Asymptotic behavior of Complete Ricci-flat Metrics on Open Manifolds

by

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Abstract

In this thesis, we describe the asymptotic behavior of complete Ricci-flat Kähler metrics on open manifolds that can be compactified by adding a smooth, ample divisor. This result provides an answer to a question addressed to by Tian and Yau in [TY1], therefore refining the main result in that paper.

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Contents

Chapter 1

Introduction

In 1978, Yau [Y] proved the Calabi Conjecture, by showing existence and uniqueness of Kähler metrics with prescribed Ricci curvature on compact complex manifolds. Here the complex manifolds in question are already supposed to admit a Kähler metric whose Ricci form satisfies the natural conditions arising from Chern-Weil theory.

Following this work, Tian and Yau [TY1] settled a non-compact version of Calabi's Conjecture on quasi-projective manifolds that can be compactified by adding a smooth, ample divisor. In a subsequent work ([TY2]), they extended their result for the case where the divisor has multiplicity greater than one, and is allowed to have orbifold singularities. This generalization was done independently by Bando [B] and Kobayashi [K]. Later, Joyce [J] provided the sharp asymptotics for the decay of the solutions provided in [TY2].

Once the existence problem is solved, an interesting question that arises is about the behavior of those complete metrics near the divisor. This question is also posed by Tian and Yau in [TY1].

The aim of this thesis is to provide an answer to the mentioned question, therefore refining the main result in [TY1]. More precisely, we shall first construct explicitly a sequence of complete Kähler metrics with special approximating properties on a quasi-projective manifold (in our case, the complement of a smooth, ample divisor on a compact complex manifold). Then by using these approximating metrics, we are going to study the solution of a complex Monge-Ampère equation on the open manifold. A careful analysis of the complex Monge-Ampère operator will allow us to describe asymptotic properties of the solution. As a matter of fact, the reader will note that our results apply equally well to divisors having orbifold-type singularities.

To state the main results of this thesis, let us consider a compact, complex manifold $\overline{M}$ of complex dimension n. Let D be an *admissible* divisor in $\overline{M}$, ie, a divisor satisfying the following conditions:

- Sing $\overline{M} \subset D$.
- D is smooth in $\overline{M} \setminus$ Sing $\overline{M}$.
- For any $x \in$ Sing $\overline{M}$, let $\Pi_x : \tilde{\mathcal{U}}_x \to \mathcal{U}_x$ be its local uniformization with $\tilde{\mathcal{U}}_x \in \mathbb{C}^n$. Then $\Pi_x^{-1}(D)$ is smooth in $\tilde{\mathcal{U}}_x$.

Let Ω be a smooth, closed $(1,1)$-form in the cohomology class $c_1(K_{\overline{M}}^{-1} \otimes L_D^{-1})$, where $K_{\overline{M}}$ stands for the canonical line bundle of $\overline{M}$, and L_D for the line bundle associated to D. Let S be a defining section of D on L_D and let M be the open manifold $M = \overline{M} \setminus D$. Consider a hermitian metric $||.||$ on L_D.

Fefferman, in his paper [F], developed inductively an n-th order approximation to a complete Kähler-Einstein metric on strictly pseudoconvex domains on $\mathbb{C}^n$ with smooth boundary, and he suggested that higher order approximations could be obtained by considering log terms in the formal expansion of the solution to a certain complex Monge-Ampère equation. This idea was used by Lee and Melrose in [LM], where they constructed the full asymptotic expansion of the solution to the Monge-Ampère equation introduced by Fefferman.

Motivated by this work, we construct inductively a sequence of rescalings $||.||_{\phi_m} := e^{\phi_m/2}||.||$ of a fixed hermitian metric $||.||$ on L_D, which will be the main ingredient of the proof of the following result.

Theorem 1.1 *Let M, Ω and D be as above. Then for any $\varepsilon > 0$, there exists an explicitly given complete Kähler metric g_ε on M such that*

$$Ric(g_\varepsilon) - \Omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} f_\varepsilon \qquad \text{on } M, \tag{1.1}$$

where f_ε *is a smooth function on* M *that decays to the order of* $O(||S||^\varepsilon)$*. Furthermore, the Riemann curvature tensor* $R(g_\varepsilon)$ *of the metric* g_ε *decays to the order of* $O((-n\log||S||^2)^{-\frac{1}{n}})$*.*

Remark: In the above statement, it should be emphasized that the metric in question is explicit. In other words, this result provides complete metrics that are "approximate solutions" to the Calabi problem, but that have the advantage of being explicitly described.

So far, there has been a large amount of work concerned with deriving asymptotic expansions for Kähler-Einstein metrics in different contexts: after Cheng and Yau [CY1] proved existence and uniqueness of Kähler-Einstein metrics on strictly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary (in addition to results on the regularity of the solution), Lee and Melrose [LM] derived an asymptotic expansion for the Cheng-Yau solution, which completely determines the form of the singularity and improves the regularity result of [CY1]. On the setting of quasi-projective manifolds, Cheng and Yau [CY2] and Tian and Yau [TY3] showed the existence of Kähler-Einstein metrics under certain conditions on the divisor, and Wu [Wu] developed the asymptotic expansion to the Cheng-Yau metric on a quasi-projective manifold (also assuming some conditions on the divisor), as the parallel part to the work of Lee and Melrose [LM].

However, in the context of quasi-projective manifolds, an asymptotic description of complete Kähler metrics with prescribed Ricci curvature was still lacking. This is provided by our results below.

In [TY1], the result of existence of a complete Kähler metric (in a given Kähler class) with prescribed Ricci curvature is achieved by solving the following complex Monge-Ampère equation

$$\begin{cases} \left(\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\right)^n = e^f\omega^n, & \\ \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u > 0, & u \in C^\infty(M,\mathbb{R}), \end{cases} \tag{1.2}$$

where f is a given smooth function satisfying the integrability condition

$$\int_M (e^f - 1)w^n = 0. \tag{1.3}$$

Our main result describes the asymptotic behavior of the solution to (1.2), by showing that the approximate metrics given in Theorem 1.1 are asymptotically as close to the actual solution as possible.

Theorem 1.2 *For each $\varepsilon > 0$, let g_ε and f_ε be given by Theorem 1.1.*

Consider the solution u_ε to the problem

$$\begin{cases} \left(\omega_{g_\varepsilon} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_\varepsilon\right)^n = e^{f_\varepsilon}\omega_{g_\varepsilon}^n, \\ \omega_{g_\varepsilon} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_\varepsilon > 0, \qquad u_\varepsilon \in C^\infty(M, \mathbb{R}). \end{cases} \tag{1.4}$$

Then the solution u_ε decays as $O(||S||^\varepsilon)$ near the divisor.

This theorem has an important, straightforward corollary.

Corollary 1.1 *Let $\overline{M}$ be a compact Kähler manifold of complex dimension n, and let D be a smooth anti-canonical divisor. Then for any $\varepsilon > 0$, there exists a complete Ricci-flat Kähler metric on $M = \overline{M} \setminus D$ that can be described as*

$$\hat{\omega} = \omega_{g_\varepsilon} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_\varepsilon,$$

where g_ε is the Kähler metric constructed in Theorem 1.1, and u is a smooth function on D, with bounded derivative, such that u decays at least to the order of $O(||S||^\varepsilon)$.

The structure of this thesis is as follows. In Chapter 3, we construct inductively a sequence of hermitian metrics $\{||.||_m\}_{m\in\mathbb{N}}$ on L_D such that the closed $(1,1)$-form

$$\omega_m = \frac{\sqrt{-1}}{2\pi}\frac{n^{1+1/n}}{n+1}\partial\bar{\partial}(-\log||S||_m^2)^{\frac{n+1}{n}} \tag{1.5}$$

is positive definite on a tubular neighborhood V_m of D in $\overline{M}$.

The Kähler form ω_m defines a Kähler metric g_m on V_m such that $\mathrm{Ric}(g_m) - \Omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} f_m$, for a smooth function f_m on M that decays to the order of $O(||S||^m)$. An important technical result for this construction is Lemma 3.2 whose proof is the object of Chapter 4.

In Chapter 5, we use the constructions of Chapter 3 to complete the proof of Theorem 1.1. First, we shall obtain the necessary estimates on the decay of the Riemann curvature tensor of the metrics g_m. Then we shall proceed to the construction of approximating metrics that are defined on the whole manifold (and not only on a neighborhood of the divisor at infinity).

Finally, Chapter 6 is devoted to the asymptotic study of the Monge-Ampère equation 1.2. By using the maximum principle for the complex Monge-Ampère operator, and the construction of a suitable barrier, we shall complete the proof of Theorem 1.2.

Chapter 2

Background

In order to understand the main problem considered in this thesis, we shall start from its original motivation.

Let M be a compact, complex manifold of complex dimension n, and consider g, a hermitian metric defined on M. Note that g is a complex-valued sesquilinear form acting on $TM \times TM$, and can therefore be written as

$$g = S - 2\sqrt{-1}\omega_g,$$

where S and $-\omega$ are real bilinear forms.

If $(z_1, \ldots, z_n)$ are local coordinates around a point $x \in M$, we can write the metric g as $\sum g_{i\bar{j}} dz^i \otimes d\bar{z}^j$. Then, it is easy to see that in these coordinates

$$\omega_g = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} g_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

The form ω_g is a real 2−form of type $(1,1)$, and is called the *fundamental form* of the metric g.

Definition 2.1 *We say that a hermitian metric on a complex manifold is* Kähler *if its associated fundamental form ω_g is closed,* ie, $d\omega_g = 0$. *A complex manifold equipped with a Kähler metric is called a* Kähler manifold.

The reader will find in the literature a number of equivalent definitions for a Kähler metric. We will keep this choice for convenience of the exposition.

We point out that on a Kähler manifold, the form ω_g is uniquely determined by the metric g, and vice-versa.

Let $R(g) = R_{i\bar{j}k\bar{l}}$ be the Riemann curvature tensor of the metric g written in the coordinates described above. We define the *Ricci curvature tensor* of the metric g as the trace of the Riemann curvature tensor. Its components in local coordinates can be written as

$$\mathrm{Ric}_{k\bar{l}} = \sum_{i,j=1}^{n} g^{i\bar{j}} R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z_k \partial \bar{z}_l} \log \det(g_{i\bar{j}}). \tag{2.1}$$

We also define the *Ricci form* associated to g as being the form given in local coordinates by

$$\mathrm{Ric} = \sum_{i,j=1}^{n} Ric_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

Now, given a metric g, we can define a matrix-valued 2-form Ω by writing its expression in local coordinates, as follows

$$\Omega_i^j = \sum_{i,p=1}^{n} g^{j\bar{p}} R_{i\bar{p}k\bar{l}} dz^k \wedge d\bar{z}^l. \tag{2.2}$$

This expression for Ω gives a well-defined $(1,1)$-form, to be called the *curvature form* of the metric g.

Next, consider the following expression

$$\det\left(\mathrm{Id} + \frac{t\sqrt{-1}}{2\pi}\Omega\right) = 1 + t\phi_1(g) + t^2\phi_2(g) + \ldots,$$

where each $\phi_i(g)$ denotes the i-th homogeneous component of the left-hand side, considered as a polynomial in the variable t.

Each of the forms $\phi_i(g)$ is a (i,i)-form, and is called the *i-th Chern form* of the metric g. It is a fact (see for example [We] for further explanations) that the cohomology class represented by each $\phi_i(g)$ is independent on the metric g, and hence is a topological invariant of the manifold M. These cohomology classes are called the

Chern classes of M and they are going to be denoted by $c_i(M)$.

There are analogous definitions for the curvature form of a hermitian metric on a general complex vector bundle E on a complex manifold, and we can also define in the same fashion the Chern class $c_i(M, E)$ of a vector bundle, which will also be independent on the choice of the metric. In fact, we say that the Chern classes $c_i(M)$ of the manifold M are the Chern classes $c_i(M, TM)$ of the tangent bundle of M.

We will restrict our attention to the first Chern class $c_1(M)$. Note that the form $\phi_1(g)$ represents the class $c_1(M)$, and that $\phi_1(M)$ is simply the trace of the curvature form:

$$\phi_1(g) = \frac{\sqrt{-1}}{2\pi} \sum_{i=1}^{n} \Omega_i^i = \frac{\sqrt{-1}}{2\pi} \sum_{i,p=1}^{n} g^{i\bar{p}} R_{i\bar{p}k\bar{l}} dz^k \wedge d\bar{z}^l. \tag{2.3}$$

On the other hand, notice that the right-hand side of (2.3) is equal to $\frac{\sqrt{-1}}{2\pi}\mathrm{Ric}_{k\bar{l}}$, in view of (2.1). Therefore, we conclude that the Ricci form of a Kähler metric represents the first Chern class of the manifold M.

A natural question that arises is: given a Kähler class $[\omega] \in H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})$ in a compact, complex manifold M, and any $(1,1)$-form Ω representing $c_1(M)$, is that possible to find a metric g on M such that $\mathrm{Ric}(g) = \Omega$? This question was addressed to by Calabi in 1960, and it was answered by Yau [Y] almost 20 years later.

Theorem 2.1 (Yau, 1978) *If the manifold M is compact and Kähler , then there exists a unique Kähler metric g on M satisfying $Ric(g) = \Omega$.*

This theorem has a large number of applications in different areas of Mathematics and Physics. Its proof amounts to solving an elliptic differential equation, as explained below.

