Topological Hochschild Homology of Twisted Group Algebras

by

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Abstract

Let G be a group and A be a ring. There is a stable equivalence of orthogonal spectra

$$\mathrm{THH}(A) \wedge N^{\mathrm{cy}}(G)_+ \xrightarrow{\sim} \mathrm{THH}(A[G])$$

between the topological Hochschild homology of the group algebra $A[G]$ and the smash product of the topological Hochschild homology of A and the cyclic bar construction of G. This thesis generalizes this result to a *twisted group algebra* $A^{\tau}[G]$. As an A-module, $A^{\tau}[G] = A[G]$, but the multiplication is given by $ag \cdot a'g' = ag(a') \cdot gg'$, where G acts on A from the left through ring automorphisms. The main result is given in terms of a variant $\mathrm{THH}^{g}(A)$ of the topological Hochschild spectrum that is equipped with a twisted cyclic structure inherited from the cyclic structure of the cyclic pointed space $\mathrm{THH}(A)[-]$. We first define a parametrized orthogonal spectrum $E(A,G)$ over the cyclic bar construction $N^{\mathrm{cy}}(G)$. We prove there is a stable equivalence of spectra between the associated Thom spectrum of $E(A,G)$ and $\mathrm{THH}(A^{\tau}[G])$. We then prove there is a stable equivalence of orthogonal spectra

$$\bigvee_{\langle g \rangle} EG_+ \wedge_{C_G(g)} \mathrm{THH}^{g}(A) \xrightarrow{\sim} \mathrm{THH}(A^{\tau}[G]),$$

where the wedge-sum on the left hand side ranges over the conjugacy classes of elements of G and the equivalence depends on a choice of representative $g \in \langle g \rangle$ of every conjugacy class of elements in G.

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Contents

Chapter 1

Introduction

Let G be a group and A be a ring. The topological Hochschild homology spectrum of the group algebra $A[G]$ is determined by the cyclic bar construction of the group G and the topological Hochschild spectrum of the ring A. More precisely, there is a stable equivalence of orthogonal spectra

$$\mathrm{THH}(A) \wedge N^{\mathrm{cy}}(G)_+ \xrightarrow{\sim} \mathrm{THH}(A[G])$$

[5, Prop. 4.1]. This thesis generalizes this result to a *twisted group algebra* $A^\tau[G]$. Let G act on A from the left through ring automorphisms. Then as an A-module, $A^\tau[G] = A[G]$, but the multiplication is given by $ag \cdot a'g' = ag(a') \cdot gg'$. The result is stated in terms of a variant $\mathrm{THH}^g(A)$ of the topological Hochschild spectrum that we now describe. The topological Hochschild spectrum is defined as the geometric realization

$$\mathrm{THH}(A; S^\lambda) = \big|[k] \mapsto \mathrm{THH}(A; S^\lambda)[k]\big|$$

of a cyclic orthogonal spectrum $\mathrm{THH}(A; S^\lambda)[k]$. Similarly,

$$\mathrm{THH}^g(A; S^\lambda) = \big|[k] \mapsto \mathrm{THH}^g(A; S^\lambda)[k]\big|$$

where

$$\mathrm{THH}^g(A; S^\lambda)[k] = \mathrm{THH}(A; S^\lambda)[k]$$

with the usual cyclic structure maps except that the zeroth face map is replaced by the composite $d_0^g = d_0 \circ (1, g, 1, \ldots, 1)_*$. Here, if $(g_0, \ldots, g_k)$ is a k-tuple of elements of the group G,

$$(g_0, \ldots, g_k)_* : \mathrm{THH}(A)[k] \longrightarrow \mathrm{THH}(A)[k]$$

is the map induced by g_i acting on the ith factor and d_0 is the usual face map. A more precise definition of the map $(g_0, \ldots, g_k)_*$ is given in §3.1. We then prove the following result.

Theorem 1.0.1. *Let G be a group that acts on a ring A, and let $A^\tau[G]$ be the twisted group algebra. Then there is a stable equivalence of orthogonal spectra*

$$\Phi : \bigvee_{\langle g \rangle} EG_+ \wedge_{C_G(g)} \mathrm{THH}^g(A) \xrightarrow{\sim} \mathrm{THH}(A^\tau[G]),$$

where the wedge-sum on the left hand side ranges over the conjugacy classes of elements of G. The map Φ depends on a choice of representative $g \in \langle g \rangle$ of every conjugacy class of elements in G.

We first prove a form of this equivalence that is independent of the choice of conjugacy class representatives. For each non-negative integer k, we define a parametrized orthogonal spectrum $E(A, G)[k]$ with λ-th space given by

$$E(A, G)[k]_\lambda = E(A, G; S^\lambda)[k] = \mathrm{THH}(A; S^\lambda)[k] \times N^{\mathrm{cy}}(G)[k].$$

We then define cyclic operators $d^\tau_{i,E}$, $s^\tau_{i,E}$, and $t^\tau_{k,E}$. For $0 \leq i \leq k$, the degeneracy operator $s^\tau_{i,E}$ is given as the product of the i-th degeneracy operator of the cyclic pointed space $\mathrm{THH}(A, S^\lambda)[-]$ and the i-th degneracy operator of the cyclic set $N^{\mathrm{cy}}(G)[-]$, respectively. Similarily, the cyclic operator $t^\tau_{k,E}$ is the product of the cyclic operator of $\mathrm{THH}(A; S^\lambda)[-]$ and the cyclic operator of $N^{\mathrm{cy}}(G)[-]$. The face operators, however, are replaced by *twisted* face operators defined in §3,1. At each simplicial level k, $E(A, G)[k]$ is a parametrized orthogonal spectrum over $N^{\mathrm{cy}}(G)[k]$ but it is not a cyclic object in a category of parametrized orthogonal spectra over a fixed base. Fixing λ

and letting k vary, we have a cyclic space, $E(A,G)[-]_\lambda$. The geometric realization of $E(A,G)[-]_\lambda$ is

$$E(A,G)_\lambda = \big|[k] \mapsto E(A,G)[k]_\lambda\big|.$$

At each level k, the projection map of the parametrized space $E(A,G)[-]_\lambda$ commutes with the operators $d^\tau_{i,E}$, $s^\tau_{i,E}$, and $t^\tau_{i,E}$. Thus for varying k the projection maps form a cyclic map. Similarly the level k section maps form a cyclic map. Since the geometric realization is a functor, it follows that $E(A,G)_\lambda$ is a parametrized space over $N^{\text{cy}}(G)$. Letting λ vary then gives a parametrized orthogonal spectrum $E(A,G)$ over $N^{\text{cy}}(G)$ in the sense of Def. 2.2.1. The space $E(A,G)_\lambda$ is also a bundle over $N^{\text{cy}}(G)$ with the fiber over a vertex $g \in \langle g \rangle$ denoted by $E(A,G)^g_\lambda$. The fiber $E(A,G)^g_\lambda$ is itself obtained as the realization of a cyclic set $E(A,G)^g[-]$. We discuss this in §3.2.

In general, if $\mathcal{T}_*$ denotes the category of pointed spaces and $\mathcal{T}_B$ the category of parametrized spaces over a base B, there exists an adjoint pair of functors $(f_!, f^*)$ between parametrized spaces and pointed spaces, $f_! : \mathcal{T}_B \longrightarrow \mathcal{T}_*$, the associated Thom space, and $f^* : \mathcal{T}_* \longrightarrow \mathcal{T}_B$, the change of base functor. Applying the Thom space functor $f_!$ levelwise to a parametrized orthogonal spectrum gives us an orthogonal spectrum [12, Thm. 11.4..1]. Since the functor $f_!$ is a left adjoint, it preserves colimits. Thus for the parametrized spectrum $E(A,G)$, the Thom spectrum $f_!E(A,G)$ is given by the realization of a cyclic orthogonal spectrum whose orthogonal spectrum in simplicial degree k has λ-th space given by $\mathrm{THH}(A;S^\lambda)[k] \wedge N^{\text{cy}}(G)[k]_+$.

Theorem 1.0.2. *There exists a canonical stable equivalence of orthogonal spectra*

$$\Psi : f_!E(A,G) \xrightarrow{\sim} \mathrm{THH}(A^\tau[G]).$$

By Connes' theory of cyclic sets, the realization of a cyclic set is a $\mathbb{T}$-space. Since the topological Hochschild homology spectrum is defined as the realization of a cyclic orthogonal spectrum, it is equipped with an action of the circle; see [8]. In particular, the spectrum $\mathrm{THH}(A^\tau[G])$ is an orthogonal $\mathbb{T}$-spectrum as defined in [10] and Thm. 1.0.2 can be extended to an equivalence of orthogonal $\mathbb{T}$-spectra. However, this extended result would require an understanding of the fixed point sets of an equivariant Thom spectrum, $f_! E(A, G)$, and these fixed sets are notoriously difficult to understand.

There is a deep connection between topologicial invariants and K-theory. An instance of this connection is the cyclotomic trace map,

$$K(A) \xrightarrow{trc} \mathrm{TC}(A)$$

from the K-theory spectrum of the ring A to the topological cyclic homology spectrum TC. See [2]. The spectrum TC is constructed from a pro-spectrum TR that is obtained from the topological Hochschild homology spectrum THH. Namely, the topological Hochschild spectrum $\mathrm{THH}(A)$ has an action of the circle and we can thus consider fixed-point spectra given by finite subgroups of the circle. For a fixed prime p the pro-spectrum TR has pro-system level n spectrum given by

$$\mathrm{TR}^n(A;p) = \mathrm{THH}(A)^{C_{p^{n-1}}},$$

where $C_{p^{n-1}} \subset \mathbb{T}$ is the cyclic group of order p^{n-1}. Between these spectra there are *restriction* maps, R, and *Frobenius* maps, F. The maps R and F are defined from TR^n to TR^{n-1}. To define the topological cyclic homology, we first define the spectrum

$$\mathrm{TR}(A;p) = \operatorname{holim} \mathrm{TR}^n(A;p).$$

Then the topological cyclic homology of A at the prime p is defined to be the homotopy fixed points of $\mathrm{TR}(A;p)$ under the action of the additive monoid $\mathbb{N}$, which acts via the powers of the Frobenius map: $\mathrm{TC}(A;p) = \mathrm{TR}(A;p)^{ho\mathbb{N}}$.

The study of group algebras is important in K-theory and indeed, group algebras are among the first objects studied within the field of algebraic K-theory. A particular example is the theorem of Dundas. Dundas theorem states that given a space X, if

$A(X)$ is Waldhausen's K-theory functor, the diagram

$$\begin{array}{ccc} A(X) & \xrightarrow{trc} & \mathrm{TC}(X) \\ \downarrow & & \downarrow \\ K(\mathbb{Z}[\pi_1 X]) & \xrightarrow{trc} & \mathrm{TC}(\mathbb{Z}[\pi_1 X]) \end{array}$$

is a homotopy Cartesian diagram after profinite completion. A proof of Dundas theorem is given in [9, Thm. 3.5.1]. Another example is given when the ring A is a finite algebra over the Witt vectors of a finite field k of characteristic p. In this case, given a finite group G, the group algebra $A[G]$ is also finite and the homotopy groups of the K-theory spectrum are isomorphic to the homotopy groups of the TC spectrum,

$$K_i(A[G]; \mathbb{Z}_p) \cong \mathrm{TC}_i(A[G]; \mathbb{Z}_p).$$

Here K_i denotes the homotopy groups of the K-theory spectrum and TC_i denotes the homotopy groups of the spectrum TC. Further development on the K-theory of (untwisted) group algebras may be found in [9, §5.1].

The proof of Thm. 1.0.2 employs the definition of THH and the base change functors $(f_!, f^*)$ from parametrized homotopy theory. It is independent of the twisting in the simplicial structure in that the equivalence does not depend on a preselected choice of element $g \in \langle g \rangle$ as is the case for Thm. 1.0.1. The proof of Thm. 1.0.1 involves an explicit analysis of the cyclic structure of the cyclic bar construction and is inspired by a study of the linear case of the Hochschild homology that we include in Appendix A. Another formula for the (ordinary) Hochschild homolgy of twisted group algebras is given in [3, §4], but note that we use a different system of coordinates for the cyclic bar construction (see §3.1).

We use the following notational conventions throughout this exposition. By a space we mean a compactly generated space (weak Hausdorff k-space) and by a pointed space we mean a compactly generated space with a choice of base-point. Let

$\mathcal{S}$ be the category of sets. We denote both a given cyclic set

$$X : \Lambda^{\mathrm{op}} \longrightarrow \mathcal{S}$$

and a given simplicial set

$$X : \Delta^{\mathrm{op}} \longrightarrow \mathcal{S}$$

by $X[-]$. We will always state whether we are considering a simplicial set or a cyclic set so that no confusion arises. We also let X denote both the geometric realization of a simplicial set $X[-]$ and the geometric realization of the cyclic set $X[-]$, defined as the geometric realization of the underlying simplicial set. Finally, whenever we make use of the functors $f_!$ and f^*, f will always be the map sending the base to the one point space unless explicitly stated otherwise.

Chapter 2

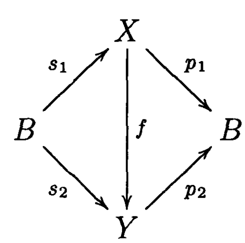
Parametrized Orthogonal Spectra

2.1 Parametrized Spaces

We discuss parametrized spaces and define the adjoint functors $f_!$ and f^* that we use in Thm. 1.0.1 and Thm. 1.0.2. We will also define a right adjoint to f^*, denoted f_*. The reference for this material is [12].

Definition 2.1.1. *Let B be a fixed base space. A parametrized space X over B consists of a space X together with projection and section maps, $p : X \longrightarrow B$ and $s : B \longrightarrow X$ respectively, such that $p \circ s = \mathrm{id}_B$.*

The category of parametrized spaces $\mathcal{T}_B$ over a fixed base space B has objects parametrized spaces over B. The morphisms of $\mathcal{T}_B$ are maps of the total spaces that commute with both the section and projection maps, or commutative diagrams:



We construct the *parametrized mapping space* $F_B(-,-)$ and *parametrized smash product* to make $\mathcal{T}_B$ a closed symmetric monoidal category with unit $S^0_B = B \times S^0$. The zero object $*_B$ in $\mathcal{T}_B$ is the space B with projection and section maps given by the identity. Let $\mathcal{T}_B(X,Y)$ be the set of all morphisms in $\mathcal{T}_B$ from X to Y. We topologize this set as a subspace of the space of all unbased maps of unbased total spaces $X \longrightarrow Y$. We note that the space $\mathcal{T}_B(X,Y)$ is a based space with basepoint the map $s_2 \circ p_1 : X \longrightarrow Y$. This is the unique map factoring through $*_B$ in $\mathcal{T}_B$. Thus the category $\mathcal{T}_B$ is enriched over $\mathcal{T}_*$. It is also *based topologically bicomplete* in the sense of [12, §1.2]. Constructing $F_B(-,-)$ requires us to first construct the *unbased* parametrized mapping space $Map_B(-,-)$. To do this, we introduce a subtle preliminary notion. For a space $Y \in \mathcal{T}$ (unbased), the *partial map classifier* is $\tilde{Y} = Y \cup \{\omega\}$ where ω is a disjoint basepoint. It is topologized as the space with basis $\{U \cup \{\omega\} : U \in \mathcal{U}\}$ where $\mathcal{U}$ is a basis for Y [12, Def. 1.3.10]. We note that the point ω is not closed and $\tilde{Y}$ is not weak Hausdorff [12, Def. 1.1.1]. Also the closure of $\{\omega\}$ is all of $\tilde{Y}$. The point ω is analagous to a generic point of a variety. The space $\tilde{Y}$ is known as the *partial map classifier* because of the bijective correspondence between maps $f : A \longrightarrow Y$ with $A \subseteq X$ a closed subset, and corresponding maps $\tilde{f} : X \longrightarrow \tilde{Y}$ defined by

$$\tilde{f}(x) = \begin{cases} \omega & \text{if } x \notin A \\ f(a) & \text{if } x \in A. \end{cases}$$

For a space $p : X \longrightarrow B$ over B we define a map $\xi : B \longrightarrow Map(X, \tilde{B})$ by

$$\xi(b)(x) = \begin{cases} b & \text{if } x \in X_b \\ \omega & \text{otherwise.} \end{cases}$$

Here X_b denotes the fiber $p^{-1}(b)$ over $b \in B$. The map ξ is the adjoint of the map $\tilde{f} : X \times B \longrightarrow B$ obtained as $f : \Delta^{-1}(p \times \mathrm{id}(X \times B)) \longrightarrow B$ where $p \times \mathrm{id} : X \times B \longrightarrow B \times B$ and $\Delta : B \longrightarrow B \times B$ denotes the diagonal embedding. We now have the following.

