Super Symmetric Vertex Algebras and Supercurves

by

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Abstract

We define and study the structure of *SUSY Lie conformal and vertex algebras*. This leads to effective rules for computations with superfields. Given a strongly conformal SUSY vertex algebra V and a supercurve X, we construct a vector bundle $\mathscr{V}_X^\tau$ on X, the fiber of which, is isomorphic to V. Moreover, the state-field correspondence of V canonically gives rise to (local) sections of these vector bundles. We also define *chiral algebras* on any supercurve X, and show that the vector bundle $\mathscr{V}_X^\tau$, corresponding to a SUSY vertex algebra, carries the structure of a chiral algebra.

Thesis Supervisor: Victor G. Kac
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Contents

Chapter 1

Introduction

1.1. Vertex algebras were introduced about 20 years ago by Borcherds [9]. They provide a rigorous definition of the chiral part of 2-dimensional conformal field theory, intensively studied by physicists. Since then they have had important applications to string theory and conformal field theory, and to mathematics, by providing tools to study the most interesting representations of infinite dimensional Lie algebras. Since their appearance, they have been extensively studied in many papers and books (for the latter we refer to [17], [18], [22], [21], [16], [5]).

Vertex algebras also appeared in algebraic geometry as *Factorization Algebras* on complex curves [5], [16]. In the last five years, numerous applications of this deep connection between factorization algebras and vertex algebras have been exploited, notably in the study of the *moduli spaces* (of curves, vector bundles, principal bundles, etc) arising in algebraic geometry. There are also connections between the theory of vertex algebras and the geometric Langlands conjecture [16, ch. 17]. Vertex algebras have also given new invariants of manifolds [25], [27] and applications to mirror symmetry [10].

Even though these approaches have been successful in formalizing 2-dimensional conformal field theories, it has been known for some time to physicists, that in order to describe super symmetric theories, similar objects should be defined on *supercurves* instead of simply curves (cf. [14], [11], [3]). With this motivation, mathematicians have studied in detail the supergeometry of manifolds, and in particular supercurves (cf. [13], [28], [29], [34] among others).

The purpose of this thesis is to generalize the above objects to describe chiral and factorization algebras over supercurves. To accomplish this, we first need to define *supersymmetric (SUSY) vertex algebras* in such a way that the *state-field correspondence* includes the odd coordinates of the supercurve as formal parameters, that is, to any vector a in a SUSY vertex algebra, we associate a *superfield*

$$\overset{s}{Y}(a, z, \theta^1, \dots, \theta^N), \tag{1.1.1}$$

such that structural properties, similar to those of ordinary vertex algebras, hold.

Given a SUSY vertex algebra V and a supercurve X, we want to assign a vector bundle $\mathscr{V}$ over X in such a way that the fiber at a point $x \in X$ is identified with

V. Moreover, we would like $\overset{s}{Y}$ to canonically define sections of this vector bundle (more precisely, its restricted dual). Here we find the first difference with the classical theory, namely, supercurves come in different flavors: general $1|n$ dimensional supercurves and superconformal $1|n$ supercurves. The latter are to the former what holomorphic curves are to compact connected 2-manifolds. The upshot is that we define two different versions of what a SUSY vertex algebra is, one which will localize to give vector bundles on a general $1|n$-dimensional supercurve (called $N_W = n$ SUSY vertex algebras) and another which gives vector bundles on superconformal supercurves (called $N_K = n$ or symply $N = n$ SUSY vertex algebras). The latter are generated by superfields in the sense studied by physicists [14], the former seem to be new objects.

There are several relations between these different SUSY vertex algebras. As a basic example, let us consider the cases with low odd dimensions. Roughly speaking, a general $N = 1$ supercurve is the data of a curve X and a line bundle $\mathscr{L}$ over it, sections of this line bundle are considered to be the *values of a coordinate in the odd direction.* Similarly, an (oriented) superconformal $N = 2$ supercurve consists of a curve X and two line bundles $\mathscr{L}$ and $\mathscr{H}$ over it such that $\mathscr{L} \otimes \mathscr{H}$ is the canonical bundle ω of X. It follows that an $N = 1$ supercurve gives rise canonically to another $N = 1$ supercurve (interchanging $\mathscr{L}$ with $\omega \otimes \mathscr{L}^{-1}$) and to a superconformal $N = 2$ supercurve (by taking $\mathscr{H} = \omega \otimes \mathscr{L}^{-1}$). On the algebraic side, any (conformal) $N_W = 1$ SUSY vertex algebra gives rise to a (conformal) $N = 2$ SUSY vertex algebra (this corresponds to the isomorphism between the superconformal Lie algebras $K(1,2)$ and $W(1,2)$) and both of them correspond to vertex algebras with $N = 2$ superconformal structure. It follows that any such vertex algebra gives vector bundles in both $N = 1$ supercurves and in the corresponding superconformal $N = 2$ supercurve. These three vector bundles are intimately related as we will see in section 4.3.

As in the ordinary vertex algebra case, the vector bundles we construct (more precisely quotients of them) are extensions of (powers of) the Berezinian bundle of X (a super analog of the canonical bundle). The algebraic properties of V reflect in geometric properties of $\mathscr{V}$ as in the ordinary vertex algebra case. We obtain thus superprojective structures, affine structures, global differential operators, etc. as splittings of these extensions. In particular, the state-field correspondence itself gives such splittings (locally).

1.2. After constructing these vector bundles, it is natural to ask if they carry the structure of a *chiral algebra* on a supercurve. It is shown that the usual definitions carry over to the super case with minor difficulties, and that the vector bundles obtained from V are indeed chiral algebras. This allows us to define the coinvariants and conformal blocks of a SUSY vertex algebra in a coordinate independent way as in [16].

1.3. The organization of this thesis is as follows: In chapter 2 we recall some well known notions about vertex algebras and supercurves. In section 2.1 we recollect notation and examples of vertex algebras and we give the basic examples of SUSY vertex algebras from the point of view of ordinary vertex algebras. We also define two families of SUSY vertex algebras, postponing their detailed study until chapter

3. In section 2.2 we collect definitions and examples of supercurves. In particular we recall the duality of $N = 1$ supercurves, the notion of superconformal curves and the relations between oriented superconformal $N = 2$ curves and general $N = 1$ supercurves.

In chapter 3 we systematically study the structure theory of SUSY vertex algebras. We define $N_W = n$ and $N_K = n$ SUSY vertex algebras, and we derive all the basic results and identities, analogous to those in the case of ordinary vertex algebras, along the lines of [22]. Though a SUSY vertex algebra is an ordinary vertex algebra with additional structure, the presence of supersymmetry considerably simplifies calculations.

In chapter 4 we construct a vector bundle with a flat connection associated to a $N_W = n$ SUSY vertex algebra, over any $N = n$ supercurve. We also construct vector bundles associated to $N_K = n$ SUSY vertex algebras over oriented superconformal $N = n$ supercurves. In this chapter we follow closely [16]. In section 4.1 we define the groups $\operatorname{Aut}\mathscr{O}$ of changes of coordinates and the $\operatorname{Aut}\mathscr{O}$-torsor Aut_X for a supercurve. In section 4.2 we construct the vector bundles themselves and their sections. In particular we show that the state-field correspondence for a SUSY vertex algebra is a section of the dual of the corresponding vector bundle. In section 4.3 we compute explicitly some examples of vector bundles over supercurves of low odd dimension.

In chapter 5 we define chiral algebras over supercurves and we prove that the vector bundles constructed from SUSY vertex algebras are examples of chiral algebras. We also define the spaces of coinvariants in a coordinate independent way.

In appendix A we give a brief description of a family of representations of the Lie algebra $\mathfrak{gl}(1|1)$ and their realizations as fibers of certain natural vector bundles over $N = 1$ supercurves.

Chapter 2

Basic notions

2.1 Vertex algebras

In this section we recall some notation and basic results on vertex algebras. We also give the first examples of SUSY vertex algebras constructed via ordinary vertex algebras. The reader is referred to [22] for an introduction to the vertex algebra theory.

Definition 2.1.1. Let $\mathscr{A}$ be a Lie superalgebra. An *$\mathscr{A}$-valued formal distribution* is a formal expression of the form:

$$B(z) = \sum_{n\in\mathbb{Z}} B_{(n)} z^{-1-n} \tag{2.1.1.1}$$

where $B_{(n)} \in \mathscr{A}$ have the same parity for all $n \in \mathbb{Z}$; this parity is called the *parity of* $B(z)$. The coefficients $B_{(n)}$ are called the *Fourier modes* of $B(z)$, and z is a formal parameter. A pair of formal distributions $B(z)$, $C(w)$ is *local* if

$$(z-w)^N [B(z), C(w)] = 0 \qquad \text{for } N \gg 0. \tag{2.1.1.2}$$

If $\mathscr{A} = \mathrm{End}(V)$, where V is a vector superspace, we say that $B(z)$ is a *field* if, for every $v \in V$, $B_{(n)}v = 0$ for large enough n.

Definition 2.1.2. A *vertex algebra* is a quadruple $(V, |0\rangle, T, Y)$ where

- V is a vector superspace,
- $|0\rangle \in V$ is an even vector,
- $T \in \mathrm{End}(V)$ is an even operator,
- Y is a parity preserving linear map from V to the space of $\mathrm{End}(V)$-valued fields: $a \mapsto Y(a, z)$.

This data should satisfy the following axioms:

- vacuum axioms:
$$Y(a,z)|0\!> = a + O(z), \quad T|0\!> = 0; \tag{2.1.2.1}$$
- translation invariance axiom:
$$[T, Y(a,z)] = \partial_z Y(a,z); \tag{2.1.2.2}$$
- locality axiom:
$$(z-w)^n[Y(a,z), Y(b,w)] = 0 \qquad \text{for } n \gg 0. \tag{2.1.2.3}$$

We will denote a vertex algebra by its underlying vector space V when there is no possible confusion.

The map Y is called the *state-field correspondence* and we will use this map to identify a vector $a \in V$ with its corresponding field $Y(a,z)$. The vector $|0\!>$ is called the *vacuum vector* and the endomorphism T is called the *translation operator*.

A *morphism* of vertex algebras $f : V_1 \mapsto V_2$ is a linear map f such that:
$$\begin{aligned} f \circ T_1 &= T_2 \circ f. \\ Y_2(f(a), z) f(b) &= f(Y_1(a,z)b) \qquad \forall a,\, b \in V_1. \end{aligned} \tag{2.1.2.4}$$

2.1.3. Given a vertex algebra V we denote
$$\begin{aligned} &a_{(n)}b = a_{(n)}(b), \\ &[a_\lambda b] = \sum_{k\geq 0} \frac{\lambda^k}{k!} a_{(k)}b, \\ &: ab : = a_{(-1)}b. \end{aligned} \tag{2.1.3.1}$$

The first operation is called the n-th product, the second is called the λ-bracket and the third the normally ordered product.

2.1.4. For each $n \in \mathbb{Z}$, define the n-th product of $\mathrm{End}(V)$-valued fields $A(z)$ and $B(z)$ as follows. Denote by $i_{z,w}$ the *expansion in the domain* $|z| > |w|$:
$$i_{z,w} z^m w^n (z-w)^k = z^{m+k} w^n i_{z,w} \left(1 - \frac{w}{z}\right)^k = \sum_{j \geq 0} (-1)^j \binom{k}{j} z^{m+k-j} w^{n+j}. \tag{2.1.4.1}$$

Define
$$\begin{aligned} A(w)_{(n)}B(w) = \mathrm{res}_z \big(&i_{z,w}(z-w)^n A(z)B(w) - \\ &- i_{w,z}(z-w)^n (-1)^{p(A)p(B)} B(w)A(z)\big), \end{aligned} \tag{2.1.4.2}$$

where $p(A)$ denotes the parity of $A(w)$. It can be shown that the following n-th product identity holds (cf. [22, prop. 4.4])
$$Y(a_{(n)}b, z) = Y(a,z)_{(n)} Y(b,z) \qquad \forall n \in \mathbb{Z}, \tag{2.1.4.3}$$

hence,

$$Y(Ta,z)=\partial_z Y(a,z). \tag{2.1.4.4}$$

2.1.5. In a vertex algebra V we have the following commutator formulas [22, pp 112]

$$\begin{aligned}
[a_{(m)},b_{(n)}] &= \sum_{j\geq 0}\binom{m}{j}\left(a_{(j)}b\right)_{(m+n-j)},\\
[a_{(m)},Y(b,w)] &= \sum_{j\geq 0}\left(\frac{\partial_w^j w^m}{j!}\right)Y(a_{(j)}b,w).
\end{aligned} \tag{2.1.5.1}$$

This formula shows that the space of Fourier modes of all fields of a vertex algebra is closed under the Lie bracket, and, moreover, the commutation relations are expressed in terms of j-th products.

Definition 2.1.6. A *Lie conformal algebra* is a super $\mathbb{C}[\partial]$-module $\mathscr{R}$ equipped with a parity preserving bilinear map

$$[\,{}_\lambda\,]:\mathscr{R}\otimes\mathscr{R}\to\mathbb{C}[\lambda]\otimes\mathscr{R}, \tag{2.1.6.1}$$

satisfying the following axioms:

- Sesquilinearity:

$$[\partial a_\lambda b]=-\lambda[a_\lambda b],\quad [a_\lambda \partial b]=(\partial+\lambda)[a_\lambda b]. \tag{2.1.6.2}$$

- Skew-commutativity:

$$[b_\lambda a]=-(-1)^{p(a)p(b)}[a_{-\partial-\lambda}b]. \tag{2.1.6.3}$$

- Jacobi identity:

$$[a_\lambda[b_\mu c]]=[[a_\lambda b]_{\lambda+\mu}c]+(-1)^{p(a)p(b)}[b_\mu[a_\lambda c]], \tag{2.1.6.4}$$

for all a, b, $c\in\mathscr{R}$.

Given a Lie conformal algebra $\mathscr{R}$, we can associate to it a vertex algebra $V(\mathscr{R})$ (cf. [22], [2]) called the *universal enveloping vertex algebra of $\mathscr{R}$*. If $\mathscr{R}$ is generated by some vectors $\{a_i\}$ as a $\mathbb{C}[\partial]$-module, we say that $V(\mathscr{R})$ is generated by the same vectors. If $C\in\mathscr{R}$ is a central element such that $\partial C=0$, given any complex number c, we denote by $V^c(\mathscr{R})$ the quotient of $V(\mathscr{R})$ by the ideal $(C-c)V(\mathscr{R})$.

One can show [22] that a vertex algebra V is canonically a Lie conformal algebra with the λ-bracket defined in (2.1.3.1) and $\partial=T$.

Example 2.1.7. The *Virasoro* vertex algebra Vir^c is generated by an even field L satisfying:

$$[L_\lambda L]=(\partial+2\lambda)L+\frac{c}{12}\lambda^3. \tag{2.1.7.1}$$

The complex number c is called the *central charge*.

The Fourier modes of L satisfy the commutation relations:

$$[L_{(m)}, L_{(n)}] = (m-n)L_{(m+n-1)} + \delta_{m+n,-2}\frac{m(m-1)(m-2)}{12}c \tag{2.1.7.2}$$

Letting $L_n = L_{(n-1)}$, we obtain the familiar commutation relations of the Virasoro Lie algebra.

Example 2.1.8. Let $\mathfrak{g}$ be a finite dimensional Lie superalgebra with a non-degenerate invariant supersymmetric bilinear form $(,)$ and let $k \in \mathbb{C}$. The *universal affine vertex algebra* $V^k(\mathfrak{g})$ at level k, corresponding to $\mathfrak{g}$, is generated by fields $a \in \mathfrak{g}$ and the commutation relations:

$$[a_\lambda b] = [a, b] + \lambda k(a, b). \tag{2.1.8.1}$$

When $\mathfrak{g}$ is simple and $k \neq -h^\vee$ (the negative of the dual Coxeter number, i.e. 1/2 of the value of the Casimir operator on $\mathfrak{g}$) we have an injective morphism of vertex algebras $\mathrm{Vir}^c \hookrightarrow V^k(\mathfrak{g})$ where

$$c = \frac{k\mathrm{sdim}\mathfrak{g}}{k + h^\vee} \tag{2.1.8.2}$$

This morphism is given by the *Sugawara construction* (cf. [22, Thm 5.7]):

$$L \mapsto \frac{1}{2(k+h^\vee)}\sum :a^i a_i: \tag{2.1.8.3}$$

where $\{a_i\}$ is a basis of $\mathfrak{g}$ and $\{a^i\}$ is its dual basis with respect to $(,)$, i.e. $(a_i, a^j) = \delta_{i,j}$.

Identifying L with its image in $V^k(\mathfrak{g})$ we see that $L_{(0)} = T \in \mathrm{End}(V^k(\mathfrak{g}))$ which follows from the fact that all the *currents* $a \in \mathfrak{g}$ satisfy the commutation relations:

$$[L_\lambda a] = (\partial + \lambda)a. \tag{2.1.8.4}$$

It follows from (2.1.8.4) and (2.1.5.1) that the operator $L_{(1)}$ acts diagonally on $V^k(\mathfrak{g})$ with non-negative integer eigenvalues.

A vertex algebra V with a vector $\nu \in V$ such that the corresponding field $L(z) = Y(\nu, z)$ satisfies (2.1.7.1) and moreover $L_{(0)} = T$ and $L_{(1)}$ acts diagonally with eigenvalues bounded from below is called a *conformal vertex algebra.* A field a, satisfying

$$[L_\lambda a] = (\partial + \Delta\lambda)a, \tag{2.1.8.5}$$

for some $\Delta \in \mathbb{C}$, is called a *primary field* of conformal weight Δ.

Example 2.1.9. A commutative associative unital superalgebra is naturally a vertex algebra with $|0> = 1$, $T = 0$ and $a_{(n)}b = \delta_{n,-1}ab$. More generally, a commutative associative unital superalgebra with an even derivation T is naturally a vertex algebra, the state field correspondence is given by:

$$Y(a, z)b = (e^{zT}a)b. \tag{2.1.9.1}$$

Example 2.1.10. Let V and W be vertex algebras, $a \in V$, $b \in W$. We define $a_{(n)} \otimes b_{(m)} \in \operatorname{End}(V \otimes W)$ by

$$\left(a_{(n)} \otimes b_{(m)}\right)(v \otimes w) = (-1)^{p(a)p(b)} a_{(n)} v \otimes b_{(m)} w. \tag{2.1.10.1}$$

With this definition we construct a vertex algebra structure in $V \otimes W$ by extending the state field correspondence as $Y(a \otimes b, z) = Y(a, z) \otimes Y(b, z)$ and defining the translation operator as $T = T_V \otimes \operatorname{Id} + \operatorname{Id} \otimes T_W$ where T_V (resp. T_W) is the translation operator in V (resp. W).

Example 2.1.11. The Neveu Schwarz (NS) vertex algebra is generated by an even virasoro field L (satisfying (2.1.7.1)) and an odd primary field G of conformal weight $3/2$ (i.e. (2.1.8.5) holds with $\Delta = 3/2$), satisfying the commutation relation:

$$[G_\lambda G] = 2L + \frac{\lambda^2}{3} c. \tag{2.1.11.1}$$

If we expand the corresponding fields

$$\begin{aligned} L(z) &= \sum_{n \in \mathbb{Z}} L_n z^{-2-n}, \\ G(z) &= \sum_{n \in 1/2 + \mathbb{Z}} G_n z^{-3/2-n}, \end{aligned} \tag{2.1.11.2}$$

then the coefficients of such expansions satisfy the following commutation relations (which can be seen from (2.1.5.1)):

$$\begin{aligned} [L_m, L_n] &= (m-n) L_{m+n} + \delta_{m,-n} \frac{m^3 - m}{12} c, \\ [G_m, L_n] &= \left(m - \frac{n}{2}\right) G_{m+n}, \\ [G_m, G_n] &= 2L_{m+n} + \delta_{m,-n} \frac{m^2 - 1/4}{3} c. \end{aligned} \tag{2.1.11.3}$$

Given $a \in \mathrm{NS}$, we define the *superfield* (cf. [22, (5.9.5)]), where θ is an odd indeterminate, $\theta^2 = 0$, $\theta z = z\theta$:

$$\overset{s}{Y}(a, z, \theta) = Y(a, z) + \theta Y(G_{(0)} a, z). \tag{2.1.11.4}$$

Note in particular that if $\nu = L_{(-1)}|0>$ and $\tau = G_{(-1)}|0>$ we have

$$\begin{aligned} \overset{s}{Y}(\nu, z, \theta) &= L(z) + \frac{1}{2} \theta \partial_z G(z), \\ \overset{s}{Y}(\tau, z, \theta) &= G(z) + 2\theta L(z). \end{aligned} \tag{2.1.11.5}$$

Using (2.1.5.1), we can prove easily that

$$[L_{(0)}, \overset{s}{Y}(a,z,\theta)] = \partial_z \overset{s}{Y}(a,z,\theta), \tag{2.1.11.6a}$$

$$[G_{(0)}, \overset{s}{Y}(a,z,\theta)] = (\partial_\theta - \theta\partial_z)\overset{s}{Y}(a,z,\theta). \tag{2.1.11.6b}$$

Similarly we have:

$$\overset{s}{Y}(L_{(0)}a,z,\theta) = \partial_z \overset{s}{Y}(a,z,\theta), \tag{2.1.11.7a}$$

$$\overset{s}{Y}(G_{(0)}a,z,\theta) = (\partial_\theta + \theta\partial_z)\overset{s}{Y}(a,z,\theta). \tag{2.1.11.7b}$$

2.1.12. Motivated by (2.1.11.6), we are ready to define an $N = 1$ *(rather* $N_K = 1$*)* *SUSY vertex algebra* (cf. 3.5.14). Let V be a vector superspace over $\mathbb{C}$. An $\mathrm{End}(V)$-valued ($N = 1$) *superfield* is a formal sum of the form

$$A(z,\theta) = \sum_{n\in\mathbb{Z}} A_{(n,1)} z^{-1-n} + \theta \sum_{n\in\mathbb{Z}} A_{(n,0)} z^{-1-n}, \tag{2.1.12.1}$$

where $A_{(n,i)} \in \mathrm{End}(V)$ are such that for each $v \in V$ we have $A_{(n,i)}v = 0$ for $n \gg 0$ and all $i = 0, 1$.

An $N = 1$ *SUSY vertex algebra* structure in V consist of

- $|0> \in V_0$ is an even vector,
- $S \in \mathrm{End}(V)_1$ is an odd endomorphism,
- $\overset{s}{Y}$ is a parity preserving $\mathbb{C}$-linear map from V to $\mathrm{End}(V)$-valued superfields $a \mapsto \overset{s}{Y}(a,z,\theta)$,

such that the following axioms hold:

- vacuum axioms:

$$S|0>= 0, \quad \overset{s}{Y}(a,z,\theta)|0>= a + O(z,\theta), \tag{2.1.12.2}$$

  where $O(z,\theta)$ denotes a V-valued formal power series in z and θ with zero constant coefficient,

- translation invariance:

$$[S, \overset{s}{Y}(a,z,\theta)] = (\partial_\theta - \theta\partial_z)\overset{s}{Y}(a,z,\theta), \tag{2.1.12.3}$$

- locality:

$$(z-w)^n[\overset{s}{Y}(a,z,\theta), \overset{s}{Y}(b,w,\zeta)] = 0 \quad \text{for } n \gg 0, \tag{2.1.12.4}$$

  were z and w commuting even indeterminates and θ and ζ are anticommuting odd indeterminates commuting with z and w.

Morphisms of $N = 1$ SUSY vertex algebras are defined in the same way as for ordinary vertex algebras.

Remark 2.1.13. Given a $N = 1$ SUSY vertex algebra V, letting $T = S^2$ and puting

$$Y(a, z) = \overset{s}{Y}(a, z, 0), \tag{2.1.13.1}$$

we obtain a vertex algebra $(V, |0>, T, Y)$.

Example 2.1.14. The vertex algebra NS constructed in 2.1.11 has an $N = 1$ SUSY structure given by (2.1.11.4) and $S = G_{-1/2}$.

Definition 2.1.15. An $N = 1$ SUSY vertex algebra V with a vector $\tau \in V$ such that the corresponding superfield $\overset{s}{Y}(\tau, z, \theta) = G(z) + 2\theta L(z)$ satisfies the commutation relations (2.1.11.3) of the Neveu-Schwarz algebra, and moreover, $\tau_{(0,1)} = S$, $\tau_{(0,0)} = 2T$ and the operator $\tau_{(1,0)}$ acts diagonally with eigenvalues bounded from below, is called a *superconformal* $N = 1$ SUSY vertex algebra. The vector τ will be called the *superconformal* vector.

Remark 2.1.16. Let V be a vertex algebra with an $N = 1$ superconformal vector τ (cf. [22, definition 5.9]). Namely, the Fourier modes of the fields

$$\begin{aligned} G(z) &= Y(\tau, z) \sum_{n \in 1/2 + \mathbb{Z}} G_n z^{-n-3/2}, \\ L(z) &= \frac{1}{2} Y(G_{-1/2}\tau, z) = \sum_{n \in \mathbb{Z}} L_n z^{-2-n}, \end{aligned} \tag{2.1.16.1}$$

satisfy the relations (2.1.11.3) of a Neveu-Schwarz algebra for some $c \in \mathbb{C}$, $L_{-1}(= G^2_{-1/2}) = T$ and the operator L_0 is diagonalizable with eigenvalues bounded below. Then V carries a structure of an $N = 1$ SUSY vertex algebra with $S = G_{-1/2}$ and superfields

$$\overset{s}{Y}(a, z, \theta) = Y(a, z) + \theta Y(G_{-1/2}a, z). \tag{2.1.16.2}$$

It is, of course, a superconformal $N = 1$ SUSY vertex algebra with the superconformal vector τ, which is automatically a superconformal vector for this $N = 1$ SUSY vertex algebra structure.

Below we give some examples of vertex algebras with a superconformal vector. By the above remark, they are automatically $N = 1$ SUSY vertex algebras with a superconformal vector.

Example 2.1.17. [22, ex. 5.9a] Let V be the universal envelopping vertex algebra of the Lie conformal algebra generated by an even vector (free boson) α and an odd vector (free fermion) φ, namely

$$\begin{aligned} [\alpha_\lambda \alpha] &= \lambda, \\ [\varphi_\lambda \varphi] &= 1, \\ [\alpha_\lambda \varphi] &= 0. \end{aligned} \tag{2.1.17.1}$$

Then V is a (simple) vertex algebra with a family of $N=1$ superconformal vectors

$$\tau = (\alpha_{(-1)}\varphi_{(-1)} + m\varphi_{(-2)})|0>, \qquad m \in \mathbb{C}, \tag{2.1.17.2}$$

of central charge $c = \frac{3}{2} - 3m^2$.

Example 2.1.18. [23] [22, thm 5.9] Let $\mathfrak{g}$ be a finite dimensional Lie algebra with a non-degenerate invariant symmetric bilinear form $(\,,)$, and let $h^\vee$ be the dual Coxeter number. We construct a vertex algebra $V^k(\mathfrak{g}_{\mathrm{super}})$ generated by the usual currents $a, b \in \mathfrak{g}$, satisfying

$$[a_\lambda b] = [a,b] + (k+h^\vee)\lambda(a,b), \tag{2.1.18.1}$$

and the odd super currents $\{\bar{a}\} \subset \mathfrak{g}$ with reversed parity, satisfying:

$$\begin{aligned} [a_\lambda \bar{b}] &= \overline{[a,b]}, \\ [\bar{a}_\lambda \bar{b}] &= (k+h^\vee)(a,b). \end{aligned} \tag{2.1.18.2}$$

Let $\{a^i\}$ and $\{b^i\}$ be dual bases of $\mathfrak{g}$. Provided that $k \neq -h^\vee$ the vertex algebra $V^k(\mathfrak{g}_{\mathrm{super}})$ admits an $N=1$ superconformal vector

$$\tau = \frac{1}{k+h^\vee}\left(\sum_i a^i_{(-1)}\bar{b}^i_{(-1)} + \frac{1}{3(k+h^\vee)}\sum_{i,j,r}([a^i,a^j],a^r)\bar{b}^i_{(-1)}\bar{b}^j_{(-1)}\bar{b}^r_{(-1)}\right)|0>, \tag{2.1.18.3}$$

of central charge

$$c_k = \frac{k\mathrm{dim}\mathfrak{g}}{k+h^\vee} + \frac{1}{2}\mathrm{dim}\mathfrak{g}. \tag{2.1.18.4}$$

This is known as the *Kac-Todorov* construction. The formulas in [22] should be corrected as above.

Example 2.1.19. [22, Thm 5.10] The $N=2$ vertex algebra is generated by a Virasoro field L of central charge c, an even field J, primary of conformal weight 1, and two odd fields $G^\pm$, primary of conformal weight 3/2. The remaining commutation relations are:

$$\begin{gathered} [J_\lambda J] = \frac{c}{3}\lambda, \quad [G^\pm_\lambda G^\pm] = 0, \quad [J_\lambda G^\pm] = \pm G^\pm, \\ [G^+_\lambda G^-] = L + \frac{1}{2}\partial J + \lambda J + \frac{c}{6}\lambda^2. \end{gathered} \tag{2.1.19.1}$$

This vertex algebra contains an $N=1$ superconformal vector:

$$\tau = G^+_{(-1)}|0> + G^-_{(-1)}|0> . \tag{2.1.19.2}$$

Also, this vertex algebra admits a $\mathbb{Z}/2\mathbb{Z}\times\mathbb{C}^*$ family of automorphisms. The generator of $\mathbb{Z}/2\mathbb{Z}$ is given by $L \mapsto L$, $J \mapsto -J$ and $G^\pm \mapsto G^\mp$. The $\mathbb{C}^*$ family is given by $G^+ \mapsto \mu G^+$ and $G^- \mapsto \mu^{-1}G^-$. Applying these automorphisms, we get a family of $N=1$ superconformal structures.

By expanding the corresponding fields

$$\begin{aligned} L(z) &= \sum_{n\in\mathbb{Z}} L_n z^{-2-n}, \\ G^{\pm}(z) &= \sum_{n\in 1/2+\mathbb{Z}} G^{\pm}_n z^{-3/2-n}, \\ J(z) &= \sum_{n\in\mathbb{Z}} J_n z^{-1-n}, \end{aligned} \tag{2.1.19.3}$$

we get the commutation relations of the Virasoro operators L_n, and the following remainning commutation relations

$$\begin{aligned} [J_m, J_n] &= \frac{m}{3}\delta_{m,-n}c, \quad [J_m, G^{\pm}_n] = \pm G^{\pm}_{m+n}, \\ [G^{\pm}_m, L_n] &= \left(m - \frac{n}{2}\right) G^{\pm}_{m+n}, \quad [L_m, J_n] = -nJ_{m+n}, \\ [G^{+}_m, G^{-}_n] &= L_{m+n} + \frac{m-n}{2}J_{m+n} + \frac{c}{6}\left(m^2 - \frac{1}{4}\right)\delta_{m,-n}. \end{aligned} \tag{2.1.19.4}$$

Sometimes it is convenient to introduce a different set of generating fields for this vertex algebra. We define $\tilde{L} = L - 1/2\partial J$. This is a Virasoro field with central charge zero, namely

$$[\tilde{L}_\lambda \tilde{L}] = (\partial + 2\lambda)\tilde{L}. \tag{2.1.19.5a}$$

With respect to this Virasoro element, G^+ is primary of conformal weight 2 and G^- is primary of conformal weight 1; J has conformal weight 1 but is no longer a primary field. To summarize the commutation relations, we write

$$\begin{aligned} Q(z) = G^{+}(z) &= \sum_{n\in\mathbb{Z}} Q_n z^{-2-n}, \\ H(z) = G^{-}(z) &= \sum_{n\in\mathbb{Z}} H_n z^{-1-n}, \\ \tilde{L}(z) &= \sum_{n\in\mathbb{Z}} T_n z^{-2-n}. \end{aligned} \tag{2.1.19.5b}$$

The corresponding λ-brackets of these fields are given by:

$$\begin{aligned} [\tilde{L}_\lambda \tilde{L}] &= (\partial + 2\lambda)\tilde{L}, \\ [\tilde{L}_\lambda J] &= (\partial + \lambda)J - \frac{\lambda^2}{6}c, \\ [\tilde{L}_\lambda Q] &= (\partial + 2\lambda)Q, \\ [\tilde{L}_\lambda H] &= (\partial + \lambda)H, \\ [H_\lambda Q] &= \tilde{L} - \lambda J + \frac{c}{6}\lambda^2. \end{aligned} \tag{2.1.19.5c}$$

The commutation relations of the coefficients in (2.1.19.5b) are:

$$[T_m, T_n] = (m-n)T_{m+n}, [Q_m, Q_n] = [H_m, H_n] = 0, \tag{2.1.19.5d}$$

$$[T_m, H_n] = -nH_{m+n}, \qquad [T_m, J_n] = -nJ_{m+n} - m(m+1)\frac{c}{12}\delta_{m,-n}, \tag{2.1.19.5e}$$

$$[T_m, Q_n] = (m-n)Q_{m+n}, [H_m, Q_n] = T_{m+n} - mJ_{m+n} + m(m-1)\frac{c}{6}\delta_{m,-n} \tag{2.1.19.5f}$$

2.1.20. The definition in 2.1.12 can be extended to the general $N = n$ case (cf. 3.5.14) by requiring the existence of n odd endomorphisms $S^1, \dots, S^n$ and changing correspondingly the translation invariance axioms, as follows.

Let V be a vector superspace over $\mathbb{C}$. Let z be an even indeterminate and $\theta^1, \dots, \theta^N$ be odd anticommuting indeterminates which commute with z. For an ordered subset $I = (i_1, \dots, i_k) \subset \{1, \dots, n\}$, we will write $\theta^I = \theta^{i_1} \dots \theta^{i_k}$ and let $N \setminus I$ be the ordered complement of I in $\{1, \dots, n\}$.

An End(V)-valued superfield is a expression of the form:

$$A(z, \theta^1, \dots, \theta^n) = \sum_{(n|I), n \in \mathbb{Z}} \theta^{N \setminus I} A_{(n|I)} z^{-1-n} \tag{2.1.20.1}$$

where I runs over all ordered subsets of the set $\{1, \dots, N\}$, $A_{(n|I)} \in \mathrm{End}(V)$, and for each I and $v \in V$ we have $A_{(n|I)}v = 0$ for n large enough. We will usually write $A(z, \theta)$ or simply $A(Z)$ for this field, where $Z = (z, \theta^1, \dots, \theta^N)$.

Definition 2.1.21. An $N_K = n$ (or simply $N = n$) SUSY vertex algebra consists of a vector superspace V, an even vector $|0> \in V$, n odd endomorphisms $S^1, \dots, S^n$ (super translation operators) satisfying $[S_i, S_j] = 2\delta_{ij}T$ for some even endormorphism T, and a parity preserving linear map $\overset{s}{Y}$ from V to the space of End(V)-valued superfields $a \mapsto \overset{s}{Y}(a, z, \theta)$, satisfying the following axioms:

- vacuum axioms:

$$\overset{s}{Y}(a, z, \theta)|0> = a + O(z, \theta), \qquad S^i|0> = 0, \quad i = 1, \dots, n, \tag{2.1.21.1}$$

- translation invariance

$$[S^i, \overset{s}{Y}(a, z, \theta)] = \left(\partial_{\theta^i} - \theta^i \partial_z\right) \overset{s}{Y}(a, z, \theta), \tag{2.1.21.2}$$

- locality

$$(z-w)^k [\overset{s}{Y}(a, z, \theta), \overset{s}{Y}(b, w, \zeta)] = 0, \quad \text{for } k \gg 0, \tag{2.1.21.3}$$

where, as before, θ^i and ζ^j anticommute, and all of them commute with z and w.

Morphisms of $N = n$ SUSY vertex algebras are defined in the same way as for $N = 1$ SUSY vertex algebras. As before, $(V, |0>, T, Y(a,z) = \overset{s}{Y}(a,z,0))$ is an ordinary vertex algebra.

Example 2.1.22. As in the $N = 1$ case, we can give the $N = 2$ vertex algebra, as defined in 2.1.19, the structure of an $N_K = 2$ SUSY vertex algebra. To do this, we define the operators $S^1 = (G^+_{(0)} + G^-_{(0)})$ and $S^2 = i(G^+_{(0)} - G^-_{(0)})$. With these definitions the state field correspondence is given by

$$\overset{s}{Y}(a,z,\theta^1,\theta^2) = Y(a,z) + \theta^1 Y(S^1 a, z) + \theta^2 Y(S^2 a, z) + \\ + \theta^2\theta^1 Y(S^1 S^2 a, z). \quad (2.1.22.1)$$

All the properties required in the definition are easy to check. We also note that

$$\begin{aligned} \overset{s}{Y}(S^i a, z, \theta^1, \theta^2) &= \left(\partial_{\theta^i} + \theta^i \partial_z\right) \overset{s}{Y}(a,z,\theta^1,\theta^2), \\ \overset{s}{Y}(L_{(0)} a, z, \theta^1, \theta^2) &= \partial_z \overset{s}{Y}(a,z,\theta^1,\theta^2). \end{aligned} \quad (2.1.22.2)$$

Also we check directly that letting

$$\tau = \sqrt{-1} J_{(-1)} |0>, \quad (2.1.22.3)$$

we get:

$$\overset{s}{Y}(\tau, z, \theta^i) = \sqrt{-1} J(z) + \theta^1 G^{(2)}(z) - \theta^2 G^{(1)}(z) + 2\theta^1\theta^2 L(z) \quad (2.1.22.4)$$

where $G^{(1)}(z) = G^+(z) + G^-(z)$ and $G^{(2)}(z) = i(G^+(z) - G^-(z))$. It follows that $\tau_{(0|0)} = 2T$, $\tau_{(0|1)} = -S^1$ and $\tau_{(0|2)} = -S^2$ (cf.3.6.6 below).

We note that $G^{(i)}$ are primary of conformal weight 3/2, and J is primary of conformal weight 1. The other commutation relations between the generating fields L, J, $G^{(i)}$ are

$$\begin{aligned} [G^{(i)}{}_\lambda G^{(i)}] &= 2L + \frac{c\lambda^2}{3}, \\ [G^{(1)}{}_\lambda G^{(2)}] &= -i\left(\partial + 2\lambda\right) J, \\ [J_\lambda G^{(1)}] &= -iG^{(2)}, \\ [J_\lambda G^{(2)}] &= iG^{(1)}, \end{aligned} \quad (2.1.22.5)$$

or equivalently

$$\begin{aligned} [G^{(i)}_m, G^{(i)}_n] &= 2L_{m+n} + \left(m^2 - \frac{1}{4}\right)\frac{c}{3}\delta_{m,-n}, \\ [G^{(1)}_m, G^{(2)}_n] &= i\left(n - m\right) J_{m+n}, \\ [J_m, G^{(1)}_n] &= -iG^{(2)}_{m+n}, \\ [J_m, G^{(2)}_n] &= iG^{(1)}_{m+n}. \end{aligned} \quad (2.1.22.6)$$

Definition 2.1.23. An $N_K = 2$ SUSY vertex algebra with a vector $\tau \in V$ such that the corresponding field is as in (2.1.22.4), satisfying (2.1.22.5), (2.1.22.6), and moreover $\tau_{(0|0)} = 2T$, $\tau_{(0|1)} = -S^1$, $\tau_{(0|2)} = -S^2$ and $\tau_{(1|0)}$ acts diagonally with eigenvalues bounded below , is called a *superconformal* $N = 2$ SUSY vertex algebra (cf. 3.6.6 below). The vector τ is called the $N = 2$ superconformal vector.

Example 2.1.24. [22, ex. 5.9d] Consider the vertex algebra generated by a pair of free charged bosons $\alpha^\pm$ and a pair of free charged fermions $\varphi^\pm$ where the only non-trivial commutation relations are:

$$\begin{aligned}[\alpha^\pm{}_\lambda \alpha^\mp] &= \lambda, \\ [\varphi^\pm{}_\lambda \varphi^\mp] &= 1.\end{aligned} \tag{2.1.24.1}$$

This vertex algebra contains the following family of $N = 2$ vertex subalgebras

$$\begin{aligned}G^\pm &=: \alpha^\pm \varphi^\pm : \pm m\partial\varphi^\pm, \qquad m \in \mathbb{C}, \\ J &=: \varphi^+\varphi^- : -m(\alpha^+ + \alpha^-), \\ L &=: \alpha^+\alpha^- : +\frac{1}{2} : \partial\varphi^+\varphi^- : + \\ &+ \frac{1}{2} : \partial\varphi^-\varphi^+ : -\frac{m}{2}\partial(\alpha^+ - \alpha^-).\end{aligned} \tag{2.1.24.2}$$

The vector τ given by (2.1.22.3) provides this vertex algebra with the structure of a superconformal $N_K = 2$ SUSY vertex algebra, by letting $T = L_{-1}$ and $S^i = G^{(i)}_{-1/2}$.

Remark 2.1.25. In the super case, a formula like (2.1.4.4) can be proved (cf. Theorem 3.5.19 below):

$$\overset{s}{Y}(S^i a, z, \theta) = (\partial_{\theta^i} + \theta^i \partial_z)\overset{s}{Y}(a, z, \theta), \tag{2.1.25.1}$$

thus, unlike in the ordinary vertex algebra case, $[S^i, \overset{s}{Y}(a, z, \theta)] \neq \overset{s}{Y}(S^i a, z, \theta)$.

2.1.26. The definition of an $N_K = n$ SUSY vertex algebra formalize the notion of *supersymmetry* in CFT as widely known in the physics literature. As we shall explain later on, this notion is closely related to the K series of superconformal Lie algebras. Now we will define the $N_W = n$ SUSY vertex algebra, related to the W series of superconformal Lie algebras, by replacing the differential operators $\partial_\theta - \theta\partial_z$ by the simpler ∂_θ.

Definition 2.1.27. Let V be a vector superspace. A $N_W = n$ *SUSY vertex algebra structure* on V consists of an even vector $|0\rangle \in V$, n anticommuting odd operators S^i (the odd translation operators), an even operator T commuting with all the S^i (the even translation operator), and a parity preserving linear map $\overset{s}{Y}$ from V to the space of End(V)-valued superfields $a \mapsto \overset{s}{Y}(a, z, \theta)$, staisfying the following axioms:

- vacuum axioms:

$$\begin{aligned}\overset{s}{Y}(a, z, \theta)|0\rangle &= a + O(z, \theta), \\ T|0\rangle = S^i|0\rangle &= 0, \qquad i = 1, \dots, n.\end{aligned} \tag{2.1.27.1}$$

- translation invariance

$$
\begin{aligned}
[S^i, \overset{s}{Y}(a,z,\theta)] &= \partial_{\theta^i} \overset{s}{Y}(a,z,\theta), \\
[T, \overset{s}{Y}(a,z,\theta)] &= \partial_z \overset{s}{Y}(a,z,\theta).
\end{aligned} \tag{2.1.27.2}
$$

- locality

$$
(z-w)^k [\overset{s}{Y}(a,z,\theta), \overset{s}{Y}(b,w,\zeta)] = 0 \quad \text{for } k \gg 0. \tag{2.1.27.3}
$$

Morphisms of $N_W = n$ SUSY vertex algebras are defined as before.

Example 2.1.28. We show here that the $N = 2$ vertex algebra carries a structure of a $N_W = 1$ SUSY vertex algebra. We will use the generating fields $\tilde{L}, Q, H$, and J with the commutation relations (2.1.19.5f). Define the superfields:

$$
\overset{s}{Y}(a,z,\theta) = Y(a,z) + \theta Y(Q_{-1}a, z), \tag{2.1.28.1}
$$

and let $T = \tilde{L}_{-1}$, $S = S^1 = Q_{-1}$. The vacuum axioms and the locality axioms are clear. For translation invariance, we have:

$$
[T, \overset{s}{Y}(a,z,\theta)] = [T, Y(a,z)] + \theta[T, Y(Q_{-1}a, z)] = \partial_z Y(a,z) + \theta\partial_z Y(Q_{-1}a, z), \tag{2.1.28.2}
$$

proving the even translation invariance axiom. For the odd translation invariance we have by (2.1.5.1) (recall that $Q_{-1} = Q_{(0)}$):

$$
[Q_{(0)}, \overset{s}{Y}(a,z,\theta)] \;=\; [Q_{(0)}, Y(a,z)] \;-\; \theta[Q_{(0)}, Y(Q_{(0)}a, z)] \;=\; Y(Q_{(0)}a, z). \tag{2.1.28.3}
$$

Note that defining the vectors $\nu = H_{(-1)}|0>$ and $\tau = -J_{(-1)}|0>$ we have in particular

$$
\begin{aligned}
\overset{s}{Y}(\nu, z, \theta) &= H(z) + \theta(\tilde{L}(z) + \partial_z J(z)), \\
\overset{s}{Y}(\tau, z, \theta) &= -J(z) + \theta Q(z).
\end{aligned} \tag{2.1.28.4}
$$

Therefore, if we consider the Fourier modes as defined in (2.1.20.1), we have

$$
\nu_{(0,0)} = T, \quad \tau_{(0,0)} = S. \tag{2.1.28.5}
$$

Moreover, it is easy to see that the field $\tilde{L}(z) + \partial_z J(z)$ is also a Virasoro field and the conformal weights of the generating fields $\tilde{L}, H, Q, J$ are positive with respect to this Virasoro field as well. It follows that the operator $\nu_{(1,0)}$ acts diagonally with non-negative eigenvalues.

This example motivates the following definition.

Definition 2.1.29. A *superconformal* $N_W = 1$ SUSY vertex algebra is a $N_W = 1$ SUSY vertex algebra with two vectors ν, τ satisfying the properties in the last example, i.e. (2.1.28.4), (2.1.28.5), where H, $\tilde{L}$, J and Q satisfy (2.1.19.5c), and $\nu_{(1,0)}$ is diagonalizable with real spectrum bounded below.

Example 2.1.30. More generally, we can define a conformal $N_W = N$ SUSY vertex algebra for each N as follows. Consider the superalgebra $A = \mathbb{C}[t, t^{-1}, \theta^1, \ldots, \theta^N]$ where t is even and θ^i are odd indeterminates. Let $W(1|N)$ be the Lie superalgebra of derivations of this algebra, and define the following collection of $W(1|N)$-valued formal distributions:

$$\mathscr{F} = \left\{ a^j(z) = \sum_{n \in \mathbb{Z}} (t^n a \partial_j) z^{-1-n},\ a \in A,\ j = 0, 1, \ldots, N \right\}, \tag{2.1.30.1}$$

where $\partial_j = \partial_{\theta^j}$ if $j > 0$ and $\partial_0 = \partial_t$. The pair $(W(1|N), \mathscr{F})$ is a *formal distribution Lie superalgebra* (cf. [15] and [22]). We can construct a Lie conformal superalgebra from a formal distribution Lie superalgebra according to [22]. In this case it is the $\mathbb{C}[\partial]$-module generated by the vectors a^j, with $a \in A$ and $j = 0, \ldots, N$, and the following λ-brackets:

$$\begin{aligned}
[a^i{}_\lambda b^j] &= (a\partial_i b)^j + (-1)^{p(a)}((\partial_j a)b)^i, \qquad i, j \geq 1, \\
[a^i{}_\lambda b^0] &= (a\partial_i b)^0 - (-1)^{p(b)}(ab)^i \lambda, \\
[a^0{}_\lambda b^0] &= -\partial(ab)^0 - 2(ab)^0 \lambda.
\end{aligned} \tag{2.1.30.2}$$

Let W_N be the associated universal enveloping vertex algebra. The field

$$L(z) = -1^0(z) + \sum_{i=1}^{n} \partial_z (\theta^i)^i(z), \tag{2.1.30.3}$$

is a Virasoro field, and the elements $(\theta^i)^j$ are primary of conformal weight 1, while the elements -1^i are primary of conformal weight 2. We will need later its Fourier modes, which are given by:

$$L_n = -t^{n+1}\partial_t - (n+1) \sum t^n \theta^i \partial_{\theta^i}. \tag{2.1.30.4}$$

We define $\nu = L_{(-1)}|0\rangle$ and $T = L_{(0)} = L_{-1} = -\partial_t$. In order to be consistent with previous notation we define the fields $Q^i(z) = -1^i(z)$ and write down their Fourier modes which are

$$Q^i_n = -t^{n+1}\partial_{\theta^i}. \tag{2.1.30.5}$$

In particular, we define $S^i = Q^i_{-1}$ for $i \geq 1$ and note that $(S^i)^2 = 0$.

In order to construct an $N_W = N$ SUSY vertex algebra from the vertex algebra W_N we proceed as before, defining the superfields

$$\overset{s}{Y}(a, z, \theta^1, \ldots, \theta^N) = \sum_I (-1)^{\frac{I(I-1)}{2}} \theta^I Y(S^{i_1} \ldots S^{i_s} a, z), \tag{2.1.30.6}$$

where the summation is taken over all ordered subsets $I = (i_1, \ldots, i_s)$ of $\{1, \ldots, N\}$.

It is straightfoward to check that the data $(W_N, T, S^i, |0\rangle, \overset{s}{Y})$ is indeed an $N_W = N$ SUSY vertex algebra. We shall return to this example in 3.6.1.

Example 2.1.31. We can similarly construct an $N_K = N$ SUSY vertex algebra for any N. For this we define a subalgebra $K(1|N)$ of $W(1|N)$, of those differential operators preserving the form $\omega = dt + \sum \theta^i d\theta^i$ up to multiplication by a function. It consists of differential operators of the form:

$$D^f = f\partial_0 + \frac{1}{2}(-1)^{p(f)} \sum_{i=1}^{N} (\theta^i\partial_0 + \partial_i)(f)(\theta_i\partial_0 + \partial_i), \tag{2.1.31.1}$$

for $f \in A$. These operators satisfy

$$[D^f, D^g] = D^{\{f,g\}}, \tag{2.1.31.2}$$

where

$$\{f,g\} = \left(f - \frac{1}{2}\sum_{i=1}^{N} \theta^i \partial_i f\right) \partial_0 g - \partial_0 f \left(g - 1\frac{1}{2}\sum_{i=1}^{N} \theta^i\partial_i g\right) + (-1)^f \frac{1}{2}\sum_{i=1}^{N} \partial_i f \partial_i g. \tag{2.1.31.3}$$

In particular $K(1|N)$ contains the operators

$$\begin{aligned} L_n &= -t^{n+1}\partial_t - \frac{n+1}{2} t^n \sum \theta^i \partial_{\theta^i}, \qquad n \in \mathbb{Z}, \\ G_n^{(i)} &= -t^{n+1/2}(\partial_{\theta^i} - \theta^i\partial_t) + \left(n + \frac{1}{2}\right) t^{n-1/2}\theta^i \sum \theta^j \partial_{\theta^j}, \qquad n \in \frac{1}{2} + \mathbb{Z}. \end{aligned} \tag{2.1.31.4}$$

It is easy to see that the operators L_n span a centerless Virasoro Lie algebra.

As in the $W(1|N)$ case, we construct the corresponding Lie conformal superalgebra as follows. It is the $\mathbb{C}[\partial]$-module generated by vectors $a \in \mathbb{C}[\theta^1, \dots, \theta^n]$, with the following λ-brackets [15]

$$[a_\lambda b] = \left(\left(\frac{r}{2} - 1\right)\partial(ab) + (-1)^r \frac{1}{2}\sum_{i=1}^{n} \partial_i a \partial_i b\right) + \lambda\left(\frac{r+s}{2} - 2\right) ab, \tag{2.1.31.5}$$

where $a = \theta_{i_1} \dots \theta_{i_r}$, $b = \theta_{j_1} \dots \theta_{j_s}$.

We denote by K_N its universal enveloping vertex algebra, and we define the operators $T = L_{-1}$ and $S^i = G_{-1/2}^{(i)}$. Now we define the state-field correspondence as in (2.1.30.6):

$$\overset{s}{Y}(a, z, \theta) = \sum_I (-1)^{\frac{I(I-1)}{2}} \theta^I Y(S^I a, z). \tag{2.1.31.6}$$

All the properties of an $N_K = N$ SUSY vertex algebra are straightforward to check as in the previous cases. In particular we note that the Lie subalgebra of regular derivations preserving ω acts in this SUSY vertex algebra as Fourier coefficients of vertex operators, we call this subalgebra the anihilation subalgebra of $K(1|N)$. We will return to this example in 3.6.5.

2.1.32. There is another infinite series of superconformal Lie algebras, namely the series s_n. The corresponding Lie algebra is the Lie algebra of divergence free elements of w_n. Recall that the divergence of the derivation $D = P_0\partial_t + \sum P^i\partial_{\theta^i}$ is defined to be

$$\mathrm{Div}D = \partial_t P_0 + \sum (-1)^{p(P_i)}\partial_{\theta^i}P_i. \tag{2.1.32.1}$$

Now we see that the following operators are in particular divergence free:

$$\begin{aligned} L_m &= -t^{m+1}\partial_t - \frac{m+1}{n}t^m\sum\theta^i\partial_{\theta^i} \qquad m\in\mathbb{Z}, \\ Q^{(i)}_m &= -t^{m+1/n}\partial_{\theta^i} \qquad m\in -\frac{1}{n}+\mathbb{Z}. \end{aligned} \tag{2.1.32.2}$$

It is easy to see that the corresponding field $L(z)$ is a Virasoro field (of zero central charge) and $Q^{(i)}(z)$ are primary fields of conformal weight $(n+1)/n$.

With the prescription in [22], we associate to this formal distribution Lie algebra a Lie conformal superalgebra s_n. We let S_n be the associated vertex algebra and we see that we get a $N_W = n$ vertex algebra by the above procedure. The difference now is that if we define $\nu = L_{(-1)}|0\rangle = L_{-2}|0\rangle$ then since we have

$$[L_{(-1)}, Q^{(i)}_{(0)}] = -\frac{1}{n}(\partial Q^{(i)})_{(-1)} \tag{2.1.32.3}$$

we get that the superfield associated to ν is given by:

$$\overset{s}{Y}(\nu,z,\theta) = L(z) + \frac{1}{n}\sum\theta^i\partial_z Q^{(i)}(z) + O(\theta^i\theta^j) \tag{2.1.32.4}$$

Remark 2.1.33. It is known that the Lie conformal superalgebra s_N constructed above admits a non-trivial central extension only when $N=2$. In this case, considering the corresponding vertex algebra we obtain what is called the $N=4$ vertex algebra. Not to confuse with the above notation we will call this algebra S_2. The even part of S_2 is given by the Virasoro field L, and three currents of conformal weight 1, J^i, $(i=0,1,2)$ generating the current algebra for $\mathfrak{sl}_2$. In terms of vector fields as above we have

$$\begin{aligned} J^1_n &= t^n\theta^1\partial^2_\theta, \\ J^2_n &= t^n\theta^2\partial_{\theta^1}, \\ J^0_n &= t^n\left(\theta^1\partial_{\theta^1} - \theta^2\partial_{\theta^2}\right). \end{aligned} \tag{2.1.33.1}$$

The odd part is generated by four fields of conformal weight 3/2, their Fourier modes are given by:

$$\begin{aligned} G^i_n &= -t^{n+1/2}\partial_{\theta^i} \qquad i=1,2, \\ H^1_n &= t^{n+1/2}\theta^1\partial_t - \left(n+\frac{1}{2}\right)t^{n-1/2}\theta^1\theta^2\partial_{\theta^2}, \\ H^2_n &= t^{n+1/2}\theta^2\partial_t - (n+1/2)\,t^{n-1/2}\theta^2\theta^1\partial_{\theta^1}, \end{aligned} \tag{2.1.33.2}$$

and the central extension of s_2 is given by the following cocycle [15]:

$$\alpha_3(L,L) = \frac{c}{2} \qquad \alpha_1(J^1,J^2) = \frac{c}{6} \tag{2.1.33.3}$$

$$\alpha_1(J^0,J^0) = \frac{c}{6} \qquad \alpha_2(G^i,H^i) = -\frac{c}{3} \tag{2.1.33.4}$$

2.2 Supercurves

2.2.1. For a general introduction to the theory of supermanifolds and schemes, the reader should refer to [29]. We are going to follow [7] for the theory of *supercurves* over a Grassmann algebra Λ. The deformation theory of superspaces and sheaves over them can be found in [34]. The relations between superconformal Lie algebras and the moduli spaces of supercurves was stated in [35]. The reader may also find useful the notes [13].

Definition 2.2.2. A *superspace* is a locally ringed space $(X, \mathscr{O}_X)$ where X is a topological space and $\mathscr{O}_X$ is a sheaf of supercommutative rings. A *morphism* of superspaces is a graded morphism of locally ringed spaces. We will use X to denote such a superspace when no confusion should arise. A *superscheme* is a superspace such that $(X, \mathscr{O}_{X,\bar{0}})$ is a scheme, where from now on $\mathscr{O}_{X,i}$ denotes the i-th graded part of $\mathscr{O}_X$, $i = \bar{0}, \bar{1}$.

2.2.3. Given a superspace $(X, \mathscr{O}_X)$ define $\mathscr{J} = \mathscr{O}_{X,\bar{1}} + \mathscr{O}_{X,\bar{1}}^2$. Clearly $\mathscr{J}$ is a sheaf of ideals in $(X, \mathscr{O}_X)$ and the corresponding subspace $(X, \mathscr{O}_X/\mathscr{J})$ will be denoted $(X_{\mathrm{rd}}, \mathscr{O}_{X_{\mathrm{rd}}})$.

