to a choice of server to be used. Given a queuing system, a controller, and some limited knowledge about the arrival process, the goal is to verify that the resulting closed loop queuing system is stable. In particular, assume that the controller picks the same server with rate π_o, regardless of the length of the queue and that the arrival process obeys the Leaky Bucket model, namely:

$$A(s,t) \leq \alpha \cdot (t-s) + \beta$$

where $A(s,t)$ is the total number of (integer valued) arrivals in time interval $[s,t]$ and α

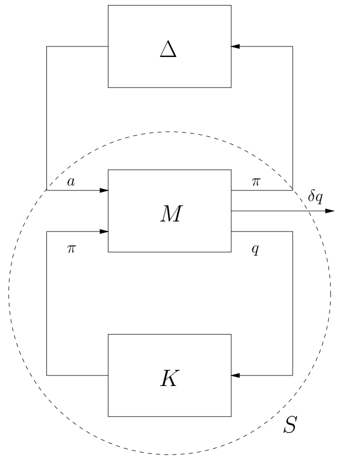

Figure 2-4: Setup for robust performance verification

and β are given positive constants. We wish to find conditions for π_o under which stability can be guaranteed. The queuing system can be modeled as a system over finite alphabets (system 'M' in Figure 2-4) with two inputs, the server rate π (control input) and the number of arrivals a (disturbance input). There is no exogenous input in this example. The internal state of M is q, the length of the queue, with the update equation described in Example 2.2. The system has output q (sensor output), π (input to the uncertainty block Δ) and δq (output used to characterize the performance objective, where $\delta q(t) = q(t+1) - q(t)$). The controller K is a system whose output is identically equal to π_o. The arrival process can be modeled by an uncertainty block Δ satisfying the gain condition:

$$\inf_{T \geq 0} \sum_{t=0}^{T} \Big(\frac{\alpha}{r^o} r(t) - a(t) \Big) > -\infty$$

The following gain condition is satisfied by S, the interconnection of M and K:

$$\inf_{T \geq 0} \sum_{t=0}^{T} \Big(- a(t) + (\delta q(t) + \pi(t)) \Big) > -\infty$$

In the terminology of Theorem 2.1, we have $\rho_S(u,a) = -a$, $\mu_S(\delta q, r) = \delta q + r$, $\rho_\Delta(r) = \frac{\alpha}{r^o} r$ and $\mu_\Delta(a) = a$. For $\rho(u) = \max_{a \in \mathcal{A}} \{-2a\} = 0$ and $\mu(\delta q) = \min_{r \in \{r^o\}} \{\delta q + r - \frac{\alpha}{r^o} r\} = \delta q + (r^o - \alpha)$, Theorem 2.1 allows us to write that:

$$\inf_{T \geq 0} \sum_{t=0}^{T} \Big(- \delta q(t) - (r^o - \alpha) \Big) > -\infty$$

which implies (2.13) if $r^o - \alpha \geq 0$. Thus, a sufficient condition for stability is $r^o \geq \alpha$, which is a well known result in queuing theory. ∇

Chapter 3

Approximating Hybrid Systems by Finite State Machines

3.1 Introduction

This chapter addresses the problem of approximating a system with finite actuation and sensing by a deterministic finite state machine for the purpose of control design.

Consider a discrete-time hybrid system P with exogenous and control inputs r and u respectively, and performance and measurement outputs v and y, respectively. Assume that alphabet sets $\mathcal{U}$ and $\mathcal{Y}$ are finite; thus P is a system with finite actuation and sensing. Assume further that the performance objective is ρ/μ gain stability for some given $\rho : \mathcal{R} \to \mathbf{R}$ and $\mu : \mathcal{V} \to \mathbf{R}$. In other words, a controller $K \subset \mathcal{Y}^{\mathbf{Z}_+} \times \mathcal{U}^{\mathbf{Z}_+}$ is sought such that the closed loop system, interconnection (P, K), is ρ/μ gain stable.

The first step of the hybrid design paradigm advocated in this thesis consists of approximating P by a deterministic finite state machine M and describing the approximation uncertainty Δ in terms of an appropriate gain stability property. The questions that arise are:

1. What does it mean for M to be an approximation of P for the purpose of controller synthesis?

2. What does the approximation uncertainty Δ represent and how to characterize it in terms of gain stability?

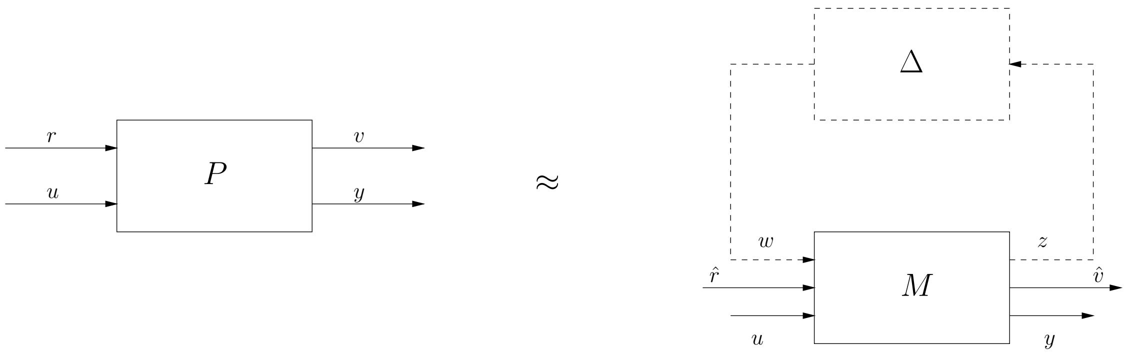

Figure 3-1: A DFM approximation of P

Answers to these questions are presented in this chapter. A notion of approximation that is compatible with control design is proposed for the class of systems of interest in Section 3.2. In the following three sections, an approximation approach, consistent with the notion of approximation defined in Section 3.2, is developed for systems with no exogenous inputs. In particular, a second notion of input/output stability, referred to as external stability, is defined in Section 3.3 and shown to be relevant to the problem of approximation. An observer structure is then proposed for the approximation uncertainty, which allows for approximation of systems that are not externally stable. Two constructive algorithms for generating finite state machine approximations are presented in Section 3.4, and corresponding algorithms for computing a-posteriori upper bounds on the resulting approximation error are presented in Section 3.5.

3.2 Approximation for Control Design

A definition of approximation for systems with finite actuation and sensing by deterministic finite state machines is proposed in this section. The definition takes into account the desired performance objective and is thus compatible with control design.

Definition 3.1. *Consider a plant* $P \subset (\mathcal{U} \times \mathcal{R})^{\mathbf{Z}_+} \times (\mathcal{Y} \times \mathcal{V})^{\mathbf{Z}_+}$, *where* $\mathcal{U}$ *and* $\mathcal{Y}$ *are finite sets. Assume that the performance objective is* ρ/μ *gain stability with* $\gamma = 1$ *for some given* $\rho : \mathcal{R} \to \mathbf{R}$ *and* $\mu : \mathcal{V} \to \mathbf{R}$. *A deterministic finite state machine:*

$$M \subset (\mathcal{U} \times \hat{\mathcal{R}} \times \mathcal{W})^{\mathbf{Z}_+} \times (\mathcal{Y} \times \hat{\mathcal{V}} \times \mathcal{Z})^{\mathbf{Z}_+}$$

is a ***ρ/μ approximation*** *of* P *if there exists a* $\Delta \subset \mathcal{Z}^{\mathbf{Z}^+} \times \mathcal{W}^{\mathbf{Z}_+}$ *and functions* $\rho_o : \hat{\mathcal{R}} \to \mathbf{R}$

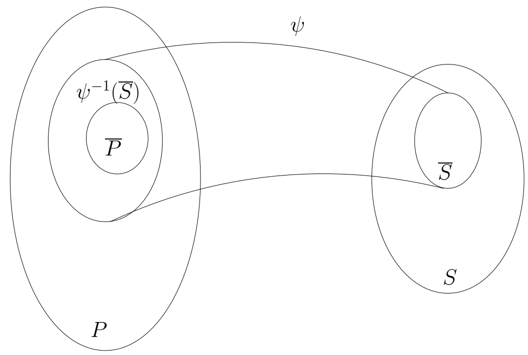

Figure 3-2: Approximation for control synthesis: finite alphabet considerations

and $\mu_o : \hat{\mathcal{V}} \to \mathbf{R}$ *such that:*

1. *There exists a surjective map* $\psi : P \to S$*, where* S *is the feedback interconnection of* M *and* Δ*, as shown in Figure 3-1, satisfying:*

   (a) $\psi^{-1}\Big(S_{\mathbf{u},\mathbf{y}}\Big) \supseteq P_{\mathbf{u},\mathbf{y}}$*, where:*

$$S_{\mathbf{u},\mathbf{y}} = \{((\mathbf{u},\hat{\mathbf{r}}),(\mathbf{y},\hat{\mathbf{v}})) \in S\}$$
$$P_{\mathbf{u},\mathbf{y}} = \{((\mathbf{u},\mathbf{r}),(\mathbf{y},\mathbf{v})) \in P\}$$

   (b) *If* $((\mathbf{u},\hat{\mathbf{r}}),(\mathbf{y},\hat{\mathbf{v}})) \in S$ *satisfies:*

$$\inf_{T\geq 0} \sum_{t=0}^{T} \rho_o(\hat{r}(t)) - \mu_o(\hat{v}(t)) > -\infty \tag{3.1}$$

   then every $((\mathbf{u},\mathbf{r}),(\mathbf{y},\mathbf{v})) \in \psi^{-1}\Big(\{((\mathbf{u},\hat{\mathbf{r}}),(\mathbf{y},\hat{\mathbf{v}}))\}\Big)$ *satisfies:*

$$\inf_{T\geq 0} \sum_{t=0}^{T} \rho(r(t)) - \mu(v(t)) > -\infty \tag{3.2}$$

2. Δ *is* ρ_Δ/μ_Δ *gain stable for some non-zero functions* $\rho_\Delta : \mathcal{Z}^{\mathbf{Z}_+} \to \mathbf{R}_+$ *and* $\mu_\Delta : \mathcal{W}^{\mathbf{Z}_+} \to \mathbf{R}_+$*.*

Ultimately, the goal of the approximation is to enable systematic synthesis of a controller $K \subset \mathcal{Y}^{\mathbf{Z}_+} \times \mathcal{U}^{\mathbf{Z}_+}$ such that the closed loop system (P, K) satisfies the performance objective in (3.2). This controller design problem can be thought of as finding a set $K \subset \mathcal{Y}^{\mathbf{Z}_+} \times \mathcal{U}^{\mathbf{Z}_+}$

such that all the elements of $\overline{P}$, defined as:

$$\overline{P} = \{((\mathbf{u}, \mathbf{r}), (\mathbf{y}, \mathbf{v})) \in P | (\mathbf{y}, \mathbf{u}) \in K\}$$

satisfy (3.2).

The definition of system approximation by deterministic finite state machines proposed here is compatible with the goal of robust controller synthesis: let M be a ρ/μ approximation of P as in Definition 3.1. If a K is found such that every element of $\overline{S}$, defined as:

$$\overline{S} = \{((\mathbf{u}, \hat{\mathbf{r}}), (\mathbf{y}, \hat{\mathbf{v}})) \in S | (\mathbf{y}, \mathbf{u}) \in K\}$$

satisfies the auxiliary performance objective (3.1), then it follows from condition 1(b) in the definition that every element of $\psi^{-1}(\overline{S})$ satisfies performance objective (3.2). Thus, in order to ensure that the closed loop system (P, K) also satisfies this performance objective, it is sufficient to ensure that $\overline{P} \subseteq \psi^{-1}(\overline{S})$, which is guaranteed by condition 1(a) in the definition (see Figure 3-2 for a pictorial illustration of this argument).

Finding a controller K such that all the elements of $\overline{S}$ satisfy (3.1) is a difficult problem in general. A simpler problem can be posed by characterizing the approximation error Δ in terms of an appropriate gain stability property (ρ_Δ/μ_Δ gain stability with gain bound γ), and then designing a controller K for the nominal DFM model M that is robust to all admissible uncertainties; in other words, K is designed such that the interconnection of M, Δ and K satisfies the auxiliary performance objective (3.1), for any Δ in the class $\underline{\Delta}$:

$$\underline{\Delta} = \{\Delta \subset \mathcal{Z}^{\mathbf{Z}_+} \times \mathcal{W}^{\mathbf{Z}_+} | \inf_{T \geq 0} \sum_{t=0}^{T} \gamma \rho_\Delta(z(t)) - \mu_\Delta(w(t)) > -\infty \text{ holds } \forall (z, w) \in \Delta\}$$

In particular, given a choice of functions ρ_Δ and μ_Δ, let $\underline{\Delta}_i$ be the uncertainty class associated with gain bound γ_i on Δ, and let $S_{\underline{\Delta}_i}$ be the family of systems consisting of the interconnection of M and every $\Delta \in \underline{\Delta}_i$. That is, $S_{\underline{\Delta}_i}$ is the set of all $((\mathbf{u}, \hat{\mathbf{r}}), (\mathbf{y}, \hat{\mathbf{v}})) \in (\mathcal{U} \times \hat{\mathcal{R}})^{\mathbf{Z}_+} \times (\mathcal{Y} \times \hat{\mathcal{V}})^{\mathbf{Z}_+}$ for which there exists $\mathbf{w}$ and $\mathbf{z}$ satisfying:

$$\inf_{T \geq 0} \sum_{t=0}^{T} \gamma_i \rho_\Delta(z(t)) - \mu_\Delta(w(t)) > -\infty$$

and such that $((\mathbf{u}, \hat{\mathbf{r}}, \mathbf{w}), (\mathbf{y}, \hat{\mathbf{v}}, \mathbf{z})) \in M$. Since by definition $S \subseteq S_{\underline{\Delta}_i}$, if a controller K is

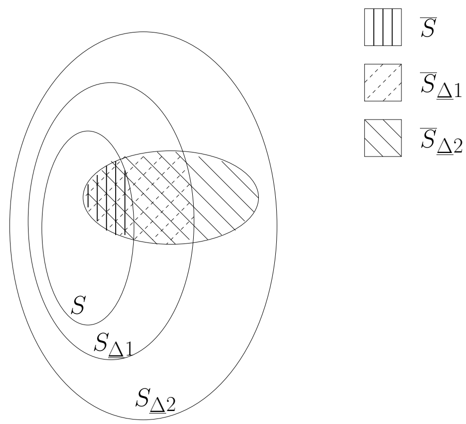

Figure 3-3: Approximation for control synthesis: description of Δ

found such that every element of $\overline{S}_{\underline{\Delta}_i}$, defined as:

$$\overline{S}_{\underline{\Delta}_i} = \{((\mathbf{u}, \hat{\mathbf{r}}), (\mathbf{y}, \hat{\mathbf{v}})) \in S_{\underline{\Delta}_i} | (\mathbf{y}, \mathbf{u}) \in K\}$$

satisfies the auxiliary performance objective (3.1), then so does every element of $\overline{S}$.

The numerical value of the gain γ, while not directly indicative of the quality of approximation[1], should reflect the trend that quality of approximation increases as γ decreases. The requirement in condition 2 of Definition 3.1 that ρ_Δ and μ_Δ are non-negative and are not identically zero allows for a well-defined notion of gain. For two gain bounds $\gamma_1 < \gamma_2$, it follows that (Figure 3-3):

$$S \subseteq S_{\underline{\Delta}_1} \subseteq S_{\underline{\Delta}_2}$$

and hence for a given controller $K \subset \mathcal{Y}^{\mathbf{Z}_+} \times \mathcal{U}^{\mathbf{Z}_+}$, we have:

$$\overline{S} \subseteq \overline{S}_{\underline{\Delta}_1} \subseteq \overline{S}_{\underline{\Delta}_2}$$

Discussion of the robust controller synthesis problem is deferred until Chapter 5; however, it is intuitively clear that the difficulty of finding a controller K to meet the auxiliary performance objective (3.1) increases as the set of feasible signals of the system considered gets larger, and hence as gain bound γ gets larger.

Effectively, when a ρ/μ approximation of P is found, there exists a 1-1 correspondence between the sets $P_{\mathbf{u},\mathbf{y}}$ and $S_{\mathbf{u},\mathbf{y}}$, as shown by the following Proposition.

[1] As is the case in the classical robust control framework, whether a given value of γ is small enough for controller synthesis is not known until an attempt is made to synthesize a controller.

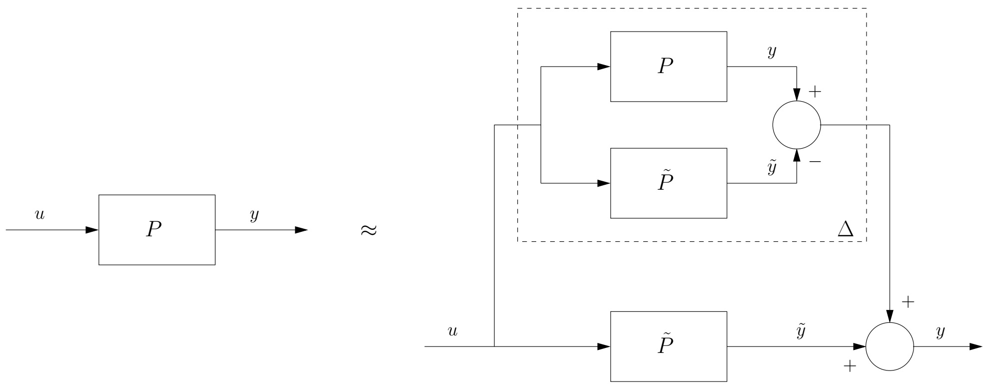

Figure 3-4: Traditional LTI model reduction

Proposition 3.1. *If there exists a map $\psi : P \to S$ satisfying condition 1(a) in Definition 3.1, then* $P_{\mathbf{u},\mathbf{y}} = \psi^{-1}\Big(S_{\mathbf{u},\mathbf{y}}\Big)$.

Proof. By contradiction. Suppose that $P_{\mathbf{u},\mathbf{y}} \not\supseteq \psi^{-1}\Big(S_{\mathbf{u},\mathbf{y}}\Big)$; thus, there exists an $x \in \psi^{-1}\Big(S_{\mathbf{u},\mathbf{y}}\Big) \setminus P_{u,y}$. $x \in P_{\mathbf{u}_1,\mathbf{y}_1}$ for some $(\mathbf{u}_1, \mathbf{y}_1) \neq (\mathbf{u}, \mathbf{y})$, and hence by assumption $x \in \psi^{-1}\Big(S_{\mathbf{u}_1,\mathbf{y}_1}\Big)$, and $\psi^{-1}\Big(S_{\mathbf{u},\mathbf{y}}\Big) \cap \psi^{-1}\Big(S_{\mathbf{u}_1,\mathbf{y}_1}\Big) \neq \emptyset$. This leads to a contradiction since $S_{\mathbf{u},\mathbf{y}} \cap S_{\mathbf{u}_1,\mathbf{y}_1} = \emptyset$. □

3.3 The Approximation Uncertainty

An approximation approach that is consistent with Definition 3.1 is developed in this section and in Sections 3.4 and 3.5. In particular, the discussion focuses on systems with no exogenous input r (or equivalently, the case where the corresponding alphabet set $\mathcal{R}$ is a singleton). In Section 3.3.3, an observer structure is proposed for the approximation error Δ. The motivation for this structure is first explained in Section 3.3.2, following a brief review of traditional model order reduction in Section 3.3.1 and a discussion of an extension of the classical paradigm to the class of systems of interest.

3.3.1 A Traditional View of Approximation Error

Traditional model order reduction techniques such as balanced truncation or Hankel model order reduction [1, 31, 57, 61] deal with stable LTI systems. Given a stable LTI plant P of order n, one possible goal is to find a stable LTI plant $\tilde{P}$ of order $m < n$ such that the $\mathcal{H}_\infty$ norm of the transfer function of the difference system $P - \tilde{P}$ is minimized. Intuitively, $\tilde{P}$ is considered to be a good approximation of P if the outputs of the two systems, when driven side by side using the same input, are not too different in the worst case scenario. Thus, in the traditional setting, the error system Δ is simply the difference between the original and the approximate systems (Figure 3-4). This formulation of the model order reduction problem implicitly assumes that the performance objective is expressed in terms of an $\mathcal{L}_2$ gain condition for the system with input u and output y.

In the traditional setup just described, the original system P and its approximation $\tilde{P}$ are both assumed to be initialized to zero. Moreover, they have the same alphabet sets, and hence the feasible signals of P and that of the interconnection $S = (\tilde{P}, \Delta)$ are identical. Thus, there exists a bijective map $\psi : P \to S$, namely the identity map, with the trivial property that a feasible signal $(\mathbf{u}, \mathbf{y}) \in S$ satisfies a given $\mathcal{L}_2$ gain condition iff $\psi^{-1}(\mathbf{u}, \mathbf{y}) = (\mathbf{u}, \mathbf{y})$ satisfies it as well (in the terminology of Section 3.2, the 'performance objective' and the 'auxiliary performance objective' are identical). The approximation uncertainty Δ is described in terms of an $\mathcal{L}_2$ gain bound. Given two lower order approximations $\tilde{P}_1$ and $\tilde{P}_2$ of P, $\tilde{P}_1$ is a better approximation than $\tilde{P}_2$ if the $\mathcal{L}_2$ gain of $\Delta_1 = P - \tilde{P}_1$ is smaller than that of $\Delta_2 = P - \tilde{P}_2$.

