Models of High Rank for Weakly Scattered Theories

by

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Abstract

The Scott rank of a countable structure $\mathcal{A}$, denoted $sr(\mathcal{A})$, was observed by Nadel to be at most $\omega_1^{\mathcal{A}} + 1$, where $\omega_1^{\mathcal{A}}$ is the least ordinal not recursive in $\mathcal{A}$. Let T be weakly scattered and $L(\alpha, T)$ be Σ_2-admissible. We give a sufficient condition, the B_α-hypothesis, under which T has model $\mathcal{A}$ with $\omega_1^{\mathcal{A}} = \alpha$ and $sr(\mathcal{A}) = \alpha + 1$. Given the B_α-hypothesis, an iterated forcing argument is used to obtain a generic $T^\alpha \supseteq T$ such that T^α has a model with the desired properties.

Thesis Supervisor: Gerald E. Sacks
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Gratitude is one of the least articulate of the emotions, especially when it is deep.

-Felix Frankfurter

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"Yeah, but if I fail math, there goes my chance at a good job and a happy life full of hard work, like you always say." -A.D.

Contents

Chapter 1

Introduction

Robert Vaught, in a 1961 paper, conjectured that a countable complete theory has, up to isomorphism, either a countable number of countable models or size continuum many. Though Knight [5] has proposed a counterexample, Vaught's Conjecture remains open as of this writing. Among the large body of work that grew out of the conjecture is the study of Scott rank in relation to weakly scattered theories. In this dissertation, we give a sufficient condition for a weakly scattered theory to have a model whose Scott rank is the highest possible.

The concept of a scattered theory was introduced by Morley [8]. In his paper, the first major breakthrough on Vaught's Conjecture, Morley showed that if T has fewer than 2^ω many countable models, then T is scattered. It follows that if T is a counterexample to Vaught's Conjecture, then T is scattered. The focus of this work is on weakly scattered theories–theories that satisfy a generalized notion of scatteredness.

Morley's proof makes use of the Scott analysis of a countable structure. Associated with the Scott analysis is the Scott rank of a structure. An ordinal invariant, Scott rank measures the model theoretic complexity of a model $\mathcal{A}$. Nadel [9] observed that the Scott rank of $\mathcal{A}$ can be as high as $\omega_1^{\mathcal{A}} + 1$, where $\omega_1^{\mathcal{A}}$ is the least ordinal not recursive in $\mathcal{A}$.

A previous result, from Sacks [11], on weakly scattered theories and models with high Scott rank is that a weakly scattered T has a model $\mathcal{A}$ of Scott rank $\omega_1^{\mathcal{A}} + 1$, if

T satisfies the *effective k-splitting hypothesis.* This was later improved by Goddard [3] who removed the assumption of the predecessor property from the k-splitting hypothesis.

Using a finite support iteration of forcing notions, we show that a weakly scattered T has a model $\mathcal{A}$ of Scott rank $\omega_1^{\mathcal{A}} + 1$ if what we call the *B_α-hypothesis* holds for T. Working in a Σ_2-admissible $L(\alpha, T)$[1] , generic theories extending T are obtained via forcing and the B_α-hypothesis, which says that it is consistent that the generic theories have models with arbitrarily high Scott rank, allows the iteration to work.

Preliminaries are covered in Chapter 1. We briefly discuss infinitary logic and admissible sets which are necessary for the definitions of Scott rank and scattered theories. In Chapter 2, we describe the forcing notions used and state the B_α-hypothesis. The main result is proved in Chapter 3. The terminology and definitions used in this work matches those in Barwise [1], Keisler [4] and Sacks [11] unless otherwise noted.

1.1 Admissible Sets

Admissible sets were introduced by Kripke [6] and Platek [10] as a general setting for recursion theory.

Definition 1.1. KP, the Kripke-Platek axiom system, is the theory over the language $\{\in, \ldots\}$ axiomatized by the universal closures of the following:

Extensionality: $\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b$;

Foundation: $\exists x \varphi(x) \rightarrow \exists x[\varphi(x) \wedge \forall y \in x \neg\varphi(y)]$ for all formulas $\varphi(x)$ in which y does not occur free;

Pairing: $\exists a(x \in a \wedge y \in a)$;

Union: $\exists b \, \forall y \in a \forall x \in y(x \in b)$;

Δ_0-separation: $\exists b \, \forall x(x \in b \leftrightarrow x \in a \wedge \varphi(x))$ for all Δ_0 formulas in which b does not occur free;

[1] $L(\alpha, T)$ is Gödel's L relativized to T and cut off at α.

Δ_0*-bounding:* $\forall x \in a\, \exists y\, \varphi(x,y) \rightarrow \exists b\, \forall x \in a \exists y \in b\, \varphi(x,y)$ for all Δ_0 formulas in which b does not occur free.

An *admissible (or Σ_1-admissible)* set is a transitive set A that is a model of KP.

It can be derived from the above definition that admissible sets also satisfy Δ_1-separation, Σ_1-bounding and Σ_1-recursion.

Theorem 1.2. *(Σ_1-recursion) Let A be admissible and $G(,)$ be a function such that for all $a, b \in A$, $G(a,b) \in A$ and $G|A$ is Σ_1 on A. Let $tc(x)$ denote the transitive closure of x. Suppose F is a function with domain A which is defined recursively by*

$$F(x) = G(x, F|tc(x)) \text{ all } x \in A.$$

Then F is Δ_1 on A and F maps A into A.

The smallest admissible set is $R(\omega)$, the hereditarily finite sets. An ordinal α is admissible if $L(\alpha)$, the set of all sets constructible before α, is admissible. We say A is Σ_n*-admissible* if A satisfies Σ_n-replacement.

1.2 Infinitary Logic

Infinitary logic, $\mathcal{L}_{\alpha,\beta}$ for infinite cardinals α, β, is an extension of first order logic that allows conjunctions and disjunctions of a set with fewer than α formulas and universal and existential quantification on a set with fewer than β variables. We focus on $\mathcal{L}_{\omega_1,\omega}$ which allows countable disjunctions and conjunctions but only finite quantifiers. If $\mathcal{L}$ is a first order language, then $\mathcal{L}_{\omega_1,\omega}$ has the same symbols as $\mathcal{L}$ but in $\mathcal{L}_{\omega_1,\omega}$, the conjunction and disjunction symbols may be applied to countable sets of formulas. (The symbol $\mathcal{L}_{\omega_1,\omega}$ will denote both the logic and the language.)

Two basic results for first order logic, *Compactness* and *Upward Lowenheim-Skolem* fail for $\mathcal{L}_{\omega_1,\omega}$. For example, let $c_0, c_1, \ldots, c_\omega$ be constants and consider Σ, the set of sentences

$$\forall x \bigvee_{n<\omega} (x = c_n),\; c_\omega \neq c_0, c_\omega \neq c_1, \ldots.$$

Every finite subset of Σ has a model but Σ has no model, and $\forall x \bigvee_{n<\omega}(x = c_n)$ has no uncountable model. Because of this and also because $\mathcal{L}_{\omega_1,\omega}$ has uncountable many formulas, well behaved countable subsets of $\mathcal{L}_{\omega_1,\omega}$ are considered instead. *Fragments* are an approximation to those well-behaved subsets.

Definition 1.3. Let $\mathcal{L}$ be a language. A *fragment* of $\mathcal{L}_{\omega_1,\omega}$ is a set $\mathcal{L}'$ of infinitary formulas and variables such that

1. every finite formula of $\mathcal{L}_{\omega,\omega}$ is in $\mathcal{L}'$,
2. if $\varphi \in \mathcal{L}'$, then every subformula and variable of φ is in $\mathcal{L}'$,
3. if $\varphi(v) \in \mathcal{L}'$, and t is a term of $\mathcal{L}$ all of whose variables lie in $\mathcal{L}'$ then $\varphi(t/v)$ is in $\mathcal{L}'$ ($\varphi(t/v)$ is the expression obtained by replacing the variable v with the term t wherever v occurs free), and
4. if φ, ψ, v are in $\mathcal{L}'$ so are

$$\neg\varphi,\ \exists v\varphi,\ \forall v\varphi,\ \varphi \vee \psi,\ \varphi \wedge \psi.$$

If $\mathcal{A}$ is admissible, then $\mathcal{L}_{\mathcal{A}} = \mathcal{L}_{\omega_1,\omega} \cap \mathcal{A}$ is called an *admissible fragment* of $\mathcal{L}_{\omega_1,\omega}$.