Example 2.2.4. Let R be a supercommutative ring, and let $J = R_1 + R_{\bar{1}}^2$ be the ideal generated by $R_{\bar{1}}$ as above, then $(\operatorname{Spec} R, R)$ is a superscheme. Note that as topological spaces $\operatorname{Spec} R = \operatorname{Spec} R/J$ since every element in J is nilpotent (we consider only homogeneous ideals with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading).

Definition 2.2.5 (cf. [29]). A *supermanifold* is a superspace $(X, \mathscr{O}_X)$ such that for every point $x \in X$ there exists an open neighborhood U of x and a locally free sheaf $\mathscr{E}$ of $\mathscr{O}_{X_{\mathrm{rd}}}|_U$-modules, of (purely odd) rank $0|q$ such that $(U, \mathscr{O}_X|_U)$ is isomorphic to $(U_{\mathrm{rd}}, S_{\mathscr{O}_{X_{\mathrm{rd}}}}(\mathscr{E})|_U)$. Here $S(\mathscr{E})$ denotes the symmetric algebra of a (purely odd) vector bundle.

2.2.6. An open sub-supermanifold of $(X, \mathscr{O}_X)$ consist of an open subset $U \subset X$ and the restriction of the structure sheaf, namely $(U, \mathscr{O}_X|_U)$.

2.2.7. In the analytic setting, the situation is easier to describe. The supermanifold $\mathbb{C}^{p|q}$ is the topological space $\mathbb{C}^p$ endowed with the sheaf of supercommutative algebras $\mathscr{O}[\theta_1, \ldots, \theta_q]$ where $\mathscr{O}$ is the sheaf of germs of holomorphic functions on $\mathbb{C}^p$ and θ_i are odd anticommuting variables. A supermanifold is a topological space $|X|$ with a sheaf of supercommutative algebras $\mathscr{O}_X$ locally isomorphic to $\mathbb{C}^{p|q}$. Morphisms of supermanifolds are continuous maps $\sigma : |X| \longrightarrow |Y|$ together with morphisms of sheaves $\sigma^\sharp : \sigma_* \mathscr{O}_Y \longrightarrow \mathscr{O}_X$.

Let $\Lambda = \mathbb{C}[\alpha_1, \dots, \alpha_n]$ be a Grassmann algebra as before. The $0|n$-dimensional superscheme $\operatorname{Spec}\Lambda$ has as underlying topological space a single point. We will work in the category of superschemes over Λ, namely super schemes S together with a structure morphism $S \to \operatorname{Spec}\Lambda$. In the case when S is a proper, smooth of relative dimension $1|q$ super-scheme, we say that S is a $N = q$ *supercurve* (over Λ).

Definition 2.2.8. More explicitly (cf. [7]), a smooth compact connected complex supercurve over Λ of dimension $1|N$ is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of supercommutative Λ-algebras over X equipped with a structure morphism $(X, \mathcal{O}_X) \to \operatorname{Spec}\Lambda$ such that:

1. $(X, \mathcal{O}_X^{\text{red}})$ is a smooth compact connected algebraic curve. Here $\mathcal{O}_X^{\text{red}}$ is the reduced sheaf of $\mathbb{C}$-algebras on X obtained by quotienting out the nilpotents in $\mathcal{O}_X$.

2. For some open sets $U_\alpha \subset X$ and some linearly independent odd elements θ_α^i of $\mathcal{O}_X(U_\alpha)$ we have:
$$\mathcal{O}_X(U_\alpha) = \mathcal{O}_X^{\text{red}} \otimes \Lambda[\theta_\alpha^1, \dots, \theta_\alpha^N]. \tag{2.2.8.1}$$

The U_α above are called *coordinate neighborhoods* of $(X, \mathcal{O}_X)$ and $Z_\alpha = (z_\alpha, \theta_\alpha^1, \dots, \theta_\alpha^N)$ are called local coordinates for $(X, \mathcal{O}_X)$ if z_α (mod nilpotents) are local coordinates for $(X, \mathcal{O}_X^{\text{red}})$. On overlaps $U_\alpha \cap U_\beta$ we have:

$$z_\beta = F_{\beta\alpha}(z_\alpha, \theta_\alpha^j), \quad \theta_\beta^i = \Psi_{\beta\alpha}^i(z_\alpha, \theta_\alpha^j), \tag{2.2.8.2}$$

where $F_{\beta\alpha}$ are even and $\Psi_{\beta\alpha}$ are odd. We will write such a change of coordinates as $Z_\beta = \rho_{\beta,\alpha}(Z_\alpha)$ with $\rho = (F, \Psi^i)$ where no confusion should arise.

2.2.9. A Λ-point of a supercurve $(X, \mathcal{O}_X)$ is a morphism $\varphi : \operatorname{Spec}\Lambda \to (X, \mathcal{O}_X)$ over Λ, namely the composition of φ with the structure morphism $(X, \mathcal{O}_X) \to \operatorname{Spec}\Lambda$ is the identity. Locally, a Λ point is given by specifying the images of the local coordinates under the even Λ-homomorphism $\varphi^\sharp : \mathcal{O}_X(U_\alpha) \to \Lambda$. These local parameters ($p_\alpha = \varphi^\sharp(z_\alpha), \pi_\alpha^i = \varphi^\sharp(\theta_\alpha^i)$) transform as the coordinates do in (2.2.8.2).

2.2.10. The $N = q$ formal superdisk is an ind superscheme as in the non-super situation, namely, let $R = \mathbb{C}[t, \theta^1, \dots, \theta^q]$ and let $\mathfrak{m}$ be the maximal ideal generated by $(t, \theta^1, \dots, \theta^q)$. We define the superschemes $D^{(n)} = \operatorname{Spec} R/\mathfrak{m}^{n+1}$ and we clearly have embeddings $D^{(n+1)} \hookrightarrow D^{(n)}$. The formal disk is then

$$D = \varinjlim_{n \to \infty} D^{(n)}. \tag{2.2.10.1}$$

If we want to emphasize the dimensions of these disks we will denote them by $D^{1|q}$.

2.2.11. Vector bundles of rank $(p|q)$ over a supermanifold $(X, \mathcal{O}_X)$ are locally free sheaves $\mathcal{E}$ of $\mathcal{O}_X$-modules over X, of rank $p|q$. That is, locally, $\mathcal{E}$ is isomorphic to $\mathcal{O}_X^p \oplus (\Pi\mathcal{O}_X)^q$ where Π is the parity change operator.

An example is the *tangent bundle* to a $p|q$-dimensional supermanifold $(X, \mathcal{O}_X)$; it is a rank $p|q$ vector bundle. Its fiber at the point $x \in X$ is given as in the non-super case as the subset of morphisms in $\mathrm{Hom}(D^{(1)}, X)$ mapping the closed point in $D^{(1)}$ to x. The *cotangent bundle* Ω^1_X of $(X, \mathscr{O}_X)$ is the dual of the tangent bundle.

Another example is the *Berezinian bundle* of a supermanifold $(X, \mathscr{O}_X)$. We will define this bundle by giving local trivializations. Recall [13, §1.10] that given a free module L of finite type over a supercommutative algebra A, the superdeterminant is a homomorphism

$$\mathrm{sdet} : \mathrm{GL}(L) \to \mathrm{GL}(1|0) = A_0^\times, \tag{2.2.11.1}$$

defined in coordinates as follows: for a parity preserving automorphism T of $A^{p|q}$ with matrix $\left(\begin{smallmatrix} K & L \\ M & N \end{smallmatrix}\right)$ we put:

$$\mathrm{sdet}(T) = \det(K - LN^{-1}M)\det(N)^{-1}. \tag{2.2.11.2}$$

With this definition we can now define the Berezinian of the module L as the following A-module denoted $\mathrm{Ber}(L)$. Let $\{e_1, \ldots, e_{p+q}\}$ be a basis of L where the first p elements are even and the last q are odd. This basis defines a one-element basis of $\mathrm{Ber}(L)$ denoted by $[e_1 \ldots e_{p+q}]$ of parity $q \mod 2$. Given an automorphism T of L we put

$$[Te_1 \ldots Te_{p+q}] = \mathrm{sdet}(T)[e_1 \ldots e_{p+q}]. \tag{2.2.11.3}$$

This makes $\mathrm{Ber}(L)$ a well defined rank $1|0$ A-module when q is even and a rank $0|1$ A-module when q is odd. Now we can define the Berezinian bundle of $(X, \mathscr{O}_X)$ as $\mathrm{Ber}_X = \mathrm{Ber}(\Omega^1_X)$.

The definition of coherent and quasi-coherent sheaves is exactly the same as in the non-super case, in particular for super manifolds it follows that the structure sheaf is coherent [34].

2.2.12. Given an $N = 1$ supercurve $(X, \mathscr{O}_X)$ and an extension of $\mathscr{O}_X$ by an invertible sheaf $\mathscr{E}$:

$$0 \to \mathscr{O}_X \to \hat{\mathscr{E}} \to \mathscr{E} \to 0, \tag{2.2.12.1}$$

we can construct an $N = 2$ supercurve $(Y, \mathscr{O}_Y)$ canonically. Its local coordinates are given by $(z_\alpha, \theta_\alpha, \rho_\alpha)$, where $(z_\alpha, \theta_\alpha)$ are local coordinates of X and ρ_α are local sections of $\mathscr{E}$. In each coordinate patch U_α we can construct the form $dz_\alpha - d\theta_\alpha \rho_\alpha$. We say that the $N = 2$ supercurve $(Y, \mathscr{O}_Y)$ is *superconformal* if this form is globally defined up to multiplication by a function.

This happens if on overlaps $U_\alpha \cap U_\beta$ we have (see (2.2.8.2))

$$\rho_\beta = \mathrm{sdet}\begin{pmatrix} \partial_z F & \partial_z \Psi \\ \partial_\theta F & \partial_\theta \Psi \end{pmatrix} \rho_\alpha + \frac{\partial_\theta F}{\partial_\theta \Psi}. \tag{2.2.12.2}$$

Here sdet is the *superdeterminant* of an automorphism defined above, which can be written as

$$\mathrm{sdet}\begin{pmatrix} \partial_z F & \partial_z \Psi \\ \partial_\theta F & \partial_\theta \Psi \end{pmatrix} = D\left(\frac{DF}{D\Psi}\right), \tag{2.2.12.3}$$

where $D = \partial_\theta + \theta\partial_z$.

Conversely, if (2.2.12.2) is satisfied on overlaps, the cocycle condition is satisfied and we have an extension as in (2.2.12.1).

Therefore to each $N = 1$ supercurve $(X, \mathcal{O}_X)$, we canonically associate a $N = 2$ superconformal curve $(Y, \mathcal{O}_Y)$.

From (2.2.12.2) we see that we have an exact sequence of sheaves on Y:

$$0 \to \mathcal{O}_X \to \mathcal{O}_Y \to \mathrm{Ber}_X \to 0, \tag{2.2.12.4}$$

where Ber_X is the Berezinian bundle on $(X, \mathcal{O}_X)$. The last map $\hat{D} : \mathcal{O}_Y \to \mathrm{Ber}_X$ is given in the above local coordinates, by the differential operator ∂_{ρ_α}.

Introducing new coordinates

$$\begin{aligned} \hat{z}_\alpha &= z_\alpha - \theta_\alpha \rho_\alpha, \\ \hat{\theta}_\alpha &= \theta_\alpha \\ \hat{\rho}_\alpha &= \rho_\alpha, \end{aligned} \tag{2.2.12.5}$$

we obtain on overlaps $U_\alpha \cap U_\beta$:

$$\begin{aligned} \hat{z}_\beta &= F(\hat{z}_\alpha, \hat{\rho}_\alpha) + \frac{DF(\hat{z}_\alpha, \hat{\rho}_\alpha)}{D\Psi(\hat{z}_\alpha, \hat{\rho}_\alpha)} \Psi(\hat{z}_\alpha, \hat{\rho}_\alpha), \\ \hat{\rho}_\beta &= \frac{DF(\hat{z}_\alpha, \hat{\rho}_\alpha)}{D\Psi(\hat{z}_\alpha, \hat{\rho}_\alpha)}, \end{aligned} \tag{2.2.12.6}$$

where $D = \partial_\theta + \theta \partial_z$ in local coordinates (z, θ, ρ) as above.

We see from (2.2.12.6) that $\mathcal{O}_Y$ contains the structure sheaf of another $N = 1$ supercurve $(\hat{X}, \mathcal{O}_{\hat{X}})$, whose local coordinates are $(\hat{z}_\alpha, \hat{\rho}_\alpha)$. We call $(\hat{X}, \mathcal{O}_{\hat{X}})$ the *dual curve* of $(X, \mathcal{O}_X)$.

Finally, we define an $N = 1$ superconformal curve as an $N = 1$ supercurve $(X, \mathcal{O}_X)$ which is self-dual. We see from (2.2.12.6) that the transition functions F, Ψ must satisfy

$$DF = \Psi D\Psi, \tag{2.2.12.7}$$

for $(X, \mathcal{O}_X)$ to be superconformal. In this case the operator $D_\alpha = \partial_{\theta_\alpha} + \theta_\alpha \partial_{z_\alpha}$ transforms as

$$D_\beta = (D\Psi)^{-1} D_\alpha \tag{2.2.12.8}$$

hence in this situation the supercurve $(X, \mathcal{O}_X)$ carries a $0|1$-dimmensional distribution D such that D^2 is nowhere vanishing (since $D^2 = \partial_z$ in local coordinates).

Remark 2.2.13. An equivalent definition of $N = 1$ and $N = 2$ superconformal curves was given by Manin [28] (under the name SUSY curves). Let X be a complex supermanifold of dimension $1|N$ ($N = 1$ or 2). When $N = 1$ we say that a locally free direct subsheaf $\mathcal{T}^1 \subset \mathcal{T}_X$ ($\mathcal{T}_X$ is the tangent sheaf of X) of rank $0|1$ for which the Frobenius form

$$(\mathcal{T}^1)^{\otimes 2} \to \mathcal{T}^0 := \mathcal{T}_X / \mathcal{T}^1, \quad t_1 \otimes t_2 \mapsto [t_1, t_2] \mod \mathcal{T}^1 \tag{2.2.13.1}$$

is an isomorphism, is a SUSY structure on X.

When $N=2$, a SUSY structure consists of two locally free direct subsheaves $\mathcal{T}'$, $\mathcal{T}''$ of $\mathcal{T}_X$ of rank $0|1$ whose sum in $\mathcal{T}_X$ is direct, they are integrable distributions and the Frobenius form

$$\mathcal{T}'\otimes\mathcal{T}''\to\mathcal{T}_X/(\mathcal{T}'\oplus\mathcal{T}''),\quad t_1\otimes t_2\mapsto[t_1,t_2]\mod(\mathcal{T}'\oplus\mathcal{T}'')\qquad(2.2.13.2)$$

is an isomorphism.

Let $(X,\mathcal{O}_X)$ be an $N=1$ supercurve and D_α be and a family of vector fields in U_α, such that D_α and D_α^2 form a basis for $\mathcal{T}_X$ on U_α and $D_\alpha=G_{\alpha\beta}D_\beta$ on $U_\alpha\cap U_\beta$, where $G_{\alpha\beta}$ is a family of invertible even functions. The sheaf defined by $\mathcal{T}^1|_{U_\alpha}=\mathcal{O}_XD_\alpha$ is a SUSY structure in $(X,\mathcal{O}_X)$ [28]. In local coordinates as above, the vector fields $D_\alpha=\partial_{\theta_\alpha}+\theta_\alpha\partial_{z_\alpha}$ satisfy these conditions when X is an $N=1$ superconformal curve (see (2.2.12.8)).

The $N=2$ case is similar. Let $(X,\mathcal{O}_X)$ be an $N=2$ supercurve and $\{D_\alpha^1,\,D_\alpha^2\}$ be a family of vector fields such that D_α^i, $[D_\alpha^1,D_\alpha^2]$ generate $\mathcal{T}_X$ in U_α and, moreover, we have:

$$(D_\alpha^1)^2=f_\alpha^1D_\alpha^1;\qquad (D_\alpha^2)^2=f_\alpha^2D_\alpha^2;\qquad(2.2.13.3)$$
$$D_\alpha^1=F_{\alpha,\beta}^1D_\beta^1\qquad D_\alpha^2=F_{\alpha\beta}^2D_\beta^2\quad\text{on } U_\alpha\cap U_\beta\qquad(2.2.13.4)$$

where f_α^i and $F_{\alpha\beta}^i$ are even functions. Putting $\mathcal{T}'|_{U_\alpha}=\mathcal{O}_XD_\alpha^1$ and $\mathcal{T}''|_{U_\alpha}=\mathcal{O}_XD_\alpha^2$ we obtain an $N=2$ superconformal structure on $(X,\mathcal{O}_X)$. If the two distributions $\mathcal{T}'$ and $\mathcal{T}''$ can be distinguished globally, the $N=2$ superconformal curve is called orientable and a choice of one of these distributions is called its orientation.

It is clear that the construction given in 2.2.12 gives an oriented $N=2$ superconformal curve; conversely, given such a curve, we can consider the functor $X\to X/\mathcal{T}'$ (recall that $\mathcal{T}'$ is integrable therefore this quotient makes sense). The duality that was explained in 2.2.12 corresponds to the duality $X/\mathcal{T}'\leftrightarrow X/\mathcal{T}''$.

2.2.14. Recall [7] that a Λ-point of an $N=1$ supercurve X transforms as an irreducible divisor of the dual curve $\hat{X}$. Indeed, an irreducible divisor of X is given in local coordinates (z_α,θ_α) by expressions of the form $P_\alpha=z_\alpha-\hat{z}_\alpha-\theta_\alpha\rho_\alpha$. Two divisors P_α and P_β are said to correspond to each other in the intersection $U_\alpha\cap U_\beta$ if in this intersection we have

$$P_\beta(z_\beta,\theta_\beta)=P_\alpha(z_\alpha,\theta_\alpha)g(z_\alpha,\theta_\alpha)\qquad(2.2.14.1)$$

for some even invertible function $g(z_\alpha,\theta_\alpha)$ (we consider Cartier divisors). It is easy to see that the parameters $\hat{z}_\alpha,\rho_\alpha$ transform as in (2.2.12.6), namely as the parameters of a Λ-point of $\hat{X}$.

2.2.15. We can define a theory of contour integration on an $N=1$ superconformal curve as in [19], [30], [32]. We describe briefly a generalization to arbitrary $N=1$ supercurves due to Bergvelt and Rabin (cf. [7]). For simplicity we will work in the analytic category. Let us define a *super contour* to be a triple $\Gamma=(\gamma,P,Q)$ consisting of an ordinary contour γ on the reduction $|X|$ and two Cartier divisors as in 2.2.14

such that their reductions to $|X|$ are the endpoints of γ. If in local coordinates

$$P = z - \hat{p} - \theta\hat{\pi}, \qquad Q = z - \hat{q} - \theta\hat{\xi}, \tag{2.2.15.1}$$

then the corresponding Λ-points of the dual curve $\hat{X}$ are given by $(\hat{p}, \hat{\pi})$ and $(\hat{q}, \hat{\xi})$. Let $z = \hat{p}_{\mathrm{rd}}$ and $z = \hat{q}_{\mathrm{rd}}$ be the equations for the reductions of these points, i.e. the endpoints for γ. We define the integral of a section $\omega_\alpha = D\hat{f}_\alpha$ of the Berezinian sheaf of X (here we recall that $D : \mathscr{O}_{\hat{X}} \twoheadrightarrow \operatorname{Ber} X$) along Γ by:

$$\int_P^Q \omega = \int_P^Q D\hat{f} = \hat{f}(\hat{q}, \hat{\xi}) - \hat{f}(\hat{p}, \hat{\pi}). \tag{2.2.15.2}$$

Here we assume that the contour connecting P and Q lies in a single simply connected open set U_α. If the contour traverses several open sets then we need to choose intermediate divisors on each overlap and we have to prove that the resulting integral is independent of these divisors. In what follows we will only need the integration in a sufficiently "small" open set U_α (the formal disk around a point).

Dually, we can integrate sections of $\operatorname{Ber}_{\hat{X}}$ along contours in X. Indeed, let γ be a path in the topological space $|X|$ and two Λ-points P, Q of X whose reduced parts are the end-points of γ. Let $\hat{\omega} \in \operatorname{Ber}_{\hat{X}}(U_\alpha)$ and suppose that γ lies in a simply connected open U_α. Then $\hat{\omega} = \hat{D}f$ for some function $f \in \mathscr{O}_X(U_\alpha)$, and we put

$$\int_P^Q \hat{\omega} = f(Q) - f(P). \tag{2.2.15.3}$$

As it is shown in [7], this theory of integration can be understood in terms of a theory of contour integration on the corresponding $N = 2$ superconformal curve (cf. [11]). For this let X and $\hat{X}$ be an $N = 1$ supercurve and its dual. Let Y be the corresponding $N = 2$ superconformal curve and denote by π and $\hat{\pi}$ the corresponding projections to X and $\hat{X}$ respectively. We have two short exact sequences of sheaves in Y:

$$\begin{aligned} 0 \to \pi^*\mathscr{O}_X \to \mathscr{O}_Y \xrightarrow{D^-} \hat{\pi}^* \operatorname{Ber}_{\hat{X}} \to 0, \\ 0 \to \hat{\pi}^*\mathscr{O}_{\hat{X}} \to \mathscr{O}_Y \xrightarrow{D^+} \pi^* \operatorname{Ber}_X \to 0. \end{aligned} \tag{2.2.15.4}$$

We can define a sheaf operator on $\mathscr{O}_Y^{\oplus 2}$ by the component-wise action of the differential operators (D^-, D^+). It is shown in [7] that for U a simply connected open in $|Y| = |X|$ and (f, g) a section of $\mathscr{O}_Y^{\oplus 2}(U)$ such that $(D^-, D^+)(f, g) = 0$, there exists a section $H \in \mathscr{O}_Y(U)$, unique up to an additive constant, such that $(f, g) = (D^- H, D^+ H)$. Let $\mathscr{M}$ be the subsheaf of $\mathscr{O}_Y^{\oplus 2}$ consisting of closed sections (f, g) as above. It follows that $\mathscr{M} = \pi^* \operatorname{Ber}_X \oplus \hat{\pi}^* \operatorname{Ber}_{\hat{X}}$. A super contour in Y consists of a triple (γ, P, Q) where P and Q are Λ-points of Y such that their reduced points are the endpoints of γ. If γ is supported on a simply connected open set U, then any section $\omega \in \mathscr{M}(U)$ can be written as $(D^- H, D^+ H)$ and we put

$$\int_P^Q \omega = H(Q) - H(P), \tag{2.2.15.5}$$

The extension to contours not lying in a single simply connected U is straightforward but we will not need it.

2.2.16. We will define in general a superconformal $N = n$ supercurve to be a curve such that in some coordinate system $Z_\alpha = (z_\alpha, \theta^i_\alpha)$ the differential form

$$\omega = dz_\alpha + \sum_i \theta^i_\alpha d\theta^i_\alpha \tag{2.2.16.1}$$

is well defined up to multiplication by a function. It is easy to show that this definition agrees with the definition above in the $N = 1$ and $N = 2$ cases (cf. §4.1.4 and §4.1.5).

A set of coordinates $Z = (z, \theta^i)$ such that the form ω has the form (2.2.16.1) (up to multiplication by a function) will be called *SUSY* coordinates (or *coordinates compatible with the superconformal structure*).

2.2.17. Let $(X, \mathcal{O}_X)$ be an $N = 1$ superconformal curve as above. Denote by $\mathscr{D}$ the sheaf of relative differential operators $\mathcal{O}_X \to \mathcal{O}_X$ over Λ. If $(z_\alpha, \theta_\alpha)$ is a local coordinate system, $\mathscr{D}$ is generated by D_α in the sense that any section of $\mathscr{D}$ can be locally written as $\sum a_i D^i$ where a_i are Λ-valued functions. If $a_d \neq 0$ is the highest non-zero coefficient in such expansion, we define the *superorder* sord of such an operator to be $d/2$; in particular $\operatorname{sord}_{(z,\theta)}(\partial_z) = 1$.

Lemma 2.2.18 ([29]Lemma 4.3). *Let (z, θ) and (z', θ') be two local coordinates on an $N = 1$ supercurve $(X, \mathcal{O}_X)$. We denote $D = \partial_\theta + \theta\partial_z$ and $D' = \partial_{\theta'} + \theta'\partial_{z'}$. The following are equivalent:*

1. *(z, θ) and (z', θ') are compatible with the same (local) superconformal structure.*

2. *$D'z = \theta D'\theta$.*

3. *For some integer $i \geq 0$*

$$\operatorname{sord}_{(z,\theta)}\left((D')^{2i+1}\right) = (2i+1) = \operatorname{sord}_{(z',\theta')}(D^{2i+1}). \tag{2.2.18.1}$$

4. *The induced filtrations of $\mathscr{D}$ by $\operatorname{sord}_{(z,\theta)}$ and $\operatorname{sord}_{(z',\theta')}$ coincide.*

2.2.19. Let (z, θ) and (z', θ') be two local coordinates compatible with a (local) superconformal structure on an $N = 1$ supercurve $(X, \mathcal{O}_X)$. Let G be the invertible function such that

$$D = GD' \tag{2.2.19.1}$$

We define the *Schwarzian derivative* of (z', θ') with respect to (z, θ) to be the (odd) function

$$\sigma = \frac{D^3 G}{G} - 2\frac{DGD^2G}{G^2}. \tag{2.2.19.2}$$

Definition 2.2.20. A *superprojective* structure on an $N = 1$ superconformal curve over Λ is a (maximal) atlas consisting of coordinates $(z_\alpha, \theta_\alpha)$ compatible with the superconformal structure (2.2.12.7) and such that its transition functions are fractional linear transformations, that is, changes of coordinates of the form:

$$\begin{aligned} z' &= \frac{az+b+\alpha\theta}{cz+d+\beta\theta}, \\ \theta' &= \frac{\gamma z+\delta+e\theta}{cz+d+\beta\theta}, \end{aligned} \tag{2.2.20.1}$$

for some even constants a, b, c, d and $e \in \Lambda$ and some odd constants α, β, γ and $\delta \in \Lambda$, such that

$$\operatorname{sdet}\begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & e \end{pmatrix} = 1 \tag{2.2.20.2}$$

Proposition 2.2.21 ([29]Proposition 4.7). *Let (z,θ) and (z',θ') be two local coordinates on $(X, \mathscr{O}_X)$. The following statements are equivalent:*

1. *(z,θ) and (z',θ') are compatible with a common superconformal structure and $\sigma = 0$.*

2. *(z,θ) and (z',θ') define the same superprojective structure.*

2.2.22. Let $(X, \mathscr{O}_X)$ be a superconformal $N = 1$ curve with a superprojective structure. Let $\mathscr{T} \subset \mathscr{T}_X$ be the locally free locally direct subsheaf of the tangent sheaf $\mathscr{T}_X$ generated by the distribution D defined above. Put $\omega^i = \mathscr{T}^{\otimes(-i)}$. We have an operator $L : \omega^{-1} \to \omega^2$ defined as

$$L : \ aD \mapsto D^3 a \cdot D^{-2}. \tag{2.2.22.1}$$

Here D^{-2} is a section of ω^2 and not an operator. D is defined in terms of a coordinate system locally as $\partial_\theta + \theta\partial_z$, and the operator L is independent of the coordinates chosen as long as they define the same superprojective structure. This operator L is called the *associated* operator to the superprojective structure.

Chapter 3

Structure theory of SUSY vertex algebras

In this chapter we develop the structure theory of SUSY Lie conformal algebras and SUSY vertex algebras along the lines of [22] (see also [12] for a better exposition). Proofs are rather straightforwards adaptations of those in the vertex algebra case, the only difficulty being the problem of signs.

3.1 Formal distribution calculus and notation

3.1.1. In what follows we fix the ground field to be the complex numbers $\mathbb{C}$ and N to be a non-negative integer. Let $\theta^1, \dots, \theta^N$ be Grassmann variables and $I = \{i_1, \dots, i_k\}$ be an *ordered k-tuple:* $1 \leq i_1 < \dots < i_k \leq N$. We will denote

$$\theta^I = \theta^{i_1} \dots \theta^{i_k}, \qquad \theta^N = \theta^1 \dots \theta^N. \tag{3.1.1.1}$$

For an element a in a vector superspace we will denote $(-1)^a = (-1)^{p(a)}$, where $p(a) \in \mathbb{Z}/2\mathbb{Z}$ is the parity of a, and, given a k-tuple I as above we will let $(-1)^I = (-1)^k$.

Given two disjoint ordered tuples I and J, we define $\sigma(I, J) = \pm 1$ by

$$\theta^I \theta^J = \sigma(I, J) \theta^{I \cup J}, \tag{3.1.1.2}$$

and we define $\sigma(I, J)$ to be zero if $I \cap J \neq \emptyset$. Also, unless noted otherwise, all "union" symbols "$\cup$" will denote disjoint unions[1]. It follows easily, by looking at $\theta^I \theta^J \theta^K$, that for three mutually disjoint tuples, I, J and K we have:

$$\sigma(I, J)\sigma(I \cup J, K) = \sigma(I, J \cup K)\sigma(J, K), \quad \sigma(I, J) = (-1)^{IJ} \sigma(J, I). \tag{3.1.1.3}$$

Here and further $(-1)^{IJ}$ stands for $(-1)^{(\sharp I)(\sharp J)}$.

We will denote by $N \setminus I$ the ordered complement of I in $\{1, \dots, N\}$ and define

[1] This will not be true in section 3.5 where we analize $N_K = n$ SUSY vertex algebras

$\sigma(I) := \sigma(I, N \setminus I)$. It follows from the definitions then that

$$\theta^I \theta^{N-I} = \sigma(I)\theta^N. \tag{3.1.1.4}$$

3.1.2. Let $Z = (z, \theta^1, \ldots, \theta^N)$ and $W = (w, \zeta^1, \ldots, \zeta^N)$ denote two sets of coordinates in the formal superdisk $D = D^{1|N}$. As before, all θ^i and ζ^j anticommute. We will denote

$$\begin{aligned} Z - W = (z - w, \theta^1 - \zeta^1, \ldots, \theta^N - \zeta^N), \qquad Z^{n|I} = z^n \theta^I, \\ (Z - W)^{j|J} = (z - w)^j \prod_{i \in J} (\theta^i - \zeta^i). \end{aligned} \tag{3.1.2.1}$$

Let $\mathbb{C}[[z]]$ be the algebra of formal power series in z, its elements are are series $\sum_{n \geq 0} a_n z^n$ with $a_n \in \mathbb{C}$. The superalgebra of regular functions in D is defined as $\mathbb{C}[[Z]] := \mathbb{C}[[z]] \otimes \mathbb{C}[\theta^1, \ldots, \theta^N]$. Similarly, we define the superalgebra $\mathbb{C}[[Z, W]] := \mathbb{C}[[z, w]] \otimes \mathbb{C}[\theta^1, \ldots, \theta^N, \zeta^1, \ldots, \zeta^N]$.

For any $\mathbb{C}$-algebra R, we denote by $R((z))$ the algebra of R-valued formal Laurent series, its elements are series of the form $\sum_{n \in \mathbb{Z}} a_n z^n$ such that $a_n \in R$ and there exists $N_0 \in \mathbb{Z}$ such that $a_n = 0$ for all $n \leq N_0$. If R is a field, so is $R((z))$. We denote $R((Z)) := R((z)) \otimes_{\mathbb{C}} \mathbb{C}[\theta^1, \ldots, \theta^N]$. Denote also by $\mathbb{C}((Z))((W))$ the superalgebra $R((W))$ where $R = \mathbb{C}((Z))$; its elements are Laurent series in W whose coefficients are Laurent series in Z. Similarly we have the superalgebra $\mathbb{C}((W))((Z))$.

Denote by $\mathbb{C}((z, w))$ the field of fractions of $\mathbb{C}[[z, w]]$ and put $\mathbb{C}((Z, W)) := \mathbb{C}((z, w)) \otimes_{\mathbb{C}} \mathbb{C}[\theta^1, \ldots, \theta^N, \zeta^1, \ldots, \zeta^N]$. One may think of this superalgebra as the algebra of meromorphic functions in the formal superdisk $D^{2|2N}$. Given such a meromorphic function, we can "expand it near the w axis", to obtain an element of $\mathbb{C}((Z))((W))$. Indeed, $\mathbb{C}[[z, w]]$ embedds naturally in $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$ respectively. Since $\mathbb{C}((z, w))$ is the ring of fractions of $\mathbb{C}[[z, w]]$ and $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$ are fields, these embedding induce respective algebra embeddings

$$\mathbb{C}((z))((w)) \overset{i_{z,w}}{\hookleftarrow} \mathbb{C}((z, w)) \overset{i_{w,z}}{\hookrightarrow} \mathbb{C}((w))((z)). \tag{3.1.2.2}$$

(A concrete example is given by (2.1.4.1)).

Tensoring with the corresponding Grassmann superalgebras, we obtain superalgebra embeddings

$$\mathbb{C}((Z))((W)) \overset{i_{z,w}}{\hookleftarrow} \mathbb{C}((Z, W)) \overset{i_{w,z}}{\hookrightarrow} \mathbb{C}((W))((Z)). \tag{3.1.2.3}$$

Let $\mathscr{U}$ be a vector superspace. An $\mathscr{U}$-valued formal distribution is an expression of the form

$$a(Z) = \sum_{(n|I), n \in \mathbb{Z}} Z^{n|I} a_{n|I}, \qquad a_{n|I} \in \mathscr{U}. \tag{3.1.2.4}$$

The space of such distributions will be denoted $\mathscr{U}[[Z, Z^{-1}]]$. We denote by $\mathbb{C}[Z, Z^{-1}] := \mathbb{C}[z, z^{-1}] \otimes \mathbb{C}[\theta^1, \ldots, \theta^N]$ the superalgebra of Laurent polynomials. A $\mathscr{U}$-valued formal distribution is canonically a linear functional $\mathbb{C}[Z, Z^{-1}] \to \mathscr{U}$. To see this, we define the *super residue* as the coefficient of $Z^{-1|N}$:

$$\operatorname{res}_Z a(Z) = a_{-1|N}. \tag{3.1.2.5}$$

This clearly satisfies

$$\operatorname{res}_Z \partial_z a(Z) = \operatorname{res}_Z \partial_\theta a(Z) = 0. \tag{3.1.2.6}$$

Given a $\mathscr{U}$-valued formal distribution $a(Z)$ we obtain a linear map $\mathbb{C}[Z, Z^{-1}] \to \mathscr{U}$ given by

$$f(Z) \mapsto \operatorname{res}_Z a(Z) f(Z). \tag{3.1.2.7}$$

Conversely, every formal distribution arises in this way. Indeed we have:

$$\operatorname{res}_Z Z^{n|I} a(Z) = \sigma(I) a_{-1-n|N \setminus I}. \tag{3.1.2.8}$$

Therefore the formal distribution $a(Z)$ can be written as

$$a(Z) = \sum_{(n|I), n \in \mathbb{Z}} Z^{-1-n|N \setminus I} a_{(n|I)}, \tag{3.1.2.9}$$

where

$$a_{(n|I)} = \sigma(I) \operatorname{res}_Z Z^{n|I} a(Z). \tag{3.1.2.10}$$

We can similarly define $\mathscr{U}$-valued formal distributions in two variables, as expressions of the form

$$a(Z, W) = \sum_{(j|J),(k|K)} Z^{j|J} W^{k|K} a_{j|J,k|K}, \qquad a_{j|J,k|K} \in \mathscr{U}. \tag{3.1.2.11}$$

The space of such formal distributions will be denoted $\mathscr{U}[[Z, Z^{-1}, W, W^{-1}]]$.

Note that in the case $\mathscr{U} = \mathbb{C}$, both algebras $\mathbb{C}((Z))((W))$ and $\mathbb{C}((W))((Z))$ are embedded in $\mathbb{C}[[Z, Z^{-1}, W, W^{-1}]]$. We will denote by $i_{z,w}$ and $i_{w,z}$ the corresponding embeddings of $\mathbb{C}((Z, W))$ in $\mathbb{C}[[Z, Z^{-1}, W, W^{-1}]]$. When $f(Z, W)$ is a Laurent polynomial (that is a polynomial in z, z^{-1}, w, w^{-1} and the odd variables) then the embeddings $i_{z,w} f$ and $i_{w,z} f$ coincide on $\mathbb{C}[[Z, Z^{-1}, W, W^{-1}]]$. Indeed, it is mmediate to see that

$$\mathbb{C}((Z))((W)) \cap \mathbb{C}((W))((Z)) = \mathbb{C}[[Z, W]][z^{-1}, w^{-1}], \tag{3.1.2.12}$$

where the intersection is taken in $\mathbb{C}[[Z, Z^{-1}, W, W^{-1}]]$. The images under these embeddings are different for other functions, as we will see below in the case $f(Z, W) = (Z - W)^{-1|N}$ (cf. 3.1.5).

A $\mathscr{U}$-valued formal distribution in two variables is called *local* if there exists a non-negative integer n such that

$$(z - w)^n a(Z, W) = 0. \tag{3.1.2.13}$$

3.1.3. Note that the differential operators $\partial_z, \partial_{\theta^i}$ and $\partial_w, \partial_{\zeta^i}$ act in the usual way on the spaces $\mathbb{C}((Z,W))$, $\mathbb{C}[[Z,Z^{-1},W,W^{-1}]]$. For $j \in \mathbb{Z}_+$ and $J = (j_1, \dots, j_k)$ we will denote

$$\partial_Z^{j|J} = \partial_z^j \partial_{\theta^{j_1}} \dots \partial_{\theta^{j_k}}.$$

We define

$$\partial_Z^{(j|J)} := \frac{(-1)^{\frac{J(J+1)}{2}}}{j!} \partial_Z^{j|J}, \qquad Z^{(j|J)} := \frac{(-1)^{\frac{J(J+1)}{2}}}{j!} Z^{j|J}. \tag{3.1.3.1}$$

One checks easily that the embeddings $i_{z,w}$ and $i_{w,z}$ defined above, commute with the action of the differential operators $\partial_Z^{j|J}$ and $\partial_W^{j|J}$.

Put

$$\partial_W = (\partial_w, \partial_{\zeta^1}, \dots, \partial_{\zeta^N}), \tag{3.1.3.2}$$

and for any $\mathscr{U}$-valued formal distribution $f(Z)$, we define its *Taylor expansion* as:

$$f(Z) = e^{(Z-W)\partial_W} f(W), \tag{3.1.3.3}$$

where

$$(Z-W)\partial_W = (z-w)\partial_w + \sum_i (\theta^i - \zeta^i)\partial_{\zeta^i}.$$

Expanding the exponential in (3.1.3.3) we obtain:

$$f(Z) = \sum_{(j|J), j \geq 0} (-1)^J (Z-W)^{j|J} \partial_W^{(j|J)} f(W). \tag{3.1.3.4}$$

Remark 3.1.4. In the definition of formal distributions and super residues, we can replace $\mathbb{C}$ by any commutative superalgebra $\mathscr{A}$, and $\mathscr{U}$ by any $\mathscr{A}$-module. We see immediately that the residue map is of parity $N \mod 2$, that is, for $\chi \in \mathscr{A}$, and $u(Z)$ an $\mathscr{U}$-valued distribution, we have:

$$\operatorname{res}_Z \chi u(Z) = (-1)^{\chi N} \operatorname{res}_Z u(Z). \tag{3.1.4.1}$$

On the other hand, this residue map is a morphism of right $\mathscr{A}$-modules, namely:

$$\operatorname{res}_Z u(Z)\chi = \Big(\operatorname{res}_Z u(Z)\Big)\chi. \tag{3.1.4.2}$$

Proposition 3.1.5. *There exists a unique $\mathbb{C}$-valued formal distribution $\delta(Z,W)$ such that for every function $f \in \mathscr{U}(Z)$ we have $\operatorname{res}_Z \delta(Z,W) f(Z) = f(W)$.*

Proof. For uniqueness, let δ and δ' be two such distributions, then $\beta = \delta - \delta'$ satisfies $\operatorname{res}_Z \beta(Z,W) f(Z) = 0$ for all functions $f(Z)$. Decomposing $\beta(Z,W) = \sum \beta_{n|I,m|J} W^{m|J} Z^{n|I}$, and multiplying by $Z^{k|L}$ we see that $\beta_{-1-k|N-L,m|J} = 0$ for all $m|J$, hence $\beta = 0$. Existence will be proved below. □

3.1.6. We define the formal δ-function as the $\mathbb{C}$-valued formal distribution in two variables, given by

$$\delta(Z,W) = (i_{z,w} - i_{w,z})(Z-W)^{-1|N} = (i_{z,w} - i_{w,z})\frac{(\theta-\zeta)^N}{z-w} \tag{3.1.6.1}$$

It follows that

$$\partial_w^{(n)}\delta(Z,W) = \frac{1}{n!}\partial_w^n\delta(Z,W) = (i_{z,w} - i_{w,z})(Z-W)^{-1-n|N}. \tag{3.1.6.2}$$

This distribution has the following properties:

1. $(Z-W)^{m|J}\partial_W^{n|I}\delta(Z,W) = 0$ whenever $m > n$ or $J \supsetneq I$,

2. $(Z-W)^{j|J}\partial_W^{(n|I)}\delta(Z,W) = \sigma(I \setminus J, J)\partial_W^{(n-j|I\setminus J)}\delta(Z,W)$ if $n \geq j$ and $I \supset J$,

3. $\delta(Z,W) = (-1)^N\delta(W,Z)$,

4. $\partial_Z^{j|J}\delta(Z,W) = (-1)^{j+N+J}\partial_W^{j|J}\delta(W,Z)$,

5. $\delta(Z,W)a(Z) = \delta(Z,W)a(W)$, where $a(Z)$ is any formal distribution,

6. $\operatorname{res}_Z \delta(Z,W)a(Z) = a(W)$,

7. $\exp\left((Z-W)\Lambda\right)\partial_W^{n|I}\delta(Z,W) = (\Lambda + \partial_W)^{n|I}\delta(Z,W)$, where $\Lambda = (\lambda, \chi^1, \dots, \chi^N)$, χ^i are odd anticommuting variables, λ is even, λ commutes with χ^i, and we write

$$\begin{aligned} (Z-W)\Lambda &= (z-w)\lambda + \sum_i (\theta^i - \zeta^i)\chi^i, \\ (\Lambda + \partial_W) &= (\lambda + \partial_w, \chi_i + \partial_{\theta^i}). \end{aligned} \tag{3.1.6.3}$$

Proof. Writting $\partial_\zeta^I = \partial_{\zeta^{i_1}} \dots \partial_{\zeta^{i_k}}$ we have

$$\begin{aligned} (Z-W)^{m|J}&\partial_W^{n|I}\delta(Z,W) = \\ &= (z-w)^m n!(i_{z,w} - i_{w,z})(z-w)^{-1-n}(\theta-\zeta)^J\partial_\zeta^I(\theta-\zeta)^N \end{aligned} \tag{3.1.6.4}$$

Now this clearly vanishes if $m \geq 1 + n$ since then the two embeddings $i_{z,w}$ and $i_{w,z}$ coincide on the regular function $(z-w)^{m-n-1}$. The other factor is clearly zero if $J \supsetneq I$ since for every $j \in J \setminus I$ we have a factor $(\theta^j - \zeta^j)$ in $\partial_\zeta^I(\theta-\zeta)^N$. This proves (1).

In order to prove (2) we write:

$$
\begin{aligned}
&(Z-W)^{j|J}\partial_W^{n|I}\delta(Z,W) = \\
&\qquad = n!(i_{z,w}-i_{w,z})(z-w)^{j-1-n}(\theta-\zeta)^J\partial_\zeta^I(\theta-\zeta)^N = \\
&\qquad = \frac{n!}{(n-j)!}(n-j)!(i_{z,w}-i_{w,z})(z-w)^{-1-(n-j)}\times \\
&\qquad \times(-1)^J\sigma(J,I\setminus J)(\theta-\zeta)^J\partial_\theta^J\partial_\zeta^{I\setminus J}(\theta-\zeta)^N = \\
&\qquad = \frac{n!}{(n-j)!}(-1)^{\frac{J(J+1)}{2}}\sigma(J,I\setminus J)(n-j)!\times \\
&\qquad \times(i_{z,w}-i_{w,z})(z-w)^{-1-(n-j)}\partial_\zeta^{I\setminus J}(\theta-\zeta)^N = \\
&\qquad = (-1)^{\frac{J(J+1)}{2}}\sigma(J,I\setminus J)\frac{n!}{(n-j)!}\partial_W^{n-j|I\setminus J}\delta(Z,W).
\end{aligned}
\tag{3.1.6.5}
$$

This implies (2).

(3) is obvious and (4) follows from (3) easily. In order to prove (5) wee see that from (1) we have $\delta z=\delta w$ therefore we get $\delta(Z,W)z^k=\delta(Z,W)w^k$. On the other hand, also from (1) it follows that $\delta(Z,W)\theta^i=\delta(Z,W)\zeta^i$. Hence $\delta(Z,W)\theta^I=\delta(Z,W)\zeta^I$ and we have proved that $\delta(Z,W)Z^{n|I}=\delta(Z,W)W^{n|I}$. The result follows by linearity now.

(6) follows by taking residue in (5). To prove (7) we first expand the exponential in power series:

$$
\begin{aligned}
&\exp\left((Z-W)\Lambda\right)\partial_W^{n|I}\delta(Z,W) = \\
&\qquad = \sum_{(j|J),j\geq 0}(-1)^J(Z-W)^{(j|J)}\Lambda^{j|J}\partial_W^{n|I}\delta(Z,W)
\end{aligned}
\tag{3.1.6.6}
$$

Now using (2) we see that this is:

$$
\sum_{(j|J),j\geq 0}\binom{n}{j}\Lambda^{j|J}\sigma(J,I\setminus J)\partial_W^{n-j|I\setminus J}\delta(Z,W) \tag{3.1.6.7}
$$

On the other hand we can expand the right hand side of (7) as:

$$
\begin{aligned}
&(\Lambda+\partial_W)^{n|I} = (\lambda+\partial_w)^n(\chi+\partial_\zeta)^I = \\
&\qquad = \sum_{(j|J),j\geq 0}\binom{n}{j}\lambda^j\partial_w^{n-j}\sigma(J,I\setminus J)\chi^J\partial_\zeta^{I\setminus J} = \\
&\qquad = \sum_{(j|J),j\geq 0}\binom{n}{j}\Lambda^{j|J}\sigma(J,I\setminus J)\partial_W^{n-j|I\setminus J}
\end{aligned}
\tag{3.1.6.8}
$$

Comparing with (3.1.6.7) we get the result. □

Lemma 3.1.7. *Let $a(Z,W)$ be a local formal distribution in two variables. Then $a(Z,W)$ can be uniquely decomposed as*

$$a(Z,W) = \sum_{(j|J),j\geq 0} \left(\partial_W^{(j|J)}\delta(Z,W)\right) c_{j|J}(W), \tag{3.1.7.1}$$

where the sum is finite. The coefficients $c_{j|J}$ are given by

$$c_{j|J}(W) = \mathrm{res}_Z (Z-W)^{j|J} a(Z,W). \tag{3.1.7.2}$$

Proof. First we note that if $a(Z,W)$ is local then the sum on the right hand side is finite. Let $b(Z,W)$ the difference between the right hand side and the left hand side of (3.1.7.1). We find:

$$\begin{aligned} \mathrm{res}_Z(Z-W)^{k|K} b(Z,W) &= \mathrm{res}_Z(Z-W)^{k|K} a(Z,W) - \\ &\quad - \mathrm{res}_Z \sum_{(j|J),j\geq 0} (Z-W)^{k|K} \left(\partial_W^{(j|J)}\delta(Z,W)\right) c_{j|J}(W) \\ &= c_{k|K}(W) - \mathrm{res}_Z \left(\partial_W^{(j-k|J\setminus K)}\delta(Z,W)\right) c_{(j|J)}(W) \\ &= c_{k|K}(W) - \mathrm{res}_Z\, \delta(Z,W) c_{k|K}(W) = 0, \end{aligned} \tag{3.1.7.3}$$

where in the second line we have used (2) of 3.1.6. It follows that $b(Z,W)$ has no negative powers of z. Moreover, $b(Z,W)$ is local, since $a(Z,W)$ is, and the right hand side of (3.1.7.1) is local by (1) of 3.1.6. We can write then

$$b(Z,W) = \sum_{(j,J),j\geq 0} Z^{j|J} b_{j|J}(W), \tag{3.1.7.4}$$

and since $(z-w)^n b(Z,W) = 0$ we obtain:

$$\sum_{\substack{(j|J)\\ j\geq k\geq 0}} \binom{n}{k} Z^{j|J} w^{n-k} b_{j-k|J}(W) = 0, \tag{3.1.7.5}$$

which easily shows that $b(Z,W) = 0$. Uniqueness is clear by taking residues on both sides of (3.1.7.1). □

3.1.8. Let $a(Z,W)$ be a formal distribution in two variables. We define its formal Fourier transform by:

$$\mathscr{F}_{Z,W}^{\Lambda} a(Z,W) = \mathrm{res}_Z \exp\left((Z-W)\Lambda\right) a(Z,W), \tag{3.1.8.1}$$

where $\Lambda = (\lambda, \chi^1, \ldots, \chi^N)$, λ is an even variable, and χ^i are odd anticommutative variables, commuting with λ.

Expanding this exponential we have (recall (3.1.6.6)) :

$$\begin{aligned}\mathscr{F}^{\Lambda}_{Z,W}a(Z,W) &= \operatorname{res}_Z \sum_{(j|J),j\geq 0} \Lambda^{j|J}(Z-W)^{(j|J)}a(Z,W) \\ &= \sum_{(j|J),j\geq 0} (-1)^{JN}\Lambda^{j|J}\operatorname{res}_Z(Z-W)^{(j|J)}a(Z,W) \\ &= \sum_{(j|J),j\geq 0} (-1)^{JN}\Lambda^{(j|J)}c_{j|J}(W)\end{aligned} \tag{3.1.8.2}$$

where $c_{j|J}$ are defined by (3.1.7.2) and we write, as before

$$\Lambda^{j|J} = \lambda^j \chi^{j_1}\dots\chi^{j_k}, \quad \Lambda^{(j|J)} := \frac{(-1)^{\frac{J(J+1)}{2}}}{j!}\Lambda^{j|J}. \tag{3.1.8.3}$$

Proposition 3.1.9. *The formal Fourier transform satisfies the following properties:*

1. *sesquilinearity:*

$$\begin{aligned}\mathscr{F}^{\Lambda}_{Z,W}\partial_z a(Z,W) &= -\lambda\mathscr{F}^{\Lambda}_{Z,W}a(Z,W) = [\partial_w, \mathscr{F}^{\Lambda}_{Z,W}]a(Z,W), \\ \mathscr{F}^{\Lambda}_{Z,W}\partial_{\theta^i} a(Z,W) &= -(-1)^N\chi^i\mathscr{F}^{\Lambda}_{Z,W}a(Z,W) = (-1)^N[\partial_{\zeta^i}, \mathscr{F}^{\Lambda}_{Z,W}]a(Z,W).\end{aligned} \tag{3.1.9.1}$$

2. *For any local formal distribution $a(Z,W)$ we have:*

$$\begin{aligned}(-1)^N\mathscr{F}^{\Lambda}_{Z,W}a(W,Z) &= \mathscr{F}^{-\Lambda-\partial_W}_{Z,W}a(Z,W), \\ &= \mathscr{F}^{\Gamma}_{Z,W}a(Z,W)|_{\Gamma=-\Lambda-\partial_W},\end{aligned} \tag{3.1.9.2}$$

   where $-\lambda-\partial_W = (-\lambda-\partial_w, -\chi^i-\partial_{\zeta^i})$.

3. *For any formal distribution in three variables $a(Z,W,X)$ we have*

$$\mathscr{F}^{\Lambda}_{Z,W}\mathscr{F}^{\Gamma}_{X,W}a(Z,W,X) = (-1)^N\mathscr{F}^{\Lambda+\Gamma}_{X,W}\mathscr{F}^{\Lambda}_{Z,X}a(Z,W,X), \tag{3.1.9.3}$$

   where $\Gamma = (\gamma, \eta^1, \dots, \eta^N)$, with η^i odd anticommutative variables and γ is even and commutes with η^i, $\Lambda+\Gamma$ is the sum $(\lambda+\gamma, \chi^i+\eta^i)$, and the superalgebra $\mathbb{C}[\Lambda,\Gamma]$ is commutative.

Proof. The proof of the first equality of the first line of (1) follows from (3.1.2.6). For the second line we have

$$\begin{aligned}\mathscr{F}^{\Lambda}_{Z,W}\partial_{\theta^i}a(Z,W) &= \operatorname{res}_Z \exp\left((Z-W)\Lambda\right)\partial_{\theta^i}a(Z,W) \\ &= -\operatorname{res}_Z\left(\partial_{\theta^i}\exp\left((Z-W)\Lambda\right)\right)a(Z,W) \\ &= -\operatorname{res}_Z \chi^i \exp\left((Z-W)\Lambda\right)a(Z,W) \\ &= -(-1)^N\chi^i\mathscr{F}^{\Lambda}_{Z,W}a(Z,W)\end{aligned} \tag{3.1.9.4}$$

For the first equality of the second equation we have:

$$\begin{aligned}[\partial_{\zeta^i}, \mathscr{F}^{\Lambda}_{Z,W}]a(Z,W) = (-1)^N \big(&\operatorname{res}_Z \partial_{\zeta^i} \exp\left((Z-W)\Lambda\right) a(Z,W) - \\ &\exp\left((Z-W)\Lambda\right) \partial_{\zeta^i} a(Z,W)\big) = \\ &= (-1)^N \operatorname{res}_Z \left(\partial_{\zeta^i} \exp\left((Z-W)\Lambda\right)\right) a(Z,W) \\ &= -\chi^i \mathscr{F}^{\Lambda}_{Z,W} a(Z,W).\end{aligned} \tag{3.1.9.5}$$

To prove (2) it is enough, by Lemma 3.1.7, to prove the statement when $a(Z,W) = \left(\partial_W^{j|J} \delta(Z,W)\right) c(W)$. In this case we have:

$$\begin{aligned}\mathscr{F}^{\Lambda}_{Z,W} a(W,Z) &= \mathscr{F}^{\Lambda}_{Z,W} \left(\partial_Z^{j|J} \delta(W,Z)\right) c(Z) \\ &= \mathscr{F}^{\Lambda}_{Z,W} (-1)^{j+J+N} \partial_W^{j|J} \delta(Z,W) c(Z).\end{aligned} \tag{3.1.9.6}$$

Now using (7) in 3.1.6 we can express the last expression in (3.1.9.6) as:

$$\begin{aligned}&(-1)^{j+J+N} \operatorname{res}_Z (\Lambda + \partial_W)^{j|J} \delta(Z,W) c(Z) = \\ &\qquad = (-1)^{j+J+JN+N} (\Lambda + \partial_W)^{j|J} \operatorname{res}_Z \delta(Z,W) c(Z) = \\ &\qquad\qquad = (-1)^{j+J+JN+N} (\Lambda + \partial_W)^{j|J} c(W).\end{aligned} \tag{3.1.9.7}$$

On the other hand we have

$$\begin{aligned}&\mathscr{F}^{\Gamma}_{Z,W} a(Z,W)|_{\Gamma = -\Lambda - \partial_W} = \\ &\qquad = (-1)^{JN} (-\Lambda - \partial_W)^{j|J} c(W) = \\ &\qquad\qquad = (-1)^{j+J+JN} (\Lambda + \partial_W)^{j|J} c(W).\end{aligned} \tag{3.1.9.8}$$

The proof of (3) is straightforward:

$$\begin{aligned}\mathscr{F}^{\Lambda}_{Z,W} \mathscr{F}^{\Gamma}_{X,W} &= \operatorname{res}_Z \exp\left((Z-W)\Lambda\right) \operatorname{res}_X \exp\left((X-W)\Gamma\right) = \\ &= \operatorname{res}_Z \operatorname{res}_X \exp\left((Z-W)\Lambda + (X-W)\Gamma\right) = \\ &= (-1)^N \operatorname{res}_X \operatorname{res}_Z \exp\left((Z-X)\Lambda + (X-W)(\Lambda+\Gamma)\right) = \\ &= (-1)^N \operatorname{res}_X \exp\left((X-W)(\Lambda+\Gamma)\right) \operatorname{res}_Z \exp\left((Z-X)\Lambda\right) = \\ &= (-1)^N \mathscr{F}^{\Lambda+\Gamma}_{X,W} \mathscr{F}^{\Lambda}_{Z,X}\end{aligned} \tag{3.1.9.9}$$

The sign $(-1)^N$ appears when we commute the residue maps (recall that they have parity $N \mod 2$). □

3.2 SUSY Lie conformal algebras

3.2.1. Let $\mathfrak{g}$ be a Lie superalgebra. A pair of $\mathfrak{g}$-valued formal distributions $a(Z), b(Z)$ is called *local* if the distribution $[a(Z), b(W)]$ is local. By the decomposition Lemma

3.1.7 we have for such a pair:

$$[a(Z), b(W)] = \sum_{(j|J), j\geq 0} \left(\partial_W^{(j|J)} \delta(Z, W)\right) c_{j|J}(W). \tag{3.2.1.1}$$

where

$$c_{j|J}(W) = \mathrm{res}_Z (Z-W)^{j|J} [a(Z), b(W)]. \tag{3.2.1.2}$$

We define $a(W)_{(j|J)} b(W) = c_{j|J}(W)$ and we call this operation the $(j|J)$ product. Let us also define the Λ-bracket of two $\mathfrak{g}$-valued formal distributions by

$$[a_\Lambda b](W) = \mathscr{F}^\Lambda_{Z,W} [a(Z), b(W)], \tag{3.2.1.3}$$

where $\mathscr{F}^\Lambda_{Z,W}$ is the formal Fourier transform defined in 3.1.8. It follows from the definitions and from (3.1.8.2) that

$$[a_\Lambda b] = \sum_{(j|J), j\geq 0} (-1)^{JN} \Lambda^{(j|J)} a_{(j|J)} b. \tag{3.2.1.4}$$

Note also that the Λ-bracket has parity $N \mod 2$ (this follows from the fact that the residue map has parity $N \mod 2$).