It is tempting to attempt to extend this setup to the class of systems and nominal models of interest. While addition and subtraction of signals are not well defined operations due to the absence of algebraic structure, a meaningful extension nonetheless exists. Given a system $P \subset \mathcal{U}^{\mathbf{Z}_+} \times (\mathcal{Y} \times \mathcal{V})^{\mathbf{Z}_+}$ and a deterministic finite state machine $\hat{M} \subset \mathcal{U}^{\mathbf{Z}_+} \times (\mathcal{Y} \times \hat{\mathcal{V}})^{\mathbf{Z}_+}$, where $\mathcal{U}$ and $\mathcal{Y}$ are finite sets with $card(\mathcal{Y}) = b$. Consider their interconnection as shown in Figure 3-5, with $w(t) \in \mathcal{W}$, where $\mathcal{W}$ is an arbitrary finite alphabet with cardinality:

$$card(\mathcal{W}) = \begin{pmatrix} b \\ 2 \end{pmatrix} + 1 = \frac{b^2 - b + 2}{2}$$

Function $\beta : \mathcal{Y} \times \mathcal{Y} \to \mathcal{W}$ is a symmetric, surjective function satisfying:

$$\beta(y, y) = w_o$$

Given a choice of β, function $\phi : \mathcal{Y} \times \mathcal{W} \to \mathcal{Y}$ is defined by:

$$\phi(y, w) \doteq y_1 \text{ such that } \beta(y, y_1) = w$$

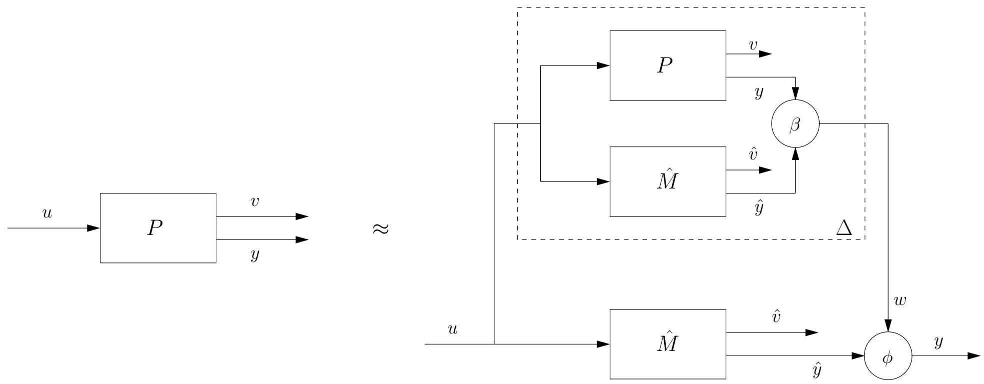

Figure 3-5: An extension of the traditional model reduction setup

In this setup, a possible characterization of Δ in terms of gain stability corresponds to a choice of functions $\rho_\Delta : \mathcal{U} \to [1, a]$ and $\mu_\Delta : \mathcal{W} \to [0, b]$, for some $a \geq 1$ and $b > 0$, with $\mu_\Delta(w) = 0$ iff $w = w_o$. In particular, when $\rho_\Delta(\mathcal{U}) = \mu_\Delta(\mathcal{W} \setminus \{w_o\}) = \{1\}$, the ρ_Δ/μ_Δ gain of Δ, γ, is such that $\gamma \in [0, 1]$. γ then represents the fraction of mismatched outputs in the worst case scenario. In general, different weights can be associated with different inputs and mismatch pairs to reflect some particular knowledge about the specific problem at hand (for instance, some mismatches may be more detrimental than others, or certain inputs need to be more strongly penalized).

Remark 3.1 In the case where $\mathcal{Y}$ is a binary alphabet, β can be interpreted as a logical XOR operation: $\mathcal{W} = \{0, 1\}$, $w_o = 0$, an output of 0 (1) indicates matched (mismatched) outputs of P and $\hat{M}$. In this case, function ϕ simply flips $\hat{y}$ whenever $w = 1$. ◇

While this setup is plausible, it will be shown in the next section that it may be too restrictive in general.

3.3.2 External Stability and the Quality of Approximation

A notion of input/output stability that is relevant to the approximation problem is presented here. Intuitively, a system is externally stable if it appears to forget its past. The following definition of external *instability* makes this notion rigorous.

Definition 3.2. *A system $S \subset \mathcal{U}^{\mathbf{Z}_+} \times \mathcal{Y}^{\mathbf{Z}_+}$ is **externally unstable** if there exists a finite constant $\tau \geq 0$ and two elements $(\mathbf{u}, \mathbf{y}_1)$ and $(\mathbf{u}, \mathbf{y}_2)$ of S such that $y_1(t') \neq y_2(t')$ for some $t' \in [t, t + \tau]$, for every $t \geq 0$.*

Remark 3.2 In particular, when the output set is $\mathcal{Y} = \mathcal{Y}_1 \times \ldots \times \mathcal{Y}_r$, a more precise notion of external stability can be defined. $S \subset \mathcal{U}^{\mathbf{Z}_+} \times \mathcal{Y}^{\mathbf{Z}_+}$ is said to be *externally unstable with respect to* $\mathcal{Y}_p$, $p \in \{1, \ldots, r\}$, if there exists a $\tau \geq 0$ and two elements $(u, (y_1^o, \ldots, y_p^o, \ldots, y_r^o))$ and $(u, (y_1^*, \ldots, y_p^*, \ldots, y_r^*))$ of S such that $y_p^o(t') \neq y_p^*(t')$ for some $t' \in [t, t+\tau]$, for every $t \geq 0$. $\diamond$

The lack of external stability in a system has an important consequence: in that case, it may not be possible to approximate the system arbitrarily well (i.e. with arbitrarily small γ) in the sense described in Section 3.3.1.

Theorem 3.2. *Consider a system* $P \subset \mathcal{U}^{\mathbf{Z}_+} \times (\mathcal{Y} \times \mathcal{V})^{\mathbf{Z}_+}$, *where* $\mathcal{U}$ *and* $\mathcal{Y}$ *are finite sets with* $card(\mathcal{Y}) = b$, *and an alphabet set* $\mathcal{W}$ *with cardinality* $\frac{b^2-b+2}{2}$. *Given functions* $\rho_\Delta : \mathcal{U} \rightarrow [1, a]$ *and* $\mu_\Delta : \mathcal{W} \rightarrow [0, b]$, $a \geq 1$ *and* $b > 0$, *with* $\mu_\Delta(w) = 0$ *iff* $w = w_o$ *for some* w_o, *and given function* $\beta : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathcal{W}$ *satisfying* $\beta(y, y) = w_o$. *If* P *is externally unstable with respect to* $\mathcal{Y}$, *there exists no deterministic finite state machine* $\hat{M} \subset \mathcal{U}^{\mathbf{Z}_+} \times (\mathcal{Y} \times \hat{\mathcal{V}})^{\mathbf{Z}_+}$ *such that the* ρ_Δ/μ_Δ *gain of* Δ *(with the structure shown in Figure 3-5) is arbitrarily small.*

Proof. The proof is by contradiction. For given functions ρ_Δ and μ_Δ, suppose that for every $\gamma_\epsilon \geq 0$, there exists a deterministic finite state machine $\hat{M}_\epsilon$ such that γ_ϵ is a valid ρ_Δ/μ_Δ gain bound for the resulting system Δ. Since P is externally unstable with respect to $\mathcal{Y}$, there exists a $\tau \geq 0$, $(\mathbf{u}, (\mathbf{y}_1, \mathbf{v}_1))$ and $(\mathbf{u}, (\mathbf{y}_2, \mathbf{v}_2))$ of P such that $y_1(t') \neq y_2(t')$ for some $t' \in [t, t+\tau]$, for every $t \geq 0$. Let τ_o be the smallest such τ, with the outputs of Δ corresponding to $(\mathbf{u}, (\mathbf{y}_1, \mathbf{v}_1))$ and $(\mathbf{u}, (\mathbf{y}_2, \mathbf{v}_2))$ and some fixed initial condition of $\hat{M}_\epsilon$ being $\mathbf{w}_1$ and $\mathbf{w}_2$, respectively. By assumption, we have:

$$\inf_{T \geq 0} \sum_{t=0}^{T} \gamma_\epsilon \rho_\Delta(u(t)) - \mu_\Delta(w_1(t)) > -\infty$$

and

$$\inf_{T \geq 0} \sum_{t=0}^{T} \gamma_\epsilon \rho_\Delta(u(t)) - \mu_\Delta(w_2(t)) > -\infty$$

We have:

$$
\begin{aligned}
& \sum_{t=0}^{T} 2\gamma_\epsilon \rho_\Delta(u(t)) - \mu_\Delta(w_1(t)) - \mu_\Delta(w_2(t)) \\
= \; & \sum_{t=0}^{T} \gamma_\epsilon \rho_\Delta(u(t)) - \mu_\Delta(w_1(t)) + \sum_{t=0}^{T} \gamma_\epsilon \rho_\Delta(u(t)) - \mu_\Delta(w_2(t)), \; \forall T \geq 0 \\
\geq \; & \inf_{T \geq 0} \sum_{t=0}^{T} \gamma_\epsilon \rho_\Delta(u(t)) - \mu_\Delta(w_1(t)) + \inf_{T \geq 0} \sum_{t=0}^{T} \gamma_\epsilon \rho_\Delta(u(t)) - \mu_\Delta(w_2(t)), \; \forall T \geq 0
\end{aligned}
$$

Hence, it follows that:

$$
\inf_{T \geq 0} \sum_{t=0}^{T} 2\gamma_\epsilon \rho_\Delta(u(t)) - \mu_\Delta(w_1(t)) - \mu_\Delta(w_2(t)) > -\infty
$$

Let $c_1 = \min_{w \in \mathcal{W} \setminus \{w_o\}} \mu_\Delta(w)$ and let $c_2 = \max_{u \in \mathcal{U}} \rho_\Delta(u)$. We have: $2\gamma_\epsilon \rho_\Delta(u(t)) \leq 2\gamma_\epsilon c_2$. Since $w_1(t_o) \neq w_2(t_o)$ for some $t_o \in [t, t+\tau_o]$, for any $t \geq 0$ (the output of $\hat{M}$ for a given initial condition and input $\mathbf{u}$ is fixed), we also have for any $k \geq 0$:

$$
\sum_{t=k}^{k+\tau_o} \mu_\Delta(w_1(t)) + \mu_\Delta(w_2(t)) \geq c_1
$$

Thus:

$$
\sum_{t=k}^{k+\tau_o} 2\gamma_\epsilon \rho_\Delta(u(t)) - \mu_\Delta(w_1(t)) - \mu_\Delta(w_2(t)) \leq 2\gamma_\epsilon c_2(\tau_o + 1) - c_1
$$

holds for all $k \geq 0$, and hence when $\gamma_\epsilon < \dfrac{c_1}{2c_2(\tau_o + 1)}$:

$$
\inf_{T \geq 0} \sum_{t=0}^{T} 2\gamma_\epsilon \rho_\Delta(u(t)) - \mu_\Delta(w_1(t)) - \mu_\Delta(w_2(t)) = -\infty
$$

thus leading to a contradiction. □

Remark 3.3 The proof of Theorem 3.2 assumed that the initial state of $\hat{M}$ is fixed, which is a reasonable assumption. In practice, either $\hat{M}$ forgets its initial condition, in which case there is no loss of generality in assuming it to be fixed, or it doesn't, in which case part of the question in the approximation problem is properly initializing the nominal model. The choice of initial condition is discussed in Section 3.4 in conjunction with the two algorithms

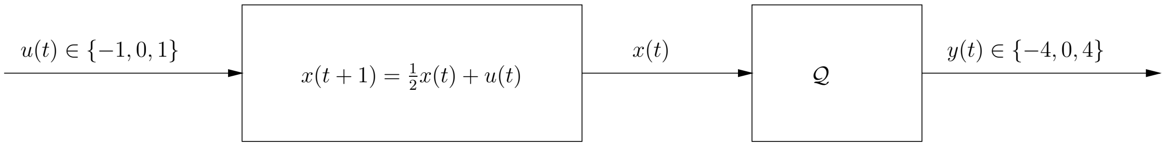

Figure 3-6: Internally stable LTI system with state quantizer

presented for constructing $\hat{M}$. ◇

What is interesting is that it is possible to start out with a discrete-time LTI system that is stable (poles within the unit disk), and where the corresponding system with discrete actuation and sensing, obtained by simply restricting the inputs of the system to a finite subset of **R** and finitely quantizing the output, is not externally stable. This is illustrated in the following simple example.

Example 3.1 Consider a stable LTI system described by:

$$x(t+1) = \frac{1}{2}x(t) + u(t)$$

and an output quantizer Q described by:

$$y(x) = \begin{cases} \vdots & \\ -4 & -6 \leq x < -2 \\ 0 & -2 \leq x < 2 \\ 4 & 2 \leq x < 6 \\ \vdots & \end{cases}$$

interconnected as shown in Figure 3-6. The input to the system is assumed to be restricted to take on the values 0, ± 1 and the initial state of the LTI system is assumed to lie in the interval $[-6, 6)$. Consequently, the output of Q takes on the values 0, ± 4. Even though the LTI system is stable (pole inside the unit disk), the system with input u and output y is not externally stable: consider the constant input $u(t) = 1$ and the two initial conditions $x_1(0) = 0$ and $x_2(0) = 4$. The corresponding constant outputs, $y_1(t) = 0$ and $y_2(t) = 4$, are unequal at every time step. ▽

In practice, the approximation setup described in Section 3.3.1 may be good enough to approximate an externally unstable system sufficiently well for successful controller design, particularly when τ_o (the smallest τ as in Definition 3.1) is large, and hence the lower bound on achievable γ given in Theorem 3.2 is small. However, in order to avoid restrictions on the class of systems that can be handled well, an alternative structure is proposed for the approximation error Δ in Section 3.3.3.

3.3.3 An Observer Based Structure

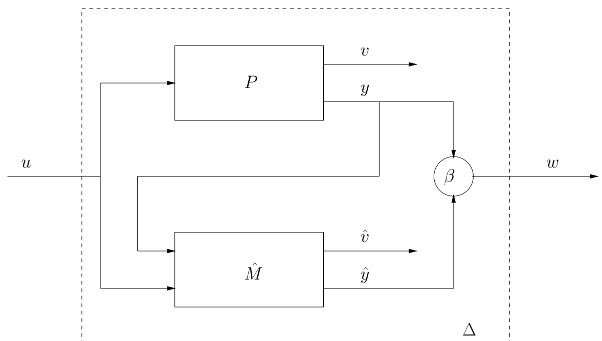

Figure 3-7: Structure of the error system Δ

The modified structure proposed for the approximation error Δ is shown in Figure 3-7. $\hat{M}$ in this setup is a deterministic finite state machine with the restriction that *direct feed-through from input y to output $\hat{y}$ is not allowed*; that is, output $\hat{y}(t)$ is only a function of state $q(t)$ and input $u(t)$. This structure, in which the output of P is fed back to $\hat{M}$, is essentially an observer structure. Intuitively, if system P does not forget its past (or its initial state, when P has a state-space description), there is a need to explicitly estimate its past (or initial state) in order to have some hope of correctly predicting its future.

This particular structure for Δ in turn imposes a particular structure on the deterministic finite state machine approximation of P. This is shown in Figure 3-8, where the output of the block ϕ, which is identical to that of the plant P by construction, is fed back to $\hat{M}$.

Given a plant P, let $\hat{M}$ be a deterministic finite state machine and let S be the interconnection (Figure 3-9) of M and Δ with the structures described above and shown in Figures 3-7 and 3-8. For any choice of $\hat{M}$ with fixed initial condition, there always exists a function $\psi : P \to S$ satisfying condition 1(a) of Definition 3.1, as shown in Proposition 3.3.

Proposition 3.3. *Consider a system $P \subset \mathcal{U}^{\mathbf{Z}_+} \times (\mathcal{Y} \times \mathcal{Z})^{\mathbf{Z}_+}$ where $\mathcal{U}$ and $\mathcal{Y}$ are finite. For any deterministic finite state machine $\hat{M}$ with fixed initial condition $q(0)$, there exists*

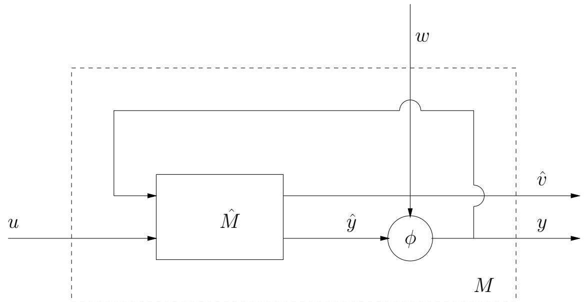

Figure 3-8: Structure of M, the finite state machine approximation of P

a $\psi : P \to S$*, where* S *is the interconnection of* $\hat{M}$ *and* P *shown in Figure 3-9, such that* ψ *is surjective and* $\psi^{-1}(S_{\mathbf{u},\mathbf{y}}) \supseteq P_{\mathbf{u},\mathbf{y}}$*.*

Proof. Consider $\psi_1 : P \to \hat{M}$ where $\psi_1\Big((\mathbf{u}, (\mathbf{y}, \mathbf{v})\Big) = ((\mathbf{u}, \mathbf{y}), (\hat{\mathbf{y}}, \hat{\mathbf{v}})) \in \hat{M}$, where $(\hat{\mathbf{y}}, \hat{\mathbf{v}})$ is the unique output response of $\hat{M}$ to input $(\mathbf{u}, \mathbf{y})$ for initial condition $q(0)$. Also consider $\psi_2 : \psi_1(P) \to S$ defined by:

$$\psi_2\Big(((\mathbf{u}, \mathbf{y}), (\hat{\mathbf{y}}, \hat{\mathbf{v}}))\Big) = (\mathbf{u}, (\mathbf{y}, \hat{\mathbf{v}}))$$

Let $\psi = \psi_2 \circ \psi_1$. ψ is surjective, since ψ_2 is surjective and $\psi_2^{-1}(S) = \psi_1(P)$ by definition. Moreover, $\psi(P_{\mathbf{u},\mathbf{y}}) \subseteq S_{\mathbf{u},\mathbf{y}}$:

$$\psi(P_{\mathbf{u},\mathbf{y}}) = \psi_2(\psi_1(P_{\mathbf{u},\mathbf{y}})) \subseteq \psi_2(\psi_1(P)_{\mathbf{u},\mathbf{y}}) \subseteq S_{\mathbf{u},\mathbf{y}}$$

where $\psi_1(P)_{\mathbf{u},\mathbf{y}} = \{((\mathbf{u}, \mathbf{y}), (\hat{\mathbf{y}}, \hat{\mathbf{v}})) \in \psi_1(P)\}$. Hence, $P_{\mathbf{u},\mathbf{y}} \subseteq \psi^{-1}(S_{\mathbf{u},\mathbf{y}})$. □

Moreover, for any choice of $\rho_\Delta : \mathcal{U} \to [1, a]$ and $\mu_\Delta : \mathcal{W} \to [0, b]$ for some $a \geq 1$ and $b > 0$, with $\mu_\Delta(w) = 0$ iff $w = w_o$, the ρ_Δ/μ_Δ gain of Δ is finite. Thus, a remaining question is: how to construct and initialize $\hat{M}$ such that the resulting deterministic finite state machine M is a ρ/μ approximation of P (i.e. so as to satisfy condition 1(b) of Definition 3.1)? The next question becomes: how to compute a gain bound for Δ? The first question is addressed in Section 3.4 and the second one in Section 3.5.

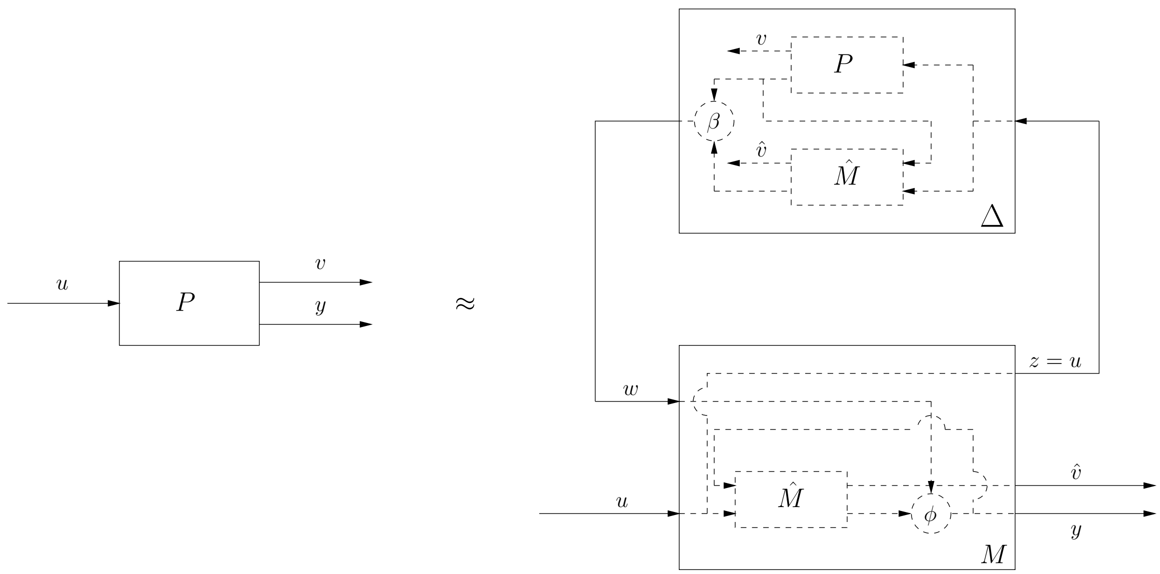

Figure 3-9: Interconnection of M and Δ

3.4 Constructing the Nominal Models

In this section, two constructive approaches for finding a ρ/μ approximation of a given system $P \subset \mathcal{U}^{\mathbf{Z}_+} \times (\mathcal{Y} \times \mathcal{V})^{\mathbf{Z}_+}$ are presented. Each approach provides a systematic procedure for constructing $\hat{M}$ (and consequently M), and for choosing function $\mu_o : \hat{\mathcal{V}} \to \mathbf{R}$ defining the auxiliary performance objective. In the first approach, presented in Section 3.4.1, the states of $\hat{M}$ are associated with finite length strings of control input and measurement output of the original system. The starting point of this approach resembles existing work, such as in [63]; however, the approach presented here is the first[2] to offer a usable quantitative measure of the accuracy of the approximation. The second approach, presented in Section 3.4.2, relies on state quantization, and hence assumes, among other things, that a state-space description of the system is available. The states of $\hat{M}$ in this approach correspond to a covering of the state space. Problems involving state quantization have also been extensively studied (see for instance [24, 47]). The approach presented here once again differs by offering a way of quantifying the approximation error in a usable manner within the framework developed in this thesis.

[2]to the author's knowledge. A preliminary version of this work was presented in [74].

3.4.1 Quantization of the Feasible Signals

Given a system $P \subset \mathcal{U}^{\mathbf{Z}_+} \times (\mathcal{Y} \times \mathcal{V})^{\mathbf{Z}_+}$ with finite alphabet sets $\mathcal{U}$ and $\mathcal{Y}$, and a desired performance objective:

$$\inf_{T \geq 0} \sum_{t=0}^{T} -\mu(v(t)) > -\infty$$

for some given function $\mu : \mathcal{V} \rightarrow \mathbf{R}$.