Theorem 1.4. *(Barwise Compactness Theorem) Let $\mathcal{L}_{\mathcal{A}}$ be a countable admissible fragment of $\mathcal{L}_{\omega_1,\omega}$. Let T be a set of sentences of $\mathcal{L}_{\mathcal{A}}$ which is Σ_1 on $\mathcal{A}$. If every $T_0 \subseteq T$ which is an element of $\mathcal{A}$ has a model, then T has a model.*

The Barwise Compactness Theorem and the Barwise Completeness Theorem (not stated here) show that admissible fragments have properties similar to ones of ordinary first-order logic.

Theorem 1.5. *(Omitting Types Theorem) Let $\mathcal{L}'$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$, and let T be a set of sentences of $\mathcal{L}'$ which has a model. For each n, let Φ_n be a set of formulas of $\mathcal{L}'$ with free variables among $v_1, \ldots, v_{k_n}$. Assume that for each n and each formula $\psi(v_1, \ldots, v_{k_n})$ of $\mathcal{L}'$: if*

$T \cup \{\exists v_1, \ldots, v_{k_n}\psi\}$ has a model, so does

$T \cup \{\exists v_1, \ldots, v_{k_n} \psi \wedge \varphi\}$ for some $\varphi(v_1, \ldots, v_{k_n}) \in \Phi_n$.

Given this hypothesis, there is a model $\mathcal{M}$ of T such that for each $n < \omega$

$$\mathcal{M} \models \forall v_1, \ldots, v_{k_n} \bigvee_{\varphi \in \Phi_n} \varphi(v_1, \ldots, v_{k_n}).$$

With the Omitting Types Theorem, one can construct models that "omit" elements not satisfying certain infinite disjunctions. As in the following theorem, we use the Omitting Types Theorem to obtain models that omit an admissible ordinal. The proof is a variation of one found in Keisler [4].

Theorem 1.6. *("Effective" Type Omitting) Let $\alpha > \omega$ be a countable admissible ordinal and let $\mathcal{A} = L(\alpha)$. Let Z be the following set of $\mathcal{L}_{\mathcal{A}}$ sentences:*

1. *The atomic diagram of $L(\alpha)$, where c_β is a constant symbol for each ordinal $\beta < \alpha$.*
2. *The axioms of Σ_1-admissibility.*

Then Z has a model that is a proper end extension of $L(\alpha)$ but omits α.

Proof. Let $S \subseteq \alpha$ be Σ but not Δ on $\mathcal{A}$ and let $\Theta(x)$ be the following set of formulas:

$$\forall y(y \in x \rightarrow y \text{ is an ordinal}),$$

$$c_\beta \in x \text{ for each } \beta \in S,$$

$$\neg c_\beta \in x \text{ for each } \beta \in \alpha - S.$$

We will use the Omitting Types Theorem to get a model of Z that omits S, ie., a model of $Z \cup \{\neg \exists x \bigwedge \Theta\}$.

Suppose $\psi(x) \in \mathcal{L}_{\mathcal{A}}$ and $Z \cup \{\exists x\, \psi(x)\}$ has a model. Let Γ be the set of formulas $\varphi(x)$ such that $Z \models \psi(x) \rightarrow \varphi(x)$. Then Γ is Σ on $\mathcal{A}$.

Assume that $\Theta \subseteq \Gamma$. Then as Γ is consistent the sets

$$S = \{\beta < \alpha \mid (c_\beta \in x) \in \Gamma\},$$

$$\alpha - S = \{\beta < \alpha \mid (\neg c_\beta \in x) \in \Gamma\}$$

are both Σ on $\mathcal{A}$, which is a contradiction as S is not Δ on $\mathcal{A}$.

Let $\theta \in \Theta - \Gamma$. Then $Z \cup \{\exists x(\psi(x) \wedge \neg\theta(x))\}$ has a model and by the Omitting Types Theorem, Z has a model $\mathcal{M}$ in which $\forall x \bigvee_{\theta\in\Theta} \neg\theta$, equivalent to $\neg\exists x \bigwedge \Theta$, holds and hence omits S.

Now we show that $\mathcal{M}$ omits α. Suppose not and that $\alpha \in \mathcal{M}$. Let $\sigma(x)$ be a Σ-definition of S in $\mathcal{A}$

$$S = \{\beta < \alpha \mid \mathcal{M} \models \sigma^{\mathcal{A}}(c_\beta)\}.$$

By Δ_0-separation in $\mathcal{M}$, there is an $s \in \mathcal{M}$ such that

$$\mathcal{M} \models \forall y(y \in s \leftrightarrow \sigma^{\mathcal{A}}(y)).$$

Now we have

$$\mathcal{M} \models c_\beta \in s \text{ for } \beta \in S,$$

$$\mathcal{M} \models \neg c_\beta \in s \text{ for } \beta \in \alpha - S$$

which contradicts S having been omitted in $\mathcal{M}$.

□

Let $\mathcal{L}'$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ and $T \subseteq \mathcal{L}'$ a set of sentences.

Definition 1.7. T is *ω-complete* in $\mathcal{L}'$ if

1. for every sentence $\varphi \in \mathcal{L}'$, either $\varphi \in T$ or $(\neg\varphi) \in T$, and
2. for any sentence of the form $\bigvee_{i<\omega} \varphi_i$ in T, there is an i such that $\varphi_i \in T$.

Definition 1.8. T is *finitarily consistent* if no contradiction can be derived from T using only the finitary rules of $\mathcal{L}_{\omega_1,\omega}$. We avoid the infinitary step that derives an infinite conjunction by deriving each of its components. A theory T is *ω-consistent* if for any sentence $\bigvee_{i<\omega} \varphi_i \in \mathcal{L}'$, if $T \cup \{\bigvee_{i<\omega} \varphi_i\}$ is finitarily consistent, then there is an i such that $T \cup \{\varphi_i\}$ is finitarily consistent.

Proposition 1.9. *If T is finitarily consistent and ω-complete, then T has a model.*

Proof. Note that T is ω-consistent. The construction is similar to a Henkin construction. We first extend $\mathcal{L}'$ to $\mathcal{L}_0$ by adding a sequence $\{c_i \mid i < \omega\}$ of constants not occurring in T. Let $\{\varphi_i(x) \mid i < \omega\}$ be an enumeration of formulas (of the extended language $\mathcal{L}_0$) with at most one free variable x. We construct an increasing sequence $\{T_i \mid i < \omega\}$ of ω-consistent sets of sentences that include Henkin axioms.

Let $T_0 = T$. Suppose that T_i has been constructed such that T_i is ω-consistent.

Case 1: x is a free variable of $\varphi_i(x)$. Choose c_k not appearing in $\varphi_i(x)$ nor in T_i. Let $T_{i+1} = T_i \cup \{\exists x \varphi_i(x) \rightarrow \varphi(c_k)\}$.

Case 2: φ_i is a sentence.

Case 2a: φ_i is not of the form $\bigvee_j \psi_j$. If $T_i \cup \{\varphi_i\}$ is finitarily consistent, $T_{i+1} = T_i \cup \{\varphi_i\}$; otherwise, $T_{i+1} = T_{\cup}\{\neg\varphi_i\}$.

Case 2b: φ_i is of the form $\bigvee_j \psi_j$. If $T_i \cup \{\varphi_i\}$ is not finitarily consistent, let $T_{i+1} = T_i \cup \{\neg\varphi_i\}$. Otherwise, $T_i \cup \{\bigvee_j \psi_j\}$ is finitarily consistent. Since T_i is ω-consistent, there exists j such that $T_i \cup \{\psi_j\}$ is consistent. Let $T_{i+1} = T_i \cup \{\varphi_i, \psi_j\}$.

Let $T_\omega = \cup\{T_i \mid i < \omega\}$.

A model $\mathcal{A}$ of T can be constructed from T_ω. The members of $\mathcal{A}$ are equivalence classes of constant terms occurring in T_ω. □

Proposition 1.10. *Suppose for all $\beta \leq \gamma < \lambda$, T_β is finitarily consistent and ω-complete in the fragment $\mathcal{L}_\beta$, $T_\beta \subseteq T_\gamma$, and $\mathcal{L}_\beta \subset \mathcal{L}_\gamma$. Then $\cup\{T_\beta \mid \beta < \lambda\}$ is finitarily consistent and ω-complete in the fragment $\cup\{\mathcal{L}_\beta \mid \beta < \lambda\}$.*

1.3 Scott Analysis

Scott [12] showed that for any countable model $\mathcal{A}$ for a language $\mathcal{L}$, there is a sentence φ of $\mathcal{L}_{\omega_1,\omega}$ (the *Scott sentence*) that characterizes $\mathcal{A}$ up to isomorphism, that is

$$\mathcal{A} \models \varphi \text{ and } \mathcal{B} \models \varphi \Rightarrow \mathcal{A} \cong \mathcal{B}.$$

The canonical Scott sentence φ is constructed by an inductive procedure that terminates at a countable ordinal $sr(\mathcal{A})$.