Definition 2.1.2. *[12, Def. 1.3.11] Let $p : X \longrightarrow B$ and $q : Y \longrightarrow B$ be spaces over B and $Map(X,Y)$ the space of unbased maps from X to Y. Then $Map_B(X,Y)$*

is defined to be the pullback of the following diagram,

$$\begin{array}{ccc} Map_B(X,Y) & \longrightarrow & Map(X,\tilde{Y}) \\ \big\downarrow & & \big\downarrow \tilde{q}_* \\ B & \xrightarrow{\xi} & Map(X,\tilde{B}). \end{array}$$

We note that as a point-set,

$$Map_B(X,Y) = \coprod_{b\in B} Map(X_b, Y_b).$$

We may now define the *parametrized mapping space,*

Definition 2.1.3. *[12, Def. 1.3.16] The parametrized mapping space $F_B(X,Y)$ of two parametrized spaces X and Y is the pullback of the following diagram,*

$$\begin{array}{ccc} F_B(X,Y) & \longrightarrow & Map_B(X,Y) \\ \big\downarrow & & \big\downarrow (s_1)_* \\ B & \xrightarrow{s_2} Y \xrightarrow{\sim} & Map_B(B,Y) \end{array}$$

where s_1 and s_2 are the sections of X and Y, respectively, and $Y \xrightarrow{\sim} Map_B(B,Y)$ is the canonical isomorphism.

Again we note that as a point-set,

$$F_B(X,Y) = \coprod_{b\in B} F(X_b, Y_b).$$

The parametrized mapping space is thus the subspace of $Map_B(X,Y)$ consisting of maps that restrict to based maps between the fibers X_b and Y_b with respective basepoints $s_1(b)$ and $s_2(b)$.

Given a map of spaces $f : A \longrightarrow B$ we define a pair of adjoint functors,

$$f_! : \mathcal{T}_A \longrightarrow \mathcal{T}_B$$

and

$$f^* : \mathcal{T}_B \longrightarrow \mathcal{T}_A$$

by the following pushout and pullback diagrams, respectively.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle s}\downarrow & & \downarrow \\ X & \longrightarrow & f_!X \end{array} \qquad\qquad \begin{array}{ccc} f^*Y & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B \end{array}$$

where $X \in \mathcal{T}_A$ and $Y \in \mathcal{T}_B$. Of particular interest is the example where $f : B \longrightarrow *$ is the map sending B to the one-point space. In this case, for a parametrized space X over B, one obtains the pointed space $f_!(X) = X/s(B)$ with basepoint provided by the class of the section. Similarly, given a pointed space Z with basepoint $z_0 \in Z$, one has the parametrized space $f^*(Z)$ over B with total space $Z \times B$ and projection provided by projecting onto the second factor. The section $s : B \longrightarrow Z \times B$ is then defined by $s(b) = (z_0, b)$. The functor f^*, has a right adjoint, $f_* : \mathcal{T}_A \longrightarrow \mathcal{T}_B$ defined as follows. Let $\iota : B \longrightarrow Map_B(A, A)$ be the adjoint of the map $A \times_B B \longrightarrow A$ sending $(a, f(a)) \mapsto a$. Then for $X \in \mathcal{T}_A$ we define f_*X as the pullback

$$\begin{array}{ccc} f_*X & \longrightarrow & Map_B(A, X) \\ \downarrow & & \downarrow \\ B & \xrightarrow{\iota} & Map_B(A, A). \end{array}$$

In the case where $f : B \longrightarrow *$, f_*X is the space of all sections on X with basepoint the section $s : B \longrightarrow X$.

Within the category of parametrized spaces, we can define parametrized versions of the wedge and smash products by taking the usual wedge and smash products fiberwise. More precisely, we have the following definition.

Definition 2.1.4. *[12, Def. 1.3.8] Let $X, Y \in \mathcal{T}_B$.*

(1) The product of spaces X and Y over B, $X \times_B Y$, is the pullback of the following

diagram,

$$\begin{CD} X \times_B Y @>>> X \\ @VVV @VV{p_1}V \\ Y @>>{p_2}> B \end{CD}$$

(2) The wedge of spaces X and Y over B, $X \vee_B Y$, is the pushout obtained from the following diagram,

$$\begin{CD} B @>{s_1}>> X \\ @V{s_2}VV @VVV \\ Y @>>> X \vee_B Y \end{CD}$$

(3) Finally, we have the inclusion map $X \vee_B Y \longrightarrow X \times_B Y$ defined by sending $x \mapsto (x, s_2p_1(x))$ and $y \mapsto (s_1p_2(y), y)$. It is easy to check that it is well-defined. We then define the smash product of X and Y over B, $X \wedge_B Y$, to be the pushout,

$$\begin{CD} X \vee_B Y @>>> X \times_B Y \\ @VVV @VVV \\ B @>>> X \wedge_B Y. \end{CD}$$

Since every fiber has a basepoint given by the section, each of the above constructions for the product, wedge product, and smash product, is given the parametrized space structure that gives us fiberwise product, wedge product, and smash product, respectively.

2.2 Parametrized Orthogonal Spectra

Let λ be a finite dimensional real inner product space and let S^λ be the one-point compactification of λ. A *topological category* is a category enriched in the symmetric monoidal category of pointed spaces and smash product. Let $\mathcal{I}$ be the topological category with objects all finite dimensional real inner product spaces λ and morphism spaces given by the pointed space of linear isometries from λ to λ',

$$\mathrm{Hom}_{\mathcal{I}}(\lambda, \lambda') = \mathcal{O}(\lambda, \lambda')_+.$$

Let

$$\begin{array}{ccc} E(\lambda,\lambda') & \hookrightarrow & \mathcal{O}(\lambda,\lambda')\times\lambda' \\ \downarrow & & \downarrow{\scriptstyle pr_1} \\ \mathcal{O}(\lambda,\lambda') & = & \mathcal{O}(\lambda,\lambda') \end{array}$$

be the sub-bundle of pairs (f,x) such that $x \in \lambda' - f(\lambda)$, the orthogonal complement. Let $\mathcal{J}$ be the topological category with the same objects as $\mathcal{I}$ but with morphism spaces $\mathrm{Hom}_{\mathcal{J}}(\lambda,\lambda')$ defined to be the Thom space of the vector bundle $E(\lambda,\lambda')$ over $\mathcal{O}(\lambda,\lambda')$. Composition is defined

$$\mathrm{Hom}_{\mathcal{J}}(\lambda',\lambda'') \wedge \mathrm{Hom}_{\mathcal{J}}(\lambda,\lambda') \longrightarrow \mathrm{Hom}_{\mathcal{I}}(\lambda,\lambda'')$$

via $((g,y);(f,x)) \mapsto (g\circ f, g(x)+y)$. The inclusion of the zero-section in $E(\lambda,\lambda')$ induces a map of Thom spaces $\mathrm{Hom}_{\mathcal{I}}(\lambda,\lambda') \longrightarrow \mathrm{Hom}_{\mathcal{J}}(\lambda,\lambda')$ and this map is an isomorphism if the dimensions of λ and λ' are equal. These maps constitute a functor $\mathcal{I} \longrightarrow \mathcal{J}$. A *pointed-topological functor* is a functor enriched over pointed spaces. By definition an *orthogonal spectrum* is a pointed-topological functor

$$X : \mathcal{J} \longrightarrow \mathcal{T}_*.$$

The topological Hochschild homology spectrum THH defined in §3.1 is an example of an orthogonal spectrum.

We recall that the category $\mathcal{T}_B$ is enriched in the symmetric monoidal category of pointed spaces and smash product. Let $S^\lambda_B = f^*(S^\lambda)$.

Definition 2.2.1. *A parametrized orthogonal spectrum over B is a pointed-topological functor*

$$X : \mathcal{J} \longrightarrow \mathcal{T}_B.$$

(Compare [12, Def. 11.2.3]). This amounts to a pointed-topological functor (that we denote by the same symbol)

$$X : \mathcal{I} \longrightarrow \mathcal{T}_B$$

together with continuous natural transformations

$$\sigma_{\lambda,\lambda'} : X(\lambda) \wedge_B S_B^{\lambda'} \longrightarrow X(\lambda \oplus \lambda')$$

of pointed-topological functors from $\mathcal{I} \times \mathcal{I}$ to $\mathcal{T}_B$ such that

$$\sigma_{\lambda,0} : X(\lambda) \wedge_B S_B^0 \longrightarrow X(\lambda \oplus 0)$$

is the canonical isomorphism, and such that the diagram

$$\begin{array}{ccc} X(\lambda) \wedge_B S_B^{\lambda'} \wedge_B S_B^{\lambda''} & \xrightarrow{\sigma_{\lambda,\lambda'} \wedge \mathrm{id}} & X(\lambda \oplus \lambda') \wedge_B S_B^{\lambda''} \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sigma_{\lambda\oplus\lambda',\lambda''}} \\ X(\lambda) \wedge_B S_B^{\lambda'\oplus\lambda''} & \xrightarrow{\sigma_{\lambda,\lambda'\oplus\lambda''}} & X(\lambda \oplus \lambda' \oplus \lambda'') \end{array}$$

commutes. Here the left-hand vertical map is the canonical isomorphism. We note that in Def. 2.2.1 when $B = *$, we obtain the usual definition of an orthogonal spectrum.

We recall from §2.1 that a map of spaces $f : A \longrightarrow B$ gives rise to adjoint functors $f_! : \mathcal{T}_A \longrightarrow \mathcal{T}_B$, $f^* : \mathcal{T}_B \longrightarrow \mathcal{T}_A$ and $f_* : \mathcal{T}_A \longrightarrow \mathcal{T}_B$, which is right adjoint to f^*. For the category of parametrized orthogonal spectra over A and the category of parametrized orthogonal spectra over B, levelwise application of the functors $f_!, f^*$, and f_* gives rise to adjoint functors (that we also denote $f_!, f^*$ and f_*) between the categories of parametrized orthogonal spectra over A and parametrized orthogonal spectra over B [12, Thm. 11.4.1]. Furthermore, if $g : B \longrightarrow C$ is another map of spaces, then there are canonical isomorphisms,

$$(g \circ f)_! E \xrightarrow{\sim} g_! f_! E, \quad (g \circ f)^* E' \xrightarrow{\sim} f^* g^* E', \quad (g \circ f)_* E \xrightarrow{\sim} g_* f_* E,$$

where E is a parametrized orthogonal spectrum over A and E' is a parametrized orthogonal spectrum over C.

Let $B[-]$ be a simplicial (or cyclic) space. Suppose for all non-negative integers

k, we have a parametrized orthogonal spectrum $E[k]$ over $B[k]$, and for every map $\theta : [m] \longrightarrow [n]$ in the simplicial index category, we have a map of parametrized orthogonal spectra over $B[n]$,

$$\theta_E : E[n] \longrightarrow \theta_B^* E[m].$$

Here θ_B^* is the base-change functor associated with the map of spaces $\theta_B : B[n] \longrightarrow B[m]$. We shall require that if $\theta : [m] \longrightarrow [n]$ and $\theta' : [n] \longrightarrow [p]$ are two composable maps in the simplicial index category, then the following diagram of parametrized orthogonal spectra over $B[p]$ commutes:

$$\begin{array}{ccc} E[p] & \xrightarrow{(\theta' \circ \theta)_E} & (\theta_B \circ \theta'_B)^* E[m] \\ \downarrow{\scriptstyle \theta'_E} & & \downarrow{\scriptstyle \sim} \\ {\theta'_B}^* E[n] & \xrightarrow{{\theta'_B}^* \theta_E} & {\theta'_B}^* \theta_B^* E[m]. \end{array}$$

Here the right-hand vertical map is the canonical isomorphism. We recall that the realization B of $B[-]$ is defined to be the coequalizer

$$\coprod_{\theta\colon [m] \to [n]} B[n] \times \Delta^m \underset{g_B}{\overset{f_B}{\rightrightarrows}} \coprod_{[k]} B[k] \times \Delta^k \xrightarrow{\epsilon_B} B$$

where, on the second summand indexed by $\theta : [m] \longrightarrow [n]$, the map f_B is the unique map that, for every map $\theta\colon [m] \longrightarrow [n]$ in the simplicial index category, makes the following diagram commute:

$$\begin{array}{ccc} B[n] \times \Delta^m & \xrightarrow{\theta_B \times \mathrm{id}} & B[m] \times \Delta^m \\ \downarrow{\scriptstyle \mathrm{in}_\theta} & & \downarrow{\scriptstyle \mathrm{in}_{[m]}} \\ \coprod_{\varphi\colon [k] \to [l]} B[l] \times \Delta^k & \xrightarrow{f_B} & \coprod_{[k]} B[k] \times \Delta^k. \end{array}$$

The map g_B is defined similarly as the unique map that, for every map $\theta\colon [m] \longrightarrow [n]$

in the simplicial index category, makes the following diagram commute:

$$\begin{array}{ccc} B[n] \times \Delta^m & \xrightarrow{\mathrm{id} \times \theta_\Delta} & B[n] \times \Delta^n \\ \Big\downarrow \mathrm{in}_\theta & & \Big\downarrow \mathrm{in}_{[n]} \\ \coprod_{\varphi\colon [k] \to [l]} B[l] \times \Delta^k & \xrightarrow{g_B} & \coprod_{[k]} B[k] \times \Delta^k. \end{array}$$

Let $\epsilon'_B = \epsilon_B \circ f_B = \epsilon_B \circ g_B$. Let $\mathrm{pr}_{m,n}\colon B[n] \times \Delta^m \longrightarrow B[n]$ and $\mathrm{pr}_k\colon B[k] \times \Delta^k \longrightarrow B[k]$ be the canonical projections. We then have the parametrized orthogonal spectrum $\mathrm{pr}^*_{m,n} E[n]$ over $B[n] \times \Delta^m$ and the parametrized orthogonal spectrum $\mathrm{pr}^*_k E[k]$ over $B[k] \times \Delta^k$. Let

$$\coprod_{[k]} \mathrm{in}_{[k]!} \mathrm{pr}^*_k E[k]$$

denote the coproduct of the parametrized orthogonal spectra $\mathrm{in}_{[k]!} \mathrm{pr}^*_k E[k]$ over $\coprod_{[k]} B[k] \times \Delta^k$ and let

$$\coprod_{\theta\colon [m] \to [n]} \mathrm{in}_{\theta!} \mathrm{pr}^*_{m,n} E[n]$$

denote the coproduct of the parametrized orthogonal spectra $\mathrm{in}_{\theta!} \mathrm{pr}^*_{m,n} E[n]$ over $\coprod_{\theta\colon [m] \longrightarrow [n]} B[n] \times \Delta^m$. We then define the parametrized spectrum E over B to be the following coequalizer of parametrized spectra over B:

$$\epsilon'_{B!}\Big(\coprod_{\theta\colon [m] \to [n]} \mathrm{in}_{\theta!} \mathrm{pr}^*_{m,n} E[n]\Big) \underset{g_E}{\overset{f_E}{\rightrightarrows}} \epsilon_{B!}\Big(\coprod_{[k]} \mathrm{in}_{[k]!} \mathrm{pr}^*_k E[k]\Big) \xrightarrow{\epsilon_E} E.$$