Deterministic finite state machine $\hat{M}$ is constructed as follows:

(1) The **state set** $\mathcal{Q}$ is defined as:

$$\mathcal{Q} \doteq \{q \in \mathcal{U}^{\mathbf{Z}_m} \times \mathcal{Y}^{\mathbf{Z}_{m+1}} | \exists t \geq m+1, (\mathbf{u}, (\mathbf{y}, \mathbf{v})) \in P \text{ s.t. } q = \mathcal{P}^+_{t-m-1}(\mathcal{P}^-_{t-1}(\mathbf{u})) \times \mathcal{P}^+_{t-m-1}(\mathcal{P}^-_{t}(\mathbf{y}))\}$$

for some positive integer m, where $\mathcal{P}^-_T$ and $\mathcal{P}^+_T$ denote truncation operators: For $\mathbf{a} \in \mathcal{A}^{\mathbf{Z}_k}$ and for $T < k$, $\mathcal{P}^+_T(\mathbf{a})$ denotes the subsequence $\{a(t)\}_{t=T+1}^{k}$, while for $T \leq k$, $\mathcal{P}^-_T(\mathbf{a})$ denotes the subsequence $\{a(t)\}_{t=0}^{T}$.

(2) For each $q = (u_1, \ldots, u_m, y_o, y_1, \ldots, y_m) \in \mathcal{Q}$, $u \in \mathcal{U}$ and $y \in \mathcal{Y}$, define the map $\mathcal{F} : \mathcal{Q} \times \mathcal{U} \times \mathcal{Y} \rightarrow \mathcal{P}(\mathcal{Q})$, where $\mathcal{P}(\mathcal{Q})$ is the power set of $\mathcal{Q}$, as follows:

$$\mathcal{F}(q, u, y) = \{q' \in \mathcal{Q} | u'_1 = u,\ y'_1 = y,\ ,u'_{k+1} = u_k,\ y'_{k+1} = y_k \text{ for } 1 \leq k \leq m-1\}$$

The **state transition function** $f : \mathcal{Q} \times \mathcal{U} \times \mathcal{Y} \rightarrow \mathcal{Q}$ is an arbitrary function satisfying:

$$f(q, u, y) \in \mathcal{F}(q, u, y)$$

(3) The **output function** $g_1 : \mathcal{Q} \times \mathcal{U} \rightarrow \mathcal{Y}$ is defined by:

$$g_1(q, u) = y_o$$

where $q = (u_1, \ldots, u_m, y_o, y_1, \ldots, y_m)$.

(4) For each $q \in \mathcal{Q}$, let $\mathcal{V}^u_q$ be the set of all $v \in \mathcal{V}$ for which there exists $t \geq m+1$ and

$(\mathbf{u}, (\mathbf{y}, \mathbf{v})) \in P$ such that:

$$
\begin{aligned}
v &= \mathcal{P}^+_{t-1}(\mathcal{P}^-_t(\mathbf{v})) \\
u &= \mathcal{P}^+_{t-1}(\mathcal{P}^-_t(\mathbf{u})) \\
q &= \mathcal{P}^+_{t-m-1}(\mathcal{P}^-_{t-1}(\mathbf{u})) \times \mathcal{P}^+_{t-m-1}(\mathcal{P}^-_t(\mathbf{y}))
\end{aligned}
$$

and let:

$$v^u_q = \sup_{v \in \mathcal{V}^u_q} \rho(v)$$

The **output set** $\hat{\mathcal{V}}$ is defined as:

$$\hat{\mathcal{V}} = \bigcup_{q \in \mathcal{Q}, u \in \mathcal{U}} \{v^u_q\}$$

The corresponding **output function** $g_2 : \mathcal{Q} \times \mathcal{U} \times \mathcal{Y} \to \hat{\mathcal{V}}$ is defined by:

$$g_2(q, u, y) = v^u_{q'}$$

where $q = (u_1, \ldots, u_m, y_o, y_1, \ldots, y_m)$ and $q' = (u_1, \ldots, u_m, y, y_1, \ldots, y_m)$.

(5) Function $\mu_o : \hat{\mathcal{V}} \to \mathbf{R}$ defining the **auxiliary performance objective** is the identity map.

Intuitively, the state of the nominal model $\hat{M}$ at the current time keeps track of the last m inputs and outputs of P, as well as the current output:

$$q(t) = \begin{pmatrix} u(t-1) \\ \vdots \\ u(t-m) \\ y(t) \\ y(t-1) \\ \vdots \\ y(t-m) \end{pmatrix}$$

If $card(\mathcal{U}) = a$ and $card(\mathcal{Y}) = b$, the cardinality of the resulting state set $\mathcal{Q}$ satisfies $card(\mathcal{Q}) \leq a^m b^{m+1}$. Given current state q and inputs u and y, the next state may not be uniquely determined; at most b states are feasible next states. Since there is a choice in picking the transition to the next state, this approach does not lead to a unique nominal model $\hat{M}$, but to many possible models. The question of how to best pick one among this family of models is not addressed here. A choice is assumed to be made, and the output functions are thus constructed accordingly. Finer approximations can be obtained by increasing the value of parameter m, and thus increasing the length of the snapshot of control input and measurement output pair recorded in memory.

Remark 3.4 (On the choice of initial state) For a given choice of m, the state of $\hat{M}$ at time $t = m + 1$ is uniquely determined, regardless of the initial condition. In other words, $\hat{M}$ forgets its initial condition in at most $m+1$ steps. Thus, the choice of initial condition here is irrelevant; the initial state can be arbitrarily fixed. $\diamond$

3.4.2 State Quantization

Given a system $P \subset \mathcal{U}^{\mathbf{Z}_+} \times (\mathcal{Y} \times \mathcal{V})^{\mathbf{Z}_+}$ with finite alphabet sets $\mathcal{U}$ and $\mathcal{Y}$, and a desired performance objective:

$$\inf_{T \geq 0} \sum_{t=0}^{T} -\mu(v(t)) > -\infty$$

for some given function $\mu : \mathcal{V} \to \mathbf{R}$. Assume that P has a state-space description of the following form:

$$\begin{aligned} x(t+1) &= f_u(x(t)) \\ y(t) &= g_u(x(t)) \\ v(t) &= h_u(x(t)) \end{aligned}$$

with given maps $f_u : \mathbf{R}^n \to \mathbf{R}^n$, $g_u : \mathbf{R}^n \to \mathcal{Y}$ and $h_u : \mathbf{R}^n \to \mathcal{V}$ for each $u \in \mathcal{U}$. Finally, assume that the state of P evolves in some bounded subset X of $\mathbf{R}^n$, and that there exists a finite partition of X, $\{X_1, \ldots, X_p\}$, such that $g_u(X_i)$ is a singleton for all $1 \leq i \leq p$ and for all $u \in \mathcal{U}$.

Deterministic finite state machine $\hat{M}$ is constructed as follows:

(1) The **state set** $\mathcal{Q}$ consists of:

- State q_o associated with set X.

- States q_i, $1 \leq i \leq p$, associated with sets X_i.

- States associated with arbitrary unions of sets X_1,..., X_p.

(2) Let

$$\eta : \mathcal{Q} \to \{X_1, \ldots, X_p, X_1 \cup X_2, X_1 \cup X_3, \ldots, X_1 \cup X_2 \cup X_3, \ldots, X_1 \cup \ldots \cup X_p\}$$

be the one-to-one correspondence between the states of $\mathcal{Q}$ and sets X_1,...,X_p and their unions. The **state transition function** $f : \mathcal{Q} \times \mathcal{U} \times \mathcal{Y} \to \mathcal{Q}$ is defined by:

$$f(q, u, y) = q'$$

where q' is the unique element of $\mathcal{Q}$ satisfying:

$$\eta(q') \supseteq f_u\Big(\eta(q)|_y\Big)$$

and

$$\eta(q'') \supseteq f_u\Big(\eta(q)|_y\Big) \Rightarrow \eta(q'') \supseteq \eta(q')$$

with $\eta(q)|_y$ defined as:

$$\eta(q)|_y \doteq \{X_i \in \eta(q) | g_u(X_i) = \{y\}\}$$

(3) The **output function** $g_1 : \mathcal{Q} \times \mathcal{U} \to \mathcal{Y}$ is an arbitrary function satisfying:

$$g_1(q, u) \in g_u\Big(\eta(q)\Big)$$

(4) The **output set** $\hat{\mathcal{V}}$ is defined as:

$$\hat{\mathcal{V}} = \bigcup_{u \in \mathcal{U}} \hat{\mathcal{V}}_u$$

where

$$\hat{\mathcal{V}}_u = \{\hat{v}_i^u \in \mathbf{R} | \hat{v}_i^u = \sup_{x \in X_i} \mu(h_u(x)),\ 1 \leq i \leq p\}$$

The corresponding **output function** $g_2 : \mathcal{Q} \times \mathcal{U} \times \mathcal{Y} \to \hat{\mathcal{V}}$ is defined by:

$$g_2(q, u, y) = \max_{i | X_i \in \eta(q)|_y} \hat{v}_i^u$$

(5) Function $\mu_o : \hat{\mathcal{V}} \to \mathbf{R}$ defining the **auxiliary performance objective** is the identity map.

This constructive procedure results in a non-unique deterministic finite state machine $\hat{M}$, due to the multitude of possible choices for output function g_1. No attempt is made at this point to identify and/or characterize the best choice.

Remark 3.5 (The Initial Condition) If $x(t) \in \eta(q(t))$ and $y(t) = y$, then it follows that:

$$x(t+1) = f_u(x(t)) \in f_u\Big(\eta(q(t))|_y\Big) \subseteq \eta(q(t+1))$$

and hence $x(t+1) \in \eta(q(t+1))$. Moreover, if $x(t) \in \eta(q(t))$, it follows that $\mu_o(\hat{v}(t)) \geq \mu(v(t))$. Hence, in order to ensure that the deterministic finite state machine $\hat{M}$ constructed as described in this section is a ρ/μ approximation of P, it is sufficient to initialize $\hat{M}$ to $q(0) = q_o$. $\diamond$

The approach proposed here will be illustrated using a simple benchmark example in Chapter 6.

3.5 Computing a Gain Bound for the Approximation Uncertainty

Consider a plant $P \subset \mathcal{U}^{\mathbf{Z}_+} \times (\mathcal{Y} \times \mathcal{V})^{\mathbf{Z}_+}$, a symmetric, surjective function $\beta : \mathcal{Y} \times \mathcal{Y} \to \mathcal{W}$ with $\beta(y, y) = w_o$, and functions $\rho_\Delta : \mathcal{U} \to [1, a]$ and $\mu_\Delta : \mathcal{W} \to [0, b]$ for some $a \geq 1$, $b > 0$ with $\mu_\Delta(w) = 0$ iff $w = w_o$. Given a deterministic finite state machine $\hat{M} \subset (\mathcal{U} \times \mathcal{Y})^{\mathbf{Z}_+} \times (\mathcal{Y} \times \hat{\mathcal{V}})^{\mathbf{Z}_+}$, the goal is to compute the ρ_Δ/μ_Δ gain of the resulting approximation error, system Δ. While this is a difficult problem in general, the problem of computing a gain *bound* for Δ when $\hat{M}$ is constructed using either of the two approaches given in Section 3.4 is much simpler; this is the problem addressed in this section.

3.5.1 Gain Bound for Δ Resulting from Feasible Signal Quantization

Suppose that $\hat{M}$ is constructed by quantizing the feasible signals of P, as described in Section 3.4.1. Associate with every $q \in \mathcal{Q}$, $q = (u_1, \ldots, u_m, y_o, y_1, \ldots, y_m)$ a subset $\mathcal{Q}_q$ of $\mathcal{Q}$, defined as:

$$Q_q = \{q' \in \mathcal{Q} | u'_k = u_k, y'_k = y_k,\ 1 \leq k \leq m\}$$

Consider a function $d_1 : \mathcal{Q} \rightarrow \{0, b\}$, where $b = \max_{w \in \mathcal{W} \backslash \{w_o\}} \mu_\Delta(w)$, defined by:

$$d_1(q) \doteq \begin{cases} 0 & \text{if } card(\mathcal{Q}_q) = 1 \\ b & \text{otherwise} \end{cases}$$

Proposition 3.4. *If $\hat{M}$ satisfies the inequality constraint:*

$$\inf_{T \geq 0} \sum_{t=0}^{T} \gamma \rho_\Delta(u(t)) - d_1(q(t)) > -\infty \tag{3.3}$$

for some $\gamma > 0$, then Δ is ρ_Δ/μ_Δ gain stable with gain bound not exceeding γ.

Proof. Note that for any $t \geq m$, inputs $u(t-1), \ldots, u(t-m)$ and $y(t-1), \ldots, y(t-m)$ are known without any ambiguity. Thus, $q(t) \in \mathcal{Q}_{q(t)}$ and $y(t)$, $\hat{y}(t)$ take their values in the following set:

$$\{y_o | (u(t-1), \ldots, u(t-m), y_o, y(t-1), \ldots, y(t-m)) \in \mathcal{Q}\}$$

If $card(\mathcal{Q}_{q(t)}) = 1$, then $y(t) = \hat{y}(t)$, $w(t) = w_o$ and $\mu_\Delta(w(t)) = 0$. Otherwise, $\mu_\Delta(w(t)) \leq b = d_1(q(t))$. Hence, $\mu_\Delta(w(t)) \leq d_1(q(t))$, and:

$$\sum_{t=0}^{T} \gamma \rho_\Delta(u(t)) - \mu_\Delta(w(t)) \geq C + \sum_{t=0}^{T} \gamma \rho_\Delta(u(t)) - d_1(q(t))$$

holds for all $T \geq 0$, for all finite constants C, which implies that:

$$\inf_{T \geq 0} \sum_{t=0}^{T} \gamma \rho_\Delta(u(t)) - \mu_\Delta(w(t)) \geq C + \inf_{T \geq 0} \sum_{t=0}^{T} \gamma \rho_\Delta(u(t)) - d_1(q(t))$$

Hence, if $\hat{M}$ satisfies equation (3.3), then Δ is ρ_Δ/μ_Δ gain stable with gain at most equal to γ. $\square$

3.5.2 Gain Bound for Δ Resulting from State Quantization

Suppose now that $\hat{M}$ is constructed by quantizing the state-space of P, as described in Section 3.4.2. Consider the function $d_2 : \mathcal{Q} \to \{0, b\}$ defined by:

$$d_2(q) \doteq \begin{cases} 0 & \text{if } \exists y \text{ such that } \eta(q)|_y = \eta(q) \\ b & \text{otherwise} \end{cases}$$

Proposition 3.5. *If $\hat{M}$ satisfies the inequality constraint:*

$$\inf_{T \geq 0} \sum_{t=0}^{T} \gamma \rho_\Delta(u(t)) - d_2(q(t)) > -\infty \tag{3.4}$$

for some $\gamma > 0$, then Δ is ρ_Δ/μ_Δ gain stable with gain bound not exceeding γ.

Proof. When $q(0) = q_o$, it follows from Remark 3.5 that $x(t) \in \eta(q(t))$ for all $t \geq 0$. Hence, $y(t) = g_u(x(t)) \in g_u(\eta(q(t)))$, and we have $\hat{y}(t) \in g_u(\eta(q(t)))$ by construction. If $\eta(q)|_y = \eta(q)$ for some y, then $g_u(\eta(q(t))) = \{y\}$, $y = \hat{y}(t) = y(t)$, $w = w_o$ and $\mu_\Delta(w(t)) = d_2(q(t)) = 0$. Otherwise, $\mu_\Delta(w(t)) \leq b = d_2(q(t))$. Hence, $d_2(q(t)) \geq \mu_\Delta(w(t))$ along all system trajectories, and thus the following inequality holds:

$$\inf_{T \geq 0} \sum_{t=0}^{T} \gamma \rho_\Delta(u(t)) - \mu_\Delta(w(t)) \geq \inf_{T \geq 0} \sum_{t=0}^{T} \gamma \rho_\Delta(u(t)) - d_2(q(t))$$

Hence, if $\hat{M}$ satisfies equation (3.4), then Δ is ρ_Δ/μ_Δ gain stable with gain at most equal to γ. $\square$

3.5.3 Comments on the Complexity and the Conservatism of the Approach

The gain bounds verified as described in Sections 3.5.1 and 3.5.2 are conservative for two reasons:

1. The algorithms assume that every time an error can occur, it does occur.

2. The algorithms assume that all pairs $(\mathbf{u}, \mathbf{y}) \in \mathcal{U}^{\mathbf{Z}_+} \times \mathcal{Y}^{\mathbf{Z}_+}$ are valid input signals to $\hat{M}$, which is clearly not the case since $\mathbf{y}$ is an output of P corresponding to input $\mathbf{u}$.

On the other hand, although the approximation uncertainty has complex hybrid internal dynamics in general, the computational cost of the proposed algorithms grows only with the size of the nominal finite state machine model. Moreover, it will be shown in Chapter 4 that the problem of verifying a given gain bound for a deterministic finite state machine can be handled very efficiently.

Chapter 4

Analyzing Stability of Deterministic Finite State Machines

4.1 Introduction

This chapter addresses the following analysis questions: given a deterministic finite state machine M, how to verify whether it is ρ/μ gain stable? How to compute its ρ/μ gain? These questions arise frequently in problems of hybrid system verification and design. In particular, the paradigm advocated in this thesis manages the complexity inherent in hybrid systems (particularly those with finite actuation and sensing) by approximating the system by a finite state machine model and quantifying the resulting approximation error. While the approximation uncertainty Δ is not a finite state machine in general, the (conservative) approach proposed in this thesis for quantifying Δ essentially reduces to verifying gain stability of the nominal DFM model for some appropriate choice of ρ and μ. The results derived in this chapter show that this can be done efficiently, hence justifying the approach in spite of its conservatism. Another class of problems where these questions arise are hybrid system verification problems, that is, verifying DFM controllers designed by some combination of ad-hoc methods and intuition about the hybrid system. Such verification problems can be addressed by approximating the hybrid plant as a DFM, evaluating the approximation error, verifying gain stability of the nominal model interconnected with the controller, and then using the 'Small Gain' Theorem to verify the performance of the actual closed loop system.

In Section 4.2, each of these questions is shown to reduce to searching for an appropriate storage function, which can be practically implemented by solving a linear program. In Section 4.3, the problems of stability and gain verification are shown to be related to the problem of checking that no negative cost cycles exist in a network. Using this insight and based on a solution approach for discrete shortest path problems, a strongly polynomial algorithm is proposed. Its worst-case computational complexity is shown to be $\mathcal{O}(n^2)$ for a DFM with n states, assuming that the cardinality of the input alphabet is much smaller than that of the state set.

4.2 Stability Verification

4.2.1 Necessary and Sufficient Conditions for Gain Stability

Necessary and sufficient conditions for verifying gain stability and gain bounds are presented in this section. The proofs of Theorems 4.1 and 4.4 are postponed until section 4.2.2.

Theorem 4.1. *Consider a deterministic finite state machine M defined by equations (2.1) and (2.2), and a function $\sigma : \mathcal{Q} \times \mathcal{U} \to \mathbf{R}$. The following two statements are equivalent:*

(a) *For any $\mathbf{u} \in \mathcal{U}^{\mathbf{Z}_+}$ and $q(0) \in \mathcal{Q}$, the following inequality is satisfied:*

$$\inf_{T \geq 0} \sum_{t=0}^{T} \sigma(q(t), u(t)) > -\infty \tag{4.1}$$

(b) *There exists a non-negative function $V : \mathcal{Q} \to \mathbf{R}_+$ such that the inequality:*

$$V(f(q,u)) - V(q) \leq \sigma(q,u) \tag{4.2}$$

holds for all $q \in \mathcal{Q}$ and $u \in \mathcal{U}$.

In the terminology of the theory of dissipative systems proposed by Willems [79, 80], V in Theorem 4.1 is the storage function of the dissipative system M with supply rate σ. The following Corollary shows that finiteness of the set of states in a DFM model results in a stronger notion of gain stability.

Corollary 4.2. *Consider a deterministic finite state machine M and functions $\rho : \mathcal{U} \to \mathbf{R}$ and $\mu : \mathcal{Y} \to \mathbf{R}$. M is ρ/μ gain stable iff there exists finite non-negative constants C, γ such that for every input $\mathbf{u} \in \mathcal{U}^{\mathbf{Z}_+}$ and initial condition $q \in \mathcal{Q}$, the corresponding output $\mathbf{y} \in \mathcal{Y}^{\mathbf{Z}_+}$ satisfies:*

$$\sum_{t=0}^{T} \mu(y(t)) \leq C + \gamma \sum_{t=0}^{T} \rho(u(t))$$

for all $T \geq 0$.

Proof. Sufficiency is straightforward. Necessity follows from Theorem 4.1 with:

$$C = \max_{q_1, q_2} V(q_1) - V(q_2)$$

□

Corollary 4.3. *Consider a deterministic finite state machine M and sets $\mathcal{U}_o \subset \mathcal{U}$, $\mathcal{Y}_o \subset \mathcal{Y}$. M is gain stable about $(\mathcal{U}_o, \mathcal{Y}_o)$ with ρ/μ gain not exceeding γ if and only if there exists a non-negative function $V : \mathcal{Q} \to \mathbf{R}_+$ such that the inequality:*

$$V(f(q,u)) - V(q) \leq \gamma\rho(u) - \mu(g(q,u))$$

holds for all $q \in \mathcal{Q}$ and $u \in \mathcal{U}$.

Proof. Follows from Theorem 4.1 with $\sigma(q,u) = \gamma\rho(u) - \mu(g(q,u))$. □

In some cases, we may be interested in simply verifying gain stability of a DFM about some $(\mathcal{U}_o, \mathcal{Y}_o)$, rather than in explicitly computing its ρ/μ gain.

Theorem 4.4. *Consider a deterministic finite state machine M defined by equations (2.1) and (2.2) and sets $\mathcal{U}_o \subset \mathcal{U}$, $\mathcal{Y}_o \subset \mathcal{Y}$. M is gain stable about $(\mathcal{U}_o, \mathcal{Y}_o)$ if and only if there exists a non-negative function $V : \mathcal{Q} \to \mathbf{R}_+$ such that the inequality:*

$$V(f(q,u)) - V(q) \leq -\mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(q,u)) \tag{4.3}$$

holds for all $q \in \mathcal{Q}$ and $u \in \mathcal{U}_o$.

4.2.2 Proof of the Necessary and Sufficient Conditions

Lemmas 4.5 through 4.7 will be used in proving Theorems 4.1 and 4.4.