Fix a Kähler form $\omega \in [\omega]$ representing the previously given Kähler class in $H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})$. In local coordinates, we can write ω as $\omega = g_{i\bar{j}} dz^i \wedge d\bar{z}^j$. Also it was seen that both $\mathrm{Ric}(\omega)$ and Ω represent the same cohomology class, namely $c_1(M)$. Therefore, due to the famous $\partial\bar{\partial}$-Lemma, there exists a function f on M such that

$$\mathrm{Ric}(\omega) - \Omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} f,$$

where f is uniquely determined after imposing the normalization

$$\int_M \left(e^f - 1\right) \omega^n = 0. \tag{2.4}$$

Notice that f is fixed once we have fixed ω and Ω.

Any other metric in the same cohomology class $[\omega]$ will be written as $\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi$, for some function $\phi \in C^\infty(M, \mathbb{R})$. Hence, we are trying to find a representative $\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi$ of the class $[\omega]$ that satisfies

$$\mathrm{Ric}\left(\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi\right) = \Omega = \mathrm{Ric}(\omega) - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f. \tag{2.5}$$

Rewriting (2.5) in local coordinates, we have

$$-\partial\bar{\partial}\log\det\left(g_{i\bar{j}} + \frac{\partial^2}{\partial z_i \partial \bar{z}_j}\right) = -\partial\bar{\partial}\log\det\left(g_{i\bar{j}}\right) - \partial\bar{\partial}f,$$

or

$$\partial\bar{\partial}\log\frac{\det\left(g_{i\bar{j}} + \frac{\partial^2}{\partial z_i \partial \bar{z}_j}\right)}{\det\left(g_{i\bar{j}}\right)} = \partial\bar{\partial}f. \tag{2.6}$$

Even though this is a local expression, the term on the left-hand side of (2.6) is well-defined globally, and gives rise to the global equation

$$\left(\omega + \partial\bar{\partial}\phi\right)^n = e^f \omega^n, \tag{2.7}$$

called the Monge-Ampère equation.

Notice that the resulting $(1,1)$-form given by $\omega' = \omega + \partial\bar{\partial}\phi$ defines a Kähler metric g', which, in turn, satisfies $\mathrm{Ric}(g') = \Omega$. So, in order to find metrics that are solutions to Calabi's problem, it suffices to determine a solution ϕ to (2.7).

The celebrated Yau's Theorem in [Y] determines a unique solution to (2.7) when f satisfies the integrability condition (2.4), and therefore provides a satisfatory answer to the problem of finding Ricci-flat metrics when the underlying manifold M is compact. Calabi's Problem, though, has a very natural generalization for the case of a special class of open manifolds. However, we will need to make minor modifications to the original conjecture.

Suppose that $\overline{M}$ is a compact, Kähler manifold, and let D be a smooth divisor in $\overline{M}$. We are now interested in constructing complete Kähler metrics with prescribed Ricci curvature on the open manifold M, defined as the complement of the divisor D in $\overline{M}$.

If g' is a metric defined on $\overline{M}$, then the metric $\det(g')$ (given locally by $\det(g') = \det(g'_{i\bar{j}})dz_1 d\bar{z}_1 \ldots dz_n d\bar{z}_n)$ is a metric defined on the canonical line bundle $K_{\overline{M}}$ of $\overline{M}$.

Consider the line bundle L_D associated to D, let S be a defining section of D in L_D, and finally, let h define a hermitian metric on L_D. Let us write h, in the previous choice of local coordinates, as a positive function a.

With the preceding notations, the line bundle given by $K_{\overline{M}} \otimes -L_D$ has a metric defined locally by $\det(g'_{i\bar{j}})a^{-1}$. Indeed the reader will note that this expression makes sense globally on M. In turn, the metric $\det(g'_{i\bar{j}})a^{-1}$ can be written as $\det(g'_{i\bar{j}})a^{-1} = \det(\frac{g'_{i\bar{j}}}{b})$, where $b^n = a$. In particular, we have a new metric g defined on M (and also on $\overline{M}$) which is given in local coordinates by $g_{i\bar{j}} = \frac{g'_{i\bar{j}}}{b}$. Naturally the Ricci form of the metric g is a representative of the first Chern class $c_1(-K_{\overline{M}} \otimes -L_D)$. On the other hand, we would like the resulting metric g to be complete on the open manifold M. Strictly speaking, this will never happen since g is also a metric on the closure $\overline{M}$. Nonetheless, this construction suggests a natural way to try to obtain complete metrics. Namely we let the metric h conveniently degenerate on the divisor D. This implies that the function a will vanish on D and thus that the metric g will become unbounded near D. So we may hope to find complete metrics on M by this procedure. Note also that the class of the Ricci form of g is not affected by the "degeneration" of h.

Summarizing what precedes, to generalize Calabi's Conjecture to open manifolds, we begin by fixing a representative $\Omega \in c_1(-K_{\overline{M}} \otimes -L_D)$. From our previous discussion, the Ricci form of a Kähler metric defined on M is a representative of $c_1(-K_{\overline{M}} \otimes -L_D)$. Now we want to study the converse problem, namely:

Question: Fixed a Kähler class $[\omega]$ in the manifold M, pick any representative Ω of the first Chern class $c_1(-K_{\overline{M}} \otimes -L_D)$. Can we construct a complete Kähler metric g on M such that $\mathrm{Ric}(g) = \Omega$?

There are some results on the existence of such metrics (to be discussed in the next chapters), and the main purpose of this thesis is to provide a better understanding of complete Ricci-flat Kähler metrics, a problem of great interest by both physicists and geometers. For that, we will discuss the behavior of such metrics near the divisor in the remainder of this work.

Chapter 3

Approximating Kähler metrics

Let $\overline{M}$ be a compact kähler manifold of complex dimension n, and let D be an admissible divisor in $\overline{M}$.

The divisor D induces a line bundle L_D on $\overline{M}$. We will assume that the restriction of L_D to D is ample, so that there exists an orbifold hermitian metric $||.||$ on L_D such that its curvature form $\tilde{\omega}$ is positive definite along D.

Consider a closed $(1,1)$-form Ω in the Chern class $c_1(-K_{\overline{M}} - L_D)$. The goal of this section is to construct a complete kähler metric g such that

$$\mathrm{Ric}(g) - \Omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} f \text{ on } M, \tag{3.1}$$

for a smooth function f with sufficiently fast decay, where $M = \overline{M} \setminus D$ and $\mathrm{Ric}(g)$ stands for the Ricci form of the metric g.

Fix an orbifold hermitian metric $||.||$ on L_D such that its curvature form $\tilde{\omega}$ is positive definite along D. We shall need to rescale the metric by a suitable factor which will be determined in the following discussion. Let us begin by observing that the restriction $\Omega|_D$ of Ω to D belongs to $c_1(D)$ since, by assumption, $\Omega \in c_1(-K_{\overline{M}} - L_D)$. Hence, there exists a function φ such that $\tilde{\omega}|_D + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ defines a metric g_D verifying $\mathrm{Ric}(g_D) = \Omega|_D$. So, by rescaling $||.||$ by an appropriate factor, we may assume that $\tilde{\omega}$, when restricted to the infinity D, defines a metric g_D such that $\mathrm{Ric}(g_D) = \Omega|_D$.

Next denote by S the defining section of D, and write $||.||_\phi = e^{-\phi/2}||.||$ for the rescaling of $||.||$, where ϕ is any smooth function on $\overline{M}$.

We define

$$\omega_\phi = \frac{\sqrt{-1}}{2\pi}\frac{n^{1+1/n}}{n+1}\partial\bar{\partial}(-\log||S||^2_\phi)^{\frac{n+1}{n}}. \tag{3.2}$$

Then, it follows that

$$\omega_\phi = (-n\log||S||^2_\phi)^{1/n}\tilde{\omega}_\phi + \frac{1}{(-n\log||S||^2_\phi)^{\frac{(n-1)}{n}}}\frac{\sqrt{-1}}{2\pi}\partial\log||S||^2_\phi\wedge\bar{\partial}\log||S||^2_\phi, \tag{3.3}$$

where $\tilde{\omega}_\phi$ is the curvature form of the metric $||.||_\phi$. From this expression, we can see that, as long as $\tilde{\omega}_\phi$ is positive definite along D, ω_ϕ is positive definite near D.

For further reference, we compute here

$$\omega^n_\phi = (-n\log||S||^2_\phi)\tilde{\omega}^{n-1}_\phi\wedge(\tilde{\omega}_\phi + n\frac{\sqrt{-1}}{2\pi}\frac{\partial\log||S||^2_\phi\wedge\bar{\partial}\log||S||^2_\phi}{(-n\log||S||^2_\phi)}). \tag{3.4}$$

We state here the main result of this chapter.

Proposition 3.1 *Let $\overline{M}$ be a compact Kähler manifold of complex dimension n, and let D be an admissible divisor in $\overline{M}$. Consider also a form $\Omega \in c_1(-K_{\overline{M}} - L_D)$, where L_D is the line bundle induced by D.*

Then there exist sequences of neighborhoods $\{V_m\}_{m\in\mathbb{N}}$ of D along with complete Kähler metrics ω_m on $(V_m \setminus D, \partial(V_m\setminus D))$ (as defined on (3.2) such that

$$Ric(\omega_m) - \Omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f_m \text{ on } V_m\setminus D \tag{3.5}$$

where f_m are smooth functions on $M = \overline{M}\setminus D$. Furthermore, each f_m decays on the order of $O(||S||^m)$. In addition, the curvature tensors $R(g_m)$ of the metrics g_m decay to the order of at least $(-n\log||S||^2)^{\frac{-1}{n}}$ near the divisor.

The remainder of this chapter will be devoted to the proof of Proposition 3.1.

If $\tilde{\omega} = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log||S||^2$ is the curvature form of $||.||$, then for any Kähler metric

g' on $\overline{M}$, $\mathrm{Ric}(g') - \tilde{\omega} \in c_1(-K_{\overline{M}} - L_D)$. Hence, up to constant, there is a unique function Ψ such that

$$\Omega = \mathrm{Ric}(g') - \tilde{\omega} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\Psi. \tag{3.6}$$

Definition 3.1 *For x in the set where ω_ϕ is positive definite (x near D), write*

$$f_\phi(x) = -\log ||S||^2 - \log(\frac{\omega_\phi^n}{\omega'^n}) - \Psi,$$

where ω' is the kähler form of g'.

Lemma 3.1 *The function $f_0(x)$ converges uniformly to a constant if and only if $Ric(g_D) = \Omega|_D$.*

Proof: The proof of this lemma is analogous to the proof of [TY2], Lemma 2.1, and for completeness, we sketch it here. Choose a coordinate system $(z_1, \ldots, z_n)$ around a point x near D such that the local defining section S of D is given by $\{z_n = 0\}$.

In these coordinates, write $\tilde{\omega} = \tilde{\omega}_0$ as $(h_{ij})_{1\le i,j\le n}$, g' as $(g'_{ij})_{1\le i,j\le n}$, and $||.||$ as a positive function a.

By definition,

$$\begin{aligned} f_0(x) &= -\log\left(\frac{||S||^2\omega_0^n}{\omega'^n}\right) - \Psi(x) = \\ &= -\log\left(\frac{a\det(h_{ij})_{1\le i,j\le n-1}}{\det(g'_{ij})_{1\le i,j\le n}}\right)(x) - \Psi(x) + O(||S(x)||), \end{aligned}$$

for x near D.

Since $a^{-1}\det(g'_{ij})_{1\le i,j\le n}|_D$ is a well defined volume form on D, it makes sense to write

$$f_0(x) = -\log(\frac{a\det(h_{ij})_{1\le i,j\le n-1}e^\Psi}{\det(g'_{ij})_{1\le i,j\le n}})(z',0) + O(||S(x)||),$$

for $x = (z', z_n)$. Hence, $\lim_{x\to D} f_0(x)$ is a constant if and only if $\frac{a\det(h_{ij})_{1\le i,j\le n-1}e^\Psi}{\det(g'_{ij})_{1\le i,j\le n}}(z',0)$ is constant. In other words, $\lim_{x\to D} f_0(x)$ is a constant if and only if

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\Psi = -\log\det(h_{ij})_{1\le i,j\le n-1} - \log a\det(g'_{ij})_{1\le i,j\le n}.$$

Since $\Omega = \mathrm{Ric}(g') - \tilde{\omega} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\Psi$, $\lim_{x\to D} f_0(x)$ is a constant if and only if $\mathrm{Ric}(g_D) = \Omega|_D$. □

An appropriate choice of Ψ allows us to assume in the sequel that $f_0(x)$ converges uniformly to zero as $x \to D$.

The function $f_0(x)$ was only defined for x near D, but we can extend it smoothly to be zero along D, because $||S||^2\omega_0^n$ is a well-defined volume form over all $\overline{M}$. Hence, there exists a $\delta_0 > 0$ such that, in the neighborhood $V_0 := \{x \in \overline{M}; ||S(x)|| < \delta_0\}$, f_0 can be written as

$$f_0 = S \cdot u_1 + \overline{S} \cdot \overline{u}_1,$$

where u_1 is a C^∞ local section in $\Gamma(V_0, L_D^{-1})$.

Our goal now would be to construct a function ϕ_1 of the form $S \cdot \theta_1 + \overline{S} \cdot \overline{\theta}_1$, so that the corresponding $f_{\phi_1} = f_1$ vanishes at order 2 along D, and then proceed successively to higher order. Unfortunately, there is an obstruction to higher order approximation that lies in the kernel of the Laplacian on L_D^{-1} restricted to D. In order to deal with this difficulty, one must introduce $(-\log ||S||^2)$ terms in the expansion of ϕ_1, as pointed out in [F] and [LM], where the similar problem of finding expansions of the solution of the Monge-Ampère equation on a strictly pseudoconvex domain was treated. Further details can be found in the next section.