A pair $(\mathfrak{g}, \mathscr{R})$ consisting of a Lie superalgebra $\mathfrak{g}$ and a family $\mathscr{R}$ of pairwise local $\mathfrak{g}$-valued formal distributions $a(Z)$, whose coefficients span $\mathfrak{g}$, stable under all $j|J$-th products and under the derivations ∂_z and ∂_{θ^i} is called an $N_W = N$ SUSY *formal distribution Lie superalgebra.*

Proposition 3.2.2. *The Λ-bracket defined in (3.2.1.3) satisfies the following properties:*

1. *Sesquilinearity for a pair $(a(Z), b(W))$:*

$$[\partial_z a_\Lambda b] = -\lambda [a_\Lambda b] \qquad [a_\Lambda \partial_w b] = (\partial_w + \lambda)[a_\Lambda b] \tag{3.2.2.1}$$

$$[\partial_{\theta^i} a_\Lambda b] = -(-1)^N \chi^i [a_\Lambda b] \qquad [a_\Lambda \partial_{\zeta^i} b] = (-1)^{a+N} (\partial_{\zeta^i} + \chi^i)[a_\Lambda b] \tag{3.2.2.2}$$

2. *Skew-symmetry for a local pair $(a(Z), b(W))$:*

$$[b_\Lambda a] = -(-1)^{ab+N} [a_{-\Lambda - \partial_W} b]. \tag{3.2.2.3}$$

3. *Jacobi identity for a triple $(a(Z), b(X), c(W))$:*

$$[a_\Lambda [b_\Gamma c]] = (-1)^{aN+N} [[a_\Lambda b]_{\Lambda+\Gamma} c] + (-1)^{(a+N)(b+N)} [b_\Gamma [a_\Lambda c]], \tag{3.2.2.4}$$

where $\Gamma = (\gamma, \eta^1, \ldots, \eta^N)$ and the superalgebra $\mathbb{C}[\Lambda, \Gamma]$ is commutative.

Proof. In order to prove the first equation in (3.2.2.2) we expand:

$$\begin{aligned}
[\partial_{\theta^i} a_\Lambda b] &= \mathscr{F}^\Lambda_{Z,W}[\partial_{\theta^i} a(Z), b(W)] \\
&= \mathscr{F}^\Lambda_{Z,W}\partial_{\theta^i}[a(Z), b(W)] \\
&= -(-1)^N \chi^i \mathscr{F}^\Lambda_{Z,W}[a(Z), b(W)] \\
&= -(-1)^N \chi^i [a_\Lambda b].
\end{aligned} \tag{3.2.2.5}$$

For the second equation we have:

$$\begin{aligned}
[a_\Lambda \partial_{\zeta^i} b] &= \mathscr{F}^\Lambda_{Z,W}[a(Z), \partial_{\zeta^i} b(W)] \\
&= \mathscr{F}^\Lambda_{Z,W}(-1)^a \partial_{\zeta^i}[a(Z), b(W)] \\
&= (-1)^a \left([\mathscr{F}^\Lambda_{Z,W}, \partial_{\zeta^i}] + (-1)^N \partial_{\zeta^i}\mathscr{F}^\Lambda_{Z,W}\right)[a(Z), b(W)] \\
&= (-1)^{a+N}(\chi^i + \partial_{\zeta^i})\mathscr{F}^\Lambda_{Z,W}[a(Z), b(W)] \\
&= (-1)^{a+N}\left(\chi^i + \partial_{\zeta^i}\right)[a_\Lambda b].
\end{aligned} \tag{3.2.2.6}$$

Skew-symmetry follows from the skew-symmetry property of the Fourier transform (3.1.9.2) as follows:

$$\begin{aligned}
[b_\Lambda a] &= \mathscr{F}^\Lambda_{Z,W}[b(Z), a(W)] \\
&= -(-1)^{ab}\mathscr{F}^\Lambda_{Z,W}[a(W), b(Z)] \\
&= -(-1)^{ab+N}\mathscr{F}^{-\Lambda-\partial_W}_{Z,W}[a(Z), b(W)] \\
&= -(-1)^{ab+N}[a_{-\Lambda-\partial_W} b].
\end{aligned} \tag{3.2.2.7}$$

Finally, to prove the Jacobi identity we write:

$$\begin{aligned}
[a_\Lambda[b_\Gamma c]] &= \mathscr{F}^\Lambda_{Z,W}[a(Z), \mathscr{F}^\Gamma_{X,W}[b(X), c(W)]] \\
&= (-1)^{aN}\mathscr{F}^\Lambda_{Z,W}\mathscr{F}^\Gamma_{X,W}[a(Z), [b(X), c(W)]] \\
&= (-1)^{aN}\mathscr{F}^\Lambda_{Z,W}\mathscr{F}^\Gamma_{X,W}[[a(Z), b(X)], c(W)] + \\
&\quad + (-1)^{ab+aN}\mathscr{F}^\Lambda_{Z,W}\mathscr{F}^\Gamma_{X,W}[b(X), [a(Z), c(W)]] \\
&= (-1)^{aN+N}\mathscr{F}^{\Lambda+\Gamma}_{X,W}\mathscr{F}^\Lambda_{Z,X}[[a(Z), b(X)], c(W)] + \\
&\quad (-1)^{ab+aN+bN+N}\mathscr{F}^\Gamma_{X,W}[b(X), \mathscr{F}^\Lambda_{Z,W}[a(Z), c(W)]] \\
&= (-1)^{aN+N}[[a_\Lambda b]_{\Gamma+\Lambda} c] + (-1)^{(a+N)(b+N)}[b_\Gamma[a_\Lambda c]].
\end{aligned} \tag{3.2.2.8}$$

□

Definition 3.2.3. Let $\mathbb{C}[T,S] := \mathbb{C}[T, S^1, \dots, S^N]$ be the commutative superalgebra freely generated by an even element T and N odd elements S^i. A $N_W = N$ *SUSY Lie conformal algebra* is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[T,S]$-module $\mathscr{R}$ with a $\mathbb{C}$-bilinear operation $[\,_\Lambda\,] : \mathscr{R} \otimes_{\mathbb{C}} \mathscr{R} \to \mathbb{C}[\Lambda] \otimes_{\mathbb{C}} \mathscr{R}$ of parity $N \mod 2$ satisfying the following three axioms:

1. Sesquilinearity:

$$[Ta_\Lambda b] = -\lambda[a_\Lambda b] \qquad [a_\Lambda Tb] = (T+\lambda)[a_\Lambda b] \tag{3.2.3.1}$$

$$[S^i a_\Lambda b] = -(-1)^N \chi^i [a_\Lambda b] \qquad [a_\Lambda S^i b] = (-1)^{a+N}\left(S^i + \chi^i\right)[a_\Lambda b] \tag{3.2.3.2}$$

2. Skew-symmetry:

$$[b_\Lambda a] = -(-1)^{ab+N}[b_{-\Lambda-\nabla} a], \tag{3.2.3.3}$$

where $\nabla = (T, S^1, \dots, S^N)$, the Λ-bracket in the RHS means compute first the Γ bracket and then let $\Gamma = -\Lambda - \nabla$.

3. Jacobi identity:

$$[a_\Lambda [b_\Gamma c]] = (-1)^{aN+N}[[a_\Lambda b]_{\Gamma+\Lambda} c] + (-1)^{(a+N)(b+N)}[b_\Gamma [a_\Lambda c]]. \tag{3.2.3.4}$$

We will drop the adjective SUSY when no confusion may arise.

Remark 3.2.4. Even though in this case the situation is simple, it is instructive to realize the Λ bracket as a morphism of $\mathbb{C}[\Lambda]$-modules. Consider the co-commutative Hopf superalgebra $\mathscr{H} = \mathbb{C}[\Lambda]$ with commultiplication $\Delta\lambda = \lambda \otimes 1 + 1 \otimes \lambda$, $\Delta\chi^i = \chi^i \otimes 1 + 1 \otimes \chi^i$. Note that $\mathbb{C}[\nabla] \simeq \mathscr{H}$. Consider $\mathscr{H}$ as a $\mathscr{H}$-module with the adjoint action (which is trivial in this case, given that $\mathscr{H}$ is super-commutative). Then we may think of $\mathscr{H} \otimes \mathscr{R}$ as an $\mathscr{H}$ module, the action is given by $h \mapsto \Delta h$. Similarly $\mathscr{R} \otimes \mathscr{R}$ is an $\mathscr{H}$-module. The Λ-bracket is then a $\mathscr{H}$-module homomorphism of degree $(-1)^N$. Namely, let ϕ denote the morphism $\mathscr{R} \otimes \mathscr{R} \to \mathscr{H} \otimes \mathscr{R}$ which is given by the Λ-bracket. Then for every $h \in \mathscr{H}$ we have

$$\phi h - (-1)^{hN} h\phi = 0, \tag{3.2.4.1}$$

as elements in $\mathrm{Hom}(\mathscr{R} \otimes \mathscr{R}, \mathscr{H} \otimes \mathscr{R})$. Similarly, the Jacobi identity is an identity in

$$\mathrm{Hom}(\mathscr{R} \otimes \mathscr{R} \otimes \mathscr{R}, \mathscr{H} \otimes \mathscr{H} \otimes \mathscr{R}). \tag{3.2.4.2}$$

We will expand on this in Remark 3.5.10.

Remark 3.2.5. According to Proposition 3.2.2, given any $N_W = N$ SUSY formal distribution Lie superalgebra $(\mathfrak{g}, \mathscr{R})$, the space $\mathscr{R}$ is a SUSY Lie conformal algebra where $T = \partial_w$ and $S^i = \partial_{\zeta^i}$, and the Λ-bracket is defined by (3.2.1.3).

Definition 3.2.6. A Lie superalgebra of degree $p \in \mathbb{Z}/2\mathbb{Z}$ is a vector superspace $\mathfrak{h}$ with a bilinear operation $\{\, , \,\} : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$ of parity p satisfying:

1. Skew-symmetry: $\{a, b\} = -(-1)^{ab+p}\{b, a\}$.

2. Jacobi identity: $\big\{a, \{b, c\}\big\} = (-1)^{ap+p}\big\{\{a, b\}, c\big\} + (-1)^{(a+p)(b+p)}\big\{b, \{a, c\}\big\}$.

Lemma 3.2.7. *Let $\mathfrak{h}$ be a Lie superalgebra of degree $p \in \mathbb{Z}/2\mathbb{Z}$. Define $\mathfrak{g}$ as a vector superspace to be $\mathfrak{h}$ if $p = 0 \mod 2$ or $\mathfrak{h}$ with the reversed parity if $p = 1 \mod 2$. Define the bilinear operation $[\,,\,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ by:*

$$[a, b] = (-1)^{ap+p}\{a, b\} \tag{3.2.7.1}$$

where the right hand side is computed in $\mathfrak{h}$ and then we reverse the parity if $p = \bar{1}$. Then $(\mathfrak{g}, [\,,\,])$ is a Lie superalgebra which we will denote as $\mathrm{Lie}(\mathfrak{h})$.

Proof. We have:

$$[b, a] = (-1)^{bp+p}\{b, a\} = -(-1)^{bp+ab}\{a, b\} = -(-1)^{(a+p)(b+p)}[a, b], \tag{3.2.7.2}$$

which is skew-symmetry for the Lie algebra provided the parity in $\mathfrak{g}$ is shifted by p. To check Jacobi identity we have:

$$\begin{aligned}
&[a, [b, c]] = \\
&\quad = (-1)^{pb+ap}\{a, \{b, c\}\} = (-1)^{pb+p}\{\{a, b\}, c\} + (-1)^{ab+p}\{b, \{a, c\}\} = \\
&\quad = (-1)^{pb+p+(a+b+p)p+ap}[[a, b], c] + (-1)^{ab+p+ap+bp}[b, [a, c]] = \\
&\quad = [[a, b], c] + (-1)^{(a+p)(b+p)}[b, [a, c]].
\end{aligned} \tag{3.2.7.3}$$

□

Lemma 3.2.8. *Let $\mathscr{R}$ be a $N_W = N$ SUSY Lie conformal algebra. Then $\mathscr{R}/\nabla\mathscr{R}$ is naturally a Lie superalgebra of degree $N \mod 2$ with bracket*

$$\{a + \nabla\mathscr{R}, b + \nabla\mathscr{R}\} = [a_\Lambda b]_{\Lambda=0} + \nabla\mathscr{R}. \tag{3.2.8.1}$$

Proof. The fact that the bilinear map $\{\,,\,\}$ is well defined follows from sesquilinearity. Skew-symmetry and the Jacobi identity follow from the corresponding axioms for the SUSY Lie conformal algebra $\mathscr{R}$. □

Lemma 3.2.9. *Let $\mathscr{R}$ be an $N_W = N$ SUSY Lie conformal algebra. Let $W = (w, \zeta^1, \dots, \zeta^N)$ be formal variables as before, and consider the superalgebra of Laurent polynomials $\mathbb{C}[W, W^{-1}]$. Then $\tilde{\mathscr{R}} := \mathscr{R} \otimes \mathbb{C}[W, W^{-1}]$ is an $N_W = N$ SUSY Lie conformal algebra with Λ-bracket:*

$$[a \otimes f_\Lambda b \otimes g] = (-1)^{fb}[a_{\Lambda+\partial_W} b] \otimes f(W)g(W')|_{W'=W}, \tag{3.2.9.1}$$

and with $\tilde{T} = T \otimes \mathrm{id} + \mathrm{id} \otimes \partial_w$ and $\tilde{S}^i = S^i \otimes \mathrm{id} + \mathrm{id} \otimes \partial_{\zeta^i}$.

Proof. We prove here skew-symmetry, the other axioms are checked in a similar way:

$$\begin{aligned}
[a\otimes f\,{}_\Lambda b\otimes g] &= (-1)^{fb}[a_{\Lambda+\partial_W}b]\otimes f(W)g(W')|_{W=W'} \\
&= -(-1)^{ab+N+fb}[b_{-\Lambda-\partial_W-\nabla}a]\otimes f(W)g(W')|_{W=W'} \\
&= -(-1)^{(a+f)(b+g)+N+ga}[b_{-\Lambda-\partial_W-\nabla-\partial_{W'}+\partial_{W'}}a]\otimes g(W')f(W)|_{W=W'} \\
&= -(-1)^{(a+f)(b+g)+N}[b\otimes g\,{}_{-\Lambda-\tilde{\nabla}}a\otimes f]
\end{aligned} \tag{3.2.9.2}$$

□

3.2.10. For any $N_W = N$ SUSY Lie conformal algebra $\mathscr{R}$, we put $L(\mathscr{R}) = \tilde{\mathscr{R}}/\tilde{\nabla}\tilde{\mathscr{R}}$ and $\mathrm{Lie}(\mathscr{R}) := \mathrm{Lie}(L(\mathscr{R}))$ (see Lemmas 3.2.7 and 3.2.8). For each $a \in \mathscr{R}$, let $a_{<n|I>} \in L(\mathscr{R})$ be the image of $a \otimes W^{n|I}$. Similarly define $a_{(n|I)} \in \mathrm{Lie}(\mathscr{R})$ as the image of the following element of $L(\mathscr{R})$

$$(-1)^{aI}\sigma(I)a_{<n|I>}, \tag{3.2.10.1}$$

and define, for each $a \in \mathscr{R}$, the following $\mathrm{Lie}(\mathscr{R})$-valued formal distribution

$$a(Z) = \sum_{j\in\mathbb{Z},J} Z^{-1-j|N\setminus J}a_{(j|J)} \in \mathrm{Lie}(\mathscr{R})[[Z,Z^{-1}]]. \tag{3.2.10.2}$$

Using (3.2.9.1) with $f = W^{n|I}$ and $g = W^{k|K}$ and putting $\Lambda = 0$ we compute explicitly the Lie bracket (of parity $N \mod 2$) in $L(\mathscr{R})$:

$$\begin{aligned}
\{a_{<n|I>}, b_{<k|K>}\} = \sum_{j\ge 0,J}(-1)^{aJ+b(I-J)}\binom{n}{j}\times \\
\times\,\sigma(J,I\setminus J)\sigma(I\setminus J,K)(a_{(j|J)}b)_{<n-j+k|K\cup(I\setminus J)>}.
\end{aligned} \tag{3.2.10.3}$$

It is straightforward to check using Lemma 3.2.7, that the Lie bracket in $\mathrm{Lie}(\mathscr{R})$ is given by:

$$\begin{aligned}
[a_{(n|I)}, b_{(k|K)}] = (-1)^{(a+N-I)(N-K)}\sum_{(j|J),j\ge 0}(-1)^{(I-J)(N-J)}\binom{n}{j}\times \\
\times\,\sigma(I)\sigma(J,I\setminus J)\sigma\left(I\setminus J,(N\setminus K)\setminus(I\setminus J)\right)\left(a_{(j|J)}b\right)_{(n+k-j|K\cup(I\setminus J))}.
\end{aligned} \tag{3.2.10.4}$$

Proposition 3.2.11. *Let $\mathscr{R}$ be an $N_W = N$ SUSY Lie conformal algebra, a, b two vectors in $\mathscr{R}$, and $a(Z)$, $b(W)$ the corresponding* $\mathrm{Lie}(\mathscr{R})$*-valued formal distributions defined by (3.2.10.2). Then*

$$[a(Z),b(W)] = \sum_{j\ge 0,J}\left(\partial_W^{(j|J)}\delta(Z,W)\right)\left(a_{(j|J)}b\right)(W). \tag{3.2.11.1}$$

Proof. First we expand

$$\partial_W^{(j|J)}\delta(Z,W) = \sum_{n\in\mathbb{Z},I} \binom{n}{j} (-1)^{I-J}\sigma(J)\sigma(N\setminus I, I\setminus J) Z^{-1-n|N\setminus I} W^{n-j|I\setminus J}. \quad (3.2.11.2)$$

Now using (3.2.10.4) we have:

$$[a(Z), b(W)] = \sum_{\substack{n\in\mathbb{Z},I\\ k\in\mathbb{Z},K}} \binom{n}{j} (-1)^{(I-J)(N-J)}\sigma(I)\sigma(J, I\setminus J)\times$$
$$\times\, \sigma(I\setminus J, (N\setminus K)\setminus(I\setminus J)) Z^{-1-n|N\setminus I} W^{-1-k|N\setminus K} \left(a_{(j|J)}b\right)_{(n+k-j|K\cup(I\setminus J)}. \quad (3.2.11.3)$$

On the other hand we have

$$W^{-1-k|N\setminus K} = \sigma(I\setminus J, (N\setminus K)\setminus(I\setminus J)) W^{n-j|I\setminus J} W^{-1-k-n+j|(N\setminus K)\setminus(I\setminus J)}, \quad (3.2.11.4)$$

and, due to (3.1.1.3),

$$\sigma(I)\sigma(J, I\setminus J) = (-1)^{(I-J)(N-I)}\sigma(N\setminus I, I\setminus J)\sigma(J). \quad (3.2.11.5)$$

Now substituting (3.2.11.4) in (3.2.11.3) and using (3.2.11.5) we obtain (3.2.11.1). □

Proposition 3.2.12. *Let $\mathscr{R}$ be an $N_W = N$ SUSY Lie conformal algebra, then the pair* $(\mathrm{Lie}(\mathscr{R}), \mathscr{R})$ *is an $N_W = N$ SUSY formal distribution Lie superalgebra.*

Proof. The fact that the family of distributions (3.2.10.2) is closed under $(j|J)$-th products and that they are pairwise local follows from Proposition 3.2.11 since $a_{(j|J)}b = 0$ for $j \gg 0$ in $\mathscr{R}$. The fact that this family is closed under the derivations $\partial_z, \partial_{\theta^i}$ follows from the following identities wich are straightforward to check

$$\begin{aligned} (Ta)_{(j|J)} &= -ja_{(j-1|J)}, \\ (S^i a)_{(j|J)} &= \sigma(e_i, N\setminus J) a_{(j|J\setminus e_i)}. \end{aligned} \quad (3.2.12.1)$$

□

3.2.13. Recall that we have defined $(j|J)$-th products of formal distributions for $j \geq 0$ in 3.2.1. In order to define these products for $j < 0$ we let for a formal distribution $a(Z) = \sum Z^{j|J} a_{j|J}$:

$$\begin{aligned} a_+(Z) &= \sum_{(j|J), j\geq 0} Z^{j|J} a_{j|J} \\ a_-(Z) &= \sum_{(j|J), j<0} Z^{j|J} a_{j|J} \end{aligned} \quad (3.2.13.1)$$

It follows easily from the definitions that

$$\begin{aligned} a_+(W) &= \mathrm{res}_Z\, i_{z,w}(Z-W)^{-1|N} a(Z), \\ a_-(W) &= -\,\mathrm{res}_Z\, i_{w,z}(Z-W)^{-1|N} a(Z). \end{aligned} \quad (3.2.13.2)$$

Indeed, we have

$$i_{z,w}(Z-W)^{-1|N} = \sum_{(m|J),m\geq 0} (-1)^J \sigma(J) W^{m|J} Z^{-1-m|N\setminus J}. \tag{3.2.13.3}$$

Hence:

$$\begin{aligned}
\operatorname{res}_Z i_{z,w}(Z-W)^{-1|N} a(Z) = \\
&= \operatorname{res}_Z \sum_{\substack{(m|J),m\geq 0\\ (n|I),n\in\mathbb{Z}}} (-1)^J \sigma(J) W^{m|J} Z^{-1-m|N\setminus J} Z^{n|I} a_{n|I} = \\
&= \operatorname{res}_Z \sum_{(m|J),m\geq 0} (-1)^J \sigma(J)\sigma(N\setminus J, J) W^{m|J} Z^{-1|N} a_{m|J} = \\
&= \sum_{(m|J),m\geq 0} W^{m|J} a_{m|J} = a_+(W).
\end{aligned} \tag{3.2.13.4}$$

The second equation in (3.2.13.2) follows similarly, or by noting that it is a consequence of the first equation in (3.2.13.2), the definition of the δ function (3.1.6.1) and property (5) in 3.1.6. Differentiating (3.2.13.2) we find:

$$\begin{aligned}
(-1)^{JN} \partial_W^{(j|J)} a(W)_+ &= \sigma(J) \operatorname{res}_Z i_{z,w}(Z-W)^{-1-j|N\setminus J} a(Z), \\
(-1)^{JN} \partial_W^{(j|J)} a(W)_- &= -\sigma(J) \operatorname{res}_Z i_{w,z}(Z-W)^{-1-j|N\setminus J} a(Z).
\end{aligned} \tag{3.2.13.5}$$

These equations (3.2.13.5) are called the super *Cauchy formulae.*

Definition 3.2.14. Let V be a vector superspace. An $\operatorname{End}(V)$-valued formal distribution $a(Z)$ is called a *field* if for every vector $v \in V$ we have $a(Z)v \in V((Z))$, i.e. there are finitely many negative powers of z in $a(Z)v$. For two such fields we define their *normally ordered product* to be

$$:a(Z)b(Z): := a_+(Z)b(Z) + (-1)^{ab} b(Z) a_-(Z) \tag{3.2.14.1}$$

3.2.15. The normally ordered product of fields is again a well defined field. Indeed, when applied to any vector $v \in V$ the first sumand in (3.2.14.1) clearly has finitely many negative powers of z since $b(Z)v \in V((Z))$ and $a_+(Z)$ has only non-negative powers of z. For the second sumand we see that $a_-(Z)v \in V[Z, Z^{-1}]$, namely it is a Laurent polynomial with values in V, therefore $b(Z)a_-(Z)v \in V((Z))$ as we wanted.

Lemma 3.2.16.

$$\begin{aligned}
:a(W)b(W): := \operatorname{res}_Z \big(& i_{z,w}(Z-W)^{-1|N} a(Z)b(W) - \\
& -(-1)^{ab} i_{w,z}(Z-W)^{-1|N} b(W)a(Z) \big).
\end{aligned} \tag{3.2.16.1}$$

Proof. By using (3.2.13.2) it follows that

$$
\begin{aligned}
: a(W)b(W) := &\left(\operatorname{res}_Z i_{z,w}(Z-W)^{-1|N} a(Z)\right) b(W) - \\
&- (-1)^{ab} b(W) \operatorname{res}_Z i_{w,z}(Z-W)^{-1|N} a(Z) = \\
&= \operatorname{res}_Z \left(i_{z,w}(Z-W)^{-1|N} a(Z) b(W) - \right. \\
&\left. -(-1)^{ab} i_{w,z}(Z-W)^{-1|N} b(W) a(Z) \right) \quad (3.2.16.2)
\end{aligned}
$$

as we wanted. □

3.2.17. Given the last lemma and the Cauchy formulae (3.2.13.5) it is natural to define

$$a(W)_{(-1-j|N\setminus J)} b(W) = \sigma(J)(-1)^{JN} : \left(\partial_W^{(j|J)} a(W)\right) b(W) : . \quad (3.2.17.1)$$

Differentiating (3.2.16.1) we find:

$$
\begin{aligned}
a(W)_{(-1-j|N\setminus J)} b(W) = \operatorname{res}_Z &\left(\left(i_{z,w}(Z-W)^{-1-j|N\setminus J} \right) a(Z) b(W) - \right. \\
&\left. -(-1)^{ab} \left(i_{w,z}(Z-W)^{-1-j|N\setminus J} \right) b(W) a(Z) \right). \quad (3.2.17.2)
\end{aligned}
$$

Similarly, from the definition of the $j|J$-th products for $j \geq 0$ in 3.2.1 we have:

$$
\begin{aligned}
\operatorname{res}_Z &\left(\left(i_{z,w}(Z-W)^{j|J} \right) a(Z) b(W) - (-1)^{ab} \left(i_{w,z}(Z-W)^{j|J} \right) b(W) a(Z) \right) = \\
&= \operatorname{res}_Z (Z-W)^{j|J} \left(a(Z) b(W) - (-1)^{ab} b(W) a(Z) \right) = \\
&= \operatorname{res}_Z (Z-W)^{j|J} [a(Z), b(W)] = a(W)_{(j|J)} b(W). \quad (3.2.17.3)
\end{aligned}
$$

Therefore we have proved that for every $j \in \mathbb{Z}$ and every tuple J we have:

$$
\begin{aligned}
a(W)_{(j|J)} b(W) = \operatorname{res}_Z &\left(\left(i_{z,w}(Z-W)^{j|J} \right) a(Z) b(W) - \right. \\
&\left. -(-1)^{ab} \left(i_{w,z}(Z-W)^{j|J} \right) b(W) a(Z) \right). \quad (3.2.17.4)
\end{aligned}
$$

Proposition 3.2.18. *The following identities analogous to sesquilinearity for all pairs $j|J$ are true:*

$$
\begin{aligned}
(\partial_w a(W))_{(j|J)} b(W) &= -j a(W)_{(-j-1|J)} b(W) \\
\partial_w \left(a(W)_{(j|J)} b(W) \right) &= (\partial_w a(W))_{(j|J)} b(W) + a(W)_{(j|J)} \partial_w b(W) \\
\left(\partial_{\zeta^i} a(W) \right)_{(j|J)} b(W) &= \sigma(J \setminus e_i, e_i) a(W)_{(j|J\setminus e_i)} b(W) \qquad (3.2.18.1) \\
\partial_{\zeta^i} \left(a(W)_{(j|J)} b(W) \right) &= (-1)^{N-J} \left(\left(\partial_{\zeta^i} a(W) \right)_{(j|J)} b(W) + \right. \\
&\left. +(-1)^a a(W)_{(j|J)} \left(\partial_{\zeta^i} b(W) \right) \right),
\end{aligned}
$$

where e_i is the tuple consisting of only one element $\{i\}$ and we recall that we are defining $\sigma(e_i, J \setminus e_i)$ to be zero if $i \notin J$.

Proof. The first two equations are standard and their proof is similar to the last two. We will prove the last two equations by using (3.2.17.4). If $i \notin J$ the result is obvious.

$$\begin{aligned}\operatorname{res}_Z i_{z,w}(Z-W)^{j|J}\partial_{\theta^i}a(Z)b(W) &= -(-1)^J \operatorname{res}_Z \left(\partial_{\theta^i} i_{z,w}(Z-W)^{j|J}\right) a(Z)b(W) \\ &= -(-1)^J \sigma(e_i, J\setminus e_i) \operatorname{res}_Z i_{z,w}(Z-W)^{j|J\setminus e_i} a(Z)b(W). \quad (3.2.18.2)\end{aligned}$$

Similarly we have:

$$\begin{aligned}&-(-1)^{(a+1)b}\operatorname{res}_Z i_{w,z}(Z-W)^{j|J} b(W)\partial_{\theta^i}a(Z) = \\ &\qquad = (-1)^{ab+J}\operatorname{res}_Z\left(\partial_{\theta^i} i_{w,z}(Z-W)^{j|J}\right) b(W)a(Z) = \\ &\qquad = (-1)^{ab+J}\sigma(e_i, J\setminus e_i)\operatorname{res}_Z i_{w,z}(Z-W)^{j|J\setminus e_i} b(W)a(Z). \quad (3.2.18.3)\end{aligned}$$

Adding (3.2.18.2) and (3.2.18.3) we obtain:

$$\left(\partial_{\zeta^i}a(W)\right)_{(j|J)} b(W) = -(-1)^J\sigma(e_i, J\setminus e_i) a(W)_{(j|J\setminus e_i)} b(W). \qquad (3.2.18.4)$$

Finally, to prove the last relation in (3.2.18.1) we expand:

$$\begin{aligned}\partial_{\zeta^i}\left(a(W)_{(j|J)}b(W)\right) &= \partial_{\zeta^i}\operatorname{res}_Z\left(i_{z,w}(Z-W)^{j|J}a(Z)b(W) -\right. \\ &\qquad \left. -(-1)^{ab} i_{w,z}(Z-W)^{j|J} b(W)a(Z)\right) = \\ &(-1)^N \operatorname{res}_Z\left(-\sigma(e_i, J\setminus e_i) i_{z,w}(Z-W)^{j|J\setminus e_i}a(Z)b(W) + \right. \\ &\qquad +(-1)^{J+a} i_{z,w}(Z-W)^{j|J} a(Z)\partial_{\zeta^i}b(W) - \\ &\qquad -(-1)^{ab}\sigma(e_i, J\setminus e_i) i_{w,z}(Z-W)^{j|J\setminus e_i} b(W)a(Z) - \\ &\qquad \left. -(-1)^{ab+J} i_{w,z}(Z-W)^{j|J}\partial_{\zeta^i}b(W)a(Z)\right) = \\ &\qquad = -(-1)^N\sigma(e_i, J\setminus e_i)a(W)_{(j|J\setminus e_i)}b(W) + \\ &\qquad + (-1)^{N+J+a} a(W)_{(j|J)}\partial_{\zeta^i}b(W) = \\ &= (-1)^{N-J}\left(\left(\partial_{\zeta^i}a(W)\right)_{(j|J)} b(W) + (-1)^a a(W)_{(j|J)}\partial_{\zeta^i}b(W)\right). \quad (3.2.18.5)\end{aligned}$$

□

Proposition 3.2.19. *The following identity holds for any $(j|J)$ and any three fields $a = a(W)$, $b = b(W)$, $c = c(W)$:*

$$\begin{aligned}&[a_\Lambda (b_{(j|J)}c)] = \\ &\qquad = \sum_{(k|K), k\geq 0} (-1)^{(a+K+N)(J+N)}\sigma(J,K)\Lambda^{(k|K)}[a_\Lambda b]_{(j+k|J\cup K)}c + \\ &\qquad\qquad + (-1)^{(a+N)(b+N-J)} b_{(j|J)}[a_\Lambda c]. \quad (3.2.19.1)\end{aligned}$$

Proof. The left hand side is

$$
\begin{aligned}
&\operatorname{res}_Z \exp\left((Z-W)\Lambda\right)\left[a(Z),(b(W)_{(j|J)}c(W))\right]=\\
&\qquad=\operatorname{res}_Z \exp\left((Z-W)\Lambda\right)\left(\left[a(Z),\operatorname{res}_X i_{x,w}(X-W)^{j|J}b(X)c(W)\right]-\right.\\
&\qquad\qquad\left.-(-1)^{bc}\left[a(Z),\operatorname{res}_X i_{w,x}(X-W)^{j|J}c(W)b(X)\right]\right)=\\
&(-1)^{a(N-J)}\operatorname{res}_Z\operatorname{res}_X \exp\left((Z-W)\Lambda\right) i_{x,w}(X-W)^{j|J}[a(Z),b(X)c(W)]-\\
&-(-1)^{bc+a(N-J)}\operatorname{res}_Z\operatorname{res}_X \exp\left((Z-W)\Lambda\right) i_{w,x}(X-W)^{j|J}[a(Z),c(W)b(X)]
\end{aligned}
\tag{3.2.19.2}
$$

Using the identity $[a,bc]=[a,b]c+(-1)^{ab}b[a,c]$ we can write the first term of the RHS of the last equality as:

$$
\begin{aligned}
&(-1)^{a(N-J)}\operatorname{res}_Z\operatorname{res}_X \exp\left((Z-X+X-W)\Lambda\right)\times\\
&\qquad\times i_{x,w}(X-W)^{j|J}[a(Z),b(X)]c(W)+\\
&\qquad+(-1)^{a(N-J+b)}\operatorname{res}_Z\operatorname{res}_X \exp\left((Z-W)\Lambda\right)\times\\
&\qquad\times i_{x,w}(X-W)^{j|J}b(X)[a(Z),c(W)]=\\
&=(-1)^{a(N-J)+N+JN}\operatorname{res}_X \exp\left((X-W)\Lambda\right) i_{x,w}(X-W)^{j|J}[a_\Lambda b](X)c(W)+\\
&\quad+(-1)^{a(N-J+b)+N+JN+bN}\operatorname{res}_X i_{x,w}(X-W)^{j|J}b(X)[a_\Lambda c](W)=\\
&=(-1)^{(a+N)(N-J)}\operatorname{res}_X\sum_{(k|K),k\geq 0}\frac{(-1)^{\frac{K(K+1)}{2}}}{k!}\Lambda^{k|K}\times\\
&\qquad\times i_{x,w}(X-W)^{k|K}(X-W)^{j|J}[a_\Lambda b](X)c(W)+\\
&\quad+(-1)^{(a+N)(N-J+b)}\operatorname{res}_X i_{x,w}(X-W)^{j|J}b(X)[a_\Lambda c](W)=\\
&=(-1)^{(a+N)(N-J)}\operatorname{res}_X\sum_{(k|K),k\geq 0}\frac{(-1)^{\frac{K(K+1)}{2}}}{k!}\sigma(K,J)\times\\
&\qquad\times \Lambda^{k|K} i_{x,w}(X-W)^{k+j|K\cup J}[a_\Lambda b](X)c(W)+\\
&\quad+(-1)^{(a+N)(N-J+b)}\operatorname{res}_X i_{x,w}(X-W)^{j|J}b(X)[a_\Lambda c](W)
\end{aligned}
\tag{3.2.19.3}
$$

Similarly the second term in the RHS of the last equality of (3.2.19.2) can be written as:

$$
\begin{aligned}
&-(-1)^{bc+a(N-J)}\operatorname{res}_Z\operatorname{res}_X\exp\left((Z-W)\Lambda\right)\times\\
&\quad\times i_{w,x}(X-W)^{j|J}[a(Z),c(W)]b(X)-(-1)^{bc+a(N-J+c)}\operatorname{res}_Z\operatorname{res}_X\times\\
&\quad\times\exp\left((Z-W)\Lambda\right)i_{w,x}(X-W)^{j|J}c(W)[a(Z),b(X)]=\\
&\quad=-(-1)^{bc+a(N-J)+N+JN}\operatorname{res}_X i_{w,x}(X-W)^{j|J}[a_\Lambda c](W)b(X)-\\
&-(-1)^{bc+(a+N)(N-J+c)}\operatorname{res}_X\exp\left((X-W)\Lambda\right)i_{w,x}(X-W)^{j|J}c(W)[a_\Lambda b](X)\\
&\quad=-(-1)^{bc+(a+N)(N-J)}\operatorname{res}_X i_{w,x}(X-W)^{j|J}[a_\Lambda c](W)b(X)-\\
&\quad-(-1)^{bc+(a+N)(N-J+c)}\operatorname{res}_X\sum_{(k|K),k\geq 0}\frac{(-1)^{\frac{K(K+1)}{2}}}{k!}\times\\
&\quad\times\sigma(K,J)\Lambda^{k|K}i_{w,x}(X-W)^{k+j,K\cup J}c(W)[a_\Lambda b](X). \qquad (3.2.19.4)
\end{aligned}
$$

Now adding (3.2.19.3) and (3.2.19.4) we get (recall that the Λ-bracket has parity $N \bmod 2$):

$$
\begin{aligned}
(-1)^{(a+N)(N-J)}\sum_{(k|K),k\geq 0}\frac{(-1)^{\frac{K(K+1+2N)}{2}}}{k!}\sigma(K,J)\Lambda^{k|K}[a_\Lambda b]_{(k+j|J\cup K)}c+\\
+(-1)^{(a+N)(N-J+b)}b_{(j|J)}[a_\Lambda c]. \qquad (3.2.19.5)
\end{aligned}
$$

□

Remark 3.2.20. If we multiply both sides of (3.2.19.1) by

$$
\frac{(-1)^{\frac{J(J+1+2a)}{2}}}{j!}\Gamma^{j|J}, \qquad (3.2.20.1)
$$

and sum over all pairs $(j|J)$ with $j \geq 0$ we obtain the Jacobi identity for the Λ-bracket that we have already proved in Proposition 3.2.2. Therefore, the identities (3.2.19.1) for $j \geq 0$ are equivalent to the Jacobi identity (3.2.3.4).

Next we note that if we replace b by $\partial_w b$ in (3.2.19.1) we obtain the same identity with j replaced by $j-1$ whenever $j \leq -1$. Similarly, replacing b by $\partial_{\zeta^i} b$ we obtain the same identity with J replaced by $J \setminus e_i$. It follows the identity (3.2.19.1) is equivalent to the Jacobi identity (3.2.3.4) and (3.2.19.1) with $(j|J) = (-1|N)$. In this case the formula (3.2.19.1) looks as follows:

$$
[a_\Lambda : bc :] = \sum_{k\geq 0}\frac{\lambda^k}{k!}[a_\Lambda b]_{(k-1|N)}c + (-1)^{(a+N)b} : b[a_\Lambda c] : \qquad (3.2.20.2)
$$

Rewriting the sum as the sum of the $k=0$ term and the rest, this becomes:

$$
[a_\Lambda : bc :] =: [a_\Lambda b]c : +(-1)^{(a+N)b} : b[a_\Lambda c] : + \int_0^\Lambda [[a_\Lambda b]_\Gamma c] d\Gamma. \qquad (3.2.20.3)
$$

Here the integral $\int_0^\Lambda$ is computed by taking the indefinite integral in the even variable γ of ∂_η^N of the integrand, and then taking the difference of the values at the limits. This is the super analogue of the *non-commutative Wick formula* [22]. Thus, the identity (3.2.19.1) is equivalent to the Jacobi identity plus this non-commutative Wick formula.

The following lemma is proved as in the ordinary vertex algebra case.

Lemma 3.2.21 (Dong's Lemma). *Given three pairwise local formal distributions a, b, c, the pair $(a, b_{(j|J)}c)$ is local for any $(j|J)$.*

3.3 Existence theorem.

In this section we define $N_W = N$ SUSY vertex algebras and prove an existence theorem as in the non-super case [22, thm. 4.5]. Recall the definition of a $N_W = N$ SUSY vertex algebra in 2.1.27:

Definition 3.3.1. An $N_W = N$ SUSY vertex algebra consists of a vector superspace V, an even vector $|0\rangle \in V$, N odd operators S^i (the odd translation operators), an even operator T (the even translation operator), and a parity preserving linear map $\overset{s}{Y}$ from V to the space of End(V)-valued superfields $a \mapsto \overset{s}{Y}(a, Z)$. The following axioms must be satisfied:

- Vacuum axioms:

$$\begin{aligned} \overset{s}{Y}(a,Z)|0\rangle &= a + O(Z), \\ T|0\rangle = S^i|0\rangle &= 0, \qquad i = 1, \dots, N. \end{aligned} \tag{3.3.1.1}$$

- Translation invariance

$$\begin{aligned} [S^i, \overset{s}{Y}(a,Z)] &= \partial_{\theta^i}\overset{s}{Y}(A,Z), \\ [T, \overset{s}{Y}(a,Z)] &= \partial_z \overset{s}{Y}(a,Z). \end{aligned} \tag{3.3.1.2}$$

- Locality

$$(z-w)^n[\overset{s}{Y}(a,Z), \overset{s}{Y}(b,W)] = 0 \quad \text{for } n \gg 0 \tag{3.3.1.3}$$

Morphisms between $N_W = N$ SUSY vertex algebras are linear maps $f : V_1 \to V_2$ such that

$$\begin{aligned} f \circ T_1 &= T_2 \circ f \\ f(\overset{s}{Y}_1(a,Z)b) &= \overset{s}{Y}(f(a),Z)f(b) \qquad \forall a, b \in V_1. \end{aligned} \tag{3.3.1.4}$$

3.3.2. Given a $N_W = n$ SUSY vertex algebra V, we can define the $(j|J)$ product of two vectors of V as follows. We expand the field $\overset{s}{Y}(a,Z)$ for $a \in V$:

$$\overset{s}{Y}(a,Z) = \sum_{(j|J),j\in\mathbb{Z}} Z^{-1-j|N\setminus J} a_{(j|J)}, \tag{3.3.2.1}$$

and define the $j|J$-product of two vectors in V as:

$$a_{(j|J)}b := a_{(j|J)}\big(b\big). \tag{3.3.2.2}$$

This is a $\mathbb{C}$-bilinear product on V of parity $N - J \mod 2$. We can rewrite the axioms of the vertex algebra in terms of these products. We will need only the vacuum axioms and translation invariance. The vacuum axioms are equivalent to:

$$|0>_{(j|J)} a = \delta_{j,-1}\delta_{J,N}a, \qquad a_{(j|J)}|0> = 0 \quad \text{if } j \geq 0, \tag{3.3.2.3}$$

$$T|0>= S^i|0> = 0, \qquad a_{(-1|N)}|0> = a. \tag{3.3.2.4}$$

Translation invariance is equivalent to:

$$\begin{aligned} &[T, a_{(j|J)}] = -j a_{(j-1|J)}, \\ &[S^i, a_{(j|J)}] = \sigma(N \setminus J, e_i) a_{(j|J\setminus e_i)} \quad \text{if } i \in J, \\ &[S^i, a_{(j|J)}] = 0 \qquad \text{if } i \notin J. \end{aligned} \tag{3.3.2.5}$$

These equations are obtained easily by expanding the fields as in (3.3.2.1) and using (3.3.1.2).

Of course the fact that $\overset{s}{Y}(a,Z)$ is a field is equivalent to $a_{(j|J)}b = 0$ for $j \gg 0$, given $a, b \in V$.

Theorem 3.3.3. *Let $\mathscr{U}$ be a vector superspace and V a space of pairwise local* $\operatorname{End}(\mathscr{U})$*-valued fields such that V contains the constant field* Id, *it is invariant under the derivations $\partial_z, \partial_{\theta^i}$ and closed under all $(j|J)$-th products. Then V is a $N_W = N$ SUSY vertex algebra with vacuum vector* Id, *translation operators $Ta(Z) = \partial_z a(Z)$ and $S^i a(Z) = \partial_{\theta^i} a(Z)$, and the $(j|J)$ products are given by the RHS of (3.2.17.4) multiplied by $\sigma(J)^2$.*

Proof. To check the vacuum axioms we have (we fix $j \geq 0$ in these equations):

$$\begin{aligned} 1_{(-1|N)}a(Z) &=: 1a(Z) := a(Z), \\ 1_{(-1-j|N\setminus J)}a(Z) &\propto: \left(\partial_W^{j|J} 1\right) a(Z) := 0, \\ 1_{(j|J)}a(W) &= \sigma(J)\operatorname{res}_Z (Z-W)^{j|J}[1, a(W)] = 0, \\ a(Z)_{(j|J)}1 &= \sigma(J)\operatorname{res}_Z (Z-W)^{j|J}[a(Z), 1] = 0, \\ a(Z)_{(-1|N)}1 &=: a(Z)1 := a(Z), \\ \partial_z 1 &= \partial_{\theta^i} 1 = 0. \end{aligned} \tag{3.3.3.1}$$

[2]This normalization becomes is necesary because of our choice in (3.2.17.1), see also theorem 3.3.9

To check translation invariance we have:

$$\partial_z(a(Z)_{(j|J)}b(Z)) - a(Z)_{(j|J)}\partial_z b(Z) = (\partial_z a(Z))_{(j|J)}b(Z), \tag{3.3.3.2}$$

but this is $-ja(Z)_{(j-1|J)}b(Z)$, according to (3.2.18.1). Therefore we see that the first equation in (3.3.2.5) holds (we recover actually this equation multiplied by $\sigma(J)$). For the odd tranlation operators we write (note that the parity of $a_{(j|J)}$ is $a+N-J$ since $\overset{s}{Y}$ is parity preserving and our choice of decomposing the field in (3.3.2.1)):

$$\begin{aligned}\sigma(J)\Big(\partial_{\theta^i}(a(Z)_{(j|J)}b(Z)) - (-1)^{a+N-J}a(Z)_{(j|J)}\partial_{\theta^i}b(Z)\Big) &= \\ = (-1)^{N-J}\sigma(J)(\partial_{\theta^i}a(Z))_{(j|J)}b(Z),\end{aligned} \tag{3.3.3.3}$$

and again by (3.2.18.1) we see that this is

$$\begin{aligned}-(-1)^N\sigma(J)\sigma(e_i, J\setminus e_i)a(Z)_{(j|J\setminus e_i)}b(Z) &= \\ = \sigma(N\setminus J, e_i)\sigma(J\setminus e_i)a(Z)_{(j|J\setminus e_i)}b(Z),\end{aligned} \tag{3.3.3.4}$$

and we have proved the second identity in (3.3.2.5). In order to check locality, we expand

$$\begin{aligned}\overset{s}{Y}(a(W),X)b(W) &= \sum_{(j|J),j\in\mathbb{Z}} \sigma(J)X^{-1-j|N\setminus J}a(W)_{(j|J)}b(W) \\ &= \mathrm{res}_Z \sum_{(j|J),j\in\mathbb{Z}} (-1)^{(N-J)N}\sigma(J)X^{-1-j|N\setminus J}\times \\ &\times \Big(i_{z,w}(Z-W)^{j|J}a(Z)b(W) - (-1)^{ab}i_{w,z}(Z-W)^{j|J}b(W)a(Z)\Big) \\ &= \mathrm{res}_Z \sum_{(j|J),j\in\mathbb{Z}} (-1)^{N-J}\sigma(J)\Big(i_{z,w}(Z-W)^{j|J}X^{-1-j|N\setminus J}\times \\ &\times a(Z)b(W) - (-1)^{ab}i_{w,z}(Z-W)^{j|J}X^{-1-j|N\setminus J}b(W)a(Z)\Big).\end{aligned} \tag{3.3.3.5}$$

We note now that

$$\begin{aligned}i_{z,w}\sum_{(j|J),j\in\mathbb{Z}} (-1)^{(N-J)}\sigma(J)(Z-W)^{j|J}X^{-1-j|N\setminus J} &= \\ = i_{z,w}\delta(Z-W,X).\end{aligned} \tag{3.3.3.6}$$

Therefore the RHS of (3.3.3.5) reads:

$$\mathrm{res}_Z\Big(i_{z,w}\delta(Z-W,X)a(Z)b(W) - (-1)^{ab}i_{w,z}\delta(Z-W,X)b(W)a(Z)\Big). \tag{3.3.3.7}$$

With this last equation we can compute then the commutator $[\overset{s}{Y}(a(W)),\overset{s}{Y}(b(W))]c(W)$.

Indeed, the product $\overset{s}{Y}(a(W),X)\overset{s}{Y}(b(W),Y)c(W)$ is given by:

$$\begin{aligned}\operatorname{res}_Z \operatorname{res}_U \Big(& i_{u,w}i_{z,w}\delta(U-W,X)\delta(Z-W,Y)a(U)b(Z)c(W)- \\ & -(-1)^{bc} i_{u,w}i_{w,z}\delta(U-W,X)\delta(Z-W,Y)a(U)c(W)b(Z)- \\ & -(-1)^{a(b+c)} i_{w,u}i_{z,w}\delta(U-W,X)\delta(Z-W,Y)b(Z)c(W)a(U)+ \\ & +(-1)^{a(b+c)+bc} i_{w,u}i_{w,z}\delta(U-W,X)\delta(Z-W,Y)c(W)b(Z)a(U)\Big), \end{aligned} \tag{3.3.3.8}$$

and we get a similar expression for the product $\overset{s}{Y}(b(W),Y)\overset{s}{Y}(a(W),X)c(W)$. Substracting we obtain:

$$\begin{aligned}&[\overset{s}{Y}(a(W),X),\overset{s}{Y}(b(W),Y)]c(W) = \\ &\qquad \operatorname{res}_Z \operatorname{res}_U \Big(i_{u,w}i_{z,w}\delta(U-W,X)\delta(Z-W,Y)[a(U),b(Z)]c(W)- \\ &\qquad -(-1)^{(a+b)c} i_{w,u}i_{w,z}\delta(U-W,X)\delta(Z-W,Y)c(W)[a(U),b(Z)]\Big). \end{aligned} \tag{3.3.3.9}$$

Let $n \in \mathbb{Z}_+$ be such tht $(u-z)^n[a(U),b(Z)] = 0$. Multliplying (3.3.3.9) by $(x-y)^n$ we obtain that the RHS vanishes. Indeed, using

$$(x-y) = (z-u) - ((z-w)-x) + ((u-w)-y), \tag{3.3.3.10}$$

we see that all terms in the expansion of $(x-y)^n$ vanish when multiplyed by δ functions, with the exception of $(z-u)^n$. But this term vanishes when multiplied by the factors $[a(U),b(Z)]$ in (3.3.3.9). Therefore we have proved locality and the theorem. □

Corollary 3.3.4. *Any identity on elements of an $N_W = N$ SUSY vertex algebra, holds for any collection of pairwise local fields.*

Lemma 3.3.5. *Let V be a vector superspace and let $|0>$ be an even vector of V. Let $a(Z), b(Z)$ be two* $\operatorname{End}(V)$*-valued fields such that $a(Z)|0> \in V[[Z]]$ and $b(Z)|0> \in V[[Z]]$. Then for all $(j|J)$, $a(W)_{(j|J)}b(W)|0> \in V[[W]]$ and the constant term is*

$$\sigma(J)a_{(j|J)}b_{(-1|N)}|0> . \tag{3.3.5.1}$$

Proof. Applying both sides of (3.2.17.4) to the vacuum, we see that the second term on the RHS of (3.2.17.4) vanishes since it contains only positive powers of z. The first term in the RHS contains only positive powers of w since $i_{z,w}(Z-W)^{j|J}$ does and $b(W)|0> \in \mathbb{C}[[W]]$. Letting $W=0$ we get

$$a(W)_{(j|J)}b(W)|0> |_{W=0} = \operatorname{res}_Z Z^{j|J}a(Z)\left(b_{(-1|N)}|0>\right). \tag{3.3.5.2}$$

It follows from (3.1.2.8) that the RHS of (3.3.5.2) is

$$\sigma(J)a_{(j|J)}\left(b_{(-1|N)}|0>\right). \tag{3.3.5.3}$$

□

The following lemma is straightforward

Lemma 3.3.6. *Let A and $B_1, \ldots, B_N$ be linear operators on a vector superspace $\mathscr{U}$. Suppose that A is even and B_i are odd and they pairwise (super) commute, i.e. $AB_i = B_iA$, $B_iB_j = -B_jB_i$. Then there exists a unique solution $f(Z) \in \mathscr{U}[[Z]]$ to the system of differential equations:*

$$\partial_z f(Z) = Af(Z) \qquad \partial_{\theta^i} f(Z) = B_i f(Z) \tag{3.3.6.1}$$

for any initial condition $f(0) = f_0$.

Proof. Using (3.3.6.1), the coefficients of $f(Z)$ can be computed by induction, given f_0. □

Proposition 3.3.7. *Let V be a $N_W = N$ SUSY vertex algebra. Then for every $a, b \in V$:*

a. $\overset{s}{Y}(a, Z)|0>= \exp(Z\nabla)a$.

b. $\exp(Z\nabla)\overset{s}{Y}(a, W)\exp(-Z\nabla) = i_{w,z}\overset{s}{Y}(a, Z+W)$.

c. $\overset{s}{Y}(a, Z)_{(j|J)}\overset{s}{Y}(b, Z)|0>= \sigma(J)\overset{s}{Y}(a_{(j|J)}b, Z)|0>$,

where $\nabla = (T, S^1, \ldots, S^N)$ *and* $Z\nabla = zT + \sum \theta^i S^i$.

Proof. We note that both sides in (a) and (c) are elements of $V[[Z]]$ whereas both sides of (b) are elements of $\mathrm{End}(V)[[W, W^{-1}]][[Z]]$. Note that by evaluating at $Z = 0$ we get equalities in all three cases, the only non-trivial case is (c) but it follows from Lemma 3.3.5. Let us denote the right hand side in each case by $X(Z)$. It is easy to show that it satisfies the following systems of equations respectively:

1. $\partial_z X(Z) = TX(Z)$, and $\partial_{\theta^i} X(Z) = S^i X(Z)$.

2. $\partial_z X(Z) = [T, X(Z)]$ and $\partial_{\theta^i} X(Z) = [S^i, X(Z)]$ by the translation axioms.

3. $\partial_z X(Z) = TX(Z)$ and $\partial_{\theta^i} X(Z) = S^i X(Z)$ by the translation axioms (recall $T|0>= S^i|0>= 0$.

In order to apply Lemma 3.3.6, we have to show that the left hand side of (a), (b) and (c) satisfies the same differential equations (1), (2) and (3) respectively;

1.
$$\partial_z \overset{s}{Y}(a,Z)|0>= [T,\overset{s}{Y}(a,Z)]|0>= T\overset{s}{Y}(a,Z)|0>,$$

and
$$\partial_{\theta^i} \overset{s}{Y}(a,Z) = [S^i,\overset{s}{Y}(a,Z)]|0>= S^i\overset{s}{Y}(a,Z)|0> .$$

by the vacuum and translation invariance axioms.

2. In the case of (b), denoting $Y(Z) = e^{Z\nabla}\overset{s}{Y}(a,W)e^{-Z\nabla}$ we have:

$$\partial_z Y(Z) = TY(Z) - Y(Z)T == [T,Y(Z)],$$

and similarly:

$$\begin{aligned}\partial_{\theta^i} Y(Z) &= S^iY(Z) + (-1)^a e^{Z\nabla}\overset{s}{Y}(a,W)(-S^i)e^{-Z\nabla} \\ &= S^iY(Z) - (-1)^a Y(Z)S^i = [S^i,Y(Z)].\end{aligned}$$

3. Denote $Y(Z) = \overset{s}{Y}(a,Z)_{(j|J)}\overset{s}{Y}(b,Z)|0>$ and recall that from Proposition 3.2.18, the derivatives ∂_z and ∂_{θ^i} are derivations of the $(j|J)$ products. To simplify notation, we will denote $a(Z) = \overset{s}{Y}(a,Z)$ and $b(Z) = \overset{s}{Y}(b,Z)$. We have:

$$\begin{aligned}S^iY(W) &= S^i \operatorname{res}_Z\Big(i_{z,w}(Z-W)^{j|J}a(Z)b(W)|0> - \\ &\quad - (-1)^{ab} i_{w,z}(Z-W)^{j|J}b(W)a(Z)|0>\Big) \\ &= (-1)^{N+J}\operatorname{res}_Z\Big(i_{z,w}(Z-W)^{j|J}[S^i,a(Z)]b(W)|0> + \\ &\quad + (-1)^a i_{z,w}(Z-W)^{j|J}a(Z)[S^i,b(W)]|0> - \\ &\quad - (-1)^{ab} i_{w,z}(Z-W)^{j|J}[S^i,b(W)]a(Z)|0> - \\ &\quad - (-1)^{ab+b} i_{w,z}(Z-W)^{j|J}b(W)[S^i,a(Z)]|0>\Big),\end{aligned}$$

and using $S^i|0>= 0$

$$\begin{aligned}&= (-1)^{N+J}\operatorname{res}_Z\Big(i_{z,w}(Z-W)^{j|J}(\partial_{\theta^i}a(Z))b(W)|0> + \\ &\quad + (-1)^a i_{z,w}(Z-W)^{j|J}a(Z)(\partial_{\varsigma^i}b(W))|0> - \\ &\quad - (-1)^{ab} i_{w,z}(Z-W)^{j|J}(\partial_{\varsigma^i}b(W))a(Z)|0> - \\ &\quad - (-1)^{ab+b} i_{w,z}(Z-W)^{j|J}b(W)(\partial_{\theta^i}a(Z))|0>\Big) \\ &= (-1)^{N+J}\Big((\partial_{\varsigma^i}a(W))_{(j|J)}b(W) + (-1)^a a(W)_{(j|J)}(\partial_{\varsigma^i}b(W))\Big)|0> \\ &= \partial_{\varsigma^i}\left(a(W)_{(j|J)}b(W)|0>\right).\end{aligned}$$

The proof for T is similar.