Lemma 4.5. *Consider a deterministic finite state machine M and a function $\sigma : \mathcal{Q} \times \mathcal{U} \to \mathbf{R}$. If there exists a function $V : \mathcal{Q} \to \mathbf{R}$ such that (4.4) holds for all $q \in \mathcal{Q}$ and $u \in \mathcal{U}$:*

$$V(f(q,u)) - V(q) \leq \sigma(q,u) \tag{4.4}$$

then for any $\tau > 0$, input sequence $\mathbf{u} \in \mathcal{U}^{\mathbf{Z}_\tau}$ and corresponding state sequence $\mathbf{q} \in \mathcal{Q}^{\mathbf{Z}_\tau}$ satisfying $q(0) = q(\tau)$, we have:

$$\sum_{t=0}^{\tau} \sigma(q(t), u(t)) \geq 0$$

Proof. By summing up (4.4) along any trajectory from $t = 0$ to $t = T$, we get:

$$\sum_{t=0}^{T} \sigma(q(t), u(t)) \geq V(f(q(T), u(T))) - V(q(0)) \geq \min_{q_1, q_2} V(q_1) - V(q_2) \tag{4.5}$$

Suppose there exists a $\tau > 0$, $u \in \mathcal{U}^{\mathbf{Z}_\tau}$ and corresponding $\mathbf{q} \in \mathcal{Q}^{\mathbf{Z}_\tau}$, with $q(0) = q(\tau)$, such that $\sum_{t=0}^{\tau} \sigma(q(t), u(t)) < 0$. We can construct a periodic input such that, for initial condition $q(0)$, the summation in eq (4.5) can be made arbitrarily negative for large enough T, thus leading to a contradiction. □

Lemma 4.6. *Consider a deterministic finite state machine M and sets $\mathcal{U}_o \subset \mathcal{U}$, $\mathcal{Y}_o \subset \mathcal{Y}$. If there exists a function $V : \mathcal{Q} \to \mathbf{R}$ such that (4.6) holds for all $q \in \mathcal{Q}$ and $u \in \mathcal{U}_o$:*

$$V(f(q,u)) - V(q) \leq -\mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(q,u)) \tag{4.6}$$

then for any $\tau > 0$, input $\mathbf{u} \in \mathcal{U}^{\mathbf{Z}_\tau}$ and corresponding state sequence $\mathbf{q} \in \mathcal{Q}^{\mathbf{Z}_\tau}$ with $q(0) = q(\tau)$, the following inequality:

$$\sum_{t=0}^{\tau} \gamma \mathbf{I}_{\mathcal{U}-\mathcal{U}_o}(u(t)) - \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(q(t), u(t))) \geq 0 \tag{4.7}$$

holds for $\gamma = card(\mathcal{Q})$.

Proof. By summing up (4.6) along any trajectory from $t = 0$ to $t = T$, we get:

$$\sum_{t=0}^{T} \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(q(t), u(t))) \leq V(q(0)) - V(q(T+1)) \leq \max_{q_1,q_2} V(q_1) - V(q_2) \qquad (4.8)$$

Now consider an input sequence $\mathbf{u} \in \mathcal{U}^{\mathbf{Z}_\tau}$ and corresponding state sequence $\mathbf{q} \in \mathcal{Q}^{\mathbf{Z}_\tau}$ satisfying $q(0) = q(\tau)$. First, note that (4.7) holds for $\tau \leq card(\mathcal{Q})$: if $u(t) \in \mathcal{U} - \mathcal{U}_o$ for at least one t in $\mathbf{Z}_\tau$, (4.7) is clearly satisfied. Otherwise, if $u(t) \in \mathcal{U}_o$ for all t, we conclude that $\sum_{t=0}^{\tau} \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(q(t), u(t))) = 0$. For if that was not the case, we can construct a periodic input sequence for which the summation on the left hand side of (4.8) can be made infinite, thus leading to a contradiction. Next, note that if $\tau > card(\mathcal{Q})$, there exists integers t', t'' in $\mathbf{Z}_\tau$ such that $t' < t''$, $q(t') = q(t'')$, $t'' - t' \leq card(\mathcal{Q})$ and $\sum_{t=t'}^{t''} \gamma \mathbf{I}_{\mathcal{U}-\mathcal{U}_o}(u(t)) - \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(q(t), u(t))) \geq 0$. Let $\tau' = \tau - (t'' - t')$ and define:

$$q'(t) := \begin{cases} q(t) & 0 \leq t < t' \\ q(t + t'' - t') & t' \leq t \leq \tau' \end{cases} \qquad u'(t) := \begin{cases} u(t) & 0 \leq t < t' \\ u(t + t'' - t') & t' \leq t \leq \tau' \end{cases}$$

Note that:

$$\sum_{t=0}^{\tau} \gamma \mathbf{I}_{\mathcal{U}-\mathcal{U}_o}(u(t)) - \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(q(t), u(t))) \geq \sum_{t=0}^{\tau'} \gamma \mathbf{I}_{\mathcal{U}-\mathcal{U}_o}(u'(t)) - \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(q'(t), u'(t)))$$

Thus for any $\tau_0 > card(\mathcal{Q})$ we can construct, as described above, a sequence of integers $\tau_0, \ldots, \tau_{k_o}$ and corresponding state and input sequences $\mathbf{q}_k \in \mathcal{Q}^{\mathbf{Z}_{\tau_k}}$, $\mathbf{u}_k \in \mathcal{U}^{\mathbf{Z}_{\tau_k}}$, with $\tau_0 > \ldots > card(\mathcal{Q}) \geq \tau_{k_o}$ and such that:

$$\sum_{t=0}^{\tau_k} \gamma \mathbf{I}_{\mathcal{U}-\mathcal{U}_o}(u_k(t)) - \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(q_k(t), u_k(t))) \geq \sum_{t=0}^{\tau_{k+1}} \gamma \mathbf{I}_{\mathcal{U}-\mathcal{U}_o}(u_{k+1}(t)) - \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(q_{k+1}(t), u_{k+1}(t)))$$

holds for $k \in \{0, \ldots, k_o - 1\}$. □

Lemma 4.7. *Consider a deterministic finite state machine M and a function $\sigma : \mathcal{Q} \times \mathcal{U} \to \mathbf{R}$. Suppose that for any $\mathbf{u} \in \mathcal{U}^{\mathbf{Z}_+}$ and $q(0) \in \mathcal{Q}$, the following inequality is satisfied:*

$$\inf_{T \geq 0} \sum_{t=0}^{T} \sigma(q(t), u(t)) > -\infty \qquad (4.9)$$

Then, for any $q(0) \in \mathcal{Q}$ and $\mathbf{u} \in \mathcal{U}^{\mathbf{Z}}$, and for all $T \geq 0$, we have:

$$\sum_{t=0}^{T} \sigma(q(t), u(t)) \geq -card(\mathcal{Q}) \max_{q,u} |\sigma(q,u)| \tag{4.10}$$

Proof. The condition given in (4.9) can be equivalently written as: for every input $\mathbf{u}$ and initial state q there exists a finite constant $C_{\mathbf{u},q}$ such that:

$$\sum_{t=0}^{T} \sigma(q(t), u(t)) \geq -C_{\mathbf{u},q}, \ \forall T \geq 0$$

We claim that $C_{\mathbf{u},q} \leq card(\mathcal{Q}) \max_{q,u} |\sigma(q,u)|$. The following argument will be used: if there exists some initial state $q(0)$ and input $\mathbf{u}$ for which $\sum_{t=0}^{T} \sigma(q(t), u(t)) < -card(\mathcal{Q}) \max_{q,u} |\sigma(q,u)|$, for some T, then $T > card(\mathcal{Q})$, because the summation is bounded below by $-T \max_{q,u} |\sigma(q,u)|$. Thus there exists two integers t' and t'' in $\mathbf{Z}_T$, $t' < t''$, such that $q(t') = q(t'')$. Moreover, it must be the case that $\sum_{t=t'}^{t''-1} \sigma(q(t), u(t)) \geq 0$, otherwise there exists at least one initial condition and periodic input sequence for which (4.9) is violated. Let $T' = T - (t'' - t')$ and consider $\overline{\mathbf{q}} \in \mathcal{Q}^{\mathbf{Z}_{T'}}$ and $\overline{\mathbf{u}} \in \mathcal{U}^{\mathbf{Z}_{T'}}$ defined by:

$$\overline{q}(t) := \begin{cases} q(t) & 0 \leq t < t' \\ q(t + t'' - t') & t' \leq t \leq T' \end{cases} \qquad \overline{u}(t) := \begin{cases} u(t) & 0 \leq t < t' \\ u(t + t'' - t') & t' \leq t \leq T' \end{cases}$$

$\overline{\mathbf{q}}$ and $\overline{\mathbf{u}}$ are valid state and input sequences, and they satisfy:

$$\sum_{t=0}^{T'} \sigma(\overline{q}(t), \overline{u}(t)) \leq \sum_{t=0}^{T} \sigma(q(t), u(t)) < -card(\mathcal{Q}) \max_{q,u} |\sigma(q,u)|$$

Now, suppose our claim is not true. That is, there exists some initial state $q(0)$ and input $\mathbf{u}$ for which $\sum_{t=0}^{T_0} \sigma(q(t), u(t)) < -card(\mathcal{Q}) \max_{q,u} |\sigma(q,u)|$, for some T_0. By the above argument, we can construct a finite sequence of integers $T_0 > \ldots > T_{k_o}$ with corresponding state and input sequences $\mathbf{q}_k \in \mathcal{Q}^{\mathbf{Z}_{T_k}}$ and $\mathbf{u}_k \in \mathcal{U}^{\mathbf{Z}_{T_k}}$, satisfying $\sum_{t=0}^{T_k} \sigma(q_k(t), u_k(t)) < -card(\mathcal{Q}) \max_{q,u} |\sigma(q,u)|$, and such that $T_{k_o} \leq card(\mathcal{Q})$, leading to a contradiction. □

We are now ready to prove Theorems 4.1 and 4.4.

Proof of Theorem 4.1: To prove that (b) $\Rightarrow$ (a), note that by summing up (4.2) along any trajectory from $t = 0$ to $t = T$, we get:

$$\sum_{t=0}^{T} \sigma(q(t), u(t)) \geq V(f(q(T), u(T))) - V(q(0)) \geq \min_{q_1, q_2} V(q_1) - V(q_2)$$

which implies (4.1).

To prove that (a) $\Rightarrow$ (b), define function $V : \mathcal{Q} \to \mathbf{R}$ as follows:

$$V(q) \doteq \max_{q,u} |\sigma(q,u)| + \sup_{T, \mathbf{u} \in \mathcal{U}^{\mathbf{Z}_T}} \sum_{t=0}^{T} -\sigma(q(t), u(t)) \tag{4.11}$$

where $\mathbf{q}$ is the state trajectory associated with initial condition q and input $\mathbf{u}$. It follows from Lemma 4.7 that the right hand side of (4.11) is bounded above and hence V is well defined. Moreover, V is non-negative by construction. Finally, we have:

$$\begin{aligned} V(f(q,u)) - V(q) &= \sup_{T, \mathbf{u} \in \mathcal{U}^{\mathbf{Z}_T}} \sum_{t=0}^{T} -\sigma(\hat{q}(t), u(t)) - \sup_{T, \mathbf{u} \in \mathcal{U}^{\mathbf{Z}_T}} \sum_{t=0}^{T} -\sigma(g(q(t), u(t))) \\ &\leq \sup_{T, \mathbf{u} \in \mathcal{U}^{\mathbf{Z}_T}} \sum_{t=0}^{T} -\sigma(\hat{q}(t), u(t)) + \Big(\sigma(q,u) - \sup_{T, \mathbf{u} \in \mathcal{U}^{\mathbf{Z}_T}} \sum_{t=0}^{T} -\sigma(\hat{q}(t), u(t))\Big) \end{aligned}$$

where $q(0) = q$, $\hat{q}(0) = f(q,u)$. □

Proof of Theorem 4.4: To prove sufficiency, we will show that for any input signal $\mathbf{u} \in \mathcal{U}^{\mathbf{Z}_+}$ and initial condition $q \in \mathcal{Q}$, and for $\gamma = card(\mathcal{Q})$ and $C = card(\mathcal{Q})card(\mathcal{U})$, the following inequality is satisfied for all $T \geq 0$:

$$\sum_{t=0}^{T} \gamma \mathbf{I}_{\mathcal{U} - \mathcal{U}_o}(u(t)) - \mathbf{I}_{\mathcal{Y} - \mathcal{Y}_o}(g(q(t), u(t))) \geq -C \tag{4.12}$$

This is clearly the case when $T \leq C$. If $T > C$, there must exist two integers t' and t'' in $\mathbf{Z}_T$, with $t' < t''$, such that $q(t') = q(t'')$. Thus, it follows from Lemma 4.6 that we can construct new state and input sequences, $\overline{\mathbf{q}} \in \mathcal{Q}^{\mathbf{Z}_{T'}}$ and $\overline{\mathbf{u}} \in \mathcal{U}^{\mathbf{Z}_{T'}}$, where $T' = T - (t'' - t')$, such that:

$$\sum_{t=0}^{T'} \gamma \mathbf{I}_{\mathcal{U}-\mathcal{U}_o}(\overline{u}(t)) - \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(\overline{q}(t), \overline{u}(t))) \geq -C$$
$$\Rightarrow \sum_{t=0}^{T} \gamma \mathbf{I}_{\mathcal{U}-\mathcal{U}_o}(u(t)) - \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(q(t), u(t))) \geq -C$$

Now, for any $T_0 > C$, by the preceding argument, we can construct a finite sequence of integers $T_0 > \ldots > C \geq T_{k_o}$ and corresponding state and input sequences $\mathbf{q}_k \in \mathcal{Q}^{\mathbf{Z}_{T_k}}$ and $\mathbf{u}_k \in \mathcal{U}^{\mathbf{Z}_{T_k}}$ such that:

$$\sum_{t=0}^{T_{k+1}} \gamma \mathbf{I}_{\mathcal{U}-\mathcal{U}_o}(u_{k+1}(t)) - \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(q_{k+1}(t), u_{k+1}(t))) \geq -C$$
$$\Rightarrow \sum_{t=0}^{T_k} \gamma \mathbf{I}_{\mathcal{U}-\mathcal{U}_o}(u_k(t)) - \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(q_k(t), u_k(t))) \geq -C$$

Hence, it follows that (4.12) is satisfied for all $T \geq 0$.

To prove necessity, suppose that M is gain stable about $(\mathcal{U}_o, \mathcal{Y}_o)$. Define function $V : \mathcal{Q} \rightarrow \mathbf{R}_+$ as follows:

$$V(q) \doteq \sup_{u \in \mathcal{U}_o^{\mathbf{Z}_+}} \sum_{t=0}^{\infty} \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(q(t), u(t)))$$

where $\mathbf{q}$ is the state trajectory corresponding to initial state q and input $\mathbf{u} \in \mathcal{U}^{\mathbf{Z}_+}$. V is non-negative by construction and finite by Lemma 4.7 (where $\sigma(q, u)$ is $\gamma \mathbf{I}_{\mathcal{U}-\mathcal{U}_o}(u) - \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(q, u))$). It also follows from the definition that:

$$V(f(q, u)) - V(q) = \sup_{\mathbf{u} \in \mathcal{U}_o^{\mathbf{Z}_+}} \sum_{t=0}^{\infty} \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(\hat{q}(t), u(t))) - \sup_{\mathbf{u} \in \mathcal{U}_o^{\mathbf{Z}_+}} \sum_{t=0}^{\infty} \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(q(t), u(t)))$$

where $\hat{q}(0) = f(q, u)$. Moreover, we have:

$$\begin{aligned} \sup_{\mathbf{u} \in \mathcal{U}_o^{\mathbf{Z}_+}} \sum_{t=0}^{\infty} \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(q(t), u(t))) &\geq \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(q, u)) + \sup_{\mathbf{u} \in \mathcal{U}_o^{\mathbf{Z}_+}} \sum_{t=1}^{\infty} \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(q(t), u(t))) \\ &= \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(q, u)) + \sup_{\mathbf{u} \in \mathcal{U}_o^{\mathbf{Z}_+}} \sum_{t=0}^{\infty} \mathbf{I}_{\mathcal{Y}-\mathcal{Y}_o}(g(\hat{q}(t), u(t))) \end{aligned}$$

Hence, (4.3) holds for every $q \in \mathcal{Q}$, $u \in \mathcal{U}_o$. □

4.2.3 A Note About the Search for Storage Functions

It follows from Theorems 4.1 and 4.4 and Corollary 4.3 in Section 4.2.1 that verifying gain stability of a DFM about some $(\mathcal{U}_o, \mathcal{Y}_o)$, as well as verifying a particular gain bound, can be done by checking feasibility of a linear program of the form:

$$Ax \geq b$$

The decision variable x is the vector of values of the storage function V we are searching for. Similarly, computing the gain of a DFM requires solving a linear program of the form:

$$\begin{aligned} \min \quad & c'x \\ \text{subject to} \quad & Ax \geq b \end{aligned}$$

Decision variable x is the vector of values of the gain and the storage function V.

Remark 4.1 It is not necessary to enforce $x \geq 0$ in the above linear programs. If a feasible solution exists, a feasible non-negative solution also exists. ◇

Any of these problems can be solved using an off the shelf LP solver. However this is not advisable for the following two reasons:

1. The linear programs in question are highly structured: Matrix A is sparse, with integer entries taking one of three values (-1, 0 or 1). It also has the property that there are at most two non-zero entries per row and the entries along each row sum up to zero. Moreover, in the linear problems associated with verifying stability or incremental stability, vector b consists of integer entries taking the values 0 or 1. Finally, vector c is an all zero vector except for a single unity entry.

2. The DFM models of interest, typically being approximate models of potentially complex dynamical systems, are expected to have a large number of states.

In view of this structure and the potential size of the problems of interest, it is important to develop specialized algorithms with better worst-case bounds on computational complexity than what generic LP solution algorithms can guarantee. This is addressed in the next section.

4.3 Strongly Polynomial Computational Algorithms

Given a DFM M defined by (2.1) and (2.2) and non-negative functions $\rho : \mathcal{U} \to \mathbf{R}_+$ and $\mu : \mathcal{Y} \to \mathbf{R}_+$. Verifying that M is ρ/μ gain stable and verifying an upper bound for its ρ/μ gain can be interpreted as verifying the non-existence of negative cost cycles in appropriately constructed and labeled networks. The underlying network problem is described in section 4.3.1, and a computational algorithm that makes use of a label-correcting algorithm for solving the all-to-one discrete shortest path problem is proposed in section 4.3.2. The algorithm is strongly polynomial, with worst-case computational complexity $\mathcal{O}(n^2)$ for a network with n nodes and at most an edges, where $a \ll n$.

4.3.1 A Related Network Problem

A directed graph $G = (\mathcal{N}, \mathcal{E})$ is a set of nodes $\mathcal{N} = \{1, \ldots, n\}$ and a set of directed edges $\mathcal{E} \subset \mathcal{N}^2$. The pair $(i, j) \in \mathcal{E}$ is an edge outgoing from node i and incoming into node j. A network is a directed graph with additional numerical information associated with its edges and/or nodes. A specific class of networks, referred to as 'γ-networks', is of particular interest.

Definition 4.1. *A γ-network G_γ is a directed graph $G = (\mathcal{N}, \mathcal{E})$ with a cost function of a scalar parameter γ, $c_{ij}(\gamma)$ associated with each edge (i, j). Each cost function is a piecewise linear function of γ, $c_{ij}(\gamma) = \min\limits_k (a_{ij}^k \gamma - b_{ij}^k)$, with $a_{ij}^k \geq 0$, $b_{ij}^k \geq 0$.*

A cycle is a sequence of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$ such that $i_1 = i_k$. A cycle is said to be a simple cycle if $i_1, \ldots, i_{k-1}$ are distinct. The cost of a cycle is the sum of the costs of its edges, and is thus also a piecewise linear function of γ. A path is a sequence of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$ whose nodes are all distinct. The cost of a path is the sum of the costs of its edges.

Definition 4.2. *Consider a γ-network G_γ. The gain of G_γ is the smallest value of γ for which all of its cycles have non-negative cost.*

In particular, if for every value of γ, there exists at least one cycle with negative cost, G_γ is said to have infinite gain.

Lemma 4.8. *Consider a γ-network G_γ as in Definition 4.1. Let $n = card(\mathcal{N})$, $b_o = \max\{b_{ij}^k\}$, and $a_o = \min\{a_{ij}^k | a_{ij}^k \neq 0\}$. The gain of G_γ is either infinite or bounded above by $n\dfrac{b_o}{a_o}$.*

Proof. First, note that all cycles of G_γ have non-negative cost iff all simple cycles of G_γ have non-negative cost. Now suppose that the gain γ_o of G_γ is finite. Note that $\gamma_o = \min\{\gamma | c_C(\gamma) \geq 0$ for all simple cycles $C\}$, where $c_C(\gamma)$ denotes the cost of C. If $c_C(\gamma)$ is a constant function, we conclude from the finite gain assumption that $c_C(\gamma) \geq 0$. Otherwise, for a simple cycle with m edges $(1 \leq m \leq n)$, we have:

$$c_C(\gamma) \geq m a_o \gamma - m b_o \geq a_o \gamma - m b_o \geq a_o \gamma - n b_o$$

Thus all simple cycles have non-negative cost for $\gamma = n\dfrac{b_o}{a_o}$, and hence $\gamma_o \leq n\frac{b_o}{a_o}$. □

Let M be a deterministic finite state machine defined by (2.1) and (2.2) with $card(\mathcal{Q}) = n$. Given non-negative functions $\rho : \mathcal{U} \to \mathbf{R}_+$ and $\mu : \mathcal{Y} \to \mathbf{R}_+$, M can be associated with a γ-network G_γ^M constructed as follows:

$$\begin{aligned} \mathcal{N} &\doteq \{1, \ldots, n\} \\ \mathcal{E} &\doteq \{(i,j) \in \mathcal{N} \times \mathcal{N} | \exists u \in \mathcal{U} \text{ such that } q_j = f(q_i, u)\} \\ c_{ij}(\gamma) &\doteq \min_{u | f(q_i,u) = q_j} \Big(\gamma\rho(u) - \mu(g(q_i,u))\Big) \end{aligned} \tag{4.13}$$

Lemma 4.9. *M is gain stable with ρ/μ gain γ_o iff G_γ^M has gain γ_o.*

Proof. : Necessity follows from Theorem 4.1 and Lemma 4.5. To prove sufficiency, suppose that G_γ^M has gain γ_o. Let $\mathcal{P}(i)$ be the set of all paths starting at node i, and let $c_p(\gamma)$ be the cost of path $p \in \mathcal{P}(i)$. Define a function $\overline{V} : \mathcal{N} \to \mathbf{R}_+$ by the following rule: $\overline{V}(i) \doteq \sup_{p \in \mathcal{P}(i)} \{-c_p(\gamma_o)\}$. $\overline{V}$ is bounded by construction (the maximum number of edges in any path is $n-1$). Moreover, $\overline{V}(i) \geq -c_{ij}(\gamma_o) + \overline{V}(j), \forall (i,j) \in \mathcal{E}$. Thus, function $V : \mathcal{Q} \to \mathbf{R}_+$ defined by $V(q_i) = \overline{V}(i) - \min_i \overline{V}(i)$ is non-negative and satisfies (4.2) with $\sigma(q,u) = \gamma_o\rho(u) - \mu(g(q,u))$. It follows from Theorem 4.1 that M is gain stable with ρ/μ gain γ_o. □

4.3.2 Algorithms for Stability and Gain Verification

We begin by briefly describing the all-to-one discrete shortest path problem. Consider a directed graph $G = (\mathcal{N}, \mathcal{E})$ with numerical cost c_{ij}, representing the 'length' of the edge, associated with each edge $(i, j) \in \mathcal{E}$. A node, say node k, is picked as the destination node. The objective is to find the shortest (i.e least costly) directed path from each of the remaining nodes to k. A class of algorithms, collectively referred to as label-correcting algorithms, solve this problem. The basic idea is to associate with each node j a distance label $d(j)$, which provides an upper bound on the length of the shortest path from node j to the destination node while the algorithm is running. Various implementations of label-correcting algorithms exist. They differ mainly in the manner in which they pick the sequence of distance labels to be updated, and consequently also in their termination condition (see [2] for a detailed discussion). The algorithms all terminate either when the lengths (i.e. costs) of all the shortest paths have been computed or when a negative cost cycle has been discovered. The following well-known optimality condition (see [2] for a proof) allows us to assess whether a given set of distance labels represents the shortest path lengths to the destination node.