Definition 1.11. The *Scott rank* of a model $\mathcal{A}$, denoted $sr(\mathcal{A})$, is defined via a Σ_1-recursion.

- $\mathcal{L}_0^{\mathcal{A}} = \mathcal{L}$.
- $T_\delta^{\mathcal{A}}$ = complete theory of $\mathcal{A}$ in $\mathcal{L}_\delta^{\mathcal{A}}$.
- $\mathcal{L}_{\delta+1}^{\mathcal{A}}$ = least fragment $\mathcal{L}'$ of $\mathcal{L}_{\omega_1,\omega}$ such that $\mathcal{L}' \supseteq \mathcal{L}_\delta^{\mathcal{A}}$, and for each $n > 0$, if $p(\bar{x})$ is a non-principal n-type of $T_\delta^{\mathcal{A}}$ realized in $\mathcal{A}$, then the conjunction $\wedge\{\varphi(\bar{x}) \mid \varphi(\bar{x}) \in p(\bar{x})\}$ is a member of $\mathcal{L}'$.
- $\mathcal{L}_\lambda^{\mathcal{A}} = \cup\{\mathcal{L}_\delta^{\mathcal{A}} \mid \delta < \lambda\}$ for λ limit.

The Scott rank of $\mathcal{A}$ is the least ordinal α such that $\mathcal{A}$ is the atomic model of $T_\alpha^{\mathcal{A}}$, and the Scott sentence is the one that asserts $\mathcal{A}$ is the atomic model of $T_\alpha^{\mathcal{A}}$.

Note that $sr(\mathcal{A})$ is also the least α such that $\mathcal{L}_\alpha^{\mathcal{A}} = \mathcal{L}_{\alpha+1}^{\mathcal{A}}$.

Proposition 1.12. *If $\mathcal{A}$ is countable, then $sr(\mathcal{A})$ exists and is a countable ordinal.*

Proof. Suppose $\mathcal{L}_{\delta+1}^{\mathcal{A}} \neq \mathcal{L}_{\delta+2}^{\mathcal{A}}$. We show that there are two n-tuples of $\mathcal{A}$ that are equivalent with respect to all $\mathcal{L}_\delta^{\mathcal{A}}$ formulas but inequivalent with respect to a $\mathcal{L}_{\delta+1}^{\mathcal{A}}$ formula. If $\mathcal{L}_{\delta+1}^{\mathcal{A}} \neq \mathcal{L}_{\delta+2}^{\mathcal{A}}$, there must exist a non-principal type $p(\bar{x})$ of $T_{\delta+1}^{\mathcal{A}}$ realized in $\mathcal{A}$. Since $p(\bar{x})$ is non-principal, there is a formula $\psi(\bar{x})$ of $\mathcal{L}_{\delta+1}^{\mathcal{A}}$ such that

$$\exists x[p(\bar{x}) \wedge \psi(\bar{x})] \text{ and } \exists x[p(\bar{x}) \wedge \neg\psi(\bar{x})]$$

are both in $T^{\mathcal{A}}_{\delta+1}$. Then there are tuples, $\bar{b}, \bar{c}$ such that

$$\mathcal{A} \models p(\bar{b}) \wedge \psi(\bar{b}) \text{ and } \mathcal{A} \models p(\bar{c}) \wedge \neg\psi(\bar{c}).$$

Hence $\bar{b}, \bar{c}$ are distinguished by an $\mathcal{L}^{\mathcal{A}}_{\delta+1}$ formula.

Since $\mathcal{A}$ is countable, there can be at most countable distinctions made and $sr(\mathcal{A})$ exists and is countable. □

Theorem 1.13. *(Nadel [9]) $\mathcal{A}$ is a homogeneous model of $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$, the complete theory of $\mathcal{A}$ in $\mathcal{L}_{\omega_1,\omega} \cap L(\omega_1^{\mathcal{A}}, \mathcal{A})$.*

Proof. Suppose $\bar{a}$ and $\bar{b}$ realize the same type n-type $p(\bar{x})$ over $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$. Let $q(\bar{x}, y) \supseteq p(\bar{x})$ be a $n+1$-type of $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$ such that

$$\mathcal{A} \models p(\bar{a}) \wedge p(\bar{b}) \wedge \exists y \; q(\bar{a}, y).$$

To establish the homogeneity of $\mathcal{A}$, we must show there exists a $c \in \mathcal{A}$ such that $\mathcal{A} \models q(\bar{b}, c)$. Suppose no such c exists. Let $q_\delta(\bar{x}, y)$ be the restriction of $q(\bar{x}, y)$ to $\mathcal{L}^{\mathcal{A}}_{\delta}$. Then the set $\{q_\delta(\bar{x}, y) \mid \delta < \omega_1^{\mathcal{A}}\}$ is $\Sigma_1^{L(\omega_1^{\mathcal{A}}, \mathcal{A})}$. For each $c \in \mathcal{A}$, there is a $\delta < \omega_1^{\mathcal{A}}$ such that $\neg q_\delta(\bar{b}, c)$ and δ can be defined as a $\Sigma_1^{L(\omega_1^{\mathcal{A}}, \mathcal{A})}$ function of c. By the Σ_1-admissibility of $L(\omega_1^{\mathcal{A}}, \mathcal{A})$, there is a $\delta_\infty < \omega_1^{\mathcal{A}}$ such that $\mathcal{A} \models \forall y \neg q_{\delta_\infty}(\bar{b}, y)$. As $\bar{a}, \bar{b}$ realize the same n-type, this implies that $\mathcal{A} \models \forall y \neg q_{\delta_\infty}(\bar{a}, y)$ and we get $\mathcal{A} \models \forall y \neg q(\bar{a}, y)$, a contradiction.

□

Let $d_{\mathcal{A}}$ be the least ordinal $\delta < \omega_1$ such that every distinction ever made between n-tuples (for all $n \geq 1$) is made by a $\mathcal{L}^{\mathcal{A}}_{\delta}$ formula. Then $\mathcal{A}$ is the atomic model of $T^{\mathcal{A}}_{d_{\mathcal{A}}+1}$. By Theorem 1.13, $d_{\mathcal{A}} \leq \omega_1^{\mathcal{A}}$ and we have the following corollary.

Corollary 1.14. $sr(\mathcal{A}) \leq \omega_1^{\mathcal{A}} + 1$.

1.4 Scattered Theories

If for some countable fragment $S_n(T)$, the set of complete n-types of T, has cardinality 2^ω, then T has 2^ω many countable models because each countable model can realize at most a countable number of types. Scattered theories, on the other hand, have as few types as possible over all countable fragments.

Definition 1.15. Let $\mathcal{L}$ be a countable first order language and $\mathcal{L}_0$ a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ extending $\mathcal{L}$. Let $T \subseteq \mathcal{L}_0$ be a complete theory and T' a complete theory in $\mathcal{L}'$ extending T. Then T is *scattered* if

1. for all $n > 0$ and all $T' \supseteq T$, $S_n(T')$ is countable, and

2. for all $\mathcal{L}'$, the set $\{T' \mid T' \subseteq \mathcal{L}'\}$ is countable.

A theory T is *weakly scattered* if only (1) holds.

While a scattered theory can have at most ω_1 many countable models, a weakly scattered theory can have up to 2^ω many such models.

Chapter 2

Notions of Forcing

Given a weakly scattered theory T satisfying the B_α-hypothesis and $L(\alpha, T)$ Σ_2-admissible, we use an α-stage iteration with finite support to obtain T^α, an extension of T, with a model $\mathcal{A}$ such that $\omega_1^{\mathcal{A}} = \alpha$ and $sr(\mathcal{A}) = \omega_1^{\mathcal{A}} + 1$. In this chapter, we describe the forcing notions used and state the B_α-hypothesis. We also show that the iteration preserves Σ_2-admissibility.

2.1 Raw Hierarchy

Before the notions of forcing can be described, the *raw hierarchy* of a weakly scattered theory needs to be introduced. Let T be a complete theory over $\mathcal{L}_0$, a countable fragment of $\mathcal{L}_{\omega_1,\omega}$.