$\square$

Proposition 3.3.8 (Uniqueness). *Let V be a $N_W = N$ SUSY vertex algebra and let $a(Z)$ be an* $\mathrm{End}(V)$*-valued field such that the pair $(a(Z), \overset{s}{Y}(b, Z))$ is local for every $b \in V$, and $a(Z)|0\rangle = 0$, then $a(Z) = 0$.*

Proof. By locality there exists $n \in \mathbb{Z}_+$ such that

$$(z-w)^n a(Z) \overset{s}{Y}(b, W)|0\rangle = (-1)^{ab}(z-w)^n \overset{s}{Y}(b, W) a(Z)|0\rangle = 0. \qquad (3.3.8.1)$$

By proposition 3.3.7 (1) the left hand side is $(z-w)^n a(Z) e^{W\nabla} b$, Letting $W = 0$ we get $z^n a(Z) b = 0$, and this holds for all b, therefore $a(Z) = 0$. □

As a simple corolary of the previous proposition and proposition 3.3.7 we obtain the following

Theorem 3.3.9. *In an $N_W = N$ SUSY vertex algebra the following identities hold*

1. $\overset{s}{Y}(a_{(j|J)} b, Z) = \sigma(J) \overset{s}{Y}(a, Z)_{(j|J)} \overset{s}{Y}(b, Z)$. *This identity is called the $(j|J)$-th product identity.*
2. $\overset{s}{Y}(a_{(-1|N)} b, Z) =: \overset{s}{Y}(a, Z) \overset{s}{Y}(b, Z) :$.
3. $\overset{s}{Y}(Ta, Z) = \partial_z \overset{s}{Y}(a, Z)$.
4. $\overset{s}{Y}(S^i a, Z) = \partial_{\theta^i} \overset{s}{Y}(a, Z)$.
5. *We have the following OPE formula:*

$$\begin{aligned}[\overset{s}{Y}(a, Z), \overset{s}{Y}(b, W)] &= \sum_{(j,J), j\geq 0} \sigma(J) (\partial_W^{(j|J)} \delta(Z, W)) \overset{s}{Y}(a_{(j|J)} b, W) \\ &= \sum_{(j|J), j \geq 0} (i_{z,w} - i_{w,z})(Z-W)^{-1-j|N\setminus J} \overset{s}{Y}(a_{(j|J)} b, W)\end{aligned} \qquad (3.3.9.1)$$

where the sum is finite.

Proof. (1) is the combined statement of Dong's lemma and Propositions 3.3.8 and 3.3.7 (c). (2) follows from (1) by letting $j|J = -1|N$. To prove (3) we write:

$$\begin{aligned}\overset{s}{Y}(Ta, Z) = \overset{s}{Y}(a_{(-2,N)}|0\rangle, Z) = \overset{s}{Y}(a, Z)_{(-2|N)} \mathrm{Id} = \\ =: \partial_z \overset{s}{Y}(a, Z) \mathrm{Id} := \partial_z \overset{s}{Y}(a, Z)\end{aligned} \qquad (3.3.9.2)$$

(4) follows similarly:

$$\begin{aligned}&\overset{s}{Y}(S^i a, Z) = \overset{s}{Y}(a_{(-1,N\setminus e_i)}|0\rangle, Z) = \\ &= -\sigma(N\setminus e_i, e_i)\sigma(e_i, N\setminus e_i)(-1)^N : \partial_{\theta^i} \overset{s}{Y}(a, Z) \mathrm{Id} := \partial_{\theta^i} \overset{s}{Y}(a, Z).\end{aligned} \qquad (3.3.9.3)$$

Finally (5) follows from (1) and the decomposition Lemma 3.1.7 $\square$

Corollary 3.3.10. *Let $e_i = \{i\}$. One has:*

$$\begin{aligned}
(Ta)_{(j|J)} &= -j a_{(j-1|J)} \\
(S^i a)_{(j|J)} &= \sigma(e_i, N \setminus J) a_{(j|J\setminus e_i)} \\
T(a_{(j|J)}b) &= (Ta)_{(j|J)}b + a_{(j|J)}T(b) \\
S^i(a_{(j|J)}b) &= (-1)^{N-J}\Big((S^i a)_{(j|J)}b + (-1)^a a_{(j|J)} S^i b\Big).
\end{aligned} \tag{3.3.10.1}$$

Compare with (3.2.18.1).

Lemma 3.3.11.

$$i_{x,z}\delta(X - Z, W) = i_{w,z}\delta(X, W + Z). \tag{3.3.11.1}$$

Proof. For simplicity let us assume $N = 0$, the general result follows easily. Denote:

$$\begin{aligned}
\psi &= i_{x,z} i_{x-z,w}(x - w - z)^{-1} \in \mathbb{C}[[x, x^{-1}, z, z^{-1}, w, w^{-1}]], \\
\varphi &= i_{w,z} i_{x,w+z}(x - w - z)^{-1} \in \mathbb{C}[[x, x^{-1}, z, z^{-1}, w, w^{-1}]]
\end{aligned} \tag{3.3.11.2}$$

It is straightforward to check that both ψ and φ are elements of $K[[z, w]]$ where $K = \mathbb{C}((x))$. On the other hand, since both compositions $i_{x,z} i_{x-z,w}$ and $i_{w,z} i_{x,w+z}$ are algebra morphisms, we have

$$(x - w - z)(\psi - \varphi) = 0, \tag{3.3.11.3}$$

hence $\psi = \varphi$, since $K[[z, w]]$ has no zero divisors. Similarly, we have:

$$(i_{x,z} i_{w,x-z} - i_{w,z} i_{w+z,x})(x - w - z)^{-1} = 0,$$

from where the lemma follows. $\square$

3.3.12. Taking the generating series in 3.3.9(1) we obtain for the left hand side:

$$\sum_{(j|J), j\in\mathbb{Z}} W^{-1-j|N\setminus J} \overset{s}{Y}(a_{(j|J)}b, Z) = \overset{s}{Y}\Big(\overset{s}{Y}(a, W)b, Z\Big). \tag{3.3.12.1}$$

On the right hand side we obtain

$$\sum_{(j|J),j\in\mathbb{Z}} W^{-1-j|N\setminus J}\sigma(J)\operatorname{res}_X\Big(i_{x,z}(X-Z)^{j|J}\overset{s}{Y}(a,X)\overset{s}{Y}(b,Z)-$$

$$-(-1)^{ab}i_{z,x}(X-Z)^{j|J}b(Z)a(X)\Big)=$$

$$=\operatorname{res}_X\sum_{(j|J),j\in\mathbb{Z}}(-1)^{N\setminus J}\sigma(J)\Big(i_{x,z}(X-Z)^{j|J}W^{-1-j|N\setminus J}\overset{s}{Y}(a,X)\overset{s}{Y}(b,Z)-$$

$$-(-1)^{ab}i_{z,x}(X-Z)^{j|J}W^{-1-j|N\setminus J}\overset{s}{Y}(b,Z)\overset{s}{Y}(a,X)\Big).\quad(3.3.12.2)$$

But, according to (3.3.3.6), this is

$$\operatorname{res}_X\Big(i_{x,z}\delta(X-Z,W)\overset{s}{Y}(a,X)\overset{s}{Y}(b,Z)-(-1)^{ab}i_{z,x}\delta(X-Z,W)\overset{s}{Y}(b,Z)\overset{s}{Y}(a,X)\Big).\quad(3.3.12.3)$$

Using Lemma 3.3.11, the first term gives

$$i_{w,z}\overset{s}{Y}(a,W+Z)\overset{s}{Y}(b,Z).\quad(3.3.12.4)$$

In order to compute the second term we expand in Taylor series (cf. 3.1.3.3)

$$i_{z,x}\delta(X-Z,W)=\sum_{(k|K),k\geq 0}(-1)^K X^{k|K}\partial_{-Z}^{(k|K)}\delta(-Z,W).\quad(3.3.12.5)$$

Hence the second term in (3.3.12.3) reads:

$$-(-1)^{ab}\operatorname{res}_X\sum_{(k|K),k\geq 0}(-1)^K X^{k|K}\partial_Z^{(k|K)}\delta(-Z,W)\overset{s}{Y}(b,Z)X^{-1-n|N\setminus I}a_{(n|I)}=$$

$$=-\operatorname{res}_X\sum_{(k|K),k\geq 0}(-1)^{ab+(N-I)(b+N-K)+K}\sigma(K,N\setminus I)\times$$

$$\times X^{k-1-n|K\cup(N\setminus I)}\partial_{-Z}^{(k|K)}\delta(-Z,W)\overset{s}{Y}(b,Z)a_{(n|I)}=$$

$$=-\sum_{(k|K),k\geq 0}(-1)^{(a+N-K)b+N}\sigma(K)\partial_{-Z}^{(k|K)}\delta(-Z,W)\overset{s}{Y}(b,Z)a_{(k|K)}.\quad(3.3.12.6)$$

Adding this to (3.3.12.4) and changing Z by $-Z$ we obtain the important formula

$$\overset{s}{Y}\Big(\overset{s}{Y}(a,W)b,-Z\Big)=i_{w,z}\overset{s}{Y}(a,W-Z)\overset{s}{Y}(b,-Z)-$$

$$-\sum_{(k|K),k\geq 0}(-1)^{(a+N-K)b+N}\sigma(K)\partial_Z^{(k|K)}\delta(Z,W)\overset{s}{Y}(b,-Z)a_{(k|K)}.\quad(3.3.12.7)$$

Note now that by acting on any vector $c\in V$ and multiplying this last equation by a sufficiently high power of $(z-w)$ the second term vanishes, therefore we obtain *associativity* for the vertex operators, namely:

$$(z-w)^n \overset{s}{Y}\left(\overset{s}{Y}(a,W)b,-Z\right)c = (z-w)^n \overset{s}{Y}(a,W-Z)\overset{s}{Y}(b,-Z)c, \quad n \gg 0. \quad (3.3.12.8)$$

As in [16, 3.2.3] we obtain an equivalent formulation which is called the *Cousin* property. Recall the embedding:

$$i_{z,w} : \mathbb{C}((Z,W)) \hookrightarrow \mathbb{C}((Z))((W)). \qquad (3.3.12.9)$$

Corollary 3.3.13 (Cousin property). *For any $N_W = n$ SUSY vertex algebra V and vectors $a, b, c \in V$, the three expressions:*

$$\begin{aligned} \overset{s}{Y}(a,Z)\overset{s}{Y}(b,W)c &\in V((Z))((W)) \\ (-1)^{ab}\overset{s}{Y}(b,W)\overset{s}{Y}(a,Z)c &\in V((W))((Z)) \\ \overset{s}{Y}\left(\overset{s}{Y}(a,Z-W)b,W\right)c &\in V((W))((Z-W)) \end{aligned} \qquad (3.3.13.1)$$

are the expansions, in the domains $|z| > |w|$, $|w| > |z|$ and $|w| > |w-z|$ respectively, of the same element of

$$V[[Z,W]][z^{-1}, w^{-1}, (z-w)^{-1}]. \qquad (3.3.13.2)$$

Proof. By the locality axiom, there exists $n \in \mathbb{Z}_+$ such that:

$$(z-w)^n \overset{s}{Y}(a,Z)\overset{s}{Y}(b,W)c = (-1)^{ab}(z-w)^n \overset{s}{Y}(b,W)\overset{s}{Y}(a,Z)c. \qquad (3.3.13.3)$$

Since the LHS is an element of $V((Z))((W))$ and the RHS is an element of $V((W))((Z))$, it follows that they are both equal to some $\varphi \in V[[Z,W]][z^{-1}, w^{-1}]$ (cf. (3.1.2.12)). Since $i_{z,w}$ and $i_{w,z}$ are algebra morphisms, we get

$$\overset{s}{Y}(a,Z)\overset{s}{Y}(b,W)c = i_{z,w}\frac{\varphi}{(z-w)^n}, \quad (-1)^{ab}\overset{s}{Y}(b,W)\overset{s}{Y}(a,Z)c = i_{w,z}\frac{\varphi}{(z-w)^n}. \qquad (3.3.13.4)$$

The rest of the corolary is proved in a similar way, using (3.3.12.8). □

Theorem 3.3.14 (Skew-symmetry). *In an $N_W = N$ SUSY vertex algebra the following identity, called* skew-symmetry, *holds*

$$\overset{s}{Y}(a,Z)b = (-1)^{ab} e^{Z\nabla}\overset{s}{Y}(b,-Z)a \qquad (3.3.14.1)$$

Proof. By the locality axiom we have for $n \gg 0$

$$(z-w)^n \overset{s}{Y}(a,Z)\overset{s}{Y}(b,W)|0> = (z-w)^n(-1)^{ab}\overset{s}{Y}(b,W)\overset{s}{Y}(a,Z)|0> \tag{3.3.14.2}$$

Now by (1) in proposition 3.3.7 we can write this as:

$$\begin{aligned}(z-w)^n\overset{s}{Y}(a,Z)e^{W\nabla}b &= (z-w)(-1)^{ab}\overset{s}{Y}(b,W)e^{Z\nabla}a \\ &= (z-w)^n(-1)^{ab}e^{Z\nabla}e^{-Z\nabla}\overset{s}{Y}(b,W)e^{Z\nabla}a \\ &= (z-w)^n(-1)^{ab}e^{Z\nabla}i_{w,z}\overset{s}{Y}(b,W-Z)a\end{aligned} \tag{3.3.14.3}$$

where in the last line we used (2) in 3.3.7. Now both sides in (3.3.14.3) are formal power series in W. Indeed, since $b_{(j|J)}a = 0$ for $j \gg 0$ we see that by making n large enough we may assume that there are no negative powers of w in the RHS. We can then let $W = 0$ in (3.3.14.3) and multiply by z^{-n} to obtain (3.3.14.1). □

3.3.15. Expanding both sides in (3.3.14.1) we have:

$$\begin{aligned}Y(a,Z)b &= (-1)^{ab}e^{Z\nabla}\overset{s}{Y}(b,-Z)a \\ \sum_{(j|J),j\in\mathbb{Z}} Z^{-1-j|N\setminus J}a_{(j|J)}b &= (-1)^{ab}\Big(\sum_{(j|J),j\geq 0}\nabla^{(j|J)}Z^{j|J}\Big)\times \\ &\quad\times\Big(\sum_{(k|K),k\in\mathbb{Z}}(-Z)^{-1-k|N\setminus K}b_{(k|K)}a\Big) \\ &= (-1)^{ab}\sum_{\substack{(j|J),j\geq 0\\(k|K),k\in\mathbb{Z}}}(-1)^{1+k+N-K}\nabla^{(j|J)}\times \\ &\quad\times\sigma(J,N\setminus K)Z^{j-1-k|J\cup(N\setminus K)}b_{(k|K)}a\end{aligned} \tag{3.3.15.1}$$

Taking the coefficient of $Z^{-1-n|N\setminus I}$ on both sides we get:

$$\begin{aligned}a_{(n|I)}b = (-1)^{ab}&\sum_{j\geq 0, J\cap I=\emptyset}(-1)^{1-n+N+J-I}\times \\ &\times(-\nabla)^{(j|J)}\sigma(N\setminus(I\cup J),J)b_{(n+j|I\cup J)}a\end{aligned} \tag{3.3.15.2}$$

In particular, when $(n|I) = (-1|N)$ in (3.3.15.2) we get:

$$: ab : -(-1)^{ab} : ba := (-1)^{ab}\sum_{j\geq 1}\frac{(-T)^j}{j!}b_{(-1+j|N)}a, \tag{3.3.15.3}$$

equivalently:

$$: ab : -(-1)^{ab} : ba := \int_{-\nabla}^{0}[a_\Lambda b]d\Lambda. \tag{3.3.15.4}$$

This last identity (3.3.15.4) is called *quasi*-commutativity of the normally ordered product.

3.3.16. Define the following *formal Fourier transform* by

$$F_Z^\Lambda a(Z) = \operatorname{res}_Z e^{Z\Lambda} a(Z). \tag{3.3.16.1}$$

It is a linear map from the space of $\mathscr{U}$-valued formal distributions in Z to $\mathscr{U}[[\Lambda]]$. It has the following properties which are immediate to check:

$$F_Z^\Lambda \partial_z a(Z) = -\lambda F_Z^\Lambda a(Z), \tag{3.3.16.2}$$

$$F_Z^\Lambda \partial_{\theta^i} a(Z) = -(-1)^N \chi^i F_Z^\Lambda a(Z), \tag{3.3.16.3}$$

$$F_Z^\Lambda \left(e^{Z\nabla} a(Z)\right) = F_Z^{\Lambda+\nabla} a(Z) \text{ if } a(Z) \in \mathscr{U}((Z)), \tag{3.3.16.4}$$

$$F_Z^\Lambda a(-Z) = -F_Z^{-\Lambda} a(Z), \tag{3.3.16.5}$$

$$F_Z^\Lambda \left(\partial_W^{(j|J)} \delta(Z,W)\right) = (-1)^{JN} e^{W\Lambda} \Lambda^{(j|J)}. \tag{3.3.16.6}$$

Theorem 3.3.17. *Let V be a $N_W = N$ SUSY vertex algebra. Then V is a $N_W = N$ SUSY Lie conformal algebra with Λ-bracket:*

$$[a_\Lambda b] = F_Z^\Lambda \overset{s}{Y}(a,Z)b = \sum_{(j|J), j\geq 0} (-1)^{JN} \sigma(J, N\setminus J) \Lambda^{(j|J)} (a_{(j|J)} b). \tag{3.3.17.1}$$

Proof. The sesquilinearity relations follow from Corolary 3.3.10 for $j \geq 0$. Applying F_Z^Λ to both sides of (3.3.14.1) and using (3.3.16.4) and (3.3.16.5) we get the skew-symmetry relation. In order to prove the Jacobi identity, apply F_Z^Λ to the OPE formula (3.3.9.1) applied to c, and use (3.3.16.6) to obtain:

$$[a_\Lambda \overset{s}{Y}(b,W)c] = (-1)^{ab+bN} \overset{s}{Y}(b,W)[a_\Lambda c] + e^{W\Lambda} \overset{s}{Y}([a_\Lambda b], W)c. \tag{3.3.17.2}$$

Applying F_W^Γ to both sides of this formula we get the Jacobi identity. □

Theorem 3.3.18. *Let V be a $N_W = N$ SUSY vertex algebra. The following identity called "quasi-associativity" of the normally ordered product holds for every $a, b, c \in V$:*

$$:: ab : c : - : a : bc ::= \sum_{j\geq 0} a_{(-2-j|N)} \left(b_{(j|N)} c\right) + (-1)^{ab} \sum_{j\geq 0} b_{(-2-j|N)} \left(a_{(j|N)} c\right). \tag{3.3.18.1}$$

Equivalently

$$:: ab : c : - : a : bc ::= \left(\int_0^\nabla d\Lambda a\right) [b_\Lambda c] + (-1)^{ab} \left(\int_0^\nabla d\Lambda b\right) [a_\Lambda c], \tag{3.3.18.2}$$

where the integral is computed as follows: expand the Λ-bracket, put the powers of Λ on the left, under the sign of integral. then take the definite integral by the usual rules inside the parenthesis.

Proof. Applying both sides of 3.3.9 (2) to c and taking the constant coefficient, the LHS is $:: ab : c ::$. By (3.2.14.1), the RHS of 3.3.9 (2) applied to c is

$$\sum_{\substack{j<0,J\\ k,K\cup J=N}} (-1)^{(N-K)(a+N-J)}\sigma(N\setminus J, N\setminus K)Z^{-2-j-k|N\setminus(J\cap K)}a_{(j|J)}\left(b_{(k|K)}c\right) + \\ + \sum_{\substack{j\geq 0,J\\ k,K\cup J=N}} (-1)^{(N-J)(b+N-K)+ab}\sigma(N\setminus K, N\setminus J)Z^{-2-j-k|N\setminus(J\cap K)}b_{(k|K)}\left(a_{(j|J)}c\right). \tag{3.3.18.3}$$

To compute the constant coefficient in the last formula, we need $K=J=N$, and $k=-2-j$, we get

$$\sum_{j\geq -1} a_{(-2-j|N)}\left(b_{(j|N)}c\right) + (-1)^{ab}\sum_{j\geq 0} b_{(-2-j|N)}\left(a_{(j|N)}c\right). \tag{3.3.18.4}$$

Noting that the term with $j=-1$ in the first summand in the last formula is $: a : bc ::$, the theorem follows. □

We thus arrive to the following equivalent definition of an $N_W = N$ SUSY vertex algebra (cf. [2])

Definition 3.3.19. An $N_W = N$ SUSY vertex algebra is a tuple $(V, T, S^i, [\cdot_\Lambda\cdot], |0>, ::)$, $i = 1, \dots, N$, where

- $(V, T, S^i, [\cdot_\Lambda\cdot])$ is an $N_W = N$ SUSY Lie conformal algebra,
- $(V, |0>, T, S^i, ::)$ is a unital quasicommutative quasiassociative differential superalgebra (i.e. T is an even derivation of $::$ and S^i are odd derivations of $::$),
- the Λ-bracket and the product $::$ are related by the non-commutative Wick formula (3.2.20.3).

Proof. We have shown that this definition follows from Definition 3.3.1. For the converse, we refer the reader to [2], the proof carries over to the SUSY case with minor modifications. □

Removing the "quantum corrections" we arrive to the following definition

Definition 3.3.20. An $N_W = N$ *Poisson* SUSY vertex algebra is tuple $(V, |0>, T, S^i, \{\cdot_\Lambda\cdot\}, \cdot)$, where

- $(V, T, S^i, \{\cdot_\Lambda\cdot\})$ is an $N_W = N$ SUSY Lie conformal algebra,
- $(V, |0>, T, S^i, \cdot)$ is an unital commutative associative differential superalgebra,

- the following *Leibniz rule* is satisfied:

$$\{a_\Lambda bc\} = \{a_\Lambda b\}c + (-1)^{(a+N)b} b\{a_\Lambda c\}. \tag{3.3.20.1}$$

Theorem 3.3.21. *Let V be an $N_W = N$ SUSY vertex algebra. For each $a, b \in V$, $k \in \mathbb{Z}$ and $K \subset \{1, \dots, N\}$, the following identity, called* Borcherds identity *holds:*

$$\begin{aligned}\left(i_{z,w}(Z-W)^{k|K}\right) \overset{s}{Y}(a,Z)\overset{s}{Y}(b,W) - (-1)^{ab}\left(i_{w,z}(Z-W)^{k|K}\right)\overset{s}{Y}(b,W)\overset{s}{Y}(a,Z) = \\ = \sum_{j\geq 0, J} \sigma(J,K)\sigma(J\cup K)\left(\partial_W^{(j|J)}\delta(Z,W)\right)\overset{s}{Y}(a_{(k+j|K\cup J)}b, W).\end{aligned} \tag{3.3.21.1}$$

Proof. The LHS of (3.3.21.1) is local since multiplied by $(z-w)^n$ for $n \gg 0$ it is equal to

$$(Z-W)^{n+k|K}[\overset{s}{Y}(a,Z), \overset{s}{Y}(b,W)] = 0, \tag{3.3.21.2}$$

by the locality axiom. Therefore we can apply the decomposition Lemma 3.1.7 to the LHS of (3.3.21.1). We have

$$\begin{aligned}c_{j|J}(W) = \sigma(J,K)\operatorname{res}_Z \Bigg(\left(i_{z,w}(Z-W)^{k+J|K\cup J}\right)\overset{s}{Y}(a,Z)\overset{s}{Y}(b,W) - \\ -(-1)^{ab}\sigma(J,K)\left(i_{w,z}(Z-W)^{k+j|K\cup J}\right)\overset{s}{Y}(b,W)\overset{s}{Y}(a,Z)\Bigg),\end{aligned} \tag{3.3.21.3}$$

therefore the theorem follows from (3.2.17.4) and Theorem 3.3.9 (1). $\square$

Proposition 3.3.22. *Let V be a $N_W = N$ SUSY vertex algebra. Then*

$$\begin{aligned}[a_{(n|I)}, \overset{s}{Y}(b,W)] = \sum_{(j|J), j\geq 0} (-1)^{JN+IN+IJ}\sigma(J)\times \\ \times \sigma(I)\left(\partial_W^{(j|J)} W^{n|I}\right)\overset{s}{Y}\left(a_{(j|J)}b, W\right).\end{aligned} \tag{3.3.22.1}$$

If, moreover, $n \geq 0$, this becomes:

$$[a_{(n|I)}, \overset{s}{Y}(b,W)] = \overset{s}{Y}(e^{-W\nabla}a_{(n|I)}e^{W\nabla}b, W). \tag{3.3.22.2}$$

Proof. Multiplying the OPE formula (3.3.9.1) by $Z^{n|I}$ and taking residues we obtain in the left hand side

$$\sigma(I)[a_{(n|I)}, \overset{s}{Y}(b,W)]. \tag{3.3.22.3}$$

While the right hand side is

$$\begin{aligned}
\operatorname{res}_Z \sum_{(j|J),j\geq 0} (-1)^{I(N-J)}\sigma(J)(\partial_W^{(j|J)}\delta(Z,W)Z^{n|I})\overset{s}{Y}(a_{(j|J)}b,W) = \\
= \operatorname{res}_Z \sum_{(j|J),j\geq 0} (-1)^{I(N-J)}\sigma(J)(\partial_W^{(j|J)}\delta(Z,W)W^{n|I})\overset{s}{Y}(a_{(j|J)}b,W) = \\
= \sum_{(j|J),j\geq 0} (-1)^{I(N-J)+JN}\sigma(J)(\partial_W^{(j|J)}W^{n|I})\overset{s}{Y}(a_{(j|J)}b,W) \quad (3.3.22.4)
\end{aligned}$$

From where (3.3.22.1) follows. To prove (3.3.22.2) we note first that from Dong's lemma 3.2.21 the commutator on the left hand side is local with all the fields of the vertex algebra. To apply the uniqueness theorem, we need to check that both sides agree when valuated at the vacuum vector. The left hand side is given by

$$[a_{(n|I)}, \overset{s}{Y}(b,W)]|0>= a_{(n|I)}e^{W\nabla}b, \quad (3.3.22.5)$$

where we used the fact that $a_{(n|I)}|0>= 0$ and Proposition 3.3.7 (1). On the other hand, by the same proposition the left hand side is

$$\overset{s}{Y}(e^{-W\nabla}a_{(n|I)}e^{W\nabla},W)|0>= e^{W\nabla}e^{-W\nabla}a_{(n|I)}e^{W\nabla}|0>, \quad (3.3.22.6)$$

and (3.3.22.2) follows. □

Remark 3.3.23. As a consequence of (3.3.22.1) we see that by taking the coefficient of $W^{-1-k|N\setminus K}$ we obtain the commutator $[a_{(n|I)}, b_{(k|K)}]$ as a linear combination of Fourier modes of fields in V. This rather complicated formula says that the linear span of Fourier modes of $\operatorname{End}(V)$-valued fields is a Lie superalgebra. In order to compute explicitly the Lie bracket, we compute the coefficient of $W^{-1-k|N\setminus K}$ on the left hand side of (3.3.22.1) to obtain:

$$(-1)^{(a+N-I)(N-K)}[a_{(n|I)}, b_{(k|K)}]. \quad (3.3.23.1)$$

To compute this coefficient on the right hand side we first expand:

$$\begin{aligned}
(\partial_W^{j|J}W^{n|I})W^{-1-l|N\setminus L} = \frac{(-1)^{\frac{J(J-1)}{2}}n!}{(n-j)!}\times \\
\times\,\sigma(J, I\setminus J)\sigma(I\setminus J, N\setminus L)W^{n-j-1-l|(I\setminus J)\cup(N\setminus L)}. \quad (3.3.23.2)
\end{aligned}$$

Note that in order for the corresponding term in (3.3.22.1) not to vanish, we must have $J\subset I$ and in order for the coefficient of $W^{-1-k|N\setminus K}$ not to be zero in (3.3.23.2) we must have $K\cap I\subset J$. Now we set then $n-j-l-1=-1-k$ and $(I\setminus J)\cup(N\setminus L) = N\setminus K$

and we obtain $l = n+k-j$ and $L = K \cup (I \setminus J)$. We get then for the right hand side

$$\sum_{(j|J), j\geq 0} (-1)^{(J+I)(N-J)} \binom{n}{j} \sigma(J)\sigma(I)\times \\ \times \sigma(J, I\setminus J)\sigma(I\setminus J, (N\setminus K)\setminus (I\setminus J))\big(a_{(j|J)}b\big)_{(n+k-j|K\cup(I\setminus J))} \tag{3.3.23.3}$$

Combining with (3.3.23.1) we obtain:

$$[a_{(n|I)}, b_{(k|K)}] = (-1)^{(a+N-I)(N-K)} \sum_{(j|J), j\geq 0} (-1)^{(I-J)(N-J)} \binom{n}{j} \sigma(J)\times \\ \times \sigma(I)\sigma(J, I\setminus J)\sigma\left(I\setminus J, (N\setminus K)\setminus (I\setminus J)\right)\left(a_{(j|J)}b\right)_{(n+k-j|K\cup(I\setminus J))} \tag{3.3.23.4}$$

3.3.24. We can define the *tensor product* of two $N_W = N$ SUSY vertex algebras in the usual way, namely, let V and W be two $N_W = N$ SUSY vertex algebras. Their tensor product is $V \otimes W$ as a vector space. The vacuum vector is $|0\rangle_V \otimes |0\rangle_W$. Let us denote $\overset{s}{Y}_V$ and $\overset{s}{Y}_W$ the corresponding state-field correspondences. We define the state field correspondence $\overset{s}{Y}$ for $V\otimes W$ as

$$\overset{s}{Y}(a\otimes b, Z) = \overset{s}{Y}_V(a, Z)\otimes \overset{s}{Y}_W(b, Z) = \\ = \sum_{(j|J),(k|K)} (-1)^{a(N-K)}\sigma(N\setminus K, N\setminus J) Z^{-2-j-k|(N\setminus(J\cap K))} a_{(j|J)}\otimes b_{(k|K)}, \tag{3.3.24.1}$$

where the endomorphism $a_{(j|J)}\otimes b_{(k|K)}$ is defined to be

$$(a_{(j|J)}\otimes b_{(k|K)})(v\otimes w) = (-1)^{(b+N-K)v} a_{(j|J)}v \otimes b_{(k|K)}w, \tag{3.3.24.2}$$

and, as usual, the $\mathbb{Z}/2\mathbb{Z}$-grading of $V\otimes W$ is the sum of those of V and W. Note that in order for σ not to vanish in (3.3.24.1) we must have $J\cup K = N$. Finally, we let the translation operators be $T = T_V \otimes Id + Id \otimes T_W$ and $S^i = S^i_V \otimes Id + Id\otimes S^i_W$. All the axioms of SUSY vertex algebra are straightforward to check.

Theorem 3.3.25 (Existence). *Let V be a vector superspace, $|0\rangle \in V$ an even vector, T an even endomorphism of V and S^i, $i = 1, \dots, N$, odd endomorphisms of V, pairwise anticommuting between themselves and commuting with T. Suppose moreover that $T|0\rangle = S^i|0\rangle = 0$. Let $\mathscr{F}$ be a family of* $\mathrm{End}(V)$*-valued fields*

$$a^\alpha(Z) = \sum_{j\in\mathbb{Z}, J} Z^{-1-j|N\setminus J} a^\alpha_{(j|J)} \tag{3.3.25.1}$$

indexed by $\alpha \in A$, such that

1. $a^\alpha(Z)|0\rangle\, |_{Z=0} = a^\alpha \in V$,

2. $[T, a^\alpha(Z)] = \partial_z a^\alpha(Z)$ *and* $[S^i, a^\alpha(Z)] = \partial_{\theta^i} a^\alpha(Z)$.

3. *all pairs* $(a^\alpha(Z), a^\beta(Z))$ *are local*

4. *all the following vectors span* V

$$a^{\alpha_s}_{(j_s|J_s)} \cdots a^{\alpha_1}_{(j_1|J_1)}|0> . \tag{3.3.25.2}$$

Then the formula

$$\overset{s}{Y}(a^{\alpha_s}_{(j_s|J_s)} \cdots a^{\alpha_1}_{(j_1|J_1)}|0>, Z) =$$
$$= \prod \sigma(J_i) a^{\alpha_s}(Z)_{(j_s|J_s)} \Big(\dots a^{\alpha_2}_{(j_2|J_2)} \big(a^{\alpha_1}_{(j_1|J_1)} \operatorname{Id} \big) \dots \Big). \tag{3.3.25.3}$$

gives a well defined structure of an $N_W = N$ *SUSY vertex algebra on* V*, with vacuum vector* $|0>$*, translation operators* T*,* S^i*, and such that*

$$\overset{s}{Y}(a^\alpha, Z) = a^\alpha(Z). \tag{3.3.25.4}$$

Such a structure is unique.

Proof. Let $\bar{\mathscr{F}}$ be the minimal family of pairwise local $\operatorname{End}(V)$-valued fields containing $\mathscr{F}$, closed under all $(j|J)$-products and closed under the derivations ∂_z and ∂_{θ^i}. By Theorem 3.3.3, $\bar{\mathscr{F}}$ is an $N_W = N$ SUSY vertex algebra. Define a map $\varphi : \bar{\mathscr{F}} \to V$ by $a(Z) \mapsto a(Z)|0>_{Z=0}$. This map is injective by the uniqueness Proposition 3.3.8 and surjective by (4). We obtain thus a *state-field correspondence* $\overset{s}{Y} : a \mapsto \overset{s}{Y}(a, Z)$. Formula (3.3.25.3) follows from the $(j|J)$-product identity in Theorem 3.3.9 (1). □

3.4 The universal enveloping SUSY vertex algebra

In this section we construct maps φ and φ' used in the next chapters, and we construct an $N_W = N$ SUSY vertex algebra attached to each $N_W = N$ SUSY Lie conformal algebra.

Definition 3.4.1. Let $\mathscr{A}$ be a unital associative commutative superalgebra with an even derivation T, odd anticommutative derivations S^i, $i = 1, \dots, N$, commuting with T. Then $\left(\mathscr{A}, |0>= 1, T, S^i, \overset{s}{Y}(a, Z)b = \left(e^{Z\nabla a}\right) b\right)$ is an $N_W = N$ SUSY vertex algebra, called *holomorphic.*

3.4.2. Let V be an $N_W = N$ SUSY vertex algebra. According to Theorem 3.3.17 it is a SUSY Lie conformal algebra. It follows by Proposition 3.2.12 that the pair $(\operatorname{Lie}(V), V)$ is an $N_W = N$ formal distribution Lie superalgebra.

Recall that $\operatorname{Lie}(V) = \tilde{V}/\tilde{\nabla}\tilde{V}$ where $\tilde{V} = V \otimes_{\mathbb{C}} \mathbb{C}[X, X^{-1}]$ and $\tilde{\nabla}\tilde{V}$ is the space spanned by vectors of the form:

$$Ta \otimes f(X) + a \otimes \partial_x f(X), \quad S^i a \otimes f(X) + (-1)^{aN} a \otimes \partial_{\eta^i} f(X), \tag{3.4.2.1}$$

for $a \in V$, $f(X) \in \mathbb{C}[X, X^{-1}]$.

Let $\varphi : \mathrm{Lie}(V) \to \mathrm{End}(V)$ be the linear map defined by

$$a_{<n|I>} = a \otimes X^{n|I} \mapsto (-1)^{aI} \sigma(I) a_{(n|I)}, \qquad a \in V. \tag{3.4.2.2}$$

Similarly, we construct $V \otimes_{\mathbb{C}} \mathbb{C}((X))$ and consider its quotient $\mathrm{Lie}'(V)$ by the vector space generated by vectors of the form (3.4.2.1). Then (3.4.2.2) defines a map $\varphi' : \mathrm{Lie}'(V) \to \mathrm{End}(V)$. Comparing (3.2.10.4) and (3.3.23.4) and noting the extra factor $\sigma(J)$ in (3.3.17.1) we obtain the following

Theorem 3.4.3. *The maps φ, and φ' are Lie algebra homomorphisms.*

3.4.4. Let $\mathscr{R}$ be an $N_W = N$ SUSY Lie conformal algebra, and let $(\mathrm{Lie}(\mathscr{R}), \mathscr{R})$ be the corresponding $N_W = N$ formal distribution Lie superalgebra (cf. Proposition 3.2.12). Recall that the Lie bracket in $\mathrm{Lie}(\mathscr{R})$ is given by (3.2.10.4). In particular we see that $\mathrm{Lie}(\mathscr{R})$ has a subalgebra $\mathrm{Lie}(\mathscr{R})_-$ spanned by all Fourier modes $a_{(j|J)}$ with $j \geq 0$. Note also that from (3.2.12.1) it follows that $\nabla \mathrm{Lie}(\mathscr{R})_- \subset \mathrm{Lie}(\mathscr{R})_-$. We extend the derivations T, S^i to the universal enveloping algebra $U(\mathrm{Lie}(\mathscr{R}))$ of $\mathrm{Lie}(\mathscr{R})$ by Leibniz rule.

Theorem 3.4.5. *Let $\mathscr{R}$ be an $N_W = N$ SUSY Lie conformal algebra. Let $V = V(\mathscr{R})$ be the quotient of $U(\mathrm{Lie}(\mathscr{R}))$ by the left ideal generated by $\mathrm{Lie}(\mathscr{R})_-$. Then V admits an $N_W = N$ SUSY vertex algebra structure whose vacuum vector is the image of* 1 *in V. This vertex algebra is called the* universal enveloping vertex algebra *of $\mathscr{R}$.*

Proof. Recall that $\mathrm{Lie}(\mathscr{R})$ is an $N_W = N$ formal distribution Lie superalgebra (cf. Proposition 3.2.12), in particular, the distributions (3.2.10.2) are pairwise local and translation invariant. We need to check that the distributions (3.2.10.2) are $\mathrm{End}(V)$-valued fields. Indeed for any $b_{(k|K)} \in \mathrm{Lie}(\mathscr{R})$ we have from (3.2.10.4) that for any $a \in \mathscr{R}$, $[a_{(n|I)}, b_{(k,K)}] \in \mathrm{Lie}(\mathscr{R})_-$ for $n \gg 0$ since $a_{(j|J)}b = 0$ for $j \gg 0$ hence we can make $n + k - j \geq 0$. It follows that $a(Z)b$ has finitely many negative powers of z. For products $b^1_{(j_1|J_1)} \dots b^k_{(j_k|J_k)}$ we proceed by induction. Now the theorem follows easily from the existence theorem 3.3.25. $\square$

3.5 $N_K = N$ SUSY vertex algebras

3.5.1. In this section we develop the structure theory of $N_K = N$ *SUSY vertex algebras*. This algebras have been studied, in some particular cases, in the physics literature. Roughly speaking an $N_K = N$ SUSY vertex algebra is an $N_W = N$ SUSY vertex algebra, but instead of ∂_{θ^i}, we consider the differential operators

$$D^i_Z = D^{e_i}_Z = \partial_{\theta^i} + \theta^i \partial_z. \tag{3.5.1.1}$$

To describe the corresponding *SUSY Lie conformal superalgebras*, perhaps the language of *H-pseudoalgebras* is more convenient [1]. On the other hand, we are interested in their universal enveloping vertex algebras and in particular we want a description along the lines of the previous sections.

In order to have a uniform notation between this section and the previous ones, given two sets of coordinates $Z=(z,\theta^i)$ and $W=(w,\zeta^i)$ we will denote

$$\begin{aligned} Z-W&=\left(z-w-\sum_{i=1}^N\theta^i\zeta^i,\theta^j-\zeta^j\right),\\ (Z-W)^{j|J}&=\left(z-w-\sum_{i=1}^N\theta^i\zeta^i\right)^j\prod_{i\in J}(\theta^i-\zeta^i). \end{aligned} \tag{3.5.1.2}$$

As before, we define

$$Z^{j|J}:=z^j\theta^J.$$

Note that

$$(Z-W)^{-1|0}=\sum_{k=0}^N\frac{\left(\sum_{i=1}^N\theta^i\zeta^i\right)^k}{(z-w)^{k+1}}, \tag{3.5.1.3}$$

therefore $(Z-W)^{-1|N}$ agrees with our previous notation:

$$(Z-W)^{-1|N}=\frac{(\theta-\zeta)^N}{z-w}. \tag{3.5.1.4}$$

The differential operators D_Z^i satisfy the commutation relations

$$[D_Z^i,D_Z^j]=2\delta_{i,j}\partial_z, \tag{3.5.1.5}$$

and, as before, we denote for $J=(j_1,\dots,j_k)$

$$D_Z=(\partial_z,D_Z^1,\dots,D_Z^N),\quad D_Z^{j|J}=\partial_z^jD_Z^{j_1}\dots D_Z^{j_k},\quad D_Z^{(j|J)}=\frac{(-1)^{\frac{J(J+1)}{2}}}{j!}D_Z^{j|J}. \tag{3.5.1.6}$$

Ocasionaly, when $j=0$ in (3.5.1.6) we will write $D_Z^{0|J}=D_Z^J$.

Finally, in this section we will consider not necessarily disjoint subsets $I,J\subset\{1,\dots,N\}$ as in the $N_W=N$ case. Given I and J, ordered subsets of $\{1,\dots,N\}$, we will write $I\triangle J=(I\setminus J)\cup(J\setminus I)$. We will use the same formal δ function $\delta(Z,W)$ as before. Remarkably, the new binomial $(Z-W)^{j|J}$, given by (3.5.1.2) "behaves" with respect to the operators $D_W^{j|J}$, in the same way as the old binomials (3.1.2.1) with respect to $\partial_W^{j|J}$.

Lemma 3.5.2. *The following identity is true:*

$$D_W^{(j|J)}\delta(Z,W)=\sigma(J)(i_{z,w}-i_{w,z})(Z-W)^{-1-j|N\setminus J}. \tag{3.5.2.1}$$

Proof. Let us assume for simplicity that $j=0$, the general case follows easily from this, differentiating by w. We will prove the lemma by induction on $\sharp J$. When $J=\emptyset$, this is the usual formula for $\delta(Z,W)$. When $J = e_i = \{i\}$, the left hand side of (3.5.2.1) is given by

$$
\begin{aligned}
-D^i_W \delta(Z,W) &= -D^i(i_{z,w} - i_{w,z}) \frac{(\theta-\zeta)^N}{z-w} \\
&= -(i_{z,w} - i_{w,z})\big(-\sigma(e_i) \frac{(\theta-\zeta)^{N\setminus e_i}}{z-w} + \zeta^i \frac{(\theta-\zeta)^N}{(z-w)^2}\big)
\end{aligned} \tag{3.5.2.2}
$$

On the other hand

$$
\begin{aligned}
(Z-W)^{-1|N\setminus e_i} &= \sum_{k\geq 0} \frac{\left(\sum \theta^i\zeta^i\right)^k}{(z-w)^{k+1}} (\theta-\zeta)^{N\setminus e_i} \\
&= \frac{(\theta-\zeta)^{N\setminus e_i}}{z-w} + \frac{\theta^i\zeta^i}{(z-w)^2}(\theta-\zeta)^{N\setminus e_i} \\
&= \frac{(\theta-\zeta)^{N\setminus e_i}}{z-w} - \sigma(e_i, N\setminus e_i) \frac{\zeta^i}{(z-w)^2}(\theta-\zeta)^N,
\end{aligned} \tag{3.5.2.3}
$$

from where (3.5.2.1) follows when $J = e_i$. To prove the general case, let us assume that the lemma is valid for $J = I \setminus e_i$. Since $D^I_W = \sigma(e_i, I\setminus e_i) D^i_W D^{I\setminus e_i}_W$ we have by the induction hypothesis

$$
\begin{aligned}
D^I_W \delta(Z,W) = \sigma(e_i, I\setminus e_i)\sigma(I\setminus e_i, N\setminus (I\setminus e_i))(-1)^{\frac{(I-1)I}{2}} \times \\
\times D^i_W (i_{z,w} - i_{w,z})(Z-W)^{-1|N\setminus(I\setminus e_i)}
\end{aligned} \tag{3.5.2.4}
$$

We expand the last factor as:

$$
\begin{aligned}
D^i(Z-W)^{-1|N\setminus(I\setminus e_i)} = -\sum_{k\geq 1} k\zeta^i \frac{\left(\sum\theta^j\zeta^j\right)^{k-1}}{(z-w)^{k+1}}(\theta-\zeta)^{N\setminus(I\setminus e_i)} - \\
-\sigma(e_i, N\setminus I)\sum_{k\geq 0}\frac{\left(\sum\theta^j\zeta^j\right)^k}{(z-w)^{k+1}}(\theta-\zeta)^{N\setminus I} + \sum_{k\geq 0}(k+1)\zeta^i\frac{\left(\sum\theta^j\zeta^j\right)^k}{(z-w)^{k+2}}(\theta-\zeta)^{N\setminus(I\setminus e_i)}.
\end{aligned} \tag{3.5.2.5}
$$

Relabeling the indexes we see that the first and last term cancel. Finally we note that, by (3.1.1.3):

$$
\sigma(e_i, I\setminus e_i)\sigma(e_i, N\setminus I)\sigma(I\setminus e_i) = (-1)^I\sigma(I). \tag{3.5.2.6}
$$

Combining (3.5.2.6), (3.5.2.5) and (3.5.2.4) we obtain the lemma for $j = 0$. □

3.5.3. Most of the results proved in the previous sections for $N_W = N$ SUSY vertex algebras carry over to this setting with the following modifications.

- replace ∂_{θ^i} by D^i_Z and ∂_Z by D_Z,

- replace $Z - W = (z - w, \theta^i - \zeta^i)$ by $Z - W = \left(z - w - \sum_{i=1}^N \theta^i \zeta^i, \theta^j - \zeta^j\right)$,
- replace $(Z - W)^{j|J} = (z - w)^j \prod_{i \in J} (\theta^i - \zeta^i)$ by

$$(Z - W)^{j|J} = \left(z - w - \sum_{i=1}^{N} \theta^i \zeta^i\right)^j \prod_{i \in J} (\theta^i - \zeta^i),$$

- replace the commutative associative "translation" superalgebra $\mathbb{C}[T, S^i]$ by the non-commutative associative "translation" superalgebra $\mathscr{H}$ generated by the set $\nabla = (T, S^1, \dots, S^N)$, where T is an even generator and S^i are odd generators, subject to the relations:

$$[T, S^i] = 0, \quad [S^i, S^j] = 2\delta_{ij} T; \tag{3.5.3.1}$$

- replace the commutative associative "parameter" superalgebra $\mathbb{C}[\lambda, \chi^i]$ by the non-commutative associative "parameter" superalgebra $\mathscr{L}$, generated by the set $\Lambda = (\lambda, \chi^1, \dots, \chi^N)$, where λ is an even generator and χ^i are odd generators, subject to the relations:

$$[\lambda, \chi^i] = 0, \quad [\chi^i, \chi^j] = -2\delta_{ij} \lambda; \tag{3.5.3.2}$$

  Note that we have an isomorphism $\mathscr{H} \to \mathscr{L}$ given by $\nabla \mapsto -\Lambda$.

Lemma 3.5.4. *The formal δ-function satisfies the properties (1)-(7) of 3.1.6 after replacing ∂_W by D_W and writing $(\Lambda + D_W) = (\lambda + \partial_w, \chi^i + D^i)$.*

Proof. (1) is clear from Lemma 3.5.2. In order to prove (2) we use Lemma 3.5.2 to write:

$$\begin{aligned}(Z - W)^{j|J} D_W^{(n|I)} \delta(Z, W) = \sigma(I) \sigma(J, N \setminus I) \times \\ \times (i_{z,w} - i_{w,z}) (Z - W)^{-1-n+j|N \setminus (I \setminus J)}.\end{aligned} \tag{3.5.4.1}$$

Applying Lemma 3.5.2 to $D_W^{(n-j|I \setminus J)} \delta(Z, W)$ the result follows from the following property of σ, which follows from (3.1.1.3):

$$\sigma(J, N \setminus I) \sigma(I \setminus J) = \sigma(I) \sigma(I \setminus J, J). \tag{3.5.4.2}$$

Properties (3)-(7) are proved as in 3.1.6. $\square$

Lemma 3.5.5. $D_Z^i (Z - W)^{j|J} = \sigma(e_i, J \setminus e_i)(Z - W)^{j|J \setminus e_i} + j \sigma(e_i, J)(Z - W)^{j-1|J \cup e_i}$

Proof. We prove the lemma by direct computation when $j \geq 0$:

$$\begin{aligned}
D_Z^i(Z-W)^{j|J} &= (\partial_{\theta^i} + \theta^i \partial_z)\left(z - w - \sum \theta^i \zeta^i\right)^j (\theta - \zeta)^J \\
&= -j\zeta^i (Z-W)^{j-1|J} + \sigma(e_i, J \setminus e_i)(Z-W)^{j|J\setminus e_i} + \\
&\quad + \theta^i j (Z-W)^{j-1|J} \\
&= \sigma(e_i, J\setminus e_i)(Z-W)^{j|J\setminus e_i} + j\sigma(e_i, J)(Z-W)^{j-1|J\cup e_i}
\end{aligned} \tag{3.5.5.1}$$

When $j = -1$ we have

$$\begin{aligned}
D_Z^i(Z-W)^{-1|J} &= (\partial_{\theta^i} + \theta^i \partial_z)\sum_{k\geq 0} \frac{\left(\sum_i \theta^i \zeta^i\right)^k (\theta-\zeta)^J}{(z-w)^{k+1}} \\
&= \sum_{k\geq 0} k\zeta^i \frac{\left(\sum_i \theta^i\zeta^i\right)^{k-1}(\theta-\zeta)^J}{(z-w)^{k+1}} + \sigma(e_i, J\setminus e_i)\times \\
&\quad \times \sum_{k\geq 0} \frac{\left(\sum_i \theta^i\zeta^i\right)^k (\theta-\zeta)^{J\setminus e_i}}{(z-w)^{k+1}} - \theta^i \sum_{k\geq 0}(k+1)\frac{\left(\sum_i \theta^i\zeta^i\right)^k(\theta-\zeta)^J}{(z-w)^{k+2}} \\
&= \sigma(e_i, J\setminus e_i)(Z-W)^{-1|J\setminus e_i} - \\
&\quad - \sigma(e_i, J)\sum_{k\geq 0}(k+1)\frac{\left(\sum_i \theta^i\zeta^i\right)^k(\theta-\zeta)^{J\cup e_i}}{(z-w)^{k+2}} \\
&= \sigma(e_i, J\setminus e_i)(Z-W)^{-1|J\setminus e_i} - \sigma(e_i, J)(Z-W)^{-2|J\cup e_i}.
\end{aligned} \tag{3.5.5.2}$$

The general case follows from these by considering $D_Z^{1|0} = (D_Z^i)^2 = \partial_z$ hence

$$(Z-W)^{-j-1|J} = \frac{1}{j!}(D_Z^i)^{2j}(Z-W)^{-1|J}. \tag{3.5.5.3}$$

Therefore we get for $j \geq 0$:

$$\begin{aligned}
D_Z^i(Z-W)^{-j-1|J} &= \frac{1}{j!}(D_Z^i)^{2j+1}(Z-W)^{-1|J} \\
&= \frac{1}{j!}(D_Z^i)^{2j}\left(\sigma(e_i, J\setminus e_i)(Z-W)^{-1|J} - \right. \\
&\quad \left. - \sigma(e_i, J)(Z-W)^{-2|J\cup e_i}\right) \\
&= \sigma(e_i, J\setminus e_i)(Z-W)^{-1-j|J\setminus e_i} - \\
&\quad - (j+1)\sigma(e_i, J)(Z-W)^{-j-2|J\cup e_i}.
\end{aligned} \tag{3.5.5.4}$$

□

The following decomposition lemma is now proved in the same manner as Lemma 3.1.7:

Lemma 3.5.6. *Let $a(Z, W)$ be a local distribution in two variables. Then $a(Z, W)$*

can be uniquely decomposed in the following finite sum:

$$a(Z,W) = \sum_{(j|J),j\geq 0} \left(D_W^{(j|J)}\delta(Z,W)\right) c_{j|J}(W). \tag{3.5.6.1}$$

The coefficients are given by

$$c_{j|J}(W) = \mathrm{res}_Z (Z-W)^{j|J} a(Z,W). \tag{3.5.6.2}$$

Let $(Z-W)\Lambda = \left(z - w - \sum_{i-1}^{N} \theta^i\zeta^i\right)\lambda + \sum_{i=1}^{N}(\theta^i - \zeta^i)\chi^i$. Note that $-\partial_w$, $-D_W^i$ satisfy the same commutation relations (3.5.3.2) as λ, χ^i, therefore $-\partial_w$, $-D_W^i$ generate an associative superalgebra isomorphic to $\mathscr{L}$. We will consider $\mathscr{L}$ as a module over itself via this identification, by defining:

$$[D_W^i, \chi^j] = 2\delta_{ij}\lambda, \quad [\partial_w, \chi^i] = [\partial_w, \lambda] = [D_W^i, \lambda] = 0. \tag{3.5.6.3}$$

Lemma 3.5.7.

$$D_Z^i \exp\left((Z-W)\Lambda\right) = \chi^i \exp\left((Z-W)\Lambda\right) = -[D_W^i, \exp\left((Z-W)\Lambda\right)].$$

Proof. Note that the exponent is a sum of non-commuting terms, hence the derivative of the exponential is not as obvious as in the $N_W = N$ case. Let

$$A = \sum_{j=1}^{N} (\theta^j - \zeta^j)\chi^j. \tag{3.5.7.1}$$

We have:

$$\begin{aligned} \exp\left((Z-W)\Lambda\right) &= \exp\left(\left(z - w - \sum_{j=1}^{N}\theta^j\zeta^j\right)\lambda\right)\exp(A), \\ \partial_{\theta^i} A^k &= \sum_{j=0}^{k-1} A^j \chi^i A^{k-j-1}. \end{aligned} \tag{3.5.7.2}$$

Since $[A, \chi^i] = -2(\theta^i - \zeta^i)\lambda$ we obtain

$$\begin{aligned} \partial_{\theta^i} A^k &= \sum_{j=0}^{k-1} \chi^i A^{k-1} - 2j\lambda(\theta^i - \zeta^i)A^{k-2} \\ &= k\chi^i A^{k-1} - k(k-1)\lambda(\theta^i - \zeta^i)A^{k-2}, \end{aligned} \tag{3.5.7.3}$$

therefore

$$\partial_{\theta^i} \exp(A) = \left(\chi^i - \lambda(\theta^i - \zeta^i)\right)\exp(A) \tag{3.5.7.4}$$

from which the first equality of the lemma follows easily. The proof of the second equality of the lemma is similar. Note that from (3.5.6.3) we have:

$$[D_W^i, A] = -\chi^i - 2\lambda(\theta^i - \zeta^i), \tag{3.5.7.5}$$

from where it follows as in (3.5.7.4) that

$$[D_W^i, \exp(A)] = -(\chi^i + \lambda(\theta^i - \zeta^i)) \exp(A), \tag{3.5.7.6}$$

and the lemma follows by a straightforward computation. □

3.5.8. Now we are in position to define the *formal Fourier transform* and $N_K = N$ *SUSY Lie conformal algebras* as we did in 3.2. We put

$$\mathscr{F}^{\Lambda}_{Z,W} a(Z,W) = \operatorname{res}_Z \exp\left((Z-W)\Lambda\right) a(Z,W). \tag{3.5.8.1}$$

which formally looks exactly like (3.1.8.1) but in this expression the variables χ^i do not commute and $(Z-W)$ has a different meaning (cf. (3.5.1.2) and (3.1.2.1)). Using this formal Fourier transform, we define the Λ-bracket of two formal distributions $a(W)$ and $b(W)$ as in (3.2.1.3). The $N_K = N$ version of Propositions 3.1.9 and 3.2.2 are proved in the same way as in the $N_W = N$ case with the aid of Lemma 3.5.7. There is only one subtlety involved in proving the Jacobi identity. Since the exponentials involved in this case do not commute, the argument in 3.1.9.9 is no longer valid. Consider the set $\Psi = (\psi, v^1, \dots, v^N)$, where ψ is an even indeterminate and v^i are odd indeterminates, subject to the relations:

$$\begin{gathered}[\psi, v^i] = 0, \quad [v^i, v^j] = -2\delta_{ij}\psi, \\ [\lambda, \psi] = [\lambda, v^i] = 0, \quad [\chi^i, \psi] = [\chi^i, v^j] = 0.\end{gathered} \tag{3.5.8.2}$$

Similarly, define $\mathscr{L}'$ to be another copy of $\mathscr{L}$ generated by another copy of standard generators $\Gamma = (\gamma, \eta^1, \dots, \eta^N)$. We define

$$\mathscr{F}^{\Lambda+\Gamma}_{X,W} = \mathscr{F}^{\Psi}_{X,W}|_{\Psi=\Lambda+\Gamma}, \tag{3.5.8.3}$$

where the Fourier transform on the RHS is computed as follows. First compute $\mathscr{F}^{\Psi}_{X,W}$, and then replace Ψ by $\Lambda + \Gamma = (\lambda+\gamma, \chi^1+\eta^1, \dots, \chi^N + \eta^N)$. In order to prove the Jacobi identity, we need to check $\mathscr{F}^{\Lambda}_{Z,W}\mathscr{F}^{\Gamma}_{X,W} = (-1)^N \mathscr{F}^{\Lambda+\Gamma}_{X,W}\mathscr{F}^{\Lambda}_{Z,X}$, or, equivalently

$$\exp((Z-W)\Lambda)\exp((X-W)\Gamma) = \exp((X-W)\Psi)\exp((Z-W)\Lambda)|_{\Psi=\Lambda+\Gamma}. \tag{3.5.8.4}$$

Note that the RHS of (3.5.8.4) can be computed as

$$\exp((X-W)(\Lambda+\Gamma))\exp((Z-W)\Lambda), \tag{3.5.8.5}$$

where we have to use the commutation relations

$$[\eta^i, \chi^j] = 2\lambda\delta_{i,j}, \quad [\gamma, \chi^i] = [\gamma, \lambda] = [\lambda, \eta^i] = 0, \tag{3.5.8.6}$$

which follow from (3.5.8.2) after replacing $\Psi = \Lambda + \Gamma$.