Lemma 4.10. *Consider a directed graph $G = (\mathcal{N}, \mathcal{E})$ with edge costs c_{ij}, and consider a function $d : \mathcal{N} \to \mathbf{R}$ with $d(k) = 0$, where node k is the destination node. Suppose that there are no outgoing edges from k, and that for every $j \in \mathcal{N} - \{k\}$, $d(j)$ denotes the length of a directed path from node j to node k. Function d defines the shortest path lengths iff the following condition is satisfied:*

$$d(i) \le d(j) + c_{ij}, \ \forall (i, j) \in \mathcal{E} \tag{4.14}$$

The first algorithm presented solves the all-to-one shortest path problem in a network $G = (\mathcal{N}, \mathcal{E})$ with n nodes, constant edge costs c_{ij}, and in which node k is picked as the destination node. $\theta(i)$ denotes the set of end nodes of all outgoing edges from i: $\theta(i) \doteq \{j \in \mathcal{N} | (i, j) \in \mathcal{E}\}$.

Shortest Path Algorithm (SPA)

Input: G, k.

(1) Add a new node numbered $n + 1$, and edge $(k, n + 1)$ with $c_{k,n+1} = 0$.

(2) Initialize the distance labels: $d_0(n+1) = 0$, $d_0(i) = +\infty$, *for* $i \neq n+1$.

(3) Initialize a set of nodes: $LIST_0 = \mathcal{N}$.

(4) At iteration $t+1$*:*

(i) For each node $i \in LIST_t$*, update the distance label according to the update law:*

$$d_{t+1}(i) \doteq \min_{j \in \theta i}\{d_t(j) + c_{ij}\}$$

(ii) Set $LIST_{t+1} \doteq \{l \in \mathcal{N} | (l,i) \in \mathcal{E}$ *and* $d_t(i) \neq d_{t+1}(i)\}$.

The algorithm is adapted from the Bellman-Ford algorithm (see [14]) and thus uses the same update law. However, it differs in that: (i) it explicitly keeps track of the predecessors of all nodes whose distance labels have been updated during iteration t (in $LIST_t$) and (ii) it does not assume existence of a feasible path from each node to the destination node. In particular, note that if $d_m(i)$ is finite, it denotes the length of the shortest directed path from i to k with at most m edges. On the other hand, if $d_m(i) = +\infty$, no feasible directed path with at most m edges has been found yet from i to k.

The termination properties and worst-case complexity of this algorithm are as follows:
(a) If $d_{n+1} = d_n$, we have:

$$d_{n+1}(i) = \min_{j \in \theta(i)}\{d_n(j) + c_{ij}\} \leq d_{n+1}(j) + c_{ij},\ \forall (i,j) \in \mathcal{E}$$

Thus, d_{n+1} satisfies (4.14). Let $\overline{\mathcal{N}} = \{i \in \mathcal{N} \mid d_{n+1}(i) = +\infty\}$. $\overline{\mathcal{N}}$ is the set of nodes for which no feasible path exists to node $n+1$ and hence k, since if such a path was to exist, it would consist of at most n edges. For every $i \in \mathcal{N} - \overline{\mathcal{N}}$, it follows from Lemma 4.10 that $d_{n+1}(i)$ is the shortest path length, and hence no negative cost cycles exist in $\mathcal{N} - \overline{\mathcal{N}}$. No conclusions can be drawn about the cost of cycles in $\overline{\mathcal{N}}$.
(b) If $d_{n+1} \neq d_n$, a negative cost cycle exists: for if that was not the case, d_n would be the shortest path distances, since all paths consist of at most n edges, and hence we would have $d_n = d_k$, for all $k > n$.
The algorithm terminates in at most $n+1$ iterations. Moreover, at each iteration after the first, each edge in $\mathcal{N} - \mathcal{N}_o$ is examined at most once. Let $\overline{n} = card(\overline{\mathcal{N}})$, and assume there are at most a outgoing edges from each node, with $a \ll n$. In this case, the worst case

computational complexity is $\mathcal{O}(na(n-\overline{n})+an) \sim \mathcal{O}(n(n-\overline{n}))$.

The following algorithm, which uses the Shortest Path Algorithm, verifies whether a give value of γ_o is an upper bound on the gain of a given γ-network, by checking whether any negative cost cycles exist in the network for $\gamma = \gamma_o$.

Gain Verification Algorithm (GVA)

Input: G_γ, γ_o.

(1) Set $c_{ij} \doteq c_{ij}(\gamma_o), \mathcal{N}_0 = \mathcal{N}, \mathcal{E}_0 = \mathcal{E}$, $G = (\mathcal{N}_0, \mathcal{E}_0)$.

(2) At iteration $t+1$*:*

(i) Pick an arbitrary destination node k *in* $\mathcal{N}_t$.

(ii) Run SPA. If $d_{n+1} \neq d_n$, *exit.*

(iii) Set $\mathcal{N}_{t+1} = \{i \in \mathcal{N}_t | d(i) = +\infty\}$, $\mathcal{E}_{t+1} = \{(i,j) \in \mathcal{E}_t | i,j \in \mathcal{N}_t\}$, $G = (\mathcal{N}_{t+1}, \mathcal{E}_{t+1})$.

If the Gain Verification Algorithm exits when $d_{n+1} \neq d_n$, γ_o is not an upper bound for the gain of G_γ since there exists a negative cost cycle. Otherwise, if it terminates when $\mathcal{N}_{t+1} = \emptyset$, γ_o is a verified upper bound for the gain since the network is free of negative cost cycles. The algorithm terminates in finite time, since there are at most $n+1$ iterations, each of which runs the Shortest Path Algorithm once, and hence terminates in finite time. Let $n_o = n = card(\mathcal{N}_o)$ and let $n_i = card(\mathcal{N}_i)$, $i \geq 1$. The worst case computational complexity of this algorithm is given by $\mathcal{O}(n(n-n_1) + n_1(n_1 - n_2) + \ldots + n_{k-1}(n_{k-1} - n_k)) \sim \mathcal{O}(n^2)$.

Remark 4.2 Given a deterministic finite state machine M defined by (2.1) and (2.2), and given functions $\rho : \mathcal{U} \to \mathbf{R}_+$ and $\mu : \mathcal{Y} \to \mathbf{R}_+$. It follows from Lemmas 4.8 and 4.9 that the Gain Verification Algorithm with input $G_\gamma = G_\gamma^M$ and $\gamma_o = n\dfrac{b_o}{a_o}$ allows us to verify whether M is ρ/μ gain stable. ◇

While the gain of a γ-network cannot be exactly computed using these algorithms, an upper bound for it can be computed up to any desired level of accuracy ϵ. This is done by running the Gain Verification Algorithm iteratively: an upper bound and a lower bound for the gain are established and iteratively refined using a bisection algorithm.

Gain Computation Algorithm (GCA)

Input: G_γ, ϵ.

(1) Run GVA with input G_γ, $\gamma_o = n\dfrac{b_o}{a_o}$. *If GVA exits with* $d_{n+1} \neq d_n$, *exit.*

(2) Set $\gamma_0^{LB} = \min\{\gamma | \max\limits_{(i,j)\in\mathcal{E}} c_{ij}(\gamma) \geq 0\}$, $\gamma_0^{UB} = n\dfrac{b_o}{a_o}$.

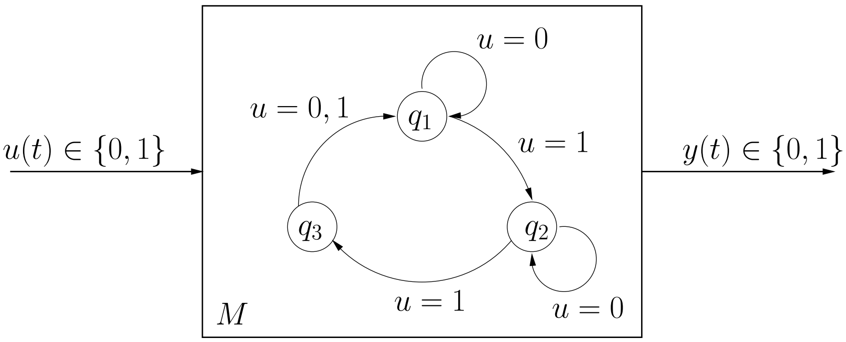

Figure 4-1: Three state deterministic finite state machine in Example 4.1

(3) At iteration $t+1$*:*

(i) If $\gamma_t^{UB} - \gamma_t^{LB} < \epsilon$*, exit.*

(ii) Set $\gamma_o = (\gamma_t^{LB} + \gamma_t^{UB})/2$.

(iii) Run GVA. If γ_o *is a verified gain bound, set* $\gamma_{t+1}^{UB} = \gamma_o, \gamma_{t+1}^{LB} = \gamma_t^{LB}$.

Else, set $\gamma_{t+1}^{UB} = \gamma_t^{UB}, \gamma_{t+1}^{LB} = \gamma_o$.

The number of iterations of the Gain Computation Algorithm grows inversely with the desired level of accuracy ϵ, with the worst case complexity of each iteration being $\mathcal{O}(n^2)$.

4.4 Examples

Example 4.1 Consider a DFM M as shown in Figure 4-1 with $\mathcal{Q} = \{q_1, q_2, q_3\}$, $\mathcal{U} = \mathcal{Y} = \{0,1\}$, state transition function f and output function g defined by:

$$f(q_i,u) = \begin{cases} q_i & \text{if } i \le 2, u = 0 \\ q_{i+1} & \text{if } i \le 2, u = 1 \\ q_1 & \text{if } i = 3 \end{cases} \qquad g(q_i,u) = \begin{cases} u & \text{if } i \le 2 \\ 0 & \text{if } i = 3 \end{cases}$$

Let $\rho : \{0,1\} \to \mathbf{R}_+$, $\mu : \{0,1\} \to \mathbf{R}_+$ be the identity maps. M is ρ/μ gain stable with gain 1. Stability can be verified using Theorem 4.4 by finding a feasible solution ($V = 0$) to the LP:

$$\begin{aligned} V_i - V_i &\le 0, \ i \in \{1,2\} \\ V_1 - V_3 &\le 0 \end{aligned}$$

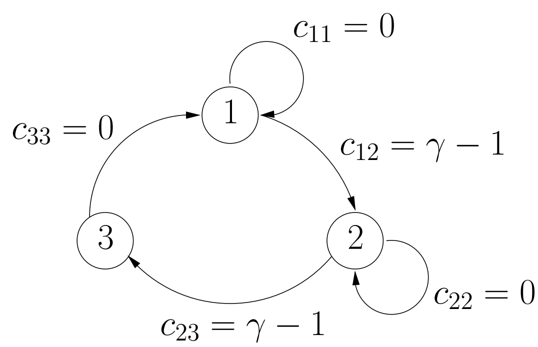

Figure 4-2: The network graph G_γ^M associated with M in Example 4.1

The gain can be computed using Corollary 4.3 by finding the optimal solution ($\gamma = 1$) of the LP:

$$\begin{array}{lll} \min & \gamma & \\ \text{subject to} & V_1 - V_1 & \leq \; 0 \\ & V_2 - V_1 - \gamma & \leq \; -1 \\ & V_2 - V_2 & \leq \; 0 \\ & V_3 - V_2 - \gamma & \leq \; -1 \\ & V_1 - V_3 & \leq \; 0 \\ & V_1 - V_3 - \gamma & \leq \; 0 \end{array}$$

Alternatively, G_γ^M with $\mathcal{N} = \{1,2,3\}$, $\mathcal{E} = \{(1,1),(1,2),(2,2),(2,3),(3,1)\}$ and $c_{11}(\gamma) = c_{22}(\gamma) = 0$, $c_{12}(\gamma) = c_{23}(\gamma) = \gamma - 1$, $c_{33}(\gamma) = \min\{0,\gamma\} = 0$ can be considered (Figure 4-2). Suppose we run the Gain Computation Algorithm with $\epsilon = 0.1$: GCA will first verify that G_γ^M has finite gain bounded above by 3. It will then run successive iterations of the GVA, in which it verifies that: $\gamma = 1.5$ is a gain bound, $\gamma = 0.75$ is not, $\gamma = 1.125$ is, $\gamma = 0.9375$ is not, $\gamma = 1.03125$ is, at which point it exits since a gain bound with the desired level of accuracy has been verified. $\triangledown$

Chapter 5

Synthesizing Robust Feedback Controllers for Finite State Machines

5.1 Introduction

This chapter addresses the problem of synthesizing controllers for a finite state machine system in order to achieve a given performance objective in the presence of admissible uncertainty, where both the objective and the uncertainty are described in terms of gain stability. In particular, the problem considered here is the full state feedback problem, due to its relevance in the context of the hybrid design paradigm advocated in this thesis. The assumption is that prior effort, as described in Chapter 3, has been put into finding a deterministic finite state machine approximation of the hybrid system of interest, and thus the state of the nominal model is available to the controller.

Problems of robust control synthesis for automata models were considered in [9, 52]. While the problems considered were slightly different in formulations (output feedback problem in the first reference, randomized full state feedback in the second), the results derived there and in this chapter are similarly rooted in principles of Dynamic Programming [12, 13]. Controller synthesis essentially reduces to solving a Bellman equation or inequality.

A formal statement of the robust control synthesis problem is given in Section 5.2. The 'Small Gain' Theorem derived in Chapter 2 is used in Section 5.3 to formulate a design objective for the nominal closed loop system so as to ensure robust performance of the actual closed loop system. The analytical description and a computational solution of the resulting design problem are presented in Section 5.4. The development in Sections 5.2-5.4 applies to arbitrary deterministic finite state machine models. In Section 5.5, nominal models with the particular structure described in Section 3.3.3 are revisited. In this case, the resulting controller for the original system with finite actuation and sensing is shown to consist of a deterministic finite state machine estimator derived in Section 3.4 and the corresponding full state feedback control law designed in Section 5.4.

5.2 Problem Statement

Consider a deterministic finite state machine M with state set $\mathcal{Q}$, control, exogenous and disturbance inputs u, $\hat{r}$ and w, and outputs $\hat{v}$ (performance) and z, defined by state transition equation (5.1) and output equations (5.2) and (5.3):

$$q(t+1) = f(q(t), u(t), \hat{r}(t), w(t)) \tag{5.1}$$

$$z(t) = g_1(q(t), u(t), \hat{r}(t), w(t)) \tag{5.2}$$

$$\hat{v}(t) = g_2(q(t), u(t), \hat{r}(t), w(t)) \tag{5.3}$$

Consider also a family of systems $\underline{\Delta}$ defined by:

$$\underline{\Delta} = \{\Delta \subset \mathcal{Z}^{\mathbf{Z}_+} \times \mathcal{W}^{\mathbf{Z}_+} | \inf_{T \geq 0} \sum_{t=0}^{T} \gamma \rho_\Delta(z(t)) - \mu_\Delta(w(t)) > -\infty \ \forall (\mathbf{z}, \mathbf{w}) \in \Delta\}$$

where γ is a given positive constant.

Given some functions $\rho_o : \mathcal{Z} \rightarrow \mathbf{R}_+$ and $\mu_o : \mathcal{W} \rightarrow \mathbf{R}_+$, the goal is to design a full state feedback controller $\varphi : \mathcal{Q} \rightarrow \mathcal{U}$ so that the closed loop system (Δ, M, φ) (Figure 5-1) satisfies the performance objective:

$$\inf_{T \geq 0} \sum_{t=0}^{T} \rho_o(\hat{r}(t)) - \mu_o(\hat{v}(t)) > -\infty \tag{5.4}$$

for any $\Delta \in \underline{\Delta}$.

This goal is achieved by:

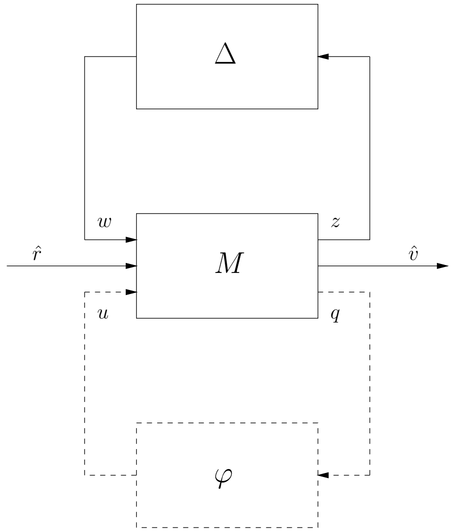

Figure 5-1: Robust DFM full state feedback control problem

1. Posing a robust control design problem for (M, φ): in other words, formulating a design objective for the nominal closed loop system (M, φ) that ensures that the objective (5.4) is met by the actual closed loop system for all admissible uncertainties.

2. Solving the design problem formulated.

These two problems are addressed in Sections 5.3 and 5.4, respectively.

5.3 Posing a Robust Control Design Problem

For the problem stated in Section 5.2, consider the functions $\rho_S : \hat{\mathcal{R}} \times \mathcal{W} \to \mathbf{R}$ and $\mu_S : \hat{\mathcal{V}} \times \mathcal{Z} \to \mathbf{R}$, parametrized by scaling factor $\tau_d > 0$, defined by:

$$\rho_S(\hat{r}, w) = \rho_o(\hat{r}) + \tau_d \; \mu_\Delta(w) \tag{5.5}$$

$$\mu_S(\hat{v}, z) = \mu_o(\hat{v}) + \tau_d \; \gamma \rho_\Delta(z) \tag{5.6}$$

It follows from the 'Small Gain' Theorem (Theorem 2.1) that if the closed loop system (M, φ) is ρ_S/μ_S gain stable, meaning all its feasible signals satisfy:

$$\inf_{T \geq 0} \sum_{t=0}^{T} \rho_S(\hat{r}(t), w(t)) - \mu_S(\hat{v}(t), z(t)) > -\infty \tag{5.7}$$

then the perturbed closed loop system (Δ, M, φ) satisfies the performance objective in (5.4) for all admissible uncertainties Δ. Thus, a solution to the following problem is sought:

Problem: Find a control law $\varphi : \mathcal{Q} \to \mathcal{U}$ and a scaling parameter $\tau_d > 0$ such that all feasible signals of the nominal closed loop system (M, φ) satisfy (5.7), for ρ_S and μ_S defined as in (5.5) and (5.6).

5.4 Solving the Robust Control Design Problem

The solution proposed to the problem posed in Section 5.3 is iterative; non-negative values of τ_d are are sampled until an appropriate scale is found, that is, one for which a corresponding switching law φ that achieves the design objective exists. The problem of verifying whether a suitable control law φ exists for a given value of τ_d reduces to solving a min-max formulation of the Bellman Inequality. This is presented in Section 5.4.1. A solution to the Bellman Inequality can be computed using a Value Iteration Algorithm. For the sake of completeness, a description of the algorithm and a proof of convergence are given in Section 5.4.2.

5.4.1 An Analytical Formulation

For the remainder of this chapter, the notation $\sigma_{\tau_d}(q, u, \hat{r}, w)$ will be used to denote the function:

$$\begin{aligned} \sigma_{\tau_d}(q, u, \hat{r}, w) &= \rho_S(\hat{r}, w) - \mu_S(\hat{v}, z) \\ &= \rho_S(\hat{r}, w) - \mu_S(g_2(q, u, \hat{r}, w), g_1(q, u, \hat{r}, w)) \\ &= \rho_o(\hat{r}) - \mu_o(g_2(q, u, \hat{r}, w)) + \tau_d\Big(\mu_\Delta(w) - \gamma\rho_\Delta(g_1(q, u, \hat{r}, w))\Big) \end{aligned}$$

parametrized by scale factor τ_d.

Theorem 5.1. *Consider a deterministic finite state machine M defined by (5.1), (5.2) and (5.3), and let $\sigma_{\tau_d} : \mathcal{Q} \times \mathcal{U} \times \hat{\mathcal{R}} \times \mathcal{W} \to \mathbf{R}$ be a given map. The following two statements are equivalent:*

(a) There exists a $\varphi : \mathcal{Q} \to \mathcal{U}$ such that the closed loop system (M, φ) satisfies:

$$\inf_{T \geq 0} \sum_{t=0}^{T} \sigma_{\tau_d}(q(t), u(t), \hat{r}(t), w(t)) > -\infty \tag{5.8}$$

(b) There exists a function $J : \mathcal{Q} \to \mathbf{R}$ such that for any $q \in \mathcal{Q}$, the following inequality holds:

$$J(q) \geq \min_{u \in \mathcal{U}} \max_{w \in \mathcal{W}, \hat{r} \in \hat{\mathcal{R}}} \{-\sigma_{\tau_d}(q, u, \hat{r}, w) + J(f(q, u, \hat{r}, w))\} \tag{5.9}$$

Proof. (b) ⇒ (a): suppose there exists a function $J : \mathcal{Q} \to \mathbf{R}$ satisfying (5.9), and let:

$$\varphi(q) = u^*(q) = \arg\min_{u \in \mathcal{U}} \{ \max_{w \in \mathcal{W}, \hat{r} \in \hat{\mathcal{R}}} \Big(-\sigma_{\tau_d}(q, u, \hat{r}, w) + J(f(q, u, \hat{r}, w)) \Big) \}$$

We have, for any q:

$$\begin{aligned} J(q) &\geq \max_{w \in \mathcal{W}, \hat{r} \in \mathcal{R}} \{-\sigma_{\tau_d}(q, \varphi(q), \hat{r}, w) + J(f(q, \varphi(q), \hat{r}, w))\} \\ &\geq -\sigma_{\tau_d}(q, \varphi(q), \hat{r}, w) + J(f(q, \varphi(q), \hat{r}, w)), \ \forall w, \hat{r} \end{aligned}$$

It follows from Theorem 4.1 that (5.8) is satisfied.