Here, on the summand indexed by $\theta : [m] \longrightarrow [n]$, f_E is defined as follows. First, the functor $f_{B!}$ commutes with coproducts since it has a right-adjoint functor f^*_B. We define a map

$$f'_E\colon \coprod_{\theta\colon [m] \to [n]} f_{B!} \mathrm{in}_{\theta!} \mathrm{pr}^*_{m,n} E[n] \longrightarrow \coprod_{[k]} \mathrm{in}_{[k]!} \mathrm{pr}^*_k E[k]$$

of parametrized orthogonal spectra over $\coprod_{[k]} B[k] \times \Delta^k$ to be the unique map such that, for every map $\theta\colon [m] \longrightarrow [n]$ in the simplicial index category, the following diagram of parametrized orthogonal spectra over $\coprod_{[k]} B[k] \times \Delta^k$ commutes:

$$\begin{array}{ccc} f_{B!}\,\mathrm{in}_{\theta!}\,\mathrm{pr}^*_{m,n}\,E[n] & \xrightarrow{f'_{E,\theta}} & \mathrm{in}_{[m]!}\,\mathrm{pr}^*_m\,E[m] \\ \big\downarrow{\scriptstyle \mathrm{in}_\theta} & & \big\downarrow{\scriptstyle \mathrm{in}_{[m]}} \\ \coprod_{\varphi\colon [k]\to[l]} f_{B!}\,\mathrm{in}_{\varphi!}\,\mathrm{pr}^*_{k,l}\,E[l] & \xrightarrow{f'_E} & \coprod_{[k]} \mathrm{pr}^*_k\,E[k]. \end{array}$$

Here the map $f'_{E,\theta}$ is defined to be the following composite map:

$$\begin{aligned} f_{B!}\,\mathrm{in}_{\theta!}\,\mathrm{pr}^*_{m,n}\,E[n] &\xrightarrow{\sim} \mathrm{in}_{[m]!}(\theta_B \times \mathrm{id})_!\,\mathrm{pr}^*_{m,n}\,E[n] \\ &\xrightarrow{\sim} \mathrm{in}_{[m]!}\,\mathrm{pr}^*_m\,\theta_{B!}E[n] \xrightarrow{\mathrm{in}_{[m]!}\,\mathrm{pr}^*_m\,\theta^\#_E} \mathrm{in}_{[m]!}\,\mathrm{pr}^*_m\,E[m]. \end{aligned}$$

The first map is the unique natural isomorphism $f_{B!}\,\mathrm{in}_{\theta!} \xrightarrow{\sim} \mathrm{in}_{[m]!}(\theta_B \times \mathrm{id})_!$ that exists because

$$f_B \circ \mathrm{in}_\theta = (\theta_B \times \mathrm{id}) \circ \mathrm{in}_{[m]}\colon B[n] \times \Delta^m \longrightarrow \coprod_{[k]} B[k] \times \Delta^k.$$

The second map is induced from the unique natural isomorphism of [12, Prop. 2.2.9], $(\theta_B \times \mathrm{id})_!\,\mathrm{pr}^*_{m,n} \xrightarrow{\sim} \mathrm{pr}^*_m\,\theta_{B!}$, that exists because the following diagram is a pull-back:

$$\begin{array}{ccc} B[n] \times \Delta^m & \xrightarrow{\theta_B \times \mathrm{id}} & B[m] \times \Delta^m \\ \big\downarrow{\scriptstyle \mathrm{pr}_{m,n}} & & \big\downarrow{\scriptstyle \mathrm{pr}_m} \\ B[n] & \xrightarrow{\theta_B} & B[m]. \end{array}$$

Finally, the last map is induced by the map $\theta^\#_E\colon \theta_{B!}E[n] \to E[m]$ that is the adjoint of the given map $\theta_E\colon E[n] \to \theta^*_B E[m]$. Then f_E is defined to be the map $\epsilon_{B!} f'_E$. To define the map g_E, again, we define a map

$$g'_E\colon \coprod_{\theta\colon [m]\longrightarrow[n]} g_{B!}\,\mathrm{in}_{\theta!}\,\mathrm{pr}^*_{m,n}\,E[n] \longrightarrow \coprod_{[k]} \mathrm{in}_{[k]!}\,\mathrm{pr}^*_k\,E[k]$$

of parametrized orthogonal spectra over $\coprod_{[k]} B[k] \times \Delta^k$ to be the unique map such that, for every map $\theta\colon [m] \longrightarrow [n]$ in the simplicial index category, the following diagram of parametrized orthogonal spectra over $\coprod_{[k]} B[k] \times \Delta^k$ commutes:

$$\begin{array}{ccc} g_{B!}\,\mathrm{in}_{\theta!}\,\mathrm{pr}^*_{m,n}\,E[n] & \xrightarrow{g'_{E,\theta}} & \mathrm{in}_{[n]!}\,\mathrm{pr}^*_n\,E[n] \\ \Big\downarrow{\scriptstyle \mathrm{in}_\theta} & & \Big\downarrow{\scriptstyle \mathrm{in}_{[n]}} \\ \coprod_{\varphi\colon [k]\to[l]} g_{B!}\,\mathrm{in}_{\varphi!}\,\mathrm{pr}^*_{k,l}\,E[l] & \xrightarrow{g'_E} & \coprod_{[k]} \mathrm{pr}^*_k\,E[k]. \end{array}$$

The map $g'_{E,\theta}$ is defined to be the following composite map:

$$g_{B!}\,\mathrm{in}_{\theta!}\,\mathrm{pr}^*_{m,n}\,E[n] \xrightarrow{\sim} \mathrm{in}_{[n]!}(\mathrm{id}\times\theta_\Delta)_!\,\mathrm{pr}^*_{m,n}\,E[n] \longrightarrow \mathrm{in}_{[n]!}\,\mathrm{pr}^*_n\,E[n].$$

The first map is the unique natural isomorphism $g_{B!}\,\mathrm{in}_{\theta!} \xrightarrow{\sim} \mathrm{in}_{[n]!}(\mathrm{id}\times\theta_\Delta)_!$ that exists because

$$g_B \circ \mathrm{in}_\theta = \mathrm{in}_{[n]} \circ (\mathrm{id}\times\theta_\Delta)\colon B[n]\times\Delta^m \longrightarrow \coprod_{[k]} B[k]\times\Delta^k.$$

The second map is induced from the map

$$(\mathrm{id}\times\theta_\Delta)_!\,\mathrm{pr}^*_{m,n}\,E[n] \longrightarrow \mathrm{pr}^*_n\,E[n]$$

that is the adjoint of the unique natural isomorphism

$$\mathrm{pr}^*_{m,n}\,E[n] \xrightarrow{\sim} (\mathrm{id}\times\theta_\Delta)^*\,\mathrm{pr}^*_n\,E[n]$$

which exists because

$$\mathrm{pr}_{m,n} = \mathrm{pr}_n \circ (\mathrm{id}\times\theta_\Delta)\colon B[n]\times\Delta^m \longrightarrow B[n].$$

Then g_E is defined to be the map $\epsilon_{B!}g'_E$. The parametrized orthogonal spectrum E over B has the following mapping property, the proof of which follows directly from the definition of the parametrized orthogonal spectrum E over B.

Proposition 2.2.2. *Let X be a parametrized orthogonal spectrum over B. Then* *giving a map $\alpha\colon E \longrightarrow X$ of parametrized orthogonal spectra over B is equivalent to giving, for every non-negative integer k, a map of parametrized orthogonal spectra over $B[k] \times \Delta^k$*

$$\alpha_k\colon \operatorname{pr}_k^* E[k] \longrightarrow (\epsilon_B \circ \operatorname{in}_{[k]})^* X$$

such that, for every map $\theta\colon [m] \longrightarrow [n]$ in the simplicial index category,

$$(\theta_B \times \operatorname{id})^* \alpha_m = (\operatorname{id} \times \theta_\Delta)^* \alpha_n\colon \operatorname{pr}_{m,n}^* E[n] \longrightarrow (\epsilon'_B \circ \operatorname{in}_\theta)^* X.$$

To understand the λ-th space E_λ of the parametrized orthogonal spectrum E over B we note that limits, colimits, and the functors $f_!$, f^*, and f_* are defined levelwise. The space E_λ is therefore given by the following coequalizer diagram of parametrized spaces over B:

$$\epsilon'_{B!}\Big(\coprod_{\theta\colon [m] \longrightarrow [n]} \operatorname{pr}_{m,n}^* E[n]_\lambda\Big) \underset{g_{E,\lambda}}{\overset{f_{E,\lambda}}{\rightrightarrows}} \epsilon_{B!}\Big(\coprod_{[k]} \operatorname{pr}_k^* E[k]_\lambda\Big) \xrightarrow{\epsilon_{E,\lambda}} E_\lambda.$$

We recall the space $\theta_B^* E[m]_\lambda$ is defined by a pullback diagram. Hence, there is a canonical map of spaces $\operatorname{pr}\colon \theta_B^* E[m]_\lambda \longrightarrow E[m]_\lambda$, and we define

$$\theta^\#_{E,\lambda} = \operatorname{pr} \circ \theta_{E,\lambda}\colon E[n]_\lambda \longrightarrow E[m]_\lambda.$$

The map $\theta^\#_{E,\lambda}$ is a map of spaces and the following diagrams commute:

$$\begin{array}{ccc} E[n]_\lambda & \xrightarrow{\theta^\#_{E,\lambda}} & E[m]_\lambda \\ \downarrow p & & \downarrow p \\ B[n] & \xrightarrow{\theta_B} & B[m] \end{array} \qquad \begin{array}{ccc} E[n]_\lambda & \xrightarrow{\theta^\#_{E,\lambda}} & E[m]_\lambda \\ \uparrow s & & \uparrow s \\ B[n] & \xrightarrow{\theta_B} & B[m]. \end{array}$$

Let E'_λ be the space defined by the following coequalizer diagram:

$$\coprod_{\theta\colon [m] \to [n]} \operatorname{pr}_{m,n}^* E[n]_\lambda \underset{g^\#_{E,\lambda}}{\overset{f^\#_{E,\lambda}}{\rightrightarrows}} \coprod_{[k]} \operatorname{pr}_k^* E[k]_\lambda \xrightarrow{\epsilon^\#_{E,\lambda}} E'_\lambda.$$

Then there is a canonical map $E'_\lambda \to E_\lambda$ and we claim that this map is a homeomorphism. Indeed, this is a special case of the following more general statement:

Proposition 2.2.3. *Let $X[-]$ and $B[-]$ be two diagrams of spaces indexed by a small category I, and let $X = \operatorname{colim}_I X[-]$ and $B = \operatorname{colim}_I B[-]$. Let $p[-]\colon X[-] \longrightarrow B[-]$ and $s[-]\colon B[-] \longrightarrow X[-]$ be natural transformations such that $p[-] \circ s[-]$ is the identity natural transformation of $B[-]$. Then each canonical map $\iota_{B,\alpha}\colon B[\alpha] \longrightarrow B$ gives rise to a parametrized space $(\iota_{B,\alpha})_! X[\alpha]$ over B and the induced map*

$$\operatorname*{colim}_I (\iota_{B,\alpha})_! X[\alpha] \longrightarrow X$$

is an isomorphism of parametrized spaces over B, where the induced maps $p\colon X \longrightarrow B$ and $s\colon B \longrightarrow X$ provide the projection and section maps for the parametrized space X over B.

Proof. We have commutative diagrams

$$\begin{array}{ccc} X[\alpha] & \xrightarrow{\iota_{X,\alpha}} & X \\ \Big\downarrow{\scriptstyle p[\alpha]} & & \Big\downarrow{\scriptstyle p} \\ B[\alpha] & \xrightarrow{\iota_{B,\alpha}} & B \end{array} \qquad \begin{array}{ccc} X[\alpha] & \xrightarrow{\iota_{X,\alpha}} & X \\ \Big\uparrow{\scriptstyle s[\alpha]} & & \Big\uparrow{\scriptstyle s} \\ B[\alpha] & \xrightarrow{\iota_{B,\alpha}} & B. \end{array}$$

It follows that we obtain a parametrized space $(\iota_{B,\alpha})_! X[\alpha]$ over B together with a map $\tilde{\iota}_{X,\alpha}\colon (\iota_{B,\alpha})_! X[\alpha] \to X$ of parametrized spaces over B. The parametrized spaces $(\iota_{B,\alpha})_! X[\alpha]$ over B form an I-diagram of parametrized spaces over B, and the maps $\tilde{\iota}_{X,\alpha}$ give rise to the following map of parametrized spaces over B:

$$\tilde{\iota}\colon \operatorname*{colim}_I (\iota_{B,\alpha})_! X[\alpha] \to X.$$

Now the general statement is that this map is an isomorphism of parametrized spaces over B. Indeed, the canonical maps

$$\varphi_\alpha\colon X[\alpha] \to (\iota_{B,\alpha})_! X[\alpha] \to \operatorname*{colim}_I (\iota_{B,\beta})_! X[\beta]$$

give rise to a map

$$\varphi\colon X = \operatorname*{colim}_I X[\alpha] \to \operatorname*{colim}_I (\iota_{B,\alpha})_! X[\alpha]$$

which is the inverse of the map $\tilde{\iota}$. $\square$

Corollary 2.2.4. *Let $\omega\colon B \longrightarrow *$ be the map from B to the one-point space and let $\omega_k\colon B[k] \longrightarrow *$ be the map from $B[k]$ to the one-point space. The Thom spectrum $\omega_! E$ of the parametrized orthogonal spectrum E over B is canonically isomorphic to the realization of the simplicial orthogonal spectrum given by the Thom spectra of the parametrized orthogonal spectra $E[k]$ over $B[k]$,*

$$\omega_! E \approx \big|[k] \mapsto \omega_{k!} E[k]\big|$$

with simplicial structure maps induced from the maps $\theta^{\#}_{E,\lambda}$.