First, we note that given two operators A, B such that their commutator $[A, B] = C$ commutes with both A and B, we have

$$e^A e^B = e^C e^B e^A. \tag{3.5.8.7}$$

Now we expand:

$$\begin{aligned}\exp\left((Z-W)\Lambda\right)\exp\left((X-W)\Gamma\right) &= \exp\left((z-w-\sum\theta^i\zeta^i)\lambda\right)\times\\ \times\exp\left(\sum(\theta^i-\zeta^i)\chi^i\right)&\exp\left((x-w-\sum\pi^i\zeta^i)\gamma\right)\exp\left(\sum(\pi^i-\zeta^i)\eta^i\right).\end{aligned} \tag{3.5.8.8}$$

Note also that we have

$$\begin{aligned}\exp\left(\sum(\theta^i-\zeta^i)\chi^i\right) &= \prod\exp\left((\theta^i-\zeta^i)\chi^i\right)\\ &= \prod(1+(\theta^i-\zeta^i)\chi^i)\\ &= \prod\left((1+\theta^i\chi^i)(1-\zeta^i\chi^i)(1+\theta^i\zeta^i\lambda)\right)\\ &= \prod e^{\theta^i\chi^i}e^{-\zeta^i\chi^i}e^{\theta^i\zeta^i\lambda}\\ &= \exp\left(\sum\theta^i\chi^i\right)\exp\left(-\sum\zeta^i\chi^i\right)\exp\left(\sum\theta^i\zeta^i\lambda\right),\end{aligned} \tag{3.5.8.9}$$

therefore (3.5.8.8) reads:

$$\begin{aligned}&\exp\left((Z-W)\Lambda\right)\exp\left((X-W)\Gamma\right) = \exp\left((z-w)\lambda\right)\times\\ &\times\exp\left(\sum\theta^i\chi^i\right)\exp\left(-\sum\zeta^i\chi^i\right)\exp\left((x-w)\gamma\right)\exp\left(\sum\pi^i\eta^i\right)\exp\left(-\sum\zeta^i\eta^i\right).\end{aligned} \tag{3.5.8.10}$$

Commuting the exponentials using (3.5.8.7) and (3.5.8.6), (3.5.8.10) can be expressed as:

$$\begin{aligned}&\exp\left((z-w)\lambda+(x-w)\gamma\right)\exp\left(-\sum\zeta^i\chi^i\right)\exp\left(\sum\pi^i\eta^i\right)\times\\ &\qquad\times\exp\left(-\sum\zeta^i\eta^i\right)\exp\left(\sum\theta^i\chi^i\right)\exp\left(-2\sum\theta^i\pi^i\lambda\right).\end{aligned} \tag{3.5.8.11}$$

Multiplying and dividing by $\exp(\sum\pi^i\chi^i)$ and using (3.5.8.9) we can express (3.5.8.11) as

$$\begin{aligned}&\exp\left((z-w)\lambda+(x-w)\gamma\right)\exp\left(-\sum\zeta^i\chi^i\right)\exp\left(\sum\pi^i\eta^i\right)\times\\ &\quad\times\exp\left(-\sum\zeta^i\eta^i\right)\exp\left(\sum\pi^i\chi^i\right)\exp\left(\sum(\theta^i-\pi^i)\chi^i\right)\exp\left(-\sum\theta^i\pi^i\lambda\right).\end{aligned} \tag{3.5.8.12}$$

Combining again the exponentials it is easy to express this as

$$\begin{aligned}\exp\left((z-w)\lambda+(x-w)\gamma\right)\exp\left(\sum(\pi^i-\zeta^i)(\chi^i+\eta^i)\right)\times\\ \times\exp\left(-\sum\pi^i\zeta^i(\lambda+\gamma)\right)\exp\left(\sum(\theta^i-\pi^i)\chi^i\right)\exp\left(-\sum\theta^i\pi^i\lambda\right),\quad(3.5.8.13)\end{aligned}$$

which is equal to (3.5.8.5). From this, the Jacobi identity folows as in the $N_W = N$ case.

Definition 3.5.9. An $N_K = N$ SUSY Lie conformal algebra is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathscr{H}$-module $\mathscr{R}$, endowed with a parity $N \mod 2$ $\mathbb{C}$-bilinear map $\mathscr{R}\otimes_{\mathbb{C}}\mathscr{R}\to\mathscr{L}\otimes_{\mathbb{C}}\mathscr{R}$ denoted (as before, we omit the symbol $\otimes$ in the Λ-bracket)

$$a\otimes b\mapsto[a_\Lambda b]=\sum_{\substack{j\geq 0,J\\ \text{finite}}}(-1)^{JN}\Lambda^{(j|J)}a_{(j|J)}b,\qquad(3.5.9.1)$$

where $a_{(j|J)}b\in\mathscr{R}$. This data should satisfy the following axioms:

1. sesquilinearity (this is an equality in $\mathscr{L}\otimes\mathscr{R}$):

$$[S^ia_\Lambda b]=-(-1)^N\chi^i[a_\Lambda b],\qquad[a_\Lambda S^ib]=(-1)^{a+N}\left(S^i+\chi^i\right)[a_\Lambda b],\qquad(3.5.9.2)$$

where in the RHS of the second equation, to obtain an element of $\mathscr{L}\otimes\mathscr{R}$, we first compute the Λ-bracket, and then we commute S^i to the right using $[S^i,\chi^j]=2\delta_{ij}\lambda$.

2. skew-symmetry (this is an equality in $\mathscr{L}\otimes\mathscr{R}$):

$$[a_\Lambda b]=-(-1)^{ab+N}[b_{-\Lambda-\nabla}a],\qquad(3.5.9.3)$$

where the commutator on the right hand side is computed as follows: first compute $[b_\Gamma a]=\sum_{j\geq 0,J}\Gamma^{j|J}c_{j|J}\in\mathscr{L}'\otimes\mathscr{R}$, where $\mathscr{L}'$ is another copy of $\mathscr{L}$ generated by the set $\Gamma=(\gamma,\eta^1,\dots,\eta^N)$, where γ is an even generator, η^i are odd generators, subject to the relations

$$[\gamma,\eta^i]=0,\quad[\eta^i,\eta^j]=-2\delta_{ij}\gamma.$$

Then replace Γ by $-\nabla-\Lambda=(-T-\lambda,-S^1-\chi^1,\dots,-S^N-\chi^N)$ and apply T and S^i to $c_{j|J}\in\mathscr{R}$.

3. Jacobi identity (this is an equality in $\mathscr{L}\otimes\mathscr{L}'\otimes\mathscr{R}$):

$$[a_\Lambda[b_\Gamma c]]=(-1)^{aN+N}\left[[a_\Lambda b]_{\Gamma+\Lambda}c\right]+(-1)^{(a+N)(b+N)}[b_\Gamma[a_\Lambda c]],\qquad(3.5.9.4)$$

where $[[a_\Lambda b]_{\Lambda+\Gamma}c]$ is computed as follows, first compute $[[a_\Lambda b]_\Psi c]\in\mathscr{L}\otimes\mathscr{L}''\otimes\mathscr{R}$, where $\mathscr{L}''$ is another copy of $\mathscr{L}$ generated by the set $\Psi=(\psi,v^1,\dots,v^N)$, where

ψ is an even generator, v^i are odd generators, subject to the relations

$$[\psi, v^i] = 0, \quad [v^i, v^j] = -2\delta_{ij}\psi.$$

Then replace Ψ by $\Lambda + \Gamma = (\lambda + \gamma, \theta^1 + \eta^1, \dots, \theta^N + \eta^N)$ to obtain an element of $\mathscr{L} \otimes \mathscr{L}' \otimes \mathscr{R}$.

Remark 3.5.10. We want to give an explanation for the commutation relations $[S^i, \chi^j] = 2\delta_{ij}\lambda$ appearing in sesquilinearity. For this, we give an abstract descriptions of the axioms of a Lie conformal algebra as follows. Let $\mathscr{H}$ be a co-commutative Hopf superalgebra with commultiplication $\Delta : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ and antipode S (note that this is the case in definition 3.5.9). Let $\mathscr{R}$ be a (left) $\mathscr{H}$-module. The spaces $\mathscr{R} \otimes \mathscr{R}$ and $\mathscr{H} \otimes \mathscr{R}$ are canonically $\mathscr{H}$ modules with $h \mapsto \Delta h$ and we consider $\mathscr{H}$ as a $\mathscr{H}$-module with the adjoint action. An *$\mathscr{H}$-Lie conformal algebra* structure in $\mathscr{R}$ is a linear map $\phi : \mathscr{R} \otimes \mathscr{R} \to \mathscr{H} \otimes \mathscr{R}$ satisfying the following axioms (see [1]):

- ϕ is an homomorphism of $\mathscr{H}$-modules, namely, the following diagram is commutative for any $h \in \mathscr{H}$:

$$\begin{tikzcd} \mathscr{R} \otimes \mathscr{R} \arrow[r, "\phi"] \arrow[d, "\Delta h"'] & \mathscr{H} \otimes \mathscr{R} \arrow[d, "\Delta h"] \\ \mathscr{R} \otimes \mathscr{R} \arrow[r, "\phi"'] & \mathscr{H} \otimes \mathscr{R} \end{tikzcd} \tag{3.5.10.1}$$

- (Sesquilinearity) Let L_h be the operator of left multiplication by h in H. The following diagram is commutative:

$$\begin{tikzcd} \mathscr{R} \otimes \mathscr{R} \arrow[r, "\phi"] \arrow[d, "h\otimes 1"'] & \mathscr{H} \otimes \mathscr{R} \arrow[d, "L_h\otimes 1"] \\ \mathscr{R} \otimes \mathscr{R} \arrow[r, "\phi"'] & \mathscr{H} \otimes \mathscr{R} \end{tikzcd} \tag{3.5.10.2}$$

- (Skew-symmetry) Let A and B be two $\mathscr{H}$-modules. Let σ_{12} be the permutation isomorphism $A \otimes B \simeq B \otimes A$. Let $\mu : \mathscr{H} \otimes \mathscr{R} \to \mathscr{R}$ be the natural multiplication comming from the $\mathscr{H}$-module structure in $\mathscr{R}$. The following diagram is commutative:

$$\begin{tikzcd} \mathscr{R} \otimes \mathscr{R} \arrow[r, "\phi"] \arrow[d, "\sigma_{12}"'] & \mathscr{H} \otimes \mathscr{R} \arrow[d, "(S\otimes\mu)\circ(\Delta\otimes 1)"] \\ \mathscr{R} \otimes \mathscr{R} \arrow[r, "-\phi"'] & \mathscr{H} \otimes \mathscr{R} \end{tikzcd} \tag{3.5.10.3}$$

- (Jacobi identity) Let us define three morphisms $\mathscr{R}^{\otimes 3} \to \mathscr{H}^{\otimes 2} \otimes \mathscr{R}$ corresponding to the three terms in the Jacobi identity. First, let $\mu_{1\{23\}}$ be the composition

$$\mathscr{R}^{\otimes 3} \xrightarrow{1\otimes\phi} \mathscr{R} \otimes \mathscr{H} \otimes \mathscr{R} \xrightarrow{\sigma_{12}(1\otimes\phi)\sigma_{12}} \mathscr{H}^{\otimes 2} \otimes \mathscr{R}. \tag{3.5.10.4}$$

Similarly, we define $\mu_{2\{13\}}$ to be the map

$$\mathscr{R}^{\otimes 3} \xrightarrow{\sigma_{12}(1\otimes\phi)\sigma_{12}} \mathscr{H}\otimes\mathscr{R}^{\otimes 2} \xrightarrow{(1\otimes\phi)} \mathscr{H}^{\otimes 2}\otimes\mathscr{R} \tag{3.5.10.5}$$

Finally, let $\nu : \mathscr{H}\otimes\mathscr{H} \to \mathscr{H}$ be the multiplication map. We define $\mu_{\{12\}3}$ to be the map:

$$\mathscr{R}^{\otimes 3} \xrightarrow{\phi\otimes 1} \mathscr{H}\otimes\mathscr{R}^{\otimes 2} \xrightarrow{1\otimes\phi} \mathscr{H}^{\otimes 2}\otimes\mathscr{R} \xrightarrow{(\nu\otimes 1\otimes 1)(1\otimes\Delta\otimes 1)} \mathscr{H}^{\otimes 2}\otimes\mathscr{R} \tag{3.5.10.6}$$

Now we can state the Jacobi identity as the following axiom:

$$\mu_{1\{23\}} = \mu_{\{12\}3} + \mu_{2\{13\}} \tag{3.5.10.7}$$

In the $N_K = N$ case, identifying:

$$S^i \mapsto -\chi^i, \qquad T \mapsto -\lambda, \qquad \gamma \mapsto \lambda, \qquad \eta^i \mapsto \chi^i, \tag{3.5.10.8}$$

and, as in 3.2.7, changing the parity of $\mathscr{R}$ if N is odd and defining

$$\phi(a\otimes b) = (-1)^{aN+N}[a_\Lambda b], \tag{3.5.10.9}$$

it is straigtforward to check that the axioms of an $N_K = N$ SUSY Lie conformal algebra, as in Definition 3.5.9, get transformed into the axioms of an $\mathscr{H}$-Lie conformal algebra.

3.5.11. Lemmas 3.2.8 and 3.2.9 hold in this setting replacing ∂_W with D_W in (3.2.9.1). For an $N_K = N$ SUSY Lie conformal algebra $\mathscr{R}$, we let $L(R) = \tilde{\mathscr{R}}/\tilde{\nabla}\tilde{\mathscr{R}}$ be the corresponding Lie superalgebra of degree $N \mod 2$ and $\mathrm{Lie}(\mathscr{R})$ be the correspoding Lie superalgebra. For $J = (j_1, \dots, j_k)$, we have

$$\begin{aligned} D_W^{j|J} &= \partial_w^j(\partial_{\zeta^{j_1}} + \zeta^{j_1}\partial_w)\dots(\partial_{\zeta^{j_k}} + \zeta^{j_k}\partial_w) \\ &= \sum_{K\subset J} \sigma(K, J\setminus K)\zeta^K \partial_W^{j+\sharp K|J\setminus K}. \end{aligned} \tag{3.5.11.1}$$

Let now $a_{<n|I>} = a\otimes W^{n|I} \in L(\mathscr{R})$ for each $a\in\mathscr{R}$. Using (3.5.11.1) and (3.2.9.1) with $f = W^{n|I}$, $g = W^{k|K}$ and letting $\Lambda = 0$, we compute the Lie bracket (of parity $N \mod 2$) in $L(\mathscr{R})$:

$$\begin{aligned} \{a_{<n|I>}, b_{<k|K>}\} = \sum_{j\geq 0, J} (-1)^{aJ+b(I-J)+\frac{(J\cap I)(J\cap I-1)}{2}+\frac{J(J-1)}{2}} \frac{n!}{(n-j-\sharp(J\setminus I))!j!}\times \\ \times\sigma(J\setminus I, J\cap I)\sigma(J\cap I, I\setminus J)\sigma(J\setminus I, I\setminus J)\sigma(I\triangle J, K)\left(a_{(j|J)}b\right)_{<n+k-j-\sharp(J\setminus I)|K\cup(I\triangle J)>}. \end{aligned} \tag{3.5.11.2}$$

Defining $a_{(n|I)}$ as the image of $(-1)^{aI}\sigma(I)a_{<n|I>}$ in $\mathrm{Lie}(\mathscr{R})$ and using (3.5.11.2) and

Lemma 3.2.7 we compute the Lie bracket in Lie($\mathscr{R}$):

$$[a_{(n|I)}, b_{(k|K)}] = (-1)^{(a+N-I)(N-K)} \sum_{j\geq 0,K} (-1)^{J(N-I)+IN+\frac{(J\cap I)(J\cap I-1)}{2}+\frac{J(J+1)}{2}} \times$$
$$\times \frac{n!}{(n-j-\sharp(J\setminus I))!j!}\sigma(I\triangle J, N\setminus (K\cup I\triangle J))\times$$
$$\times \sigma(I)\sigma(J\setminus I, J\cap I)\sigma(J\cap I, I\setminus J)\times$$
$$\times \sigma(J\setminus I, I\setminus J)\left(a_{(j|J)}b\right)_{(n+k-j-\sharp(J\setminus I)|K\cup(I\triangle J))}. \quad (3.5.11.3)$$

Substituting (3.2.11.2) in (3.5.11.1) we find:

$$D_W^{(j|J)}\delta(Z,W) = \sum_{n\in\mathbb{Z},I} (-1)^{\frac{(J\cap I)(J\cap I-1)}{2}+\frac{J(J+1)}{2}+I+(N-I)(J-I)}\times$$
$$\times \frac{n!}{(n-j-\sharp(J\setminus I))!j!}\sigma(J\setminus I, J\cap I)\sigma(J\cap I)\times$$
$$\times \sigma(I\setminus J, N\setminus I)\sigma(J\setminus I, I\setminus J)Z^{-1-n|N\setminus I}W^{n-j-\sharp(J\setminus I)|I\triangle J}. \quad (3.5.11.4)$$

For each $a\in\mathscr{R}$ define the following Lie($\mathscr{R}$)-valued formal distribution:

$$a(Z) = \sum_{n\in\mathbb{Z},I} Z^{-1-n|N\setminus I}a_{(n|I)}. \quad (3.5.11.5)$$

Using (3.5.11.3) and (3.5.11.4) we obtain:

$$[a(Z), b(W)] = \sum_{j\geq 0,J}\left(D_W^{(j|J)}\delta(Z,W)\right)\left(a_{(j|J)}b\right)(W), \quad (3.5.11.6)$$

Hence Proposition 3.2.12 is valid in this setting.

3.5.12. We can now prove analogous versions for most of the results of section 3.2 with the prescription of 3.5.3. We define the normally ordered product of fields by the same formula as in the $N_W = N$ case (recall that $(Z-W)^{-1|N}$ is the same in both situations) and all the other products by using the derivations D_W instead of ∂_W.

In particular, given an $N_K = N$ SUSY formal distribution Lie superalgebra $(\mathfrak{g}, \mathscr{R})$, it follows that $\mathscr{R}$ is a $N_K = N$ SUSY Lie conformal algebra with $S^i = D_W^i$.

Even though the general formula of proposition 3.2.19 is no longer true in this situation, we see that the proof works in the particular case when $(j|J) = (-1|N)$. Therefore the non-commutative Wick formula (3.2.20.3) is still true in this context.

We will point out in this section the mayor differences in the proofs, leaving the particular details for the reader.

Proposition 3.5.13. *The following identities analogous to sesquilinearity for all* *pairs* $(j|J)$ *are true:*

$$\begin{aligned}\left(D^i_W a(W)\right)_{(j|J)} b(W) &= -(-1)^J \left(\sigma(e_i, J) a(W)_{(j|J\setminus e_i)} b(W) + \right.\\ &\left. + j\sigma(e_i, J) a(W)_{(j-1|J\cup e_i)} b(W)\right)\\ D^i_W \left(a(W)_{(j|J)} b(W)\right) &= (-1)^{N-J} \left(\left(D^i_W a(W)\right)_{(j|J)} b(W) + \right.\\ &\left. + (-1)^a a(W)_{(j|J)} \left(D^i_W b(W)\right)\right).\end{aligned} \quad (3.5.13.1)$$

Proof. According to lemma 3.5.5 we have:

$$\begin{aligned}&\operatorname{res}_Z i_{z,w}(Z-W)^{j|J} D^i_Z a(Z) b(W) =\\ &\quad = -(-1)^J \operatorname{res}_Z \left(D^i_Z i_{z,w}(Z-W)^{j|J}\right) a(Z) b(W) =\\ &\quad = -(-1)^J \operatorname{res}_Z \left(\sigma(e_i, J\setminus e_i) i_{z,w}(Z-W)^{j|J\setminus e_i} + \right.\\ &\qquad \left. + j\sigma(e_i, J) i_{z,w}(Z-W)^{j-1|J\cup e_i}\right) a(Z) b(W) \quad (3.5.13.2)\end{aligned}$$

Similarly we have:

$$\begin{aligned}&-(-1)^{(a+1)b} \operatorname{res}_Z i_{w,z}(Z-W)^{j|J} b(W) \left(D^i_Z a(Z)\right) =\\ &\quad = (-1)^{ab+J} \operatorname{res}_Z \left(D^i_Z i_{w,z}(Z-W)^{j|J}\right) b(W) a(Z) =\\ &\quad = (-1)^{ab+J} \operatorname{res}_Z \left(\sigma(e_i, J\setminus e_i) i_{w,z}(Z-W)^{j|J\setminus e_i} + \right.\\ &\qquad \left. + j\sigma(e_i, J) i_{w,z}(Z-W)^{j-1|J\cup e_i}\right) b(W) a(Z) \quad (3.5.13.3)\end{aligned}$$

Adding (3.5.13.2) and (3.5.13.3) we obtain:

$$\begin{aligned}\left(D^i_W a(W)\right)_{(j|J)} b(W) &= -(-1)^J \left(\sigma(e_i, J\setminus e_i) a(W)_{(j|J\setminus e_i)} b(W) + \right.\\ &\left. + j\sigma(e_i, J) a(W)_{(j-1|J\cup e_i)} b(W)\right) \quad (3.5.13.4)\end{aligned}$$

The fact that D^i_W is a derivation of all $(j|J)$-products is proved in the same way as in (3.2.18.5).

$$\begin{aligned}
D^i_W\left(a(W)_{(j|J)}b(W)\right) &= D^i_W \operatorname{res}_Z\left(i_{z,w}(Z-W)^{j|J}a(Z)b(W)-\right.\\
&\quad\left.-(-1)^{ab}i_{w,z}(Z-W)^{j|J}b(W)a(Z)\right) = \\
&(-1)^N \operatorname{res}_Z\left(\left(-\sigma(e_i, J\setminus e_i)i_{z,w}(Z-W)^{j|J\setminus e_i} - j\sigma(e_i,J)i_{z,w}(Z-W)^{j-1|J\cup e_i}\right)\times\right.\\
&\times a(Z)b(W) + (-1)^{J+a}i_{z,w}(Z-W)^{j|J}a(Z)D^i_W b(W)+\\
&+(-1)^{ab}\left(\sigma(e_i,J\setminus e_i)i_{w,z}(Z-W)^{j|J\setminus e_i} + j\sigma(e_i,J)i_{w,z}(Z-W)^{j-1|J\cup e_i}\right)b(W)a(Z)-\\
&\left.-(-1)^{ab+J}i_{w,z}(Z-W)^{j|J}D^i_W b(W)a(Z)\right) = \\
&= -(-1)^N\sigma(e_i,J\setminus e_i)a(W)_{(j|J\setminus e_i)}b(W) - (-1)^N j\sigma(e_i,J)a(W)_{(j-1|J\cup e_i)}b(W)+\\
&+(-1)^{N+J+a}a(W)_{(j|J)}D^i_W b(W) = \\
&= (-1)^{N-J}\left(\left(D^i_W a(W)\right)_{(j|J)}b(W) + (-1)^a a(W)_{(j|J)}D^i_W b(W)\right)
\end{aligned} \tag{3.5.13.5}$$

□

3.5.14. There is a slight change when defining the corresponding $N_K = N$ SUSY vertex algebras. Let $\bar{D}^i_Z = \partial_{\theta^i} - \theta^i\partial_z$. We define an $N_K = N$ SUSY vertex algebra as the data consisting of a super vector space V, an even vector $|0>\in V$, N odd endomorphisms S^i and a parity preserving linear map $\overset{s}{Y}$ from V to $\operatorname{End}(V)$-valued superfields $a\mapsto \overset{s}{Y}(a,Z)$, satisfying the following axioms:

- vacuum axioms:

$$\begin{aligned}\overset{s}{Y}(a,Z)|0> &= a + O(Z),\\ S^i|0> &= 0, \qquad i=1,\dots,N,\end{aligned} \tag{3.5.14.1}$$

- translation invariance:

$$[S^i, \overset{s}{Y}(a,Z)] = \bar{D}^i_Z\overset{s}{Y}(a,Z), \tag{3.5.14.2}$$

- locality:

$$(z-w)^n[\overset{s}{Y}(a,Z), \overset{s}{Y}(b,W)] = 0, \qquad \text{for some} n\in\mathbb{Z}_+. \tag{3.5.14.3}$$

3.5.15. We define the $(j|J)$-products for a $N_K = N$ SUSY vertex algebra, as in the $N_W = N$ case, by (3.3.2.2).

As in 3.3.2 we see easily that the vacuum axioms may be formulated as (3.3.2.4) and translation invariance is equivalent to:

$$[S^i, a_{(j,J)}] = \begin{cases}\sigma(N\setminus J, e_i)a_{(j,J\setminus e_i)} & e_i\in J\\ j\sigma(N\setminus(J\cup e_i), e_i)a_{(j-1|J\cup e_i)} & e_i\notin J\end{cases} \tag{3.5.15.1}$$

3.5.16. It follows easily from (3.5.15.1) and the vacuum axioms that

$$[S^i, S^j] = 2\delta_{i,j}T, \qquad [S^i, T] = 0, \tag{3.5.16.1}$$

where T is an even operator satisfying:

$$[T, a_{(j|J)}] = -ja_{(j-1|J)} \qquad \forall a, (j|J), \tag{3.5.16.2}$$

or equivalently:

$$[T, \overset{s}{Y}(a, Z)] = \partial_z \overset{s}{Y}(a, Z). \tag{3.5.16.3}$$

With these results we can prove the $N_K = N$ version of theorem 3.3.3.

Theorem 3.5.17. *Let $\mathscr{U}$ be a vector superspace and V a space of pairwise local* $\operatorname{End}(\mathscr{U})$*-valued fields such that V contains the constant field* Id, *it is invariant under the derivations $D^i_Z = \partial_{\theta^i} + \theta^i \partial_z$ and closed under all $(j|J)$-th products. Then V is an $N_K = N$ SUSY vertex algebra with vacuum vector* Id, *the translation operators are $S^i a(Z) = D^i_Z a(Z)$, the $(j|J)$ product is the one for distributions multiplied by $\sigma(J)$.*

Proof. The proof goes like the proof of 3.3.3. To check translation invariance we see that $D^i_Z 1 = 0$ and that

$$\begin{aligned}\sigma(J) D^i_Z \left(a(Z)_{(j|J)} b(Z)\right) - (-1)^{a+N-J} a(Z)_{(j|J)} D^i_Z b(Z) = \\ = (-1)^{N-J} \sigma(J) \left(D^i_Z a(Z)\right)_{(j|J)} b(Z).\end{aligned} \tag{3.5.17.1}$$

But in view of (3.5.13.1) this is:

$$\begin{aligned}&- (-1)^N \sigma(J) \left(\sigma(e_i, J) a(Z)_{(j|J\setminus e_i)} b(Z) + \right.\\ &\qquad \left. + j\sigma(e_i, J) a(W)_{(j-1|J\cup e_i)} b(Z)\right) = \\ &= \sigma(N\setminus J, e_i)\sigma(J\setminus e_i) a(Z)_{(j|J\setminus e_i)} b(Z) \\ &\quad + j\sigma(N \setminus (J\cup e_i), e_i)\sigma(J\cup e_i) a(Z)_{(j-1|J\cup e_i)} b(Z),\end{aligned} \tag{3.5.17.2}$$

proving equation (3.5.15.1).

Locality is proved in the same way as in 3.3.3. □

Lemma 3.3.5 is still valid for $N_K = N$ SUSY vertex algebras. Its proof parallels the proof for $N_W = N$ SUSY vertex algebras. The proof of proposition 3.3.7 in this context is more subtle:

Proposition 3.5.18. *Let V be a $N_K = N$ SUSY vertex algebra. Then for every $a, b \in V$ we have:*

1. $\overset{s}{Y}(a, Z)|0\rangle = \exp(Z\nabla) a$,

2. $\exp(Z\nabla) \overset{s}{Y}(a, W) \exp(-Z\nabla) = i_{w,z} \overset{s}{Y}(a, W + Z)$,

3. $\overset{s}{Y}(a,Z)_{(j|J)}\overset{s}{Y}(b,Z)|0> = \sigma(J)\overset{s}{Y}(a_{(j|J)}b,Z)|0>$.

where $\nabla = (T, S^1, \dots, S^N)$, *and* $Z\nabla = zT + \sum \theta^i S^i$. *Finally we define* $W + Z = W - (-Z) = (z + w + \sum \zeta^i \theta^i, \theta^j + \zeta^j)$[3].

Proof. As in the proof of proposition 3.3.7 we note that both sides of (1) and (3) are elements of $V[[Z]]$, whereas both sides of (2) are elements of $\mathrm{End}(V)[[W, W^{-1}]][[Z]]$. By evaluating at $Z = 0$ we get equalities in all three cases. Indeed (1) and (2) are trivial, and (3) follows from the $N_K = N$ version of lemma 3.3.5. We need to show that both sides in each equation satisfy the same system of differential equations.

(1) Similarly to the proof of lemma 3.5.7 we expand:

$$\begin{aligned} \bar{D}^i_Z \exp(Z\nabla) &= \left(\partial_{\theta^i} - \theta^i \partial_z\right) \exp(zT) \sum_{k \geq 0} \frac{(\sum_i \theta^i S^i)^k}{k!} \\ &= (S^i + T\theta^i) \exp(Z\nabla) - \theta^i T \exp(Z\nabla) \\ &= S^i \exp(Z\nabla), \end{aligned} \tag{3.5.18.1}$$

from where the right hand side $X(Z)$ of (1) satisfies the system of differential equations:

$$\bar{D}^i_Z X(Z) = S^i X(Z). \tag{3.5.18.2}$$

Similarly by translation invariance we have for the left hand side of (1):

$$\bar{D}^i_Z \overset{s}{Y}(a,Z)|0> = [S^i, \overset{s}{Y}(a,Z)]|0> = S^i \overset{s}{Y}(a,Z)|0>\,. \tag{3.5.18.3}$$

We also point out that a similar computation to (3.5.18.1) shows that

$$\bar{D}^i_Z \exp(-Z\nabla) = -\exp(-Z\nabla) S^i, \tag{3.5.18.4}$$

which is not entirely obvious since S^i does not commute with the exponential.

(2) By translation invariance we have:

$$\begin{aligned} \bar{D}^i_Z \overset{s}{Y}(a, W+Z) = \big(&-\zeta^i \partial_{w+z+\sum\zeta^i\theta^i} + \\ &+\partial_{\zeta^i+\theta^i} - \theta^i \partial_{w+z+\sum \zeta^i\theta^i}\big) \overset{s}{Y}(a, W+Z) = \\ &= \bar{D}^i_{W+Z} \overset{s}{Y}(a, Z+W) = [S^i, \overset{s}{Y}(a, Z+W)]. \end{aligned} \tag{3.5.18.5}$$

On the other hand, letting $Y(Z) = e^{Z\nabla} \overset{s}{Y}(a,W) e^{-Z\nabla}$ we have (cf. (3.5.18.4)):

$$\begin{aligned} \bar{D}^i_Z Y(Z) &= S^i Y(Z) - (-1)^a Y(Z) S^i \\ &= [S^i, Y(Z)]. \end{aligned} \tag{3.5.18.6}$$

(3) For the right hand side we have by translation invariance and the vacuum

[3] Note that $Z + W \neq W + Z$

axioms:

$$S^i\overset{s}{Y}(a_{(j|J)}b,Z)|0>=[S^i,\overset{s}{Y}(a_{(j|J)}b,Z)]|0>=\bar{D}^i_Z\overset{s}{Y}(a_{(j|J)}b,Z)|0>. \tag{3.5.18.7}$$

To prove that the left hand side satisfies the same differential equation we proceed exactly in the same way as in the proof of proposition 3.3.7. We only need the fact that $\bar{D}^i_Z$ is a derivation of all $(j|J)$-products. But $\partial_z=(D^i_Z)^2$ is a derivation since:

$$\begin{aligned}\partial_z a(Z)_{(j|J)}b(Z)&=(-1)^{N-J}D^i_Z\left((D^i_Za(Z))_{(j|J)}b(Z)+(-1)^a a_{(j|J)}D^i_Zb(Z)\right)=\\&(\partial_z a(Z))_{(j|J)}b(Z)+(-1)^{a+1}(D^i_Za(Z))_{(j|J)}D^i_Zb(Z)+\\&+(-1)^a(D^i_Za(Z))_{(j|J)}D^i_Zb(Z)+a(Z)_{(j|J)}\partial_zb(Z)\end{aligned} \tag{3.5.18.8}$$

therefore $\bar{D}^i_Z=D^i_Z-2\theta^i\partial_z$ is a derivation of all $(j|J)$-products. □

The uniqueness proposition 3.3.8 is still valid in this context, As its corollary, we obtain an analogous version of theorem 3.3.9, namely

Theorem 3.5.19. *On an $N_K=N$ SUSY vertex algebra the following identities hold*

1. $\overset{s}{Y}(a_{(j|J)}b,Z)=\sigma(J)\overset{s}{Y}(a,Z)_{(j|J)}\overset{s}{Y}(b,Z)$.
2. $\overset{s}{Y}(a_{(-1|N)}b,Z)=:\overset{s}{Y}(a,Z)\overset{s}{Y}(b,Z):$.
3. $\overset{s}{Y}(S^ia,Z)=D^i_Z\overset{s}{Y}(a,Z)$.
4. *We have the following OPE formula:*

$$[\overset{s}{Y}(a,Z),\overset{s}{Y}(b,W)]=\sum_{(j,J),j\geq 0}\sigma(J)(D_W^{(j|J)}\delta(Z,W))\overset{s}{Y}(a_{(j|J)}b,W) \tag{3.5.19.1}$$

Remark 3.5.20. Note that as a consequence of (3) we obtain

$$[S^i,\overset{s}{Y}(a,Z)]\neq\overset{s}{Y}(S^ia,Z), \tag{3.5.20.1}$$

in contrast to the $N_W=N$ and, in particular, the ordinary vertex algebra case.

Corollary 3.5.21.

$$\begin{aligned}(S^ia)_{(j|J)}&=\sigma(e_i,N\setminus J)a_{(j|J\setminus e_i)}-j\sigma(e_i,N\setminus(J\cup e_i))a_{(j-1|J\cup e_i)}\\&=\begin{cases}\sigma(e_i,N\setminus J)a_{(j|J\setminus e_i)} & \text{for } e_i\in J\\ -j\sigma(e_i,N\setminus(J\cup e_i))a_{(j-1|J\cup e_i)} & \text{for } e_i\notin J\end{cases}\end{aligned} \tag{3.5.21.1}$$

$$S^i\left(a_{(j|J)}b\right)=(-1)^{N-J}\left((S^ia)_{(j|J)}b+(-1)^a a_{(j|J)}S^ib\right)$$

3.5.22. We note that

$$\delta(Z,W)=\sum_{(j|J)}(-1)^{N-J}\sigma(J)Z^{j|J}W^{-1-j|N\setminus J} \tag{3.5.22.1}$$

therefore equation (3.3.3.6) is still valid in this context. Also according to our prescription to add coordinates we see that

$$(X-Z)-W=X-(W+Z)=X-(W-(-Z)). \tag{3.5.22.2}$$

We also have a *Taylor expansion* in this context, namely, for any formal power series $a(Z)\in\mathbb{C}[[Z]]$ we have:

$$a(W+Z)=\sum_{\substack{(j|J)\\ j\geq 0}}(-1)^{\frac{J(J-1)}{2}}\frac{W^{j|J}}{j!}D_Z^{j|J}a(Z)=e^{WD_Z}a(Z). \tag{3.5.22.3}$$

Indeed, the usual Taylor expansion is:

$$\begin{aligned} a(W+Z)=a\left(w+z+\sum\zeta^i\theta^i,\zeta^j+\theta^j\right)= \\ =\sum(-1)^{\frac{J(J-1)}{2}}\frac{\partial_W^{j|J}}{j!}a(W+Z)|_{W=0}. \end{aligned} \tag{3.5.22.4}$$

In this case:

$$\begin{aligned} \partial_W^{1|0}a(W+Z)|_{W=0}&=D_Z^{1|0}a(Z), \\ \partial_W^{0|i}a(W+Z)|_{W=0}&=(\theta^i\partial_z+\partial_{\theta^i})a(Z)=D_Z^ia(Z), \end{aligned} \tag{3.5.22.5}$$

proving (3.5.22.3).

We can now proceed exactly as in 3.3.12 by taking the generating series of 3.5.19 (1) to get the analogous versions of the *associativity* formulas (3.3.12.7) and (3.3.12.8).

The proofs for *skew-symmetry* in theorem 3.3.14, quasi-commutativity for the normally ordered product as in 3.3.15 and *quasi-associativity* for the normally ordered product as in theorem 3.3.18 carry over verbatim to the $N_K=N$ case. We obtain then an analogous result to theorem 3.3.17, namely an $N_K=N$ SUSY vertex algebra gives rise to an $N_K=N$ SUSY Lie conformal algebra.

Proposition 3.3.22 is proved in the same way for $N_K=N$ SUSY vertex algebras. Following the argument in remark 3.3.23 and using (3.5.11.1) we obtain the commutator formula (cf. (3.5.11.3)) for the Fourier coefficients of the $\mathrm{End}(V)$ valued fields of the SUSY vertex algebra.

As in the $N_W=N$ case, we have the following equivalent definition:

Definition 3.5.23. An $N_K=N$ SUSY vertex algebra is a tuple $(V,T,S^i,[\cdot_\Lambda\cdot],|0\rangle,::)$, $i=1,\dots,N$, where

- $(V,T,S^i,[\cdot_\Lambda\cdot])$ is an $N_K=N$ SUSY Lie conformal algebra,

- $(V, |0>, T, S^i, ::)$ is a unital quasicommutative quasiassociative differential superalgebra (i.e. T is an even derivation of $::$ and S^i are odd derivations of $::$),
- the Λ-bracket and the product $::$ are related by the non-commutative Wick formula (3.2.20.3).

The rest of section 3.3 carries over to the $N_K = N$ case with minor modifications, in particular we define tensor products of $N_K = N$ SUSY vertex algebras as in 3.3.24 and we have an *existence theorem* as in 3.3.25 that we restate here:

Theorem 3.5.24 (Existence of $N_K = N$ SUSY vertex algebras). *Let V be a vector space, $|0>\in V$ an even vector, T an even endomorphism of V and S^i, $i = 1, \dots, N$ odd endomorphisms of V, satisfying $[S^i, S^j] = 2\delta_{i,j}T$. Suppose moreover that $S^i|0>= 0$. Let $\mathscr{F}$ be a family of fields of the form*

$$a^\alpha(Z) = \sum Z^{-1-j|N\setminus J} a^\alpha_{(j|J)} \tag{3.5.24.1}$$

indexed by $\alpha \in A$, and such that

1. $a^\alpha(Z)|0>|_{Z=0} = a^\alpha \in V$,
2. $[S^i, a^\alpha(Z)] = \bar{D}^i_Z a^\alpha(Z)$,
3. *all pairs $(a^\alpha(Z), a^\beta(Z))$ are local,*
4. *the following vectors span V*

$$a^{\alpha_s}_{(j_s|J_s)} \dots a^{\alpha_1}_{(j_1|J_1)}|0> . \tag{3.5.24.2}$$

Then the formula

$$\overset{s}{Y}(a^{\alpha_s}_{(j_s|J_s)} \dots a^{\alpha_1}_{(j_1|J_1)}|0>, Z) = \\ = \prod \sigma(J_i) a^{\alpha_s}(Z)_{(j_s|J_s)}\Big(\dots a^{\alpha_2}_{(j_2|J_2)}(Z)\big(a^{\alpha_1}_{(j_1|J_1)}(Z)\,\mathrm{Id}\big)\dots\Big) \tag{3.5.24.3}$$

defines a structure of an $N_K = N$ SUSY vertex algebra on V, with vacuum vector $|0>$, translation operators S^i and such that

$$\overset{s}{Y}(a^\alpha, Z) = a^\alpha(Z). \tag{3.5.24.4}$$

Such a structure is unique.

3.5.25. The results in section 3.4 generalize to this context without difficulty. In particular, we obtain the *universal enveloping $N_K = N$ SUSY vertex algebra* of an $N_K = N$ SUSY Lie conformal algebra. Note that the definition of a holomorphic $N_K = N$ SUSY vertex algebra is straightforward: a unital associative commutative superalgebra $\mathscr{A}$ with an even derivation T and N odd derivations S^i, commuting with T and satisfying $[S^i, S^j] = 2\delta_{ij}T$ defines an $N_K = N$ SUSY vertex algebra, with $|0\rangle = 1$, and the state field correspondence given by $\overset{s}{Y}(a, Z)b = \left(e^{Z\nabla}a\right) b$. In particular $\mathbb{C}[W, W^{-1}]$ and $\mathbb{C}((W))$ are $N_K = N$ SUSY vertex algebras with $S^i = D^i_W$. We obtain therefore, in the same way as Theorem 3.4.3

Theorem 3.5.26. *Let V be an $N_K = N$ SUSY vertex algebra. Let $\mathscr{A} = \mathbb{C}[X, X^{-1}]$, define $L(V)$ to be the quotient of $\tilde{V} = \mathscr{A} \otimes_{\mathbb{C}} V$ by the linear span of vectors of the form:*

$$S^i a \otimes f(X) + (-1)^a a \otimes D^i_X f(X). \tag{3.5.26.1}$$

and let $L'(V)$ be its completion with respect to the natural topology in $\mathscr{A}$. Then $L(V)$ (resp. $L'(V)$) carries a natural Lie superalgebra of degree $N \mod 2$ structure. Let $\mathrm{Lie}(V)$ (resp. $\mathrm{Lie}'(V)$) be the corresponding Lie superalgebra. The map $\varphi : \mathrm{Lie}(V) \to \mathrm{End}(V)$ (resp. $\varphi' : \mathrm{Lie}'(V) \to \mathrm{End}(V)$), given by formula (3.4.2.2), is a Lie superalgebra homomorphism.

3.6 Examples

Example 3.6.1 (W_N series). Let $X = (x, \xi^1, \dots, \xi^N)$, where x is even and ξ^i are odd anticommuting variables commuting with x. Consider the Lie algebra $\mathfrak{g} = W(1|N)$ of derivations of $\mathbb{C}[X, X^{-1}]$, it is spanned by elements of the form $X^{j|J}\partial_x$ and $X^{j|J}\partial_{\xi^i}$ (cf. Example 2.1.30). Define the following $\mathfrak{g}$-valued formal distributions:

$$L(Z) = -\delta(Z, X)\partial_x, \qquad Q^i(Z) = -\delta(Z, X)\partial_{\xi^i}, \quad i = 1, \dots, N. \tag{3.6.1.1}$$

A long but straightforward computation shows that these distributions satisfy the following commutation relations:

$$\begin{aligned}
[L(Z), L(W)] &= \delta(Z, W)\partial_w L(W) + 2\left(\partial_w \delta(Z, W)\right) L(W), \\
[L(Z), Q^i(W)] &= \delta(Z, W)\partial_w Q^i(W) + \left(\partial_{\zeta^i}\delta(Z, W)\right) L(W) + \\
&\quad + \left(\partial_w \delta(Z, W)\right) Q^i(W), \\
[Q^i(Z), Q^j(W)] &= \delta(Z, W)\partial_{\zeta^i} Q^j(W) + (-1)^N \left(\partial_{\zeta^i}\delta(Z, W)\right) Q^j(W) - \\
&\quad - (-1)^N \left(\partial_{\zeta^j}\delta(Z, W)\right) Q^i(W),
\end{aligned} \tag{3.6.1.2}$$

in particular, the distributions (3.6.1.1) are pairwise local. Let $\mathscr{F}$ be the family of $\mathfrak{g}$-valued formal distributions

$$\mathscr{F} = \left\{ \partial_Z^{j|J} L(Z), \, \partial_Z^{j|J} Q^i(Z) \middle| \quad j \geq 0, \, J \subset \{1, \dots, N\}, \, i = 1, \dots, N \right\}.$$

Then $(\mathfrak{g}, \mathscr{F})$ is an $N_W = N$ SUSY formal distribution Lie superalgebra.

Let $\mathscr{W}(1|N)$ be the corresponding $N_W = N$ SUSY Lie conformal algebra. It is generated as a $\mathbb{C}[T, S^i]$-module by a vector L of parity $N \mod 2$ and N-vectors Q^i, $i = 1, \dots, N$ of parity $N + 1 \mod 2$ satisfying the following Λ-brackets (which can be easily obtained from (3.6.1.2))

$$\begin{aligned}[][L_\Lambda L] &= (T+2\lambda)L \\ [Q^i_\Lambda Q^j] &= (S^i+\chi^i)Q^j - \chi^j Q^i \\ [L_\Lambda Q^i] &= (T+\lambda)Q^i + (-1)^N \chi^i L.\end{aligned} \tag{3.6.1.3}$$

When $N=0$, this is the centerless Virasoro conformal algebra. It is well known that it admits a central extension defined by:

$$[L_\lambda L] = (T+2\lambda)L + \frac{\lambda^3}{12}C \tag{3.6.1.4}$$

where C is even, central, and satisfies $TC=0$.

Translating the formulas in [15], it follows that when $N=1$, $\mathscr{W}(1|1)$ admits a central extension of the form:

$$\begin{aligned}[][L_\Lambda L] &= (T+2\lambda)L \\ [Q_\Lambda Q] &= SQ + \frac{\lambda\chi}{3}C \\ [L_\Lambda Q] &= (T+\lambda)Q - \chi L + \frac{\lambda^2}{6}C,\end{aligned} \tag{3.6.1.5}$$

where C is even, central, and satisfies $TC=SC=0$.

When $N=2$, $\mathscr{W}(1|2)$ admits a central extension given by:

$$\begin{aligned}[][L_\Lambda L] &= (T+2\lambda)L \\ [Q^i_\Lambda Q^i] &= S^i Q^i \\ [Q^1_\Lambda Q^2] &= (S^1+\chi^1)Q^2 - \chi^2 Q^1 + \frac{\lambda}{6}C \\ [L_\Lambda Q^i] &= (T+\lambda)Q^i + \chi^i L\end{aligned} \tag{3.6.1.6}$$

where C is as above. It follows from [24] that these algebras do not admit central extensions for $N\geq 3$.

If $N\leq 2$, we let $\tilde{W}_N$ be the universal enveloping $N_W=N$ SUSY vertex algebra of the central extension of $\mathscr{W}(1|N)$ as given by theorem 3.4.5, and let W^c_N be the quotient of $\tilde{W}_N$ by the ideal $(C-c|0\rangle)_{(-1|N)}\tilde{W}_N$, where $c\in\mathbb{C}$ is called the *central charge.*

When $N=1$, expanding the superfields as

$$Q(Z) = -J(z) + \theta G^+(z), \qquad L(Z) = G^-(z) + \theta\left(L(z) + \frac{1}{2}\partial_z J(z)\right), \tag{3.6.1.7}$$

we check that the fields $L, J, G^\pm$ satisfy the commutation relations of the $N=2$ vertex algebra as defined in example 2.1.19.

When $N\geq 3$, we let W_N be the universal envelopping $N_W=N$ SUSY vertex algebra of $\mathscr{W}(1|N)$. It follows from the definitions that $\mathrm{Lie}(\mathscr{W}(1|N)) = W(1|N)$, the

Lie superalgebra of derivations of $\mathbb{C}[X, X^{-1}]$. Also, $\mathrm{Lie}(\mathscr{W}(1|N))_- = W(1|N)_-$ is the Lie superalgebra of derivations of $\mathbb{C}[X]$. Denote by $W(1|N)_< = \mathrm{Lie}(\mathscr{W}(1|N))_< \subset \mathrm{Lie}(\mathscr{W}(1|N))_-$ the Lie superalgebra of vector fields vanishing at the origin, it is spanned by vectors of the form $X^{j|J}\partial_x$ and and $X^{j|J}\partial_{\xi^i}$, with $j + \sharp J > 0$.

Definition 3.6.2. An $N_W = N$ SUSY vertex algebra V is called *conformal* if there exists $N+1$ vectors $\nu, \tau^1, \dots, \tau^N$ in V such that their associated superfields $L(Z) = \overset{s}{Y}(\nu, Z)$ and $Q^i(Z) = \overset{s}{Y}(\tau^i, Z)$ satisfy (3.6.1.3) (or possibly a central extension) and moreover:

- $\nu_{(0|0)} = T$,
- $\tau^i_{(0|0)} = S^i$,
- The operator $\nu_{(1|0)}$ acts diagonally with eigenvalues bounded below and with finite dimensional eigenspaces.

If moreover, the action of $\mathrm{Lie}(\mathscr{W}(1|N))_<$ on V can be exponentiated to the group of automorphisms of the disk $D^{1|N}$, we will say that V is *strongly conformal.* This ammounts to the following extra condition

- The operators $\nu_{(1|0)}$ and $\sum_{i=1}^N \sigma(e_i)\tau^i_{(0|e_i)}$ have integer eigenvalues.

If $a \in V$ is an eigenvector of $\nu_{(1|0)}$ of eigenvalue Δ, we say that a has *conformal weight* Δ. This happens if a satisfies

$$[L_\Lambda a] = (T + \Delta\lambda)a + O(\lambda^2) + O(\chi^1, \dots, \chi^N), \tag{3.6.2.1}$$

If, moreover, a satisfies $[L_\Lambda a] = (T + \Delta\lambda)a$ we say that a is *primary.*

As in the ordinary vertex algebra case, the conformal weight $\Delta(a)$ is an important *book-keeping device*:

$$\Delta(Ta) = \Delta(a) + 1, \quad \Delta(S^i a) = \Delta(a), \quad \Delta(:ab:) = \Delta(a) + \Delta(b).$$

Furthermore, if we let $\Delta(\chi^i) = 0$ and $\Delta(\lambda) = 1$, all terms in $[a_\Lambda b]$ have conformal weight $\Delta(a) + \Delta(b) - 1$.

Remark 3.6.3. It is clear from this definition that the $N_W = N$ SUSY vertex algebra W_N^c defined in example 3.6.1 is indeed strongly conformal.

Example 3.6.4 (Free Fields). As an example of a strongly conformal $N_W = N$ SUSY vertex algebra, we will compute explicitly the free fields case, namely, let α, C be even vectors and let φ be an odd vector. Consider the $N_W = N$ SUSY Lie conformal algebra generated by these three vectors, where C is central and anihilated by ∇, and the other commutation relations are:

$$[\alpha_\Lambda \varphi] = C. \tag{3.6.4.1}$$

Let $\tilde{F}_N$ be its universal envelopping $N_W = N$ SUSY vertex algebra and F_N its quotient by the ideal $(C - |0>)_{(-1|N)}\tilde{F}_N$.

Expanding the superfields

$$\alpha(Z) = a(z) + \theta\psi(z), \qquad \varphi(Z) = \phi(z) + \theta b(z) \tag{3.6.4.2}$$

we find that the fields a, b, ψ and ϕ generate the well known $bc - \beta\gamma$-system, namely, the non-trivial λ-brackets are (up to skew-symmetry):

$$[b_\lambda a] = [\psi_\lambda \phi] = 1. \tag{3.6.4.3}$$

When $N = 1$, this SUSY vertex algebra admits a $N_W = 1$ strongly conformal structure with:

$$\begin{aligned} \nu &= \alpha_{(-2|1)}\varphi_{(-1|1)}|0>, \\ \tau &= -\alpha_{(-1|0)}\varphi_{(-1|1)}|0>, \end{aligned} \tag{3.6.4.4}$$

and central charge $c = 3$. The associated fields $L = \overset{s}{Y}(\nu, Z)$ and $Q = \overset{s}{Y}(\tau, Z)$ are:

$$L =: (T\alpha)\varphi : \qquad Q = - : (S\alpha)\varphi : . \tag{3.6.4.5}$$

In order to check the commutation relations (3.6.1.5) we use the non-commutative Wick formula (3.2.20.3) to find:

$$[\alpha_\Lambda L] = T\alpha \qquad [\varphi_\Lambda L] = \lambda\varphi \tag{3.6.4.6}$$
$$[\alpha_\Lambda Q] = S\alpha \qquad [\varphi_\Lambda Q] = -\chi\varphi. \tag{3.6.4.7}$$

And now by skew-symmetry and sesquilinearity we obtain:

$$[L_\Lambda \alpha] = T\alpha \qquad [L_\Lambda \varphi] = (\lambda + T)\varphi \tag{3.6.4.8}$$
$$[L_\Lambda T\alpha] = (T + \lambda)T\alpha \qquad [L_\Lambda S\alpha] = (S + \chi)T\alpha \tag{3.6.4.9}$$
$$[Q_\Lambda \alpha] = S\alpha \qquad [Q_\Lambda \varphi] = (S + \chi)\varphi \tag{3.6.4.10}$$
$$[Q_\Lambda S\alpha] = -\chi S\alpha. \tag{3.6.4.11}$$

Formula (3.6.4.8) says that α and φ are primary fields of conformal weight 0 and 1 respectively. With these formulas and using again the Wick formula (3.2.20.3) we obtain

$$\begin{aligned} [L_\Lambda L] = [L_\Lambda : (T\alpha)\varphi :] &=: ((T + \lambda)T\alpha)\varphi : + : T\alpha(\lambda + T)\varphi := \\ &= 2\lambda L + : (T(T\alpha))\varphi : + : T\alpha T\varphi := (T + 2\lambda)L, \end{aligned} \tag{3.6.4.12}$$

since the integral term obviously vanishes. For the other commutation relations we compute:

$$[Q_\Lambda Q] = -[Q_\Lambda : (S\alpha)\varphi :] = \chi : (S\alpha)\varphi : + : S\alpha(\chi + S)\varphi : + \int_0^\Lambda [\chi S\alpha_\Gamma \varphi] d\Gamma =$$
$$=: S\alpha S\varphi : + \int_0^\Lambda (\eta - \chi)\eta d\Gamma = SQ + \lambda\chi. \quad (3.6.4.13)$$

Finally for the last commutator we find:

$$[L_\Lambda Q] = -[L_\Lambda : (S\alpha)\varphi :] = - : ((S+\chi)T\alpha)\varphi : - : S\alpha(\lambda + T)\varphi : -$$
$$- \int_0^\Lambda [(S+\chi)T\alpha_\Gamma\varphi] d\Gamma = TQ - \chi : (T\alpha)\varphi : + \lambda Q + \int_0^\Lambda (\eta - \chi)\gamma d\Gamma =$$
$$= (T+\lambda)Q - \chi L + \frac{\lambda^2}{2}. \quad (3.6.4.14)$$

According to (3.6.1.5) this is a conformal $N_W = 1$ SUSY vertex algebra with central charge 3. It is easy to check that this SUSY vertex algebra is indeed strongly conformal.

Example 3.6.5 (K_N series). Consider now the Lie subsuperalgebra $K(1|N) \subset W(1|N)$ consisting of those derivations of $\mathbb{C}[X, X^{-1}]$ preserving the form

$$\omega = dx + \sum_{i=1}^N \xi^i d\xi^i,$$

up to multiplication by a function (cf. Example 2.1.31). Define the following $\mathfrak{g}$-valued formal distribution:

$$G(Z) = -2\delta(Z,X)\partial_x - (-1)^N \sum_{i=1}^N \left(D^i_X \delta(Z,X)\right) D^i_X. \quad (3.6.5.1)$$

It follows from (2.1.31.1) that its Z-coefficients form a basis of $K(1|N)$. A long but straightforward computation shows that this formal distribution satisfies the following commutation relation:

$$[G(Z), G(W)] = 2\delta(Z,W)\partial_w G(W) + (4-N)\left(\partial_w \delta(Z,W)\right) G(W) +$$
$$+ (-1)^N \sum_{i=1}^N \left(D^i_W \delta(Z,W)\right) D^i_W G(W), \quad (3.6.5.2)$$

in particular, the pair of $\mathfrak{g}$-valued formal distributions $(G(Z), G(Z))$ is local. Letting

$$\mathscr{F} = \left\{ \partial_Z^{j|J} G(Z),\ j \geq 0,\ J \subset \{1, \dots, N\} \right\},$$

we see that $(\mathfrak{g}, \mathscr{F})$ is an $N_K = N$ SUSY formal distribution Lie superalgebra.