(a) ⇒ (b): suppose there exists a φ such that the closed loop system satisfies (5.8). Equivalently, there exists a $J : \mathcal{Q} \to \mathbf{R}$ such that:

$$J(f(q, \varphi(q), \hat{r}, w)) - J(q) \leq \sigma_{\tau_d}(q, \varphi(q), \hat{r}, w), \ \forall q, w, \hat{r}$$

We then have:

$$\begin{aligned} J(q) &\geq -\sigma_{\tau_d}(q, \varphi(q), \hat{r}, w) + J(f(q, \varphi(q), \hat{r}, w)), \ \forall q, w, \hat{r} \\ &\geq \max_{w \in \mathcal{W}, \hat{r} \in \hat{\mathcal{R}}} \{-\sigma_{\tau_d}(q, \varphi(q), \hat{r}, w) + J(f(q, \varphi(q), \hat{r}, w))\}, \ \forall q \\ &\geq \min_{u \in \mathcal{U}} \max_{w \in \mathcal{W}, \hat{r} \in \hat{\mathcal{R}}} \{-\sigma_{\tau_d}(q, u, \hat{r}, w) + J(f(q, u, \hat{r} w))\}, \ \forall q \end{aligned}$$

□

Thus, for a given value of $\tau_d > 0$, a switching law with the desired properties exists iff there exists a storage function (or cost-to-go function in the terminology of Dynamic Programming) satisfying (5.9).

5.4.2 A Computational Solution

Consider $T : \mathbf{R}^{\mathcal{Q}} \to \mathbf{R}^{\mathcal{Q}}$ defined by:

$$T(J(q)) \doteq \min_{u \in \mathcal{U}} \max_{w \in \mathcal{W}, \hat{r} \in \hat{\mathcal{R}}} \{-\sigma_{\tau_d}(q, u, \hat{r}, w) + J(f(q, u, \hat{r}, w))\} \tag{5.10}$$

Theorem 5.2. ***(Value Iteration)*** *Consider a deterministic finite state machine M defined by (5.1), (5.2) and (5.3), and let $\sigma_{\tau_d} : \mathcal{Q} \times \mathcal{U} \times \hat{\mathcal{R}} \times \mathcal{W} \to \mathbf{R}$ be a given map. The following two statements are equivalent:*

(a) There exists a function $J : \mathcal{Q} \to \mathbf{R}$ such that for any $q \in \mathcal{Q}$, the following inequality holds:

$$J(q) \geq \min_{u \in \mathcal{U}} \max_{w \in \mathcal{W}, \hat{r} \in \hat{\mathcal{R}}} \{-\sigma_{\tau_d}(q, u, \hat{r}, w) + J(f(q, u, \hat{r}, w))\} \tag{5.11}$$

(b) The sequence of functions $J_k : \mathcal{Q} \to \mathbf{R}$, $k \in \mathbf{Z}_+$, defined recursively by:

$$\begin{aligned} J_0 &= 0 \\ J_{k+1} &= \max\{0, T(J_k)\} \end{aligned} \tag{5.12}$$

converges.

The Value Iteration algorithm defined in (5.12) is used to solve the Bellman inequality in (5.11) for the cost-to-go function J; the desired switching law is then simply the optimizing argument. While in theory, the Value Iteration algorithm can take infinitely long to converge even when a solution exists, in practice it seems to perform quite well.

The following intermediate statements will be used in proving Theorem 5.2.

Lemma 5.3. *$T : \mathbf{R}^{\mathcal{Q}} \to \mathbf{R}^{\mathcal{Q}}$ defined in (5.10) is monotonic:*

$$J_1 \leq J_2 \Rightarrow T(J_1) \leq T(J_2)$$

Proof. Suppose that $J_1 \leq J_2$. We have:

$$\begin{aligned}
& \sigma_{\tau_d}(q,u,\hat{r},w) + J_1(f(q,u,\hat{r},w)) \leq \sigma_{\tau_d}(q,u,\hat{r},w) + J_2(f(q,u,\hat{r},w)), \forall q,u,w,\hat{r} \\
\Rightarrow\ & \max_{w,\hat{r}}\{\sigma_{\tau_d}(q,u,\hat{r},w) + J_1(f(q,u,\hat{r},w))\} \leq \max_{w,\hat{r}}\{\sigma_{\tau_d}(q,u,\hat{r},w) + J_2(f(q,u,\hat{r},w))\}, \forall q,u \\
\Rightarrow\ & \min_u \max_{w,\hat{r}}\{\sigma_{\tau_d}(q,u,\hat{r},w) + J_1(f(q,u,\hat{r},w))\} \leq \min_u \max_{w,\hat{r}}\{\sigma_{\tau_d}(q,u,\hat{r},w) + J_2(f(q,u,\hat{r},w))\} \\
\Rightarrow\ & T(J_1) \leq T(J_2)
\end{aligned}$$

□

Lemma 5.4. *Sequence $\{J_k\}$ is monotonically increasing.*

Proof. By induction on k. We have:

$$J_1 = \max\{0, T(J_0)\} \geq 0 = J_0$$

Suppose $J_k \geq J_{k-1}$. Then:

$$\begin{aligned}
J_{k+1} &= \max\{0, T(J_k)\} \\
&\geq \max\{0, T(J_{k-1})\} \\
&\geq J_k
\end{aligned}$$

□

For any $c \in \mathbf{R}$, denote by $J+c$ the map from $\mathbf{R}^{\mathcal{Q}}$ to $\mathbf{R}^{\mathcal{Q}}$ defined by: $(J+c)(q) \doteq J(q)+c$.

Lemma 5.5. *$T(J+c) = T(J) + c$, for any $c \in \mathbf{R}$.*

Proof. We have, for any $q \in \mathcal{Q}$:

$$\begin{aligned}
T((J+c)(q)) &= \min_{u\in\mathcal{U}} \max_{w\in\mathcal{W},\hat{r}\in\hat{\mathcal{R}}} \{\sigma_{\tau_d}(q,u,\hat{r},w) + (J+c)(f(q,u,\hat{r},w))\} \\
&= \min_{u\in\mathcal{U}} \max_{w\in\mathcal{W},\hat{r}\in\hat{\mathcal{R}}} \{\sigma_{\tau_d}(q,u,\hat{r},w) + J(f(q,u,\hat{r},w)) + c\} \\
&= \min_{u\in\mathcal{U}} \max_{w\in\mathcal{W},\hat{r}\in\hat{\mathcal{R}}} \{\sigma_{\tau_d}(q,u,\hat{r},w) + J(f(q,u,\hat{r},w))\} + c \\
&= T(J(q)) + c
\end{aligned}$$

□

We are now ready to prove Theorem 5.2.

Proof of Theorem 5.2: (a) $\Rightarrow$ (b): suppose there exists a function $J : \mathcal{Q} \to \mathbf{R}$ satisfying (5.11). Then $J + c$ also satisfies (5.11) for any choice of $c \in \mathbf{R}$. Since the set $\mathcal{Q}$ is finite, we can assume, without loss of generality, that $J \geq 0$ with $\min_q J(q) = 0$ and $\max_q J(q) = m$, for some $m \geq 0$. Moreover, the sequence $\{J_k\}$ is bounded above by J. The proof is by induction on k. We have:

$$J_0 = 0 \leq J$$

Suppose that $J_k \leq J$. Then:

$$T(J_k) \leq T(J) \leq J$$

where the first inequality follows from Lemma 5.3 and the second inequality follows from (5.11). Thus,

$$J_{k+1} = \max\{0, T(J_k)\} \leq J$$

where the inequality follows from the non-negativity of J. Thus, sequence $\{J_k\}$ is (pointwise) monotonically increasing (Lemma 5.4) and bounded above by J. Hence, it converges and $J^* = \lim_{k\to\infty} J_k$ satisfies $J^* \leq J$.

(b) $\Rightarrow$ (a): suppose that the sequence $\{J_k\}$ converges and let $J^* = \lim_{k\to\infty} J_k$. Since $\mathcal{Q}$ is finite, for every k there exists an ϵ_k such that:

$$J_k(q) \geq J^*(q) - \epsilon_k, \text{ for all } q$$

Thus, it follows from Lemmas 5.3 and 5.5 that:

$$T(J_k) \geq T(J^* - \epsilon_k) = T(J^*) - \epsilon_k$$

But:

$$\begin{aligned} J_{k+1} &\geq T(J_k) \\ &\geq T(J^*) - \epsilon_k, \text{ for all } k \end{aligned}$$

Thus:

$$\lim_{k\to\infty} J_{k+1} \geq T(J^*) - \lim_{k\to\infty} \epsilon_k$$

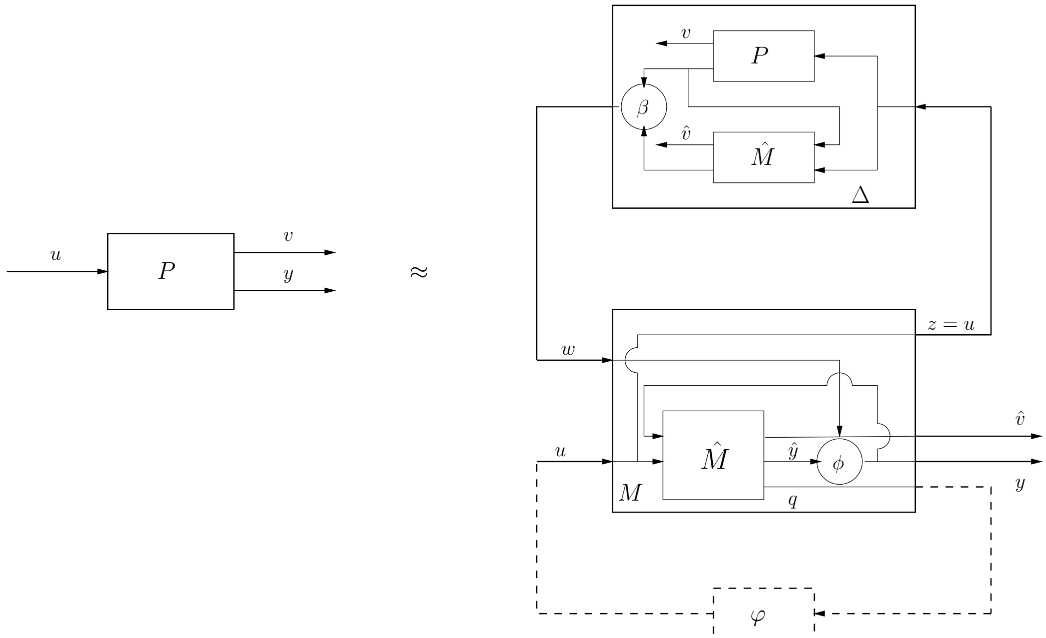

Figure 5-2: Full state feedback controller for ρ/μ approximation of P

and hence: $J^* \geq T(J^*)$. □

5.5 Structure of the Robust Controller

Consider the setup shown in Figure 5-2, where M is a ρ/μ approximation of a given system P as in Definition 3.1, with $\hat{M}$ constructed using either of the two approaches described in Section 3.4, and where Δ is the resulting approximation uncertainty. Suppose that a controller $\varphi : \mathcal{Q} \rightarrow \mathcal{U}$, where $\mathcal{Q}$ is the state set of $\hat{M}$, has been designed so that the closed loop system satisfies a given auxiliary performance objective, as in (3.1).

The block diagram of interconnection (M, Δ, φ) can be equivalently re-drawn as shown in Figure 5-3 (where the two copies of $\hat{M}$ are assumed to be identically initialized in line with the recommendations in Section 3.4). The internal structure of the resulting controller $K \subset \mathcal{Y}^{\mathbf{Z}_+} \times \mathcal{U}^{\mathbf{Z}_+}$, for which the closed loop system (P, K) satisfies the performance objective (3.2) by design, thus consists of a finite state estimator $\hat{M}$ for the hybrid system and a full state feedback switching law φ.

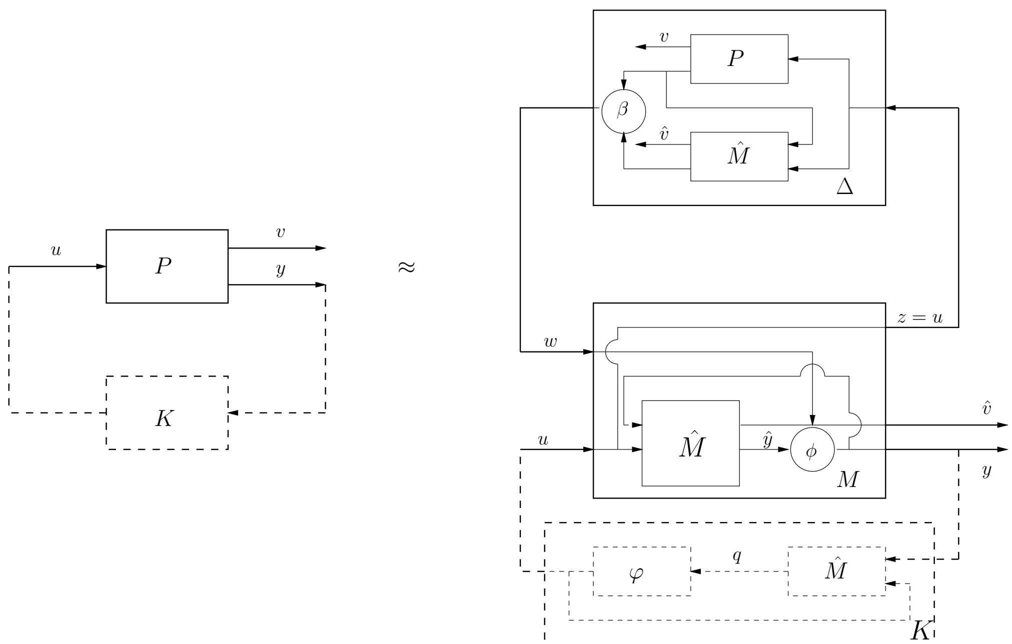

Figure 5-3: The deterministic finite state machine controller K

Chapter 6

A Simple Benchmark Example

6.1 Introduction

Switched systems are a special class of hybrid systems, consisting of a family of plants and a law for switching among them. The problem of finding a stabilizing switching law for a family of unstable linear systems has received much attention in the past decade [43, 66, 78, 82]. An overview of recent results in this area can be found in [44]. Typically, the controllers implementing the switching strategy are either assumed to have full access to the state or to have access to a sensor output that is a linear function of the state. However, in many practical applications, there may be a need or a desire to base the design and implementation of the switching controller on coarse, discrete sensing. For instance, coarse sensors may be used to keep operating costs low, or to minimize power consumption when power is scarce. When control is done over a network, the use of coarse sensors decreases the amount of information to be encoded and transmitted over communication channels. The problem of stabilizing a family of switched unstable linear systems where only very coarse sensing is available has not been as extensively studied to date. In this Chapter, an academic benchmark problem of this type is addressed using the paradigm developed in this thesis.

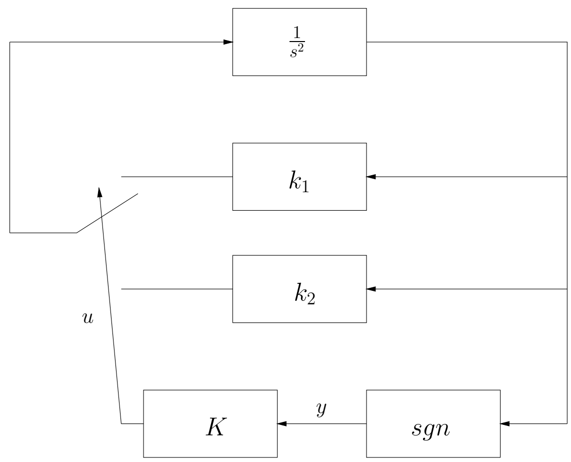

Figure 6-1: Double integrator with switched static feedback

Consider a double integrator described by the following state-space equations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ y &= x_1 \end{aligned}$$

This system cannot be stabilized using static output feedback ($u = ky$) regardless of the choice of gain $k \in \mathbf{R}$. However, it can be stabilized by cleverly switching between two appropriately chosen static output feedback laws [7, 43], assuming the switching controller has *full* access to the state. The question addressed here is the following (Figure 6-1): is it possible to stabilize the system by switching between two appropriately chosen static output feedback laws *based on very coarse sensing, namely only the sign of the position measurement*? Related questions were considered in [52, 65], where it was shown that stabilization based on coarse sensing is in fact possible, assuming the sensor output is available at fixed time intervals of length T (a design parameter) and that a gain is chosen at the beginning of each interval and held until the next measurement becomes available. In this Chapter, a constructive procedure for synthesizing a switching controller with guaranteed performance is presented, based on the framework developed in the thesis.

6.2 Problem Statement

Consider a second order discrete-time switched system P described by:

$$\begin{aligned} x(t+1) &= A_T(u(t))x(t) \\ y(t) &= sgn(x_1(t)) \\ v(t) &= log\Big(\frac{\|x(t+1)\|_2}{\|x(t)\|_2}\Big) \end{aligned} \tag{6.1}$$

where $t \in \mathbf{Z}_+$ (the set of non-negative integers) and $\|\cdot\|_2$ denotes the euclidean norm. State $x(t) \in \mathbf{R}^2$ and output $v(t) \in \mathbf{R}$, while control input u and sensor output y are binary with $u(t) \in \mathcal{U} = \{0, 1\}$ and $y(t) \in \mathcal{Y} = \{-1, 1\}$[1]. $A_T(1)$ and $A_T(2) \in \mathbf{R}^{2\times 2}$ are of the form $A_T(u) = e^{\overline{A}(u)T}$ where:

$$\overline{A}(u) = \begin{bmatrix} 0 & 1 \\ k(u) & 0 \end{bmatrix}$$

System P is thus simply a sampled version of the switched analog system with binary input u (corresponding to a binary choice of the gain) and output y shown in Figure 6-1. The sensor measurement y is assumed to be available at the beginning of each sampling interval $[tT, (t+1)T)$, $t \in \mathbf{Z}_+$, at which point a gain is chosen by the controller K and held until the end of the current sampling interval.

The objective is to design a sampling rate $T > 0$, a map $k : \{0, 1\} \to \mathbf{R}$ and a controller $K \subset \mathcal{Y}^{\mathbf{Z}_+} \times \mathcal{U}^{\mathbf{Z}_+}$ such that the closed loop system (P, K) with output v (Figure 6-2) is exponentially stable:

$$\|x(T)\|_2 \le 10^{-RT}\|x(0)\|_2,\ \forall T \in \mathbf{Z}_+ \tag{6.2}$$

Ideally, we want to maximize the rate of exponential convergence of the actual closed loop system[2].

[1] The following convention will be adopted: $y(t) = 1$ if $x_2(t) > 0$ and $y(t) = -1$ if $x_2(t) < 0$

[2] Note that the rate of convergence of the sampled system is R, with the corresponding rate of the actual closed loop system being R/T.

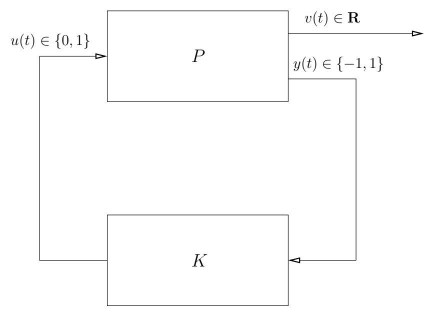

Figure 6-2: Closed loop system

6.3 Preliminary Observations

6.3.1 Gain Stability to Describe Performance Objectives

Note that the performance objective (exponential stability) given in (6.2), can be equivalently described in terms of gain stability of the closed loop system, where $\rho(r) = 0$ (since the system has no exogenous input) and $\mu(v) = v + R$. In order for this gain stability property to be physically meaningful, we require $R > 0$. The performance objective is then:

$$\inf_{T \geq 0} \sum_{t=0}^{T} -(v(t) + R) < \infty \tag{6.3}$$

6.3.2 Homogeneity of the System

Some useful properties of the system under consideration become clearer when the dynamics are expressed in polar coordinates. Consider a coordinate transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by:

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} \sqrt{x_1^2 + x_2^2} \\ \tan^{-1}(\frac{x_2}{x_1}) \end{bmatrix}$$

with $r \in [0, +\infty)$, $\theta \in [0, 2\pi)$.

The dynamics of the plant P described in (6.1) in the new coordinate system are given by:

$$\theta(t+1) = \tan^{-1}\left(\frac{a_{21}(u(t)) + a_{22}(u(t))\tan(\theta(t))}{a_{11}(u(t)) + a_{12}(u(t))\tan(\theta(t))}\right) \tag{6.4}$$

$$r(t+1) = r(t)\sqrt{\beta'(t)A'_T(u(t))A_T(u(t))\beta(t)}$$

$$y(t) = sgn(\cos(\theta(t))) \tag{6.5}$$

$$v(t) = log\Big(\sqrt{\beta'(t)A'_T(u(t))A_T(u(t))\beta(t)}\Big) \tag{6.6}$$

where

$$A_T(u) = \begin{bmatrix} a_{11}(u) & a_{12}(u) \\ a_{21}(u) & a_{22}(u) \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{\infty} k^n(u)\frac{T^{2n}}{2n!} & \sum_{i=0}^{\infty} k^n(u)\frac{T^{2n+1}}{(2n+1)!} \\ \sum_{i=0}^{\infty} k^{n+1}(u)\frac{T^{2n+1}}{(2n+1)!} & \sum_{i=0}^{\infty} k^n(u)\frac{T^{2n}}{2n!} \end{bmatrix}$$

$$\beta(t) = \begin{bmatrix} cos(\theta(t)) \\ sin(\theta(t)) \end{bmatrix}$$

This highlights an important property of the system that will be useful when designing the switching controller K, namely that the evolution of the angular coordinate, as well as the values of both outputs of system P, are independent of the radial coordinate. Effectively, the dynamics of the system evolve along the unit circle.