Chapter 3

Topological Hochschild Homology

3.1 Definition and Structure of the Parametrized Orthogonal Spectrum $E(A, G)$

In general, for any symmetric ring spectrum E as defined in [7], we can define the topological Hochschild homology spectrum $\mathrm{THH}(E)$ following the approach of Bökstedt [1]. Further details of this construction may be found in [5, §1–§2]. We now recall the definition of $\mathrm{THH}(E)$. Initially we define an index category I by declaring the objects $ob(I)$ to be the class of all finite sets, $\underline{i} = \{1, 2, \cdots, i\}, i \geq 1$ and $\underline{0} = \emptyset$. We then declare our morphisms to be *all* injective maps. We note that every morphism is the composite of the standard inclusion $m \mapsto m$ and an automorphism of the target set, albeit non-uniquely. We now have, for each symmetric ring spectrum E and pointed space X, a functor $G_k(E; X) : I^{k+1} \longrightarrow \mathcal{T}_*$. This is defined on objects as

$$G_k(E; X)(\underline{i}_0, \cdots, \underline{i}_k) = F(S^{i_0} \wedge \cdots \wedge S^{i_k}, E_{i_0} \wedge \cdots \wedge E_{i_k} \wedge X)$$

the space of based maps in $\mathcal{T}_*$. We define a cyclic pointed space with k-simplices

$$\mathrm{THH}(E; X)[k] = \operatorname*{hocolim}_{I^{k+1}} G_k(E; X).$$

We define the pointed space $\mathrm{THH}(E;X)$ as the realization of the above cyclic pointed space

$$\mathrm{THH}(E;X) = \big|[k] \mapsto \mathrm{THH}(E;X)[k]\big|.$$

Let λ be a finite dimensional inner product space and let S^λ be the one point compactification of λ. Then we define the orthogonal spectrum $\mathrm{THH}(E)$ with λ-th space

$$\mathrm{THH}(E)_\lambda = \mathrm{THH}(E;S^\lambda) = \big|[k] \mapsto \mathrm{THH}(E;S^\lambda)[k]\big|$$

where

$$\mathrm{THH}(E;S^\lambda)[k] = \operatorname*{hocolim}_{I^{k+1}} G_k(E;S^\lambda).$$

For fixed k and varying λ, each $\mathrm{THH}(E)[k]$ forms an orthogonal spectrum and $\mathrm{THH}(E)$ is the geometric realization of the resulting cyclic orthogonal spectrum. To define the face operators $d_r : \mathrm{THH}(E;S^\lambda)[k] \longrightarrow \mathrm{THH}(E;S^\lambda)[k-1]$, let

$$I \times I \xrightarrow{\sqcup} I$$

be the concatenation functor sending $(\underline{i}, \underline{i}')$ to the set $\underline{i} \sqcup \underline{i}'$. Let $\partial_r : I^{k+1} \longrightarrow I^k$ be the functor defined by

$$\partial_r(\underline{i}_0, \dots, \underline{i}_k) = \begin{cases} (\underline{i}_0, \dots, \underline{i}_r \sqcup \underline{i}_{r+1}, \dots, \underline{i}_k) & \text{if } 0 \le r < k \\ (\underline{i}_k \sqcup \underline{i}_0, \underline{i}_1, \dots, \underline{i}_{k-1}) & \text{if } r = k. \end{cases}$$

Similarly for the degeneracies and cyclic operator, let $s_r : I^{k+1} \longrightarrow I^{k+2}$ be the functor defined on objects by

$$s_r(\underline{i}_0, \dots, \underline{i}_k) = (\underline{i}_0, \dots, \underline{i}_r, \underline{0}, \underline{i}_{r+1}, \dots, \underline{i}_k)$$

for $0 \le r \le k$, and $t_k : I^k \longrightarrow I^k$ by

$$t_k(\underline{i}_0, \dots, \underline{i}_k) = (\underline{i}_k, \underline{i}_0, \dots, \underline{i}_{k-1}).$$

We then define natural transformations

$$\delta_r : G_k(E; S^\lambda) \longrightarrow G_{k-1}(E; S^\lambda) \circ \partial_r$$
$$\sigma_r : G_k(E; S^\lambda) \longrightarrow G_{k+1}(E; S^\lambda) \circ s_r$$
$$\tau_k : G_k(E; S^\lambda) \longrightarrow G_k(E; S^\lambda) \circ t_k$$

as follows. The natural transformation δ_r takes the map $f \in G_k(E; S^\lambda)(\underline{i}_0, \ldots, \underline{i}_k)$ given by

$$S^{i_0} \wedge \cdots \wedge S^{i_k} \xrightarrow{f} E_{i_0} \wedge \cdots \wedge E_{i_k}$$

to the map $\delta_r(f) \in G_{k-1}(E; S^\lambda)(\partial_r(\underline{i}_0, \ldots, \underline{i}_k))$ given by the composition

$$\begin{aligned} S^{i_0} \wedge \cdots \wedge S^{i_r + i_{r+1}} \wedge \cdots \wedge S^{i_k} &\xrightarrow{\sim} S^{i_0} \wedge \cdots \wedge S^{i_r} \wedge S^{i_{r+1}} \wedge \cdots \wedge S^{i_k} \\ &\xrightarrow{f} E_{i_0} \wedge \cdots \wedge E_{i_r} \wedge E_{i_{r+1}} \wedge \cdots \wedge E_{i_k} \wedge S^\lambda \\ &\xrightarrow{\mu} E_{i_0} \wedge \cdots \wedge E_{i_r + i_{r+1}} \wedge \cdots \wedge E_{i_k} \wedge S^\lambda \end{aligned}$$

if $0 \leq r < k$ and

$$\begin{aligned} S^{i_k + i_0} \wedge S^{i_1} \wedge \cdots \wedge S^{i_{k-1}} &\xrightarrow{\sim} S^{i_0} \wedge \cdots \wedge S^{i_{k-1}} \wedge S^{i_k} \\ &\xrightarrow{f} E_{i_0} \wedge \cdots \wedge E_{i_{k-1}} \wedge E_{i_k} \wedge S^\lambda \\ &\xrightarrow{\sim} E_{i_k} \wedge E_{i_0} \cdots \wedge E_{i_{k-1}} \wedge S^\lambda \\ &\xrightarrow{\mu} E_{i_k + i_0} \wedge E_{i_1} \cdots \wedge E_{i_{k-1}} \wedge S^\lambda \end{aligned}$$

if $r = k$. The natural transformations σ_r and τ_k are defined similarly. The face map $d_r : \mathrm{THH}(E; S^\lambda)[k] \longrightarrow \mathrm{THH}(E; S^\lambda)[k-1]$ is then defined as the composite

$$\operatorname*{hocolim}_{I^{k+1}} G_k(E; S^\lambda) \xrightarrow{\delta_r} \operatorname*{hocolim}_{I^{k+1}} G_{k-1}(E; S^\lambda) \circ \partial_r \xrightarrow{(\partial_r)_*} \operatorname*{hocolim}_{I^k} G_{k-1}(E; S^\lambda).$$

For any ring A, we have an associated symmetric ring spectrum $\tilde{A}$, the *Eilenberg-MacLane spectrum*, with level n obtained as the realization of the following simplicial set:

$$\tilde{A}_n = \big|[k] \mapsto A\{S^n[k]\}/A\{s_0[k]\}\big|,$$

where we put the usual simplicial structure on the sphere,

$$S^n[-] = (\Delta^1[-]/\partial\Delta^1[-]) \wedge \cdots \wedge (\Delta^1[-]/\partial\Delta^1[-])$$

with n smash factors and basepoint $s_0[-] \in S^n[-]$. We then define the *topological Hochschild spectrum of the ring* A to be the topological Hochschild spectrum of the Eilenberg–MacLane spectrum $\tilde{A}$ associated to the ring A, and simply write $\mathrm{THH}(A)$ for this spectrum.

Definition 3.1.1. *Let E be a connective symmetric ring spectrum and let G be a discrete group. A left action of G on the spectrum E is a continuous map*

$$\alpha : G_+ \wedge E_n \longrightarrow E_n$$

such that the following diagrams commute

$$\begin{array}{ccc}
G_+ \wedge E_m \wedge E_n & \xrightarrow{id \wedge \mu} & G_+ \wedge E_{m+n}\,, \\
\Big\downarrow {\scriptstyle \Delta \wedge id \wedge id} & & \Big\downarrow {\scriptstyle \alpha} \\
G_+ \wedge G_+ \wedge E_m \wedge E_n & & E_{m+n} \\
\Big\downarrow {\scriptstyle \sim} & & \Big\uparrow {\scriptstyle \mu} \\
G_+ \wedge E_m \wedge G_+ \wedge E_n & \xrightarrow{\alpha \wedge \alpha} & E_m \wedge E_n
\end{array}$$

and

$$\begin{array}{ccc}
G_+ \wedge S^m & \xrightarrow{id \wedge \eta} & G_+ \wedge E_m \\
\Big\downarrow {\scriptstyle pr} & & \Big\downarrow {\scriptstyle \alpha} \\
S^m & \xrightarrow{\eta} & E_m .
\end{array}$$

We note in the definition that E is not necesarily the Eilenberg–MacLane spectrum of a ring and that we do not require the ring spectrum to be connected (0-connected) but only *connective* (-1-connected). Also, the commutativaty of the these two diagrams implies the diagram

$$\begin{array}{ccc} G_+ \wedge E_n \wedge S^m & \xrightarrow{\alpha \wedge id} & E_n \wedge S^m \\ \downarrow{\scriptstyle id \wedge \sigma_{m,n}} & & \downarrow{\scriptstyle \sigma_{m,n}} \\ G_+ \wedge E_{m+n} & \xrightarrow{\alpha} & E_{m+n} \end{array}$$

commutes. Here $\sigma_{m,n}$ is the structure map in the symmetric ring spectrum E. We now define the *twisted group ring spectrum* to be the symmetric ring spectrum with level n space given by

$$(E^\tau[G])_n = E_n \wedge G_+.$$

We define the multiplication $(E^\tau[G])_m \wedge (E^\tau[G])_n \longrightarrow (E^\tau[G])_{m+n}$ as the composition

$$\begin{aligned} E_m \wedge G_+ \wedge E_n \wedge G_+ &\xrightarrow{id \wedge \Delta \wedge id \wedge id} E_m \wedge G_+ \wedge G_+ \wedge E_n \wedge G_+ \\ &\xrightarrow{id \wedge id \wedge \alpha \wedge id} E_m \wedge G_+ \wedge E_n \wedge G_+ \\ &\xrightarrow{id \wedge tw \wedge id} E_m \wedge E_n \wedge G_+ \wedge G_+ \\ &\xrightarrow{\mu_E \wedge \mu_G} E_{m+n} \wedge G_+ \end{aligned}$$

where μ_E and μ_G are the multiplication maps for E and G respectively. We define the unit map $\eta : S^m \longrightarrow (E^\tau[G])_m$ to be the composite

$$S^m \xrightarrow{\sim} S^m \wedge S^0 \xrightarrow{\eta_E \wedge 1_G} E_m \wedge G_+$$

where $\eta_E : S^m \longrightarrow E_m$ is the unit map for the ring spectrum E and $1_G : S^0 \longrightarrow G_+$ is the constant map to the identity of the group G.

We relate this definition of a twisted group ring spectrum to the usual Eilenberg–MacLane spectrum of a twisted group algebra by the following proposition.

Proposition 3.1.2. *Let A be a ring, let G be a discrete group, and let $\tilde{A}$ be the Eilenberg–Lane spectrum associated to A. Then there exists a canonical weak equivalence of ring spectra*

$$\tilde{A}^\tau[G] \longrightarrow \widetilde{A^\tau[G]}.$$

Proof. The canonical map is given by the composition

$$(\tilde{A}^{\tau}[G])_n = \tilde{A}_n \wedge G_+ \xrightarrow{\tilde{\phi}\wedge id} (\widetilde{A^{\tau}[G]})_n \wedge G_+ \xrightarrow{\tilde{r}} (\widetilde{A^{\tau}[G]})_n$$

where $\tilde{\phi}$ is induced from the ring homomorphism $\phi : A \longrightarrow A^{\tau}[G]$ defined by $\phi(a) = a \cdot 1$ and the map $\tilde{r}(- \wedge h)$ is induced from the ring homomorphism $r_h : A^{\tau}[G] \longrightarrow A^{\tau}[G]$ defined by $r_h(a \cdot g) = a \cdot gh$. This composition induces isomorphisms of spectrum homotopy groups and is thus a weak equivalence. □

Let $(g_0, \ldots, g_k)$ be a tuple of elements of G. We define a map

$$(g_0, \ldots, g_k)_* : \mathrm{THH}(A)[k] \longrightarrow \mathrm{THH}(A)[k].$$

Let $f \in G_k(A)(\underline{i}_0, \ldots, \underline{i}_k)$ be given by

$$S^{i_0} \wedge \cdots \wedge S^{i_k} \xrightarrow{f} \tilde{A}_{i_0} \wedge \cdots \wedge \tilde{A}_{i_k}.$$

We define a natural transformation γ by sending f to the composite

$$\begin{aligned} S^{i_0} \wedge \cdots \wedge S^{i_k} &\xrightarrow{f} \tilde{A}_{i_0} \wedge \cdots \wedge \tilde{A}_{i_k} \\ &\xrightarrow{\iota_g} G_+ \wedge \tilde{A}_{i_0} \wedge \cdots \wedge G_+ \wedge \tilde{A}_{i_0} \\ &\longrightarrow \tilde{A}_{i_0} \wedge \cdots \wedge \tilde{A}_{i_k} \end{aligned}$$

where $\iota_g(a_{i_0} \wedge \cdots \wedge a_{i_k}) = g_0 \wedge a_{i_0} \wedge \cdots g_k \wedge a_{i_k}$, and the last map is $\alpha_{i_0} \wedge \cdots \wedge \alpha_{i_k}$. Then

$$(g_0, \ldots, g_k)_* : \mathrm{THH}(A)[k] \longrightarrow \mathrm{THH}(A)[k]$$

is the induced map

$$\operatorname*{hocolim}_{I^{k+1}} G_k(A) \xrightarrow{\gamma_*} \operatorname*{hocolim}_{I^{k+1}} G_k(A).$$

The topological Hochschild spectrum is defined as the geometric realization

$$\mathrm{THH}(A)_{\lambda} = \big|[k] \mapsto \mathrm{THH}(A; S^{\lambda})[k]\big|$$

of the cyclic orthogonal spectrum $\mathrm{THH}(A, S^\lambda)[-]$ with cyclic operators d_r, s_r, and t_k as defined above. We now define the cyclic orthogonal spectrum $\mathrm{THH}^g(A)$ that depends on a choice of element $g \in G$. For $g \in G$ let $\mathrm{THH}^g(A)[-]$ be the cyclic orthogonal spectrum with k-simplices

$$\mathrm{THH}^g(A)[k]_\lambda = \mathrm{THH}(A; S^\lambda)[k]$$

and cyclic structure maps

$$d_r^g = \begin{cases} d_0 \circ (1, g, 1, \dots 1)_* & \text{if } r = 0, \\ d_r & \text{if } 0 < r \leq k, \end{cases}$$

$s_r^g = s_r$, and $t_k^g = t_k$. The geometric realization is the orthogonal spectrum with λ-th space

$$\mathrm{THH}^g(A)_\lambda = \big|[k] \mapsto \mathrm{THH}^g(A)[k]_\lambda\big|.$$

The cyclic bar construction of a group G is the cyclic set $N^{\mathrm{cy}}(G)[-]$ with k-simplices

$$N^{\mathrm{cy}}(G)[k] = \underbrace{G \times \cdots \times G}_{k+1}$$

and face and degeneracy maps given by

$$d_i(g_0, \dots, g_k) = \begin{cases} (g_0, \dots, g_i g_{i+1}, \dots, g_k) & \text{if } 0 \leq i < k \\ (g_k g_0, g_1, \dots, g_{k-1}) & \text{if } i = k \end{cases}$$

$$s_i(g_0, \dots, g_k) = (g_0, \dots, g_i, 1, g_{i+1}, \dots, g_k) \text{ for } 0 \leq i \leq k.$$