3.1 Inductive construction of the metrics $\{||.||_m\}_{m>0}$

Following the tecniques in [TY2], we now construct inductively a sequence of hermitian metrics $\{||.||_m\}_{m>0}$ on L_D such that, for any $m > 0$, there exists a $\delta_m > 0$ such that:

1. The corresponding kähler form ω_m associated to $||.||_m$ (as defined in (3.2)) is positive definite in $V_m := \{x \in \overline{M}; ||S(x)|| < \delta_m\}$; and

2. The function f_m associated to ω_m (as in the definition (3.1)) can be written in

V_m as

$$f_m = \sum_{k \geq m+1} \sum_{\ell=0}^{\ell_k} u_{k\ell}(-\log ||S||_m^2)^\ell, \tag{3.7}$$

where $u_{k\ell}$ are smooth functions on $\overline{V}_m$ that vanish to order k on D. In particular, the functions $u_{k\ell}$ can be written as

$$u_{k\ell} = \sum_{i+j=k} S^i \overline{S}^j \theta_{ij} + S^j \overline{S}^i \overline{\theta}_{ij}, \tag{3.8}$$

for $\theta_{ij} \in \Gamma(V_m, L_D^{-i} \bigotimes \overline{L_D}^{-j})$.

For simplicity, we will refer to functions that can be written in the form (3.8) as functions decaying to the order of $O(||S||^k)$.

We define $||.||_0 = ||.||$, and it is clear that $||.||_0$ satisfies the Conditions 1 and 2 above. Now we proceed on the inductive step: assuming the existence of $||.||_m$, we construct $||.||_{m+1}$. The next lemma gives a relation between f_m and f_ϕ, where $||.||_\phi = e^{-\phi/2}||.||_m$, and f_ϕ is associated to a smooth function ϕ on V_m of the form

$$\phi = (\sum_{i+j=k} S^i \overline{S}^j \theta_{ij} + S^j \overline{S}^i \overline{\theta}_{ij})(-\log ||S||_m^2)^k, \qquad \text{for } k \geq m+1.$$

Lemma 3.2 *Let $f_\phi(x)$ be defined as in Definition 3.1, associated to*

$$\omega_\phi = \frac{\sqrt{-1}}{2\pi} \frac{n}{n+1} n^{1/n} \partial\bar{\partial}(-\log ||S||_m^2 . \phi)^{\frac{n+1}{n}},$$

and f_m associated to ω_m. Then

$$\begin{aligned} f_\phi = f_m + nm\phi + \frac{k\phi}{(-\log ||S||_m^2)} \left(\frac{k-1}{(-\log ||S||_m^2)} + (m-1) \right) + \\ + (-\log ||S||_m^2)^{k-1} \sum_{i+j=m+1} \left\{ ij \left(S^i \overline{S}^j \theta_{ij} + S^j \overline{S}^i \overline{\theta}_{ij} \right) + \right. \\ \left. +(-\log ||S||_m^2) \left[-2(n+1)j \left(S^i \overline{S}^j \theta_{ij} + S^j \overline{S}^i \overline{\theta}_{ij} \right) + S^i \overline{S}^j \Box_m \theta_{ij} + S^j \overline{S}^i \overline{\Box}_m \overline{\theta}_{ij} \right] \right\} \\ + \sum_{k' \geq m+2} \sum_{\ell=0}^{\ell_{k'}} u_{k'\ell}(-\log ||S||_m^2)^\ell \end{aligned} \tag{3.9}$$

where $\Box_m = tr_{\omega_D}(\overline{D_m}D_m)$ *is the laplacian of the bundle* $L_D^{-1} \bigotimes L_D^{-j}$ *on* D *with respect to the hermitian metric* $||.||_m$*, and the functions* $u_{k'\ell}$ *decay as* $O(||S||^{k'})$.

The proof of the lemma will be postponed to the next chapter, so that we can now proceed to our inductive construction.

Proof of Proposition 3.1:

We want to find a function ϕ such that $||.||^2_{m+1} = e^{-\phi/2}||.||^2_m$ satisfies the Conditions 1 and 2, *ie*, we need to eliminate the terms $\sum_{\ell=0}^{\ell_{m+1}} u_{m+1,\ell}(-\log ||S||^2_m)^\ell$ from the expansion of f_m. Each of the $u_{m+1,\ell}$, $0 \le \ell \le m+1$ will be eliminated successively, as follows.

Step 1: Write $u_{m+1,\ell_{m+1}}$ as

$$u_{m+1,\ell_{m+1}} = \sum_{i+j=m+1} S^i\overline{S}^j(v_{ij} + v'_{ij}) + S^j\overline{S}^i(\overline{v}_{ij} + \overline{v'}_{ij}),$$

where $v'_{ij}|_D \in \mathrm{Ker}(\Box_m + n(m+1) - 1 - 2(n+1)j)$ and $v_{ij}|_D$ is perpendicular to that kernel.

If there is some i, j $(i+j = m+1)$ such that $v'_{ij}|_D \neq 0$, we use Lemma 3.2 with $k = \ell_{m+1} + 1$ and $\theta_{ij} = \frac{v'_{ij} \cdot k(m+1)}{-1-ij}$. Note that the constant $\frac{k(m+1)}{-1-ij}$ was chosen so as to eliminate the kernel term from the expression of $u_{m+1,\ell_{m+1}}$.

Now, Lemma 3.2 implies

$$f'_m := f_\phi = \left(\sum_{i+j=m+1} S^i\overline{S}^j v_{ij} + S^j\overline{S}^i\overline{v}_{ij}\right)(-\log ||S||^2_m)^{\ell_{m+1}} + \\ + \sum_{\ell=0}^{\ell_{m+1}-1} u_{m+1,\ell}(-\log ||S||^2_m)^\ell + \sum_{k' \ge m+1} \sum_{\ell=0}^{\ell_{k'}} u_{k'\ell}(-\log ||S||^2_m)^\ell. \quad (3.10)$$

After Step 1, we can assume (by replacing f_m by f'_m in (3.10)) that f_m has an expansion of the form

$$u_{m+1,\ell_{m+1}} = \sum_{i+j=m+1} S^i\overline{S}^j(v_{ij}) + S^j\overline{S}^i(\overline{v}_{ij}).$$

Step 2: Now we can solve

$$(\Box_m + n(m+1) - 1 - 2(n+1)j)\,\theta_{ij} = v_{ij}|_D \text{ on } D,$$

for $\theta_{ij} \in \Gamma(V_m, L_D^{-i} \bigotimes \overline{L_D}^{-j})$. Next, let us extend θ_{ij} to $\overline{M}$, and then apply again Lemma 3.2 with $k = \ell_{m+1}$ and θ_{ij} as above.

The new f_m will have an expansion of the form

$$f_m = \sum_{\ell=0}^{\ell_{m+1}-1} u_{m+1,\ell}(-\log ||S||_m^2)^\ell + O(||S||^{m+2}).$$

By repeating Steps 1 and 2 above, we are able to eliminate all the terms $\sum_{\ell=0}^{\ell_{m+1}} u_{m+1,\ell}(-\log ||S||_m^2)^\ell$ from the expansion of f_m.

Finally, let ϕ_m be the sum of all functions used in Steps 1 and 2, and define the new metric $||.||_{m+1}$ by $||.||_{m+1} = e^{-\phi/2}||.||_m$. Clearly the resulting metric satisfies Conditions 1 and 2 of (3.7). This completes the proof of the proposition. $\Box$.

Chapter 4

Proof of Lemma 3.2

This chapter is entirely devoted to proving Lemma 3.2, therefore completing the inductive construction of the metrics $||.||_m$.

According to Definition 3.1, we have

$$f_\phi(x) \quad = \quad -\log||S||^2 \;-\; \log\left(\frac{\omega_\phi^n}{\omega'^n}\right) \;-\; \Psi \quad = \quad f_m \;-\; \log\left(\frac{\omega_\phi^n}{\omega_m^n}\right). \quad (4.1)$$

Hence, we just need to compute the quotient $\frac{\omega_\phi^n}{\omega_m^n}$.

Denote by D_m (resp. D_ϕ) the covariant derivative of the metric $||.||_m$ (resp. $||.||_\phi$). Similarly, let $\tilde{\omega}_m$ and $\tilde{\omega}_\phi$ denote the corresponding curvature forms. The following relations are well-known:

$$\begin{aligned} D_\phi S &= D_m S - S\partial\phi \\ \tilde{\omega}_\phi &= \tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi \end{aligned} \quad (4.2)$$

For simplicity, set $\alpha_m = (-n\log||S||_m^2)$ and $\alpha_\phi = (-n\log||S||_\phi^2) = \alpha_m + n\phi$. By (3.4), we obtain

$$\begin{aligned} \omega_m^n &= \alpha_m\tilde{\omega}_m^{n-1}\wedge\left(\tilde{\omega}_m + \frac{n\sqrt{-1}}{2\pi\alpha_m}\frac{D_mS\wedge\overline{D_mS}}{|S|^2}\right) \\ &= \alpha_m\tilde{\omega}_m^n\left(1+\frac{1}{\alpha_m}\frac{||D_mS||_m^2}{||S||_m^2}\right), \end{aligned} \quad (4.3)$$

and

$$\omega_\phi^n = \alpha_\phi \tilde{\omega}_\phi^{n-1} \wedge \left(\tilde{\omega}_\phi + \frac{n\sqrt{-1}}{2\pi\alpha_\phi} \frac{D_\phi S \wedge \overline{D_\phi S}}{|S|^2} \right) =$$

$$= (\alpha_m + n\phi) \left(\tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi \right)^{n-1} \wedge \left[\left(\tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi \right) + \right.$$
$$\left. + \frac{n\sqrt{-1}}{2\pi\alpha_\phi} \left(\frac{D_m S - S\partial\phi}{S} \wedge \frac{\overline{D_m S} - \overline{S}\bar{\partial}\phi}{\overline{S}} \right) \right] =$$

$$= (\alpha_m + n\phi) \left(\tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi \right)^{n-1} \wedge \left\{ \left(\tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi \right) + \right.$$
$$\left. + \frac{n\sqrt{-1}}{2\pi\alpha_\phi} \left[\frac{D_m S \wedge \overline{D_m S}}{|S|^2} - \partial\phi \wedge \frac{\overline{D_m S}}{\overline{S}} - \bar{\partial}\phi \wedge \frac{D_m S}{S} + \partial\phi \wedge \bar{\partial}\phi \right] \right\} \quad (4.4)$$

Using the definition of ϕ, we get

$$\partial\phi = \sum_{i+j=m+1} (-\log ||S||_m^2)^k \left(D_m S^i \overline{S}^j \theta_{ij} + D_m S^j \overline{S}^i \overline{\theta}_{ij} + S^i \overline{S}^j D_m \theta_{ij} + \right.$$
$$\left. + S^j \overline{S}^i D_m \overline{\theta}_{ij}) \right) + k(-\log ||S||_m^2)^{k-1} (S^i \overline{S}^j \theta_{ij} + S^j \overline{S}^i \overline{\theta}_{ij}) \left(-\frac{D_m S}{S} \right), \quad (4.5)$$

so that

$$\partial\,\phi \wedge \frac{\overline{D_m S}}{\overline{S}} = \sum_{i+j=m+1} (-\log ||S||_m^2)^k \left\{ \overbrace{\left(i S^i \overline{S}^j \theta_{ij} + j S^j \overline{S}^i \overline{\theta}_{ij} \right)}^{(i)} \cdot \right.$$
$$\left. \left(\frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right) + \left(S^i \overline{S}^j D_m \theta_{ij} + S^j \overline{S}^i D_m \overline{\theta}_{ij} \right) \wedge \frac{\overline{D_m S}}{\overline{S}} \right\}$$
$$- \left(\frac{k\phi}{(-\log ||S||_m^2)} \right) \cdot \left(\frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right) + O(||S||^{2m+2}). \quad (4.6)$$

We will also need the expression for $\partial\bar{\partial}\phi$. After some computations using (4.5), it follows that

$$\partial\bar{\partial}\phi = (-\log||S||_m^2)^k \sum_{i+j=m+1} \Big\{ ij(S^i\overline{S}^j\theta_{ij} + S^j\overline{S}^i\overline{\theta}_{ij}) \left(\frac{D_mS\wedge\overline{D_mS}}{|S|^2}\right) +$$
$$+(jS^i\overline{S}^jD_m\theta_{ij} + iS^j\overline{S}^iD_m\overline{\theta_{ij}}) \wedge \frac{\overline{D_mS}}{\overline{S}} + \frac{D_mS}{S}\wedge(iS^i\overline{S}^j\overline{D_m}\theta_{ij} + jS^j\overline{S}^i\overline{D_m\theta_{ij}})+$$
$$+(S^i\overline{S}^jD_m\overline{D_m}\theta_{ij} + S^j\overline{S}^iD_m\overline{D_m\theta_{ij}})\Big\} + \left(\frac{k(m+1)\phi}{(-\log||S||_m^2)} + \frac{k(k-1)\phi}{(-\log||S||_m^2)^2}\right)\frac{D_mS\wedge\overline{D_mS}}{|S|^2}$$
$$-k(-\log||S||_m^2)^{k-1}\sum_{i+j=m+1}\Big\{(S^i\overline{S}^j\overline{D_m}\theta_{ij} + S^j\overline{S}^i\overline{D_m\theta}_{ij})\wedge\frac{D_mS}{S}$$
$$+(S^i\overline{S}^jD_m\theta_{ij} + S^j\overline{S}^iD_m\overline{\theta}_{ij})\wedge\frac{\overline{D_mS}}{\overline{S}}\Big\} \quad (4.7)$$

We can therefore conclude from a simple analysis of (4.7) that

$$||S||_m^2(\partial\bar{\partial}\phi)^\ell \wedge \tilde{\omega}_m^{n-\ell} = \tilde{\omega}_m^n O(||S||^{2m+2}) \text{ for } \ell \geq 2. \quad (4.8)$$

The above ingredients are going to be used in the completion of the proof of Lemma 3.2.