Let $\mathscr{K}(1|N)$ be the associated $N_K = N$ SUSY Lie conformal algebra. It is generated as an $\mathscr{H}$ module by a vector G of parity $N \mod 2$ satisfying the following Λ-bracket (for the definition of the algebra $\mathscr{H}$ see 3.5.3)

$$[G_\Lambda G] = \left(2T + (4-N)\lambda + \sum_{i=1}^{N} \chi^i S^i \right) G. \tag{3.6.5.3}$$

When $N \leq 3$, $\mathscr{K}(1|N)$ admits a non-trivial central extension adding a term

$$\frac{\lambda^{3-N}\chi^N}{3} C$$

to (3.6.5.3), where C is even, central and satisfies $TC = S^i C = 0$.

When $N = 4$, $\mathscr{K}(1|4)$ admits a central extension by adding a term

$$\lambda C$$

to (3.6.5.3). It follows from [24] that this algebra does not admit central extensions when $N > 4$.

When $N \leq 4$, we let $\tilde{K}_N$ be the universal enveloping $N_K = N$ SUSY vertex algebra of the central extension of $\mathscr{K}(1|N)$ and define K_N^c to be its quotient by the ideal $(C - c|0>)_{(-1|N)}\tilde{K}_N$, where $c \in \mathbb{C}$ is called the *central charge*. When $N \geq 5$, we let K_N be the universal enveloping $N_K = N$ SUSY vertex algebra of $\mathscr{K}(1|N)$.

In the case $N = 1$, if we expand the corresponding superfield as

$$G(z, \theta) = G(z) + 2\theta L(z), \tag{3.6.5.4}$$

we find that the fields $G(z)$ and $L(z)$ generate a Neveu Schwarz vertex algebra of central charge c as in example 2.1.11.

When $N = 2$ expanding the corresponding superfield as (cf. 2.1.22.4)

$$G(z, \theta^1, \theta^2) = \sqrt{-1} J(z) + \theta^1 G^2(z) - \theta^2 G^1(z) + 2\theta^1\theta^2 L(z) \tag{3.6.5.5}$$

We find that the corresponding fields $J, L, G^\pm$ satisfy the commutation relations of the $N = 2$ vertex algebra as in example 2.1.19.

When $N = 4$ the corresponding $N_K = 4$ SUSY vertex algebra is not simple. Indeed the SUSY Lie conformal superalgebra $\mathscr{K}'(1|4) \subset \mathscr{K}(1|4)$ generated by $S^i G$, $i = 1, \ldots, 4$ is an ideal. The central extension of $\mathscr{K}(1|4)$ described above restrict to a central extension of $\mathscr{K}'(1|4)$ whose cocycle is given by:

$$\alpha(S^i G, S^j G) = -\chi^i \chi^j C. \tag{3.6.5.6}$$

This SUSY Lie conformal algebra admits another central extension given by (cf. [15]):

$$\alpha(S^i G, S^j G) = \chi^1 \chi^2 \chi^3 \chi^4 C. \tag{3.6.5.7}$$

We let $K^{c\prime}_4$ (resp. $K^{c\prime\prime}$) be the corresponding $N_K = 4$ SUSY vertex algebras when we use the central extension (3.6.5.6) (resp. (3.6.5.7)).

Note that $\mathrm{Lie}(\mathscr{K}(1|N)) = K(1|N)$ by definition, while $\mathrm{Lie}(\mathscr{K}(1|N))_- = K(1|N)_-$ is the Lie superalgebra of regular vector fields preserving ω up to multiplication by a function. We will denote by $K(1|N)_< = \mathrm{Lie}(\mathscr{K}(1|N))_< \subset \mathrm{Lie}(\mathscr{K}(1|N))_-$ the Lie superalgebra of regular vector fields, preserving ω up to multiplication by a function, and vanishing at the origin, namely

$$\mathrm{Lie}(\mathscr{K}(1|N))_< = \mathrm{Lie}(\mathscr{K}(1|N))_- \cap \mathrm{Lie}(\mathscr{W}(1|N))_<.$$

Finally, a field G satisfying the commutation relations (3.6.5.3) (or a central extension of it) will be called a *super Virasoro* field.

Definition 3.6.6. Let $N \leq 4$, an $N_K = N$ SUSY vertex algebra V is called *conformal* if there exists a vector $\tau \in V$ (called the conformal vector) such that the corresponding field $G(Z) = \overset{s}{Y}(\tau, Z)$ satisfies (3.6.5.3) (or posibly a central extension) and moreover

- $\tau_{(0|0)} = 2T$,
- $\tau_{(0|e_i)} = \sigma(N \setminus e_i, e_i) S^i$,
- the operator $\tau_{(1|0)}$ acts diagonally with eigenvalues bounded below and finite dimensional eigenspaces.

If moreover, the representation of $\mathrm{Lie}(\mathscr{K}(1|N))_<$ can be exponentiated to the group of automorphisms of the disk $D^{1|N}$ preserving the SUSY structure

$$\omega = dx + \sum_{i=1}^{N} \xi^i d\xi^i, \tag{3.6.6.1}$$

we will say that V is *strongly conformal.* This amounts to the extra condition

- the operator $2\tau_{(1|0)}$ has integer eigenvalues.

If a vector a in a conformal $N_K = N$ SUSY vertex algebra V is an eigenvector of $\tau_{(1|0)}$ with eigenvalue 2Δ, we say that a has *conformal weight* Δ. This happens iff a satisfies

$$[G_\Lambda a] = \left(2T + 2\Delta\lambda + \sum_{i=1}^{N} \chi^i S^i\right) a + O(\Lambda^2), \tag{3.6.6.2}$$

where $O(\Lambda^2)$ denotes a polynomial in Λ with vanishing constant and linear terms. If, moreover, a satisfies

$$[G_\Lambda a] = \left(2T + 2\Delta\lambda + \sum_{i=1}^{N} \chi^i S^i\right) a,$$

we say that a is *primary.* For example, formula (3.6.5.3) says that G has conformal weight $2 - N/2$, and it is primary if the central extension is trivial. As in the $N_W = N$

case, the conformal weight is an important *book-keeping device*

$$\Delta(Ta) = \Delta(a) + 1, \quad \Delta(S^i a) = \Delta(a) + \frac{1}{2}, \quad \Delta(:ab:) = \Delta(a) + \Delta(b).$$

Furthermore, letting $\Delta(\lambda) = 1$ and $\Delta(\chi^i) = 1/2$, all terms in $[a_\Lambda b]$ have conformal weight $\Delta(a) + \Delta(b) - 1 + N/2$.

Remark 3.6.7. The $N_K = N$ SUSY vertex algebra K_N^c defined in Example 3.6.5 is strongly conformal when $N < 4$, moreover, $K^{c'}_4$ and $K^{c''}_4$ are strongly conformal.

Example 3.6.8. (Free fields) The well-known *boson-fermion* system is an $N_K = N$ vertex algebra generated by one superfield. Let Ψ be a vector of parity $(-1)^N$, C an even vector, and define a $N_K = N$ SUSY Lie conformal algebra generated by Ψ and C where C is central, satisfies $TC = S^i C = 0$ and the remaining commutation relations are:

$$[\Psi_\Lambda \Psi] = \Lambda^{1|N} C, \tag{3.6.8.1}$$

when N is even, and

$$[\Psi_\Lambda \Psi] = \Lambda^{0|N} C, \tag{3.6.8.2}$$

when N is odd. Skew symmetry is clear and the Jacobi identity is obvious since all triple brackets vanish. We let $\tilde{B}_N$ be the corresponding universal enveloping $N_K = N$ SUSY vertex algebra, and let B_N be its quotient by the ideal $(C - |0>)_{(-1|N)}\tilde{B}_N$.

To show an application of the above formalism as well as the subtleties involved in calculations we will show explicitly that the $N_K = 1$ SUSY vertex algebra B_1 is conformal, the corresponding super Virasoro field being

$$G =: (S\Psi)\Psi : + mT\Psi, \qquad m \in \mathbb{C}. \tag{3.6.8.3}$$

Indeed, from sesquilinearity (3.5.9.2) and skew-symmetry (3.5.9.3) we find

$$[\Psi_\Lambda S\Psi] = (S + \chi)\chi = \lambda, \qquad [S\Psi_\Lambda \Psi] = -\lambda, \tag{3.6.8.4}$$

where we used $[S, \chi] = 2\lambda$ and $\chi^2 = -\lambda$. Using sesquilinearity once more we get:

$$[S\Psi_\Lambda S\Psi] = \chi\lambda, \qquad [\Psi_\Lambda T\Psi] = \lambda\chi. \tag{3.6.8.5}$$

Now we can use the super version of the non-commutative Wick formula (3.2.20.3) to find:

$$\begin{aligned} [\Psi_\Lambda G] &= (\lambda + \chi S)\Psi + m\lambda\chi \\ [S\Psi_\Lambda G] &= \lambda(\chi - S)\Psi - m\lambda^2 \\ [T\Psi_\Lambda G] &= -\lambda(\lambda + \chi S)\Psi - m\lambda^2\chi, \end{aligned} \tag{3.6.8.6}$$

where we note that all the integral terms vanish. Using skew-symmetry again we get:

$$\begin{aligned} [G_\Lambda \Psi] &= (\lambda + 2T + \chi S)\Psi - m\lambda\chi \\ [G_\Lambda S\Psi] &= (\lambda + T)(\chi + 2S)\Psi - m\lambda^2 \\ [G_\Lambda T\Psi] &= (T + \lambda)(\lambda + 2T + \chi S)\Psi - m\lambda^2\chi. \end{aligned} \tag{3.6.8.7}$$

With these formulas we can use again (3.2.20.3) to get:

$$[G_\Lambda G] =: \left((\lambda+T)(\chi+2S)\Psi - m\lambda^2\right)\Psi : + \\ + : S\Psi\,(\lambda+2T+\chi S)\Psi - m\lambda\chi) : + m(T+\lambda)(\lambda+2T+\chi S)\Psi - m^2\lambda^2\chi \quad (3.6.8.8)$$

where again the integral term is easily seen to vanish. Note that from quasi-commutativity of the normally ordered product we find $:\Psi\Psi:=0$, from where the expression above reduces to:

$$2\lambda : (S\Psi)\Psi : + \chi : (T\Psi)\Psi : +2 : (S^3\Psi)\Psi : - m\lambda^2\Psi + \\ + : S\Psi\,((\lambda+2T+\chi S)\Psi - m\lambda\chi) : + m(T+\lambda)(\lambda+2T+\chi S)\Psi - m^2\lambda^2\chi. \quad (3.6.8.9)$$

Expanding this expression and after a simple cancellation we find

$$[G_\Lambda G] = (2T+3\lambda+\chi S)\,G - m^2\lambda^2\chi. \quad (3.6.8.10)$$

Therefore B_1 is a strongly conformal $N_K=1$ SUSY vertex algebra with central charge $-3m^2$. Note that, by (3.6.8.7), Ψ has conformal weight $1/2$ (but it is not primary).

Note that in this example, if we expand the superfield

$$\Psi(Z) = \varphi(z) + \theta\alpha(z), \quad (3.6.8.11)$$

we find easily that

$$[\varphi_\lambda\varphi] = 1, \qquad [\alpha_\lambda\alpha] = \lambda, \quad (3.6.8.12)$$

hence the name *boson-fermion* system.

Example 3.6.9. (Super Currents) Let $\mathfrak{g}$ be a simple or abelian Lie superalgebra with a non-degenerate invariant supersymmetric bilinear form $(\,,\,)$. Let N be even, then we define a SUSY Lie conformal algebra (either $N_K=N$ or $N_W=N$) generated by $\mathfrak{g}$ with commutation relations:

$$[a_\Lambda b] = [a,b] + (k+h^\vee)(a,b)\lambda \qquad \forall a,\, b\,\in\mathfrak{g}, \quad (3.6.9.1)$$

where $2h^\vee$ is the eigenvalue of the Casimir operator on $\mathfrak{g}$.

When N is odd we let $\bar{\mathfrak{g}}$ be $\mathfrak{g}$ with reversed parity, and for each element $a\in\mathfrak{g}$ we let $\bar a$ be the same element thought in $\bar{\mathfrak{g}}$. In this case we define a SUSY Lie conformal algebra generated by $\bar{\mathfrak{g}}$ with commutation relations:

$$[\bar a_\Lambda \bar b] = (-1)^a\left(\overline{[a,b]} + (k+h^\vee)(a,b)\sum_{i=1}^N \chi^i\right). \quad (3.6.9.2)$$

We let $V^k(\mathfrak{g})$ be the associated universal enveloping SUSY vertex algebra[4], either the

[4]Here as before, we are considering a central extension of a SUSY Lie conformal algebra, and then we identify the central element with a multiple of the vacuum vector in the universal enveloping SUSY vertex algebra.

$N_K = N$ or the $N_W = N$ vertex algebra, the choice will be clear in each context, as well as the value of N.

When $N = 1$ the corresponding $N_K = 1$ SUSY vertex algebra is strongly conformal, the corresponding conformal vector is

$$\tau = \frac{1}{k+h^\vee}\left(\sum(-1)^{a^i} : (S\bar{a}^i)\bar{b}^i : + \frac{1}{3(k+h^\vee)}\sum([a^i, a^j], a^r) : \bar{b}^i : \bar{b}^j\bar{b}^r ::\right), \tag{3.6.9.3}$$

where $\{a^i\}$ and $\{b^i\}$ are dual bases for $\mathfrak{g}$ with respect to $(\, ,\,)$. This is known as the *Kac-Todorov* construction [23]. This super Virasoro field has central charge

$$c = \frac{k\mathrm{sdim}\mathfrak{g}}{k+h^\vee} + \frac{\mathrm{sdim}\mathfrak{g}}{2},$$

and the fields $\bar{a} \in \bar{\mathfrak{g}}$ have conformal weight $1/2$.

Example 3.6.10. ($N = 2$ vertex algebra) As a vertex algebra it is generated by 4 fields (cf. example 2.1.19). As we have seen in example 3.6.5, this is a $N_K = 2$ SUSY vertex algebra generated by one field G. On the other hand, the $N = 2$ vertex algebra admits an embedding of the $N = 1$ vertex algebra. Therefore we can view the $N = 2$ vertex algebra as an $N_K = 1$ SUSY vertex algebra. As such, this algebra is generated by two superfields G and J, where G is a super Virasoro field of central charge c and J is even, primary of conformal weight 1. The remaining Λ bracket is given by:

$$[J_\Lambda J] = -G - \frac{c}{3}\lambda\chi. \tag{3.6.10.1}$$

Example 3.6.11. ($N = 4$ vertex algebra) As a vertex algebra it is generated by 8 fields: a Virasoro field, three currents (for the Lie algebra $\mathfrak{sl}_2$) and four fermions [22]. This vertex algebra admits and embedding of the Neveu Schwarz vertex algebra, therefore we can consider a $N_K = 1$ SUSY vertex algebra structure on it. As a $N_K = 1$ vertex algebra, it is of rank $3|1$, generated by an $N = 1$ conformal vector G with central charge c and three even vectors J^i, $i = 1, 2, 3$. Each pair (G, J^i) generates an $N = 2$ vertex algebra, viewed as an $N_K = 1$ SUSY vertex algebra, as in the previous example. The remaining commutation relations are:

$$[J^i{}_\Lambda J^j] = \varepsilon^{ijk}(S + 2\chi)J^k, \qquad i \neq j, \tag{3.6.11.1}$$

where ε is the totally antisymmetric tensor.

Example 3.6.12. ($bc - \beta\gamma$ system) This is a $N_K = 1$ SUSY vertex algebra generated by n even fields B^i and n odd fields Ψ^i. The only non-vanishing Λ-brackets (up to skew-symmetry) are:

$$[B^i{}_\Lambda \Psi^j] = \delta_{ij}. \tag{3.6.12.1}$$

This SUSY vertex algebra is strongly conformal with super Virasoro field

$$G = \sum_{i=1}^n \left(: (SB^i)(S\Psi^i) : + : (TB^i)\Psi^i :\right), \tag{3.6.12.2}$$

and central charge $3n$. The fields B^i (resp. Ψ^i) are primary of conformal weight 0 (resp. $1/2$).

Let σ^s_{ij}, $s = 1, 2, 3$, be three $n \times n$ matrices satisfying

$$\sigma^i \sigma^j = \varepsilon^{ijk} \sigma^k \qquad (\sigma^s)^2 = -\operatorname{Id} \tag{3.6.12.3}$$

The fields

$$J^i = \sum_{j,k=1}^{n} \sigma^i_{jk} : SB^j \Psi^k :, \quad i = 1,\, 2,\, 3, \tag{3.6.12.4}$$

together with G generate an $N = 4$ vertex algebra as in the previous example (cf. [6]).

Example 3.6.13. Here we explain briefly the construction of the chiral de Rham complex of a smooth manifold introduced in [27], using the formalism of $N_K = 1$ SUSY vertex algebras [6]. Let U be a differentiable manifold. Let $\mathscr{T}$ be the tangent bundle of U and $\mathscr{T}^*$ be its cotangent bundle. We let $T = \Gamma(U, \mathscr{T})$ be the space of vector fields on U and $A = \Gamma(U, \mathscr{T}^*)$ be the space of differentiable 1-forms on U. We let $\mathscr{C} = \mathscr{C}^\infty(U)$ be the space of differentiable functions on U. Denote by

$$<,>: A \otimes T \to \mathscr{C} \tag{3.6.13.1}$$

the natural pairing. Finally, we denote by Π the functor of *change of parity.*

Consider now an $N_K = 1$ SUSY Lie conformal algebra $\mathscr{R}$ generated by the vector superspace

$$\mathscr{C} \oplus \Pi T \oplus A \oplus \Pi A. \tag{3.6.13.2}$$

That is, we consider differentiable functions (to be denoted $f, g, \dots$) as even elements, vector fields $X, Y, \dots$ as odd elements, and finally we have two copies of the space of differential forms. For differential forms, which we consider to be even elements, $\alpha, \beta, \dots \in A$, we will denote the corresponding elements of ΠA by $\bar{\alpha}, \bar{\beta}, \dots$. The nonvanishing Λ-brackets in $\mathscr{R}$ are given by (up to skew-symmetry):

$$\begin{aligned}
&[X_\Lambda f] = X(f)\\
&[X_\Lambda Y] = [X, Y]_{\text{Lie}}\\
&[X_\Lambda \alpha] = \operatorname{Lie}_X \alpha + \lambda < \alpha, X >\\
&[X_\Lambda \bar{\alpha}] = \overline{\operatorname{Lie}_X \alpha} + \chi < \alpha, X >
\end{aligned} \tag{3.6.13.3}$$

where $[,]_{\text{Lie}}$ is the Lie bracket of vector fields and Lie_X is the action of X on the space of differential forms by the Lie derivative. The fact that (3.6.13.3) satisfies the Jacobi identity is a (long but) straightforward computation.

We let $V(U)$ be the corresponding universal enveloping $N_K = 1$ SUSY vertex algebra of $\mathscr{R}$. This algebra is too big. We want to impose some relations in $V(U)$. We let 1_U denote the constant function 1 in U. Let $d : \mathscr{C} \to A$ be the de Rham differential. Define $I(U) \subset V'(U)$ to be the ideal generated by elements of the form:

$$:fg: -(fg), \quad :fX: -(fX), \quad :f\alpha: -(f\alpha), \quad :f\bar{\alpha}: -(\overline{f\alpha}), \tag{3.6.13.4}$$
$$1_U - |0>, \quad Tf - df, \quad Sf - \overline{df} \tag{3.6.13.5}$$

Finally we define the $N_K = 1$ SUSY vertex algebra

$$\Omega^{\text{ch}}(U) := V(U)/I(U). \tag{3.6.13.6}$$

The following theorem is a reformulation of the corresponding result in [26]:

Theorem 3.6.14.

1. *Let $M \subset \mathbb{R}^n$ be an open submanifold. The assignment $U \mapsto \Omega^{\text{ch}}(U)$ defines a sheaf of SUSY vertex algebras Ω^{ch}_M on M.*
2. *For any diffeomorphism of open sets $M' \xrightarrow{\varphi} M$ we obtain a canonical isomorphism of SUSY vertex algebras $\Omega^{\text{ch}}(M) \xrightarrow{\Omega^{\text{ch}}(\varphi)} \Omega^{\text{ch}}(M')$. Moreover, given diffeomorphisms $M'' \xrightarrow{\varphi'} M' \xrightarrow{\varphi} M$, we have $\Omega^{\text{ch}}(\varphi \circ \varphi') = \Omega^{\text{ch}}(\varphi') \circ \Omega^{\text{ch}}(\varphi)$.*

This theorem allows us to construct a sheaf of SUSY vertex algebras in the Grothendieck topology on $\mathbb{R}^n$ (generated by open embeddings). This in turn lets us attach to any smooth manifold M, a sheaf of SUSY vertex algebras Ω^{ch}_M. We call this sheaf the *chiral de Rham complex of M*.

Example 3.6.15. (Free Fields) We can generalize Examples 3.6.8, 3.6.12, and 3.6.4 as follows. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a vector superspace, and let $(,)$ be a non-degenerate bilinear form in A. Recall that the bilinear form $(,)$ is said to be of parity $p \in \mathbb{Z}/2\mathbb{Z}$ if $(a, b) = 0$ unless $p(a) + p(b) = p$, and it is supersymmetric (resp. skew-supersymmetric) if $(a, b) = (-1)^{ab}(b, a)$ (resp $(a, b) = -(-1)^{ab}(b, a)$).

Let $\mathscr{H} = \mathbb{C}[T, S^i]$ in the $N_W = N$ case, and let $\mathscr{H}$ be defined as in 3.5.3 in the $N_K = N$ case. Let

$$\mathscr{R} = \mathscr{H} \otimes A \oplus \mathbb{C}C,$$

where C is an even element such that $TC = S^iC = 0$. Given a non-zero homogeneous polynomial $Q(\Lambda)$ of degree s (in PBW basis) and parity p, define the following Λ-bracket on $A \oplus \mathbb{C}C$:

$$[a_\Lambda b] = Q(\Lambda)(a, b)C, \quad a, b \in A, \text{ and } C \text{ central}, \tag{3.6.15.1}$$

and extend it to $\mathscr{R}$ by sesquilinearity. Then the Jacobi identity automatically holds since all triple brackets are zero. Skewsymmetry holds if and only if

$$(a, b) = -(-1)^{ab}(-1)^{N+s}(b, a). \tag{3.6.15.2}$$

Thus, (3.6.15.1) defines a structure of a SUSY Lie conformal algebra, provided that (3.6.15.2) holds together with the following parity condition:

$$p + p((,)) = N \mod 2. \tag{3.6.15.3}$$

Thus, $\mathscr{R}$ is a SUSY Lie conformal algebra if and only if $N+s$ is even (resp. odd) and the bilinear form $(\,,\,)$ is supersymmetric (resp. skew-supersymmetric) of parity $(N-p) \mod 2$.

The corresponding *free field* SUSY vertex algebra $F(A,Q)$ is the quotient of the universal enveloping vertex algebra $V(\mathscr{R})$ by the ideal $(C-|0\rangle)_{(-1|N)}V(\mathscr{R})$.

Example 3.6.16. (Spin-7) In [33], Shatashvili and Vafa constructed a vertex algebra associated to any manifold with Spin-7 holonomy. This algebra comes equipped with an $N=1$ superconformal vector, therefore we can view it as an $N_K=1$ SUSY vertex algebra. As such, it is generated by a super Virasoro field G of central charge $1/2$, and a field X of conformal weight 2. The corresponding Λ-brackets are:

$$\begin{aligned}
[G_\Lambda X] &= (2T+\chi S+4\lambda)\,X+\frac{\chi\lambda}{2}G+\frac{2}{3}\lambda^3,\\
[X_\Lambda X] &= \left(\frac{5}{2}TSX+\frac{5}{4}T^2G+6:GX:\right)+\\
&\quad+8\,(\chi T+\lambda S+2\lambda\chi)\,X+\frac{15}{4}\lambda(T+\lambda)G+\frac{8}{3}\lambda^3\chi
\end{aligned}\tag{3.6.16.1}$$

Note that this Λ bracket is quadratic in the generating fields. This in turns is due to the fact that this SUSY vertex algebra is not the universal enveloping SUSY vertex algebra of a SUSY Lie conformal algebra.

Expanding these superfields as:

$$G(Z)=G(z)+2\theta T(z),\qquad X(Z)=\tilde{X}(z)+\theta\tilde{M}(z)\tag{3.6.16.2}$$

we obtain the generating fields as in [33].

Chapter 4

The associated vector bundles

4.1 The groups $\operatorname{Aut}\mathscr{O}$

4.1.1. We start this section by describing the groups of changes of coordinates in the formal superdisk $D^{1|N}$. We analize in detail their corresponding Lie superalgebras in the cases $N=1$ and $N=2$. We then define principal bundles for these groups over any smooth supercurve.

In this section, we let Λ be a Grassman algebra over $\mathbb{C}$. We will work in the category of superschemes over Λ unless explicitly stated. When we work with a supergroup G, we will be interested in its Λ-points.

4.1.2. Let SSch/k be the category of superschemes over a field k and let Set be the category of sets. Fix a non negative integer N and a separated superscheme X of finite type over k (cf. 2.2.2). Let $D^{(m)}$ be as in 2.2.10 and $D^{1|N}$ be the formal superdisk. Define a family of contravariant functors $F_m : \mathrm{SSch}/k \to \mathrm{Set}$

$$F_m(Y) = \operatorname{Hom}_k(Y \times_k D^{(m)}, X) \tag{4.1.2.1}$$

Then these functors are representable by superschemes X_m over k.

Proof. The statement is local in Y therefore we can reduce to the case when $Y = \operatorname{Spec} A$, where A is a local k-superalgebra. Now the data of a morphism $Y \times_k D^{(m)} \to X$ is local in X so we may replace X by $\operatorname{Spec} R$. We may suppose then that $X \subset \mathbb{A}^{r|s}$ and $R = k[x_1, \ldots, x_{r+s}]/(f_1, \ldots, f_{u+w})$, where the first u polynomials f_i are even and the last w polynomials are odd; similarly the first r coordinates x_i are even and the last s are odd. Now an element of $\operatorname{Hom}(\operatorname{Spec} A \times D^{(m)}, X)$ is given by a morphism $\gamma : k[x_1, \ldots, x_{r+s}] \to A[t, \theta^1, \ldots, \theta^N]/\mathfrak{m}^{m+1}$ such that $\gamma(f_i) = 0$. These conditions are equivalent to the vanishing of each of the coefficients of $t^i\theta^I$ in $\gamma(f_j)$ where θ^I denotes as usually the monomial $\theta^{i_1} \ldots \theta^{i_j}$ for $I = (i_1, \ldots, i_j) \subset \{1, \ldots, N\}$. These give

$$(u+w) \sum_{j=0}^{m} \sum_{k=0}^{m-j} \binom{N}{k} \tag{4.1.2.2}$$

equations. Therefore the map $\gamma \to (\gamma_{i,j,I})$ where

$$\gamma(x_i) = \sum_{|I|+j\le m} \gamma_{i,j,I} t^j \theta^I \tag{4.1.2.3}$$

gives a closed immersion $X_m \hookrightarrow \mathbb{A}^{\alpha|\beta}$ where α (resp. β) is the number of even (resp. odd) monomials in $\{\gamma(x_i)\}$. $\square$

Note in particular that $X_0 = X$, and when $N = 1$ we see that X_1 is the total tangent space of X.

The embeddings $D^{(m)} \hookrightarrow D^{(m+1)}$ induce projections $X_{m+1} \to X_m$ and we define the *Jet superscheme* of X as

$$JX = \varprojlim_{m\to\infty} X_m \tag{4.1.2.4}$$

4.1.3. Let us analyze first the case $N = 1$. Consider the group of continuous (even) automorphisms of the topological commutative superalgebra $\Lambda[[Z]]$, where $Z = (z, \theta)$ are topological generators. Such an automorphism is given by a pair of power series

$$\begin{aligned} z &\mapsto a_{1,0}z + a_{0,1}\theta + a_{1,1}z\theta + \dots \\ \theta &\mapsto b_{1,0}z + b_{0,1}\theta + b_{1,1}z\theta + \dots, \end{aligned} \tag{4.1.3.1}$$

where the matrix $\left(\begin{smallmatrix} a_{1,0} & a_{0,1} \\ b_{1,0} & b_{0,1}\end{smallmatrix}\right)$ is in $GL(1|1)^1$. Denote this supergroup by $\operatorname{Aut}\mathscr{O}^{1|1}$. In what follows we will analyze its $\mathbb{C}$-points.

This supergroup is a semidirect product of $GL(1|1)$ and a pro-unipotent super group, namely, the subgroup $\operatorname{Aut}_+\mathscr{O}^{1|1}$ of automorphisms where $\left(\begin{smallmatrix} a_{1,0} & a_{0,1} \\ b_{1,0} & b_{0,1}\end{smallmatrix}\right) = \mathrm{Id}$. In fact,

$$\operatorname{Aut}_+\mathscr{O}^{1|1} = \varprojlim_{n\to\infty} \operatorname{Spec}\mathbb{C}[a_{1,1}, b_{1,1}, a_{2,0}, b_{2,0}, \dots, a_{n,1}, b_{n,1}]. \tag{4.1.3.2}$$

Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{C}[Z]$ generated by (z, θ). We have

$$\operatorname{Aut}_+\mathscr{O}^{1|1} = \varprojlim_{n\to\infty} \operatorname{Aut}(\mathbb{C}[Z]/\mathfrak{m}^n). \tag{4.1.3.3}$$

Similarly for its Lie superalgebra $\operatorname{Der}_+\mathscr{O}^{1|1}$, we have

$$\operatorname{Der}_+\mathscr{O}^{1|1} = \varprojlim_{n\to\infty} \operatorname{Der}(\mathbb{C}[Z]/\mathfrak{m}^n), \tag{4.1.3.4}$$

where for each $\mathbb{C}$-superalgebra R, we denote $\operatorname{Der}(R)$ the Lie superalgebra of derivations of R. The exponential map is an isomorphism at each step, giving an isomorphism $\exp : \operatorname{Der}_+\mathscr{O}^{1|1} \to \operatorname{Aut}_+\mathscr{O}^{1|1}$.

The linearly compact Lie superalgebra $\operatorname{Der}_0\mathscr{O}^{1|1} = \operatorname{Lie}(\operatorname{Aut}\mathscr{O}^{1|1})$ has the following

[1]Here and further, $GL(p|q)$ is the group of even automorphisms of a $p|q$ dimensional module over Λ.

topological basis:

$$z^n\partial_z \quad (n \geq 1) \qquad\qquad z^n\partial_\theta \quad (n \geq 1) \tag{4.1.3.5}$$
$$z^n\theta\partial_z \quad (n \geq 0) \qquad\qquad z^n\theta\partial_\theta(\quad n \geq 0), \tag{4.1.3.6}$$

or the following one ($n \geq 0$):

$$T_n = -z^{n+1}\partial_z - (n+1)z^n\theta\partial_\theta \qquad\qquad J_n = -z^n\theta\partial_\theta \tag{4.1.3.7a}$$
$$Q_n = -z^{n+1}\partial_\theta \qquad\qquad H_n = z^n\theta\partial_z \tag{4.1.3.7b}$$

It is straightforward to check that these elements satisfy the commutation relations of the $N = 2$ algebra (2.1.19.5) (see also Example 3.6.1) for $n \geq 0$. In particular, we see that $\mathrm{Der}_0\mathcal{O}^{1|1}$ is the formal completion of the Lie algebra $W(1|1)_<$ (cf. Example 3.6.1 for its definition). The Lie subalgebra $\mathrm{Der}_+\mathcal{O}$ is topologicaly generated by the same vectors with $n \geq 1$.

4.1.4. We now turn our attention to the superconformal $N = 1$ case. Consider the differential form $\omega = dz + \theta d\theta$ on the formal superdisk $D^{1|1}$, and the supergroup $\mathrm{Aut}^\omega\mathcal{O}^{1|1}$ of automorphisms of $D^{1|1}$ preserving this form, up to multiplication by a scalar function. This is a subgroup of $\mathrm{Aut}\mathcal{O}^{1|1}$ whose Lie superalgebra $\mathrm{Der}_0^\omega\mathcal{O}^{1|1}$ consist of derivations X in $\mathrm{Der}_0\mathcal{O}^{1|1}$ such that $L_X\omega = f\omega$ for some formal power series f (here L_X denotes the Lie derivative). More explicitly, the linearly compact Lie superalgebra $\mathrm{Der}_0^\omega\mathcal{O}^{1|1}$ is topologically generated by

$$\begin{aligned} L_n &= -\frac{n+1}{2}z^n\theta\partial_\theta - z^{n+1}\partial_z, \qquad n \in \mathbb{Z}_+ \\ G_n &= -z^{n+1/2}(\partial_\theta - \theta\partial_z), \qquad n \in \frac{1}{2} + \mathbb{Z}_+. \end{aligned} \tag{4.1.4.1}$$

We check easily that these generators satisfy the commutation relation of the Neveu-Schwarz algebra as defined in (2.1.11.3) (see also Example 3.6.5). In particular, we see that $\mathrm{Der}_0^\omega\mathcal{O}^{1|1}$ is the formal completion of the Lie superalgebra $K(1|1)_<$.

An automorphism of the formal superdisk is determined by two power series $F(Z), \Psi(Z)$ which are the images of the generators $Z = (z, \theta)$. Under this transformation we have (recall ∂_θ is an odd derivation)

$$\begin{aligned} dz + \theta d\theta &\mapsto \partial_z F dz - \partial_\theta F d\theta + \Psi(\partial_z\Psi dz + \partial_\theta\Psi d\theta) \\ &= (\partial_z F + \Psi\partial_z\Psi)dz - (\partial_\theta F - \Psi\partial_\theta\Psi)d\theta, \end{aligned} \tag{4.1.4.2}$$

therefore we get that in order for ω to be preserved up to multiplication by a function, we need

$$(\partial_\theta F - \Psi\partial_\theta\Psi) = -\theta(\partial_z F + \Psi\partial_z\Psi), \tag{4.1.4.3}$$

and this is equivalent to (2.2.12.7).

4.1.5. Finally we turn our attention to the (oriented) superconformal $N = 2$ case. For this we consider the differential form $\omega = dz + \theta^1 d\theta^1 + \theta^2 d\theta^2$ on the formal superdisk $D^{1|2}$. We want to analyze the group of automorphisms of $D^{1|2}$ preserving this form in the sense of the previous paragraph 4.1.4. Such an automorphism is determined by an even power series $F(Z)$ and two odd power series $\Psi^1(Z)$ and $\Psi^2(Z)$, where $Z = (z, \theta^1, \theta^2)$ are the coordinates on $D^{1|2}$. Under such a change of coordinates, the differential form ω changes to:

$$\begin{aligned}\partial_z F dz - \partial_{\theta^1} F d\theta^1 - \partial_{\theta^2} F d\theta^2 + \Psi^1 \left(\partial_z \Psi^1 dz + \partial_{\theta^1}\Psi^1 d\theta^1 + \partial_{\theta^2}\Psi^1 d\theta^2\right) + \\ + \Psi^2 \left(\partial_z \Psi^2 dz + \partial_{\theta^1}\Psi^2 d\theta^1 + \partial_{\theta^2}\Psi^2 d\theta^2\right) = \\ = \left(\partial_z F + \Psi^1 \partial_z \Psi^1 + \Psi^2 \partial_z \Psi^2\right) dz + \left(-\partial_{\theta^1} F + \Psi^1 \partial_{\theta^1}\Psi^1 + \Psi^2 \partial_{\theta^1}\Psi^2\right) d\theta^1 + \\ + \left(-\partial_{\theta^2} F + \Psi^1 \partial_{\theta^2}\Psi^1 + \Psi^2 \partial_{\theta^2}\Psi^2\right) d\theta^2. \quad (4.1.5.1)\end{aligned}$$

Collecting terms, imposing that the form ω is preserved up to multiplication by a function, and defining the differential operators $D^i = \partial_{\theta^i} + \theta^i \partial_z$ we obtain that the automorphisms we are considering satisfy the equations:

$$D^i F = \Psi^1 D^i \Psi^1 + \Psi^2 D^i \Psi^2, \qquad i = 1, 2. \qquad (4.1.5.2)$$

Note also that a particular case of (4.1.5.1) when $F = z - \frac{1}{2}\theta^1\theta^2$, $\Psi^1 = \frac{i}{2}(\theta^2 - \theta^1)$ and $\Psi^2 = \frac{1}{2}(\theta^1 + \theta^2)$ transforms the form

$$\omega \mapsto dz + \theta^2 d\theta^1 = dz - d\theta^1 \theta^2, \qquad (4.1.5.3)$$

hence the supergroup of automorphisms of $D^{1|2}$ preserving the latter form is exactly the supergroup of changes of coordinates preserving an $N = 2$ superconformal structure as in 2.2.12.

The linearly compact Lie superalgebra $\mathrm{Der}_0^\omega \mathscr{O}^{1|2} = \mathrm{Lie}(\mathrm{Aut}^\omega \mathscr{O}^{1|2})$ is topologicaly generated by:

$$\begin{aligned} L_n &= -z^{n+1}\partial_z - \frac{n+1}{2} z^n \left(\theta^1 \partial_{\theta^1} + \theta^2 \partial_{\theta^2}\right), \quad n \in \mathbb{Z}_+ \\ G^{(2)}_n &= +z^{n+1/2}\left(\theta^2 \partial_z - \partial_{\theta^2}\right) - \left(n + \frac{1}{2}\right) z^{n-1/2} \theta^1\theta^2 \partial_{\theta^1}, \quad n \in \frac{1}{2} + \mathbb{Z}_+ \\ G^{(1)}_n &= +z^{n+1/2}\left(\theta^1 \partial_z - \partial_{\theta^1}\right) + \left(n + \frac{1}{2}\right) z^{n-1/2} \theta^1\theta^2 \partial_{\theta^2}, \quad n \in \frac{1}{2} + \mathbb{Z}_+ \\ J_n &= -iz^n \left(\theta^2 \partial_{\theta^1} - \theta^1 \partial_{\theta^2}\right) \quad n \in \mathbb{Z}_+. \end{aligned} \qquad (4.1.5.4)$$

We easily check that these operators satisfy the commutation relations of the $N = 2$ generators as in (2.1.22.6) (see also Example 3.6.5) for $n \geq 0$. We see that the Lie superalgebra $\mathrm{Der}_0^\omega \mathscr{O}^{1|2}$ is the formal completion of the Lie superalgebra $K(1|2)_<$.

It is useful to consider *complex* coordinates $\theta^\pm = \theta^1 \pm i\theta^2$, and derivations $D^\pm = \frac{1}{2}(D^1 \pm iD^2)$. In these coordinates (z, θ^+, θ^-), these derivations are expressed as:

$$D^\pm = \partial_{\theta^\mp} + \frac{1}{2}\theta^\pm \partial_z. \qquad (4.1.5.5)$$

If we change coordinates by $\rho = (F, \Psi^+, \Psi^-)$, with $\Psi^\pm = \Psi^1 \pm i\Psi^2$, the superconformal condition (4.1.5.2) reads

$$D^{\pm}F = \frac{1}{2}\Psi^{+}D^{\pm}\Psi^{-} + \frac{1}{2}\Psi^{-}D^{\pm}\Psi^{+}. \tag{4.1.5.6}$$

With this we can easily see that under a change of coordinates $(z_\alpha, \theta^{\pm}_\alpha) \mapsto (z_\beta, \theta^{\pm}_\beta)$, the operators $D^{\pm}$ transform as

$$D^{\pm}_\alpha = (D^{\pm}_\alpha \Psi^{-}_{\beta,\alpha})D^{+}_\beta + (D^{\pm}_\alpha \Psi^{+}_{\beta,\alpha})D^{-}_\beta. \tag{4.1.5.7}$$

In the following sections, we will consider only *oriented* superconformal $N = 2$ supercurves (cf. remark 2.2.13), namely those for which there exists a coordinate atlas $(U_\alpha, z_\alpha, \theta^{\pm}_\alpha)$ such that on overlaps we have [11]:

$$D^{\pm}_\alpha \Psi^{\pm}_{\beta,\alpha} = 0. \tag{4.1.5.8}$$

In these coordinates, the topological generators of the Lie superalgebra $\mathrm{Der}^{\omega}_0 \mathscr{O}^{1|2}$ are expressed as:

$$\begin{aligned}
L_n &= -z^{n+1}\partial_z - \frac{n+1}{2}z^n(\theta^+\partial_{\theta^+} + \theta^-\partial_{\theta^-}), \quad n \in \mathbb{Z}_+ \\
J_n &= -z^n(\theta^+\partial_{\theta^+} - \theta^-\partial_{\theta^-}), \quad n \in \mathbb{Z}_+ \\
G^{\pm}_n &= -z^{n+1/2}\left(\partial_{\theta^{\pm}} - \frac{1}{2}\theta^{\mp}\partial_z\right) - \frac{n+1/2}{2}z^{n-1/2}\theta^{\pm}\theta^{\mp}\partial_{\theta^{\pm}}, \quad n \in \frac{1}{2} + \mathbb{Z}_+
\end{aligned} \tag{4.1.5.9}$$

where as before we have $G^{\pm} = \frac{1}{2}(G^{(1)} \mp iG^{(2)})$.

Recall from 2.2.12 that an oriented superconformal $N = 2$ supercurve $(Y, \mathscr{O}_Y)$ projects onto two $N = 1$ supercurves X and its dual $\hat{X}$. Defining new coordinates on (u, θ^+, θ^-), where $u = z - \frac{1}{2}\theta^+\theta^-$, we see that equations (4.1.5.6), for a change of coordinates $\rho = (G = F + \frac{1}{2}\Psi^+\Psi^-, \Psi^+, \Psi^-)$ are expressed in these coordinates as:

$$\begin{aligned}
D^- G &= \Psi^- D^- \Psi^+ \\
D^+ G &= 0.
\end{aligned} \tag{4.1.5.10}$$

Moreover, the operators $D^{\pm}$ are expressed as

$$D^+ = \partial_{\theta^-} \qquad D^- = \partial_{\theta^+} + \theta^-\partial_u. \tag{4.1.5.11}$$

Note that the coordinate θ^- does not appear in the transition functions for u, θ^+, therefore these coordinates give the topological space $|Y|$ the structure of an $N = 1$ supercurve. Let us call this curve X. Similarly, if we define $u' = z - \frac{1}{2}\theta^+\theta^-$ we obtain that u', θ^- defines the dual curve $(\hat{X}, \mathscr{O}_{\hat{X}})$.

It follows from the above discussion, that given a change of coordinates $\rho = (G, \Psi^+) \in \mathrm{Aut}\mathscr{O}^{1|1}$, we obtain uniquely a change of coordinates $\rho = (G, \Psi^+, \Psi^-) \in \mathrm{Aut}^{\omega}\mathscr{O}^{1|2}$, where $\Psi^- = D^-G/D^-\Psi^+$. This map induces an isomorphism of supergroups from $\mathrm{Aut}\mathscr{O}^{1|1}$ to the identity component of $\mathrm{Aut}^{\omega}\mathscr{O}^{1|2}$. This isomorphism corresponds to the isomorphism of Lie superalgebras $K(1|2) \equiv W(1|1)$ (cf. [24]), and has a geometric counterpart (cf. [35]) relating the moduli space of (oriented) superconformal $N = 2$ supercurves and the moduli space of $N = 1$ supercurves.

Remark 4.1.6. Let X be a superconformal $N = n$ supercurve. Then for some coordinate atlas $Z_\alpha = (z_\alpha, \theta^1_\alpha, \dots, \theta^n_\alpha)$ the form $\omega = d_z + \sum_{i=1}^n \theta^i d\theta^i$ is globally defined up to a scalar factor. Let ω be that form on the superdisk $D^{1|N}$ and on $D^{(m)}$ as well. Define the functors $F^\omega_m : \mathrm{SSch}/k \to \mathrm{Set}$ by

$$F_m(Y) = \mathrm{Hom}^\omega_k(Y \times_k D^{(m)}, X) \tag{4.1.6.1}$$

where Hom^ω denotes the set of morphisms preserving the form ω (up to multiplication by a function). Then it follows in the same way as in 4.1.2 that the functors F_m are representable by super-schemes X^ω_m. This allows us to define the superscheme

$$JX^\omega = \varprojlim_{m \to \infty} X^\omega_m, \tag{4.1.6.2}$$

parametrizing maps $D \to X$ preserving the superconformal structure.

4.1.7. Let X be an $N = n$ supercurve and let $x \in X$. If $Z = (z, \theta^1, \dots, \theta^n)$ is a local coordinate at x and $\mathscr{O}_x$ denotes the completion of the local ring at x, we have an isomorphism

$$\mathscr{O}_x \equiv \mathbb{C}[[Z]], \tag{4.1.7.1}$$

where we should replace $\mathbb{C}$ by Λ if X is defined over Λ. For the purposes of this section it is enough to consider curves over $\mathbb{C}$, the relative case follows easily. Let Aut_x denote the set of local coordinates $Z = (z, \theta^i)$ at x. In the algebraic setting we mean by coordinates an étale map $Z : X \to \mathbb{A}^{1|n}$. The set Aut_x is a torsor for the group $\mathrm{Aut}\,\mathscr{O}^{1|n}$. The torsors Aut_x glue to form an $\mathrm{Aut}\,\mathscr{O}^{1|n}$-torsor Aut_X. Indeed Aut_X consists of pairs (x, Z) where x is a point in X and $Z = (z, \theta^i)$ is a local coordinate at x. The action of $\mathrm{Aut}\,\mathscr{O}^{1|n}$ on the fibers is by change of coordinates. The torsor Aut_X may be described as an open subscheme of JX consisting of jets of maps $D \to X$ such that their 1-jet is in $GL(1|n)$. Since we can cover X by Zariski open subschemes U_α and étale maps $f_\alpha : U_\alpha \to \mathbb{A}^{1|n}$ we see that the $\mathrm{Aut}\,\mathscr{O}^{1|n}$-torsor Aut_X is locally trivial in the Zariski topology (cf. [16, §5.4.2]).

4.1.8. Similarly, let X be a (oriented) superconformal $N = n$ supercurve and $x \in X$. Let Aut^ω_x be the set of SUSY coordinates Z at x (that is, compatible with the superconformal structure). It follows that this set is an $\mathrm{Aut}^\omega \mathscr{O}^{1|n}$-torsor. Moreover these torsors glue to form an $\mathrm{Aut}^\omega \mathscr{O}^{1|n}$-torsor $\mathrm{Aut}^\omega_X \to X$. As in the previous paragraph, Aut^ω_X is an open sub superscheme of JX^ω (cf. 4.1.6) consisting of jets of maps $D \to X$ compatible with the superconformal structure and with invertible 1-jet.

Remark 4.1.9. Let V a finite rank $\mathrm{Aut}\,\mathscr{O}$-module (resp. a finite rank $\mathrm{Aut}^\omega \mathscr{O}$-module), and let X be an $N = n$ supercurve (resp. a superconformal $N = n$ supercurve). We define a vector bundle on X by

$$\mathscr{V}_X = \mathrm{Aut}_X \overset{\mathrm{Aut}\,\mathscr{O}}{\times} V \qquad (\text{resp. } \mathrm{Aut}^\omega_X \overset{\mathrm{Aut}^\omega \mathscr{O}}{\times} V), \tag{4.1.9.1}$$

consisting of pairs $(\tilde{x}, v)$ with $\tilde{x}$ in Aut_X (resp. Aut^{ω}_X) and $v \in V$ with the identification $(\tilde{x} \cdot g, v) \sim (\tilde{x}, g \cdot v)$ for $g \in \mathrm{Aut}\mathscr{O}$ (resp. $g \in \mathrm{Aut}^{\omega}\mathscr{O}$). We call $\mathscr{V}_X$ the Aut_X (resp. Aut^{ω}_X) *twist of* V.

4.2 Vector bundles, sections and connections

4.2.1. In this section we construct vector bundles on supercurves associated with SUSY vertex algebras following [16]. Briefly, the idea is the same as in [16], namely, a strongly conformal $N_W = n$ algebra is a module for the Harish-Chandra pair $(\mathrm{Der}_0\mathscr{O}^{1|n}, \mathrm{Aut}\mathscr{O}^{1|n})$, therefore we can apply the Beilinson-Bernstein localization construction [4] to get a vector bundle with a flat connection over any $N = n$ supercurve. Similarly, an $N_K = n$ strongly conformal SUSY vertex algebra ($n \leq 4$), is a module for the Harish-Chandra pair $(\mathrm{Der}^{\omega}_0\mathscr{O}^{1|n}, \mathrm{Aut}^{\omega}\mathscr{O}^{1|n})^2$, therefore we can construct vector bundles with flat connections over any oriented superconformal $N = n$ supercurve.

As in [16], it turns out that the state-field correspondence in all these cases can be thought of as a section of the corresponding bundles. The corresponding *change of coordinates* formula (a generalization of Huang's formula [21]) is proved in this section.

4.2.2. Let V be a strongly conformal $N_W = n$ SUSY vertex algebra. Therefore we have $N + 1$ vectors ν and $\tau^1, \dots, \tau^N$ such that their Fourier modes $\nu_{(m,I)}$ and $\tau^j_{(m,I)}$ with $m \geq 0$ generate a Lie superalgebra isomorphic to $\mathrm{Der}\mathscr{O}^{1|n}$ (cf. example 3.6.1). The derivation ∂_z (corresponding to $\nu_{(0,0)}$) cannot be exponentiated to the group $\mathrm{Aut}\mathscr{O}^{1|n}$ and the Lie superalgebra spanned by $\nu_{(m,I)}$, and $\tau^j_{(m,I)}$ for $m \geq 1$ if $I \neq 0$ is isomorphic to $\mathrm{Der}_0\mathscr{O}^{1|n}$.

In order to exponentiate the representation V of $\mathrm{Der}_0\mathscr{O}^{1|n}$ to a representation of the group $\mathrm{Aut}\mathscr{O}^{1|n}$ we note as before that this Lie algebra is a semidirect product of $\mathfrak{gl}(1|n)$ with the pro-nilpotent Lie subalgebra $\mathrm{Der}_+\mathscr{O}^{1|n}$. Namely, the subalgebra spanned by $z\partial_z$, $\theta^i\partial_{\theta^j}$, $z\partial_{\theta^i}$ and $\theta^j\partial_z$ is isomorphic to $\mathfrak{gl}(1|n)$. It follows from the definition of strongly conformal SUSY vertex algebras in 3.6.2 and 3.6.6, that we can exponentiate this representation of $\mathfrak{gl}(1|n)$ (the fact that the nilpotent part of the Lie algebra exponentiates follows easily from the OPE formula and the locality axiom).

4.2.3. Let X be an $N = n$ supercurve over a Grassman algebra Λ, let $x \in X$ and $\mathscr{O}_x$ be the completion of the local super ring at x. Let $Z = (z, \theta^i)$ be local coordinates at x (recall that in the formal setting Z is an étale map $X \to \mathbb{A}^{1|n}$). With such a choice of coordinates we get an isomorphism $\mathscr{O}_x \equiv \Lambda[[Z]]$ and the set of coordinates at x, Aut_x is an $\mathrm{Aut}\mathscr{O}^{1|n}$-torsor. Let us work in the analytic setting first for the sake of simplicity as in [16]. Let D_x be a small disk around x. Let p be a Λ-point given in the local coordinates $Z = (z, \theta^i)$ by $Y = (y, \alpha^i)$. The coordinates Z induce coordinates $Z - Y = (z - y, \theta^i - \alpha^i)$ at p. Now let $\rho \in \mathrm{Aut}\mathscr{O}^{1|n}$ be a change of coordinates. Recall that this change of coordinates is given by power series $(F(Z), \Psi^i(Z))$, where $F(Z) \in \Lambda[[Z]]$ is even and $\Psi^i \in \Lambda[[Z]]$ are odd. This change of coordinates induce

[2]From now on, we will abuse notation and denote by $\mathrm{Aut}^{\omega}\mathscr{O}^{1|n}$ its identity component

new coordinates at p, given by

$$\rho(Z) - \rho(Y) = (F(Z) - F(Y), \Psi^i(Z) - \Psi^i(Y)). \tag{4.2.3.1}$$

The coordinates $Z - Y = (z - y, \theta^i - \alpha^i)$ and (4.2.3.1) at p are related by a change of coordinates $\rho_Y = (F_Y, \Psi^i_Y)$ satisfying

$$\rho_Y(Z - Y) = \rho(Z) - \rho(Y). \tag{4.2.3.2}$$

Therefore, letting $W = (w, \zeta^i) = Z - Y$, we get:

$$\rho_Y(W) = \rho(W + Y) - \rho(Y). \tag{4.2.3.3}$$

In the formal setting we can not consider a small disk, but given a point x and coordinates Z at x, we can still define $\rho_Z \in \operatorname{Aut}\mathscr{O}^{1|n}$ for any $\rho \in \operatorname{Aut}\mathscr{O}^{1|n}$ by formula (4.2.3.3) with Y replaced by Z.

Let V be a strongly conformal $N_W = n$ SUSY vertex algebra, so that V is an $\operatorname{Aut}\mathscr{O}^{1|n}$-module. We will call this representation R.

Theorem 4.2.4. *Let V be a strongly conformal $N_W = n$ SUSY vertex algebra, so that V is a $\operatorname{Aut}\mathscr{O}^{1|n}$ module. Let $\rho = (F, \Psi^j) \in \operatorname{Aut}\mathscr{O}^{1|n}$ and $a \in V$. The following* change of coordinates *formula is true:*

$$\overset{s}{Y}(a, Z) = R(\rho)\overset{s}{Y}\left(R(\rho_Z)^{-1}a, \rho(Z)\right) R(\rho)^{-1} \tag{4.2.4.1}$$

where by $\rho(Z)$ we understand the images of z, θ^j under ρ, namely $F(z, \theta^i)$, $\Psi^j(z, \theta^i)$.

Proof. The proof is similar to the analogous formula in the ordinary vertex algebra case. Namely, the state-field correspondence $\overset{s}{Y}(\cdot, Z)$ is an element in the vector space $\operatorname{Hom}(V, \mathscr{F}(V))$, where $\mathscr{F}(V)$ is the space of all $\operatorname{End}(V)$-valued superfields. For each $\rho \in \operatorname{Aut}\mathscr{O}^{1|n}$ consider the linear operator in $\operatorname{Hom}(V, \mathscr{F}(V))$ given by

$$(T_\rho X)(a, Z) = R(\rho)X(R(\rho_Z)^{-1}a, \rho(Z))R(\rho)^{-1}. \tag{4.2.4.2}$$

It is easy to check that $T_\rho X \in \operatorname{Hom}(V, \mathscr{F}(V))$. Moreover, this action defines a representation of $\operatorname{Aut}\mathscr{O}^{1|n}$ in $\operatorname{Hom}(V, \mathscr{F}(V))$. Recall that the group structure in $\operatorname{Aut}\mathscr{O}^{1|n}$ is given by composition, namely, if $\rho = (F, \Psi^j)$ and $\tau = (G, \Theta^j)$ then $\rho \star \tau$ is given by H, Σ^j where

$$\begin{aligned} H(z, \theta^j) &= G(F(z, \theta^j), \Psi^k(z, \theta^j), \\ \Sigma^i(z, \theta^j) &= \Theta^i(F(z, \theta^j), \Psi^k(z, \theta^j)). \end{aligned} \tag{4.2.4.3}$$

It follows that

$$\rho_Z \star \tau_{\rho(Z)} = (\rho \star \tau)_Z. \tag{4.2.4.4}$$

Indeed, the left hand side, when evaluated in W is given by

$$\tau_{\rho(Z)}\left(\rho(W + Z) - \rho(Z)\right) = \tau\left(\rho(W + Z) - \rho(Z) + \rho(Z)\right) - \tau(\rho(Z)), \tag{4.2.4.5}$$

which is the right hand side.