When T is scattered, it is possible to give a Σ_1 enumeration of the models of T (Sacks [11]). A tree can be constructed in $L(\omega_1, T)$ with height at most ω_1 and with at most countably many nodes on each level. Each node is a finitarily consistent and ω-complete theory in a fragment $\mathcal{L}_{T'}$ with $T \subseteq T'$ and $\mathcal{L}_0 \subseteq \mathcal{L}_{T'}$. The countable models of T are exactly the atomic models of the nodes.

Because a weakly scattered theory can have up to 2^ω many models, it may not be possible to enumerate in $L(\omega_1, T)$ all the theories whose atomic models are exactly the countable models of T. Still, it is possible to arrange the countable models of T in a tree hierarchy.

For notational purposes, define

$$\delta- = \begin{cases} \delta - 1 & \text{if } \delta \text{ is a successor,} \\ \delta & \text{if } \delta \text{ is not a successor (i.e., 0 or a limit ordinal).} \end{cases}$$

Definition 2.1. Let T be a weakly scattered theory in $\mathcal{L}_0$, a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ for some first order language $\mathcal{L}$. The *raw hierarchy* of T, denoted $\mathcal{RH}(T)$, is defined as follows:

Level 0: Every $T_0 \supseteq T$ that is a finitarily consistent, ω-complete theory of $\mathcal{L}_0$ is a node on level 0. Define $\mathcal{L}_0(T_{0-})$ to be $\mathcal{L}_0$.

Level $\delta + 1$: Assume T_δ extends a unique theory $T_{\delta-}$ on level $\delta-$ and $\mathcal{L}_\delta(T_{\delta-})$ is countable. If all n-types (for $n > 0$) are principal, then $\mathcal{L}_{\delta+1}(T_\delta)$ is undefined and T_δ has no extensions on level $\delta + 1$. Otherwise, let $\mathcal{L}_{\delta+1}(T_\delta)$ be the least fragment of $\mathcal{L}_{\omega_1,\omega}$ extending $\mathcal{L}_\delta(T_{\delta-})$ and having as a member the conjunction $\wedge\{\varphi(\bar{x}) \mid \varphi(\bar{x}) \in p(\bar{x})\}$ for every non-principal n-type $p(\bar{x})$ $(n \geq 1)$ of T_δ.

$T_{\delta+1}$ is on level $\delta + 1$ of $\mathcal{RH}(T)$ if $T_{\delta+1}$ is a finitarily consistent, ω-complete theory of $\mathcal{L}_{\delta+1}(T_\delta)$ extending T_δ.

Level λ limit: T_λ is on level λ if there is a sequence $\langle T_\delta \mid \delta < \lambda \rangle$ such that:

1. T_δ is on level δ,
2. $T_\beta \subseteq T_\delta$ if $\beta \leq \delta \leq \lambda$,
3. $T_\lambda = \cup\{T_\delta \mid \delta < \lambda\}$.

Define $\mathcal{L}_\lambda(T_\lambda)$ to be $\cup\{\mathcal{L}_\delta(T_{\delta-}) \mid \delta < \lambda\}$.

$\mathcal{A}$ is a countable model of T if and only if $\mathcal{A}$ is the atomic model of T_δ for some δ. The *raw tree rank* of a model $\mathcal{A}$ is defined as

$$rtr(\mathcal{A}) = \text{least } \delta[\mathcal{A} \text{ is the atomic model of } T_\delta].$$

An analysis of a model $\mathcal{A}$ can be done with respect to the raw hierarchy of T. This analysis is very similar the Scott analysis of $\mathcal{A}$.

- $T(0, \mathcal{A})$ = theory of $\mathcal{A}$ in $\mathcal{L}_0$ and $\mathcal{L}_{T(0,\mathcal{A})} = \mathcal{L}_0$. Contrast $\mathcal{L}_{T(0,\mathcal{A})} = \mathcal{L}_0$ with $\mathcal{L}_0^{\mathcal{A}} = \mathcal{L}$ in the definition of Scott rank (Definition 1.11).

- $\mathcal{L}_{T(\delta+1,\mathcal{A})}$= least fragment of $\mathcal{L}_{\omega_1,\omega}$ extending $\mathcal{L}_{T(\delta,\mathcal{A})}$ and containing the conjunction $\wedge\{\varphi(\bar{x}) \mid \varphi(\bar{x}) \in p(\bar{x})\}$ for every non-principal n-type $p(\bar{x})$ ($n \geq 1$) of $T(\delta, \mathcal{A})$.

- $T(\delta+1, \mathcal{A})$ = theory of $\mathcal{A}$ in $\mathcal{L}_{T(\delta+1,\mathcal{A})}$.

- $T(\lambda, \mathcal{A}) = \cup\{T(\beta, \mathcal{A}) \mid \beta < \lambda\}$.

- $\mathcal{L}_{T(\lambda,\mathcal{A})} = \cup\{\mathcal{L}_{T(\beta,\mathcal{A})} \mid \beta < \lambda\}$.

The following relationships between $rtr(\mathcal{A})$ and $sr(\mathcal{A})$ were established by Sacks in [11].

Proposition 2.2. $rtr(\mathcal{A}) \leq sr(\mathcal{A})$.

Proof. The proposition follows if we show $\mathcal{A}$ is the atomic model of $T(sr(\mathcal{A}), \mathcal{A})$. The fragment $\mathcal{L}_\delta^{\mathcal{A}}$ and theory $T_\delta^{\mathcal{A}}$ were defined in Definition 1.11. By induction on δ, $\mathcal{L}_\delta^{\mathcal{A}} \subseteq \mathcal{L}_{T(\delta,\mathcal{A})}$ and, consequently, $T_{sr(\mathcal{A})}^{\mathcal{A}} \subseteq T(sr(\mathcal{A}), \mathcal{A})$. By definition, $\mathcal{A}$ is the atomic model of $T_{sr(\mathcal{A})}^{\mathcal{A}}$ and, hence, a homogeneous model of $T_{sr(\mathcal{A})}^{\mathcal{A}}$. $\mathcal{A}$ is also a homogeneous model of $T(sr(\mathcal{A}), \mathcal{A})$. If φ is an atom of $T_{sr(\mathcal{A})}^{\mathcal{A}}$, then φ is an atom of $T(sr(\mathcal{A}), \mathcal{A})$. It follows that $\mathcal{A}$ is an atomic model of $T(sr(\mathcal{A}), \mathcal{A})$. □

Proposition 2.3. *If $L(\alpha, \langle T, \mathcal{A}\rangle)$ is Σ_1-admissible, then $rtr(\mathcal{A}) < \alpha \rightarrow sr(\mathcal{A}) < \alpha$.*

Proof. For each $\delta < \alpha < \omega_1$, $\mathcal{L}_{T(\delta,\mathcal{A})}$ and $T(\delta, \mathcal{A})$ are in $L(\alpha, \langle T, A\rangle)$. Suppose the proposition fails and that $rtr(\mathcal{A}) < \alpha$ and $sr(\mathcal{A}) \geq \alpha$. Then the set D of all distinctions between n-tuples of $\mathcal{A}$ made by formulas of $\mathcal{L}_{T(rtr(\mathcal{A}),\mathcal{A})}$ belongs in $L(\alpha, \langle T, \mathcal{A}\rangle)$. Let f be the map that carries each distinction $d \in D$ to the least δ such that d is made by some formula of $\mathcal{L}_\delta^{\mathcal{A}}$. Then f is an unbounded $\Sigma_1^{L(\alpha,\langle T,\mathcal{A}\rangle)}$ map of D into α, which violates the Σ_1-admissibility of $L(\alpha, \langle T, \mathcal{A}\rangle)$. □

By Proposition 2.2, if T has an extension on level α of its raw hierarchy, then T has a model with Scott rank at least α.

The set of sentences B_α is designed so that every model of B_α constitutes a node on level α of $\mathcal{RH}(T)$. The axioms of B_α are:

1. $T \subseteq T_0$ and T_0 is a finitarily consistent, ω-complete theory of $\mathcal{L}_0$.

2. For all $\delta < \alpha$, T_δ has a non-principal n-type for some n.

3. For all $\delta < \alpha$, $T_\delta \subseteq T_{\delta+1}$ and $T_{\delta+1}$ is a finitarily consistent, ω-complete theory of $\mathcal{L}_{\delta+1}(T_\delta)$.

4. For all limit ordinals λ, $T_\lambda = \cup\{T_\delta | \delta < \lambda\}$ and $\mathcal{L}_\lambda(T_\lambda) = \cup\{\mathcal{L}_\delta(T_{\delta-}) \mid \delta < \lambda\}$.