We also have the cyclic operator, t_k, defined by

$$t_k(g_0, \dots, g_k) = (g_k, g_0, \dots, g_{k-1}).$$

For each non-negative integer k, let $E(A,G)[k]$ be the parametrized orthogonal spectrum over $N^{\text{cy}}(G)[k]$ given by

$$E(A,G;S^\lambda)[k] = \omega_k{}^*(\text{THH}(A;S^\lambda)[k]) = \text{THH}(A;S^\lambda)[k] \times N^{\text{cy}}(G)[k]$$

where ω_k is the unique map from $N^{\text{cy}}(G)$ to the one-point space. To define the structure maps of the parametrized orthogonal spectrum $E(A,G)$, we first recall the spectrum structure maps of the orthogonal spectrum $\text{THH}(A)[k]$. The space $\text{THH}(A;S^\lambda)[k]$ is obtained as the homotopy colimit of spaces of the form $F(X, Y \wedge S^\lambda)$, and the spectrum structure map

$$\sigma_{\lambda,\lambda'}\colon \text{THH}(A;S^\lambda)[k] \wedge S^{\lambda'} \longrightarrow \text{THH}(A;S^{\lambda\oplus\lambda'})[k]$$

is then obtained from the canonical map

$$F(X, Y \wedge S^\lambda) \wedge S^{\lambda'} \longrightarrow F(X, Y \wedge S^\lambda \wedge S^{\lambda'})$$

and various canonical isomorphisms. It is clear that this makes $\text{THH}(A)[k]$ an orthogonal spectrum. We now define the *twisted* structure maps

$$\theta_E^\tau\colon E(A,G)[n] \longrightarrow \theta_B^* E(A,G)[m]$$

of parametrized orthogonal spectra over $B[n] = N^{\text{cy}}(G)[n]$. First we note that we have corresponding untwisted structure maps

$$\theta_E\colon E(A,G)[n] \longrightarrow \theta_B^* E(A,G)[m]$$

defined by $\theta_E = \omega_n^* \theta_{\text{THH}(A)}$. We define

$$\begin{aligned} s_{r,E}^\tau = s_{r,E}\colon E(A,G)[k] &\longrightarrow s_{r,B}^* E(A,G)[k+1], \quad 0 \leqslant r \leqslant k, \\ t_{k,E}^\tau = t_{k,E}\colon E(A,G)[k] &\longrightarrow E(A,G)[k] \end{aligned}$$

to be the untwisted maps, and define

$$d^{\tau}_{r,E} = d_{r,E} \circ \varphi_r \colon E(A,G)[k] \longrightarrow d^{*}_{r,B}E(A,G)[k-1], \quad 0 \leqslant r \leqslant k,$$

to be the composition of the untwisted map and the automorphism

$$\varphi_r \colon E(A,G)[k] \longrightarrow E(A,G)[k]$$

of parametrized orthogonal spectra over $N^{\mathrm{cy}}(G)[k]$ defined by

$$\varphi_r(f,(g_0,\ldots,g_k)) = ((t_k^{r+1}(g_r,1,\ldots,1))_*(f),(g_0,\ldots,g_k)).$$

We verify that the map φ_r is a map of parametrized orthogonal spectra over $B[k] = N^{\mathrm{cy}}(G)[k]$; that is, we check that the following diagram of parametrized spaces over $B[k]$ commutes:

$$\begin{array}{ccc} E(A,G;S^{\lambda})[k] \wedge_{B[k]} S^{\lambda'}_{B[k]} & \xrightarrow{\varphi_{r,\lambda}\wedge \mathrm{id}} & E(A,G;S^{\lambda})[k] \wedge_{B[k]} S^{\lambda'}_{B[k]} \\ \downarrow{\scriptstyle \sigma_{\lambda,\lambda'}} & & \downarrow{\scriptstyle \sigma_{\lambda,\lambda'}} \\ E(A,G;S^{\lambda\oplus\lambda'})[k] & \xrightarrow{\varphi_{r,\lambda\oplus\lambda'}} & E(A,G;S^{\lambda\oplus\lambda'})[k]. \end{array}$$

Since the commutativity of the above diagram is established by verifying the commutativity of the maps at the point-set level, we can check the diagram one fiber at a time. The induced diagram of fibers over $(g_0,\ldots,g_k) \in B[k]$ takes the form

$$\begin{array}{ccc} \mathrm{THH}(A;S^{\lambda})[k] \wedge S^{\lambda'} & \xrightarrow{(t_k^{r+1}(g_r,1,\ldots,1))_*\wedge \mathrm{id}} & \mathrm{THH}(A;S^{\lambda})[k] \wedge S^{\lambda'} \\ \downarrow{\scriptstyle \sigma_{\lambda,\lambda'}} & & \downarrow{\scriptstyle \sigma_{\lambda,\lambda'}} \\ \mathrm{THH}(A;S^{\lambda\oplus\lambda'})[k] & \xrightarrow{(t_k^{r+1}(g_r,1,\ldots,1))_*} & \mathrm{THH}(A;S^{\lambda\oplus\lambda'})[k]. \end{array}$$

It commutes since the canonical map

$$F(X, Y \wedge S^{\lambda}) \wedge S^{\lambda'} \longrightarrow F(X, Y \wedge S^{\lambda} \wedge S^{\lambda'})$$

is natural in the variables X and Y. This completes the definition of the twisted structure maps.

We next show that given composable maps $\theta\colon [m] \longrightarrow [n]$ and $\theta'\colon [n] \longrightarrow [p]$ in the simplicial index category, the following diagram of parametrized orthogonal spectra over $B[p]$ commutes:

$$\begin{array}{ccc} E(A,G)[p] & \xrightarrow{\theta_E^\tau \circ \theta_E'^\tau} & (\theta_B \circ \theta'_B)^* E(A,G)[m] \\ \Big\downarrow{\scriptstyle \theta_E'^\tau} & & \Big\downarrow{\scriptstyle \sim} \\ {\theta'_B}^* E(A,G)[n] & \xrightarrow{{\theta'_B}^* \theta_E^\tau} & {\theta'_B}^* \theta_B^* E(A,G)[m]. \end{array}$$

Again, this can be easily be checked on fibers. Hence, we obtain a parametrized orthogonal spectrum $E(A,G)$ over $N^{\mathrm{cy}}(G)$ with λth space

$$E(A,G)_\lambda = E(A,G;S^\lambda) = \big|[k] \mapsto E(A,G;S^\lambda)[k]\big|$$

where the simplicial structure maps in the simplicial space on the right-hand side are the twisted maps $\theta_E^{\tau\#}$. We now present the proof of Thm. 1.0.2.

Proof of Theorem 1.0.2. Let ω be the unique map from $B = N^{\mathrm{cy}}(G)$ to the one-point space. We wish to construct a map of parametrized orthogonal spectra over B

$$\tilde{\Psi}\colon E(A,G) \longrightarrow \omega^* \,\mathrm{THH}(A^\tau[G])$$

and show that the adjoint map

$$\Psi\colon \omega_! E(A,G) \longrightarrow \mathrm{THH}(A^\tau[G])$$

is a stable equivalence of orthogonal spectra. These maps exist for every symmetric ring spectrum R with a G-action in the sense of Def. 3.1.1 and with the symmetric ring spectrum $R^\tau[G]$ as defined in the paragraph immediately following Def. 3.1.1. We shall work in this generality. As we noted above, the orthogonal spectrum $\omega_! E(R,G)$

is the realization of the simplicial orthogonal spectrum with k-th term

$$\omega_{k!}E(R,G)[k] = \mathrm{THH}(R)[k] \wedge N^{\mathrm{cy}}(G)[k]_+$$

and with simplicial structure maps given by the compositions

$$\theta^{\tau}_{\omega_! E} \colon \omega_{n!}E(R,G)[n] \xrightarrow{\omega_{n!}\theta^{\tau}_{E}} \omega_{n!}\theta^{*}_{B}E(R,G)[m] \xrightarrow{\omega_{m!}\epsilon\omega^{*}_{m}} \omega_{m!}E(R,G)[m].$$

Here, we recall, $\omega_{n!} = \omega_{m!}\theta_{B!}$. The map $\epsilon : \theta_{B!}\theta^{*}_{B} \longrightarrow \mathrm{id}$ is the counit of the adjunction $(\theta_{B!}, \theta^{*}_{B})$ and is given by the map

$$\epsilon = \mathrm{id} \wedge \theta_B \colon \ \mathrm{THH}(R)[m] \wedge N^{\mathrm{cy}}(G)[n]_+ \longrightarrow \mathrm{THH}(R)[m] \wedge N^{\mathrm{cy}}(G)[m]_+.$$

The map Ψ is defined to be the map of realizations obtained from a map of simplicial orthogonal spectra

$$\Psi_k \colon \ \mathrm{THH}(R)[k] \wedge N^{\mathrm{cy}}(G)[k]_+ \longrightarrow \mathrm{THH}(R^{\tau}[G])[k]$$

that we define below. The definition of this map is given in the proof of [6, Thm. 7.1]. It is also shown there that the map is a stable equivalence of orthogonal spectra (provided that the symmetric ring spectrum R converges; this is the case for $R = \tilde{A}$). Hence, it suffices to show that the maps Ψ_k are compatible with the simplicial structure maps. We first recall that the map

$$\Psi_{k,\lambda} \colon \ \mathrm{THH}(R; S^{\lambda})[k] \wedge N^{\mathrm{cy}}(G)[k]_+ \longrightarrow \mathrm{THH}(R^{\tau}[G]; S^{\lambda})[k].$$

is the map of homotopy colimits over I^{k+1} obtained from the composite map

$$\begin{aligned}
&F(S^{i_0} \wedge \cdots \wedge S^{i_k}, R_{i_0} \wedge \ldots R_{i_k} \wedge S^{\lambda}) \wedge G_+ \wedge \cdots \wedge G_+ \\
&\longrightarrow F(S^{i_0} \wedge \cdots \wedge S^{i_k}, R_{i_0} \wedge \cdots \wedge R_{i_k} \wedge G_+ \wedge \cdots \wedge G_+ \wedge S^{\lambda}) \\
&\longrightarrow F(S^{i_0} \wedge \cdots \wedge S^{i_k}, R_{i_0} \wedge G_+ \wedge \cdots \wedge R_{i_k} \wedge G_+ \wedge S^{\lambda})
\end{aligned}$$

where the first map is the same canonical map that was used to define the spectrum structure maps, and where the second map is induced from the permutation

$$R_{i_0} \wedge \cdots \wedge R_{i_k} \wedge G_+ \wedge \cdots \wedge G_+ \longrightarrow R_{i_0} \wedge G_+ \wedge \cdots \wedge R_{i_k} \wedge G_+$$

that maps $(r_0, \ldots, r_k, g_0, \ldots, g_k)$ to $(r_0, g_0, \ldots, r_k, g_k)$. We show the following diagram commutes:

$$\begin{array}{ccc} \omega_{k!}E(R,G) & \xrightarrow{\Psi_k} & \mathrm{THH}(R^\tau[G])[k] \\ \downarrow{\scriptstyle d^\tau_{r,\omega_! E}} & & \downarrow{\scriptstyle d_r} \\ \omega_{(k-1)!}E(R,G)[k-1] & \xrightarrow{\Psi_{k-1}} & \mathrm{THH}(R^\tau[G])[k-1]. \end{array}$$

Let $\partial_r \colon I^{k+1} \longrightarrow I^k$ be the functor defined in the beginning of the section and let $G_k(R;X)$ be the functor from I^{k+1} to the category of pointed spaces also defined at beginning of the section. Let $\delta_r \colon G_k(R;X) \longrightarrow G_{k-1}(R;X) \circ \partial_r$ be the natural transformation used to define the face map of the cyclic pointed space $\mathrm{THH}(R;X)[-]$, again, defined at the beginning of the section. Then the right-hand vertical map in the above diagram is given by the composite map

$$\begin{aligned} \operatorname*{hocolim}_{I^{k+1}} G_k(R^\tau[G]; S^\lambda) &\xrightarrow{\delta_{r*}} \operatorname*{hocolim}_{I^{k+1}} G_{k-1}(R^\tau[G]; S^\lambda) \circ \partial_r \\ &\xrightarrow{\partial_{r*}} \operatorname*{hocolim}_{I^k} G_{k-1}(R^\tau[G]; S^\lambda). \end{aligned}$$

The left-hand vertical map in the diagram above is also a composition, given by

$$\begin{aligned} \operatorname*{hocolim}_{I^{k+1}} G_k(R; S^\lambda) \wedge N^{\mathrm{cy}}(G)[k]_+ &\xrightarrow{\delta^\tau_{r*}} \operatorname*{hocolim}_{I^{k+1}} G_{k-1}(R; S^\lambda) \wedge N^{\mathrm{cy}}(G)[k-1]_+ \circ \partial_r \\ &\xrightarrow{\partial_{r*}} \operatorname*{hocolim}_{I^k} G_{k-1}(R; S^\lambda) \wedge N^{\mathrm{cy}}(G)[k-1]_+ \end{aligned}$$

where the natural transformation

$$\delta^\tau_r \colon G_k(R;X) \wedge N^{\mathrm{cy}}(G)[k]_+ \longrightarrow (G_{k-1}(R;X) \wedge N^{\mathrm{cy}}(G)[k-1]_+) \circ \partial_r$$

is defined by

$$\delta_r^\tau(f,(g_0,\ldots,g_k)) = (\delta_r((t_k^{r+1}(g_r,1,\ldots,1))_*(f)), d_r(g_0,\ldots,g_k)).$$

Hence, it suffices to show that the diagram of natural transformations

$$\begin{array}{ccc} G_k(R;S^\lambda)\wedge N^{\mathrm{cy}}(G)[k]_+ & \xrightarrow{\Psi_{k,\lambda}} & G_k(R^\tau[G];S^\lambda) \\ \Big\downarrow{\scriptstyle \delta_r^\tau} & & \Big\downarrow{\scriptstyle \delta_r} \\ G_{k-1}(R;S^\lambda)\wedge N^{\mathrm{cy}}(G)[k-1]_+\circ\partial_r & \xrightarrow{\Psi_{k-1,\lambda}\circ\partial_r} & G_{k-1}(R^\tau[G];S^\lambda)\circ\partial_r \end{array}$$

commutes. But this follows immediately from the definitions of the natural transformations involved and from the naturality of the canonical map

$$F(X,Y)\wedge Z \longrightarrow F(X,Y\wedge Z).$$

We therefore have the desired map Ψ of orthogonal spectra and its adjoint $\tilde{\Psi}$ of parametrized orthogonal spectra over $N^{\mathrm{cy}}(G)$. As we previously stated, it is proved in [6, Thm. 7.1] (see also [5, Prop. 4.1]) that the maps of orthogonal spectra Ψ_k are stable equivalences, provided that the symmetric ring spectrum R converges. We wish to also conclude that the induced map of realizations Ψ is a stable equivalence. It is proved in [11] that this holds, provided the simpicial spaces $\omega_! E(R,G)[-]$ and $\mathrm{THH}(R^\tau[G])[-]$ are *proper* in the sense of [11, Def. 11.2]. This, in turn, is the case, if the unit maps $\eta_i\colon S^i \longrightarrow R_i$ are Hurewicz cofibrations. If $R=\tilde{A}$, then both properties hold, and hence, the map Ψ is a stable equivalence of orthogonal spectra. □