6.3.3 The Choice of Feedback Gains

The following two observations about harmonic oscillators will be useful in guiding the choice of $k : \{0, 1\} \to \mathbf{R}$. Consider a harmonic oscillator described by:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= kx_1 \end{aligned}$$

with corresponding equations in polar coordinates given by:

$$\begin{aligned} \dot{r} &= (1+k)r\sin(\theta)\cos(\theta) \\ \dot{\theta} &= (1+k)\cos^2(\theta) - 1 \end{aligned}$$

Remark 6.1 For $k = -1$, $\dot{r} = 0$ and the state trajectories are concentric circles centered at the origin. Thus, the output v of the corresponding sampled system is identically zero, regardless of the choice of sampling rate T. $\diamond$

Remark 6.2 For any $k \neq -1$, there exists two quadrants of the state space, corresponding to:

$$\{\theta \in [0, 2\pi) | sgn(\cos(\theta)\sin(\theta)) + sgn(1+k) = 0)\}$$

such that $\dot{r} < 0$. ◇

Based on these two remarks, it should always be possible in principle to stabilize the analog system by appropriately switching between gains $k_1 = -1$ and $k_2 \in \mathbf{R}\backslash\{-1\}$. In this case, switching to k_1 corresponds to "passive control" in anticipation of switching to k_2, corresponding to "aggressive stabilization", in the appropriate sectors. However, the problem is much harder when switching can only occur at discrete time instants and when the problem is not a full state feedback problem, as is the case here. In particular, design of the discrete switching law will be based on the assumption that $k : \{0, 1\} \to \mathbf{R}$ satisfies $k(0) = -1$ and $k(1) \in \mathbf{R}\backslash\{-1\}$.

6.3.4 Other Remarks

This system is externally unstable. Consider two initial conditions $\theta_1(0) = \pi/2$ and $\theta_2(0) = -\pi/2$, and input u that is identically zero. (i.e. the gain $k = -1$ is used). The outputs of the system are unequal at every time step. Thus without the observer structure proposed in Section 3.3.3, there is no hope for finding a ρ/μ approximation of this system.

6.4 A Finite State Machine Approximation of the Plant

Design of the stabilizing controller is an iterative procedure, with each iteration consisting of several steps. First, P is approximated by the feedback interconnection of a finite state machine M and a system Δ representing the approximation error (Figure 6-3), using the procedure described in Section 3.4.2. Next, a gain bound is established for the error system Δ using the approach described in Section 3.5.2. Finally, an attempt is made to synthesize a full state feedback switching law for the nominal finite state model M that is robust to approximation errors, as described in Chapter 5. If synthesis is successful, the resulting controller is guaranteed to stabilize the system at some verified exponential rate R. Otherwise, a better approximation (meaning a finite state machine with a larger number of states) is sought for the switched system and the above process is repeated.

In the following sections, the details of the procedure for generating a finite state machine $\hat{M}$, and hence a nominal model M of the sampled system P described by (6.4), (6.5) and (6.6), are briefly described, as well as the algorithm for computing an upper bound on the gain of the resulting error system Δ. A guideline for choosing the sampling rate T is

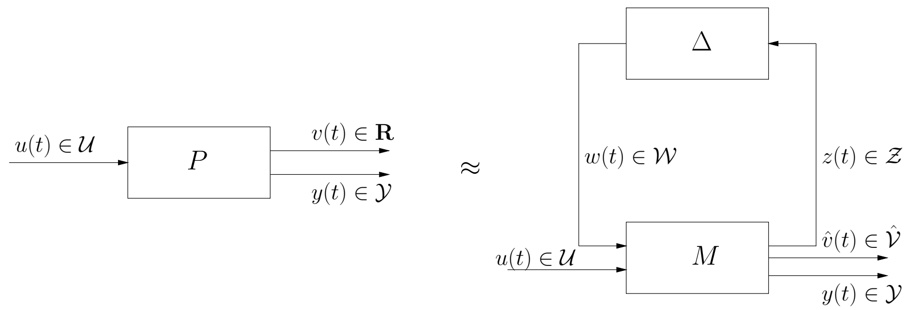

Figure 6-3: Plant P and its finite state machine approximation M

presented, with the goal of mitigating the conservatism of the gain bound that is verified for Δ.

6.4.1 Construction of the Nominal Model

As was pointed out in Section 6.3.2, the state of plant P effectively evolves on the unit circle. $\hat{M}$ is constructed by uniformly quantizing the unit circle, taking care that no quantization interval crosses the vertical axis. The quantization intervals are then the partition $X_1,...,X_p$ as described in Section 3.4.2. The states of $\hat{M}$ are then defined to be the quantization intervals and unions of adjacent quantization intervals. Due to the continuity of the system, non-adjacent quantization intervals can be immediately ruled out as potential states. The set of states $\mathcal{Q}$ of $\hat{M}$ is then:

$$\mathcal{Q} \doteq \cup_{k=1}^{n-1} \Big(\cup_{i=1}^{n} \{q_{i,k}\} \Big) \cup \{q_{n,n}\}$$

where state $q_{i,k}$ is associated with interval $I_{i,k}$ defined as:

$$I_{i,k} \doteq [(i-1)\frac{2\pi}{n} + \frac{\pi}{2}, (i+k-1)\frac{2\pi}{n} + \frac{\pi}{2})$$

Positive integer n is a design parameter; the total number of states of $\hat{M}$ is at most $n(n-1)+1$.

Let $I_q = [\theta_q^1, \theta_q^2)$ be the interval associated with $q \in \mathcal{Q}$, and let $|I_q|$ denote the length of interval I_q, that is $|I_q| \doteq |\theta_q^2 - \theta_q^1|$. $\mathcal{Q}$ can be partitioned into three sets:

$$\begin{aligned}
\mathcal{Q}_1 &\doteq \{q \in \mathcal{Q} | I_q \subset [\alpha, \alpha + \pi)\} \\
\mathcal{Q}_2 &\doteq \{q \in \mathcal{Q} | I_q \subset [\pi - \alpha, \alpha)\} \\
\mathcal{Q}_3 &\doteq \mathcal{Q} \backslash (\mathcal{Q}_1 \cup \mathcal{Q}_2)
\end{aligned}$$

$\mathcal{Q}_1$ and $\mathcal{Q}_2$ are thus the sets of states with sensor outputs -1 and 1 respectively, while $\mathcal{Q}_3$ is the set of states with ambiguous output. Consider functions $P_1 : \mathcal{Q}_3 \to \mathcal{Q}_1$ and $P_2 : \mathcal{Q}_3 \to \mathcal{Q}_2$ defined by $P_1(q) = q_1$ and $P_2(q) = q_2$, where (q_1, q_2) are the unique pair of elements in $\mathcal{Q}_1 \times \mathcal{Q}_2$ with the property that $I_q = I_{q_1} \cup I_{q_2}$. Let:

$$\mathcal{A}_u^q = \{q \in \mathcal{Q} | I_q \supset [\tan^{-1}\Big(\frac{a_{21}(u) + a_{22}(u)\tan(\theta_q^1)}{a_11(u) + a_{12}(u)\tan(\theta_q^1)}\Big), \tan^{-1}\Big(\frac{a_{21}(u) + a_{22}(u)\tan(\theta_q^2)}{a_11(u) + a_{12}(u)\tan(\theta_q^2)}\Big))\}$$

Consider function $\phi : (\hat{Q}_1 \cup \hat{Q}_2) \times \mathcal{U} \to \hat{Q}$ defined by:

$$\phi(q, u) = \arg \min_{q \in \mathcal{A}_u^q} |I_q|$$

The dynamics of $\hat{M}$ can thus be described by:

$$\begin{aligned}
q(t+1) &= f(q(t), u(t), \hat{r}(t)) \\
\hat{y}(t) &= g_1(q(t)) \qquad (6.7) \\
\hat{v}(t) &= g_2(q(t), u(t))
\end{aligned}$$

where the state transition function $f : \mathcal{Q} \times \mathcal{U} \times \mathcal{Y} \to \mathcal{Q}$ and the output functions $g_1 : \mathcal{Q} \to \mathcal{Y}$ and $g_2 : \mathcal{Q} \times \mathcal{U} \to \mathcal{V}$, constructed as described in Section 3.4.2, are defined by:

$$f(q, u, \hat{r}) \doteq \begin{cases} \phi(q, u) & \text{if } q \in \hat{Q}_1 \cup \hat{Q}_2 \\ \phi(P_1(q), u) & \text{if } q \in \hat{Q}_3, r = -1 \\ \phi(P_2(q), u) & \text{if } q \in \hat{Q}_3, r = 1 \end{cases}$$

$$g_2(q, u) \doteq \sup_{\theta \in I_q} \Big(log \sqrt{\beta'(\theta) A_T'(u) A_T(u) \beta(\theta)} \Big)$$

where $\beta(\theta) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$.

$$g_1(q) \doteq \begin{cases} -1 & \text{if } q \in \mathcal{Q}_1 \\ 1 & \text{if } q \in \mathcal{Q}_2 \\ -1 & \text{if } q \in \mathcal{Q}_3 \text{ and } |I_{P_1(q)}| \geq |I_{P_2(q)}| \\ 1 & \text{if } q \in \mathcal{Q}_3 \text{ and } |I_{P_1(q)}| < |I_{P_2(q)}| \end{cases}$$

The state of $\hat{M}$ is initialized to $q(0) = q_{n,n}$, in line with Remark 3.5.

6.4.2 Description of the Approximation Error

The procedure for computing an upper bound on the gain of the error system Δ associated with a given plant P and a corresponding nominal model M, constructed as described in the previous section, consists of searching for an appropriate storage function, as shown in the following proposition.

Proposition 6.1. *If there exists a function $V : \mathcal{Q} \to \mathbf{R}$ and a γ such that:*

$$V(f(q,u,r)) - V(q) \leq \gamma \rho_\Delta(u) - d(q) \tag{6.8}$$

holds for all $q \in \mathcal{Q}$, $u \in \mathcal{U}$ and $r \in \mathcal{Y}$, where $d : \mathcal{Q} \to \{0,1\}$ defined by:

$$d(q) = \begin{cases} 0 & q \in \mathcal{Q}_1 \cup \mathcal{Q}_2 \\ 1 & q \in \mathcal{Q}_3 \end{cases}$$

then the error system Δ is ρ/μ gain stable with gain not exceeding γ, for $\mu_\Delta : \mathcal{W} \to \{0,1\}$.

Proof. Follows from Proposition 3.5 and Theorem 4.1. □

An upper bound for the gain of Δ can thus be computed by solving a linear program in which we minimize γ such that (6.8) holds. If $N = card(\mathcal{Q})$, the linear program has $N + 1$ decision variables and $4N$ inequality constraints, and hence the algorithm grows polynomially with the size of the nominal model.

6.4.3 Choice of the Sampling Rate T

As discussed in Section 3.5, the approach for computing a gain bound for Δ is conservative. A possible way of counteracting the conservatism introduced in the computation of an upper bound for the gain of Δ is by matching the sampling rate to the quantization interval; that is, by setting $T = \frac{2\pi}{n}$.

Proposition 6.2. *When $k = -1$, state transition equation (6.4) reduces to:*

$$\theta(t+1) = \theta(t) - T$$

Proof. When $k = k(0) = -1$, the state transition matrix reduces to:

$$A_T(0) = \begin{bmatrix} \sum_{i=0}^{\infty}(-1)^n \frac{T^{2n}}{2n!} & \sum_{i=0}^{\infty}(-1)^n \frac{T^{2n+1}}{(2n+1)!} \\ \sum_{i=0}^{\infty}(-1)^{n+1} \frac{T^{2n+1}}{(2n+1)!} & \sum_{i=0}^{\infty}(-1)^n \frac{T^{2n}}{2n!} \end{bmatrix} = \begin{bmatrix} cos(T) & sin(T) \\ -sin(T) & cos(T) \end{bmatrix}$$

and the corresponding state transition equation for the angular coordinate is:

$$\begin{aligned} \theta(t+1) &= tan^{-1}\Big(\frac{-sin(T) + cos(T)tan(\theta(t))}{cos(T) + sin(T)tan(\theta(T))}\Big) \\ &= tan^{-1}\Big(\frac{tan(\theta(t)) - tan(T)}{1 + tan(T)tan(\theta(T))}\Big) \\ &= tan^{-1}(tan(\theta(t) - T)) \\ &= \theta(t) - T \end{aligned}$$

□

6.5 Design of the Stabilizing Controller

The objective is to design a switching law $\varphi : \mathcal{Q} \to \mathcal{U}$ such that the closed loop system consisting of the interconnection (M, Δ) in feedback with φ (Figure 6-4) satisfies the following performance objective:

$$\sup_{T \geq 0} \sum_{t=0}^{T} \tilde{v}(t) + R < \infty \qquad (6.9)$$

for some $R > 0$. The largest value of R for which (6.9) holds for a given switching law φ is the largest *guaranteed* logarithmic convergence rate; the actual convergence rate is generally better.

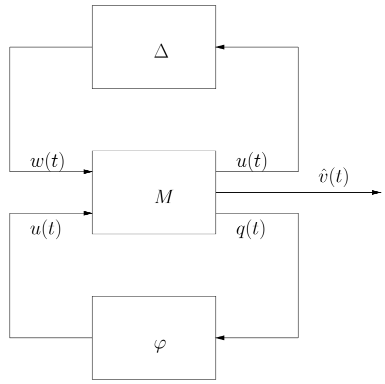

Figure 6-4: Robust control setup

For the particular control problem of interest, the cost function σ_{τ_d} depends on two parameters, R and the scale τ_d. The goal is to maximize R for which there exists a $J : \mathcal{Q} \rightarrow \mathbf{R}$ and a $\tau_d > 0$ such that (5.11) holds. However, it is not possible to directly compute R^*, the optimal value of R. Instead, a search is carried out resulting in a suboptimal value of R: first, the range of values of τ_d for which 'stability' with $R = 0$ is possible is computed. Then, different values of τ_d are sampled in this range, and the largest value of R is computed for each sampled value τ_d, with the largest of those being a suboptimal guaranteed rate of convergence.

6.6 Numerical Examples and Simulations

Example 6.1 Consider the case where the system can switch between gain $k(0) = -1$ (passive control) and gain $k(1) = -3$ (aggressive control). For the case where $\rho : \mathcal{U} \rightarrow \mathbf{R}$ is defined by $\rho(0) = 1$, $\rho(1) = 2$, the smallest value of design parameter n for which we can guarantee convergence is $n = 10$. The number of states of the corresponding stabilizing controller is 91, and the corresponding sampling rate is $T = 0.6283$. The best provable logarithmic rate is $R = 0.016$, based on a computed gain bound $\gamma = 0.375$ for the approximation error Δ. The value iteration algorithm (τ_d and R values are fixed) converges very quickly in this case, in 13 iterations. The result of the implementation of this controller in the system in shown in Figure 6-5.

When the value of n is increased to $n = 12$, with corresponding 133 state controller and sampling rate $T = 0.5236$, the computed value of the gain bound is $\gamma = 0.3333$, and

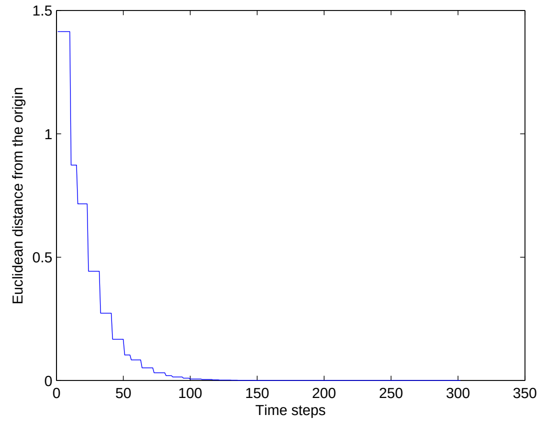

Figure 6-5: Implementation of the DFM controller in Example 6.1 (Sampling time T=0.6283)

the best provable rate of exponential convergence is $R = 0.024$. Again, the value iteration converges quickly, in 16 iterations. The result of the implementation of this controller in the system in shown in Figure 6-6.

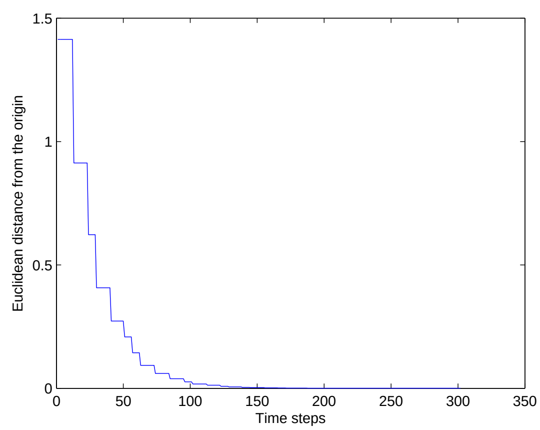

Figure 6-6: Implementation of the DFM controller in Example 6.1 (Sampling time T=0.5236)

When the value of n is further increased to $n = 18$, for instance, with a corresponding 307 state controller and sampling rate $T = 0.3491$, and computed gain bound $\gamma = 0.3571$, the best provable rate is $R = 0.0275$. Again, the value iteration converges relatively quickly, in 24 iterations. The result of the implementation of this controller in the system in shown in Figure 6-7. ∇

Example 6.2 Consider the case where the system can switch between gain $k(0) = -1$

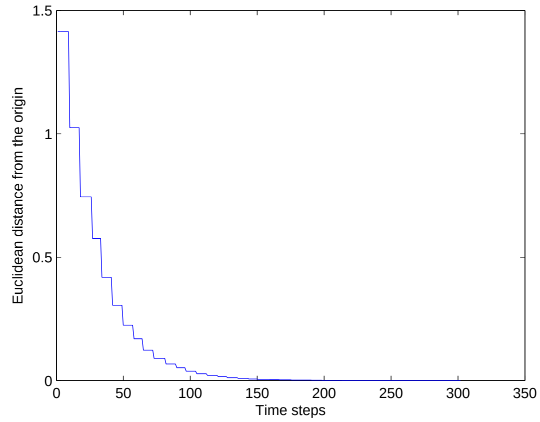

Figure 6-7: Implementation of the DFM controller in Example 6.1 (Sampling time T=0.3491)

(passive control) and gain $k(1) = -0.5$ (aggressive control). Once again, consider the case where $\rho : \mathcal{U} \to \mathbf{R}$ is defined by $\rho(0) = 1$, $\rho(1) = 2$. In this case, the smallest value of design parameter n for which we can guarantee convergence is $n = 8$. The number of states of the corresponding stabilizing controller is 56, and the corresponding sampling rate is $T = 0.7854$. The best provable logarithmic rate is $R = 0.0153$, based on a computed gain bound $\gamma = 0.333$ for the approximation error Δ. The implementation of this controller in the system is shown in Figure 6-8. ∇

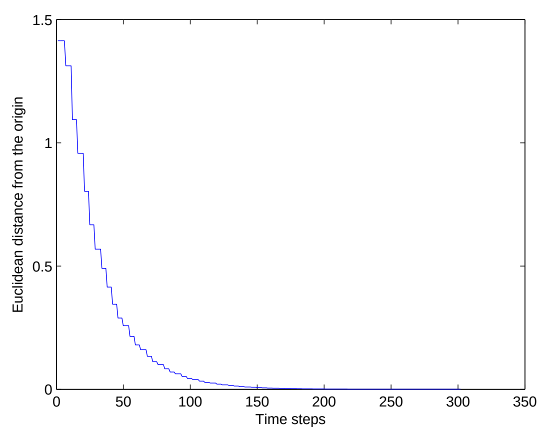

Figure 6-8: Implementation of the DFM controller in Example 6.2 (Sampling time T=0.7854)

Chapter 7

An Incremental Stability Approach

7.1 Introduction

It has long been recognized that finite gain stability, while useful in an LTI setting, is often too weak to be a useful property for general nonlinear systems within a robust analysis framework. Various incremental notions of input/output stability have been proposed and studied as possible alternatives applicable to nonlinear systems within a robust control framework [6, 27, 29, 70]. In a parallel development, this chapter considers incremental stability properties for *systems over finite alphabets*.

7.2 Incremental Stability

A real valued function $d : \mathcal{A} \times \mathcal{A} \to \mathbf{R}$ is said to be *symmetric* if $d(a,b) = d(b,a)$, $\forall a, b \in \mathcal{A}$ and *positive definite* if:

$$d(a,b) \geq 0,\ \forall a, b \in \mathcal{A} \text{ and } d(a,b) = 0 \Leftrightarrow a = b$$

Definition 7.1. *A system over finite alphabets $S \subset \mathcal{U}^{\mathbf{Z}_+} \times \mathcal{Y}^{\mathbf{Z}_+}$ is* ***incrementally stable*** *if there exists a finite non-negative constant γ and a pair of symmetric positive definite functions, $d_{\mathcal{U}} : \mathcal{U} \times \mathcal{U} \to \mathbf{R}_+$ and $d_{\mathcal{Y}} : \mathcal{Y} \times \mathcal{Y} \to \mathbf{R}_+$, such that for any two pairs $(\mathbf{u}_1, \mathbf{y}_1)$ and $(\mathbf{u}_2, \mathbf{y}_2)$ in S, the following inequality is satisfied:*

$$\inf_{T \geq 0} \sum_{t=0}^{T} \gamma d_{\mathcal{U}}(u_1(t), u_2(t)) - d_{\mathcal{Y}}(y_1(t), y_2(t)) > -\infty \tag{7.1}$$

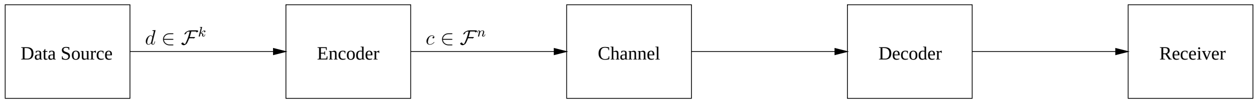

Figure 7-1: Coding setup

Given a particular choice of symmetric positive definite functions $d_{\mathcal{U}}$ and $d_{\mathcal{Y}}$, the greatest lower bound of γ such that (7.1) is satisfied is called the $d_{\mathcal{U}}/d_{\mathcal{Y}}$ ***incremental gain of*** *S.*

By an argument similar to that made in Remark 2.3, it follows that incremental stability (or lack of it) is an intrinsic property of a given system. However, the numerical value of the incremental gain of a stable system depends on the choice of functions $d_{\mathcal{U}}$ and $d_{\mathcal{Y}}$.