It is possible to construct $\mathcal{L}_\delta(T_{\delta-})$ from $T_{\delta-}$ via an ordinal defined by a $\Sigma_1^{L(\alpha,T)}$ recursion on $\delta < \alpha$ ([11, Section 8]). Because of this, B_α is $\Delta_1^{L(\alpha,T)}$.

2.2 Initial Stage

2.2.1 Set Forcing

We give a streamlined review of some forcing terminology. The definitions matches those in Baumgartner [2] and Kunen [7].

Definition 2.4. Let $(P, <)$ be a partial ordering. $(P, <)$ is called a *notion of forcing* and the elements of P are *forcing conditions.* If $p, q \in P$, then p *extends* q or p is *stronger than* q if $p \leq q$. Two forcing conditions p and q are *compatible* if there exists $r \in P$ such that $r \leq p, q$; and otherwise they are *incompatible.* The maximum element of P is denoted by 1. A set $D \subseteq P$ is *dense in* P if $\forall p \in P\, \exists q \in D\, q \leq p$.

Forcing is always considered to be taking place over V, the universe of all sets, or some transitive model M.

Definition 2.5. A set $G \subseteq P$ is P-generic over a class M if

1. $\forall p, q \in G\, \exists r \in G\, r \leq p, q$;

2. $\forall p \in G \, \forall q \in P$ if $p \leq q$ then $q \in G$; and

3. if $D \in M$ and D is dense in P then $G \cap D \neq \emptyset$.

A *name* is a set $\dot{x} \in M$ consisting of pairs $\langle \dot{y}, p \rangle$ where $\dot{y}$ is a name and $p \in P$. If G is generic, let $M[G]$ denote the generic extension. $M[G] = \{\dot{x}^G \mid \dot{x} \text{ is a name}\}$ where $\dot{x}^G = \{\dot{y}^G \mid \langle \dot{y}, p \rangle \in \dot{x}, p \in G\}$.

The *forcing language* consists of $\in$ plus constant symbols $\dot{x}$ for all names $\dot{x}$ in M. If $\varphi(\dot{x}_1, \ldots, \dot{x}_n)$ is a sentence of the forcing language then $M[G] \models \varphi$ if φ is true in $M[G]$ when $\dot{x}_i$ is interpreted as $\dot{x}_i^G$.

Definition 2.6. We define the *(strong) forcing relation* $\Vdash$ between $p \in P$ and sentences φ of the forcing language as follows:

1. $p \Vdash \dot{x} \in \dot{y}$ if and only if for some $q \geq p$ and for some $\dot{z}$, $\langle \dot{z}, q \rangle \in \dot{y}$ and $p \Vdash \dot{x} = \dot{z}$.

2. $p \Vdash \dot{x} \neq \dot{y}$ if and only if for some $q \geq p$ and some $\dot{z}$, either $\langle \dot{z}, q \rangle \in \dot{x}$ and $p \Vdash \dot{z} \notin \dot{y}$, or $\langle \dot{z}, q \rangle \in \dot{y}$ and $p \Vdash \dot{z} \notin \dot{x}$.

3. $p \Vdash \neg\varphi$ if and only if $\forall q \leq p$ it is not the case that $q \Vdash \varphi$.

4. $p \Vdash \varphi \wedge \psi$ if and only if $p \Vdash \varphi$ and $p \Vdash \psi$.

5. $p \Vdash \exists x \varphi(x)$ if and only if $p \Vdash \varphi(\dot{y})$ for some $\dot{y}$.

If $p \Vdash \varphi$, we say that *p forces φ*. The symbol $\Vdash_P$ denotes forcing with respect to P and $\Vdash_P \varphi$ means that for all $p \in P$ $p \Vdash_P \varphi$ (or $1 \Vdash_P \varphi$).

Proofs of the following lemmas can be found in Kunen [7].

Lemma 2.7. *(Extension) If $p \Vdash \varphi$ and $q \leq p$, then $q \Vdash \varphi$.*

Lemma 2.8. *(Definability) For any formula $\varphi(x_1, \ldots, x_n)$, the set*

$$\{\langle p, \dot{x}_1, \ldots \dot{x}_n \rangle \mid p \Vdash \varphi(\dot{x}_1, \ldots \dot{x}_n)\}$$

is definable over M.

Lemma 2.9. *(Truth) For all P-generic filters $G \subseteq P$,*

$$M[G] \models \varphi \leftrightarrow \exists p \in G\, p \Vdash \varphi.$$

2.2.2 Tree of Sentences

Let $\mathcal{L}_0$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ for some countable first order language $\mathcal{L}$ such that $T \subseteq \mathcal{L}_0$ is a theory with a model. Suppose T is consistent with B_α. We approximate a theory on level 0 of $\mathcal{RH}(T)$ by augmenting T with finite sets of sentences. The sentences are arranged in a tree Γ_0, shown in Figure 2-1. We now describe the construction of Γ_0.

As $\mathcal{L}_0$ is countable, enumerate its sentences as $\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots$. The initial node of Γ_0 is T (i.e., all sentences of T). There are two branches extending from the initial nodes, and on the first level there are two nodes, one for each of φ_0 and $\neg\varphi_0$. At each step of the construction, we add branches for a sentence and its negation. Extra care is taken in the $2s+1$ step if $\varphi_j = \bigvee_{i<\omega} \psi_i$.

$2s$ *step:* Take the next sentence φ_j in the enumeration and, at every terminal node, add branches for φ_j and $\neg\varphi_j$.

$2s+1$ *step:* If the φ_j used at the $2s$-step is of the form $\bigvee_{i<\omega} \psi_i$, then, at each positive φ_j node, add an infinite number of branches, one for each ψ_i. The $\neg\varphi_j$ nodes are left untouched.

A node p on the tree can be thought of as the set of the sentences along the finite path from the initial node to p.

Proposition 2.10. *The relation "p is consistent with B_α" is $\Pi_1^{L(\alpha,T)}$.*

Proof. The predicate "P is a deduction from $B_\alpha \cup \{p\}$ of φ" is defined by a Σ_1-recursion and is $\Delta_1^{L(\alpha,T)}$. So

$$\exists P[P \text{ is a deduction from } B_\alpha \cup \{p\} \text{ of } \varphi]$$

is $\Sigma_1^{L(\alpha,T)}$.

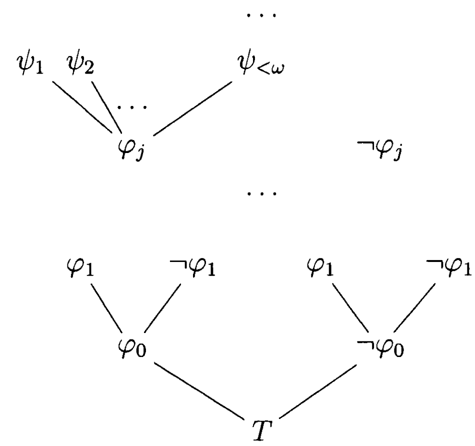

Figure 2-1: Γ_0 when $\varphi_j = \bigvee_{i<\omega} \psi_i$

The relation "p is consistent with B_α" holds if and only if

$$\neg\exists P[P \text{ is a deduction from } B_\alpha \cup \{p\} \text{ of } (\psi \wedge \neg\psi)]$$

□

The notion of forcing is the collection Q_0 of nodes on Γ_0 consistent with B_α, along with the partial ordering $q \leq p$ if and only if $p \subseteq q$. The maximal element in the ordering is the initial node of Γ_0. If $L(\alpha, T)$ is Σ_2-admissible, then $Q_0 \in L(\alpha, T)$ by Δ_2-separation.

Proposition 2.11. *For every sentence $\varphi \in \mathcal{L}_0$ the set $D_\varphi = \{p \in Q_0 \mid \varphi \in p \text{ or } \neg\varphi \notin p\}$ is dense in Q_0.*

Proof. Q_0 is non-empty because T is consistent with B_α by supposition. If $p \in Q_0$ and p is on level δ of Γ_0, then p can be extended to a node p' on level $\delta + 1$ such that $p' \in Q_0$. From p there are either branches for $\varphi, \neg\varphi$ for some $\mathcal{L}_0$-sentence φ or branches for ψ_i $(i < \omega)$, if the sentence associated with p is of the form $\bigvee_{i<\omega} \psi_i$. Since p is consistent with B_α, p can be extended by one of φ or $\neg\varphi$ or some ψ_i and remain consistent with B_α.