3.2 The Topological Hochschild Spectrum and Cyclic Bar Construction

The cyclic set $EG[-]$ is defined by $EG[k]=\mathrm{Map}([k],G)$. The face and degeneracy operators $d_i: EG[k]\longrightarrow EG[k-1]$ and $s_i: EG[k]\longrightarrow EG[k+1]$, for $0\le i\le k$, are given by

$$d_i(g_0, \dots, g_k) = (g_0, \dots, \hat{g}_i, \dots, g_k)$$
$$s_i(g_0, \dots, g_k) = (g_0, \dots, g_i, g_i, \dots, g_k),$$

where the hat symbol indicates that the i-th term is omitted. The cyclic operator is defined

$$t_k(g_0, \dots, g_k) = (g_k, g_0, \dots, g_{k-1}).$$

We follow [13], and let G^{ad} denote the set G with the group G acting from the left by conjugation. We note that an element $g \in G$ determines an isomorphism of sets between $G/C_G(g)$ and the conjugacy class $\langle g \rangle$ of the element g given by mapping the class $aC_G(g)$ to $g^a = aga^{-1}$. Here $C_G(g)$ denotes the centralizer of g. Therefore,

$$G^{\text{ad}} = \coprod_{\langle g \rangle} G/C_G(g) \cdot g.$$

Then the map

$$\phi : EG[-] \times_G G^{\text{ad}} \xrightarrow{\sim} N^{\text{cy}}(G)[-]$$

defined on level k by

$$\phi([(g_0, \dots, g_k); g]) = (g_k g g_0^{-1}, g_0 g_1^{-1}, g_0 g_1^{-1}, \dots, g_{k-1} g_k^{-1})$$

is an isomorphism. The inverse sends a k-simplex $(g_0, \dots, g_k)$ in $N^{\text{cy}}(G)[k]$ to the class $[(g_1 \cdots g_k, g_2 \cdots g_k, \dots, g_k, 1); g_0 g_1 \cdots g_k]$. The adjoint of the composition

$$\mathbb{T} \times N^{\text{cy}}(G) \xrightarrow{\mu} N^{\text{cy}}(G) \xrightarrow{\pi} BG$$

is a map to the free-loop space of the classifying space of the group,

$$N^{\text{cy}}(G) \longrightarrow \Lambda BG$$

and this map is a weak equivalence [2, Prop 2.6]. The map

$$\pi : N^{\mathrm{cy}}(G) \longrightarrow N(G) = BG$$

in the composition is given by the projection

$$(g_0, \dots, g_k) \mapsto (g_1, \dots, g_k).$$

Also, the set of connected components of $N^{\mathrm{cy}}(G)$ is in one-to-one correspondence with the set of conjugacy classes of elements in the group G. Given the cyclic set $E(A,G)[-]_\lambda$ defined in §3.1, we now describe the geometric fiber over the connected component of $E(A,G)_\lambda$ corresponding to the conjugacy class of the element $g \in G$. To understand the fiber over the point given by the 0-simplex $g \in N^{\mathrm{cy}}(G)[0]$, we evaluate the pullback

$$\begin{array}{ccc} E(A,G)^g_\lambda & \longrightarrow & E(A,G)_\lambda \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{\ g\ } & N^{\mathrm{cy}}(G) \end{array}$$

where the right-hand vertical map is the projection of the cyclic space $E(A,G)[-]_\lambda$ onto the cyclic bar construction and the bottom horizantal map takes the unique non-degenerate 0-simplex to g. Recall that geometric realization preserves finite limits [4] in the sense that the canonical map

$$\big|[k] \mapsto \lim_{\alpha} X_\alpha[-]\big| \longrightarrow \lim_{\alpha} \big|[k] \mapsto X_\alpha\big|$$

is a homeomorphism, provided that the index category for the limit system is finite. Hence the geometric fiber $E(A,G)^g$ is the space obtained as the realization of the following pullback diagram of simplicial spaces

$$\begin{array}{ccc} E(A,G)^g[-]_\lambda & \longrightarrow & E(A,G)[-]_\lambda \\ \downarrow & & \downarrow \\ \Delta^0[-] & \xrightarrow{\ g\ } & N^{\mathrm{cy}}(G)[-]. \end{array}$$

Here $\Delta^0[-]$ is the standard 0-simplex

$$\Delta^0 = \mathrm{Hom}_\Delta(-,[0])$$

and the bottom map takes the identity, $[0] \xrightarrow{1} [0]$ to the 0-simplex $g \in N^{\mathrm{cy}}(G)[0]$. Explicitly, this map takes the map $\theta : [k] \longrightarrow [0]$ to $\theta^* g \in N^{\mathrm{cy}}(G)[k]$, and since for each k, there is only one such map θ, $\theta^* g = (g, 1, \ldots, 1)$ ($k+1$ factors). Thus the fiber $E(A,G)^g[-]_\lambda$ is given simplical degree-wise by

$$E(A,G)^g[k]_\lambda = \mathrm{THH}(A; S^\lambda)[k] \times \{(g,1,\ldots,1)\} \subset E(A,G)_\lambda[k].$$

The cyclic structure of the fiber is that of $E(A,G)[-]_\lambda$ restricted at each k to the subset $\{(g,1,\ldots,1)\} \subset N^{\mathrm{cy}}(G)[k]$. Thus as a simplicial set the fiber is canonically isomorphic to the simplicial set $\mathrm{THH}^g(A)[-]$ defined in §3.1. The connected component of the 0-simplex g is obtained as the realization of the cyclic spectrum $E(A,G)\big|_{\langle g\rangle}[-]_\lambda$ given on level k by

$$E(A,G)\big|_{\langle g\rangle}[k]_\lambda = \mathrm{THH}(A; S^\lambda)[k] \times \big\{(g_0, g_1, \ldots, g_k) : g_0 g_1 \cdots g_k \in \langle g\rangle\big\},$$

a subset of $E(A,G)[k]_\lambda$. Indeed, a path from g to the zeroth vertex of the k-simplex $(g_0, \ldots, g_k)$ is given by the 1-simplex $(gh^{-1}, h) \in N^{\mathrm{cy}}(G)[1]$.

Lemma 3.2.1. *There exists an equivalence of orthogonal spectra*

$$\tilde{\phi}_g : EG \times_{C_G(g)} \mathrm{THH}^g(A) \xrightarrow{\sim} E(A,G)\big|_{\langle g\rangle},$$

between the Borel construction and the spectrum corresponding to the connected component indexed by $\langle g\rangle$. The equivalence depends on the choice of representative $g \in \langle g\rangle$.

Proof. We in fact show the stronger statement that we have a degree-wise isomorphism between the Borel construction $EG[-] \times_{C_G(g)} \mathrm{THH}^g(A)[-]_\lambda$ and $E(A,G)\big|_{\langle g\rangle}[-]_\lambda$.

We define the isomorphism

$$\tilde{\phi}_g : EG[-] \times_{C_G(g)} \mathrm{THH}^g(A)[-] \longrightarrow E(A,G)\big|_{\langle g\rangle}[-]_\lambda$$

by

$$\tilde{\phi}_g([(g_0,\ldots,g_k);[f]]) = ((g_k,g_0,\ldots,g_{k-1})_*[f];\phi(g_0,\ldots,g_k)),$$

where $\phi(g_0,\ldots,g_k) = (g_k g g_0^{-1}, g_0 g_1^{-1},\ldots,g_{k-1}g_k^{-1})$ as above and $[f]$ the class in the homotopy colimit represented by the map

$$S^{i_0} \wedge \cdots S^{i_k} \xrightarrow{f} \tilde{A}_{i_0} \wedge \cdots \wedge \tilde{A}_{i_k}.$$

Since $C_G(g)$ acts on $EG[-]$ diagonally from the right and on the fiber diagonally from the left, we have

$$[(g_0h,\ldots,g_kh);[f]] = [(g_0,\ldots,g_k);(h,\ldots,h)_*[f]].$$

We will first show that the map $\tilde{\phi}_g$ is well-defined. Given

$$\tilde{\phi}_g([g_0h,\ldots,g_kh;[f]]) = ((g_kh,g_0h,\ldots,g_{k-1}h)_*[f];\phi(\bar{g}))$$

we have

$$\begin{aligned}\phi(\bar{g}) &= (g_khg(g_0h)^{-1}, g_0h(g_1h)^{-1},\ldots,g_{k-1}h(g_kh)^{-1})\\ &= (g_kg^hg_0^{-1}, g_0g_1^{-1},\ldots,g_{k-1}g_k^{-1})\\ &= (g_kgg_0^{-1}, g_0g_1^{-1},\ldots,g_{k-1}g_k^{-1})\end{aligned}$$

since $h \in C_G(g)$. On the other hand, $\tilde{\phi}_g([(g_0,\ldots,g_k);(h,\ldots,h)_*[f]])$

$$\begin{aligned}&= ((g_k,g_0,\ldots,g_{k-1})_* \circ (h,\ldots,h)_*[f];\phi(g'))\\ &= ((g_kh,g_0h,\ldots,g_{k-1}h)_*[f];\phi(g'))\end{aligned}$$

where we use that G acts on the spectrum A from the left in the sense of Def. 3.1.1. We also have $\phi(g') = (g_k g g_0^{-1}, g_0 g_1^{-1}, \ldots, g_{k-1} g_k^{-1}) = \phi(\bar{g})$. Thus the map $\tilde{\phi}_g$ is well-defined. It only remains to demonstrate the commutativity with the cyclic operators. We check the commutavity of the zeroth operators $d_0 \times d_0^g$ and $d_{0,E}^{\mathcal{T}}$. The other operators of $EG[-] \times_{C_G(g)} \mathrm{THH}^g(A)[-]$ are the standard product operators of $EG[-]$ and $\mathrm{THH}(A)[-]_\lambda$. These operators do not involve the element $g \in G$, and their commutativity with their corresponding maps $d_{r,E}^{\mathcal{T}}$, $s_{r,E}^{\mathcal{T}}$ and $t_{k,E}^{\mathcal{T}}$ can be checked similarly to how we check for $d_{0,E}^{\mathcal{T}}$. On level k we wish to show that the following diagram commutes

$$\begin{array}{ccc}
EG[k] \times_{C_G(g)} \mathrm{hocolim}_{I^{k+1}} G_k(A) & \xrightarrow{\tilde{\phi}_g} & \mathrm{hocolim}_{I^{k+1}} G_k(A) \times N^{\mathrm{cy}}(G)[k] \\
\downarrow{\scriptstyle (1,g,1,\ldots,1)_* \times \mathrm{id}} & & \downarrow{\scriptstyle (1,g_k g g_0^{-1},\ldots,1)_* \times \mathrm{id}} \\
EG[k] \times_{C_G(g)} \mathrm{hocolim}_{I^{k+1}} G_k(A) & & \mathrm{hocolim}_{I^{k+1}} G_k(A) \times N^{\mathrm{cy}}(G)[k] \\
\downarrow{\scriptstyle \delta_0 \times \mathrm{id}} & & \downarrow{\scriptstyle \delta_0 \times \mathrm{id}} \\
EG[k] \times_{C_G(g)} \mathrm{hocolim}_{I^{k+1}} G_{k-1}(A) \circ \partial_0 & & \mathrm{hocolim}_{I^{k+1}} G_{k-1}(A) \circ \partial_0 \times N^{\mathrm{cy}}(G)[k] \\
\downarrow{\scriptstyle (\partial_0)_* \times d_0} & & \downarrow{\scriptstyle (\partial_0)_* \times d_0} \\
EG[k-1] \times_{C_G(g)} \mathrm{hocolim}_{I^k} G_{k-1}(A) & \xrightarrow{\tilde{\phi}_g} & \mathrm{hocolim}_{I^k} G_{k-1}(A) \times N^{\mathrm{cy}}(G)[k-1].
\end{array}$$

We first verify the commutativity of this diagram for the factors involving $EG[-]$ and $N^{\mathrm{cy}}(G)[-]$. For the tuple $(g_0, \ldots, g_k)$ in $EG[k]$,

$$\begin{aligned}
&d_0 \phi(g_0, \ldots, g_k) \\
&\qquad = d_0(g_k g g_0^{-1}, g_0 g_1^{-1}, \ldots, g_{k-1} g_k^{-1}) \\
&\qquad = (g_k g g_1^{-1}, \ldots, g_{k-1} g_k^{-1}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\phi d_0(g_0, \ldots, g_k) \\
&\qquad = \phi(g_1, g_2 \ldots, g_{k-1}, g_k) \\
&\qquad = (g_k g g_1^{-1}, \ldots, g_{k-1} g_k^{-1}).
\end{aligned}$$

We thus only need to verify the commutativity of the factors involving the homotopy colimits in the above diagram. Since G acts on the spectrum from the left, we have

$$(1, g_k g g_0^{-1}, 1, \ldots, 1)_* \circ (g_k, g_0, \ldots, g_{k-1})_* = (g_k, g_k g, \ldots, g_{k-1})_*.$$

For a representative

$$S^{i_0} \wedge \cdots \wedge S^{i_k} \xrightarrow{f} \tilde{A}_{i_0} \wedge \cdots \wedge \tilde{A}_{i_k}$$

of $[f]$, the natural transformation corresponding to the composite

$$d_0 \circ (g_k, g_k g, \ldots, g_{k-1})_*[f]$$

takes f to the composition

$$\begin{aligned} S^{i_0+i_1} \wedge S^{i_1} \wedge \cdots \wedge S^{i_k} &\xrightarrow{\sim} S^{i_0} \wedge \cdots \wedge S^{i_{k-1}} \wedge S^{i_k} \\ &\xrightarrow{f} \tilde{A}_{i_0} \wedge \cdots \wedge \tilde{A}_{i_{k-1}} \wedge \tilde{A}_{i_k} \\ &\xrightarrow{\iota} G_+ \wedge \tilde{A}_{i_0} \wedge \cdots \wedge G_+ \wedge \tilde{A}_{i_l} \\ &\xrightarrow{\alpha} \tilde{A}_{i_0} \wedge \cdots \wedge \tilde{A}_{i_k} \\ &\xrightarrow{\mu} \tilde{A}_{i_0+i_1} \wedge \tilde{A}_{i_2} \cdots \wedge \tilde{A}_{i_k}. \end{aligned}$$

where

$$\iota(a_{i_0} \wedge \cdots \wedge a_{i_k}) = g_k \wedge a_0 \wedge g_k g \wedge a_{i_1} \wedge \cdots \wedge g_{k-1} \wedge a_{i_k},$$

and α is the smash product of each action, $\alpha_{i_0} \wedge \cdots \wedge \alpha_{i_k}$. Via the composition going the other way, the representative

$$S^{i_0} \wedge \cdots \wedge S^{i_k} \xrightarrow{f} \tilde{A}_{i_0} \wedge \cdots \wedge \tilde{A}_{i_k},$$

is taken to

$$\begin{aligned}
S^{i_0+i_1}\wedge S^{i_1}\wedge\cdots\wedge S^{i_k} &\xrightarrow{\sim} S^{i_0}\wedge\cdots\wedge S^{i_{k-1}}\wedge S^{i_k}\\
&\xrightarrow{f} \tilde{A}_{i_0}\wedge\cdots\wedge\tilde{A}_{i_{k-1}}\wedge\tilde{A}_{i_k}\\
&\xrightarrow{\iota} G_+\wedge\tilde{A}_{i_0}\wedge\cdots\wedge G_+\wedge\tilde{A}_{i_l}\\
&\xrightarrow{\alpha} \tilde{A}_{i_0}\wedge\cdots\wedge\tilde{A}_{i_k}\\
&\xrightarrow{\mu} \tilde{A}_{i_0+i_1}\wedge\tilde{A}_{i_2}\cdots\wedge\tilde{A}_{i_k}.
\end{aligned}$$

Here

$$\iota(a_{i_0}\wedge\cdots\wedge a_{i_k}) = 1\wedge a_0\wedge g\wedge a_{i_1}\wedge\cdots\wedge 1\wedge a_{i_k}.$$