Proof of Lemma 3.2:

Recall that we only need to compute the quotient $\frac{\omega_\phi^n}{\omega_m^n}$. Formulas (4.3) and (4.4) then provide

$$\frac{\omega_\phi^n}{\omega_m^n} = \frac{||S||_m^2\alpha_\phi}{\alpha_m(||S||_m^2 + 1/\alpha_m||D_mS||_m^2)\tilde{\omega}_m^n}\cdot$$
$$\cdot\left\{\left(\tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi\right)^{n-1}\wedge\left\{\left(\tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi\right)+\right.\right.$$
$$\left.\left.+\frac{n\sqrt{-1}}{2\pi\alpha_\phi}\left[\frac{D_mS\wedge\overline{D_mS}}{|S|^2} - \partial\phi\wedge\frac{\overline{D_mS}}{\overline{S}} - \bar{\partial}\phi\wedge\frac{D_mS}{S} + \partial\phi\wedge\bar{\partial}\phi\right]\right\}\right\} \quad (4.9)$$

Recall the relation (4.8), that allows us to simplify the expression above to

$$\frac{\omega_\phi^n}{\omega_m^n} = \frac{||S||_m^2 \alpha_\phi}{(\alpha_m ||S||_m^2 + ||D_m S||_m^2)\tilde{\omega}_m^n} \cdot \left\{ \left[\tilde{\omega}_m^{n-1} + (n-1)\tilde{\omega}_m^{n-2} \wedge \partial\bar{\partial}\phi\right] \wedge \right.$$
$$\left[\tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi + \frac{n\sqrt{-1}}{2\pi\alpha_\phi}\left(\frac{D_m S \wedge \overline{D_m S}}{|S|^2} - \partial\phi \wedge \frac{\overline{D_m S}}{\overline{S}} - \right.\right.$$
$$\left.\left.\left. -\bar{\partial}\phi \wedge \frac{D_m S}{S} + \partial\phi \wedge \bar{\partial}\phi\right)\right]\right\} + O(||S||^{m+2}) \quad (4.10)$$

Notice that

$$\partial\phi \wedge \frac{\overline{D_m S}}{\overline{S}} + \bar{\partial}\phi \wedge \frac{D_m S}{S} = \left((m+1)\phi + \frac{2k\phi}{\log ||S||_m^2}\right)\frac{D_m S \wedge \overline{D_m S}}{|S|^2} + O(m+1),$$

since the term (i) overbraced in (4.6) (that appears reflected with respect to i and j for the conjugate expression) will give rise to the term involving $(m+1)\phi$. Hence,

$$\frac{\omega_\phi^n}{\omega_m^n} = \frac{||S||_m^2 \alpha_\phi}{\alpha_m(||S||_m^2 + 1/\alpha_m ||D_m S||_m^2)\tilde{\omega}_m^n} \cdot$$
$$\left\{ \left[\tilde{\omega}_m^{n-1} + (n-1)\tilde{\omega}_m^{n-2}\partial\bar{\partial}\phi)\right] \wedge \left[\tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi + \right.\right.$$
$$\left.\left.\frac{n\sqrt{-1}}{2\pi\alpha_\phi}\left(1 - (m+1)\phi - \frac{2k\phi}{\log ||S||_m^2}\right)\frac{D_m S \wedge \overline{D_m S}}{|S|^2} + \underbrace{\partial\phi \wedge \bar{\partial}\phi}_{(*)}\right]\right\} +$$
$$+ O(||S||^{m+2}). \quad (4.11)$$

Notice that the term $(*)$ underbraced above is of order of at least $O(||S||^{m+2})$. Thus,

$$\frac{\omega_\phi^n}{\omega_m^n} = \frac{||S||_m^2 \alpha_\phi}{\alpha_m(||S||_m^2 + 1/\alpha_m ||D_m S||_m^2)\tilde{\omega}_m^n} \cdot \left\{\tilde{\omega}_m^{n-1} \wedge \right.$$
$$\left[\tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi + \frac{n\sqrt{-1}}{2\pi\alpha_\phi}\left(1 - (m+1)\phi - \frac{2k\phi}{\log ||S||_m^2}\right)\frac{D_m S \wedge \overline{D_m S}}{|S|^2}\right] +$$
$$\left. +(n-1)\tilde{\omega}_m^{n-2} \wedge \left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi\right) \wedge \left(\tilde{\omega}_m + \frac{n\sqrt{-1}}{2\pi\alpha_\phi}\frac{D_m S \wedge \overline{D_m S}}{|S|^2}\right)\right\} +$$
$$+ O(||S||^{m+2}). \quad (4.12)$$

Recall that

$$
\begin{aligned}
||S||_m^2\alpha_\phi\tilde{\omega}_m^{n-1} \quad & \wedge \quad \left(\tilde{\omega}_m + \frac{n\sqrt{-1}}{2\pi\alpha_\phi}\frac{D_mS\wedge\overline{D_mS}}{|S|^2}\right) = \\
& = \quad ||S||_m^2\tilde{\omega}_m^n\left(\alpha_\phi + \frac{||D_mS||_m^2}{||S||_m^2}\right) \\
& = \quad \tilde{\omega}_m^n\left(\alpha_m||S||_m^2 + ||D_mS||_m^2 - \overbrace{n\phi||S||_m^2}^{O(||S||^{m+2})}\right) \\
& = \quad ||S||_m^2\tilde{\omega}_m^n\left(\alpha_m + \frac{||D_mS||_m^2}{||S||_m^2}\right) + O(||S||^{m+2}). \qquad (4.13)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{\omega_\phi^n}{\omega_m^n} = 1 + & \frac{||S||_m^2\alpha_\phi}{\alpha_m(||S||_m^2 + 1/\alpha_m||D_mS||_m^2)\tilde{\omega}_m^n}\cdot \\
& \left\{n\tilde{\omega}_m^{n-1}\wedge\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi\right) + \phi\left(-(m+1) - \frac{2k}{(-\log||S||_m^2)}\right)\cdot\right. \\
& \left[\tilde{\omega}_m^{n-1}\wedge\frac{n\sqrt{-1}}{2\pi\alpha_\phi}\frac{D_mS\wedge\overline{D_mS}}{|S|^2}\right] + \\
& \left. +(n-1)\tilde{\omega}_m^{n-2}\wedge\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi\right)\wedge\left(\frac{n\sqrt{-1}}{2\pi\alpha_\phi}\frac{D_mS\wedge\overline{D_mS}}{|S|^2}\right)\right\} + O(||S||^{m+2}) = \\
& = 1 + \left(-(m+1) - \frac{2k}{(-\log||S||_m^2)}\right)\phi + \\
& \frac{||S||_m^2\alpha_\phi}{\alpha_m(||S||_m^2 + 1/\alpha_m||D_mS||_m^2)\tilde{\omega}_m^n}\cdot\left\{\overbrace{n\tilde{\omega}_m^{n-1}\wedge\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi\right)}^{(a)} + \right. \\
& \left.\overbrace{(n-1)\tilde{\omega}_m^{n-2}\wedge\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi\right)\wedge\left(\frac{n\sqrt{-1}}{2\pi\alpha_\phi}\frac{D_mS\wedge\overline{D_mS}}{|S|^2}\right)}^{(b)}\right\} + O(||S||^{m+2}). \qquad (4.14)
\end{aligned}
$$

So, we have

$$(a) = \left\{ (-\log ||S||_m^2)^k \sum_{i+j=m+1} \left[ij(S^i \overline{S}^j \theta_{ij} + S^j \overline{S}^i \overline{\theta}_{ij}) \right] \right.$$
$$\left. + \frac{(m+1)k\phi}{(-\log ||S||_m^2)} + \frac{(k-1)k\phi}{(-\log ||S||_m^2)^2} \right\} \tilde{\omega}_m^{n-1} \wedge \left(\frac{n\sqrt{-1}}{2\pi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right) \quad (4.15)$$

and

$$(b) = n(n-1)\tilde{\omega}_m^{n-2} \wedge \left(\frac{n\sqrt{-1}}{2\pi\alpha_\phi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right) \wedge$$
$$\left((-\log ||S||_m^2)^k \sum_{i+j=m+1} \cdot \left\{ \left(S^i \overline{S}^j D_m \overline{D_m} \theta_{ij} + S^j \overline{S}^i D_m \overline{D_m} \overline{\theta}_{ij} \right) + \right.\right.$$
$$\left.\left. + \left(S^i D_m \overline{D_m S}^j \theta_{ij} + D_m \overline{D_m} S^j \overline{S}^i \overline{\theta}_{ij} \right) \right\} \right) \quad (4.16)$$

So,

$$\frac{\omega_\phi^n}{\omega_m^n} = 1 + \left(-(m+1) - \frac{2k}{(-\log ||S||_m^2)} \right) \phi + \frac{||S||_m^2 \alpha_\phi}{\alpha_m(||S||_m^2 + 1/\alpha_m ||D_m S||_m^2)\tilde{\omega}_m^n} \cdot$$
$$\cdot (-\log ||S||_m^2)^k \sum_{i+j=m+1} \left[ij(S^i \overline{S}^j \theta_{ij} + S^j \overline{S}^i \overline{\theta}_{ij}) \right] \left(\tilde{\omega}_m^{n-1} \wedge \right.$$
$$\left. \wedge \left(\frac{n\sqrt{-1}}{2\pi\alpha_\phi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right) \right) + \frac{||S||_m^2 \alpha_\phi}{\alpha_m(||S||_m^2 + 1/\alpha_m ||D_m S||_m^2)\tilde{\omega}_m^n} \cdot$$
$$\overbrace{\cdot \phi \left(\frac{n(m+1)k}{(-\log ||S||_m^2)} + \frac{n(k-1)k}{(-\log ||S||_m^2)^2} \right)}^{(\star)} \left(\tilde{\omega}_m^{n-1} \wedge \frac{n\sqrt{-1}}{2\pi\alpha_\phi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right) + (b), \quad (4.17)$$

and the term $(\star)$ can be simplified via (4.13), giving

$$\frac{\omega_\phi^n}{\omega_m^n} = 1 + \left(-(m+1) + \frac{k(m-1)}{(-\log ||S||_m^2)} + \frac{(k-1)k}{(-\log ||S||_m^2)^2} \right) \phi +$$
$$\frac{||S||_m^2 \alpha_\phi}{\alpha_m(||S||_m^2 + 1/\alpha_m ||D_m S||_m^2)\tilde{\omega}_m^n} \left\{ (-\log ||S||_m^2)^k \cdot \right.$$
$$\left. \sum_{i+j=m+1} \left[ij(S^i \overline{S}^j \theta_{ij} + S^j \overline{S}^i \overline{\theta}_{ij}) \right] \right\} \tilde{\omega}_m^{n-1} \wedge \left(\frac{n\sqrt{-1}}{2\pi\alpha_\phi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right) + (b) \quad (4.18)$$

Observe that the relations

$$\begin{aligned} \overline{D_m} D_m S^j &= -D_m \overline{D_m} S^j + j S^j \tilde{\omega}_m \\ \overline{D_m} D_m \theta_{ij} &= -D_m \overline{D_m} \theta_{ij} - (i-j)\theta_{ij}\tilde{\omega}_m \end{aligned} \tag{4.19}$$

imply that

$$\sum_{i+j=m+1} \left(S^i \overline{S}^j D_m \overline{D_m} \theta_{ij} + S^j \overline{S}^i D_m \overline{D_m \theta_{ij}} \right) =$$
$$\sum_{i+j=m+1} \left\{ \left(S^i \overline{S}^j \overline{D_m} D_m \theta_{ij} + S^j \overline{S}^i \overline{D_m} D_m \overline{\theta}_{ij} \right) - \left(S^i \overline{S}^j \theta_{ij} - S^j \overline{S}^i \overline{\theta}_{ij} \right) (i-j)\tilde{\omega}_m \right\}. \tag{4.20}$$