Fix φ and let $q \in Q_0$. Then q can be extended to a node p consistent with B_α such that either $\varphi \in p$ or $\neg\varphi \in p$. $\square$

If T is consistent with B_α, then a Q_0-generic will be a path through the tree by Proposition 2.11 and will be ω-complete by step $2s+1$ of the tree construction. Let T^1 be a Q_0-generic and let $L(\alpha, T, T^1)$ denote the generic extension of $L(\alpha, T)$ by T^1.

2.3 Iteration

In Section 2.2.2, we described how to obtain a generic theory on level 0 of $\mathcal{RH}(T)$. By an iterated forcing argument, this process is repeated to get theories on levels $\delta \leq \alpha$.

2.3.1 Iterated Forcing

It is possible to express the generic extension of a generic extension as a single generic extension. Suppose P is a partial ordering and $\Vdash_P \dot{Q}$ is a partial ordering. Let $P * \dot{Q} = \{(p, \dot{q}) \mid p \in P \text{ and } \Vdash_P \dot{q} \in \dot{Q}\}$. $(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2)$ if and only if $p_1 \leq p_2$ and $p_1 \Vdash \dot{q}_1 \leq \dot{q}_2$. Then forcing with $P * \dot{Q}$ is the same as forcing with P and then with $\dot{Q}$. This can be extended to an α-stage iteration. We will use $\Vdash_\alpha$ as an abbreviation for $\Vdash_{\mathbb{P}_\alpha}$.

Definition 2.12. Let $\alpha \geq 1$. A partial ordering $\mathbb{P}_\alpha$ is an *α-stage iteration* if $\mathbb{P}_\alpha$ is a set of α-sequences satisfying the following conditions:

1. If $\alpha = 1$, then there is a partial ordering Q_0 such that $p \in \mathbb{P}_1$ if and only $p(0) \in Q_0$ and $p \leq q$ if and only $p(0) \leq q(0)$. So $\mathbb{P}_1 \cong Q_0$.

2. If $\alpha = \beta + 1$, $\beta \geq 1$, then $\mathbb{P}_\beta = \{p|\beta : p \in \mathbb{P}_\alpha\}$ is a β-stage iteration and there is $\dot{Q}_\beta$ such that $\Vdash_\beta \dot{Q}_\beta$ is a partial ordering; and $p \in \mathbb{P}_\alpha$ if and only if $p|\beta \in \mathbb{P}_\beta$ and $\Vdash_\beta p(\beta) \in \dot{Q}_\beta$. Moreover, $p \leq q$ if and only if $p|\beta \leq q|\beta$ and $p|\beta \Vdash_\beta p(\beta) \leq q(\beta)$. Thus $\mathbb{P}_\alpha \cong \mathbb{P}_\beta * \dot{Q}_\beta$.

3. If α is a limit ordinal, then $\forall\beta < \alpha\ \mathbb{P}_\beta = \{p|\beta : p \in \mathbb{P}_\alpha\}$ is a β-stage iteration, and

(a) $\bar{1} \in \mathbb{P}_\alpha$, where $\bar{1}(\gamma) = \dot{1}$ for all $\gamma < \alpha$ (recall that $\dot{1}$ is the maximal element of $\dot{Q}_\gamma$).

(b) if $\beta < \alpha$, $p \in \mathbb{P}_\alpha, q \in \mathbb{P}_\beta$ and $q \leq p|\beta$, then $r \in \mathbb{P}_\alpha$, where $r|\beta = q$ and $r(\gamma) = p(\gamma)$ for $\beta \leq \gamma < \alpha$.

(c) for all $p, q \in \mathbb{P}_\alpha$, $p \leq q$ if and only if for all $\beta < \alpha, p|\beta \leq q|\beta$.

At limit stages λ, $\mathbb{P}_\lambda$ is not uniquely determined. The types of limits taken needs to be specified. We say that $\mathbb{P}_\alpha$ is the direct limit of $\langle \mathbb{P}_\beta : \beta < \alpha \rangle$ if $p \in \mathbb{P}_\alpha$ if and only if there is some $\beta < \alpha$ such that $p|\beta \in \mathbb{P}_\beta$ and $\forall \gamma (\beta \leq \gamma < \alpha \Rightarrow p(\gamma) = \dot{1})$.

If $p \in \mathbb{P}_\lambda$, the *support* of p is defined by $support(p) = \{\beta < \lambda : p(\beta) \neq \dot{1}\}$.

2.3.2 α-stage Iteration

We now use an α-stage iteration with finite support to get a theory T^α on level α of $\mathcal{RH}(T)$. As before, $\mathcal{L}_0$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ and $T \subseteq L_0$.

We define $\dot{Q}_\delta$ by induction on δ. Suppose T is consistent with B_α. When $\delta = 0$, $Q_0 \cong \mathbb{P}_1$ is the notion of forcing described in Section 2.2.2.

Let T^δ be a $\mathbb{P}_\delta$-generic and let B_{α,T^δ} be the set of sentences whose axioms are:

1. $T^\delta = T_\delta$ and $\mathcal{L}_\delta(T_{\delta-})$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ such that $T^\delta \subseteq \mathcal{L}_\delta(T_{\delta-})$.

2. For all ξ such that $\delta \leq \xi < \alpha$, T_ξ has a non-principal n-type for some n.

3. For all ξ such that $\delta \leq \xi < \alpha$, $T_\xi \subseteq T_{\xi+1}$ and $T_{\xi+1}$ is a finitarily consistent, ω-complete theory of $\mathcal{L}_{\xi+1}(T_\xi)$, the least fragment of $\mathcal{L}_{\omega_1,\omega}$ extending $\mathcal{L}_\xi(T_{\xi-})$ containing the conjunction $\wedge\{\varphi(\bar{x}) \mid \varphi(\bar{x}) \in p(\bar{x})\}$ for every non-principal n-type $p(\bar{x})$ of T_ξ $(n \geq 1)$.

4. $T_\lambda = \cup\{T_\xi \mid \delta \leq \xi < \lambda\}$ and $\mathcal{L}_\lambda(T_\lambda) = \cup\{\mathcal{L}_\xi(T_{\xi-}) \mid \delta \leq \xi < \lambda\}$, for all limit ordinals λ such that $\delta \leq \lambda < \alpha$.

The sentences of B_{α,T^δ} say that T^δ has models of arbitrarily high rank.

Now suppose that T^δ is consistent with B_{α,T^δ}, that is

$$\Vdash_\delta \text{“}\dot{T}^\delta \text{ is consistent with } B_{\alpha,T^\delta}\text{”};$$

and that the language $\mathcal{L}_\delta$ is such that

$\Vdash_\delta$ “$\dot{\mathcal{L}}_\delta$ is the least fragment of $\mathcal{L}_{\omega_1,\omega}$ extending the language of $\dot{T}^\delta$ and having as a member $\bigwedge\{\varphi(\bar{x} \mid \varphi(\bar{x}) \in p(\bar{x})\}$ for every non-principal type $p(\bar{x})$ of $\dot{T}^\delta$”.

In $L(\alpha, T, T^\delta)$, construct Γ_δ, a tree of $\mathcal{L}_\delta$ sentences. The construction of Γ_δ is nearly identical to the construction of Γ_0 save for the initial node and the sentences that are used at each step.

Enumerate the sentences of $\mathcal{L}_\delta$ as $\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots$. The initial node of Γ_δ is T^δ. On the first level, there are two nodes, one for φ_0 and one for $\neg\varphi_0$. At each step of the construction, we add branches for a sentence and its negation.

$2s$ *step:* Take the next sentence φ_j in the enumeration and, at every terminal node, add branches for φ_j and $\neg\varphi_j$.

$2s+1$ *step:* If the φ_j used at the $2s$-step is of the form $\bigvee_{i<\omega} \psi_i$, then at each positive φ_j node, add an infinite number of branches, one for each ψ_i. The $\neg\varphi_j$ nodes are untouched.

Let Q_δ be the collection of nodes on Γ_δ that are consistent with B_{α,T^δ} along with the ordering $q \leq p$ if and only if $p \subseteq q$. The maximal element is the initial node of Γ_δ. Let $\dot{Q}_\delta$ be a $\mathbb{P}_\delta$-name for Q_δ and so $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{Q}_\delta$. If $T^{\delta+1}$ is a $\mathbb{P}_{\delta+1}$-generic, then $T^{\delta+1}$ will be a theory on the next level of $\mathcal{RH}(T)$ extending T^δ.

