Computational Methods for Higher Real K-Theory with Applications to tmf.

by

Michael Anthony Hill

A.B. Harvard University 2002

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2006

Author ....
Department of Mathematics
May 1st, 2006

Certified by ....
Michael J. Hopkins
Professor of Mathematics, Harvard University
Thesis Supervisor

Accepted by ....
Pavel Etingof
Chairman, Department Committee on Graduate Students

Computational Methods for Higher Real K-Theory with Applications to tmf.

by

Michael Anthony Hill

Submitted to the Department of Mathematics
on May 1st, 2006, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Abstract

We begin by present a new Hopf algebra which can be used to compute the *tmf* homology of a space or spectrum at the prime 3. Generalizing work of Mahowald and Davis, we use this Hopf algebra to compute the *tmf* homology of the classifying space of the symmetric group on three elements. We also discuss the Σ_3 Tate spectrum of *tmf* at the prime 3.

We then build on work of Hopkins and his collaborators, first computing the Adams-Novikov zero line of the homotopy of the spectrum eo_4 at 5 and then generalizing the Hopf algebra for *tmf* to a family of Hopf algebras, one for each spectrum eo_{p-1} at p. Using these, and using a $K(p-1)$-local version, we further generalize the Davis-Mahowald result, computing the eo_{p-1} homology of the cofiber of the transfer map $B\Sigma_p \to S^0$.

We conclude by computing the initial computations needed to understand the homotopy groups of the Hopkins-Miller real K-theory spectra for heights large than $p-1$ at p. The basic computations are supplemented with conjectures as to the collapse of the spectral sequences used herein to compute the homotopy.

Thesis Supervisor: Michael J. Hopkins
Title: Professor of Mathematics, Harvard University

Acknowledgments

I begin by thanking my advisor, Mike Hopkins, for putting up with me for almost 6 years now. From helping me with my undergrad thesis, to helping me decide what to do after graduation, to never suggesting that a project I was interested in was not neat or interesting enough, Mike has been a fantastic advisor from the start. I will certainly miss our meeting come next year.

I next wish to thank my parents, Drs Ellen and Kelly Hill. While they possibly never understood my fascination with all things mathematical, they were nonetheless incredibly supportive of it, driving me to math competitions and classes at LSU when I was younger and spending inordinate sums on math texts. I would not be here without their support.

My dear and incredible friends also deserve recognition. First to my Harvard friends, notably the Amys, the Adams, Fancy, and Cliff. They definitely got me sanely through Harvard. Next comes my new MIT crowd: Teena, V, Mark, Bridget, JNKF, Marco, Chris, Andre, Nora and Francesca. MIT has been a wonderful experience, largely due to you. I wish also to acknowledge the help and support of Haynes Miller who always allowed me to crash other people's meetings with him because I might find it interesting and who carefully read this thesis.

Lastly I come to perhaps the most important acknowledgments: Bridge and Tim. Bridge and I have technically known each other since college, but it was only in graduate school that I learned what a fun and fantastic person she really is. Oddly enough, she accepted my suggestion that we live together, and she has been the best roommate I could ask for.

It is to Tim that I dedicate this thesis. He has been unfailingly supportive, he has been incredibly compassionate during this ordeal, and he has made me blissfully happy these past three years. I would not have lasted here without him.

Contents

List of Figures

Chapter 1

Introduction and Applications

1.1 Introduction

In this thesis, we will develop and analyze various computational tools to better understand the Hopkins-Miller higher real K-theories EO_n. The Hopkins-Miller theorem produces for each finite subgroup G of the extended Morava stabilizer group, $\mathbb{G}_n$, a spectrum $\mathrm{EO}_n(G)$ which sits between the Lubin-Tate spectrum E_n and the $K(n)$-local sphere [18]. These spectra serve as useful approximations to the very complicated $K(n)$-local sphere, and for small values of n, they have been beneficial in producing small resolutions of $L_{K(n)}S^0$, allowing for a relatively complete understanding of the homotopy [6, 13, 16]. However, for $n > 2$, the homotopy groups of EO_n are largely mysterious. One of the goal of this thesis is to provide a complete description of the homotopy ring of EO_4 at the prime 5, indicating how the computations work at larger primes. Building on this, we provide a new Hopf algebra suitable for computing not only the homotopy ring of EO_{p-1} at p, but also the EO_{p-1} homology of any space, knowing only the homology of the space as a comodule over the dual Steenrod algebra.

1.1.1 Chromatic Height 2 and *tmf*

The case of $n = 2$ is well studied. Using the machinery of elliptic curves, Hopkins and his collaborators produced a global spectrum *tmf* that $K(2)$-localizes to EO_2 at 2 and at 3, where in each case, we take a maximal finite subgroup of $\mathbb{G}_2$ which contains a maximal p-subgroup [19]. The spectrum *tmf* has several advantages over the spectra EO_2, in that it is an f.p. spectrum in the sense of Mahowald and Rezk [24] and the homotopy ring is finitely generated over $\mathbb{Z}$. Moreover, the close connection between elliptic curves and *tmf* allows one to show that there is a Hopf algebroid for computing *tmf* homology using the Weierstrass form of an elliptic curve. However, in practice, this is difficult to use at best.

Hopkins and Mahowald showed for *tmf* at 2 there is an Adams spectral sequence for computing *tmf* homology similar to that for *ko*.

Theorem (Hopkins-Mahowald). *There is a spectral sequence of the form*

$$\operatorname{Ext}_{\mathcal{A}(2)_*}\left(\mathbb{F}_2, H_*(X)\right) \Rightarrow tmf_*(X_2^\wedge)$$

for cell complexes X.