Hence,

$$\begin{aligned} \frac{\omega_\phi^n}{\omega_m^n} = 1 + \left(-(m+1) + \frac{k(m-1)}{(-\log\|S\|_m^2)} + \frac{k(k-1)}{(-\log\|S\|_m^2)^2} \right) \phi + \\ \frac{\|S\|_m^2 \alpha_\phi}{\alpha_m (\|S\|_m^2 + 1/\alpha_m \|D_m S\|_m^2)\tilde{\omega}_m^n} \cdot \{(-\log\|S\|_m^2)^k \cdot \\ \sum_{i+j=m+1} \underbrace{[\alpha_\phi ij (S^i \overline{S}^j \theta_{ij} + S^j \overline{S}^i \overline{\theta}_{ij})]}_{(\star_1)} \tilde{\omega}_m^{n-1} \wedge \left(\frac{n\sqrt{-1}}{2\pi\alpha_\phi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right) + \\ +(n-1)\tilde{\omega}_m^{n-2} \wedge \left(\frac{n\sqrt{-1}}{2\pi\alpha_\phi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right) \wedge \\ \wedge (-\log\|S\|_m^2)^k \left[- \sum_{i+j=m+1} \cdot \left(S^i \overline{S}^j \overline{D_m} D_m \theta_{ij} + S^j \overline{S}^i \overline{D_m} D_m \overline{\theta}_{ij} \right) + \right. \\ \left. - \sum_{i+j=m+1} \cdot \left(S^i \overline{S}^j \theta_{ij} + S^j \overline{S}^i \overline{\theta}_{ij} \right) (i-j)\tilde{\omega}_m \cdot \right] \Bigg\} + O(\|S\|^{m+2}) = \\ = 1 + \left(-(m+1) + \frac{k(m-1)}{(\log\|S\|_m^2)} + \frac{k(k-1)}{(\log\|S\|_m^2)^2} \right) \phi - \\ - (-\log\|S\|_m^2)^k \cdot \sum_{i+j=m+1} \cdot \left(S^i \overline{S}^j \Box_m \theta_{ij} + S^j \overline{S}^i \overline{\Box}_m \overline{\theta}_{ij} \right) + (-\log\|S\|_m^2)^k \cdot \\ \cdot \sum_{i+j=m+1} \underbrace{\alpha_m ij}_{(\star_2)} + (n-1)((m+1)-2j) \left(S^i \overline{S}^j \theta_{ij} + S^j \overline{S}^i \overline{\theta}_{ij} \right) + O(\|S\|^{m+2}). \end{aligned} \tag{4.21}$$

Notice that we actually can replace the term $(\star_1)$ by $(\star_2)$, since the function ϕ is assumed to be of order $O(||S||^{m+1})$, and $\alpha_\phi = \alpha_m + n\phi$. This implies that the residual term from this substitution will lie in $O(||S||^{m+2})$. Therefore, we conclude

$$\begin{aligned}\frac{\omega_\phi^n}{\omega_m^n} = 1 + n(m+1)\phi + \frac{k}{(-\log||S||_m^2)}\left(\frac{k-1}{(-\log||S||_m^2)} + (m-1)\right) + \\ + \sum_{i+j=m+1} \cdot \left(S^i\overline{S}^j\Box_m\theta_{ij} + S^j\overline{S}^i\overline{\Box}_m\overline{\theta}_{ij}\right) + \\ + 2(n-1)(-\log||S||_m^2)^k \sum_{i+j=m+1} j\left(S^i\overline{S}^j\theta_{ij} + S^j\overline{S}^i\overline{\theta}_{ij}\right) + \\ + (\log||S||_m^2)^{k-1} \sum_{i+j=m+1} ij\left(S^i\overline{S}^j\theta_{ij} - S^j\overline{S}^i\overline{\theta}_{ij}\right) + O(||S||^{m+2}). \quad (4.22)\end{aligned}$$

Finally,

$$\begin{aligned}f_\phi = f_m - \log\left(\frac{\omega_\phi^n}{\omega_m^n}\right) = \\ = nm\phi + \frac{k}{(-\log||S||_m^2)}\left(\frac{k-1}{(-\log||S||_m^2)} - (m-1)\right) + \\ + (-\log||S||_m^2)^k \sum_{i+j=m+1} \cdot \left(S^i\overline{S}^j\Box_m\theta_{ij} + S^j\overline{S}^i\overline{\Box}_m\overline{\theta}_{ij}\right) + \\ + (-\log||S||_m^2)^{k-1} \sum_{i+j=m+1} ij\left(S^i\overline{S}^j\theta_{ij} - S^j\overline{S}^i\overline{\theta}_{ij}\right) - \\ - 2(n+1)(-\log||S||_m^2)^k \sum_{i+j=m+1} j\left(S^i\overline{S}^j\theta_{ij} - S^j\overline{S}^i\overline{\theta}_{ij}\right) + O(||S||^{m+2}), \quad (4.23)\end{aligned}$$

which proves the lemma. □

The inductive construction of the metrics $||.||_m$ is now completed.

Chapter 5

Complete Kähler Metrics on M

In this section, we shall complete the proof of Theorem 1.1. In particular, it is going to be necessary to consider the asymptotic behavior of the Riemann Curvature tensor of the metrics constructed in the last chapter.

For each $m \geq 1$, consider the function f_m constructed in Chapter 3. For this choice, let the corresponding

$$\omega_m = \frac{\sqrt{-1}}{2\pi(n+1)}\partial\bar{\partial}(-n\log||S||_m^2)^{\frac{n+1}{n}}$$

define a $(1,1)-$form on M. If δ_m is sufficiently small, ω_m is positive definite on $V_m = \{||S(x)|| \leq \delta_m\}$, and defines a Kähler metric g_m.

Lemma 5.1 *The Kähler manifolds $(V_m, \partial V_m, g_m)$ are all complete, equivalent to each other near D, and for each $m > 0$, the function*

$$\rho = \frac{2}{n+1}(-n\log||S||_m^2)^{\frac{n+1}{2n}}$$

is equivalent to any distance function from a fixed point in V_m near D.

Proof: Fix $m > 0$. We have

$$|\nabla_m\rho|_{g_m}^2 = \frac{\sqrt{-1}}{2\pi}\frac{\partial\rho\wedge\bar{\partial}\rho\wedge\omega_m^{n-1}}{\omega_m^n}.$$

Since

$$\partial\rho = n^{\frac{n+1}{2n}}(-\log ||S||_m^2)^{\frac{1-n}{2n}}\frac{D_mS}{S}$$

we have that

$$\frac{\sqrt{-1}}{2\pi}\partial\rho\wedge\bar{\partial}\rho\wedge\omega_m^{n-1} = (-n\log ||S||_m^2)^{\frac{1-n}{n}}(-n\log ||S||_m^2)^{\frac{n-1}{n}}\tilde{\omega}_m^{n-1}\wedge\frac{\sqrt{-1}}{2\pi}\frac{D_mS\wedge\overline{D_mS}}{|S|^2}. \tag{5.1}$$

So, using (3.4), we have

$$\begin{aligned} |\nabla_m\rho|^2_{g_m} &= \frac{1}{n}\frac{\tilde{\omega}_m^{n-1}\wedge\frac{n\sqrt{-1}}{2\pi}\frac{D_mS\wedge\overline{D_mS}}{|S|^2}}{(-n\log ||S||_m^2)\tilde{\omega}_m^n+\tilde{\omega}_m^{n-1}\wedge\frac{n\sqrt{-1}}{2\pi}\frac{D_mS\wedge\overline{D_mS}}{|S|^2}} = \\ &= \frac{1}{n}\frac{||D_mS||_m^2}{(-n\log ||S||_m^2)+||D_mS||_m^2}. \end{aligned} \tag{5.2}$$

Recall that $||D_mS||_m^2$ is never zero, and $\lim_{||S||_m\to 0} -\log ||S||_m^2 \cdot ||S||_m^2 = 0$, hence

$$|\nabla_m\rho|^2_{g_m} \xrightarrow{||S||_m\to 0} \frac{1}{n},$$

proving that ρ is equivalent to any distance function from the boundary near D.

Also, since $\rho\to\infty$ when $x\to D$, the Kähler manifold $(V_m, \partial V_m, g_m)$ is complete.

We claim that all the metrics g_m are equivalent near D. To check the claim, note first that each $\tilde{\omega}_m$ is the curvature form of the metric $||.||_m$, hence, for every $m,\ell\in\mathbb{N}$, $\tilde{\omega}_m$ is equivalent to $\tilde{\omega}_\ell$ near D. The claim then follows from Equation (3.3), that relates the expressions for ω_m and $\tilde{\omega}_m$.

Finally here is a remark about the volume growth of $(V_m, \partial V_m, g_m)$: since ω_m^n is equivalent to $\tilde{\omega}_m^n(-n\log ||S||_m^2)$, it suffices to consider the integral

$$\int_{||S(x)||_m\geq e^{-1/2n}\rho^{\frac{2n}{n+1}}}(-n\log ||S||_m^2)\tilde{\omega}_m^n,$$

which is of order $\rho^{\frac{2n}{n+1}}$. $\square$

5.1 Decay of the curvature tensor

In the sequel we are going to carry out the estimates of the Riemann curvature tensor $R(g_m)$ corresponding to the metric g_m which are involved in the statement of Theorem (1.1). Let us begin with the following lemma:

Lemma 5.2 *Let $(V_m, \partial V_m, g_m)$ be complete, Kähler manifolds with boundary defined as in Lemma 5.1. Then the norm of $R(g_m)$ with respect to the metric g_m decays at the order of at least $(-n\log ||S||_m^2)^{\frac{-1}{n}}$ near D.*

Proof: We will prove the statement in local coordinates, as follows:

There exists a finite covering U_t of D in $\overline{M}$ such that for each t, there is a local uniformization $\Pi_t : \tilde{U}_t \to U_t$ such that ${\Pi_t}^{-1}(D)$ is smoooth in $\tilde{U}_t$, and for some local coordinate system $(z_1, \ldots, z_n)$ in $\tilde{U}_t$ with $S = z_n$ and $z' = (z_1, \ldots, z_{n-1})$ coordinates along D, we have

$$\sum_{i,j,k,l=1}^{n} R({\Pi_t}^*(g_m))_{i\bar{j}k\bar{l}}(z', z_n)\xi^i\xi^{\bar{j}}\xi^k\xi^{\bar{l}} =$$

$$= (-n\log|z_n|^2)^{1/n} \sum_{i,j,k,l=1}^{n} R({\Pi_t}^*(g_D|_{{\Pi_t}^{-1}(D)}))_{i\bar{j}k\bar{l}}\xi^i\xi^{\bar{j}}\xi^k\xi^{\bar{l}} + \\ + O((-n\log|z_n|^2)^{-1/n}), \quad (5.3)$$

for any g_m-unit vector $(\xi^1, \ldots, \xi^n)$, where g_D is the kähler metric defined by the restriction of the curvature form $\tilde{\omega}$ to the divisor.

Without loss of generality, assume $U_t \cap \overline{M}$ is smooth.

For any $x \in U_t \cap M$, we will choose local coordinates $(z_1, \ldots, z_n)$ for a neighborhood of x such that

- The defining section S of the divisor is given by z_n.
- The curvature form $\tilde{\omega}_m$ of $||.||_m$ is represented by the tensor $(h_{i\bar{j}})$ in those coordinates, and $(h_{i\bar{j}})$ satisfy

$$h_{i\bar{j}}(x) = \delta_{ij}; \quad \frac{\partial h_{i\bar{j}}}{\partial z_k}(x) = 0 \ \text{ if } j < n; \quad \frac{\partial h_{i\bar{j}}}{\partial \bar{z}_l}(x) = 0 \ \text{ if } i < n;$$

- The hermitian metric $||.||_m$ is represented by a positive funcion a with $a(x) = 1$, $da(x) = 0$ and $d(\frac{\partial a}{\partial z_k})(x) = 0$.

In order to simplify notation, let us write $B = B(|z_n|) = (-n \log |z_n|^2)$, and let us drop the subscripts for the metric g_m, to be denoted by g from now on.

Formula (3.3) implies that

$$\sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge d\bar{z}_j = B^{1/n} \left(\sum_{i,j=1}^{n-1} h_{i\bar{j}} dz_i \wedge d\bar{z}_j + \frac{dz_n \wedge d\bar{z}_n}{|z_n|^2 B} \right), \tag{5.4}$$

and hence

$$g^{i\bar{j}}(x) = \begin{cases} O(B^{-1/n}) & \text{if } i = j \text{ and } i < n \\ o(B^{-1/n}) & \text{if } i \neq j \text{ and } i, j < n \\ O(|z_n|^2 B^{-1/n}) & \text{if } i = j = n \end{cases}$$

Computations involving (5.4) lead to

$$\frac{\partial g_{i\bar{j}}}{\partial z_k} = B^{1/n} \left[\frac{\partial h_{i\bar{j}}}{\partial z_k} - \frac{1}{z_n B} \left(\delta_{kn} h_{i\bar{j}} + \delta_{in} h_{k\bar{j}} + \delta_{in}\delta_{jn}\delta_{kn} \left(\frac{n-1}{B} + \frac{1}{|z_n|^2} \right) \right) \right]$$

and

$$\begin{aligned} \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} = B^{1/n} \Bigg\{ & \frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} - \frac{1}{\bar{z}_n B} \left(\delta_{ln} \frac{\partial h_{i\bar{j}}}{\partial z_k} + \delta_{jn} \frac{\partial h_{i\bar{l}}}{\partial z_k} \right) - \\ & - \frac{1}{z_n B} \left(\delta_{kn} \frac{\partial h_{i\bar{j}}}{\partial \bar{z}_l} + \delta_{in} \frac{\partial h_{k\bar{j}}}{\partial \bar{z}_l} \right) + \frac{1-n}{|z_n|^2 B^2} \left(\delta_{ln}(\delta_{kn} h_{i\bar{j}} + \delta_{in} h_{k\bar{j}}) + \right. \\ + \; & \left. \delta_{jn}(\delta_{kn} h_{i\bar{l}} + \delta_{in} h_{k\bar{l}}) \right) + \frac{\delta_{in}\delta_{jn}\delta_{kn}\delta_{ln}}{|z_n|^4 B^3} \left(|z_n|^2 (n-1)(1-2n) + (1-n)B + B^2 \right) \Bigg\}. \end{aligned} \tag{5.5}$$

If $(\xi^1, \ldots, \xi^n)$ is a g-unit tangent vector, then

$$\begin{cases} |\xi^i|^2 \leq C B^{-1/n} & \text{if } i < n \\ |\xi^i|^2 \leq C |z_n|^2 B^{(n-1)/n} & \text{if } i = n, \end{cases}$$

where C is a constant that does not depend neither on the unit vector $(\xi^1, \ldots, \xi^n)$ nor on the point $x \in D$.