It follows from this formula that $\rho \mapsto T_\rho$ defines a representation of $\mathrm{Aut}\mathscr{O}^{1|n}$. In fact, we have

$$\begin{aligned}(T_{\rho\star\tau}X)(a,Z) &= R(\rho\star\tau)X(R((\rho\star\tau)_Z)^{-1}a, \tau(\rho(Z)))R(\rho\star\tau)^{-1} = \\ &= R(\rho)R(\tau)X(R(\rho_Z \star \tau_{\rho(Z)})^{-1}a, \tau(\rho(Z)))R(\tau)^{-1}R(\rho)^{-1} = \\ &= R(\rho)\left[R(\tau)X(R(\tau_{\rho(Z)})^{-1}R(\rho_Z)^{-1}a, \tau(\rho(Z)))R(\tau)^{-1}\right]R(\rho)^{-1} = \\ &= \left[T_\rho(T_\tau X)\right](a,Z). \quad (4.2.4.6)\end{aligned}$$

We have reduced the proof of the theorem to show that $\overset{s}{Y}(\cdot, Z)$ is fixed under this action. Since the exponential map $\exp : \mathrm{Der}_0\mathscr{O}^{1|n} \to \mathrm{Aut}\mathscr{O}^{1|n}$ is surjective, we need only to show that $\overset{s}{Y}(\cdot, Z)$ is stable under the induced infinitesimal action of $\mathrm{Der}_0\mathscr{O}^{1|n}$. For this we let $\rho = \exp(\varepsilon \mathbf{v})$ where $\mathbf{v} = v(Z)\partial_Z \in \mathrm{Der}_0\mathscr{O}^{1|n}$, $v(Z) = (f(Z), g^1(Z), \dots, g^n(Z))$ with $f(Z)$ an even function and $g^i(Z)$ odd functions of Z. As before $\partial_Z = (\partial_z, \partial_{\theta^1}, \dots, \partial_{\theta^n})$ and the product $v(Z)\partial_Z$ denotes the scalar product $f(Z)\partial_z + \sum_{i=1}^n g^i(Z)\partial_{\theta^i}$. We want to compute ρ_Z. For this we put $\rho_Z = \exp(\varepsilon\mathbf{u})$. Expanding $\rho_Z(W)$ in powers of ε, we get

$$\mathbf{u} = v(Z+W)\partial_W - v(Z)\partial_W = \left(e^{Z\partial_W}v(W)\right)\partial_W - v(Z)\partial_W. \qquad (4.2.4.7)$$

Noting that the operators corresponding to $\partial_W = (\partial_w, \partial_{\zeta^1}, \dots, \partial_{\zeta^n})$ are $-\nabla = (-T, -S^1, \dots, -S^n)$, we obtain

$$R(\mathbf{u}) = e^{-Z\nabla}R(\mathbf{v})e^{Z\nabla} + v(Z)\nabla. \qquad (4.2.4.8)$$

The action of T_ρ on $\overset{s}{Y}(a,Z)$ is given then by $\overset{s}{Y}(a,Z)$ plus the linear term in ε, which in turn is:

$$[R(\mathbf{v}), \overset{s}{Y}(a,Z)] - \overset{s}{Y}(R(\mathbf{u})a, Z) + v(Z)\nabla_Z\overset{s}{Y}(a,Z). \qquad (4.2.4.9)$$

The first term comes from the adjoint action of $R(\rho)$, the second term is the ε-linear term in $R(\rho_Z)^{-1}$, and the last term comes from the Taylor expansion of the change of coordinates. The result follows from (4.2.4.7), Proposition 3.3.22 and Theorem 3.3.9. □

4.2.5. Now we can define a vector bundle associated to an $N_W = n$ SUSY super vertex algebra over any $N = n$ supercurve. Moreover, we will define a canonical section of this bundle and a flat connection on it. First recall that from any finite dimensional $\mathrm{Aut}\mathscr{O}^{1|n}$-module we can construct a vector bundle over an $N = n$ supercurve X by twisting this $\mathrm{Aut}\mathscr{O}^{1|n}$-module by the $\mathrm{Aut}\mathscr{O}^{1|n}$-torsor Aut_X (see Remark 4.1.9). Given a strongly conformal $N_W = n$ SUSY vertex algebra V, we have a filtration $V_{\leq i}$ by finite dimensional submodules, namely $V_{\leq i}$ is the span of fields of conformal weight less or equal than i. By our assumptions these are finite dimensional $\mathrm{Aut}\mathscr{O}$-submodules of V. Let $\mathscr{V}_{\leq i}$ be the corresponding Aut_X twist. These vector bundles come equipped with embeddings $\mathscr{V}_{\leq i} \hookrightarrow \mathscr{V}_{\leq i+1}$. The limit of this directed system is a $\mathscr{O}_X$-module

$\mathscr{V}_X$[3]. That is

$$\mathscr{V}_X = \varinjlim_{i \to \infty} \mathscr{V}_{\leq i}. \tag{4.2.5.1}$$

This $\mathscr{O}_X$-module is quasi-coherent by definition.

On the other hand, the dual modules $V^*_{\leq i}$ come equipped with surjections $V^*_{\leq i+1} \twoheadrightarrow V^*_{\leq i}$ therefore we get a projective system of $\mathscr{O}_X$-modules $\mathscr{V}^*_{\leq i+1} \twoheadrightarrow \mathscr{V}^*_{\leq i}$. The inverse limit of this system is by definition $\mathscr{V}^*_X$, namely:

$$\mathscr{V}^*_X = \varprojlim_{i \to \infty} V^*_{\leq i}. \tag{4.2.5.2}$$

Thus, we have defined $\mathscr{O}_X$-modules associated with the SUSY vertex algebra V. We will call these modules the *SUSY vertex algebra bundle* and its dual. By construction, the fiber of the bundle $\mathscr{V}$ at a point $x \in X$ is isomorphic as a vector space, to V.

Similar constructions can be applied when X is replaced by a formal superdisk near a point $x \in X$. Namely, let D_x be such a formal superdisk, we have as before an $\operatorname{Aut}\mathscr{O}^{1|n}$-torsor Aut_{D_x} over D_x. Then $\mathscr{V}_{D_x}$ is the twist of V by this torsor. It is easy to see that in this case we get $\mathscr{V}_X|_{D_x} = \mathscr{V}_{D_x}$.

Let Aut_x be the torsor of coordinates at x as before. Then the fiber of $\mathscr{V}$ at x is given by

$$\mathscr{V}_x = \operatorname{Aut}_x \overset{\operatorname{Aut}\mathscr{O}}{\times} V. \tag{4.2.5.3}$$

Let $D^\times_x$ be the punctured disk at x, that is the formal completion

$$D^\times_x = \varinjlim_{i \to \infty} \operatorname{Spec}(\tilde{\mathscr{K}_x}/\mathfrak{m}^{i+1}), \tag{4.2.5.4}$$

where $\tilde{\mathscr{K}_x}$ is the ring of fractions of the local ring at x and $\mathfrak{m}$ is the maximal ideal defining x. If $Z = (z, \theta^i)$ are coordinates at x then this is isomorphic to the formal spectrum of $\Lambda((Z))$.

We will define an $\operatorname{End}\mathscr{V}_x$-valued section of $\mathscr{V}^*$ on $D^\times_x$. In order to define such a section it is enough to give its matrix coefficients, namely, for each $\varphi \in \mathscr{V}^*_x$, $v \in \mathscr{V}_x$ and s a section of $\mathscr{V}|_{D_x}$ we assign a function on $D^\times_x$, that is an element of $\mathscr{K}_x$, the fraction field of $\mathscr{O}_x$. This assignment is denoted by

$$\varphi, v, s \mapsto < \varphi, \mathscr{Y}_x(s) \cdot v >, \tag{4.2.5.5}$$

and should be linear in v and φ and $\mathscr{O}_x$ linear in s.

Let $Z = (z, \theta^i)$ be coordinates at x, we obtain a trivialization of $\mathscr{V}|_{D_x}$

$$i_Z : V[[Z]] \xrightarrow{\sim} \Gamma(D_x, \mathscr{V}). \tag{4.2.5.6}$$

This induces isomorphisms $V \xrightarrow{\sim} \mathscr{V}_x$ and $V^* \xrightarrow{\sim} \mathscr{V}^*_x$, where V^* is the *restricted dual* of V. Let $v \in V$ and $\varphi \in V^*$. Denote their images in $\mathscr{V}_x$ and $\mathscr{V}^*_x$, under these

[3]When there is no possible confusion, we will denote this bundle simply by $\mathscr{V}$.

isomorphisms by (Z,v) and (Z,φ) respectively. Let $s \in V[[Z]]$, its image under the isomorphism i_Z is a regular section of $\mathscr{V}$ in D_x. By $\mathscr{O}_x$ linearity, we may assume that $s = a \in V$. To this data, we assign the function

$$< (Z,\varphi), \mathscr{Y}_x(i_Z(a)) \cdot (Z,v) >=< \varphi, \overset{s}{Y}(a,Z)v > . \tag{4.2.5.7}$$

Theorem 4.2.6. *The assignment (4.2.5.7) is independent of the coordinates $Z = (z,\theta^1,\dots,\theta^n)$ chosen, i.e. $\mathscr{Y}_x$ is a well defined* $\mathrm{End}(\mathscr{V}_x)$*-valued section of $\mathscr{V}^*$ on $D_x^\times$.*

Proof. The proof follows the lines of the ordinary vertex algebra case in [16]. Let $W = (w,\zeta^i)$ be another set of coordinates at x. Then W and Z are related by $\rho \in \mathrm{Aut}\mathscr{O}$, $\rho(Z) = W$. Given these new coordinates, we construct another assignment by the same formula (4.2.5.7), namely

$$< (W,\varphi), \tilde{\mathscr{Y}}(i_W(a)) \cdot (W,v) >=< \varphi, Y(a,W)v > . \tag{4.2.6.1}$$

We need to show that this assignment coincides with $\mathscr{Y}$. By definition of the bundle $\mathscr{V}$ we have

$$(Z,v) = (\rho^{-1}(W),v) = (W, R(\rho)^{-1}v), \tag{4.2.6.2}$$

where $R(\cdot)$ is the representation of $\mathrm{Aut}\mathscr{O}^{1|n}$ in V. Similarly $(Z,\varphi) = (W,\varphi R(\rho))$. We need to find how does the section $i_Z(a)$ transform by this change of coordinates. Recall from 4.2.3 that in the analytic setting, if we trivialize $\mathscr{V}|_{D_x}$ with the coordinates Z, we can use the coordinates $(Z-Y) := (z-y, \theta^i - \alpha^i)$ at $Y = (y,\alpha^i)$ to identify $\mathscr{V}_y$ with V. We obtain:

$$(Z-Y,a) = (W-\rho(Y), R(\rho_Y)^{-1}a), \tag{4.2.6.3}$$

therefore the section $i_Z(a)$ is $i_W(R(\rho_Z)^{-1}a)$ in the W-trivialization.

In the formal setting, we can replace the coordinates by their n-jets, but these in turn can be extended by definition to a small Zariski open neighborhood of x, in this case the formula (4.2.6.3) is true as we have shown.

We have reduced then the problem to prove:

$$< \varphi, R(\rho)\overset{s}{Y}(R(\rho_Z)^{-1}a, W)R(\rho)^{-1}v >=< \varphi, Y(a,Z)v >, \tag{4.2.6.4}$$

thus, the theorem follows from Theorem 4.2.4 ∎

4.2.7. In the superconformal case, the situation is slightly more complicated. Roughly, the only changes that we have to make in the above prescription are the induced coordinates at a Λ-point and consequently the definition of ρ_Z.

Like in the $N_K = n$ SUSY vertex algebra situation, given two set of coordinates $Z = (z,\theta^1,\dots,\theta^n)$ and $W = (w,\zeta^1,\dots,\zeta^n)$ we will write

$$Z - W = (z - w - \sum_{i=1}^n \theta^i\zeta^i, \theta^1 - \zeta^1, \dots, \theta^n - \zeta^n).$$

Let V be a strongly conformal $N_K = n$ SUSY vertex algebra ($n \leq 4$), hence V is an $\mathrm{Aut}^{\omega}\mathscr{O}^{1|n}$-module. Moreover, V has a filtration by finite dimensional submodules $V_{\leq i}$ given by conformal weight as above. Let X be an oriented superconformal $N = n$ supercurve over Λ. We constructed an $\mathrm{Aut}^{\omega}\mathscr{O}$-torsor Aut_X over X (see 4.1.8). As above we can define the vertex algebra bundles $\mathscr{V}$ and $\mathscr{V}^*$. Similarly, we can define the $N_K = n$ SUSY vertex algebra bundles over the superconformal disks D_x^{ω}. The fibers $\mathscr{V}_x$ of these bundles are the Aut_x^{ω}-twists of V, where Aut_x^{ω} is the torsor of coordinates at x, compatible with the superconformal structure (see Remark 4.1.9). We define and $\mathrm{End}(\mathscr{V}_x)$-valued section $\mathscr{Y}$ of $\mathscr{V}^*$ on the punctured disk $D_x^{\times}$ by formula (4.2.5.7).

Theorem 4.2.8. *The assignment $\mathscr{Y}$ is independent of the coordinates $Z = (z, \theta^i)$ chosen as long as they are compatible with the superconformal structure on X.*

Proof. Let us first work in the analytic setting. If p is a Λ-point in D_x (now a small analytic disk near $x \in X$) given by local parameters $Y = (y, \alpha^i)$, then Z induces local coordinates at $T = (t, \eta^i) = Z - Y$ near p. The coordinates T are compatible with the superconformal structure. Indeed, we have

$$\begin{aligned} dt &= dz + \sum_{i=1}^{n} \alpha^i d\theta^i, \\ d\eta^i &= d\theta^i. \end{aligned} \tag{4.2.8.1}$$

Therefore

$$dt + \sum_{i=1}^{n} \eta^i d\eta^i = dz + \sum_{i=1}^{n} \alpha^i d\theta^i + (\theta^i + \alpha^i) d\theta^i = dz + \sum_{i=1}^{n} \theta^i d\theta^i. \tag{4.2.8.2}$$

If $W = (w, \zeta^i) = \rho(Z)$ is another set of coordinates compatible with the superconformal structure at x, for $\rho = (F, \Psi^i) \in \mathrm{Aut}^{\omega}\mathscr{O}^{1|n}$, then W induces another set of coordinates at p, namely

$$\rho(Z) - \rho(Y) = \left(F(z,\theta) - F(y,\alpha) - \sum_{i=1}^{n} \Psi^i(z,\theta)\Psi^i(y,\alpha), \Psi^j(z,\theta) - \Psi^j(y,\alpha) \right). \tag{4.2.8.3}$$

These are related with the coordinates T by a change of coordinates

$$\rho_Y = (F_Y, \Psi_Y^i) \in \mathrm{Aut}^{\omega}\mathscr{O}^{1|n}. \tag{4.2.8.4}$$

We have:

$$\rho_Y(T) = \rho(T + Y) - \rho(Y), \tag{4.2.8.5}$$

where, as in the $N_K = n$ SUSY vertex algebra case, we write $T + Y = T - (-Y)$.

The theorem will follow if we prove formula (4.2.4.1) for $\rho \in \mathrm{Aut}^{\omega}\mathscr{O}^{1|n}$. This is achieved as in the proof of theorem 4.2.4 by first showing that the action T_ρ (cf. proof of theorem 4.2.4) is a representation of $\mathrm{Aut}^{\omega}\mathscr{O}^{1|n}$ in $\mathrm{Hom}(V, \mathscr{F}(V))$. For this we first note that $(\rho \star \tau)_Z = \rho_Z \star \tau_{\rho(Z)}$ in exactly the same way as in the $N_W = n$ case. Again we just have to prove then that $\overset{s}{Y}$ is fixed under this action, and we check this at the level of Lie algebras. Denote $D_W = (\partial_w, D^1_W, \dots, D^n_W)$, where $D^i_W = \partial_{\zeta^i} + \zeta^i \partial_w$. Similarly, denote $\bar{D}_W = (\partial_w, \bar{D}^1_W, \dots, \bar{D}^n_W)$, where $\bar{D}^i_W = \partial_{\zeta^i} - \zeta^i \partial_w$. Let $\rho = \exp(\varepsilon \mathbf{v})$ where $\mathbf{v} = v(W)\bar{D}_W \in \operatorname{Der}^{\omega}_0 \mathscr{O}^{1|n}$, and put $\rho_Z = \exp(\varepsilon \mathbf{u})$. Expanding ρ_Z in powers of ε we find:

$$\mathbf{u} = v(W+Z)\bar{D}_W - v(Z)\bar{D}_W. \tag{4.2.8.6}$$

Note that in this context we have two *different* Taylor expansions:

$$e^{ZD_W} f(W) = f(Z+W), \qquad e^{Z\bar{D}_W} f(W) = f(W+Z), \tag{4.2.8.7}$$

using the second, we see that

$$\mathbf{u} = \left(e^{Z\bar{D}_W} v(W) \right) \bar{D}_W - v(Z)\bar{D}_W. \tag{4.2.8.8}$$

From this and the fact that the operators corresponding to $\bar{D}_W$ are

$$-\nabla = (-T, -S^1, \dots, -S^n),$$

we obtain:

$$R(\mathbf{u}) = e^{-Z\nabla} R(\mathbf{v}) e^{Z\nabla} + v(Z)\nabla. \tag{4.2.8.9}$$

The theorem now follows as in the $N_W = n$ case. □

We will construct connections on the vector bundles $\mathscr{V}$ from the previous paragraphs.

Theorem 4.2.9. *Let X be a $(1|N)$ dimensional supercurve. Let $U \subset X$ be open and Z be coordinates in U defining the vector fields ∂_z and ∂_{θ^i}. Let V be a strongly conformal $N_W = N$ SUSY vertex algebra and $\mathscr{V}$ the associated bundle. Define the connection operators $\nabla_\chi : \mathscr{V}_{|U} \to \mathscr{V}_{|U}$ for each vector field χ in U by*

$$\nabla_{\partial_z} = \partial_z + T, \qquad \nabla_{\partial_{\theta^i}} = \partial_{\theta^i} + S^i. \tag{4.2.9.1}$$

Then ∇ is a well defined (left) connection on $\mathscr{V}$ (independent of the coordinates chosen). Moreover, this connection is flat.

Proof. The proof is verbatim the proof of the analogous statement in [16, 16.1]. Indeed, strongly conformal SUSY vertex algebras are modules for the Harish Chandra pair $(\operatorname{Der}\mathscr{O}^{1|N}, \operatorname{Aut}\mathscr{O}^{1|N})$ and this in turn acts simply transitively on the torsor $\operatorname{Aut}_X \to X$. The localization procedure of formal geometry applies without difficulties. □

Remark 4.2.10. Note that this connection endowes $\mathscr{V}$ with a structure of a left $\mathscr{D}_X$-module for any supercurve X and any strongly conformal $N_W = N$ SUSY vertex algebra V.

Let V be a strongly conformal $N_K = N$ SUSY vertex algebra, and let $\mathscr{V}$ be the associated vector bundle over an oriented superconformal curve X. For an open U as before, and superconformal coordinates Z in U we will define the superconformal differential operators $\mathscr{D}_X(U)$ to be the super ring of differential operators generated by all the D^i_Z. This defines a sheaf of algebras of superconformal differential operators $\mathscr{D}_X$ over any (oriented) superconformal curve X. The asignement

$$D^i_Z \cdot f(Z)a = (D^i_Z f(Z))a + (-1)^f f(Z) S^i a \tag{4.2.10.1}$$

gives $\mathscr{V}$ the structure of a left $\mathscr{D}_X$-module.

4.3 Examples

4.3.1. In this section we give the first non-trivial examples of the super vector bundles that arise with the construction of the previous sections. To simplify the notation, we will use the ordinary description of the involved vertex algebras. For example, when we analyze the *boson-fermion* system (cf. example 4.3.2) we will work with the fermion φ and the boson α instead of the superfields Ψ and $S\Psi$. Note that the Grassman algebra Λ is a SUSY vertex algebra (either $N_W = N$ or $N_K = N$) with $T = S^i = 0$ and $|0> = 1$. In this section, given a SUSY vertex algebra V, we will consider the tensor product $W = \Lambda \otimes V$ (either of $N_W = N$ or $N_K = N$ SUSY vertex algebras), therefore we can view W as a SUSY vertex algebra over Λ, namely, W is a Λ-module and the vertex operators are Λ-linear.

Let us start with $N_K = 1$ bundles. For this let X be a super conformal $N = 1$ supercurve over Λ. Let U_α and U_β be open in X and $p = (t, \zeta)$ a Λ-point in the intersection. Let V be a strongly conformal $N_K = 1$ SUSY vertex algebra, so that V carries a representation of $Der^\omega_0 \mathscr{O}^{1|1}$ that exponentiates to a representation of $\mathrm{Aut}^\omega \mathscr{O}^{1|1}$. Suppose we have coordinates $(z_\alpha, \theta_\alpha)$ in U_α and (z_β, θ_β) in U_β that are compatible with the superconformal structure. They are related by a change of coordinates $\rho_{\beta\alpha} = (F(z_\alpha, \theta_\alpha), \Psi(z_\alpha, \theta_\alpha))$ satisfying $DF = \Psi D\Psi$ where $D = \partial_{\theta_\alpha} + \theta_\alpha \partial_{z_\alpha}$. These coordinates define coordinates at the point p therefore we obtain different trivializations of the bundle $\mathscr{V}$. The transition functions for the structure sheaf give us transition functions for $\mathscr{V}$, in particular, they act in the fiber at the point p as $R(\rho_p)^{-1}$ (cf. 4.2.6.3).

In order to compute $R(\rho_p)$ we need only to look at the odd coordinate, namely expand in Taylor series

$$\begin{aligned}
\Psi_{z,\theta}(t,\zeta) &= \Psi(t + z + \zeta\theta, \zeta + \theta) - \Psi(z, \theta) \\
&= \zeta D\Psi + tD^2\Psi + \zeta t D^3\Psi + \frac{t^2}{2} D^4\Psi + \dots \\
&= \exp\left(-\sum_{i\geq 1} (v_i L_i + w_i G_{(i)}) \right) A^{-2L_0} \cdot \zeta
\end{aligned} \tag{4.3.1.1}$$

where as in (4.1.4.1) we have

$$L_n = -\frac{n+1}{2} t^n \zeta \partial_\zeta - t^{n+1}\partial_t,$$
$$G_{(n+1/2)} = G_n = -t^{n+1/2}(\partial_\zeta - \zeta\partial_t), \tag{4.3.1.2}$$

where $v_i = v_i(z,\theta)$ are even functions and $w_i = w_i(z,\theta)$ are odd functions. Truncating the series in (4.3.1.1) at order 2 we have:

$$\Psi_{(z,\theta)}(t,\zeta) = A\left(\zeta + tw_1 + \zeta t v_1 + t^2(w_2 + v_1 w_1) + \dots\right) \tag{4.3.1.3}$$

From where we get the equations:

$$A = D\Psi \qquad\qquad w_1 A = D^2\Psi \tag{4.3.1.4}$$

$$v_1 A = D^3\Psi \qquad\qquad (w_2 + v_1 w_1)A = \frac{1}{2}D^4\Psi. \tag{4.3.1.5}$$

We can solve this system to get:

$$\begin{aligned} v_1 &= \frac{D^3\Psi}{D\Psi} \\ w_1 &= \frac{D^2\Psi}{D\Psi} \\ w_2 &= \frac{1}{2}\left(\frac{D^4\Psi}{D\Psi} - 2\frac{D^3\Psi D^2\Psi}{(D\Psi)^2}\right) = \frac{1}{2}\sigma(D\Psi) \end{aligned} \tag{4.3.1.6}$$

where σ is the $N=1$ *super-schwarzian* defined in (2.2.19.2).

Example 4.3.2 (Free Fields). Recall the strongly conformal $N_K = 1$ SUSY vertex algebra B_1 defined in Example 3.6.8 (see also Example 2.1.17). We will denote this vertex algebra as $B(1)$. As an ordinary vertex algebra, it is graded with respect to conformal weight. The fermion φ is primary of conformal weight $1/2$ and the boson α has conformal weight 1 but it is not primary unless $m=0$. It follows easily from Wick formulas that the only non-trivial relations with the fermion φ are given by:

$$G_{(1)}\varphi = -m|0> \qquad L_0\varphi = \frac{1}{2}\varphi. \tag{4.3.2.1}$$

Therefore the subspace $B(1)_{\leq 1/2}$ of $B(1)$ spanned by $\{|0>, \varphi\}$ is an $\operatorname{Aut}^\omega \mathscr{O}^{1|1}$ submodule. For a given change of coordinates $\rho = (F, \Psi)$ we can compute the action of $R(\rho_{(z,\theta)})^{-1}$. For this we write in the basis $\{|0>, \varphi\}$:

$$\begin{aligned} R(\rho_{(z,\theta)})^{-1} &= A^{2L_0} \cdot \exp\left(\sum_{i\geq 1}(v_i L_i + w_i G_{(i)})\right) \\ &= \begin{pmatrix} 1 & -mw_1 \\ 0 & A \end{pmatrix} = \begin{pmatrix} 1 & -m\frac{D^2\Psi}{D\Psi} \\ 0 & D\Psi \end{pmatrix} \end{aligned} \tag{4.3.2.2}$$

Hence, if $\mathscr{B}(1)$ is the vector bundle associated to the $\mathrm{Aut}^{\omega}\mathscr{O}^{1|1}$ module $B(1)$ and $\mathscr{B}(1)_{\leq 1/2}$ is the vector bundle corresponding to $B(1)_{\leq 1/2}$ we see that the transition functions that define $\mathscr{B}(1)_{\leq 1/2}$ are given on the intersections $U_\alpha \cap U_\beta$ by the functions (4.3.2.2).

Dually, sections of the bundle $\mathscr{B}(1)^*_{\leq 1/2}$ transform by (note that we use the *super-transpose* instead of the transpose, as defined in [29, ch. 3 § 3.1])

$$\begin{pmatrix} 1 & 0 \\ m\frac{D^2\Psi}{D\Psi} & D\Psi. \end{pmatrix} \tag{4.3.2.3}$$

In particular we have a section $\mathscr{Y}$ of $\mathscr{B}(1)^*$ which projects to a section of $\mathscr{B}(1)^*_{\leq 1/2}$. In the basis $\{|0>, \varphi\}$ this section is given by

$$\begin{pmatrix} \mathrm{Id} \\ \varphi(z,\theta) \end{pmatrix}, \tag{4.3.2.4}$$

where, according to example 2.1.17, we have

$$\varphi(z,\theta) = Y(\varphi_{-1/2}|0>, z) + \theta Y(\tau_{-1/2}\varphi_{-1/2}|0>, z). \tag{4.3.2.5}$$

According to (4.3.2.3) and theorem 4.2.8 we see that the field $\varphi(z,\theta)$ transforms as:

$$\varphi(z,\theta) = R(\rho)\varphi(\rho(z,\theta))R(\rho)^{-1}D\Psi + m\frac{D^2\Psi}{D\Psi}\mathrm{Id} \tag{4.3.2.6}$$

where $\rho = (F, \Psi)$. In particular, since X is a superconformal $N = 1$ curve we have[4]

$$D\Psi = D\left(\frac{DF}{D\Psi}\right) = \mathrm{sdet}\begin{pmatrix} F_z & \Psi_z \\ F_\theta & \Psi_\theta \end{pmatrix}. \tag{4.3.2.7}$$

Therefore when $m = 0$, $\varphi(z,\theta)[dzd\theta]$ transforms as an $\mathrm{End}\,\mathscr{B}(1)_p$-valued section of the Berezinian bundle of X on the punctured disk $D_p^\times$ for any Λ-point $p \in X$. When $m \neq 0$ this bundle is not split and $\varphi(z,\theta)$ gives rise to an $\mathrm{End}\,\mathscr{B}(1)_p$-valued section of $\mathscr{B}(1)^*_{\leq 1/2}$ that projects onto the section $1 \otimes \mathrm{Id}$ of the quotient $\mathscr{O}_X \otimes \mathrm{End}\,\mathscr{B}(1)_p$ and transforms according to (4.3.2.6) with changes of coordinates. In other words, the bundle $\mathscr{B}_{\leq 1/2}(1)^*$ is an extension

$$0 \to \mathrm{Ber}_X \to \mathscr{B}_{\leq 1/2}(1)^* \to \mathscr{O}_X \to 0 \tag{4.3.2.8}$$

which is non-split unless $m = 0$. In the case when $m \neq 0$ the section $\mathscr{Y}$ projects into the constant section 1 of $\mathscr{O}_X$.

In analogy to [16] we want to understand the geometric meaning of these sections. Equivalently, we want to find the set of splittings of the extension (4.3.2.8). This set, if non-empty, is a torsor over the space of sections of Ber_X. Recall also that the operator $D = \partial_\theta + \theta\partial_z$ takes values in Ber_X for a superconformal $N = 1$ curve. We

[4]Here and further, the subscripts z and θ denote partial derivatives.

have then

Theorem 4.3.3. *The superfield $\varphi(z,\theta)$ transforms as an odd differential operator $\nabla : \mathrm{Ber}_X \to \mathrm{Ber}_X^{\otimes 2}$ locally of the form $\nabla = -mD_\alpha + g_\alpha(z_\alpha,\theta_\alpha)$, where on the open subset U_α with coordinates (z_α,θ_α) we have $D_\alpha = \partial_{\theta_\alpha} + \theta_\alpha\partial_{z_\alpha}$ and g_α is an odd function.*

Proof. Recall that in a superconformal $N=1$ curve the generators $[dz_\alpha d\theta_\alpha]$ of the Berezinian bundle transform as

$$[dz_\beta d\theta_\beta] = (D_\alpha\Psi_{\beta,\alpha})[dz_\alpha d\theta_\alpha], \tag{4.3.3.1}$$

where the change of coordinates is $\theta_\beta = \Psi_{\beta,\alpha}(z_\alpha,\theta_\alpha)$.

Since $\nabla : \mathrm{Ber}_X \to \mathrm{Ber}_X^{\otimes 2}$ we have

$$\nabla_\alpha f_\alpha = (D_\alpha\Psi_{\beta,\alpha})^2\nabla_\beta\left((D_\alpha\Psi_{\beta,\alpha})^{-1}f_\alpha\right) \tag{4.3.3.2}$$

Therefore we get

$$\begin{aligned}
\nabla_\alpha f_\alpha &= -mD_\alpha f_\alpha + g_\alpha f_\alpha = -m(D_\alpha\Psi_{\beta,\alpha})D_\beta f_\alpha + g_\alpha f_\alpha\\
&= (D_\alpha\Psi_{\beta,\alpha})^2\left(-m\left(D_\beta\left(D_\alpha\Psi_{\beta,\alpha}\right)^{-1}f_\alpha\right) + g_\beta\left(D_\alpha\Psi_{\beta,\alpha}\right)^{-1}f_\alpha\right)\\
&= -m\left(D_\alpha\Psi_{\beta,\alpha}\right)D_\beta f_\alpha - m\left(D_\alpha\Psi_{\beta,\alpha}\right)^2 D_\beta\left(\left(D_\alpha\Psi_{\beta,\alpha}\right)^{-1}\right)f_\alpha +\\
&\quad + g_\beta\left(D_\alpha\Psi_{\beta,\alpha}\right)f_\alpha\\
&= -m\left(D_\alpha\Psi_{\beta,\alpha}\right)D_\beta f_\alpha + m\left(D_\alpha\Psi_{\beta,\alpha}\right)^{-1}D_\alpha^2\Psi_{\beta,\alpha}f_\alpha + g_\beta\left(D_\alpha\Psi_{\beta,\alpha}\right)f_\alpha\\
g_\alpha &= \left(D_\alpha\Psi_{\beta,\alpha}\right)g_\beta + m\frac{D_\alpha^2\Psi_{\beta,\alpha}}{D_\alpha\Psi_{\beta,\alpha}}
\end{aligned} \tag{4.3.3.3}$$

Hence we find finally

$$\begin{pmatrix}1\\ g_\alpha\end{pmatrix} = \begin{pmatrix}1 & 0\\ m\frac{D^2\Psi}{D\Psi} & D\Psi\end{pmatrix}\begin{pmatrix}1\\ g_\beta\end{pmatrix} \tag{4.3.3.4}$$

thus proving the theorem. □

4.3.4. Given that we can integrate a section of Ber_X along a super contour as in 2.2.15, we can state [16, 7.1.9] in this situation. We define an *affine structure* on a superconformal $N=1$ curve to be a (equivalence class of) coordinate atlas U_α with coordinates (z_α,θ_α) such that the transition functions on overlaps satisfy[5]:

$$\begin{aligned}
z_\beta &= F_{\beta,\alpha}(z_\alpha,\theta_\alpha) = a^2 z_\alpha + \theta_\alpha\xi a + b\\
\theta_\beta &= \Psi_{\beta,\alpha}(z_\alpha,\theta_\alpha) = \theta_\alpha a + \xi,
\end{aligned} \tag{4.3.4.1}$$

where a, b are even constants with a invertible and ξ is an odd constant (these constants may change with α and β). Given such an atlas, we can define $\nabla_\alpha = -mD_\alpha$ and we get from:

$$-mD_\alpha = -mD_\alpha\Psi_{\beta,\alpha}D_\beta \tag{4.3.4.2}$$

[5]These are SUSY changes of coordinates where the odd coordinate changes by affine transformations.

and the fact that $D^2\Psi = 0$ for these transition functions, that ∇_α is a well defined operator as in theorem 4.3.3.

On the other hand, suppose we have such a differential operator $\nabla_\alpha = -mD_\alpha + g_\alpha$. Consider $f_\alpha[dz_\alpha d\theta_\alpha]$ to be a section of Ber_X in U_α such that f_α is an even function and $\nabla_\alpha \cdot f_\alpha = 0$. Choose a Λ-point $P = (x, \pi)$ of U_α and for any other point Q in U_α we define the function ξ_α to be

$$\xi_\alpha(Q) = \int_P^Q f_\alpha. \tag{4.3.4.3}$$

From the definition of this integral we see that ξ is an odd function, indeed, to compute this integral we need to find $D\omega = f$ and then this integral becomes $\omega(Q) - \omega(P)$. By shrinking if necessary the open cover U_α we may assume that f_α does not vanish everywhere (it is an even function), therefore it follows that $D\xi$ is invertible everywhere. We now solve the differential equation $Dw = \xi D\xi$ (we may need to shrink U_α even more) and obtain thus a coordinate atlas U_α with new coordinates (w_α, ξ_α). We claim that this atlas is indeed an affine structure on X. We have made some choices. One is the reference point P which shifts the function ξ_α by an odd constant. The other choice was the solution ξ, which is unique up to an invertible even multiple (for this we can apply a version of Cauchy's theorem in super geometry). Therefore ξ is well defined up to affine transformations of the form $\xi \mapsto a\xi + \zeta$. This forces w to change to $\tilde{w}$ with

$$D\tilde{w} = (a\xi + \zeta)D(a\xi + \zeta) = a^2\xi D\xi + a\zeta D\xi \tag{4.3.4.4}$$

hence $\tilde{w} = a^2 w + w'$ with $Dw' = Da\zeta\xi$. Finally we see that $w' = a\zeta\xi + w"$ where $Dw" = 0$, namely the choices made combine into changes of the form:

$$\begin{aligned} \xi &\mapsto a\xi + \zeta \\ w &\mapsto a^2 w + a\zeta\xi + b \end{aligned} \tag{4.3.4.5}$$

where a, b are even constants (a is invertible) and ζ is odd. Since these changes of coordinates are of the form (4.3.4.1), we have proved:

Theorem 4.3.5. *Let X be an $N = 1$ superconformal curve. For every $m \neq 0$ the set of differential operators $\nabla : \mathrm{Ber}_X \to \mathrm{Ber}_X^{\otimes 2}$ locally defined as $\nabla_\alpha = -mD_\alpha + g_\alpha$ for odd functions g_α are in one to one correspondence with the set of affine structures on the curve X. These in turn are in one to one correspondence with the set of splittings of the extension (4.3.2.8).*

Example 4.3.6. The Neveu Schwarz algebra Recall the strongly conformall $N_K = 1$ SUSY vertex algebra K_1^c defined in Example 3.6.5 (see also Example 2.1.11). Denote this vertex algebra by $K(1)$. We note that the sub-vector space spanned by the primary elements of conformal weight less or equal to $3/2$, namely the vacuum vector and the $N = 1$ vector τ, is $\mathrm{Aut}^\omega \mathscr{O}^{1|1}$-invariant. In order to compute the transition functions we see easily that the relevant relations in this case are:

$$L_{(1)}\tau = \frac{3}{2}\tau, \qquad G_{(2)}\tau = \frac{2}{3}c. \tag{4.3.6.1}$$

Therefore we can compute in the basis $\{|0>, \tau\}$

$$\begin{aligned} R(\rho_{(z,\theta)})^{-1}\begin{pmatrix}|0>\\ \tau\end{pmatrix} &= A^{2L_0}\exp\left(\sum_{i\geq 1} v_i L_i + w_i G_{(i)}\right)\begin{pmatrix}|0>\\ \tau\end{pmatrix} \\ &= \begin{pmatrix}1 & \frac{2}{3}cw_2\\ 0 & A^3\end{pmatrix}\begin{pmatrix}|0>\\ \tau\end{pmatrix}\end{aligned} \tag{4.3.6.2}$$

It follows from (4.3.1.6) and (4.3.1.5) that the transition functions for the corresponding bundle $\mathscr{K}_{\leq 3/2}(1)$ are given by:

$$R(\rho_{(z,\theta)})^{-1} = \begin{pmatrix}1 & \frac{c}{3}\sigma(D\Psi)\\ 0 & (D\Psi)^3\end{pmatrix} \tag{4.3.6.3}$$

where, as before, $\sigma(D\Psi)$ is the *super shwarzian* derivative. Dualizing, we obtain an extension:

$$0 \to \mathrm{Ber}_X^{\otimes 3} \to \mathscr{K}_{\leq 3/2}(1)^* \to \mathscr{O}_X \to 0. \tag{4.3.6.4}$$

This extension is not split if $c \neq 0$ and, as for the free fields, we see that the section $\mathscr{Y}$ of $\mathscr{K}_{\leq 3/2}(1)^*$ projects onto $\mathscr{O}_X$ in this case. Denote by $\tau(z,\theta) = G(z) + 2\theta L(z)$ the superfield $\overset{s}{Y}(\tau, z, \theta)$. By taking the super transpose of (4.3.6.3) we find that $\tau(z,\theta)$ transforms as:

$$\tau(z,\theta) = R(\rho)\tau(\rho(z,\theta))R(\rho)^{-1}(D\Psi)^3 - \frac{c}{3}\sigma(D\Psi) \tag{4.3.6.5}$$

which in turn implies, according to Proposition 2.2.21 the following:

Theorem 4.3.7. *The set of splittings of (4.3.6.4) is in one to one correspondence with the set of superprojective structures in X.*

4.3.8. Now we turn our attention to the oriented superconformal $N = 2$ case. In a similar way as in 4.3.1 we expand in Taylor series the odd coordinates as:

$$\begin{aligned}\Psi^i_{(z,\theta^1,\theta^2)}(t,\zeta^1,\zeta^2) = \Big(&\zeta^1 D_1 + \zeta^2 D_2 + tD_1^2 + \\ &+ \zeta^1 t D_1^3 + \zeta^2 t D_2^3 - \zeta^1\zeta^2 D_1 D_2 + \frac{t^2}{2}D_1^4 + \dots\Big)\Psi^i(z,\theta^1,\theta^2)\end{aligned} \tag{4.3.8.1}$$

We want to express these functions as

$$\exp\left(-\sum_{\substack{i\geq 1\\ j=1,2}} v_i L_i + u_i J_i + w_i^{(j)} G_{(i)}^{(j)}\right)\exp(-BJ_0)A^{-2L_0}\cdot\zeta^i, \tag{4.3.8.2}$$

where $D^i = \partial_{\theta^i} + \theta^i \partial_z$ and we have, as in (4.1.5.4):

$$
\begin{aligned}
L_n &= -t^{n+1}\partial_t - \frac{n+1}{2}t^n\left(\zeta^1\partial_{\zeta^1} + \zeta^2\partial_{\zeta^2}\right),\\
G_n^{(2)} &= +t^{n+1/2}\left(\zeta^2\partial_t - \partial_{\zeta^2}\right) - \left(n+\frac{1}{2}\right)t^{n-1/2}\zeta^1\zeta^2\partial_{\zeta^1}\\
G_n^{(1)} &= +t^{n+1/2}\left(\zeta^1\partial_t - \partial_{\zeta^1}\right) + \left(n+\frac{1}{2}\right)t^{n-1/2}\zeta^1\zeta^2\partial_{\zeta^2}\\
J_n &= -it^n\left(\zeta^2\partial_{\zeta^1} - \zeta^1\partial_{\zeta^2}\right)
\end{aligned}
\tag{4.3.8.3}
$$

Using the coordinates $(z, \theta^\pm = \theta^1 \pm i\theta^2)$ and the change of coordinates $\rho = (F, \Psi^\pm = \Psi^1 \pm i\Psi^2)$ (cf. 4.1.5) it follows that

$$
\Psi^\pm_{(z,\theta^+,\theta^-)}(t,\zeta^+,\zeta^-) = \Psi^\pm\left(t + z + \frac{1}{2}(\zeta^+\theta^- + \zeta^-\theta^+), \zeta^+ + \theta^+, \zeta^- + \theta^-\right) - \\
- \Psi^\pm(z,\theta^+,\theta^-), \quad (4.3.8.4)
$$

which we want to expand in Taylor series. Let us do that in detail (here Ψ denotes either Ψ^+ or Ψ^-):

$$
\begin{aligned}
\Psi_{(z,\theta^\pm)} &= \left(1 + \frac{1}{2}(\zeta^+\theta^- + \zeta^-\theta^+)\partial_z + \frac{1}{8}(\zeta^+\theta^- + \zeta^-\theta^+)^2\partial_z^2\right)\cdot\\
&\qquad\cdot\Psi(t+z, \zeta^+ + \theta^+, \zeta^- + \theta^-) - \Psi(z,\theta^+,\theta^-)\\
&= \left(1 + \frac{1}{2}(\zeta^+\theta^- + \zeta^-\theta^+)\partial_z + \frac{1}{4}\zeta^+\zeta^-\theta^+\theta^-\partial_z^2\right)\cdot\\
&\qquad\cdot\Psi(t+z, \zeta^+ + \theta^+, \zeta^- + \theta^-) - \Psi(z,\theta^+,\theta^-)\\
&= \left[\zeta^+\left(\partial_{\theta^+} + \frac{1}{2}\theta^-\partial_z\right) + \zeta^-\left(\partial_{\theta^-} + \frac{1}{2}\theta^+\partial_z\right) + \right.\\
&\quad + t\partial_z + \zeta^+ t\left(\partial_{\theta^+} + \frac{1}{2}\theta^-\partial_z\right)\partial_z + \zeta^- t\left(\partial_{\theta^-} + \frac{1}{2}\theta^+\partial_z\right)\partial_z +\\
&\quad \left. + \zeta^+\zeta^-\left(\partial_{\theta^-,\theta^+} - \frac{1}{2}\theta^-\partial_{z,\theta^-} + \frac{1}{2}\theta^+\partial_{z,\theta^+} + \frac{1}{4}\theta^+\theta^-\partial_z^2\right) + \frac{1}{2}t^2\partial_z^2\right]\Psi + \dots\\
&= \left(\zeta^+D^- + \zeta^-D^+ + t\partial_z + \zeta^+tD^-\partial_z + \zeta^-tD^+\partial_z + \right.\\
&\qquad \left. + \zeta^+\zeta^-(D^+D^- - \frac{1}{2}\partial_z) + \frac{1}{2}t^2\partial_z^2\right)\Psi + \dots,
\end{aligned}
\tag{4.3.8.5}
$$

where $D^{\pm} = \partial_{\theta^{\mp}} + \frac{1}{2}\theta^{\pm}\partial_z$. Since the curve is oriented, this reduces to:

$$\begin{aligned}\Psi^{+}_{(z,\theta^{\pm})} &= \Big(\zeta^{+}D^{-} + tD^{+}D^{-} + \zeta^{+}tD^{-}D^{+}D^{-} + \\ &\qquad + \zeta^{+}\zeta^{-}\frac{1}{2}(D^{+}D^{-}) + \frac{1}{2}t^2(D^{+}D^{-})^2\Big)\Psi^{+} + \dots \\ \Psi^{-}_{(z,\theta^{\pm})} &= \Big(\zeta^{-}D^{+} + tD^{-}D^{+} + \zeta^{-}tD^{+}D^{-}D^{+} + \\ &\qquad + \zeta^{-}\zeta^{+}\frac{1}{2}(D^{-}D^{+}) + \frac{1}{2}t^2(D^{-}D^{+})^2\Big)\Psi^{-} + \dots\end{aligned} \tag{4.3.8.6}$$

We want to express these as the exponential of a vector field. For this we compute

$$\begin{aligned}&\exp\Big(-\sum_{i\geq 1} v_iL_i + u_iJ_i + w^{\pm}G^{\pm}_{(i)}\Big) B^{-J_0}A^{-2L_0}\cdot\zeta^{\pm} = \\ &\qquad = B^{\pm 1}A\Big[\zeta^{\pm} + tw_1^{\pm} + \\ &+ \zeta^{\pm}t(v_1 \pm u_1 + \frac{1}{2}w_1^{\mp}w_1^{\pm}) + t^2(w_2^{\pm} + \frac{1}{2}w_1^{\pm}(2v_1 \pm u_1)) + \frac{1}{2}\zeta^{\pm}\zeta^{\mp}w_1^{\pm}\Big] + \dots,\end{aligned} \tag{4.3.8.7}$$

from where get the equations:

$$\begin{aligned}B^{\pm 1}A &= D^{\mp}\Psi^{\pm} \\ w_1^{\pm} &= \frac{D^{\pm}D^{\mp}\Psi^{\pm}}{D^{\mp}\Psi^{\pm}} = \frac{\Psi^{\pm}_z}{D^{\mp}\Psi^{\pm}} = \frac{\Psi^{\pm}_z}{\Psi^{\pm}_{\theta^{\pm}}} \\ v_1 \pm u_1 + \frac{1}{2}w_1^{\pm}w_1^{\mp} &= \frac{D^{\mp}D^{\pm}D^{\mp}\Psi^{\pm}}{D^{\mp}\Psi^{\pm}} \\ w_2^{\pm} + \frac{1}{2}w_1^{\pm}(2v_1 \pm u_1) &= \frac{1}{2}\frac{(D^{\pm}D^{\mp})^2\Psi^{\pm}}{D^{\mp}\Psi^{\pm}}.\end{aligned} \tag{4.3.8.8}$$

We can solve this system to get

$$\begin{aligned}v_1 &= \frac{1}{2}\left(\frac{D^{-}\Psi^{+}_z}{D^{-}\Psi^{+}} + \frac{D^{+}\Psi^{-}_z}{D^{+}\Psi^{-}}\right) \\ u_1 &= \frac{1}{2}\left(\frac{D^{-}\Psi^{+}_z}{D^{-}\Psi^{+}} - \frac{D^{+}\Psi^{-}_z}{D^{+}\Psi^{-}}\right) - \frac{1}{2}\frac{\Psi^{+}_z\Psi^{-}_z}{D^{-}\Psi^{+}D^{+}\Psi^{-}} = -\sigma_2(\Psi^{+},\Psi^{-}) \\ w_2^{\pm} &= \frac{1}{2D^{\mp}\Psi^{\pm}}\left(\Psi^{\pm}_{z,z} - \frac{1}{2}\left(\frac{D^{\pm}\Psi^{\mp}_z}{D^{\pm}\Psi^{\mp}} + 3\frac{D^{\mp}\Psi^{\pm}_z}{D^{\mp}\Psi^{\pm}}\right)\right),\end{aligned} \tag{4.3.8.9}$$

where σ_2 is the $N = 2$ *schwarzian derivative* (cf. [11]).

Example 4.3.9. Free Fields. With the results of the previous sections we can compute now explicitly some vector bundles over oriented superconformal $N = 2$ curves. Let Y be such a curve and let $B(2)$ be the strongly conformal $N_K = 2$ SUSY vertex algebra described in Example 2.1.24. Let $\mathscr{B}(2)$ be the associated vector bundle over Y. The vector subspace spanned by the vacuum vectors and the two fermions (namely the fields with conformal weight less or equal to 1/2) is an $\mathrm{Aut}^{\omega}\mathscr{O}^{1|2}$-submodule. Let us denote these vectors, as in 2.1.24, by $\{|0>, \varphi^{\pm}\}$ respectively, and let $\mathscr{B}_{\leq 1/2}(2)$ be the associated rank $1|2$ vector bundle over Y. In order to compute its transition functions explicitly we see first that the only nontrivial relations (for our purposes) are:

$$G^{\pm}_{(1)}\varphi^{\mp} = \mp m \quad J_0\varphi^{\pm} = \pm\varphi^{\pm} \quad L_0\varphi^{\pm} = \frac{1}{2}\varphi^{\pm} \tag{4.3.9.1}$$

We can therefore compute the transition functions to be

$$\begin{aligned} R(\rho)^{-1}\begin{pmatrix}|0>\\ \varphi^{+}\\ \varphi^{-}\end{pmatrix} &= A^{2L0}B^{J_0}\exp\left(\sum_{i\geq 1} v_1 L_i + u_i J_i + w_i^{\pm}G^{\pm}_{(i)}\right)\begin{pmatrix}|0>\\ \varphi^{+}\\ \varphi^{-}\end{pmatrix} \\ &= \begin{pmatrix}1 & mw_1^{-} & -mw_1^{+}\\ 0 & BA & 0\\ 0 & 0 & B^{-1}A\end{pmatrix}\begin{pmatrix}|0>\\ \varphi^{+}\\ \varphi^{-}\end{pmatrix} \\ &= \begin{pmatrix}1 & m\frac{\Psi_z^{-}}{D^{+}\Psi^{-}} & -m\frac{\Psi_z^{+}}{D^{-}\Psi^{+}}\\ 0 & D^{-}\Psi^{+} & 0\\ 0 & 0 & D^{+}\Psi^{-}\end{pmatrix}\begin{pmatrix}|0>\\ \varphi^{+}\\ \varphi^{-}\end{pmatrix}. \end{aligned} \tag{4.3.9.2}$$

Recall now that an oriented superconformal $N = 2$ curve projects onto two $N = 1$ supercurves X and its dual $\hat{X}$ (cf. 2.2.12). Using the coordinates (cf. 4.1.5)

$$\left(u = z + \frac{1}{2}\theta^{+}\theta^{-}, \theta^{+}, \theta^{-}\right),$$

we obtain from (4.1.5.11) and (4.1.5.10) that

$$\begin{aligned} D^{+}\Psi^{-} &= D^{+}\left(\frac{D^{-}G}{D^{-}\Psi^{+}}\right) \\ &= D^{+}\left(\frac{1}{(\Psi^{+}_{\theta^{+}})^2}(\Psi^{+}_{\theta^{+}} - \theta^{-}\Psi^{+}_{u})(G_{\theta^{+}} + \theta^{-}G_u)\right) \\ &= \frac{\Psi^{+}_{\theta^{+}}G_u - \Psi^{+}_{u}G_{\theta^{+}}}{(\Psi^{+}_{\theta^{+}})^2} \\ &= \operatorname{sdet}\begin{pmatrix}G_u & \Psi^{+}_{u}\\ G_{\theta^{+}} & \Psi^{+}_{\theta^{+}}\end{pmatrix}. \end{aligned} \tag{4.3.9.3}$$

Similarly we find

$$D^{-}\Psi^{+} = \operatorname{sdet}\begin{pmatrix}G'_{u'} & \Psi^{-}_{u'}\\ G'_{\theta^{-}} & \Psi^{-}_{\theta^{-}}.\end{pmatrix} \tag{4.3.9.4}$$

Let us call π and $\hat{\pi}$ the projections from Y onto X and $\hat{X}$ respectively. We see from (4.3.9.4) and (4.3.9.3) that taking the super-transpose in (4.3.9.2) we obtain an extension (of sheaves on Y):

$$0 \to \pi^{*}\operatorname{Ber}_X \oplus \hat{\pi}^{*}\operatorname{Ber}_{\hat{X}} \to \mathscr{B}(2)^{*}_{\leq 1/2} \to \mathscr{O}_Y \to 0. \tag{4.3.9.5}$$

As in the $B(1)$ case, this extension is not split unless m vanishes. It follows in the same way as in the $N=1$ case that the set of splittings of this extension corresponds to *affine structures* on the $N=2$ superconformal curve Y. Indeed, we see that the pair of fields (φ^+, φ^-) transforms as a differential operator $\nabla : \mathrm{Ber}_{\hat{X}} \oplus \mathrm{Ber}_X \to \mathrm{Ber}_{\hat{X}}^{\otimes 2} \oplus \mathrm{Ber}_X^{\otimes 2}$ which is locally of the form $(mD^+ + g^+, -mD^- + g^-)$ for $g^\pm$ odd functions of (u, θ^+) and (u', θ^-) respectively. We note that according to 2.2.15 sections of $\mathrm{Ber}_X \oplus \mathrm{Ber}_{\hat{X}}$ can be integrated in Y up to an additive constant. The argument in the proof of theorem 4.3.5 generalizes to this setting without difficulty.

We will return to this example below (cf. 4.3.12).

Example 4.3.10. The $N=2$ vertex superalgebra. Let $K(2) := K_2^c$ be the strongly conformal $N_K = 2$ SUSY vertex algebra described in Example 3.6.5 (see also Example 2.1.22), and let $\mathscr{K}(2)$ be the associated vector bundle over Y. The vector subspace spanned by primary fields of conformal weight 0 or 1 is an $\mathrm{Aut}^\omega \mathscr{O}^{1|2}$ submodule. Let us denote these vectors as above by $\{|0>, J\}$ respectively, and let $\mathscr{K}(2)_{\leq 1}$ be the associated rank $2|0$ vector bundle over Y. To compute the transition functions we note that the only non-trivial relations we need are

$$L_0 J = J \qquad J_1 J = \frac{c}{3}|0> . \tag{4.3.10.1}$$

Therefore it follows that the transition functions are given by:

$$\begin{aligned} R(\rho)^{-1} \begin{pmatrix} |0> \\ J \end{pmatrix} &= \begin{pmatrix} 1 & \frac{c}{3}u_1 \\ 0 & A^2 \end{pmatrix} \begin{pmatrix} |0> \\ J \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\frac{c}{3}\sigma_2(\Psi^+, \Psi^-) \\ 0 & D^+\Psi^- D^-\Psi^+ \end{pmatrix} \begin{pmatrix} |0> \\ J \end{pmatrix}. \end{aligned} \tag{4.3.10.2}$$

It follows as before, by taking the super-transpose, that when $c=0$ the superfield $J(z, \theta^+, \theta^-)$ transforms as a section of $\pi^* \mathrm{Ber}_X \otimes \hat{\pi}^* \mathrm{Ber}_{\hat{X}}$, namely in this case we get an extension

$$0 \to \pi^* \mathrm{Ber}_X \otimes \hat{\pi}^* \mathrm{Ber}_{\hat{X}} \to \mathscr{K}(2)^*_{\leq 1} \to \mathscr{O}_Y \to 0, \tag{4.3.10.3}$$

which is split if and only if $c=0$. When $c \neq 0$ the extension is not split and the superfield $J(z, \theta^+, \theta^-)$ transforms as[6]

$$J(z, \theta^+, \theta^-) = (D^+\Psi^-)(D^-\Psi^+) J(\rho(z, \theta^+, \theta^-)) + \frac{c}{3}\sigma_2(\Psi^+, \Psi^-). \tag{4.3.10.4}$$

We see that the section $\mathscr{Y}$ is an even section projecting onto $1 \in \mathscr{O}_Y$, therefore giving a splitting of (4.3.10.3). The set of such splittings if non-empty is a torsor for the even part of $\pi^* \mathrm{Ber}_X \otimes \hat{\pi}^* \mathrm{Ber}_{\hat{X}}$.