It then suffices to prove the following diagram

$$\begin{array}{ccc}
G_+^{\wedge k}\wedge A_{i_0}\wedge A_{i_1}\wedge\cdots\wedge A_{i_k} & \xrightarrow{(id,\tilde{\mu},id,\ldots,id)\wedge id^{\wedge k}} & G_+^{\wedge k}\wedge A_{i_0}\wedge A_{i_1}\wedge\cdots\wedge A_{i_k}\\
\Big\downarrow{\scriptstyle \sim} & & \Big\downarrow{\scriptstyle \sim}\\
G_+\wedge A_{i_0}\wedge\cdots\wedge G_+\wedge A_{i_k} & & G_+\wedge A_{i_0}\wedge\cdots\wedge G_+\wedge A_{i_k}\\
\Big\downarrow{\scriptstyle id\wedge\alpha\wedge id\cdots\wedge id} & & \Big\downarrow{\scriptstyle \alpha\wedge\cdots\wedge\alpha}\\
G_+\wedge A_{i_0}\wedge A_{i_1}\wedge G_+\wedge A_{i_2}\wedge\cdots\wedge G_+\wedge A_{i_k} & & A_{i_0}\wedge\cdots\wedge A_{i_k}\\
\Big\downarrow{\scriptstyle id\wedge\mu\wedge id\wedge\cdots\wedge id} & & \Big\downarrow{\scriptstyle \mu\wedge id\wedge\cdots\wedge id}\\
G_+\wedge A_{i_0+i_1}\wedge G_+\wedge A_{i_2}\wedge\cdots\wedge G_+\wedge A_{i_k} & \xrightarrow[\alpha\wedge\cdots\wedge\alpha]{} & A_{i_0+i_1}\wedge A_{i_2}\wedge\cdots\wedge A_{i_k}
\end{array}$$

commutes. Here $(id,\tilde{\mu},id,\ldots,id): G^{\wedge k}\longrightarrow G^{\wedge k}$ sends the class $g_k\wedge g\wedge g_1\wedge\cdots\wedge g_{k-1}$ to the class $g_k\wedge g_k g\wedge g_1\wedge\cdots\wedge g_{k-1}$. and this map is an isomorphism. Hence there exists a unique isomorphism

$$\zeta : G_+\wedge A_{i_0}\wedge\cdots\wedge G_+\wedge A_{i_k}\xrightarrow{\sim} G_+\wedge A_{i_0}\wedge\cdots\wedge G_+\wedge A_{i_k}$$

such that the top four terms form a commutative diagram,

$$\begin{array}{ccc}
G_+^{\wedge k}\wedge A_{i_0}\wedge A_{i_1}\wedge\cdots\wedge A_{i_k} & \xrightarrow{(id,\tilde{\mu},id,\ldots,id)\wedge id^{\wedge k}} & G_+^{\wedge k}\wedge A_{i_0}\wedge A_{i_1}\wedge\cdots\wedge A_{i_k}\\
\Big\downarrow{\scriptstyle \sim} & & \Big\downarrow{\scriptstyle \sim}\\
G_+\wedge A_{i_0}\wedge\cdots\wedge G_+\wedge A_{i_k} & \xrightarrow{\sim} & G_+\wedge A_{i_0}\wedge\cdots\wedge G_+\wedge A_{i_k}.
\end{array}$$

We also have a map

$$\kappa : G_+ \wedge A_{i_0} \wedge A_{i_1} \wedge G_+ \wedge A_{i_2} \wedge \cdots \wedge G_+ \wedge A_{i_k} \longrightarrow A_{i_0} \wedge \cdots \wedge A_{i_k}$$

given by the composition

$$\begin{aligned} & G_+ \wedge A_{i_0} \wedge A_{i_1} \wedge G_+ \wedge A_{i_2} \wedge \cdots \wedge G_+ \wedge A_{i_k} \\ & \xrightarrow{\Delta \wedge id^{\wedge k+1}} G_+ \wedge G_+ \wedge A_{i_0} \wedge A_{i_1} \wedge G_+ \wedge A_{i_2} \wedge \cdots \wedge A_{i_k} \\ & \xrightarrow{\sim} G_+ \wedge A_{i_0} \wedge G_+ \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} \\ & \xrightarrow{\alpha^{\wedge k}} A_{i_0} \wedge \cdots \wedge A_{i_k}. \end{aligned}$$

Then the diagram

$$\begin{array}{ccc} G_+ \wedge A_{i_0} \wedge A_{i_1} \wedge G_+ \wedge A_{i_2} \wedge \cdots \wedge G_+ \wedge A_{i_k} & \xrightarrow{\kappa} & A_{i_0} \wedge \cdots \wedge A_{i_k} \\ \downarrow{\scriptstyle id \wedge \mu \wedge id \wedge \cdots \wedge id} & & \downarrow{\scriptstyle \mu \wedge id \wedge \cdots \wedge id} \\ G_+ \wedge A_{i_0+i_1} \wedge G_+ \wedge A_{i_2} \wedge \cdots \wedge G_+ \wedge A_{i_k} & \xrightarrow[\alpha \wedge \cdots \wedge \alpha]{} & A_{i_0+i_1} \wedge A_{i_2} \wedge \cdots \wedge A_{i_k} \end{array}$$

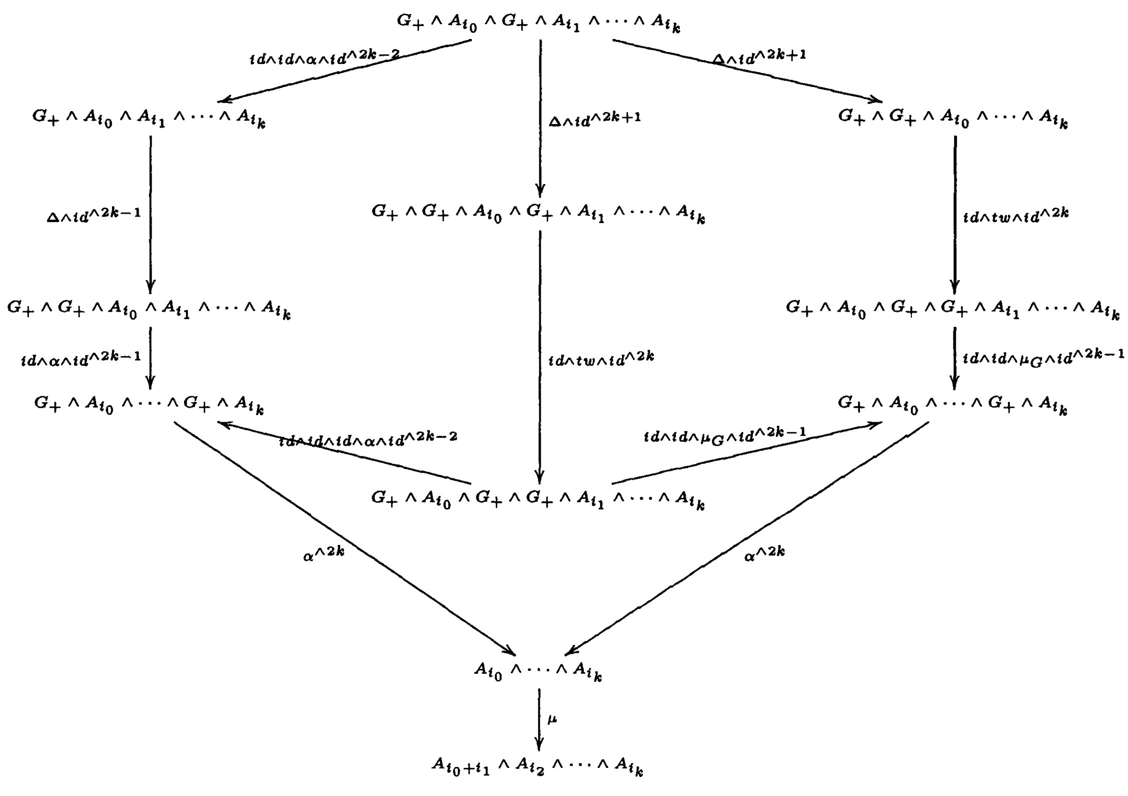
commutes by Def. 3.1.1 and the coherence of smash product, i.e. that the iterated smash product of commutative diagrams is again a commutative diagram. It only remains to show that the following diagram commutes:

$$\begin{array}{ccc} G_+ \wedge A_{i_0} \wedge \cdots \wedge G_+ \wedge A_{i_k} & \xrightarrow{\zeta} & G_+ \wedge A_{i_0} \wedge \cdots \wedge G_+ \wedge A_{i_k} \\ \downarrow{\scriptstyle id \wedge \alpha \wedge id \cdots \wedge id} & & \downarrow{\scriptstyle \alpha \wedge \cdots \wedge \alpha} \\ G_+ \wedge A_{i_0} \wedge A_{i_1} \wedge G_+ \wedge A_{i_2} \wedge \cdots \wedge G_+ \wedge A_{i_k} & \xrightarrow{\kappa} & A_{i_0} \wedge \cdots \wedge A_{i_k}. \end{array}$$

Explicitly, the isomorphism ζ can be factored as the composition

$$
\begin{aligned}
&G_+ \wedge A_{i_0} \wedge G_+ \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} \\
&\qquad\xrightarrow{\Delta \times id^{\wedge k}} G_+ \wedge G_+ \wedge A_{i_0} \wedge G_+ \wedge A_{i_1} \cdots \wedge \cdots \wedge A_{i_k} \\
&\qquad\xrightarrow{id \wedge tw \wedge id^{\wedge k}} G_+ \wedge A_{i_0} \wedge G_+ \wedge G_+ \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} \\
&\qquad\xrightarrow{id \wedge id \wedge \mu \wedge id^{\wedge k}} G_+ \wedge A_{i_0} \wedge G_+ \wedge A_{i_1} \wedge \cdots \wedge A_{i_k}.
\end{aligned}
$$

We then have the following diagram



that commutes by the following argument. The commutativity of the top right trapezoid is clear since the compositions are identical. The bottom portion of the diagram commutes by the first diagram of Def. 3.1.1 and the coherence of the smash product. It only remains to show the commutativity of the top left diagram. Namely, we have the following diagram

$$\begin{array}{ccc}
G_+ \wedge A_{i_0} \wedge G_+ \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} & \longrightarrow & G_+ \wedge A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} \\
\downarrow{\scriptstyle \Delta\wedge id^{\wedge 2k+1}} & & \downarrow{\scriptstyle \Delta\wedge id^{\wedge 2k+1}} \\
G_+ \wedge G_+ \wedge A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} & & G_+ \wedge G_+ \wedge A_{i_0} \wedge G_+ \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} \\
\downarrow{\scriptstyle id\wedge tw\wedge id^{\wedge 2k}} & & \downarrow{\scriptstyle id\wedge tw\wedge id^{\wedge 2k}} \\
G_+ \wedge A_{i_0} \wedge G_+ \wedge G_+ \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} & \longrightarrow & G_+ \wedge A_{i_0} \wedge G_+ \wedge G_+ \wedge A_{i_1} \wedge \cdots \wedge A_{i_k}
\end{array}$$

where the top horizontal map is $id \wedge id \wedge \alpha \wedge id^{\wedge 2k-2}$ and the bottom horizontal map is $id \wedge id \wedge id \wedge \alpha \wedge id^{\wedge 2k-2}$. The commutativity of this diagram follows from the smash product being a functor in both variables, that is from the commutativity of the more general diagram of spaces

$$\begin{array}{ccc}
X \wedge X' & \xrightarrow{f\wedge id} & Y \wedge X' \\
\downarrow{\scriptstyle id\wedge f'} & & \downarrow{\scriptstyle id\wedge f'} \\
X \wedge Y' & \xrightarrow[f\wedge id]{} & Y \wedge Y'
\end{array}$$

giving

$$(f \wedge id) \circ (id \wedge f') = f \wedge f' = (id \wedge f') \circ (f \wedge id).$$

Namely, the commutativity follows since the diagrams

$$\begin{array}{ccc}
G_+ \wedge A_{i_0} & \xrightarrow{id\wedge id} & G_+ \wedge A_{i_0} \\
{\scriptstyle \Delta\wedge id}\downarrow & & {\scriptstyle \Delta\wedge id}\downarrow \\
G_+ \wedge G_+ \wedge A_{i_0} & & G_+ \wedge G_+ \wedge A_{i_0} \\
{\scriptstyle id\wedge tw}\downarrow & & {\scriptstyle id\wedge tw}\downarrow \\
G_+ \wedge A_{i_0} \wedge G_+ & \xrightarrow{id^{\wedge 3}} & G_+ \wedge A_{i_0} \wedge G_+
\end{array}$$

and

$$\begin{array}{ccc}
G_+ \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} & \longrightarrow & A_{i_1} \wedge \cdots \wedge A_{i_k} \\
{\scriptstyle id^{\wedge 2k}}\downarrow & & {\scriptstyle id^{\wedge 2k-1}}\downarrow \\
G_+ \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} & & A_{i_0} \wedge \cdots \wedge A_{i_k} \\
{\scriptstyle id^{\wedge 2k}}\downarrow & & {\scriptstyle id^{\wedge 2k-1}}\downarrow \\
G_+ \wedge A_{i_1} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} & \longrightarrow & A_{i_1} \wedge G_+ \wedge \cdots \wedge A_{i_k}
\end{array}$$

both commute. In the last diagram the top and bottom horizontal map is $\alpha \wedge id^{\wedge 2k-2}$. Thus the map $\tilde{\phi}_g$ commutes with the operators $d_0 \times d_0^g$ and $d_{0,E}^{\tau}$. The commutavity with the other operators is proved similarly. $\square$

We state Lemma 3.2.1 globally as follows. Let $X[-]_\lambda$ be the cyclic space with k-simplices $X[k]_\lambda = G^{\mathrm{ad}} \times \mathrm{THH}^g(A)[k]_\lambda$ and cyclic operators those of $\mathrm{THH}^g(A)[-]_\lambda$. As a set, $X[k]_\lambda$ is the disjoint union of all fibers

$$X[k]_\lambda = \coprod_{\langle g \rangle} \mathrm{THH}^g(A)[k]_\lambda.$$

Now given the Borel construction $EG[-] \times_G X[-]_\lambda$ with the usual product simplicial structure, we give a cyclic structure by defining the cyclic operator

$$t(g_0, \ldots, g_k; g, [f]) = (g_k g, g_0, g_1, \ldots, g_{k-1}; g, \tau_k \circ (1, g, \cdots, 1)_*[f]),$$

where τ_k is the natural transformation from §3.1. For a k-simplex in $EG[-] \times_G X[-]_\lambda$, we define the map

$$\tilde{\phi} : EG[-] \times_G X[-]_\lambda \longrightarrow E(A, G)[-]_\lambda$$

via

$$\tilde{\phi}(\bar{g}; g, [f]) = ((g_k, g_0, g_1, \ldots, g_{k-1})_*[f]; \phi_g),$$

where $\bar{g} = (g_0, \ldots, g_k)$ and $\phi_g = (g_k g g_0^{-1}, g_0 g_1^{-1}, g_0 g_1^{-1}, \ldots, g_{k-1} g_k^{-1})$. We note that the restriction of this map to the connected component corresponding to $\langle g \rangle$ and a choosen representative $g \in G$ is the map $\tilde{\phi}_g$ of Lemma 3.2.1. This map is an isomorphism and we see at once we have the following proposition.