For primes bigger than 3, there are similar results, using the splitting

$$tmf_p^\wedge = \bigvee \Sigma^{p(k)} BP\langle 2\rangle,$$

where $p(k)$ and the number of wedge summand are determined by the combinatorics of tmf_*.

Davis and Mahowald have computed $\operatorname{Ext}_{\mathcal{A}(2)_*}$ for a large number of spaces, including truncated projective spaces [8]. At the time, many of these computations were viewed as academic exercises, since Davis and Mahowald thought that there was no spectrum with cohomology $\mathcal{A}//\mathcal{A}(2)$ [10].

The computational machinery established by Davis and Mahowald can also be modified using filtration arguments similar to those of Chapter 2 to prove results similar to the following.

Proposition. *As graded groups and as modules over* $\mathbb{Z}[c_4]$,

$$\pi_*(tmf^{t\Sigma_2}) = \prod \Sigma^{8k} ko_*.$$

The missing piece of the computability puzzle for *tmf* is what happens at the prime 3. The form of the Hopf algebra required for the Adams spectral sequence was conjectured by Henriques and the author and is proved in Chapter 2. Results similar to those of Davis and Mahowald are also proved in Chapter 2, together with a result analogous to the previous proposition.

1.1.2 Height $p-1$ at p

For $n > 2$, there is a geometric model similar to that of elliptic curves which was developed by Hopkins, Mahowald, and Gorbounov. It provides a Hopf algebroid analogous to the Weierstrass Hopf algebroid and will be discussed in Chapter 3. This Hopf algebroid was used by Hopkins to show that the higher Adams-Novikov filtration elements of $\pi_* \mathrm{EO}_{p-1}$ are very simple. Moreover, it can be used to compute the entire Adams-Novikov zero line, producing a complete description of the homotopy algebra. However, this computation is quite lengthy and is worked out in full only for the prime 5 in Chapter 3.

While the Hopkins-Gorbounov-Mahowald Hopf algebroid is useful in proving results about the homotopy of EO_{p-1} and has been used by others to prove results as diverse as the non-existence of certain Smith-Toda complexes [28], it is not well suited to doing actual computations of the EO_{p-1} homology of spaces or spectra. Additionally, the spectra are $K(p-1)$-local, making their homotopy algebras complete local rings. In Chapter 4, we discuss an analogue of *tmf* for height $p-1$ at the prime p.

We then prove results analogous to those of Chapter 2 for both a conjectural connective f. p. spectrum eo_{p-1} and for the non-connective, non-$K(p-1)$-local spectrum $eo_{p-1}[\Delta^{-1}]$. Applications of such a computation are also included, demonstrating the ease of use of the techniques.

1.1.3 Higher Heights

Most of the previous discussion has involved the spectra EO_{p-1} at the prime p. For larger heights divisible by $p-1$, very little is known about the spectra EO_n. The maximal finite subgroups of $\mathbb{G}_n$ are known by a theorem of Hewett [15], but the complexity of the action of $\mathbb{G}_n$ on $\pi_* E_n$ has prevented actual computations. In Chapter 5, we work out some of the higher cohomology of $\mathbb{Z}/p$ with coefficients in a distinguished module, the symmetric powers of a direct sum of copies of the reduced regular representation. Devinatz and Hopkins has shown that as a $\mathbb{Z}/p^k$-module, $\pi_* E_{p^{k-1}(p-1)}$ has a filtration such that the associated graded is essentially the symmetric algebra on the reduced regular representation for this group [12]. Restricting to the copy of $\mathbb{Z}/p$ reduces the computation required to the computation we present. This computation should provide a basis for future work on the higher homotopy of EO_n beyond the current knowledge of EO_{p-1}.

1.2 Applications of the Computations

1.2.1 The *tmf* and EO_{p-1} Hopf Algebras

Mahowald's computation of $ko_*(\mathbb{R}P^\infty)$ has proved useful in a variety of contexts at the prime 2. In particular, Mahowald used $ko_*(\mathbb{R}P^n)$ and $ko_*(\mathbb{R}P^\infty/\mathbb{R}P^k)$ to get information about v_1 metastable homotopy theory in the EHP sequence [23]. Mahowald has also used $ko_*(\mathbb{R}P^\infty)$ to detect elements in his η_j family [22]. At the prime 3, the role of the spectrum *ko* is most naturally played by the spectrum *tmf*. To generalize these results of Mahowald's, the initial piece of data needed is the *tmf* homology of $B\Sigma_3$.

A theorem of Arone and Mahowald shows that v_n periodic information is captured by the first p^n stages of the Goodwillie tower [3]. This recasts Mahowald's result from [23] into a more readily generalizable form. To get v_2 periodic information at the prime 3, the initial data needed comes in part from QS^0 and $Q(B\Sigma_3{}^\nu)$, where $B\Sigma_3{}^\nu$ is a particular Thom spectrum of $B\Sigma_3$. Just as Mahowald uses knowledge of the *ko* homology of stunted projective spaces to reduce the questions involved to ones of J homology, we hope that a similar analysis, using Behrens' $Q(2)$, spectrum will allow an analysis of the v_2 primary Goodwillie tower at 3 [6].

Minami shows that the 3 primary η_j family will be detectable in the Hurewicz image of the *tmf* homology of the n-skeleton of $B\Sigma_3$ for appropriate choices of n [27]. While determining the full Hurewicz image is a trickier task, understanding the groups and simple *tmf* operations on them could help determine if the conjectural η_j elements actually survive at the prime 3.

Minami actually shows that for all primes $p > 2$, the η_j family will be detectable in the Hurewicz image of the eo_{p-1} homology of an appropriate skeleton of $B\Sigma_p$. The computations in Chapter 4 provide a starting point for applying this program.