Now we have all the ingredients to estimate the decay of the Riemann curvature tensor. In local coordinates,

$$R(g)_{i\bar{j}k\bar{l}}(x)(\xi^i\bar{\xi}^j\xi^k\bar{\xi}^l)(x) = \left[\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l}(x) + \sum_{u,v=1}^{n} g^{u\bar{v}}(x)\frac{\partial g_{i\bar{v}}}{\partial z_k}(x)\frac{\partial g_{u\bar{j}}}{\partial \bar{z}_l}(x)\right](\xi^i\bar{\xi}^j\xi^k\bar{\xi}^l) =$$

$$= -B^{1/n}\left\{\overbrace{\frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l}(x)(\xi^i\bar{\xi}^j\xi^k\bar{\xi}^l)}^{(a)} + + \overbrace{\frac{4(1-n)|\xi^n|^2}{|z_n|^2 B^2}\left(\sum_{i=1}^{n}|\xi^i|^2\right)}^{(b)}\right.$$

$$\left. + \overbrace{\frac{|\xi^n|^4}{|z_n|^4 B^3}\left(|z_n|^2(n-1)(1-2n) + (1-n)B + B^2\right)}^{(c)}\right\} +$$

$$+ B^{2/n}\sum_{u,v=1}^{n} g^{u\bar{v}}(x)\left[\overbrace{-\frac{\xi^n}{z_n B}\left(2\xi^i h_{i\bar{v}} + \xi^n\delta_{i\bar{v}}\left(\frac{n-1}{B} + \frac{1}{|z_n|^2}\right)\right)}^{(d)}\right] \times$$

$$\times \left[\overbrace{-\frac{\bar{\xi}^n}{z_n B}\left(2\bar{\xi}^j h_{u\bar{j}} + \bar{\xi}^n\delta_{u\bar{n}}\left(\frac{n-1}{B} + \frac{1}{|z_n|^2}\right)\right)}^{(e)}\right] \quad (5.6)$$

We proceed on bounding each of the terms separately.

Using the estimates (5.1) for $|\xi^n|^2$ and our previous choice of local coordinates, we obtain, when z_n approaches zero,

$$(b) \le \frac{C|z_n|^2 B^{(n-1)/n}}{|z_n|^2 B^2}(B^{-1/n} + |z_n|^2 B^{(n-1)/n}) \le C B^{-(n+2)/n},$$

where C denotes a uniform constant.

The term (c) can be bounded as follows, as $z_n \to 0$:

$$(c) \leq \frac{C|z_n|^4 B^{(n-1)/n}}{|z_n|^4 B^3}(B^2+(1-n)B+|z_n|^2(n-1)(1-2n)) \leq CB^{-1/n}(B^{-1}+1) \leq CB^{-1/n}.$$

Now, notice that the expression (d) needs special attention,

$$\begin{aligned} (d) &\leq C\frac{|z_n|B^{(n-1)/2n}}{|z_n|B}\left(B^{-1/n}+|z_n|B^{(n-1)/2n}\left(\frac{n-1}{B}+\frac{1}{|z_n|^2}\right)\right) & (5.7)\\ &\leq C(B^{-(n+2)/2n}+|z_n|^{-1}B^{-1/n}), & (5.8)\end{aligned}$$

due to the presence of a term involving $|z_n|^{-1}$. However, our estimates for

$$g^{u\bar{v}} = B^{-1/n}\left[(1-\delta_{un}\delta_{vn})O(1)+\delta_{un}\delta_{vn}O(|z_n|^2)\right]$$

show that this term is compensated by the last term of the above expression.

The estimate for the decay of (e) is analogous to the case of (d), and will henceforth be omitted.

In conclusion, we have

$$R(g)_{i\bar{j}k\bar{l}}(x)(\xi^i\bar{\xi}^j\xi^k\bar{\xi}^l)(x) = B^{1/n}\frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l}(x)(\xi^i\bar{\xi}^j\xi^k\bar{\xi}^l) + O(B^{-1/n}),$$

which implies the expression (5.3), and concludes the proof of the lemma. □

The reader may also notice that Lemma 5.2 completes the proof of Proposition 3.1.

The following result is a trivial consequence of Lemma 5.1 combined to Lemma 5.2.

Corollary 5.1 *Let $(V_m, \partial V_m, g_m)$ be the complete, kähler manifolds with boundary as in Lemma 5.2. Then the norm of the cuvature tensor $R(g_m)$ with respect to the metric g_m decays at the order of $\rho^{-\frac{2}{n+1}}$, where ρ is any distance function from a fixed point in V_m near D.*

We are finally able to complete the proof of Theorem 1.1.

5.2 Proof of Theorem 1.1

In what follows, we keep the preceding setting and notations.

Since the divisor is assumed to be ample in $\overline{M}$, there exists a hermitian metric $||.||'$ on L_D with its curvature form $\tilde{\omega}'$ positive definite on D.

Fix an integer $k \geq \varepsilon$, and write, for $\varepsilon > 0$,

$$\omega_{g_\varepsilon} = \omega_k + C_\varepsilon \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(-||S||')^\varepsilon, \qquad C_\varepsilon > 0, \tag{5.9}$$

where ω_k is the Kähler form defined on Section 3.1. The Kähler form ω_{g_ε} is positive definite on M, and gives rise to a complete kähler metric g on M.

Let $\delta > 0$ be such that $V_\delta = \{||S(x)|| < \delta\} \subset V_k$. On V_δ, ω_{g_ε} satisfies $\mathrm{Ric}(g_\varepsilon) - \Omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f$, and we want to estimate the decay of f at infinity.

Also, on V_δ, $\mathrm{Ric}(g_k) - \Omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f_k$, which implies that

$$\begin{aligned} f = f_k - \log\frac{\omega_{g_\varepsilon}^n}{\omega_k^n} &= f_k - \log\left(\frac{w_k^n + C_\varepsilon||S||'^{(\varepsilon-1)}w_k^{n-1}\wedge\frac{n\sqrt{-1}}{2\pi}(D'S\wedge\bar{D}'\bar{S})}{\omega_k^n}\right) = \\ &= f_k - C_\varepsilon||S||'^{2(\varepsilon-1)}||D'S||_{g_k}, \end{aligned} \tag{5.10}$$

where $D'S$ denotes the covariant derivative of the metric $||.||'$. Hence, in order to estimate the decay of f, it suffices to study the decay of $||D'S||_{g_k}$. To do this, we are going to introduce a suitable new coordinate system on V_δ.

Because D is admissible, it follows that total space of the unit sphere bundle of $L_D|_D$ (with respect to the metric $||.||_k$) is a smooth manifold of real dimension $2n+1$, to be denoted by M_1.

Since L_D is simply the normal bundle of D in $\overline{M}$, there exists a diffeomorphism

$$\Psi : M_1 \times (0,\delta) \to V_\delta$$

induced by the exponential map of $(\overline{M}, ||.||_k)$ along D.

It is also known that the Kähler form of g_k is given by

$$\omega_k = \frac{\sqrt{-1}}{2\pi}\frac{n^{1+1/n}}{n+1}\partial\bar{\partial}(-\log(||S||^2 e^{2\phi_k}))^{\frac{n+1}{n}},$$

where ϕ_k is a smooth function on $\overline{M}$, that can be written as $\sum_{\kappa\geq\varepsilon}\sum_{\ell=0}^{\ell_\kappa} u_{\kappa\ell}(-\log||S||_k^2)^\ell$, where $u_{\kappa\ell}$ are smooth functions on $\overline{V}_k$ that vanish to order κ on D.

Combining the facts above, the pullback of g_k under Ψ on $M_1 \times (0,\delta)$ is given by

$$\begin{aligned}\Psi^* g_k &= (-n\log(||S||^2))^{\frac{1}{n}} g(||S||, ||S||\log(||S||)) + \\ &+ (-n\log(||S||^2))^{\frac{1-n}{n}} d\left((-n\log(||S||^2))^{\frac{1}{n}}\right) h(||S||, ||S||\log(||S||)) + \\ &+ (-n\log(||S||^2))^{\frac{1-2n}{n}} d\left((-n\log(||S||^2))^{\frac{1}{n}}\right)^2 u(||S||, ||S||\log(||S||))\end{aligned} \quad (5.11)$$

Here $g(.,.)$, $h(.,.)$ and $u(.,.)$ are C^∞ families of metrics, $1-$tensors and functions on M_1, such that for each fixed integer $\ell > 0$, there exists a constant K_ℓ that bounds all covariant derivatives (with respect to a fixed metric $\tilde{h}$ on M_1) of $g(t_0,t_1)$, $h(t_0,t_1)$ and $u(t_0,t_1)$ up to order ℓ, for all $t_0 \in [0,\delta]$, $t_1 \in [0, \delta\log(\delta)]$.

Setting $\rho = (-n\log(||S||^2))^{\frac{n+1}{2n}}$, (5.11) becomes

$$\Psi^* g_k = \rho^{\frac{2}{n+1}} g(.,.) + \rho^{\frac{2(1-n)}{n+1}} d(\rho^{\frac{2}{n+1}}) h(.,.) + \rho^{\frac{2(1-2n)}{n+1}} d(\rho^{\frac{2}{n+1}})^2 u(.,.), \quad (5.12)$$

and hence we can regard $\Psi^* g_3$ as being a metric defined on $M_1 \times \left((-n\log\delta^2)^{\frac{n+1}{2n}}, \infty\right)$.

Let the function $\gamma := \Psi^*(||S||')^{2\varepsilon}$ be defined on $M_1 \times \left((-n\log\delta^2)^{\frac{n+1}{2n}}, \infty\right)$. Our goal is to understand the decay of $||D'S||_{g_k}$, which is equivalent of studying the decay of $|\tilde{\nabla}\gamma|_{\rho^{-2/(n+1)}\Psi^* g_k}(\Psi^{-1}(x))$, where $\tilde{\nabla}$ denotes the covariant derivative of the metric $\rho^{-2/(n+1)}\Psi^* g_k$.

Notice that on $M_1 \times \left((-n\log\delta^2)^{\frac{n+1}{2n}}, \infty\right)$, the function γ can be written on the form

$$e^{\tilde{\gamma}}(., \exp\{\rho^{\frac{2n}{n+1}}\}) \exp\{\frac{\varepsilon}{n}\rho^{\frac{2n}{n+1}}\},$$

where $\tilde{\gamma}$ is a smooth function on $M_1 \times \left((-n\log\delta^2)^{\frac{n+1}{2n}}, \infty\right)$ with all derivatives bounded in terms of a fixed product metric.

Hence, from the expression (5.12), it follows that

$$|\tilde{\nabla}\gamma|_{\rho^{-2/(n+1)}\Psi^* g_k}(\Psi^{-1}(x)) = O(\exp\{\frac{\varepsilon}{n}\rho^{\frac{2n}{n+1}}\}), \tag{5.13}$$

since the curvature tensor of $\rho^{-2/(n+1)}\Psi^* g_k$ is bounded near $\Psi^{-1}(x)$.

Notice also that the equation (5.13) is equivalent to

$$||D'S||_{g_k} = O(||S||^{\varepsilon}),$$

which shows that the metric g_ε, with corresponding Kähler form ω_{g_ε} defined by (5.9), satisfies the equation $\mathrm{Ric}(g_\varepsilon) - \Omega = \partial\bar{\partial} f_\varepsilon$, for f_ε a smooth function that decays on the order of at least $O(||S||^{\varepsilon})$.

In order to complete the proof of Theorem 1.1, it only remains to note that the curvature estimates for the new metric g_ε will follow trivially from the estimates on the curvature tensor $R(g_m)$, described on Lemma 5.2. □

Chapter 6

Asymptotics of the Monge-Ampère equation on M

This last chapter is intended to provide the proof to Theorem 1.2.

Let (M, g) be a complete Kähler manifold, with Kähler form ω. Consider the following Monge-Ampère equation on M:

$$\begin{cases} \left(\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\right)^n = e^f\omega^n, \\ \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u > 0, \qquad u \in C^\infty(M, \mathbb{R}), \end{cases} \tag{6.1}$$

where f is a given smooth function satisfying the integrability condition

$$\int_M (e^f - 1)w^n = 0. \tag{6.2}$$

As discussed in Chapter 2, if u is a solution to (6.1), then the $(1,1)$-form $\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u$ satisfies $Ric(\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u) = f$. So, in order to define metrics with prescribed Ricci curvature, it is enough to solve equation (6.1).

In [TY1], Tian and Yau proved that (6.1) has, in fact, solutions modulo assuming certain conditions on the volume growth of g as well as on the decay of f at infinity. For the convenience of the reader, we state here their main result.

Theorem 6.1 (Tian, Yau, [TY1]) *Let (M, g) be a complete Kähler manifold, sat-*

isfying:

- *Sectional curvature of g bounded by a constant K;*

- $Vol_g(B_R(x_0)) \leq CR^2$ *for all* $R > 0$ *and* $Vol_g(B_1(x_0)) \geq C^{-1}(1+\rho(x))^{-\beta}$, *for a constant* β, *where* Vol_g *denotes the volume associated to the metric* g, $B_R(x_0)$ *is the geodesic ball of radius* R *around a fixed point* $x_0 \in M$, *and* $\rho(x)$ *denotes the distance (with respect to* g*) from* x_0 *to* x.