When λ is a limit, $\mathbb{P}_\lambda$ is the direct limit of the $\mathbb{P}_\delta$ $(\delta < \lambda)$. The condition $p = \langle p_\delta \mid \delta < \lambda\rangle$ is in $\mathbb{P}_\lambda$ if for each $\delta < \lambda$, $p|\delta \in \mathbb{P}_\delta$ and *support*(p) is finite.

2.3.3 B_α-hypothesis

At each stage δ, we need to make the assumption that T^δ is consistent with B_{α,T^δ} because there is no reason that T^δ should satisfy that condition. We say that the

B_α*-hypothesis* holds for T if T is consistent with B_α and for all $\delta < \alpha$,

$$\Vdash_\delta \text{“}\dot{T}^\delta \text{ is consistent with } B_{\alpha,T^\delta}\text{”}.$$

2.4 Preserving Σ_2-admissibility

The aim of this section is to show that Σ_2-admissibility is preserved at every stage of the iteration. To that end, we first show that when φ is a Σ_2 sentence, $p \Vdash \varphi$ is a Σ_2 property of p and φ.

2.4.1 The Forcing Relation

Let $\mathbb{P}_\delta$ be a notion of forcing described earlier and let $p \in \mathbb{P}_\delta$. We study the complexity of the forcing relation.

Proposition 2.13. *If φ is a Δ_0 sentence, then the set*

$$\{(p, \varphi) \mid p \Vdash \varphi\}$$

is $\Delta_1^{L(\alpha,T)}$.

Proof. A sentence φ is Δ_0 if it is constructed from atomic formulas by applications of negation, conjunction and bounded quantification. If φ is Δ_0, then by induction on the complexity of φ, the forcing relation is defined by a Σ_1-recursion and hence a $\Delta_1^{L(\alpha,T)}$ property of p and φ.

□

Proposition 2.14. *For $n \geq 1$, the set*

$$\{(p, \varphi) : \varphi \textit{ is a } \Sigma_n \textit{ sentence and } p \Vdash \varphi\}$$

is $\Sigma_n^{L(\alpha,T)}$ and the set

$$\{(p, \varphi) : \varphi \textit{ is a } \Pi_n \textit{ sentence and } p \Vdash \varphi\}$$

is $\Pi_n^{L(\alpha,T)}$.

Proof. We prove the proposition by induction on n.

Suppose that φ is a Σ_1 sentence $\exists x\psi(x)$ in which ψ is a Δ_0 formula. Then

$$p \Vdash \exists x\ \psi(x)$$

$$\Leftrightarrow \exists \dot{c}\ p \Vdash \psi(c).$$

The sentence $\psi(\dot{c})$ is bounded and, from the previous proposition, forcing a Δ_0 sentence is a $\Delta_1^{L(\alpha,T)}$ property of p and $\psi(\dot{c})$. Whether p forces φ is therefore a $\Sigma_1^{L(\alpha,T)}$ property. If φ is a Π_1 sentence $\forall x\psi(x)$, then

$$p \Vdash \forall x\ \psi(x)$$

$$\Leftrightarrow p \Vdash \neg\exists x\neg\psi(x)$$

$$\Leftrightarrow \forall q \leq p\ \forall \dot{c}\ \neg(q \Vdash \neg\psi(\dot{c})).$$

Whether q forces $\neg\psi(\dot{c})$ is a $\Delta_1^{L(\alpha,T)}$ property. Hence forcing a Π_1 sentence is a $\Pi_1^{L(\alpha,T)}$ property.

Suppose the proposition holds for n. If φ is of the form $\exists x\psi(x)$ where $\psi(x)$ is Π_n, then $p \Vdash \varphi$ if and only if there is a $\dot{c}$ such that $p \Vdash \psi(\dot{c})$. By induction, $p \Vdash \psi(\dot{c})$ is a $\Pi_n^{L(\alpha,T)}$ property and therefore $p \Vdash \varphi$ is a $\Sigma_{n+1}^{L(\alpha,T)}$ property. And if φ is $\neg\exists x\neg\psi(x)$ where $\psi(x)$ is Σ_n, then

$$p \Vdash \neg\exists x\neg\psi(x)$$

$$\Leftrightarrow \forall q \leq p\forall \dot{c}\ \neg(q \Vdash \neg\psi(\dot{c}))$$

$$\Leftrightarrow \forall q \leq p\forall \dot{c}\ \neg[\forall r \leq q\neg(r \Vdash \psi(\dot{c}))]$$

$$\Leftrightarrow \forall q \leq p\ \forall \dot{c}\ \exists r \leq q\ r \Vdash \psi(\dot{c}).$$

By induction, $r \Vdash \psi(\dot{c})$ is $\Sigma_n^{L\alpha,T)}$ and so whether p forces φ is a $\Pi_{n+1}^{L(\alpha,T)}$ property.

□

2.4.2 Σ_2-admissibility

Assume the B_α-hypothesis holds for T. Let $\mathbb{P}_\delta$ be a notion of forcing described previously and let T^δ be a $\mathbb{P}_\delta$-generic. We now show that at every stage of the iteration, Σ_2-admissibility is preserved.

Lemma 2.15. *If $L(\alpha, T)$ is Σ_2-admissible, then for all $\delta < \alpha$ $L(\alpha, T, T^\delta)$ is Σ_2-admissible.*

Proof. Suppose $\varphi(x, y)$ is a Σ_2 formula and that $L(\alpha, T, T^\delta) \models \forall x \in \dot{a}\ \exists y\ \varphi(x, y)$. This holds if and only if there exists a $p \in T^\delta$ such that $p \Vdash \forall x \in \dot{a}\ \exists y\ \varphi(x, y)$. Now we unravel the definition of $\Vdash$:

$$p \Vdash \forall x \in \dot{a} \exists y\ \varphi(x, y)$$

$$\Leftrightarrow p \Vdash \neg(\exists x \in \dot{a} \forall y\ \neg\varphi(x, y))$$

$$\Leftrightarrow \forall q \leq p\ \neg[q \Vdash \exists x \in \dot{a} \forall y\ \neg\varphi(x, y)]$$

$$\Leftrightarrow \forall q \leq p\ \neg[\exists \dot{c} \in \dot{a}\ q \Vdash \forall y \neg\varphi(\dot{c}, y)]$$

$$\Leftrightarrow \forall q \leq p\ \neg[\exists \dot{c} \in \dot{a} q \Vdash \neg(\exists y \varphi(\dot{c}, y))]$$

$$\Leftrightarrow \forall q \leq p\ \neg[\exists \dot{c} \in \dot{a} \forall r \leq q\ \neg(r \Vdash \exists y \varphi(\dot{c}, y))]$$

$$\Leftrightarrow \forall q \leq p\ \neg[\exists \dot{c} \in a \forall r \leq q \neg(\exists \dot{d}\ r \Vdash \varphi(\dot{c}, \dot{d}))]$$

$$\Leftrightarrow \forall q \leq p\ \forall \dot{c} \in \dot{a}\ \exists r \leq q\ \exists \dot{d}\ r \Vdash \varphi(\dot{c}, \dot{d}). \tag{2.1}$$

Since $L(\alpha, T)$ is Σ_2-admissible, the lemma follows if we show that the collection of conditions $\mathbb{P}_\delta$ is a set in $L(\alpha, T)$. This is because $r \Vdash \varphi(\dot{c}, \dot{d})$ in equation (2.1) is a $\Sigma_2^{L(\alpha,T)}$ property of r and φ by Proposition 2.14. In the case that $\mathbb{P}_\delta \in L(\alpha, T)$, the quantifiers over q and r will be bounded in $L(\alpha, T)$ and as $L(\alpha, T)$ is Σ_2-admissible there is a bound for $\dot{d}$ in $L(\alpha, T)$. Letting $\dot{b}$ be the canonical $\mathbb{P}_\delta$-name for the bound on $\dot{d}$ in $L(\alpha, T)$, we have

$$p \Vdash \forall x \in \dot{a}\ \exists y \in \dot{b}\ \varphi(x, y).$$

Hence $L(\alpha, T, T^\delta) \models \forall x \in \dot{a}\ \exists y \in \dot{b}\ \varphi(x,y)$ and $L(\alpha, T, T^\delta)$ is Σ_2-admissible.

We show $\mathbb{P}_\delta \in L(\alpha, T)$ by induction on δ.