1.2.2 The Homotopy of eo_4

The computation has two main immediate applications. The first is the interest in its own right: this solves an invariant problem considered "bad" by algebraists in a way that allows similar analysis for other metacyclic groups. The second, perhaps more interesting, application is to the existence of self maps realizing multiplication by v_3^k on the Smith-Toda complex $V(2)$ at the prime 5.

This story has many antecedents. Hopkins and Mahowald used the spectrum *tmf* and computations in its homotopy to correct a result of Davis and Mahowald, showing that the complex $M(1,4)$ at the prime two has a self map that induces v_2^{32} multiplication in $K(2)$-homology [17, 9]. Behrens and Pemmaraju demonstrated the similar results at the prime 3, again using *tmf* to show that $V(1)$ has a self-map inducing multiplication by v_1^9 in $K(1)$-homology [7]. The methods of Chapter 3 lend themselves to computing the eo_4 homology of $V(2)$ at the prime 5. By using tricks similar to those employed by Hopkins-Mahowald and Behrens-Pemmaraju, we should be able to compute the appropriate power of v_3 which exists on $V(2)$ at 5.

Chapter 2

The 3-local *tmf* homology of $B\Sigma_3$

2.1 Organization of Chapter

In §2.2, we introduce the main computational Hopf algebra $\mathcal{A}$, Ext over which is the Adams E_2 term for computing *tmf* homology. In §2.3, we review Mahowald's computation of the *ko* homology of $\mathbb{R}P^\infty$, presenting it in a manner which can be most readily generalized. In §2.4, we carry out one of the computational steps analogous to Mahowald's, computing the *tmf* homology of the cofiber of the transfer map, and in §2.5, we complete the computation of $tmf_*(B\Sigma_3)$. Rounding out the computations, in §2.6, we compute the *tmf* homology of the finite skeleta of R, giving additional results about that of the finite skeleta of $B\Sigma_3$. The last section presents conjectures as to further results. A computation of the homotopy of the Σ_3 Tate spectrum for *tmf* is presented in §2.7.

2.1.1 Conventions and Notation

We restrict attention to the prime 3 and assume that all spaces and spectra are 3-completed except in §2.3. For ease of readability, let H be $H\mathbb{Z}/3$. If X is a space or spectrum, let $X^{[n]}$ denote its n-skeleton.

For ease of readability, we also will write P^∞ for $B\Sigma_3$. If we are dealing with a truncated classifying space with cells between dimensions n and m, we will write the spectrum as P_n^m.

Finally, we need some *tmf* specific notation. To describe it, we begin with a picture of the Adams E_2 term which we will derive in §2.2 in which all of the elements in question will be labeled (Figure 2-1).

Let I denote the ideal of the Adams E_2 term for tmf_* generated by v_0, c_4 and c_6. Let $\bar{I}$ denote the ideal of tmf_* generated by 3, c_4, c_6, and their Δ and Δ^2 translates. I is the annihilator ideal of the elements α and β. For brevity, the reader is asked to always assume the relations $I\alpha = 0$ and $I\beta = 0$ in all Adams E_2 terms, unless explicitly stated otherwise. Moreover, the relation $c_4^3 - c_6^2 = 27\Delta$ always holds and will not be explicitly stated.

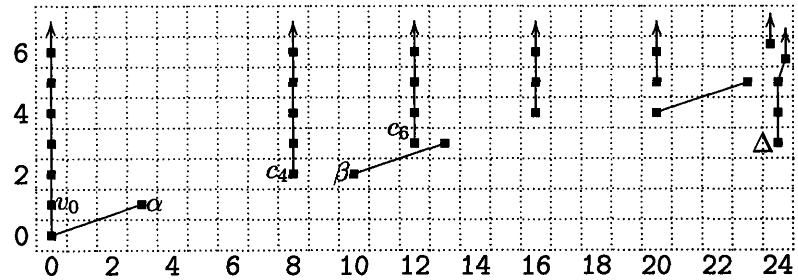

Figure 2-1: The Adams E_2 term for tmf_*

2.2 Fundamental Hopf Algebra

2.2.1 The Adams Spectral Sequence for R-modules

We begin by quickly reviewing the variant of the Adams spectral sequence we will use. Most of the statements are immediately provable using Adams' original work, and full details can be found in [4].

Let R be an E_∞ ring spectrum, and let E be an A_∞ R algebra (ie an A_∞ monoid in the category of R modules). For any R module M, we can cosimplicially resolve M using E in the category of R modules, just as with the ordinary cosimplicial Adams resolution over the sphere spectrum. In other words, we can form the cosimplicial spectrum

$$E^{\wedge_R \bullet} \wedge_R M := E \wedge_R M \rightrightarrows E \wedge_R E \wedge_R M \cdots .$$

The totalization of $E^{\wedge_R \bullet} \wedge_R M$ is the E nilpotent completion of M, $M_E^\wedge$, just as with the ordinary Adams resolution. This cosimplicial resolution gives rise to a Bousfield-Kan spectral sequence of the form

$$Tot\big(\pi_*(E^{\wedge_R \cdot} \wedge_R M)\big) \Rightarrow \pi_*(M_E^\wedge).$$

We again call this spectral sequence the Adams spectral sequence. Again, just as with the ordinary Adams spectral sequence, if we have certain flatness assumptions, then we can identify the E_2 term. To cleanly state the result, we need a small bit of notation: let $E_*^R M$ denote $\pi_*(E \wedge_R M)$.