- *There are positive numbers* $r > 0$, $r_1 > r_2 > 0$ *such that for any* $x \in M$, *there exists a holomorphic map* $\phi_x : \mathcal{U}_x \subset (\mathbb{C}^n, 0) \to B_r(x)$ *such that* $\phi_x(0) = x$; $B_{r_2} \subset \mathcal{U}_x \subset B_{r_1}$, *where* $B_r := \{z \in \mathbb{C}^n; |z| \leq r\}$; *and* $\phi_x^* g$ *is a Kähler metric in* $\mathcal{U}_x$, *such that its metric tensor has derivatives up to order 2 bounded and 1/2-Hölder-continuously bounded.*

Let f *be a smooth function, satisfying the integrability condition (6.2) and such that*

$$\sup_M \{|\nabla_g f|, |\Delta_g f|\} \leq C \qquad |f(x)| \leq C(1+\rho(x))^{-N}, \tag{6.3}$$

for some constant C, *for all* x *in* M, *where* $N \geq 4 + 2\beta$.

Then there exists a bounded, smooth solution u *for (6.1), such that* $\omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u$ *defines a complete Kähler metric equivalent to* g.

An interesting question posed by Tian and Yau in the same paper is that whether we can prove that the resulting metric is asymptotically as close to g as possible if we assume further conditions on the decay of f. We provide an answer to this problem in the remainder of this thesis.

We are interested in studying the Monge-Ampére Equation (6.1) for the Kähler manifold $(M, \omega_{g_\varepsilon})$ constructed in Chapter 5. More precisely, for any $\varepsilon > 0$, we want to understand the asymptotic behavior of a solution u to the problem

$$\begin{cases} \left(\omega_{g_\varepsilon} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\right)^n = e^{f_{g_\varepsilon}}\omega_{g_\varepsilon}^n, \\ \omega_{g_\varepsilon} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u > 0, \qquad u \in C^\infty(M, \mathbb{R}). \end{cases} \tag{6.4}$$

In order to guarantee existence of solution to (6.4), we need to check that the function f_{g_ε} (defined on Theorem 1.1) satisfies the integrability condition (6.2).

Lemma 6.1 *There exists a number $\lambda > 0$ such that, by replacing ϕ by $\phi + \lambda$ in the definition (3.1) of f_ϕ, we have*

$$\int_M (e^{f_{g_\varepsilon}} - 1)\omega_{g_\varepsilon}^n = 0. \tag{6.5}$$

Proof: Recall the definition of ω_{g_ε}:

$$\omega_{g_\varepsilon} = \underbrace{\frac{\sqrt{-1}}{2\pi}\frac{n^{1+1/n}}{n+1}\partial\bar{\partial}(-\log||S||_\phi^2)^{\frac{n+1}{n}}}_{\omega_\phi} + C_\varepsilon\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(-||S||')^{2\varepsilon}, \tag{6.6}$$

for ϕ chosen as in Section 3.1, so that the corresponding f_ϕ decays faster than $O(||S||^\varepsilon)$.

A direct computation using integration by parts shows that $\int_M \omega_{g_\varepsilon}^n - \omega_\phi^n = 0$. Also, the definitions of ω_{g_ε} and ω_ϕ imply that $e^{f_{g_\varepsilon}}\omega_{g_\varepsilon}^n = e^{f_\phi}\omega_\phi^n$. Therefore,

$$\int_M (e^{f_{g_\varepsilon}} - 1)\omega_{g_\varepsilon}^n = \int_M (e^{f_\phi} - 1)\omega_\phi^n.$$

On the other hand, Definition 3.1 gives

$$e^{f_\phi}\omega_\phi^n = \frac{e^{-\Psi}\omega'^n}{||S||^2}. \tag{6.7}$$

Notice that the function Ψ remains unchanged if we replace ϕ by $\phi+\lambda$, since $\tilde{\omega}_\phi = \tilde{\omega}_{\phi+\lambda}$. Therefore, the right-hand side of (6.7) is invariant under the transformation $\phi \mapsto \phi + \lambda$.

On the other hand, a direct computation using (3.4) shows that

$$\omega_{\phi+\lambda}^n = \left(\frac{\sqrt{-1}}{2\pi}\frac{n^{1+1/n}}{n+1}\partial\bar{\partial}(-\log||S||_{\phi+\lambda}^2)^{\frac{n+1}{n}}\right) = \tag{6.8}$$

$$= \left(\frac{\sqrt{-1}}{2\pi}\frac{n^{1+1/n}}{n+1}\partial\bar{\partial}(-\log||S||_\phi^2)^{\frac{n+1}{n}}\right) - n\lambda\tilde{\omega}_\phi^n, \tag{6.9}$$

where we recall that $\tilde{\omega}_\phi^n$ is the curvature form of the hermitian metric $||.||_\phi$.

Therefore, by redefining f_ϕ by

$$f_\phi = -\log ||S||^2 - \log \frac{\omega_{\phi+\lambda}^n}{\omega'^n} - \Psi,$$

we have that

$$\int_M (e^{f_\phi} - 1)\omega_{\phi+\lambda}^n = \int_M \left(\frac{e^{-\Psi}\omega'^n}{||S||^2} - \omega_\phi^n \right) - n\lambda \int_M \tilde{\omega}_\phi^n. \tag{6.10}$$

Since the first integral in the above expression is finite, and independent of λ, we can choose the number λ so as to make the right-hand side of (6.10) equals to zero. This establishes the lemma. □

The previous lemma shows that each f_{g_ε} satisfies the conditions on the existence theorem of Tian and Yau. Also, the estimates on the decay of the Riemann curvature tensor (Lemma 5.2) and the observation on the volume growth of the metric g_ε (see the remark after Lemma 5.1) show that (M, g_ε) is a complete Kähler manifold in which Theorem 6.1 can be applied.

Therefore, for each $\varepsilon > 0$, there exists a bounded and smooth solution u_ε to the problem (6.4). Our goal now is to understand the asymptotic behavior of u_ε.

Denote by ω_Ω the Kähler form on M given by Theorem 6.1, when we use g_ε (given by Theorem 1.1) as the ambient metric:

$$\omega_\Omega = \omega_{g_\varepsilon} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_\varepsilon.$$

Clearly, it suffices to prove the asymptotic assertions on u_ε for a small tubular neighborhood of D in $\overline{M}$. Recall from the proof of Theorem 1.1 that on $V_\varepsilon \setminus D$,

$$\omega_{g_\varepsilon} = \omega_m + C_\varepsilon \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\left(||S||'\right)^\varepsilon,$$

for some $m \geq \varepsilon$ fixed.

Since ω_m and ω_{g_ε} are cohomologous, there exists a function u_m such that we can write, on the neighborhood $V_m \setminus D$,

$$\omega_\Omega = \omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_m. \tag{6.11}$$

On the other hand, if f_m is the function defined by (3.1), (6.11) implies that u_m satisfies

$$\left(\omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_m\right)^n = e^{f_m}\omega_m^n \qquad \text{on } V_m \setminus D, \tag{6.12}$$

where we remind the reader that $|f_m|_{g_m}$ is of order of $O(||S||_m^m)$.

Therefore, in order to study the desired asymptotics, we will turn our attention to solving (6.12).

The following lemma is a necessary ingredient in the proof of Theorem 1.2, providing barriers to the solution u_m of (6.12).

Lemma 6.2 *On the neighborhood $V_m \setminus D = \{0 < ||S||_m < \delta_m\}$, we have*

$$\begin{aligned}
&\left\{\omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\left(C\left[S^i\overline{S}^j\theta_{ij} + \overline{S}^iS^j\bar{\theta}_{ij}\right](-n\log(||S||_m^2))^k\right)\right\}^n = \\
&= \omega_m^n\left[1 + C(-n\log(||S||_m^2))^{k-\frac{n+1}{n}}\left\{ij(-n\log(||S||_m^2))^2\left[S^i\overline{S}^j\theta_{ij} + \overline{S}^iS^j\bar{\theta}_{ij}\right] - \right.\right.\\
&-(-n\log(||S||_m^2))\left[(k(i+j)+j(n-1))\,S^i\overline{S}^j\theta_{ij} + (k(i+j)+i(n-1))\,\overline{S}^iS^j\bar{\theta}_{ij}\right]\\
&\left.\left.+k(k-n)\right\} + O(||S||_m^{i+j+1})\right], \qquad (6.13)
\end{aligned}$$

where θ_{ij} is a C^∞ local section of $L_D^{-i} \otimes \overline{L_D}^{-j}$ on V_m.

Proof: In order to simplify notation, define $B = (-n\log(||S||_m^2))$. Computations lead to

$$
\begin{aligned}
&\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\left(CB^k\left[S^i\overline{S}^j\theta_{ij}+\overline{S}^iS^j\bar{\theta}_{ij}\right]\right)= \\
&\qquad =\tilde{\omega}_m\left\{CB^{k-1}\left[(-jB+k)\,S^i\overline{S}^j\theta_{ij}\right]+\left[(-iB+k)\,\overline{S}^iS^j\bar{\theta}_{ij}\right]\right\}+ \\
&+\frac{\sqrt{-1}}{2\pi}\frac{D_mS\wedge \bar{D_m}S}{|S|^2}\left\{CB^{k-2}\left[S^i\overline{S}^j\theta_{ij}+\overline{S}^iS^j\bar{\theta}_{ij}\right]\left[ijF^2-k(i+j)F+k(k-1)\right]\right\}+ \\
&\qquad +CB^{k-1}\frac{\sqrt{-1}}{2\pi}\left[(jB-k)\,S^i\overline{S}^jD_m\theta_{ij}+(iB-k)\,\overline{S}^iS^jD_m\bar{\theta}_{ij}\right]\wedge\frac{\bar{D_m}S}{\overline{S}}+ \\
&\qquad +CB^{k-1}\frac{\sqrt{-1}}{2\pi}\frac{D_mS}{S}\wedge\left[(iB-k)\,S^i\overline{S}^j\bar{D}_m\theta_{ij}+(jB-k)\,\overline{S}^iS^j\bar{D}_m\bar{\theta}_{ij}\right]+ \\
&\qquad\qquad +CB^k\frac{\sqrt{-1}}{2\pi}\left[S^i\overline{S}^jD_m\bar{D}_m\theta_{ij}+\overline{S}^iS^jD_m\bar{D}_m\bar{\theta}_{ij}\right], \quad (6.14)
\end{aligned}
$$

where D_m stands for the covariant derivative with respect to the hermitian metric $||.||_m$, and where $\tilde{\omega}_m$ is its corresponding curvature form.

Using (3.3), we may conclude that

$$
\begin{aligned}
&\left[\omega_m+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\left(CB^k\left[S^i\overline{S}^j\theta_{ij}+\overline{S}^iS^j\bar{\theta}_{ij}\right]\right)\right]^n=\left[a\tilde{\omega}_m+b\frac{\sqrt{-1}}{2\pi}\frac{D_mS\wedge\bar{D}_m\overline{S}}{|S|^2}\right. \\
&\quad+\frac{\sqrt{-1}}{2\pi}\left(c_1D_m\theta_{ij}+c_2D_m\bar{\theta}_{ij}\right)\wedge\frac{\bar{D}_mS}{\overline{S}}+\frac{\sqrt{-1}}{2\pi}\frac{D_mS}{S}\wedge\left(d_1\bar{D}_m\theta_{ij}+d_2\bar{D}_m\bar{\theta}_{ij}\right)+ \\
&\qquad\qquad\left.+\,e\frac{\sqrt{-1}}{2\pi}\bar{D}_mD_m\theta_{ij}\right]^n, \quad (6.15)
\end{aligned}
$$

where

$$
\begin{aligned}
a &= B^{\frac{1}{n}}\left[1-CB^{k-\frac{n+1}{n}}\left[(jB-k)S^i\overline{S}^j\theta_{ij}(iB-k)\overline{S}^iS^j\bar{\theta}_{ij}+\right]\right] \\
b &= B^{\frac{1-n}{n}}\left[1+CB^{k-\frac{n+1}{n}}\left[S^i\overline{S}^j\theta_{ij}+\overline{S}^iS^j\bar{\theta}_{ij}\right]\left[ijF^2-k(i+j)F+k(k-1)\right]\right] \\
c_1 &= CS^i\overline{S}^jB^{k-1}[jB-k], \qquad c_2=C\overline{S}^iS^jB^{k-1}[iB-k] \qquad (6.16) \\
d_1 &= CS^i\overline{S}^jB^{k-1}[iB-k], \qquad d_2=C\overline{S}^iS^jB^{k-1}[jB-k] \\
e &= CS^i\overline{S}^jB^k.
\end{aligned}
$$

Now, we proceed on estimating each of the terms on (6.15).

$$a^n\tilde{\omega}^n_m = B^{\frac{1}{n}}\left[1 - CnB^{k-\frac{n+1}{n}}\left[(jB-k)S^i\overline{S}^j\theta_{ij} + (iB-k)\overline{S}^iS^j\bar{\theta}_{ij}\right] + \right.$$
$$\left. +O(||S||_m^{i+j+1})\right]\tilde{\omega}^n_m. \quad (6.17)$$

Also,

$$na^{n-1}b = \left[1 - C(n-1)B^{k-\frac{n+1}{n}}\left[(jB-k)S^i\overline{S}^j\theta_{ij} + (iB-k)\overline{S}^iS^j\bar{\theta}_{ij} + \right]\right]\cdot$$
$$\cdot\left[1 + CB^{k-\frac{n+1}{n}}\left[S^i\overline{S}^j\theta_{ij} + \overline{S}^iS^j\bar{\theta}_{ij}\right]\left[ijF^2 - k(i+j)F + k(k-1)\right]\right] =$$
$$= 1 + CB^{k-\frac{n+1}{n}}\left\{ijB^2\left[S^i\overline{S}^j\theta_{ij} + \overline{S}^iS^j\bar{\theta}_{ij}\right] - \right.$$
$$\left. -B\left[(k(i+j)+j(n-1))\,S^i\overline{S}^j\theta_{ij} + (k(i+j)+i(n-1))\,\overline{S}^iS^j\bar{\theta}_{ij}\right] + k(k-n)\right\}$$
$$+\,O(||S||_m^{i+j+1}) \quad (6.18)$$

The expressions for the other terms are analogous, and will henceforth be omitted.