Example 7.1 Convolutional codes are widely used to add redundancy to data transmitted over noisy channels so as to enable error free decoding at the receiver end (Figure 7-1). A convolutional encoder is a map $E : (\mathcal{F}^k)^{\mathbf{Z}_+} \to (\mathcal{F}^n)^{\mathbf{Z}_+}$, where $\mathcal{F}$ is a finite field and n and k are integers with $n > k$, such that $\mathcal{C} = E((\mathcal{F}^k)^{\mathbf{Z}_+})$ is a right shift-invariant linear subspace of $(\mathcal{F}^n)^{\mathbf{Z}_+}$. Given a convolutional code $\mathcal{C}$, the problem of finding an encoder for it can be formulated as the problem of finding a state-space realization for an invertible map $\phi_E : \mathcal{C} \to (\mathcal{F}^k)^{\mathbf{Z}_+}$. A good encoder is one that is 'non-catastrophic', among other properties. An encoder is said to be catastrophic if two codewords $\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{C}$ differing by a finite number of terms correspond to two data sequences $\mathbf{d}_1, \mathbf{d}_2 \in (\mathcal{F}^k)^{\mathbf{Z}_+}$ differing by an infinite number of terms. Ensuring that the system over finite alphabets S with feasible signals set $\mathcal{D} = \{(\mathbf{c}, \mathbf{d}) \in \mathcal{C} \times (\mathcal{F}^k)^{\mathbf{Z}_+} | \ \mathbf{d} = \phi_E(\mathbf{c})\}$ is incrementally stable, and hence satisfies:

$$\inf_{T \geq 0} \sum_{t=0}^{T} \gamma d_{\mathcal{U}}(c_1(t), c_2(t)) - d_{\mathcal{Y}}(d_1(t), d_2(t)) > -\infty$$

for some finite $\gamma \geq 0$ and some symmetric positive definite functions $d_{\mathcal{U}} : \mathcal{F}^n \times \mathcal{F}^n \to \mathbf{R}_+$ and $d_{\mathcal{Y}} : \mathcal{F}^k \times \mathcal{F}^k \to \mathbf{R}_+$, allows us to ensure that the corresponding encoder ($E = \phi^{-1}$) is non-catastrophic. $\triangledown$

7.3 An 'Incremental Small Gain' Theorem

The following theorem describes the incremental gain stability properties of an interconnected system in terms of those of its component subsystems. No assumptions are made about systems S and Δ, beyond incremental stability in some particular sense (i.e. the alphabet sets are arbitrary, not necessarily finite in this formulation).

Theorem 7.1. *(An 'Incremental Small Gain' Theorem) Suppose that system S is incrementally stable with $d_{\mathcal{U}_S}/d_{\mathcal{Y}_S}$ incremental gain not exceeding 1, for some symmetric positive definite functions $d_{\mathcal{U}_S} : (\mathcal{U}\times\mathcal{W})\times(\mathcal{U}\times\mathcal{W}) \rightarrow \mathbf{R}_+$ and $d_{\mathcal{Y}_S} : (\mathcal{Y}\times\mathcal{Z})\times(\mathcal{Y}\times\mathcal{Z}) \rightarrow \mathbf{R}_+$. Suppose also that system Δ is incrementally stable with $d_{\mathcal{U}_\Delta}/d_{\mathcal{Y}_\Delta}$ incremental gain not exceeding 1, for some symmetric positive definite functions $d_{\mathcal{U}_\Delta} : \mathcal{Z}\times\mathcal{Z} \rightarrow \mathbf{R}_+$ and $d_{\mathcal{Y}_\Delta} : \mathcal{W}\times\mathcal{W} \rightarrow \mathbf{R}_+$. If functions $d_\mathcal{U} : \mathcal{U}\times\mathcal{U} \rightarrow \mathbf{R}$ and $d_\mathcal{Y} : \mathcal{Y}\times\mathcal{Y} \rightarrow \mathbf{R}$ given by:*

$$d_\mathcal{U}(u_1,u_2) \doteq \sup_{w_1,w_2\in\mathcal{W}} \{d_{\mathcal{U}_S}((u_1,w_1),(u_2,w_2)) - d_{\mathcal{Y}_\Delta}(w_1,w_2)\} \tag{7.2}$$

$$d_\mathcal{Y}(y_1,y_2) \doteq \inf_{z_1,z_2\in\mathcal{Z}} \{d_{\mathcal{Y}_S}((y_1,z_1),(y_2,z_2)) - d_{\mathcal{U}_\Delta}(z_1,z_2)\} \tag{7.3}$$

are positive definite, the interconnected system (S,Δ) with input u and output y is incrementally stable, and its $d_\mathcal{U}/d_\mathcal{Y}$ incremental gain does not exceed 1.

Proof. First, note by inspection that functions $d_\mathcal{U}$ and $d_\mathcal{Y}$ defined in (7.2) and (7.3) are symmetric. Now, by assumption, all feasible signals of S satisfy:

$$\inf_{T\geq 0} \sum_{t=0}^{T} d_{\mathcal{U}_S}((u_1(t),w_1(t)),(u_2(t),w_2(t))) - d_{\mathcal{Y}_S}((y_1(t),z_1(t)),(y_2(t),z_2(t))) > -\infty \tag{7.4}$$

and those of Δ satisfy:

$$\inf_{T\geq 0} \sum_{t=0}^{T} d_{\mathcal{U}_\Delta}(z_1(t),z_2(t)) - d_{\mathcal{Y}_\Delta}(w_1(t),w_2(t)) > -\infty \tag{7.5}$$

For functions $d_\mathcal{U}$ and $d_\mathcal{Y}$ defined by (7.2) and (7.3), (7.4) implies that:

$$\inf_{T\geq 0} \sum_{t=0}^{T} d_\mathcal{U}(u_1(t),u_2(t)) + d_{\mathcal{Y}_\Delta}(w_1(t),w_2(t)) - d_\mathcal{Y}(y_1(t),y_2(t)) - d_{\mathcal{U}_\Delta}(z_1(t),z_2(t)) > -\infty \tag{7.6}$$

Adding (7.5) and (7.6), and noting that the infimum of the sum of two functions is larger than or equal to the sum of the infimums of the functions, we get that:

$$\inf_{T\geq 0} \sum_{t=0}^{T} d_\mathcal{U}(u_1(t),u_2(t)) - d_\mathcal{Y}(y_1(t),y_2(t)) > -\infty \tag{7.7}$$

If $d_{\mathcal{U}}$ and $d_{\mathcal{Y}}$, which are symmetric by definition, are also positive definite, the interconnection (S, Δ) is incrementally stable and its $d_{\mathcal{U}}/d_{\mathcal{Y}}$ incremental gain does not exceed 1. $\square$

7.4 Analysis of Deterministic Finite State Machines

7.4.1 Necessary and Sufficient Conditions for Incremental Stability

Necessary and sufficient conditions for incremental stability and incremental gain bound verification are presented in this section. The proofs of Theorems 7.2 and 7.3 are postponed until section 7.4.2.

Theorem 7.2. *A deterministic finite state machine M defined by (2.1) and (2.2) is incrementally stable iff there exists a non-negative function $V : \mathcal{Q} \times \mathcal{Q} \to \mathbf{R}_+$ such that:*

$$V(f(q_1, u), f(q_2, u)) - V(q_1, q_2) \leq -\mathbf{I}_{\hat{\mathcal{Y}} - \hat{\mathcal{Y}}_o}(g(q_1, u), g(q_2, u)) \tag{7.8}$$

holds for all $(q_1, q_2) \in \mathcal{Q} \times \mathcal{Q}$ and $u \in \mathcal{U}$, where $\hat{\mathcal{Y}} = \mathcal{Y} \times \mathcal{Y}$ and $\hat{\mathcal{Y}}_o = \{(y, y) |\ y \in \mathcal{Y}\}$.

Theorem 7.3. *Given a deterministic finite state machine M defined by (2.1) and (2.2), and symmetric positive definite functions $d_{\mathcal{U}} : \mathcal{U} \times \mathcal{U} \to \mathbf{R}_+$ and $d_{\mathcal{Y}} : \mathcal{Y} \times \mathcal{Y} \to \mathbf{R}_+$. The following two statements are equivalent:*

(a) *M is incrementally stable and its $d_{\mathcal{U}}/d_{\mathcal{Y}}$ incremental gain is bounded above by γ.*

(b) *There exists a non-negative function $V : \mathcal{Q} \times \mathcal{Q} \to \mathbf{R}_+$ and a non-negative constant γ such that:*

$$V(f(q_1, u_1), f(q_2, u_2)) - V(q_1, q_2) \leq \gamma d_{\mathcal{U}}(u_1, u_2) - d_{\mathcal{Y}}(g(q_1, u_1)), g(q_2, u_2)) \tag{7.9}$$

holds for all $(q_1, q_2) \in \mathcal{Q} \times \mathcal{Q}, (u_1, u_2) \in \mathcal{U} \times \mathcal{U}$.

The following Corollary shows that incremental stability is a stronger notion for DFM models than it is for arbitrary systems over finite alphabets, again due to finiteness of the state set.

Corollary 7.4. *A deterministic finite state machine M is incrementally stable iff there exists finite non-negative constants C, γ and symmetric positive definite functions $d_{\mathcal{U}} : \mathcal{U} \times \mathcal{U} \rightarrow \mathbf{R}_+$ and $d_{\mathcal{Y}} : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbf{R}_+$, such that for any two pairs of input signals/initial state, $(\mathbf{u}_1, q_1)$ and $(\mathbf{u}_2, q_2)$ in $\mathcal{U}^{\mathbf{Z}_+} \times \mathcal{Q}$, the corresponding outputs, $\mathbf{y}_1$ and $\mathbf{y}_2$ respectively, satisfy:*

$$\sum_{t=0}^{T} d_{\mathcal{Y}}(y_1(t), y_2(t)) \leq C + \gamma \sum_{t=0}^{T} d_{\mathcal{U}}(u_1(t), u_2(t))$$

for all time $T \geq 0$.

Proof. Sufficiency is straightforward. Necessity follows from Theorem 7.3 with C defined as $C \doteq \max_{q_1,q_2,q_3,q_4} V(q_1, q_2) - V(q_3, q_4)$. □

7.4.2 Proof of the Necessary and Sufficient Conditions

Let M be a deterministic finite state machine defined by (2.1) and (2.2). Consider $\hat{M}$ with input and output alphabets $\hat{\mathcal{U}} = \mathcal{U} \times \mathcal{U}$ and $\hat{\mathcal{Y}} = \mathcal{Y} \times \mathcal{Y}$, state set $\hat{\mathcal{Q}} = \mathcal{Q} \times \mathcal{Q}$, state transition function $\hat{f} : \hat{\mathcal{Q}} \times \hat{\mathcal{U}} \rightarrow \hat{\mathcal{Q}}$ and output function $\hat{g} : \hat{\mathcal{Q}} \times \hat{\mathcal{U}} \rightarrow \hat{\mathcal{Y}}$ defined by:

$$\hat{f}(q, u) \doteq (f(q_1, u_1), f(q_2, u_2)) \tag{7.10}$$

$$\hat{g}(q, u) \doteq (g(q_1, u_1), g(q_2, u_2)) \tag{7.11}$$

Thus, $\hat{M}$ is described by the following state transition (7.12) and output (7.13) equations:

$$q(t+1) = \hat{f}(q(t), u(t)) \tag{7.12}$$

$$y(t+1) = \hat{g}(q(t), u(t)) \tag{7.13}$$

The following statements are true about deterministic finite state machine M and the corresponding $\hat{M}$ constructed as described above (where $\mathcal{D}_M$ and $\mathcal{D}_{\hat{M}}$ are their feasible signals sets).

Lemma 7.5. $(\mathbf{u}, \mathbf{y}) \in \mathcal{D}_{\hat{M}}$ *iff* $(\mathbf{u}_1, \mathbf{y}_1), (\mathbf{u}_2, \mathbf{y}_2) \in \mathcal{D}_M$.

Proof.

$$
\begin{aligned}
(\mathbf{u},\mathbf{y}) \in \mathcal{D}_{\hat{M}} \quad &\Leftrightarrow \quad \exists \mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2) \text{ such that } \mathbf{q}, \mathbf{u}, \mathbf{y} \text{ satisfy (7.12) and (7.13)}, \forall t \in \mathbf{Z}_+ \\
&\Leftrightarrow \quad \exists \mathbf{q}_1, \mathbf{q}_2 \text{ such that } \mathbf{q}_1, \mathbf{u}_1, \mathbf{y}_1 \text{ and } \mathbf{q}_2, \mathbf{u}_2, \mathbf{y}_2 \text{ satisfy (2.1) and (2.2)}, \forall t \in \mathbf{Z}_+ \\
&\Leftrightarrow \quad (\mathbf{u}_1, \mathbf{y}_1), (\mathbf{u}_2, \mathbf{y}_2) \in \mathcal{D}_M
\end{aligned}
$$

□

Lemma 7.6. *Let* $\hat{\mathcal{U}}_o = \{(u,u) | u \in \mathcal{U}\}$ *and* $\hat{\mathcal{Y}}_o = \{(y,y) | y \in \mathcal{Y}\}$. M *is incrementally stable iff* $\hat{M}$ *is gain stable about* $(\hat{\mathcal{U}}_o, \hat{\mathcal{Y}}_o)$.

Proof. M is incrementally stable

$\Leftrightarrow$ $\exists \gamma \geq 0$, symmetric positive definite functions $\rho : \mathcal{U} \times \mathcal{U} \rightarrow \mathbf{R}_+, \mu : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbf{R}_+$ such that any two pairs $(\mathbf{u}_1, \mathbf{y}_1), (\mathbf{u}_2, \mathbf{y}_2)$ in $\mathcal{D}_M$ satisfy:

$$
\inf_{T \geq 0} \sum_{t=0}^{T} \gamma \rho(u_1(t), u_2(t)) - \mu(y_1(t), y_2(t)) > -\infty
$$

$\Leftrightarrow$ $\exists \gamma \geq 0$ and functions $\rho : \mathcal{U} \times \mathcal{U} \rightarrow \mathbf{R}_+$ and $\mu : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbf{R}_+$ zero on $\hat{\mathcal{U}}_o$ and $\hat{\mathcal{Y}}_o$ respectively, and positive elsewhere such that any feasible signal $(\mathbf{u}, \mathbf{y})$ in $\mathcal{D}_M$ satisfies:

$$
\inf_{T \geq 0} \sum_{t=0}^{T} \gamma \rho(u(t)) - \mu(y(t)) > -\infty
$$

$\Leftrightarrow$ $\hat{M}$ is gain stable about $(\hat{\mathcal{U}}_o, \hat{\mathcal{Y}}_o)$

(The second equivalence follows from Lemma 7.5). □

We are now ready to prove Theorems 7.2 and 7.3.

Proof of Theorem 7.2: It follows from Lemma 7.6 that M is incrementally stable iff $\hat{M}$ is gain stable about $(\hat{\mathcal{U}}_o, \hat{\mathcal{Y}}_o)$. Moreover, it follows from Theorem 4.4 that $\hat{M}$ is gain stable about $(\hat{\mathcal{U}}_o, \hat{\mathcal{Y}}_o)$ iff there exists a non-negative function $V : \mathcal{Q} \times \mathcal{Q} \rightarrow \mathbf{R}_+$ such that the following inequality holds for all $q \in \hat{\mathcal{Q}} = \mathcal{Q} \times \mathcal{Q}$ and $u \in \hat{\mathcal{U}}_o$:

$$
V(\hat{f}(q,u)) - V(q) \leq -\mathbf{I}_{\hat{\mathcal{Y}} - \hat{\mathcal{Y}}_o}(\hat{g}(q,u)) \tag{7.14}
$$

Using the definitions in (7.10) and (7.11), (7.14) can be rewritten as the condition in (7.8). □

Proof of Theorem 7.3: Consider the following statement:

(c) $\hat{M}$ is gain stable about $(\hat{\mathcal{U}}_o, \hat{\mathcal{Y}}_o)$ and its $d_{\mathcal{U}}/d_{\mathcal{Y}}$ gain is bounded above by $\hat{\gamma}$.

We have (c) $\Leftrightarrow$ (a) by Lemma 7.6. Moreover, it follows from Corollary 4.3 that (c) holds iff there exists a non-negative function $V : \mathcal{Q} \times \mathcal{Q} \rightarrow \mathbf{R}_+$ such that:

$$V(\hat{f}(q,u)) - V(q) \leq \gamma\rho(u) - \mu(\hat{g}(q,u))$$

holds for all $q \in \hat{\mathcal{Q}}$ and $u \in \hat{\mathcal{U}}$, which is equivalent to (b) for $\hat{f}$ and $\hat{g}$ defined in (7.10) and (7.11). Thus, (c) $\Leftrightarrow$ (b). □

7.5 Relating Incremental and External Stability

An interesting question that arises is: how are the various input/output stability properties defined in Chapters 2 and 3 and this chapter related? It is shown in this section that while incremental stability is generally a stronger notion than external stability (Remark 7.1), the two notions are equivalent for DFM models (Theorem 7.7).

Remark 7.1 Incremental stability is a stronger notion than external stability. To see that, suppose that a system S is incrementally stable. Then for any pair of elements $(\mathbf{u}, \mathbf{y}_1)$ and $(\mathbf{u}, \mathbf{y}_2)$ in $\mathcal{D}$, we have the following inequality:

$$\sup_{T \geq 0} \sum_{t=0}^{T} d_{\mathcal{Y}}(y_1(t), y_2(t)) < \infty$$

Since function $d_{\mathcal{Y}}$ only takes on a finite number of values ($\mathcal{Y}$ is finite), the above inequality allows us to conclude that the system is externally stable. ◇

Theorem 7.7. *A deterministic finite state machine M defined by (2.1) and (2.2) is externally stable iff it is incrementally stable.*

Proof. Sufficiency was shown in Remark 7.5. To prove necessity, suppose that M is externally stable and consider $V : \mathcal{Q} \times \mathcal{Q} \to \mathbf{R}_+$ defined as follows:

$$V(q_1, q_2) \doteq \sup_{u \in \mathcal{U}^{\mathbf{Z}_+}} \sum_{t=0}^{\infty} \phi(q_1(t), q_2(t), u(t))$$

where $\phi : \mathcal{Q} \times \mathcal{Q} \times \mathcal{U} \to \{0, 1\}$ is defined by:

$$\phi(q_1, q_2, u) := \begin{cases} 0 & g(q_1, u) = g(q_2, u) \\ 1 & \text{otherwise} \end{cases}$$

where $\mathbf{q}_1$ and $\mathbf{q}_2$ are the state trajectories corresponding to initial states q_1 and q_2 respectively and to input $\mathbf{u}$. It follows that the following inequality holds for any $u \in \mathcal{U}$:

$$V(q_1, q_2) \geq \phi(q_1, q_2, u) + V(f(q_1, u), f(q_2, u))$$

What is left to show is that V is bounded. We claim that $V \leq an^2$, where $a = card(\mathcal{U})$ and $n = card(\mathcal{Q})$. The proof is by contradiction: suppose that this is not the case, then there exists a pair of initial conditions q_1, q_2, an input sequence $\mathbf{u}$, and two integers t' and t'' such that $q_1(t') = q_1(t'')$, $q_2(t') = q_2(t'')$, $u(t') = u(t'')$ and $\phi(q_1(t'), q_2(t'), u(t')) = 1$. We can then construct two periodic feasible signals $(\mathbf{u}_o, \mathbf{y}_1)$ and $(\mathbf{u}_o, \mathbf{y}_2)$ corresponding to initial conditions $q_1(t')$ and $q_2(t')$ and to periodic input $\mathbf{u}_o$ with $u_o(k(t''-t')+l) \doteq u(t'+l)$, for $k \in \mathbf{Z}_+$ and $l \in \{0, \ldots, t''-t'-1\}$, which violate the condition in Definition 3.2, hence contradicting our assumption of external stability. It follows from Theorem 7.2 that M is incrementally stable. $\square$

Chapter 8

Conclusions

8.1 Summary

A finite state machine based paradigm for design of controllers for a class of hybrid systems, namely systems with finite actuation and sensing, was advocated and presented in this thesis. The development of this paradigm addressed three complementary problems:

1. Approximation of hybrid systems by finite state machine models, with useful descriptions of the resulting approximation error (Chapter 3).
2. Analysis of stability and performance of finite state machines (Chapter 4).
3. Synthesis of robust controllers for finite state machine nominal models (Chapter 5).

The starting point of this development was a unified input/output view of systems, performance and robustness, in which systems were viewed as sets of signals and performance objectives were described in terms of specific classes of constraints on the signals (Chapter 2). The design paradigm was demonstrated using a simple class of academic benchmark problems (Chapter 6). Extensions of the analysis tools to incremental descriptions of performance objectives were also given (Chapter 7).

8.2 Directions for Future Work

Several research directions would be interesting to pursue in the future, with the goal of improving the computational efficiency and the scalability of this approach, extending it to a wider class of systems and reducing its conservatism.

- While the notion of approximation proposed in Section 3.2 is very general, the approximation approach developed in the remainder of Chapter 3 deals with systems with no exogenous input. Extensions of this approach and/or alternative approaches are desirable for general systems with finite actuation and sensing.

- The approach used in Chapter 4 to compute a gain bound on the approximation error is conservative but efficient. This trade off between conservatism and efficiency may not always be acceptable. Thus, it would be interesting to look into mitigating the conservatism, possibly by using more descriptive gain stability conditions to characterize Δ or by developing efficient algorithmic approaches for constructing storage functions for systems with discrete and continuous state spaces such as system Δ.

- The bound on the approximation error was computed a-posteriori in this thesis, meaning after construction of the finite state machine nominal model. It would be interesting to identify particular internal structures for the hybrid system and corresponding approximation procedures for which it is possible to derive a-priori upper and lower bounds on the quality of approximation, and to quantify fundamental limitations on approximating classes of hybrid systems by finite state machines of a given size.

- Controller synthesis reduces to solving an appropriate Bellman inequality for a system with discrete state and input sets, as shown in Chapter 5. Although it appears to perform well in practice, the value iteration algorithm used can potentially take infinitely long to converge. Thus, it would be interesting to try to find practically meaningful structures for which a closed form solution of the Bellman inequality can be found.

- The results developed in Chapter 7 are mainly analysis results for systems with discrete actuation and sensing whose performance objectives are described in terms of incremental gain properties. Development of a notion of approximation and a corresponding approximation paradigm for this setup is a natural next step.

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