Proposition 2.2.1. *If $E_*^R E$ is flat as an E_* module, then $(E_*, E_*^R E)$ is a Hopf algebroid and the Adams E_2 term is*

$$\mathrm{Ext}_{(E_*, E_*^R E)}(E_*, E_*^R M).$$

As we shall see, the Hopf algebroid $(E_*, E_*^R E)$ is often quite simple to work with.

2.2.2 The *tmf* Hopf Algebra

We apply the machinery of the previous section to the case $R = tmf$, $E = H$, and $M = tmf \wedge X$. The spectrum H is made into an E_∞ tmf algebra by composing the zeroth Postnikov section of *tmf* with the reduction modulo 3. In other words, we take the composite

$$tmf \to H\mathbb{Z} \to H.$$

Since each of these is a map of E_∞ ring spectra, the composite is. Moreover, since every module is flat over H_*, we need only identify

$$\mathcal{A} := H_*^{tmf} H.$$

Theorem 2.2.2 (Henriques-Hill). *As a Hopf algebra,*

$$\mathcal{A} = \mathcal{A}(1)_* \otimes E(a_2),$$

where $|a_2| = 9$, *and* $\mathcal{A}(1)_* = \mathbb{F}_3[\xi_1]/\xi_1^3 \otimes E(\tau_0, \tau_1)$ *is dual to the subalgebra of the Steenrod algebra generated by* β *and* $\mathcal{P}^1$*. The elements in* $\mathcal{A}(1)_*$ *have their usual coproducts, and*

$$\Delta(a_2) = 1 \otimes a_2 + \xi_1 \otimes \tau_1 - \xi_1^2 \otimes \tau_0 + a_2 \otimes 1.$$

Proof. We begin with an observation of Hopkins and Mahowald, as formulated by Behrens [6]. If we let

$$C = S^0 \cup_{\alpha_1} e^4 \cup_{\alpha_1} e^8,$$

then smashing with *tmf* gives

$$tmf \wedge C = tmf_0(2),$$

where $tmf_0(2)$ is an E_∞ ring spectrum corresponding to elliptic curves together with a choice of an order 2 subgroup. As an algebra,

$$\pi_*\big(tmf_0(2)\big) = \mathbb{Z}_3[a_2, a_4],$$

where $a_2 = v_1$ and $a_4^2 = v_2$ modulo $(3, v_1)$ [6]. The ideal $(3, a_2, a_4)$ is a regular ideal, and we can pass to the quotient of $tmf_0(2)$ by it in an A_∞ way, realizing H as a $tmf_0(2)$ spectrum [2].

Spelled out more cleanly, we have realized H as the cofiber of the map a_4 on the spectrum $tmf_0(2) \wedge V(1)$.

To finish the proof, we smash this cofiber sequence with H over *tmf*, giving the cofiber sequence

$$\Sigma^8 H \wedge_{tmf} \big(tmf_0(2) \wedge V(1)\big) \xrightarrow{a_4} H \wedge_{tmf} \big(tmf_0(2) \wedge V(1)\big) \to H \wedge_{tmf} H.$$

We begin by analyzing the homotopy of the first two *tmf* modules in this resolution:

$$\pi_*\Big(H \wedge_{tmf} \big(tmf_0(2) \wedge V(1)\big)\Big) = H_*\big(C \wedge V(1); \mathbb{Z}/3\big).$$

The structure of this as a graded vector space is that of $\mathcal{A}(1)_*$. Since $\mathcal{A}$ is a commutative Hopf algebra, the Borel classification of Hopf algebras over a finite field ensures both that a_4 is zero in homology and that the structure of this as an algebra is as listed [26]. This follows from considering the degrees of the elements, since odd elements must be exterior classes and the element in degree 4 must be the generator of a truncated polynomial algebra.

Since the unit map $S^0 \to tmf$ is a 6-equivalence, the natural map

$$H \wedge_{S^0} H \to H \wedge_{tmf} H$$

is a 6-equivalence. This implies that the induced map in homotopy is a Hopf algebra isomorphism in the same range, and this gives the coproducts on the elements τ_0, τ_1 and ξ.

To determine the coproduct on a_2, we will endow $\mathcal{A}$ with a filtration such that a_2 is primitive in the associated graded. This filtration gives rise to a spectral sequence

$$\mathrm{Ext}_{Gr(\mathcal{A})}(\mathbb{F}_3, \mathbb{F}_3) \Rightarrow \mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_3, \mathbb{F}_3)$$

converging to the E_2 term of the Adams spectral sequence which computes $\pi_*(tmf)$. We shall use the known computation of $\pi_*(tmf)$ to deduce differentials in this algebraic spectral sequence, and this will determine the coproduct on a_2.

We first filter $\mathcal{A}$ by letting $\mathcal{A}(1)_*$ have filtration 0 and letting a_2 have filtration 1. The initial piece of data needed is the cohomology of $\mathcal{A}(1)_*$. As an algebra

$$\mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_3, \mathbb{F}_3) = \mathbb{F}_3[v_0, v_1^3, \beta] \otimes E(\alpha_1, \alpha_2)/\big(v_0\alpha_1 = v_0\alpha_2 = 0,\ \alpha_1\alpha_2 = v_0\beta\big).$$

This is pictorially represented in Figure 2-2.