Analyzing this algebra further we can consider the space spanned by vectors of conformal weight less than or equal to 3/2, namely $K(2)_{\leq 3/2}$. This space is spanned

[6] Note that this superfield is $-iG(Z)$ in the notation of Example 3.6.5.

by $\{|0>, J, G^{\pm}\}$. In addition to (4.3.10.1) we have the following relations:

$$\begin{aligned}
&L_0G^- = \frac{3}{2}G^- \quad J_0G^- = G^- \quad G^+_{(1)}G^- = J \quad G^+_{(2)}G^- = \frac{c}{3}|0> \\
&L_0G^+ = \frac{3}{2}G^+ \quad J_0G^+ = G^+ \quad G^-_{(1)}G^+ = -J \quad G^-_{(2)}G^+ = \frac{c}{3}|0> .
\end{aligned} \tag{4.3.10.5}$$

With these we can compute the transition functions in the basis $\{|0>, J, G^-, G^+\}$ explicitly:

$$R(\rho)^{-1} = \begin{pmatrix} 1 & \frac{c}{3}u_1 & \frac{c}{3}w_2^+ & \frac{c}{3}w_2^- \\ 0 & A^2 & A^2w_1^+ & -A^2w_1^- \\ 0 & 0 & A^3B^{-1} & 0 \\ 0 & 0 & 0 & A^3B \end{pmatrix}, \tag{4.3.10.6}$$

the first three by three block being:

$$\begin{pmatrix} 1 & -\frac{c}{3}\sigma_2(\Psi^+, \Psi^-) & \frac{c}{6D^-\Psi^+}\left(\Psi^+_{z,z} - \frac{1}{2}\left(\frac{D^+\Psi^-_z}{D^+\Psi^-} + 3\frac{D^-\Psi^+_z}{D^-\Psi^+}\right)\right) \\ 0 & (D^+\Psi^-)(D^-\Psi^+) & (D^+\Psi^-)\Psi^+_z \\ 0 & 0 & (D^-\Psi^+)(D^+\Psi^-)^2 \end{pmatrix}, \tag{4.3.10.7}$$

and the 4,4 entry in (4.3.10.6) is $(D^+\Psi^-)(D^-\Psi^+)^2$. Taking the super-transpose of (4.3.10.6) it follows that $\mathscr{K}(2)^*_{Y,\leq 3/2}$ fits in a short exact sequence of the form:

$$0 \to \pi^* \operatorname{Ber}_X \otimes (\hat{\pi}^* \operatorname{Ber}_{\hat{X}})^{\otimes 2} \to \mathscr{K}(2)^*_{Y,\leq 3/2} \to \mathscr{N}^* \to 0. \tag{4.3.10.8}$$

The bundle $\mathscr{N}$ in turn fits in the exact sequence:

$$0 \to (\pi^* \operatorname{Ber}_X)^{\otimes 2} \otimes \hat{\pi}^* \operatorname{Ber}_{\hat{X}} \to \mathscr{N}^* \to \mathscr{K}(2)^*_{Y,\leq 1} \to 0. \tag{4.3.10.9}$$

4.3.11. We turn our attention now to the $N_W = 1$ case. For this let X be a general $N = 1$ supercurve. As before, given a change of coordinates $\rho = (F, \Psi)$ we expand in Taylor series:

$$\begin{aligned}
F_{(z,\theta)}(t,\zeta) &= F(t+z, \zeta+\theta) - F(z,\theta) \\
&= tF_z + \zeta F_\theta + \zeta t F_{\theta,z} + \frac{t^2}{2}F_{z,z} + \dots \\
\Psi_{(z,\theta)}(t,\zeta) &= \Psi(t+z, \zeta+\theta) - \Psi(z,\theta) \\
&= t\Psi_z + \zeta\Psi_\theta + \zeta t\Psi_{z,\theta} + \frac{t^2}{2}\Psi_{z,z}.
\end{aligned} \tag{4.3.11.1}$$

We need to express these as:

$$\begin{aligned}
\begin{pmatrix} F_{(z,\theta)} \\ \Psi_{(z,\theta)} \end{pmatrix} &= \exp\left(-\sum_{i\geq 1} v_iT_i + u_iJ_i + q_iQ_i + h_iH_i\right) \times \\
&\times \exp(-q_0Q_0)\exp(-h_0H_0)B^{-J_0}A^{-T_0}\begin{pmatrix} t \\ \zeta \end{pmatrix},
\end{aligned} \tag{4.3.11.2}$$

where, as in (4.1.3.7), we have

$$T_n = -t^{n+1}\partial_t - (n+1)t^n\zeta\partial_\zeta \qquad J_n = -t^n\zeta\partial_\zeta \tag{4.3.11.3a}$$

$$Q_n = -t^{n+1}\partial_\zeta \qquad H_n = t^n\zeta\partial_t. \tag{4.3.11.3b}$$

Expanding (4.3.11.2) up to second order, we find:

$$\begin{aligned} F_{(z,\theta)} &= tA(1+q_0h_0) + \zeta A h_0 + t^2(v_1(A + Aq_0h_0) + Aq_1h_0) + \\ &\quad + \zeta t(A(1+q_0h_0)h_1 + 2Av_1h_0 + Au_1h_0) + \dots \\ \Psi_{(z,\theta)} &= \zeta BA + tq_0BA + t\zeta BA(2v_1 + u_1 + h_1q_0) + t^2BA(q_1 + v_1q_0) + \dots, \end{aligned} \tag{4.3.11.4}$$

and we get the equations:

$$A(1+q_0h_0) = F_z \qquad BA = \Psi_\theta \tag{4.3.11.5}$$

$$Ah_0 = F_\theta \qquad q_0BA = \Psi_z \tag{4.3.11.6}$$

$$v_1F_z + q_1F_\theta = \frac{1}{2}F_{z,z} \qquad h_1\Psi_z + (2v_1+u_1)\Psi_\theta = \Psi_{\theta,z} \tag{4.3.11.7}$$

$$h_1F_z + (2v_1+u_1)F_\theta = F_{z,\theta} \qquad v_1\Psi_z + q_1\Psi_\theta = \frac{1}{2}\Psi_{z,z}. \tag{4.3.11.8}$$

From this we find:

$$A = \frac{F_z\Psi_\theta - \Psi_zF_\theta}{\Psi_\theta} \qquad B = \frac{\Psi_\theta^2}{F_z\Psi_\theta - \Psi_zF_\theta} \tag{4.3.11.9}$$

$$h_0 = \frac{F_\theta\Psi_\theta}{F_z\Psi_\theta - \Psi_zF_\theta} \qquad q_0 = \frac{\Psi_z}{\Psi_\theta} \tag{4.3.11.10}$$

$$v_1 = \frac{1}{2}\frac{F_{z,z}\Psi_\theta - \Psi_{z,z}F_\theta}{F_z\Psi_\theta - \Psi_zF_\theta} \qquad q_1 = \frac{1}{2}\frac{F_{z,z}\Psi_z - \Psi_{z,z}F_z}{F_\theta\Psi_z - \Psi_\theta F_z} \tag{4.3.11.11}$$

$$h_1 = \frac{F_{z,\theta}\Psi_\theta - \Psi_{z,\theta}F_\theta}{F_z\Psi_\theta - \Psi_zF_\theta} \qquad u_1 = \frac{F_{z,\theta}\Psi_z - \Psi_{z,\theta}F_z}{F_\theta\Psi_z - \Psi_\theta F_z} + \frac{\Psi_{z,z}F_\theta - F_{z,z}\Psi_\theta}{F_z\Psi_\theta - \Psi_zF_\theta}. \tag{4.3.11.12}$$

Example 4.3.12. Free Fields Consider the vertex algebra $B(2)$ as in example 4.3.9 but as a $N_W = 1$ SUSY vertex algebra. As such, for each $N = 1$ supercurve X we obtain a vector bundle $\mathscr{B}(2)_X$. Recall that with respect to the Virasoro field $\tilde{L}$, the vector φ^- has conformal weight 0. Therefore the vector space spanned by $|0>$ and φ^- is an $\mathrm{Aut}\,\mathscr{O}^{1|1}$-submodule. We obtain then a rank $1|1$ vector bundle over X, to be denoted $\mathscr{B}(2)_{X,\leq 0}$. Let us compute explicitly the transition functions for this bundle. The relevant relations are in this case:

$$J_0\varphi^- = -\varphi^- \qquad Q_0\varphi^- = -m|0> . \tag{4.3.12.1}$$

Hence we obtain

$$R(\rho)^{-1}\begin{pmatrix}|0>\\ \varphi^-\end{pmatrix} = A^{T_0}B^{J_0}\exp(h_0H_0)\exp(q_0Q_0)\begin{pmatrix}|0>\\ \varphi^-\end{pmatrix} =$$

$$= \begin{pmatrix}1 & -mq_0\\ 0 & B^{-1}\end{pmatrix}\begin{pmatrix}|0>\\ \varphi^-\end{pmatrix}, \quad (4.3.12.2)$$

which implies

$$R(\rho)^{-1} = \begin{pmatrix}1 & -m\frac{\Psi_z}{\Psi_\theta}\\ 0 & \frac{F_z\Psi_\theta - \Psi_zF_\theta}{\Psi_\theta^2}\end{pmatrix}. \quad (4.3.12.3)$$

Noting that

$$\operatorname{sdet}\begin{pmatrix}F_z & \Psi_z\\ F_\theta & \Psi_\theta\end{pmatrix} = \frac{F_z\Psi_\theta - \Psi_zF_\theta}{\Psi_\theta^2}, \quad (4.3.12.4)$$

we see that by taking the super-transpose in (4.3.12.3) we obtain an extension

$$0 \to \mathrm{Ber}_X \to \mathscr{B}(2)^*_{X,\le 0} \to \mathscr{O}_X \to 0. \quad (4.3.12.5)$$

This short exact sequence is split if and only if $m = 0$. In that case, we see that $\varphi^-(z,\theta)[dzd\theta]$ transforms as a section of Ber_X. On the other hand, when $m \neq 0$, (4.3.12.5) is not split and the section $\mathscr{Y}$ projects into $1 \in \mathscr{O}_X$, giving a splitting of (4.3.12.5). In order to analyze the splittings of these sequences, we introduce maps of sheaves $\nabla : \mathrm{Ber}_{\hat{X}} \to \mathrm{Ber}_X \otimes \mathrm{Ber}_{\hat{X}}$ which are locally of the form $\nabla_\alpha = -mD^+_\alpha + g_\alpha$. Here we consider X with coordinates u, θ^+ and $\hat{X}$ with coordinates u', θ^- as in 4.1.5. We will consistently write $\hat{f}$ to denote a function of u', θ^-. It follows from (4.3.9.4), (4.3.9.5) and the fact that ∇ maps $\mathrm{Ber}_{\hat{X}} \to \mathrm{Ber}_X \otimes \mathrm{Ber}_{\hat{X}}$ that on overlaps we must have:

$$\nabla_\alpha \hat{f}_\alpha = (D^+\Psi^-)(D^-\Psi^+)\nabla_\beta\left((D^-\Psi^+)^{-1}\hat{f}_\alpha\right). \quad (4.3.12.6)$$

Replacing ∇ in both sides by its local form and using (4.1.5.7) (recall that the superconformal $N = 2$ curve associated to X is oriented), we get:

$$-\, mD^+_\alpha \hat{f}_\alpha + g_\alpha \hat{f}_\alpha \;=\; -mD^+\alpha\hat{f}_\alpha + m(D^-\Psi^+)^{-1}D^+_\alpha D^-_\alpha\Psi^+\hat{f}_\alpha + (D^+\Psi^-)g_\beta f_\alpha. \quad (4.3.12.7)$$

Now noting that $D^+D^-\Psi^+ = \Psi^+_u$ and that

$$\frac{\Psi^+_u}{D^-\Psi^+} = \frac{\Psi^+_u}{\Psi^+_{\theta^+} + \theta^-\Psi^+_u} = \frac{\Psi^+_u}{\Psi^+_{\theta^+}}, \quad (4.3.12.8)$$

we get

$$g_\alpha = \operatorname{sdet}\begin{pmatrix}G_u & \Psi^+_u\\ G_{\theta^+} & \Psi^+_{\theta^+}\end{pmatrix} + m\frac{\Psi^+_u}{\Psi^+_{\theta^+}} \quad (4.3.12.9)$$

therefore proving the following

Theorem. The set of splittings of (4.3.12.5) for $m \neq 0$ is in one to one correspondence with operators $\nabla : \mathrm{Ber}_{\hat{X}} \to \mathrm{Ber}_X \otimes \mathrm{Ber}_{\hat{X}}$ locally of the form $-mD^+_\alpha + g_\alpha$.

Let ∇ be such an operator, and let $\psi_\alpha \mathrm{Ber}_{\hat{X}}(U_\alpha)$ be a flat even section, namely $\nabla_\alpha \psi_\alpha = 0$. As a section of $\mathrm{Ber}_{\hat{X}}$ it can be integrated along any contour in X (cf. 2.2.15), namely, let P be a reference Λ-point in U_α, then for any other Λ-point in U_α we put

$$\zeta = \int_P^Q \psi. \tag{4.3.12.10}$$

The solution ψ is unique up to an even multiplicative constant, whilst changing the reference point P changes ζ by an additive odd constant, shrinking U_α we may assume that $D_\alpha \zeta$ is invertible. Choosing any other even function t with invertible differential, we obtain charts $U_\alpha, (t_\alpha, \zeta_\alpha)$. The transition functions between these charts are clearly affine functions for the odd coordinates, namely $\zeta_\beta = a_{\beta,\alpha}\zeta_\alpha + \varepsilon_{\beta,\alpha}$ for some even constants a and odd constants ε. Conversely, given such a covering of X, we define $\nabla_\alpha = -mD^+_\alpha$, where we take ζ instead of θ^+ and t instead of u in the definition of D^+. It follows from (4.3.12.9) that ∇ is well defined globally since the second term in the right hand side of (4.3.12.9) vanishes.

Combining the above paragraph with the previous theorem we have

Theorem. The set of splittings of (4.3.12.5) for $m \neq 0$ is in one to one correspondence with (equivalence classes of) atlases $U_\alpha, z_\alpha, \theta_\alpha$, such that the transition functions are affine in the odd coordinate, namely $\theta_\beta = a\theta_\alpha + \varepsilon$ for some even constant a and some odd constant ε.

Let Y be the superconformal $N = 2$ curve associated to X. Note that from (4.3.12.3) and (4.3.9.2) it follows that the following sequences are exact

$$\begin{aligned} &0 \to \hat{\pi}^* \mathrm{Ber}_{\hat{X}} \to \mathscr{B}(2)^*_{Y,\leq 1/2} \to \pi^* \mathscr{B}(2)^*_{X,\leq 0} \to 0 \\ &0 \to \pi^* \mathrm{Ber}_X \to \mathscr{B}(2)^*_{Y,\leq 1/2} \to \hat{\pi}^* \mathscr{B}(2)_{\hat{X},\leq 0} \to 0. \end{aligned} \tag{4.3.12.11}$$

The bundle $\mathscr{B}(2)_Y$ is the corresponding bundle constructed in Example 4.3.9 from this vertex algebra, but viewed as an $N_K = 2$ SUSY vertex algebra. These two extensions show how the different vector bundles constructed from the same vertex algebras in these three different curves (X, $\hat{X}$ and Y) are related.

It is instructive to analyze the next graded component of $B(2)$. For this we note that the vectors of conformal weight 1 are α^- and φ^+. The relevant relations are:

$$\begin{aligned} &H_0\varphi^+ = \alpha^- \qquad H_1\varphi^+ = m|0\rangle \qquad J_0\varphi^+ = \varphi^+ \\ &Q_0\alpha^- = \varphi^+ \qquad J_1\alpha^- = -m|0\rangle\,. \end{aligned} \tag{4.3.12.12}$$

It follows that the vector space spanned by $\{|0\rangle, \varphi^+, \alpha^-\}$ is an $\mathrm{Aut}\mathscr{O}^{1|1}$-submodule. Let $\mathscr{V}$ be the associated rank $2|1$ vector bundle over X. We compute easily the transition functions to be in this case

$$R(\rho)^{-1} = \begin{pmatrix} 1 & mh_1 & -mu_1 \\ 0 & AB & ABq_0 \\ 0 & Ah_0 & A(1-h_0q_0) \end{pmatrix}, \tag{4.3.12.13}$$

which, according to 4.3.11.12, is

$$R(\rho)^{-1} = \begin{pmatrix} 1 & m\frac{F_{z,\theta}\Psi_\theta - \Psi_{z,\theta}F_\theta}{F_z\Psi_\theta - \Psi_z F_\theta} & -m\left(\frac{F_{z,\theta}\Psi_z - \Psi_{z,\theta}F_z}{F_\theta\Psi_z - \Psi_\theta F_z} + \frac{\Psi_{z,z}F_\theta - F_{z,z}\Psi_\theta}{F_z\Psi_\theta - \Psi_z F_\theta}\right) \\ 0 & \Psi_\theta & \Psi_z \\ 0 & F_\theta & F_z \end{pmatrix} \tag{4.3.12.14}$$

It is clear now that taking the super transpose in (4.3.12.14) we get that $\mathscr{V}^*$ is an extension

$$0 \to \Omega^1_X \to \mathscr{V}^* \to \mathscr{O}_X \to 0. \tag{4.3.12.15}$$

This extension is non-split when $m \neq 0$. Now let $B(2)_{\leq 1}$ be the subspace of $B(2)$ spanned by vectors of conformal weight less or equal than 1. It follows easily from the above paragraphs that the associated rank $2|2$ vector bundle over X fits in a short exact sequence of the form:

$$0 \to \Omega^1_X \oplus \mathrm{Ber}_X \to \mathscr{B}(2)^*_{\leq 1} \to \mathscr{O}_X \to 0. \tag{4.3.12.16}$$

Example 4.3.13. The $N = 2$ vertex algebra. Let, as before $K(2)$ be the $N = 2$ super vertex algebra defined in Example 2.1.22, but considered as an $N_W = 1$ SUSY vertex algebra. Let X be an $N = 1$ supercurve. The vector space spanned by the vacuum vector, the current J, and the fermion H, is an $\mathrm{Aut}\mathscr{O}^{1|1}$-submodule. Indeed, with respect to the Virasoro field $\tilde{L}$, the fermion H has conformal weight 1. Denote the corresponding rank $2|1$ vector bundle over X by $\mathscr{K}(2)_{X,\leq 1}$. It follows from the general considerations in appendix A, that the dual of this vector bundle fits in a short exact sequence of the form:

$$0 \to \Omega^1_X \otimes \mathrm{Ber}_X \to \mathscr{K}(2)^*_{X,\leq 1} \to \mathscr{O}_X \to 0. \tag{4.3.13.1}$$

Indeed, the relevant relations are in this case:

$$\begin{aligned} &T_0 J = J \quad T_1 J = \frac{-c}{3}|0> \quad J_1 J = \frac{c}{3}|0> \quad H_0 J = H \\ &T_0 H = H \quad Q_0 H = J \qquad J_0 H = -H \quad Q_1 H = \frac{c}{3}|0>, \end{aligned} \tag{4.3.13.2}$$

therefore the vector space $K(2)_1$ spanned by $\{J, H\}$ is isomorphic (as a $\mathfrak{gl}(1|1)$-module) to $\pi_+(1,0)$ (cf.appendix A), and its dual module is then $\pi_-(-1,0) \equiv \pi_+(-1,0) \otimes \pi_-(1)$. Also we know that the $\mathrm{Aut}\mathscr{O}^{1|1}$-twist of $\pi_+(-1,0)$ (resp. $\pi_-(1)$) is Ω^1_X (resp. Ber_X).

We can actually compute this transition functions explicitly as before by exponentiating vector fields:

$$
\begin{aligned}
R(\rho)^{-1}\begin{pmatrix}|0\rangle\\ J\\ H\end{pmatrix} &= A^{T_0}B^{J_0}\exp(h_0H_0)\exp(q_0Q_0)\times\\
&\quad\times\exp\left(\sum_{i\geq 1} v_iL_i+u_iJ_i+q_iQ_i+h_iH_i\right)\cdot\begin{pmatrix}|0\rangle\\ J\\ H\end{pmatrix}\\
&= A^{T_0}B^{J_0}\exp(h_0H_0)\exp(q_0Q_0)\cdot\begin{pmatrix}|0\rangle\\ J+(u_1-v_1)\frac{c}{3}|0\rangle\\ H+q_1\frac{c}{3}|0\rangle\end{pmatrix}\\
&= \begin{pmatrix}1 & \frac{c}{3}(v_1-u_1) & \frac{c}{3}q_1\\ 0 & A & Aq_0\\ 0 & B^{-1}Ah_0 & B^{-1}A(1-h_0q_0)\end{pmatrix}\begin{pmatrix}|0\rangle\\ J\\ H\end{pmatrix},
\end{aligned}
\tag{4.3.13.3}
$$

which, according to (4.3.11.12), implies

$$
R(\rho)^{-1}=\begin{pmatrix}1 & \frac{c}{3}\left(\frac{F_{z,\theta}\Psi_z-\Psi_{z,\theta}F_z}{F_\theta\Psi_z-\Psi_\theta F_z}+\frac{3}{2}\frac{\Psi_{z,z}F_\theta-F_{z,z}\Psi_\theta}{F_z\Psi_\theta-\Psi_zF_\theta}\right) & \frac{c}{6}\frac{F_{z,z}\Psi_z-\Psi_{z,z}F_z}{F_\theta\Psi_z-\Psi_\theta F_z}\\ 0 & \frac{F_z\Psi_\theta-\Psi_zF_\theta}{\Psi_\theta} & \frac{F_z\Psi_z}{\Psi_\theta}\\ 0 & \frac{F_\theta}{\Psi_\theta} & \frac{F_z^2\Psi_\theta-\Psi_zF_\theta F_z}{\Psi_\theta^2}\end{pmatrix}. \tag{4.3.13.4}
$$

Taking the super-transpose of the lower two by two block we easily see that this block corresponds to the transition functions in $\mathrm{Ber}_X\otimes\Omega^1$, proving thus that $\mathscr{K}(2)^*_{X,\leq 1}$ is given by an extension as in (4.3.13.1). This extension is non-split unless $c=0$, in which case the pair of fields $\{J(z,\theta),H(z,\theta)\}$ transforms as a section of $\mathrm{Ber}_X\otimes\Omega^1_X$. In order to study the splittings of this extension we need to understand the differential operators appearing in the first row of (4.3.13.4). We leave this to the reader.

Chapter 5

Chiral algebras on supercurves

In this chapter we follow closely the treatment in chapter 18 of [16]. We note that most definitions carry over to the "super" case with minor technical changes. In particular we give a sheaf theoretical interpretation of the OPE formula (3.3.9.1) and its $N_K = N$ analog. We define the superconformal blocks in section 5.2

We will restrict our analysis to the $(1|1)$ dimensional case for simplicity. All the results in this chapter can be generalized to arbitrary odd dimensions without difficulty.

For the definitions of chiral algebras over non-supercurves the reader is referred to [16] and the original work of Beilinson and Drinfeld [5]. The reader may find useful the treatement of $\mathscr{D}$-modules by Bernstein [8] and [31] in the supermanifold case.

5.1 Chiral algebras

5.1.1. When trying to define chiral algebras on supercurves the first problem that we encounter is that given a $(1|N)$ dimensional supercurve X over S, the diagonal embedding $\Delta \hookrightarrow X \times_S X$ has relative codimension $(1|N)$. In particular, the diagonal is not a divisor in $X \times_S X$ unless $N = 0$.

The situation is much simpler in the superconformal case (corresponding to $N_K = N$ SUSY vertex algebras). In this case, we can define canonically a divisor in $X \times_S X$. Basically, all the arguments in the classical case work without change in the superconformal case, given that we have replaced the diagonal by a *super diagonal.*

Since we can carry explicitly the computations in the $N = 1$ case, without introducing extra notation, we will assume that this is the case in the following.

Lemma 5.1.2 (6.3 [28]). *(cf. 5.1.7 below) Let X be a superconformal $N = 1$ supercurve. Let J be the ideal defining the diagonal $i : \Delta \hookrightarrow X \times_S X$. In local coordinates J is defined by $(z - w, \theta - \zeta)$. Let $\Delta^{(1)}$ be defined by J^2. Let I be the kernel of the natural map $\Omega^1_{X/S} \to \mathrm{Ber}_{X/S}$. Finally we define Δ^s by:*

$$\mathscr{O}_{\Delta^s} = \mathscr{O}_{\Delta^{(1)}} / i_*(I). \tag{5.1.2.1}$$

Then Δ^s is a $(1|0)$ codimensional divisor in $X \times_S X$, locally defined by the equation

$$0 = z - w - \theta\zeta. \qquad (5.1.2.2)$$

This divisor will be called the super diagonal *and we will simply call it the* diagonal *when no confusion should arise.*

5.1.3. Given an $\mathscr{O}_X$-module $\mathscr{M}$, we define two extensions of $\mathscr{M}$ along the super diagonal: *extension by principal parts in the transversal direction* and *extension by delta functions in the transversal direction.* The former is given by

$$\Delta^s_+\mathscr{M} := \frac{\mathscr{O} \boxtimes \mathscr{M}(\infty\Delta^s)}{\mathscr{O} \boxtimes \mathscr{M}}, \qquad (5.1.3.1)$$

and the latter by

$$\Delta^s_!\mathscr{M} := \frac{\omega \boxtimes \mathscr{M}(\infty\Delta^s)}{\omega \boxtimes \mathscr{M}}, \qquad (5.1.3.2)$$

where ω is the Berezinian bundle of X defined in 2.2.11.

5.1.4. As in the non-super case, we have a sheaf-theoretical interpretation of the OPE formula. For this we let X be a superconformal $N = 1$ curve over Λ. Let V be a strongly conformal $N_K = 1$ SUSY vertex algebra and let $\mathscr{V}$ be the associated vector bundle over X (cf. 4.2.7). Recall that, given any Λ-point x in X, we have defined a local section $\mathscr{Y}_x$ (cf. 4.2.8). Choose local coordinates Z at x compatible with the superconformal structure. Using this coordinates we trivialize the bundle $\mathscr{V}$ in the formal superdisk D_x around x, namely we have an isomorphism $i_Z : V[[Z]] \to \mathscr{V}|_{D_x}$. Let W be another copy of Z, so that D^2_x is identified with $\operatorname{Spec}\Lambda[[Z, W]]$. The bundle $\mathscr{V} \boxtimes \mathscr{V}(\infty\Delta^s)$, when restricted to D^2_x, is the sheaf associated to the $\Lambda[[Z, W]]$-module $V \otimes V[[Z, W]][(z - w - \theta\zeta)^{-1}]$. Similarly, the restriction of the sheaf $\Delta^s_+\mathscr{V}$ to D^2_x is associated to the $\Lambda[[Z, W]]$-module $V[[Z, W]][(z - w - \theta\zeta)]/V[[Z, W]]$.

Theorem 5.1.5. *Define a map of $\mathscr{O}_{D^2_x}$-modules $\mathscr{Y}_{2,x} : \mathscr{V} \boxtimes \mathscr{V}(\infty\Delta^s) \to \Delta^s_+\mathscr{V}$ by the formula*

$$\mathscr{Y}_{2,x}(f(Z, W)a \boxtimes b) = f(Z, W)\overset{s}{Y}(a, Z - W)b \mod V[[Z, W]]. \qquad (5.1.5.1)$$

Then $\mathscr{Y}_{2,x}$ is independent of the choice of the coordinates Z as long as they are compatible with the superconformal structure induced in D_x from that of X.

Proof. Exactly as in the non-super case, we reduce the proof of this theorem to the identity:

$$\overset{s}{Y}(a, Z - W) = R(\mu_W)\overset{s}{Y}\left(R(\mu_Z)^{-1}a, \mu(Z) - \mu(W)\right)R(\mu_W)^{-1}, \qquad a \in V, \quad (5.1.5.2)$$

for any $\mu \in \operatorname{Aut}^{\omega}\mathscr{O}^{1|1}$. This identity is equivalent to 4.2.4.1 by substituting $Z - W$ instead of Z and $\mu_W(Z - W) = \mu(Z) - \mu(W)$ instead of $\rho(Z)$. $\square$

Remark 5.1.6. In order to prove a similar statement for a general $N = 1$ supercurve X over Λ, we could define a "super-diagonal" as follows. Recall that any such curve X gives rise to an oriented superconformal $N = 2$ super curve Y (cf. 2.2.12). Recall also that the curve Y comes equipped with two maps $\pi_X : Y \to X$ and $\hat{\pi} : Y \to \hat{X}$, where $\hat{X}$ is the *dual* curve. In local coordinates these maps are described by (cf. 4.3.10)

$$
\begin{aligned}
(z, \theta^+, \theta^-) &\xrightarrow{\pi} \left(z + \frac{1}{2}\theta^+\theta^-, \theta^+ \right), \\
(z, \theta^+, \theta^-) &\xrightarrow{\hat{\pi}} \left(z - \frac{1}{2}\theta^+\theta^-, \theta^- \right)
\end{aligned}
\tag{5.1.6.1}
$$

It is easy to show that Y embedds as a $(1|0)$ codimensional divisor in $X \times_\Lambda \hat{X}$. Indeed, for a Λ-point x in X given by local parameters $Z = (z, \theta)$ the preimage in Y is given by local parameters $(z - \frac{1}{2}\theta\zeta, \theta, \zeta)$. Similarly, for a point $W = (w, \zeta)$ in $\hat{X}$ we have its preimage in Y given by local parameters $(w + \frac{1}{2}\theta\zeta, \theta, \zeta)$. Then the point (Z, W) in $X \times_\Lambda \hat{X}$ is in the image of Y if and only if $z - w - \theta\zeta = 0$. Note in particular that when X is superconformal, namely $X \equiv \hat{X}$ this "diagonal" $Y \hookrightarrow X \times_\Lambda \hat{X}$ agrees with Manin's super-diagonal given in Lemma 5.1.2.

We could try to repeat the argument given above for superconformal curves, but the operation $\mathscr{Y}_2$ turns out to be coordinate-dependent[1].

5.1.7. Instead of using the approach in the previous remark, note that we can define the push-forward Δ_+ and $\Delta_!$ even when Δ is not a divisor. In our case these are easy to describe. Let Δ be the diagonal $\Delta \hookrightarrow X \times_S X$. Even though Δ is not a divisor in $X \times_S X$, it reduction $|\Delta|$ is a divisor in $|X \times_S X| = |X| \times_{|S|} |X|$. We have then an open immersion $j : X \times X \setminus \Delta \hookrightarrow X \times X$, where $X \times X \setminus \Delta$ is $U = |X| \times |X| \setminus |\Delta|$ as a topological space and the structure sheaf is the restriction of $\mathscr{O}_{X^2}$ to U. We can now define the cooresponding push-forwards of an $\mathscr{O}_X$-module $\mathscr{M}$ as:

$$
\begin{aligned}
\Delta_+ \mathscr{M} &= \frac{j_* j^* (\mathscr{O}_X \boxtimes \mathscr{M})}{\mathscr{O}_X \boxtimes \mathscr{M}} \\
\Delta_! \mathscr{M} &= \frac{j_* j^* (\omega \boxtimes \mathscr{M})}{\omega \boxtimes \mathscr{M}}
\end{aligned}
\tag{5.1.7.1}
$$

When no confusion can arise, for any sheaf $\mathscr{F}$, we will denote by $\mathscr{F}(\infty\Delta)$ the sheaf $j_* j^* \mathscr{F}$.

Remark 5.1.8. As in the non-super case, these pushforwards are in fact the push forward of left (resp. right) $\mathscr{D}_X$-modules along the diagonal, where in the superconformal case we understand for a $\mathscr{D}_X$ module, a module over the ring of differential operators preserving the contact structure ω (see also 5.1.13).

5.1.9. We construct now a morphism of sheaves on $D_x \times_\Lambda D_x$, $\mathscr{Y}_{2,x} : j_* j^* (\mathscr{V}_X \boxtimes \mathscr{V}_X) \to$

[1] It will be nice to find a way of describing the vertex algebra multiplication as an expression when a point $x \in X$ "collides" with a point $\hat{x} \in \hat{X}$ along the "diagonal" $\Delta^s \subset X \times \hat{X}$.

$\Delta^s_+\mathscr{V}_X$ by the formula:

$$\mathscr{Y}_{2,x}(f(Z,W)a\boxtimes b) = f(Z,W)\overset{s}{Y}(a, Z-W)b \mod V[[Z,W]]. \tag{5.1.9.1}$$

As in 5.1.5 we have

Theorem 5.1.10. *The map $\mathscr{Y}_{2,x}$ defined by (5.1.9.1) is a well defined map of sheaves on $D_x \times_\Lambda D_x$, i.e. $\mathscr{Y}_{2,x}$ does not depend on the coordinates Z chosen.*

5.1.11. We can now generalize all the results in [16, chapter 18] on chiral algebras without difficulty. For simplicity let us assume that X is a general $1|N$-dimensional supercurve. Suppose that the sheaf $\mathscr{M}$ on X carries a (left) action of the sheaf of differential operators $\mathscr{D}_X$. Let $\sigma_{12} : X^2 \to X^2$ be the transposition of the two factors. We obtain a canonical isomorphism of sheaves $\Delta_+\mathscr{M} \simeq \sigma_{12}^*\Delta_+\mathscr{M}$ given in local coordinates by the formula

$$\frac{1\otimes\psi}{(Z-W)^{k|K}} \mapsto e^{(Z-W)\nabla}\cdot\frac{\psi\otimes 1}{(Z-W)^{k|K}} \mod \mathscr{M}\boxtimes\mathscr{O}_X, \tag{5.1.11.1}$$

Where ψ is a local section of $\mathscr{M}$ and ∇ is the connection that we obtain from the $\mathscr{D}$-module structure in $\mathscr{M}$. When $\mathscr{M}$ carries a *right* action of $\mathscr{D}_X$, we obtain similarly an isomorphism $\Delta_!\mathscr{M} \simeq \sigma_{12}^*\Delta_!\mathscr{M}$. Note that the Berezinian bundle is of rank $(0|1)$ if N is odd, hence in the above formula we need to multiply by $(-1)^{\psi N}$.

Similarly, if X is a superconformal curve and $\mathscr{M}$ carries a (left) action of the sheaf of superconformal differential operators $\mathscr{D}_X$ (cf. 4.2.10), the above formula defines isomorphisms as in the general case.

5.1.12. The Berezinian bundle ω_X is a right D_X-module, the action given by the Lie derivative [13]. Therefore for any left $\mathscr{D}_X$-module $\mathscr{F}$ we obtain a right $\mathscr{D}_X$-module $\mathscr{F}^r := \omega\otimes\mathscr{F}$. This operation establishes an equivalence of categories between left and right $\mathscr{D}_X$-modules [31]. The same results hold for $\mathscr{D}_X$-modules over superconformal curves in the sense of 4.2.10.

Let X be a supercurve, the sheaf $\omega_X \boxtimes \omega_X$ on X^2 is isomorphic to ω_{X^2}. The natural map is expressed in local coordinates as:

$$dZ\boxtimes dW \mapsto [dZdW], \tag{5.1.12.1}$$

where as before dZ denotes the section $[dzd\theta^1\dots d\theta^N]$ of ω_X and $[dZdW]$ denotes the section $[dzdwd\theta^1d\zeta^1\dots d\theta^Nd\zeta^N]$ of ω_{X^2}. We note the skew-symmetry in (5.1.12.1) since (recall the definition of the Berezinian in 2.2.11)

$$dZ\boxtimes dW \mapsto -(-1)^N[dWdZ]. \tag{5.1.12.2}$$

We obtain thus $\Delta_!\omega_Z \simeq \omega_{X^2}(\infty\Delta)/\omega_{X^2}$. Let μ_ω denote the composition of the identification $\omega\boxtimes\omega(\infty\Delta) \simeq \omega_{X^2}(\infty\Delta)$ with the projection onto $\Delta_!\omega_X$. This map is clearly a morphism of right $\mathscr{D}_X$-modules satisfying the skew-symmetry condition:

$$\mu_\omega\circ\sigma_{12} = -\mu_\omega. \tag{5.1.12.3}$$

Note that this formula differs from (5.1.12.2) by a factor $(-1)^N$. Indeed this factor appears when applying σ_{12}, namely the composition in the right hand side of (5.1.12.3) is given by:

$$dZ \boxtimes dW \xrightarrow{\sigma_{12}} (-1)^N dW \boxtimes dZ \xrightarrow{\mu_\omega} (-1)^N [dW dZ] = -[dZ dW] = -\mu_\omega dZ \boxtimes dW. \tag{5.1.12.4}$$

Remark 5.1.13. Let X be a supercurve and $Z \hookrightarrow X$ a closed embedding, We define the functor $\underline{\Gamma}_Z$ from the category of sheaves on X to itself by letting sections of $\underline{\Gamma}_Z(\mathscr{F})$ be sections of $\mathscr{F}$ supported on Z. This functor is left exact. Let $\mathscr{H}^i_Z$ be the higher derived functors. In this sense the basic definitions of local cohomologies in [20] extend in a straightforward way to the super case. Similarly we can define the *relative local cohomologies* as the higher derived functors of $\underline{\Gamma}_{Z/Z'}$ where $Z' \hookrightarrow Z$ is another closed embedding and $\underline{\Gamma}_{Z/Z'}$ is defined in the usual way as the quotient of sections supported in Z modulo those supported in Z' [20]. From the exact sequence

$$0 \to \underline{\Gamma}_Z(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_U) \to \mathscr{H}^1_Z(\mathscr{F}) \to 0, \tag{5.1.13.1}$$

where $U = X \setminus Z$ and $j : U \hookrightarrow X$ is the open immersion, we obtain that

$$\Delta_! \omega_X = \mathscr{H}^1_\Delta(\omega_{X^2}). \tag{5.1.13.2}$$

This identification of sheaves extended by delta functions on the diagonals with local cohomology sheaves shows that indeed these are push-forwards of $\mathscr{D}_X$-modules[2].

5.1.14. We have also a dictionary between $\mathscr{D}_X$-modules and delta functions. The space $\mathbb{C}[[Z^{\pm 1}, W^{\pm 1}]]$ carries a structure of a module over the algebra of differential operators $\mathbb{C}[[Z, W]][\nabla_Z, \nabla_W]$ (here $\nabla_Z = (\partial_z, \partial_{\theta^i})$ in the general case and $\nabla_Z = (\partial_z, D^i_Z)$ in the superconformal case). The formal delta-function $\delta(Z, W)$ satisfies the relations:

$$\begin{aligned} (Z-W)^{1|0}\delta(Z,W) &= 0 \\ (Z-W)^{0|e_i}\delta(Z,W) &= 0 \\ (\nabla_Z + \nabla_W)\cdot\delta(Z,W) &= 0 \end{aligned} \tag{5.1.14.1}$$

Therefore the $\mathbb{C}[[Z, W]][\nabla_Z, \nabla_W]$-submodule of $\mathbb{C}[[Z^{\pm 1}, W^{\pm 1}]]$ generated by $\delta(Z, W)$ is spanned by $\nabla_W^{j|K}\delta(Z, W)$ with $j \geq 0$. This module gives rise to a $\mathscr{D}$-module on the disk $D^2 = \operatorname{Spec}\mathbb{C}[[Z, W]]$ supported on $z = w$ (note that this is also the case in the superconformal case, where the poles are in $z - w - \sum \theta^i \zeta^i$).

The assignment

$$(Z-W)^{-1-j|N\setminus J} dW \mapsto \sigma(J)\partial_W^{(j|J)}\delta(Z, W) \tag{5.1.14.2}$$

induces an isomorphism of left $\mathscr{D}$-modules on D^2 between $\Delta_+\omega$ and the left $\mathscr{D}$-module generated by $\delta(Z, W)$. Similarly, tensoring with ω we obtain an isomorphism of right

[2]Indeed, holonomic $\mathscr{D}_X$-modules.

$\mathscr{D}$-modules. In the superconformal case the situation is analogous, the proof follows from (3.5.2.1).

5.1.15. Recall that from Theorem 4.2.9 and (4.2.10.1), we have a natural (left) action of differential operators on $\mathscr{V}$. It follows then that the push-forward $\Delta_+\mathscr{V}$ is also a (left) $\mathscr{D}$-module. Indeed, the action of vector fields locally is given by ($a \in V$):

$$\partial_z : f(Z,W)a \to (\partial_z f(Z,W))a, \tag{5.1.15.1}$$
$$\partial_w : f(Z,W)a \to (\partial_w f(Z,W))a + f(Z,W)(Ta) \tag{5.1.15.2}$$
$$\partial_{\theta^i} : f(Z,W)a \to (\partial_{\theta^i} f(Z,W))a \tag{5.1.15.3}$$
$$\partial_{\zeta^i} : f(Z,W)a \to (\partial_{\zeta^i} f(Z,W))a + (-1)^f f(Z,W)S^i a, \tag{5.1.15.4}$$

and similarly in the superconformal case.

Also, we obtain a $\mathscr{D}$-module structure on the sheaves $\mathscr{V} \boxtimes \mathscr{V}(\infty\Delta)$ where ∂_{θ^i} acts as $\partial_{\theta^i} + S^i$ and ∂_{ζ^i} acts as $\partial_{\zeta^i} + S^i$.

Proposition 5.1.16. *The map $\mathscr{Y}_{2,x}$ commutes with the action of differential operators on D_x^2, making this map a morphism of $\mathscr{D}$-modules.*

Proof. For a general supercurve X the proof is the same as in the non-super case. We sketch the proof in the superconformal case where a subtlety arises. Let $X = (x, \eta^1, \ldots, \eta^N)$. The identity

$$\overset{s}{Y}(S^i a, Z-W)b = D^i_X \overset{s}{Y}(a,X)b|_{X=Z-W} = D^i_Z \overset{s}{Y}(a, Z-W)b \tag{5.1.16.1}$$

translates into

$$\mathscr{Y}_{2,x}(D^i_Z \cdot f(Z,W)a \boxtimes b) = D^i_Z \cdot \mathscr{Y}_{2,x}(f(Z,W)a \boxtimes b). \tag{5.1.16.2}$$

On the other hand, consider translation invariance:

$$\begin{aligned}
[S^i, \overset{s}{Y}(a, Z-W)]b &= (\partial_{\eta^i} - \eta^i \partial_x)\overset{s}{Y}(a,X)b|_{X=Z-W} \\
&= (-\partial_{\zeta^i} + \theta^i\partial_x - \eta^i\partial_x)\overset{s}{Y}(a, Z-W)b|_{X=Z-W} \\
&= (-\partial_{\zeta^i} - \zeta^i\partial_w)\overset{s}{Y}(a, Z-W)b \\
&= -D^i_W \overset{s}{Y}(a, Z-W)b.
\end{aligned} \tag{5.1.16.3}$$

From where we obtain:

$$\overset{s}{Y}(a, Z-W)S^i b = (-1)^a S^i \overset{s}{Y}(a, Z-W)b + (-1)^a D^i_W \overset{s}{Y}(a, Z-W)b, \tag{5.1.16.4}$$

and this translates into:

$$\mathscr{Y}_{2,x}(D^i_W \cdot f(Z,W)a \boxtimes b) = D^i_W \cdot \mathscr{Y}_{2,x}(f(Z,W)a \boxtimes b). \tag{5.1.16.5}$$

□

Remark 5.1.17. Since $\Delta_+\mathscr{V}$ is supported on the diagonal, we obtain a global version $\mathscr{Y}^s$ of $\mathscr{Y}_{2,x}$ by gluing these morphisms in the diagonal with the zero morphism outside of the diagonal. By the previous proposition, this morphism is a map of $\mathscr{D}$-modules on X^2.

Proposition 5.1.18. *The map $\mathscr{Y}^2 : \mathscr{V} \boxtimes \mathscr{V}(\infty\Delta) \to \Delta_+\mathscr{V}$ satisfies $\mathscr{Y}^2 = \sigma_{12} \circ \mathscr{Y}^2$ under the canonical identification $\Delta_+ \simeq \sigma_{12}^*\Delta_+\mathscr{V}$.*

Proof. From the skew-symmetry property of SUSY vertex algebras (3.3.14.1) it follows:

$$\overset{s}{Y}(a, Z - W)b = (-1)^{ab} e^{(Z-W)\nabla} \overset{s}{Y}(b, W - Z)a \tag{5.1.18.1}$$

and the exponential $e^{(Z-W)\nabla}$ is the coordinate expression for the parallel translation, using the $\mathscr{D}$-module structure on $\mathscr{V}$, from W to Z (see 5.1.11). □

5.1.19. In order to define *chiral algebras* over supercurves, we need to understand the *composition* of morphisms like $\mathscr{Y}^2$. For this we need to understand $\Delta_{123!}\mathscr{A}$ for any right $\mathscr{D}$-module $\mathscr{A}$ over X, where Δ_{123} is the small diagonal in X^3 where the three points collide. As in the non-super case, we can write this as a composition

$$\Delta_{123!}\mathscr{A} \simeq \Delta_{23!}\Delta_!\mathscr{A}. \tag{5.1.19.1}$$

This identity follows from the fact that the push-forward of right $\mathscr{D}$-modules is exact for closed embeddings (cf. [8]).

Now let $\mu : \mathscr{A} \boxtimes \mathscr{A}(\infty\Delta) \to \Delta_!\mathscr{A}$ be a morphism of $\mathscr{D}$-modules on X^2. We define a *composition* of μ:

$$\mu_{1\{23\}} : j_*\mathscr{A} \boxtimes \mathscr{A} \boxtimes \mathscr{A}|_U \to \Delta_{123!}\mathscr{A} \tag{5.1.19.2}$$

where $U = X^3 \setminus \cup\Delta_{ij}$ and $j : U \to X^3$ is the open immersion. In order to define such a composition we first apply μ to the second and third argument, and then we apply μ to the first argument and the result (cf. [16, 18.3.1]). We define other compositions of μ by changing the order in which we group the points. As in [16] we denote these compositions in the following way: given local sections a, b and c of $\mathscr{A}$ and a meromorphic function $f(X, Y, Z)$ with poles along the diagonals, we have:

$$\begin{aligned}
\mu_{1\{23\}}(f(X,Y,Z)a \boxtimes b \boxtimes c) &= \mu(f(X,Y,Z)a \boxtimes \mu(b \boxtimes c)) \\
\mu_{\{12\}3}(f(X,Y,Z)a \boxtimes b \boxtimes c) &= \mu(\mu(f(X,Y,Z)a \boxtimes b) \boxtimes c) \\
\mu_{2\{13\}}(f(X,Y,Z)a \boxtimes b \boxtimes c) &= (-1)^{ab}\sigma_{12} \circ \mu(f(X,Y,Z)b \boxtimes \mu(a \boxtimes c)).
\end{aligned} \tag{5.1.19.3}$$

With these compositions defined, we can now define a *chiral algebra* in the usual way:

Definition 5.1.20. A *chiral algebra* on a $1|N$ dimensional supercurve X is a right $\mathscr{D}$-module $\mathscr{A}$ equipped with a morphism of $\mathscr{D}$-modules:

$$\mu : \mathscr{A} \boxtimes \mathscr{A}(\infty\Delta) \to \Delta_!\mathscr{A} \tag{5.1.20.1}$$

satisfying the following conditions:

- (skew-symmetry) $\mu = -\mu \circ \sigma_{12}$.
- (Jacobi identity) $\mu_{1\{23\}} = \mu_{\{12\}3} + \mu_{2\{13\}}$.
- (Unit) We are given a canonical embedding $\omega_X \hookrightarrow \mathscr{A}$ of the Berezinian bundle compatible with the homomorphism μ_ω defined in 5.1.12.

Remark 5.1.21. Note that this definition is exactly the same as in the non-super case, namely, the signs appearing when anticommuting odd-elements are taken care by the symmetric structure of the category of modules over super-rings. That is, given a super-ring R and two R-modules M and N, the isomorphism $\sigma : M \otimes N \simeq N \otimes M$ is given by:

$$\sigma : \; m \otimes n \mapsto (-1)^{mn} n \otimes m. \tag{5.1.21.1}$$

Indeed the only difference with the non-super case is the fact that the unit ω is a rank $(0|1)$-bundle when N is odd. From the SUSY vertex algebra point of view, this is translated into the fact that the Λ-bracket has parity $N \mod 2$.

In the superconformal case there is a subtlety. We note that the intersection of two different diagonals in the sense of 5.1.2 depends on the diagonals chosen, namely:

$$\Delta^s_{12} \cap \Delta^s_{23} = \Delta^s_{13} \cap \Delta^s_{23}. \tag{5.1.21.2}$$

But despite this fact, the pushforward $\Delta_{123!}$ is still well defined, independent of the composition chosen as in (5.1.19.1).

Using the equivalence between left $\mathscr{D}$-modules and right $\mathscr{D}$-modules, we obtain a right $\mathscr{D}$-module $\mathscr{V}^r = \omega_X \otimes \mathscr{V}$ from any strongly conformal SUSY vertex algebra. Similarly, this sheaf carries a multiplication $\mu = (\mathscr{Y}^2)^r$ obtained from $\mathscr{Y}^2$.

Theorem 5.1.22. *The pair $(\mathscr{V}^r, \mu)$ carries a structure of a chiral algebra over X.*

Proof. The proof of this fact is the same as the proof in the non-super case [16, Thm 18.3.3]. Indeed, this follows by considering the *Cousin resolution* of the Berezinian bundle in X^3 and the corresponding Cousin property of SUSY vertex algebras that we proved in 3.3.13. □

5.2 Conformal blocks

In this section we define the sheaves of coinvariants of SUSY vertex algebras. The treatment follows [16]. In fact, most results carry over without change to our situation. We only mention the major differences.

5.2.1. Recall from Theorem 3.3.17 and its $N_K = N$ analog that the polar part of a SUSY vertex algebra is naturally a conformal algebra. We can consider then the operator $\mathscr{Y}_{x,-}$ which is the polar part of $\mathscr{Y}_x$. The notion of Lie* algebra over a super curve is generalized in a straightforward manner from the non-super case.

5.2.2. Let $\mathscr{A}$ be a right $\mathscr{D}$-module, the de Rham sequence of $\mathscr{A}$ is the sequence:

$$0 \to \mathscr{A} \otimes \mathscr{T} \to \mathscr{A} \to 0 \tag{5.2.2.1}$$

placed in cohomological degrees 0 and -1, where $\mathscr{T}$ is the tangent sheaf of X. In the superconformal case, we do not have an action of the entire tangent sheaf, but we can act by the subsheaf $\mathscr{T}^s$ generated by the derivations D_Z^i (i.e. the subsheaf $\mathscr{T}^1$ of remark 2.2.13 in the $1|1$ dimensional case, and the sheaf $\mathscr{T}' \oplus \mathscr{T}''$ in the $1|2$ dimensional case). We define then *the de Rham sheaf* $h(\mathscr{A})$ of $\mathscr{A}$ as

$$h(\mathscr{A}) = \mathscr{A}/(\mathscr{A} \cdot \mathscr{T}). \tag{5.2.2.2}$$

whereas in the superconformal case we put $h(\mathscr{A}) = \mathscr{A}/(\mathscr{A} \cdot \mathscr{T}^s)$.

Proposition 5.2.3. *Let $(\mathscr{A}, \mu)$ be a chiral algebra. Then*

1. *$h(\mathscr{A})(D_x^\times)$ and $h(\mathscr{A})(\Sigma)$, for any open $x \notin \Sigma \subset X$ are Lie superalgebras, and there is a natural homomorphism of Lie superalgebras $h(\mathscr{A})(\Sigma) \to h(\mathscr{A})(D_x^\times)$.*

2. *$h(\mathscr{A})(D_x^\times)$ acts on the fiber $\mathscr{A}_x$.*

3. *If $(\mathscr{A}, \mu)$ is associated to a SUSY vertex algebra V, then there is a canonical isomorphism $h(\mathscr{A})(D_x^\times) \simeq \mathrm{Lie}'(V)$ (see Theorem 3.4.3 for the definition of $\mathrm{Lie}'(V)$ in the $N_W = N$ case, resp. Theorem 3.5.26 in the superconformal case).*

Proof. We can think of $\mathscr{A} \simeq \omega \otimes \mathscr{A}^l$, where $\mathscr{A}^l$ is a left $\mathscr{D}$-module. Since we can integrate sections of the Berezinian bundle, we see immediately that we have $h(\Delta_! \mathscr{A}) = \Delta_* h(\mathscr{A})$. On the other hand the map $\mu : \mathscr{A} \boxtimes \mathscr{A}(\infty\Delta) \to \Delta_! \mathscr{A}$ induces

$$h(\mu) : h(\mathscr{A}) \boxtimes h(\mathscr{A})(\infty\Delta) \to h(\Delta_! \mathscr{A}). \tag{5.2.3.1}$$

Restricting to regular sections and pulling back along the diagonal we obtain:

$$[\,,\,] : h(\mathscr{A}) \otimes h(\mathscr{A}) \to h(\mathscr{A}). \tag{5.2.3.2}$$

The fact that $[\,,\,]$ satisfies the axioms of a Lie superalgebra follows from the skew-symmetry and Jacobi identity of chiral algebras. The rest of the theorem is proved in the same way as [16, prop 18.4.12].

(3) follows from the definitions, in formulas (3.4.2.1) for the $N_W = N$ case and (3.5.26.1) for the $N_K = N$ case. Indeed, these formulas are the equivalent of the corresponding formulas for the action of vector fields on $\mathscr{A}^l$ as defined in 4.2.9 and in (4.2.10.1). □

Remark 5.2.4. As in the non-super case, for a strongly conformal SUSY vertex algebra V, we have a natural map

$$\mathscr{Y}_x^\vee : \mathscr{V}^r(D_x^\times) \to \mathrm{End}(\mathscr{V}_x) \simeq \mathrm{End}\, \mathscr{V}_x^r, \tag{5.2.4.1}$$

on $D_x^\times$. Namely, given a section $s \in \mathscr{V}^r(D_x^\times)$ we obtain the endomorphism $\mathscr{Y}_x^\vee(s) = \operatorname{res}_X < \mathscr{Y}_x, s >$ on $\mathscr{V}_x$. If s is a total derivative, this residue vanishes and the map $\mathscr{Y}_x^\vee$ factors through $h(\mathscr{V}^r)(D_x^\times)$. The resulting Lie superalgebra homomorphism $h(\mathscr{V}^r)(D_x^\times) \to \operatorname{End}(\mathscr{V}_x^r)$ coincides with the homomorphism of Proposition 5.2.3 (2) and with the homomorphism φ' of Theorem 3.4.3 and 3.5.26.

5.2.5. We can now define the *spaces of coinvariants* for a super vertex algebra. For this let X be a supercurve and $x \in X$ a point. We have a Lie superalgebra $U_\Sigma = h(\mathscr{V}^r)(\Sigma)$, where $\Sigma = X \setminus \{x\}$ and this Lie superalgebra acts in $\mathscr{V}_x$.

Definition 5.2.6. The *space of coinvariants* associated to (V, X, x) is

$$H(V, X, x) = \mathscr{V}_x / (U_\Sigma \cdot \mathscr{V}_x). \tag{5.2.6.1}$$

Remark 5.2.7. The extension of this definition to the multiple point case with arbitrary module insertions is straightforward and we leave it for the reader.

Fix $N \geq 0$. Let $\mathfrak{g}$ be the Lie superalgebra of vector fields on the $1|N$ dimensional punctured superdisk $D^\times$. Let $\mathfrak{g}^\omega$ be the Lie subalgebra of $\mathfrak{g}$ consisting of vector fields preserving the form $\omega = dt + \sum \zeta^i d\zeta^i$. Let $\mathscr{M}_{g,1}$ be the moduli space of smooth $1|N$ dimensional genus g, pointed supercurves (here the genus of a supercurve X is the genus of X_{rd}). Let $\hat{\mathscr{M}}_{g,1}$ be the moduli space of triples (X, x, Z), where $(X, x) \in \mathscr{M}_{g,1}$ and Z is a coordinate system at x. Let $\mathscr{M}^\omega_{g,1}$ and $\hat{\mathscr{M}}^\omega_{g,1}$ be the superconformal analogous.

Theorem 5.2.8 ([35]). *The Lie algebra $\mathfrak{g}$ (resp. $\mathfrak{g}^\omega$) acts (infinitesimally) transitivelly on $\hat{\mathscr{M}}_{g,1}$ (resp. $\hat{\mathscr{M}}^\omega_{g,1}$). This action preserves the fibers of the projection $\hat{\mathscr{M}}_{g,1} \to \mathscr{M}_{g,1}$ (resp. $\hat{\mathscr{M}}^\omega_{g,1} \to \mathscr{M}^\omega_{g,1}$).*

It follows from this theorem, by repeating the *localization* construction in [16, ch. 16] that, given an $N_W = n$ SUSY vertex algebra (resp. an $N_K = n$ SUSY vertex algebra) V, we obtain a left $\mathscr{D}$-module $\Delta(V)$ on $\mathscr{M}_{g,1}$ (resp. $\mathscr{M}^\omega_{g,1}$), whose fiber at (X, x) is the space of coinvariants $H(X, x, V)$.

Appendix A

Representations of $\mathfrak{gl}(1|1)$

Let us pick a basis of $\mathfrak{gl}(1|1)$ such that

$$\begin{aligned} T &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ J &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ Q &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ H &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{1.0.8.1}$$

Then the irreducible representations such that T and J act diagonally are classified by

- $1|0$ or $0|1$ dimensional: these are representation on $\mathbb{C}^{1|0}$ or $\mathbb{C}^{0|1}$ generated by an even (resp. odd) vector $\bar{1} \in \mathbb{C}$ such that in this basis we have $T = Q = H = 0$ and $J = j$ we call these representations $\pi_{\pm}(j)$.

- $1|1$ dimensional: for each numbers $t, j \in \mathbb{C}$ there are two irreducible representations of dimension $1|1$. These are either of highest or lowest weight:

$$T = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \quad J = \begin{pmatrix} j & 0 \\ 0 & j-1 \end{pmatrix} \quad Q = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{1.0.8.2}$$

$$T = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \quad J = \begin{pmatrix} j & 0 \\ 0 & j+1 \end{pmatrix} \quad Q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \tag{1.0.8.3}$$

We note that by taking minus the super transpose we get that the duals of these representations are given (in the dual basis $\{v^*, \omega^*\}$) by

$$T = \begin{pmatrix} -t & 0 \\ 0 & -t \end{pmatrix} \quad J = \begin{pmatrix} -j & 0 \\ 0 & -j+1 \end{pmatrix} \quad Q = \begin{pmatrix} 0 & 0 \\ -t & 0 \end{pmatrix} \quad H = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.0.8.4}$$

$$T = \begin{pmatrix} -t & 0 \\ 0 & -t \end{pmatrix} \quad J = \begin{pmatrix} -j & 0 \\ 0 & -j-1 \end{pmatrix} \quad Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 0 & 0 \\ -t & 0 \end{pmatrix} \tag{1.0.8.5}$$

which in the basis $\{-t^{-1}v, \omega\}$ show that $\pi_\pm(t,j)^\vee \equiv \pi_\mp(-t,-j)$

Finally we note that the parity changed modules are $\Pi\pi_\pm(t,j) = \pi_\mp(t, j \mp 1)$.

On the formal $1|1$ dimensional superdisk with coordinates (z, θ) we have the following realization of these representations. Consider the basis for this Lie algebra $-T = z\partial_z + \theta\partial_\theta$, $J = -\theta\partial_\theta$, $Q = -z\partial_\theta$ and $H = \theta\partial_z$ acting on sections of a vector bundle by the Lie derivative. By analizyng the action of these derivations on the fibers of the corresponding bundles we obtain:

$$\begin{aligned}
\wedge^m\Omega^1 &= \mathrm{Aut}_{\mathrm{D}} \overset{Aut\mathcal{O}}{\times} \pi_+(-m, -m+1) \qquad m \equiv 1(2) \\
\wedge^m\Omega^1 &= \mathrm{Aut}_{\mathrm{D}} \overset{Aut\mathcal{O}}{\times} \pi_-(-m, -m) \qquad m \equiv 0(2) \\
S^m\Omega^1 &= \mathrm{Aut}_{\mathrm{D}} \overset{Aut\mathcal{O}}{\times} \pi_+(-m, 0) \\
\mathrm{Ber}_D &= \mathrm{Aut}_{\mathrm{D}} \overset{Aut\mathcal{O}}{\times} \pi_-(1)
\end{aligned} \tag{1.0.8.6}$$

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