Proposition 3.2.2. *There exist canonical homeomorphisms of spaces ϕ and $\tilde{\phi}$ such that the following diagram commutes:*

$$\begin{array}{ccc} EG \times_G X_\lambda & \xrightarrow{\tilde{\phi}} & E(A, G)_\lambda \\ \downarrow & & \downarrow \\ EG \times_G G^{\mathrm{ad}} & \xrightarrow{\phi} & N^{\mathrm{cy}}(G). \end{array}$$

Proof. The cyclic isomorphism ϕ is covered by the cyclic isomorphism $\tilde{\phi}$; that is we have a diagram of cyclic spaces:

$$\begin{array}{ccc} EG[-] \times_G X[-]_\lambda & \xrightarrow{\tilde{\phi}} & E(A,G)[-]_\lambda \\ \downarrow & & \downarrow \\ EG[-] \times_G G^{\mathrm{ad}} & \xrightarrow{\phi} & N^{\mathrm{cy}}(G)[-] \end{array}$$

where the two horizontal maps are cyclic isomorphisms. After taking geometric realization, the result follows. □

Corollary 3.2.3. *There are canonical homeomorphisms of spaces:*

$$\begin{array}{ccc} \coprod_{\langle g \rangle} EG \times_G \mathrm{THH}^g(A)_\lambda & \xrightarrow{\simeq} & E(A,G)_\lambda \\ \downarrow & & \downarrow \\ EG \times_G \coprod_{\langle g \rangle} G/C_G(g) \cdot g & \xrightarrow{\simeq} & N^{\mathrm{cy}}(G) \end{array}$$

where the two horizontal maps are the homeomorphisms corresponding to the cyclic isomorphisms.

Remark 3.2.4. *We note that the space $E(A,G)_\lambda$ is actually a fiber bundle over $N^{\mathrm{cy}}(G)$. In particular, the fibers over two points in the same connected component are homeomorphic.*

We now prove Thm. 1.0.1.

Proof of Theorem 1.0.1. Applying the functor $f_!$ to the parametrized spaces over $N^{\mathrm{cy}}(G)$,

$$\coprod_{\langle g \rangle} EG \times_G E(A,G)^g_\lambda \longrightarrow EG \times_G \coprod_{\langle g \rangle} G/C_G(g) \xrightarrow{\simeq} N^{\mathrm{cy}}(G)$$

and

$$E(A,G)_\lambda \longrightarrow N^{\mathrm{cy}}(G)$$

gives a map of spaces

$$\bigvee_{\langle g \rangle} EG \wedge_{C_G(G)} E(A,G)^g_\lambda \xrightarrow{\sim} f_! E(A,G)_\lambda \longrightarrow \mathrm{THH}(A^\tau[G])_\lambda.$$

The first map is an isomorphism by Cor. 3.2.3. As λ varies, the second map is an equivalence by Thm. 1.0.2. Hence for varying λ, we obtain a stable equivalence of orthogonal spectra. □

Appendix A

The Hochcshild Complex and Cyclic Bar Construction

The proof of Thm. 1.0.1 is inspired by a study of the linear case concerning Hochschild homology. We briefly present the result for ordinary Hochschild homology. Given a commutative ring with unity, A, the *Hochschild complex of the ring* A is defined to be the cyclic set $HH(A)[-]$ with k-simplices,

$$HH(A)[k] = \underbrace{A \otimes \cdots \otimes A}_{k+1}$$

together with face and degeneracy operators defined on generators by

$$d_i(a_0 \otimes \cdots \otimes a_k) = \begin{cases} (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_k) & \text{if } 0 \leq i < k \\ (a_k a_0 \otimes a_1 \cdots \otimes a_{k-1}) & \text{if } i = k \end{cases}$$

$$s_i(a_0 \otimes \cdots \otimes a_k) = (a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_k) \text{ for } 0 \leq i \leq k.$$

The cyclic operator is given by

$$t_k(a_0 \otimes \cdots \otimes a_k) = (a_k \otimes a_0 \otimes \cdots \otimes a_{k-1}).$$

The geometric realization of the cyclic set $HH(A)[-]$, defined as the geometric realization of the underlying simplicial set, is denoted by $HH(A)$. The homotopy groups of this space are then the *Hochschild homology groups* of the ring A.

The twisted group ring $A^\tau[G]$, has Hochschild complex, $HH(A^\tau[G])[-]$ with k-simplices

$$HH(A^\tau[G])[k] = \underbrace{A^\tau[G] \otimes \cdots \otimes A^\tau[G]}_{k+1}$$

and face maps defined on generators by

$$d_i(a_0g_0 \otimes \cdots \otimes a_kg_k) = \begin{cases} (a_0g_0 \otimes \cdots \otimes a_ig_i(a_{i+1})g_ig_{i+1} \otimes \cdots \otimes a_kg_k) & \text{if } 0 \leq i < k \\ (a_kg_k(a_0)g_kg_0 \otimes a_1g_1 \cdots \otimes a_{k-1}g_{k-1}) & \text{if } i = k. \end{cases}$$

The degeneracies and cyclic operator are the same as the untwisted group algebra.

Let $E[-]$ be the cyclic set with k-simplices,

$$E[k] = HH(A)[k] \times N^{cy}(G)[k]$$

and with *twisted* face maps and degenerices given by

$$\begin{aligned}
&d_i(a_0 \otimes \cdots \otimes a_k; g_0, \ldots, g_k) \\
&= \begin{cases} (a_0 \otimes \cdots \otimes a_ig_i(a_{i+1}) \otimes \cdots \otimes a_k; g_0, \ldots, g_ig_{i+1}, \ldots g_k) \\ \text{if } 0 \leq i < k \\ (a_kg_k(a_0) \otimes a_1 \otimes \cdots \otimes a_{k-1}); g_kg_0, g_1, \ldots, g_{k-1}) \\ \text{if } i = k \end{cases} \\
&s_i(a_0 \otimes \cdots \otimes a_k; g_0, \ldots, g_k) \\
&= (a_0 \otimes \cdots a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_k; g_0, \ldots g_i, 1, g_{i+1}, \ldots, g_k) \\
&t_k(a_0 \otimes \cdots \otimes a_k; g_0, \ldots, g_k) \\
&= (a_k \otimes a_0 \otimes \cdots \otimes a_{k-1}; g_k, g_1, \ldots, g_{k-1}).
\end{aligned}$$

Taking geometric realization, we obtain a space E parameterized over $N^{cy}(G)$, with projection map the realization of projecting onto the cyclic bar construction. We apply the functor f^* to the space $HH(A^\tau[G])$ obtaining a space over $N^{cy}(G)$, $f^*HH(A^\tau[G])$. We then define a map of cyclic sets

$$\varphi : E[-] \longrightarrow HH(A^\tau[G])[-] \times N^{cy}(G)[-] = f^*(HH(A^\tau[G]))$$

that is given on level k by

$$\varphi_k(a_0 \otimes \cdots \otimes a_k; g_0, \ldots, g_k) = (a_0 g_0 \otimes \cdots \otimes a_k g_k; g_0, \ldots, g_k).$$

The face operators for $E[-]$ are defined so that φ is indeed a map of cyclic sets. By the adjunction $f_! \dashv f^*$, we obtain the map

$$f_! E \longrightarrow HH(A^\tau[G]).$$

This map is the linear analog of the stable weak equivalence in Thm. 1.0.2.

As in §3.2 the the geometric fiber E_g is the realization of the following pullback of simplicial sets,

$$\begin{array}{ccc} E_g[-] & \longrightarrow & E[-] \\ \downarrow & & \downarrow \\ \Delta^0[-] & \xrightarrow{g} & N^{cy}(G)[-] \end{array}$$

where $\Delta^0[-]$ is the standard 0-simplex

$$\Delta^0 = \mathrm{Hom}_\Delta(-, [0])$$

and the bottom map takes the identity, $[0] \xrightarrow{1} [0]$ to the 0-simplex $g \in N^{cy}(G)[0]$. Explicitly, this takes the map $\theta : [k] \longrightarrow [0]$ to $\theta^* g \in N^{cy}(G)[k]$ and since for each k, there is only one such map θ, $\theta^* g = (g, 1, \ldots, 1)$ ($k+1$ factors). Thus the simplicial set $E_g[-]$ is given by

$$E_g[k] = HH(A)[k] \times \{(1, \ldots, 1, g)\} \subset E[k] = HH(A)[k] \times N^{cy}(G)[k].$$

The connected component of the 0-simplex g is then obtained as the realization of the simplical set $E[-]\big|_{\langle g\rangle}$ given on level k by

$$E[k]\big|_{\langle g\rangle} = HH(A)[k] \times \Big\{(g_0, g_1, \ldots, g_k) : hgh^{-1} = \prod_{i=0}^{n} g_i, \text{ for some } h \in G\Big\} \subset E[k].$$

Indeed, a path from g to the zeroth vertex of the k-simplex $(g_0, \ldots, g_k)$ is given by the 1-simplex $(gh^{-1}, h) \in N^{cy}(G)[1]$.

Proposition A.0.5. *The simplicial set corresponding to the connected component indexed by* $\langle g\rangle$,$E[-]\big|_{\langle g\rangle}$, *is isomorphic to the Borel construction,*

$$\tilde{\phi}_g : EG[-] \times_{C_G(g)} E_g[-] \longrightarrow E[-]\big|_{\langle g\rangle},$$

where

$$\tilde{\phi}_g([(g_0, \ldots, g_k); a_0 \otimes \cdots \otimes a_k]) = (g_k(a_0) \otimes g_0(a_1) \otimes \cdots \otimes g_{k-1}(a_k); \phi(g_0, \ldots, g_k)),$$

Hence, the realization $E\big|_{\langle g\rangle}$ *is homeomorphic to the principal bundle* $EG \times_{C_G(g)} E_g$. *The isomorphism depends on the choice of* $g \in \langle g\rangle$.

Proof. We define the isomorphism

$$\tilde{\phi}_g : EG[-] \times_{C_G(g)} E_g[-] \longrightarrow E[-]\big|_{\langle g\rangle}$$

on level k, by the formula,

$$\tilde{\phi}_g([(g_0, \ldots, g_k); a_0 \otimes \cdots \otimes a_k]) = (g_k(a_0) \otimes g_0(a_1) \otimes \cdots \otimes g_{k-1}(a_k); \phi(g_0, \ldots, g_k)),$$

where $\phi(g_0, \ldots, g_k) = (g_k g g_0^{-1}, g_0 g_1^{-1}, \ldots, g_{k-1} g_k^{-1})$ as above. Since $C_G(g)$ acts on $EG[-]$ diagonally from the right and on the fiber diagonally from the left, we have

$$[(g_0 h, \ldots, g_k h); a_0 \otimes \cdots \otimes a_k] = [(g_0, \ldots, g_k); h(a_0) \otimes \cdots \otimes h(a_k)].$$

We will first show that the map $\tilde{\phi}_g$ is well-defined. Given

$$\tilde{\phi}_g([g_0h, \ldots, g_kh; a_0 \otimes \cdots \otimes a_k]) = (g_kh(a_0) \otimes g_0hg^{-1}(a_1) \otimes \cdots \otimes g_{k-1}hg^{-1}(a_k); \phi(\bar{g}))$$

we have

$$\begin{aligned}\phi(\bar{g}) &= (g_khg(g_0h)^{-1}, g_0h(g_1h)^{-1}, \ldots, g_{k-1}h(g_kh)^{-1}) \\ &= (g_kg^hg_0^{-1}, g_0g_1^{-1}, \ldots, g_{k-1}g_k^{-1}) \\ &= (g_kgg_0^{-1}, g_0g_1^{-1}, \ldots, g_{k-1}g_k^{-1})\end{aligned}$$

since $h \in C_G(g)$. On the other hand, $\tilde{\phi}_g([(g_0, \ldots, g_k); h(a_0) \otimes \cdots \otimes h(a_k)])$

$$\begin{aligned}&= (g_kh(a_0) \otimes g_0h(a_1) \otimes \cdots \otimes g_{k-1}gh(a_k); \phi(g')) \\ &= (g_kh(a_0) \otimes g_0h(a_1) \otimes \cdots \otimes g_{k-1}h(a_k); \phi(g'))\end{aligned}$$

again using that h is in the centralizer of g. We also have $\phi(g') = \phi(\bar{g})$. Thus the map $\tilde{\phi}_g$ is well-defined. It only remains to demonstrate the commutativity with the cyclic operators. We check the commutavity of d_0. The other operators are standard and can be checked similarly. Again, on level k, $[(g_0, \ldots, g_k); a_0 \otimes \cdots \otimes a_k]$

$$\begin{aligned}&\overset{\tilde{\phi}_g}{\mapsto} (g_k(a_0) \otimes g_0(a_1) \otimes \cdots \otimes g_{k-1}(a_k); \phi(g_0, \ldots, g_k)) \\ &\overset{d_0}{\mapsto} (g_k(a_0)g_kgg_0^{-1}g_0(a_1) \otimes \cdots \otimes g_{k-1}(a_k); d_0\phi(\bar{g})) \\ &= ((g_k(a_0)g(a_1)) \otimes g_1(a_2) \otimes \cdots \otimes g_{k-1}(a_k); d_0\phi(\bar{g}))\end{aligned}$$

where $d_0\phi(\bar{g}) = (g_kgg_1^{-1}, g_1g_2^{-1} \ldots, g_{k-1}g_k^{-1})$. On the other hand, $[(g_0, \ldots, g_k); a_0 \otimes \cdots \otimes a_k]$

$$\begin{aligned}&\overset{d_0}{\mapsto} [(g_1, g_0, \ldots, g_k), a_0g(a_1) \otimes \cdots \otimes a_{k-1}] \\ &\overset{\tilde{\phi}_g}{\mapsto} (g_k((a_k)g(a_0)) \otimes g_1(a_2) \otimes \cdots \otimes g_{k-1}(a_k); \phi(d_0\bar{g}))\end{aligned}$$

where $\phi(d_0\bar{g}) = \phi(g_1, \ldots, g_k) = (g_k g g_1^{-1}, g_1 g_2^{-1} \ldots, g_{k-1} g_k^{-1})$. Thus the map $\tilde{\phi}_g$ commutes with the operator d_k. The commutavity with the other operators is proved similarly.

□

We state the global linear analog to Lemma 3.2.1 as follows. Let $X[-]$ be the cyclic set with k-simplices $X[k] = G^{\mathrm{ad}} \times A^{\otimes k+1}$ and cyclic operators those of $E[-]$. As a set, $X[k]$ is the disjoint union of all fibers

$$X[k] = \coprod_{\langle g \rangle} E_g[k].$$

Now given the Borel construction $EG[-] \times_G X[-]$ with the usual product simplicial structure, we give a cyclic structure by defining the cyclic operator

$$t(g_0, \ldots, g_k; g, a_0 \otimes \cdots \otimes a_k) = (g_k g, g_0, g_1, \ldots, g_{k-1}; g, a_k \otimes g(a_0) \otimes a_1 \otimes \cdots \otimes a_{k-1}).$$

For a k-simplex in $EG[-] \times_G X[-]$, we define the map $\tilde{\phi} : EG[k] \times_G X[-] \longrightarrow E[-]$ via

$$\tilde{\phi}(\bar{g}; g, a_0 \otimes \cdots \otimes a_k) = (g_k(a_0) \otimes g_0(a_1) \otimes \cdots \otimes g_{k-1}(a_k); \phi_g),$$

where $\bar{g} = (g_0, \ldots, g_k)$ and $\phi_g = (g_k g g_0^{-1}, g_0 g_1^{-1}, g_0 g_1^{-1}, \ldots, g_{k-1} g_k^{-1})$. This map is an isomorphism and hence after realization, we have a homeomorphism between the corresponding spaces.

Proposition A.0.6. *The following diagram of spaces commutes*

$$\begin{array}{ccc} EG \times_G X & \xrightarrow{\tilde{\phi}} & E \\ \downarrow & & \downarrow \\ EG \times_G G^{\mathrm{ad}} & \xrightarrow{\phi} & N^{cy}(G). \end{array}$$

Here the two horizontal maps are the homeomorphisms corresponding to the cyclic isomorphisms.

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