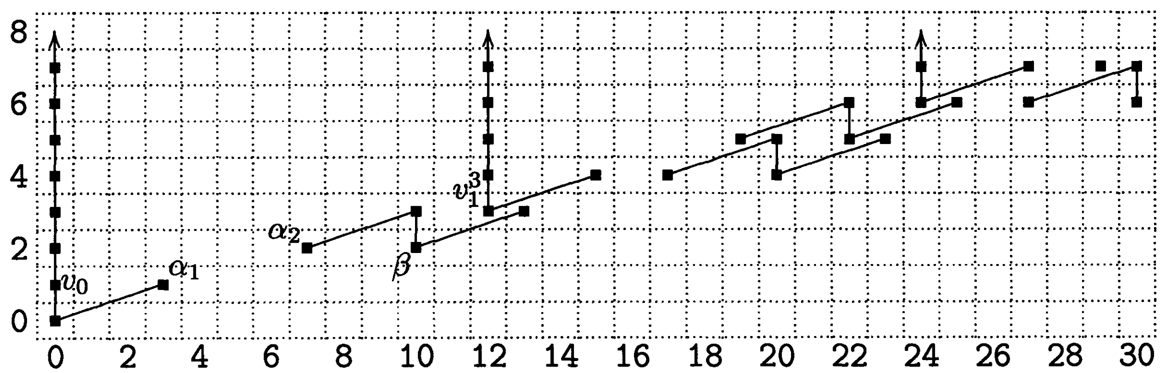

Figure 2-2: $\mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_3, \mathbb{F}_3)$

Since a_2 is primitive in the associated graded Hopf algebra, we know that

$$\mathrm{Ext}_{Gr(\mathcal{A})}(\mathbb{F}_3, \mathbb{F}_3) = \mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_3, \mathbb{F}_3)[\tilde{c}_4].$$

This Ext group is the E_1 page of a spectral sequence converging to the Adams E_2 term $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_3, \mathbb{F}_3)$. Since there is nothing in dimension 7 in tmf_*, we know that the element α_2 must be killed. The only possible way for to achieve this is for $d_1(\tilde{c}_4) = \alpha_2$. This E_1 page is given together with this necessary d_1 differential in Figure 2-3.

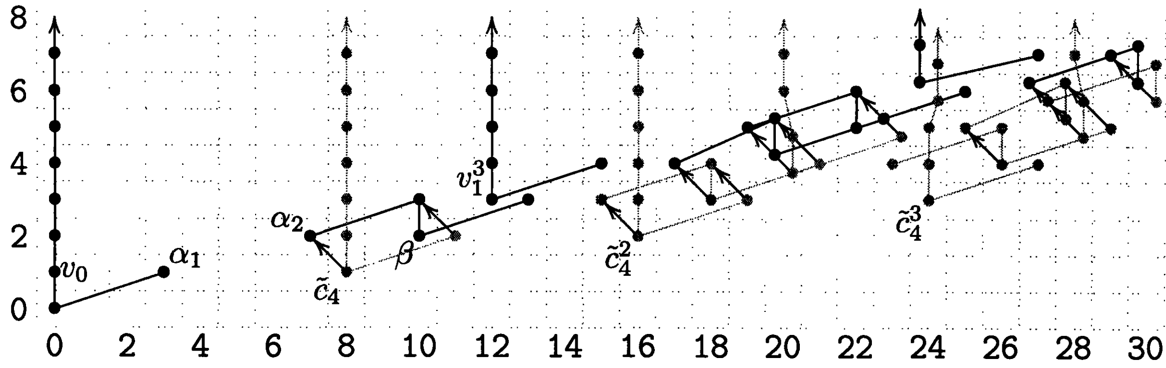

Figure 2-3: $\mathrm{Ext}_{Gr(\mathcal{A})}(\mathbb{F}_3, \mathbb{F}_3)$

At this point, we rename some of the remaining elements:

$$c_4 = v_0 \tilde{c}_4, \quad c_6 = v_1^3, \quad \Delta = \tilde{c}_4^3.$$

Lemma 6.2.1 gives the d_2 differentials:

$$d_2([\alpha_2 \tilde{c}_4^2]) = v_1^3 \beta, \text{ and } d_2([v_0 \tilde{c}_4^2]) = v_1^3 \alpha.$$

The E_2 page with the d_2 differentials is included as Figure 2-4.

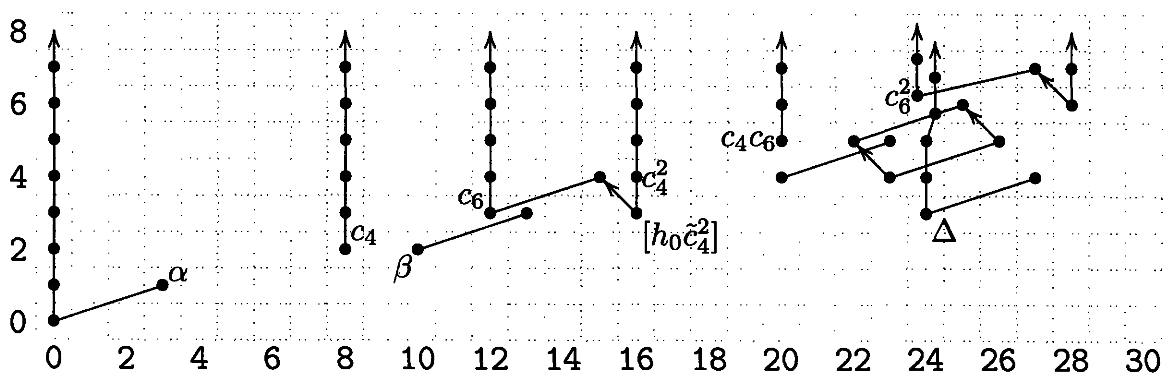

Figure 2-4: May E_2 page for $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_3, \mathbb{F}_3)$

For the d_1 to have the appropriate form, we must have

$$\psi(a_2) = 1 \otimes a_2 + a_2 \otimes 1 \pm (\xi_1 \otimes \tau_1 - \xi_1^2 \otimes \tau_0).$$

If the sign is negative, then we can simply replace a_2 by $-a_2$ to correct this. □

One can ask if there is a formal group interpretation to the Hopf algebra given in Theorem 2.2.2, similar to the interpretation of the Steenrod algebra as the automorphisms of the super additive formal group. This seems to be the case. If E is an elliptic spectrum, then the homotopy groups of $E \wedge_{tmf} E$ are the automorphisms of the formal group of E that extend to automorphisms of the associated elliptic curve. For the case $E = H$, we can proceed only by analogy, since the additive elliptic curve is not in the moduli stack used in the construction of *tmf*. However, if we consider the automorphisms of the additive formal group which extend to automorphisms of the additive elliptic curve, then we reconstruct the truncated polynomial part of Theorem 2.2.2. We conjecture that a full results can be recovered by considering super formal groups and super elliptic curves.