From (3.4), we deduce that

$$\tilde{\omega}^n_m = \omega^n_m\frac{||S||^2_mB^{-1}}{||S||^2_m + B^{-1}||D_mS||^2_m},$$

and since

$$\frac{||S||^2_m}{||S||^2_m + B^{-1}||D_mS||^2_m} = \frac{||S||^2_mB^{-1}}{||D_mS||^2_m}\left(\frac{1}{1+\frac{B||S||^2_m}{||D_mS||^2_m}}\right) = O(||S||^2_mB^{-1}),$$

all the terms in (6.15) will decay at the order of at least $O(||S||_m^{i+j+1})$, with the exception of the term (6.18), which will be written as:

$$a^{n-1}b\left\{\tilde{\omega}_m^{n-1}\wedge\frac{n\sqrt{-1}}{2\pi}\frac{D_mS\wedge \bar{D_m S}}{|S|^2}\right\}=a^{n-1}b\frac{||D_mS||_m^2}{||S||_m^2}\tilde{\omega}_m^n=$$

$$=a^{n-1}b\left(\frac{||D_mS||_m^2}{||S||_m^2}\right)\frac{\omega_m^n||S||_m^2B^{-1}}{||S||_m^2+B^{-1}||D_mS||_m^2}=$$

$$=\omega_m^n\left[1+CB^{k-\frac{n+1}{n}}\left\{ijB^2\left[S^i\overline{S}^j\theta_{ij}+\overline{S}^iS^j\bar{\theta}_{ij}\right]-\right.\right.$$

$$\left.-B\left[(k(i+j)+j(n-1))\,S^i\overline{S}^j\theta_{ij}+(k(i+j)+i(n-1))\,\overline{S}^iS^j\bar{\theta}_{ij}\right]+k(k-n)\right\}$$

$$\left.+O(||S||_m^{i+j+1})\right] \quad (6.19)$$

Therefore,

$$\left[\omega_m+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\left(CS^i\overline{S}^j\theta_{ij}B^k\right)\right]^n=\omega_m^n\left[1+CB^{k-\frac{n+1}{n}}\left\{ijB^2\left[S^i\overline{S}^j\theta_{ij}+\overline{S}^iS^j\bar{\theta}_{ij}\right]-\right.\right.$$

$$\left.-B\left[(k(i+j)+j(n-1))\,S^i\overline{S}^j\theta_{ij}+(k(i+j)+i(n-1))\,\overline{S}^iS^j\bar{\theta}_{ij}\right]+k(k-n)\right\}$$

$$\left.+O(||S||_m^{i+j+1})\right], \quad (6.20)$$

completing the proof of the lemma. □

Proposition 6.1 *Let u_m be a solution to the Monge-Ampère equation (6.12). If $u_m(x)$ converges uniformly to zero as x approaches the divisor, then there exists a constant $C=C(m)$ such that*

$$|u_m(x)|\leq C||S||_m^{m+1} \qquad \textit{on } V_m\setminus D. \qquad (6.21)$$

Proof: It suffices to prove (6.21) in a neighborhood of D. Apply Lemma 6.2 for $i=m+2$ and $j=-1$, and choose the section θ_{ij} so that the function $S^i\overline{S}^j\theta_{ij}+\overline{S}^iS^j\bar{\theta}_{ij}$ is positive on $V_m\setminus D$. Note that there is, in fact, a C^∞-section θ_{ij} satisfying this condition. Indeed, a local section on a trivializing coordinate can clearly be constructed by means of a bump function. In particular we can consider finitely many local sections as above such that the union of their supports covers the all of D. Since the positivity condition is naturally respected by the cocycle relations arising from the change of coordinates, the desired section θ_{ij} can simply be obtained by adding these local sections.

With the above choices, we have

$$\left[\omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}C\left[S^i\overline{S}^j\theta_{ij} + \overline{S}^i S^j\bar{\theta}_{ij}\right](-n\log(||S||^2_m))^k\right]^n =$$
$$= \omega_m^n\left[1 - C(m+2)(-n\log(||S||^2_m))^{k+\frac{n-1}{n}}\left\{[1+o(1)]\left[S^i\overline{S}^j\theta_{ij} + \overline{S}^i S^j\bar{\theta}_{ij}\right]\right\} + \right.$$
$$\left. + O(||S||_m^{m+2})\right]. \quad (6.22)$$

On the other hand,

$$e^{f_m}\omega_m^n = [1 + O(||S||_m^{m+1})]\omega_m^n \qquad \text{on } V_m \setminus D. \qquad (6.23)$$

More precisely, we can write on $V_m \setminus D$

$$e^{f_m}\omega_m^n = [1 + \sum_{\ell=0}^{\ell_{m+1}}\{S^i\overline{S}^j\theta_{ij} + \overline{S}^i S^j\bar{\theta}_{ij}\}(-\log||S||^2_m)^\ell + O(||S||_m^{m+2})]\omega_m^n, \qquad (6.24)$$

for sections $\theta_{ij} \in \Gamma(V_m \setminus D, L_D^{-i} \otimes \bar{L}_D^{-j})$.

Let $\varepsilon > 0$, and define $C_i = \frac{C_i'}{\varepsilon}$, where $C_1' := \sup_{x \in V_m \setminus D}(|u_m| + 1)$, and $C_2' = -C_1'$. Then, for all $x \in V_m \setminus D$ verifying

$$\left(S^{m+2}\overline{S}^{-1}\theta_{m+2,-1} + \overline{S}^{m+2}S^{-1}\bar{\theta}_{m+2,-1}(-n\log(||S||^2_m))^{\ell_{m+1}}\right)(x) = \varepsilon, \qquad (6.25)$$

it follows that

$$C_1\left(S^{m+2}\overline{S}^{-1}\theta_{m+2,-1} + \overline{S}^{m+2}S^{-1}\bar{\theta}_{m+2,-1}(-n\log(||S||^2_m))^{\ell_{m+1}}\right)(x) > |u_m(x)|.$$

Furthermore, if ε is sufficiently small, then on the subset

$$\{x \in V_m \setminus D; \left(S^{m+2}\overline{S}^{-1}\theta_{m+2,-1} + \overline{S}^{m+2}S^{-1}\bar{\theta}_{m+2,-1}(-n\log(||S||^2_m))^{\ell_{m+1}}\right)(x) \leq \varepsilon\},$$

we have (for $i = m+2$ and $j = -1$)

$$\left[\omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} C_1 \left[S^i \overline{S}^j \theta_{ij} + \overline{S}^i S^j \bar{\theta}_{ij}\right] (-n \log(||S||_m^2))^{\ell_{m+1} - \frac{n-1}{n}}\right]^n \leq e^{f_m}\omega_m^n, \quad \text{and}$$

$$\left[\omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} C_2 \left[S^i \overline{S}^j \theta_{ij} + \overline{S}^i S^j \bar{\theta}_{ij}\right] (-n \log(||S||_m^2))^{\ell_{m+1} - \frac{n-1}{n}}\right]^n \geq e^{f_m}\omega_m^n.$$

Finally, by using the hypothesis on the uniform vanishing of u_m on D, the proposition follows from the maximum principle for the complex Monge-Ampère operator: we obtain the following bound

$$|u_m| \leq C \left[S^i \overline{S}^j \theta_{ij} + \overline{S}^i S^j \bar{\theta}_{ij}\right] (-n \log(||S||_m^2))^{\ell_{m+1} - \frac{n-1}{n}} \tag{6.26}$$

on the neighborhood given in (6.25), where $C = \max\{C_1, -C_2\}$. This completes the proof of the proposition. □

Finally, the last step in the proof of Theorem 1.2, which consists of showing that the solution to the Monge-Ampère equation (6.12) actually converges uniformly to zero.

Proposition 6.2 *For a fixed $m \geq 2$, let u_m be a solution to (6.12). Then $u_m(x)$ converges uniformly to zero as x approaches the divisor D.*

Proof: In [TY1], the solution u_m to the Monge-Ampère equation (6.1) is obtained as the uniform limit, as ε goes to *zero*, of solutions $u_{m,\varepsilon}$ of the perturbed Monge-Ampère equations

$$\begin{cases} \left(\omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} u\right)^n = e^{f_m + \varepsilon u}\omega_m^n, & \\ \omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} u > 0, & u \in C^\infty(M, \mathbb{R}). \end{cases} \tag{6.27}$$

On the neighborhood V_m, Lemma (6.2) applied for $i = 2$, $j = -1$ and $k = 0$, gives

$$\left\{\omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\left(C\left[S^2\overline{S}^{-1}\theta_{2,-1} + \overline{S}^2 S^{-1}\bar{\theta}_{2,-1}\right]\right)\right\}^n =$$
$$= \omega_m^n \Big[1 - 2C(-n\log(||S||_m^2))^{-\frac{n+1}{n}}\Big\{(-n\log(||S||_m^2))^2\left[S^2\overline{S}^{-1}\theta_{2,-1} + \overline{S}^2 S^{-1}\bar{\theta}_{2,-1}\right] -$$
$$-(-n\log(||S||_m^2))\left[(1-n)S^2\overline{S}^{-1}\theta_{2,-1} + 2(n-1)\overline{S}^2 S^{-1}\bar{\theta}_{2,-1}\right]\Big\} + O(||S||_m^2)\Big]\,. \tag{6.28}$$

Again, we can choose appropriate local C^∞-sections $\theta_{2,-1}$ such that

$$\left[S^2\overline{S}^{-1}\theta_{2,-1} + \overline{S}^2 S^{-1}\bar{\theta}_{2,-1}\right]$$

is a positive function on a neighborhood of the divisor, and use this function as a uniform barrier to the sequence of solutions $\{u_{m,\varepsilon}\}$.

Note that $e^{f_m+\varepsilon u_{m,\varepsilon}} = 1 + O(||S||_m)$. Hence, as in the proof of Lemma 6.2, we can define, for a fixed $\delta > 0$, $C_i = \frac{C_i'}{\delta}$, where $C_1' := \sup_{x\in V_m\setminus D}(|u_{m,\varepsilon}|+1)$ and $C_2' = -C_1'$. A priori, C_i' could depend on ε, but it turns out (see [TY1] for details) that $\sup_M |u_{m,\varepsilon}|$ can be bounded uniformly by a constant independent on ε. Then, for all $x \in V_m \setminus D$ such that

$$\left(S^2\overline{S}^{-1}\theta_{2,-1} + \overline{S}^2 S^{-1}\bar{\theta}_{2,-1}\right)(x) = \delta,$$

we have that

$$C_1\left(S^2\overline{S}^{-1}\theta_{2,-1} + \overline{S}^2 S^{-1}\bar{\theta}_{2,-1}\right)(x) > |u_{m,\varepsilon(x)}|.$$

In addition, in the neighborhood $\{x \in V_m \setminus D; \left(S^2\overline{S}^{-1}\theta_{2,-1} + \overline{S}^2 S^{-1}\bar{\theta}_{2,-1}\right)(x) \leq \delta\}$, for a fixed δ sufficiently small, we have

$$\left[\omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}C_1\left[S^2\overline{S}^{-1}\theta_{2,-1} + \overline{S}^2 S^{-1}\bar{\theta}_{2,-1}\right]\right]^n \leq e^{f_m+\varepsilon C_1\left[S^2\overline{S}^{-1}\theta_{2,-1}+\overline{S}^2 S^{-1}\bar{\theta}_{2,-1}\right]}\omega_m^n$$

and

$$\left[\omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}C_2\left[S^2\overline{S}^{-1}\theta_{2,-1} + \overline{S}^2 S^{-1}\bar{\theta}_{2,-1}\right]\right]^n \geq e^{f_m+\varepsilon C_1\left[S^2\overline{S}^{-1}\theta_{2,-1}+\overline{S}^2 S^{-1}\bar{\theta}_{2,-1}\right]}\omega_m^n.$$

Since $u_{m,\varepsilon}$ vanishes at D (see [CY1]), we can apply the maximum principle to conclude that there exists a C independent of ε such that, near D,

$$-C\left[S^2\overline{S}^{-1}\theta_{2,-1}+\overline{S}^2S^{-1}\bar{\theta}_{2,-1}\right]\leq u_{m,\varepsilon}\leq C\left[S^2\overline{S}^{-1}\theta_{2,-1}+\overline{S}^2S^{-1}\bar{\theta}_{2,-1}\right]$$

Now, since the neighborhood $\{x\in V_m\setminus D;\left(S^2\overline{S}^{-1}\theta_{2,-1}+\overline{S}^2S^{-1}\bar{\theta}_{2,-1}\right)(x)\leq\delta\}$ is a fixed set, independent of ε, and the constant C above is also independent of ε, we can pass to the limit when ε goes to zero, obtaining the claim. □

Proof of Theorem (1.2): It follows immediately from the combination of Propositions (6.1) and (6.2). □

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