$\mathbb{P}_1 \in L(\alpha, T)$ by Proposition 2.10. The case when $\delta = \beta + 1$ is similar to the base case. Suppose that $\mathbb{P}_\beta \in L(\alpha, T)$. Then $L(\alpha, T, T^\beta)$ is Σ_2-admissible. The relation "p is consistent with B_{α, T^β}" is $\Pi_1^{L(\alpha,T,T^\beta)}$, so $Q_\beta \in L(\alpha, T, T^\beta)$ by Δ_2-separation. It follows that $\mathbb{P}_{\beta+1} = \mathbb{P}_\beta * \dot{Q}_\beta$ is a set in $L(\alpha, T)$.

If $\delta = \lambda$ where λ is a limit, and $\mathbb{P}_\beta \in L(\alpha, T)$ for all $\beta < \lambda$, then $\mathbb{P}_\lambda \in L(\alpha, T)$ because we are iterating with finite support.

□

Lemma 2.16. *If $L(\alpha, T)$ is Σ_2-admissible, then $L(\alpha, T, T^\alpha)$ is Σ_2-admissible.*

Proof. By the above lemma, Σ_2-admissibility is preserved at each stage $\delta < \alpha$. Because the collection of conditions $\mathbb{P}_\alpha$ is too big to be a set in $L(\alpha, T)$, we cannot argue as in the previous lemma. Our solution is to make forcing with $\mathbb{P}_\alpha$ look like set forcing.

Let $\varphi(x,y)$ be a Σ_2 formula. Then $L(\alpha, T, T^\alpha) \models \forall x \in \dot{a}\ \exists y\ \varphi(x,y)$ if and only if there exists $p \in T^\alpha$ such that

$$p \Vdash \forall x \in \dot{a}\ \exists y\ \varphi(x,y)$$

$$\Leftrightarrow \forall \dot{c} \in \dot{a}\ \forall q \leq p\ \exists r \leq q\ \exists \dot{d}\ r \Vdash \varphi(\dot{c}, \dot{d}). \tag{2.2}$$

By Proposition 2.14, the relation $r \Vdash \varphi(\dot{c}, \dot{d})$ is $\Sigma_2^{L(\alpha,T)}$. Let $L(\gamma, T)$ ($\gamma < \alpha$) be an initial segment of $L(\alpha, T)$ that contains $\dot{a}, p$ and the parameters of φ. Consider the $\Sigma_2^{L(\alpha,T)}$ function f that, given $\langle q, \dot{c}\rangle$ ($q \leq p, \dot{c} \in \dot{a}$), returns the least such $\langle r, \dot{d}\rangle$ satisfying (2.2) above:

$$f(\langle q, \dot{c}\rangle) = \text{least } \langle r, \dot{d}\rangle [r \leq q\ \wedge r \Vdash \varphi(\dot{c}, \dot{d})].$$

The Σ_2-admissibility of $L(\alpha, T)$ implies the closure of $L(\gamma, T)$ under f is contained in $L(\lambda, T)$ for some $\lambda < \alpha$. Let λ be the least such. We show that $L(\lambda, T)$ is a bound for $\dot{d}$ in equation (2.2).

Let $q \leq p$. If q is in $L(\lambda, T)$, then f maps $\langle q, \dot{c}\rangle$ to the least $\langle r, \dot{d}\rangle \in L(\lambda, T)$ such that

$$r \leq q \text{ and } r \Vdash \varphi(\dot{c}, \dot{d}).$$

If q is not contained in $L(\lambda, T)$, we break q up into q_a, its part above $L(\lambda, T)$, and q_b, its part in $L(\lambda, T)$, so that

$$q = q_a \wedge q_b.$$

Apply f to $\langle q_b, \dot{c}\rangle$ to get the least $\langle s, \dot{d}\rangle \in L(\lambda, T)$ such that

$$s \leq q_b \text{ and } s \Vdash \varphi(\dot{c}, \dot{d}).$$

Let $r = q_a \wedge s$. Then r extends s and therefore

$$r \leq q \text{ and } r \Vdash \varphi(\dot{c}, \dot{d}).$$

Let $\dot{b}$ be the canonical $\mathbb{P}_\alpha$-name for $L(\lambda, T)$. We then have

$$p \Vdash \forall x \in \dot{a}\ \exists y\ \varphi(x, y)$$

$$\Leftrightarrow \forall \dot{c} \in \dot{a}\ \forall q \leq p\ \exists r \leq q\ \exists \dot{d} \in \dot{b}\ r \Vdash \varphi(\dot{c}, \dot{d})$$

$$\Leftrightarrow p \Vdash \forall x \in \dot{a}\ \exists y \in \dot{b}\ \varphi(x, y)$$

and $L(\alpha, T, T^\alpha)$ is Σ_2-admissible. □

Chapter 3

A Model of High Rank

We now prove our main result.

Theorem 3.1. *Suppose T is weakly scattered and $L(\alpha, T)$ is countable and Σ_2-admissible. If the B_α-hypothesis holds for T, then T has a countable model $\mathcal{A}$ such that $\omega_1^{\mathcal{A}} = \alpha$ and $sr(\mathcal{A}) = \alpha + 1$.*

Proof. Since the B_α-hypothesis holds for T, we do an finite support iteration of length α with the forcing notions described in Chapter 2. The $\mathbb{P}_\alpha$-generic T^α is a theory on level α of the raw hierarchy of T and $L(\alpha, T, T^\alpha)$ is Σ_2-admissible by Lemma 2.16. To show that T^α has a model $\mathcal{A}$ with $\omega_1^{\mathcal{A}} = \alpha$ and $sr(\mathcal{A}) = \alpha + 1$, we apply a type omitting argument.

The argument involves a proper end extension of $L(\alpha, T, T^\alpha)$. Let Z be the following set of sentences:

1. The atomic diagram of $L(\alpha, T, T^\alpha)$ in the sense of $\mathcal{L}_{\omega_1,\omega}$.

2. $(\bar{d} > \bar{\beta})$ for all $\beta < \alpha$. $\bar{d}$ is a constant not appearing in $(Z1)$.

3. Let $T^{\bar{d}}$ be a theory on level $\bar{d}$ of $\mathcal{RH}(T)$. Add $\mathcal{A}$ is the countable atomic model of $T^{\bar{d}}$ and $\varphi \in T^{\bar{d}}$ for each sentence $\varphi \in T^\alpha$.

4. $(\varphi(\bar{x})$ is an atom of $T^{\bar{d}})$ for each $\varphi(\bar{x})$ that is an atom of T^α; $\varphi(\bar{x})$ is an atom if $\varphi(\bar{x})$ generates a non-principal type of T^α.

5. The axioms of Σ_1-admissibility.

Any model of Z will be a proper end extension of $L(\alpha, T, T^\alpha)$ and will contain a model of T^α. The set Z is $\Sigma_2^{L(\alpha,T,T^\alpha)}$ since the set of T^α atoms is $\Pi_1^{L(\alpha,T,T^\alpha)}$.

By Barwise Compactness and "effective" type omitting, there is a model $\mathcal{M}$ of Z that is a proper end extension of $L(\alpha, T, T^\alpha)$ but omits α. If $\alpha < \omega_1^{\mathcal{A}}$, then α is recursive in $\mathcal{A}$ and $\alpha \in \mathcal{M}$; hence $\omega_1^{\mathcal{A}} \leq \alpha$. By Corollary 1.14, $sr(\mathcal{A}) \leq \alpha + 1$.

The structure $\mathcal{A}$ is a model of T^β for all $\beta < \alpha$, so $rtr(\mathcal{A}) \geq \alpha$. It follows that $sr(\mathcal{A}) \geq \alpha$ because, by Proposition 2.2, $rtr(\mathcal{A}) \leq sr(\mathcal{A})$.

We now show that $sr(\mathcal{A}) = \alpha + 1$. Suppose $sr(\mathcal{A}) = \alpha$. Then $\mathcal{A}$ is the atomic model of T^α. Define the rank of an atom $\varphi(\bar{x})$ to be the least $\beta < \alpha$ such that $\varphi(\bar{x})$ is an atom of T^β. Let f be the function that takes each n-tuple of $\mathcal{A}$ to the rank of an atom of T^α realized by the tuple. By ($Z4$), the atoms of T^α are atoms of $T^{\bar{d}}$. Therefore f is definable from $T^{\bar{d}}$ and $f \in \mathcal{M}$. But $lub(range(f)) = \alpha$ implies $\alpha \in \mathcal{M}$, a contradiction. So $sr(\mathcal{A}) = \alpha + 1 \leq \omega_1^{\mathcal{A}} + 1$. From above, $\omega_1^{\mathcal{A}} \leq \alpha$; consequently, $\omega_1^{\mathcal{A}} = \alpha$.

□

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