Corollary 2.2.3. *There is a spectral sequence with E_2 term*

$$\mathrm{Ext}_{\mathcal{A}}\big(\mathbb{F}_3, H_*(X)\big)$$

converging to the 3*-completed tmf homology of a space or spectrum* X.

2.3 Review of $ko_*(\mathbb{R}P^\infty)$

In [21], Mahowald uses the homology of cofiber R of the transfer map $B\Sigma_2 \to S^0$ to compute its *ko* homology and the *ko* homology of $\mathbb{R}P^\infty$. Since the method we will employ to handle $tmf_*(B\Sigma_3)$ is similar, we quickly review Mahowald's technique here. For this section only, all computations will be done at the prime 2.

2.3.1 General Results and Definitions

The homology of R sits as an extension of the homology of $\Sigma\mathbb{R}P^\infty$ by the homology of S^0, and let e_i denote the generator of $H_i(R)$. The coaction of the dual Steenrod algebra on $H_*(R)$ is determined by the comodule structure on $H_*(\Sigma\mathbb{R}P^\infty)$ and the coaction formula

$$\psi(e_2) = \xi_1^2 \otimes e_0 + 1 \otimes e_2.$$

Let $A(1)$ be a spectrum whose cohomology is a free $\mathcal{A}(1)$-module of rank 1. Smashing $A(1)$ with *ko* gives a presentation of $H\mathbb{F}_2$ as a *ko*-module spectrum. Applying the Adams spectral sequence machinery introduced earlier reestablishes the following classical result, normally proved using a change of rings argument.

Proposition 2.3.1. *There is a spectral sequence converging to the ko homology of a space X with E_2 term* $\mathrm{Ext}_{\mathcal{A}(1)_*}\big(\mathbb{F}_2, H_*(X)\big)$.

2.3.2 The *ko* homology of R

Mahowald's key observation was that there is a filtration of $H_*(R)$ such that the associated graded is a sum of comodules over $\mathcal{A}(1)_*$ whose Ext groups are easy to compute.

Proposition 2.3.2. *There is a filtration of $H_*(R)$ such that the associated graded is*

$$Gr = Gr\big(H_*(R)\big) = \bigoplus_{k=0}^{\infty} \Sigma^{4k} M,$$

where M is the $\mathcal{A}(1)_*$ comodule $\mathcal{A}(1)_*\Box_{\mathcal{A}(0)_*}\mathbb{F}_2$.

The proposition shows that if we compute Ext of Gr, then we see that it is torsion free, with a $\mathbb{Z}$ in dimensions congruent to $0 \mod 4$ (Figure 2-5).

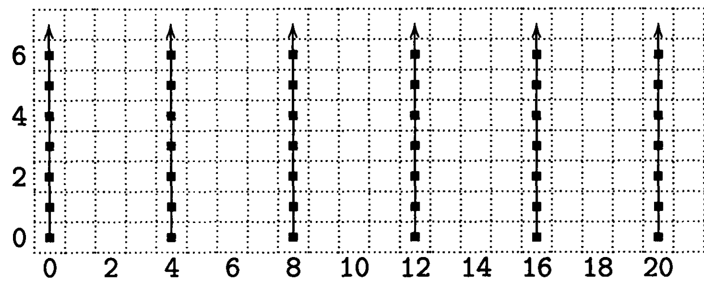

Figure 2-5: $\mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_2, Gr)$

Since this is concentrated in even degrees, both the algebraic extension spectral sequence and the Adams spectral sequence collapse. There are non-trivial extensions, though, as a ko_*-module.

Lemma 2.3.3. *As a module over ko_*,*

$$ko_*(R) = \mathbb{Z}_2\left[\frac{v_1^2}{4}\right].$$

Remark. This lemma shows that Mahowald and Davis' result in [11] that $ko \wedge R$ splits as a wedge of copies of $H\mathbb{Z}$ is not true in the category of ko-module spectra.

2.3.3 Computing $ko_*(\mathbb{R}P^\infty)$

Computing $ko_*(\mathbb{R}P^\infty)$ requires looking at the long exact sequence in ko homology for the cofiber sequence

$$S^0 \to R \to \Sigma\mathbb{R}P^\infty.$$

The first map is the inclusion of the zero cell, and takes 1 to 1. From this, the result is easily determined (Figure 2-6).

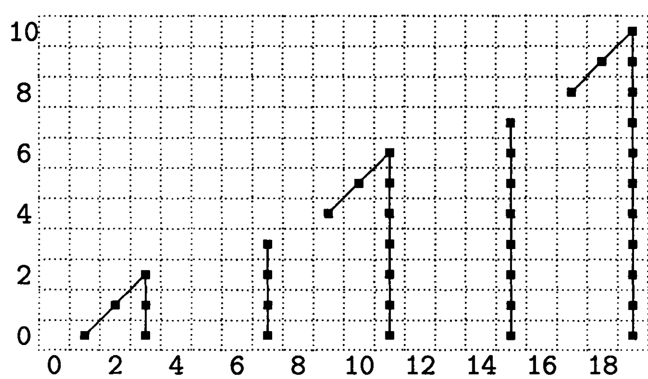

Figure 2-6: $ko_*(\mathbb{R}P^\infty)$

2.4 The *tmf* Homology of the Cofiber of the Transfer $P^\infty \to S^0$

Homologically, the situation at the prime 3 is analogous to the computation at 2. Let R denote the cofiber of the transfer map $P^\infty \to S^0$. The homology of R sits as an extension of the homology of ΣP^∞ by the homology of S^0, and again let e_i denote the generator of $H_i(R)$. The coaction of the dual Steenrod algebra on $H_*(R)$ is determined by the comodule structure on $H_*(\Sigma P^\infty)$ and the coaction formula

$$\psi(e_4) = -\xi_1 \otimes e_0 + 1 \otimes e_4.$$

The *tmf* analogue M is again the comodule $\mathcal{A}(1)_* \Box_{\mathcal{A}(0)_*} \mathbb{F}_3$, where $\mathcal{A}(0)$ is the exterior algebra on the Bockstein.

Lemma 2.4.1. *$H_*(R)$ admits a filtration for which the associated graded is*

$$Gr\big(H_*(R)\big) = \bigoplus_{k=0}^{\infty} \Sigma^{12k} M.$$

Proof. The $-k^{\text{th}}$ stage of the filtration is given by taking the subcomodule generated by the classes in dimensions $12n+1$ for all $n > k$. An elementary computation in the cohomology of the symmetric group shows that the associated graded is exactly what is claimed. $\square$

Lemma 2.4.2.

$$\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_3, M) = \mathbb{F}_3\left[v_0, \tilde{c_4}\right].$$

Proof. To prove this lemma we apply a long sequence of spectral sequences. First filter $\mathcal{A}$ as before by letting $\mathcal{A}(1)_*$ have filtration 0 and a_2 have filtration 1. This filtration extends to a filtration of M in an obvious way, letting M have filtration 0, and we have a spectral sequence

$$\mathrm{Ext}_{Gr(\mathcal{A})}(\mathbb{F}_3, M) \Rightarrow \mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_3, M).$$

As a Hopf algebra, $Gr(\mathcal{A})$ is very simple: the algebra structure stays the same, and now a_2 is primitive. Now we can use the two short exact sequences of Hopf algebras

$$\mathcal{A}(1)_* \to Gr(\mathcal{A}) \to E(a_2) \quad \text{and} \quad E(a_2) \to Gr(\mathcal{A}) \to \mathcal{A}(1)_*$$

to get a spectral sequence that converges to this Ext group and starts with

$$\mathrm{Ext}_{E(a_2)}\big(\mathbb{F}_3, \mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_3, M)\big).$$

A final change of rings argument shows that

$$\mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_3, M) = \mathrm{Ext}_{\mathcal{A}(0)_*}(\mathbb{F}_3, \mathbb{F}_3) = \mathbb{F}_3[v_0],$$

and this forces the result in question, since the target of any differential on the polynomial generator is zero for degree reasons. □

Since this algebra is concentrated in even degrees and since each of the graded pieces starts an even number of steps apart, the spectral sequence starting with Ext of the associated graded for $H_*(R)$ collapses (Figure 2-7). There are non-trivial extensions.

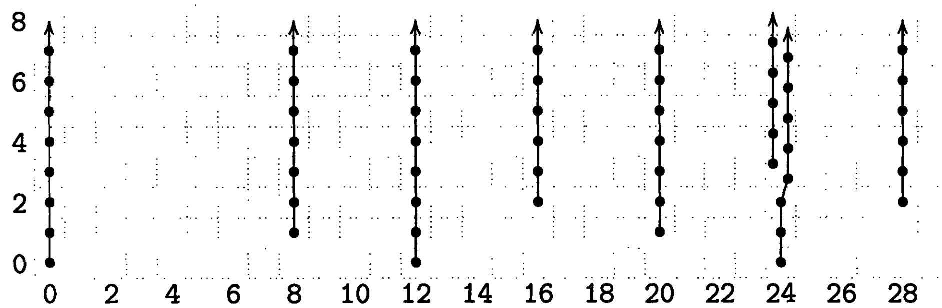

Figure 2-7: $\mathrm{Ext}_{\mathcal{A}}\left(\mathbb{F}_3, H_*(R)\right)$

Lemma 2.4.3.

$$\mathrm{Ext}_{\mathcal{A}}\left(\mathbb{F}_3, H_*(R)\right) = \bigoplus_{k=0}^{\infty} \mathbb{F}_3\left[v_0, \tilde{c_4}\right] e_{12k}/c_6 e_{12k} = v_0^3 e_{12(k+1)}.$$

Proof. We show this by returning to the cobar complex. Since the homology of R has the very simple pattern of copies of M connected by a τ_0 comultiplication on the top class in each hitting the bottom class in the next, it will suffice to show that in the first copy, c_6 on the 0 cell is cohomologous to 27 on the 12 cell.

For simplicity, we will let i_n denote the class in dimension n in M. In the cobar complex for $\mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_3, M)$, there is an element x_{16} such that

$$x_{16} = \tau_0 \otimes \tau_0 \otimes i_{13} + \dots \text{ and } d(x_{16}) = c_6 \otimes i_0.$$

The class x_{16} can be found by considering the Ext implications of the short exact sequence of comodules:

$$\mathbb{F}_3\{i_0, i_4, i_8\} \to M \to \mathbb{F}_3\{i_5, i_9, i_{13}\}.$$

When we add in the next copy of M, we change the coproduct on i_{13} to

$$\psi(i_{13}) = \left(\xi_1 \otimes i_9 + \xi_1^2 \otimes i_5 + \tau_1 \otimes i_8 + \xi_1\tau_1 \otimes i_4 + \xi_1^2\tau_1 \otimes i_0 + 1 \otimes i_{13}\right) + \tau_0 \otimes i_{12}.$$

This is the only change to the coproducts in our comodule, so when we consider again x_{16} and take its boundary, the only change is the addition of terms coming from this new term in the coproduct. However, the only instance of i_{13} in x_{16} is the one coming from $\tau_0 \otimes \tau_0 \otimes i_{13}$, so the real boundary is

$$d(x_{16}) = c_6 \otimes i_0 + \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes i_{12}.$$

In other words, c_6 on the base class is (up to a sign) v_0^3 times the class in dimension 12. □

Theorem 2.4.4. *The Adams spectral sequence for the tmf homology of R collapses, and as a tmf_*-module,*

$$tmf_*(R) = \mathbb{Z}_3\left[\frac{c_4}{3}, \frac{c_6}{27}\right].$$

Proof. The Adams E_2 term is concentrated in even topological degrees, and this implies the collapse of the Adams spectral sequence. The previous lemma solved the extension problem, and the proof of Theorem 2.2.2 shows that $\tilde{c}_4$ gives the element $\frac{c_4}{3}$. □

2.5 The *tmf* Homology of P^∞

The most difficult of the computations now behind us, we can compute the *tmf* homology of P^∞ by simply considering the long exact sequence induced by applying tmf_* to the cofiber sequence

$$S^0 \to R \to \Sigma P^\infty.$$

The first map is the inclusion of the zero cell into R, and so this map in *tmf*-homology just takes 1 to 1. Since this is a map of tmf_*-modules, we see immediately that this map is injective on elements of Adams-Novikov filtration 0, with image

$$\mathbb{Z}_3[c_4, c_6, [3\Delta], [3\Delta^2], [c_4\Delta], [c_4\Delta^2], [c_6\Delta], [c_6\Delta^2], \Delta^3]/(27\Delta = c_4^3 - c_6^2) \subset \mathbb{Z}_3[\frac{c_4}{3}, \frac{c_6}{27}].$$

Additionally, since α and β act as zero on all of the classes in $tmf_*(R)$, the kernel of this first map is the submodule of tmf_* generated by α, β and their Δ translates. These together establish the following theorem about the *tmf* homology of ΣP^∞.

Theorem 2.5.1. *The tmf homology ΣP^∞ sits in a short exact sequence*

$$0 \to G_n \to tmf_n(\Sigma P^\infty) \to \widehat{tmf}_{n-1} \to 0,$$

where $\widehat{tmf}_{n-1}$ is the subgroup of tmf_{n-1} of Adams-Novikov filtration at least 1 and G_n, the cokernel of the map $tmf_n \to tmf_n(R)$, is given by

$$G_{24k+12j+8i} = \begin{cases} \mathbb{Z}/3 \oplus \bigoplus_{m=1}^{k} \mathbb{Z}/3^{6m} & k \equiv 1,2 \bmod 3, i+j=0 \\ \bigoplus_{m=0}^{k} \mathbb{Z}/3^{6m+3j+i} & k \equiv 0 \bmod 3 \\ \bigoplus_{m=0}^{k} \mathbb{Z}/3^{6m+3j+i} & k \equiv 1,2 \bmod 3, i+j>0 \\ 0 & \textit{otherwise} \end{cases},$$

where $j < 2$, and $i < 3$. The sequence is split as a sequence of groups. There is a hidden α extension originating on the copy of β^2 in $\widehat{tmf}_{20}$ and hitting the $\mathbb{Z}/3$ summand of G_{24}.

Proof. This short exact sequence is just a restatement of the earlier comments about the long exact sequence in *tmf* homology. It is split because the elements coming from G_n have Adams-Novikov filtration 0, and the convergence of the Adams-Novikov spectral sequence ensures a map of groups from $tmf_*(\Sigma P^\infty)$ to G_n which is a left inverse to this inclusion.

The structure of the groups G_n is easy to show. A basis for $tmf_*(R)$ is given by the collection of monomials of the form $\Delta^k \tilde{c}_6^j \tilde{c}_4^i$, where $i < 3$, and $27\tilde{c}_6 = c_6$, $3\tilde{c}_4 = c_4$. This is simply because if we can solve the relation on Δ in $tmf_*(R)$. A basis for the Adams-Novikov filtration 0 subring of tmf_* is given by the monomials

$$\Delta^k c_6^j c_4^i \text{ for } k \equiv 0 \mod 3 \text{ or } k \equiv 1,2 \mod 3, i+j>0, \quad [3\Delta]\Delta^k, \text{ and } [3\Delta^2]\Delta^k.$$

Recalling that

$$\Delta^k c_6^j c_4^i = 3^{3j+i} \Delta^k \tilde{c}_6^j \tilde{c}_4^i$$

and collecting all terms of the same degree yields G_n.

The hidden extension can most readily been seen by considering the long exact sequence in Ext induced by the cofiber sequence. In this situation, Δ from the ground sphere kills Δ in the Adams E_2 term for $tmf_*(R)$, and $\alpha\beta^2$ on the ground sphere survives. □

Remark. The proof of this theorem also shows that the transfer induces a bijection between the elements of higher Adams-Novikov filtration of tmf_* and the elements of $tmf_*(P^\infty)$ of Adams-Novikov filtration at least one (together with the $\mathbb{Z}/3$ coming from the 3-cell). This exactly repeats the situation at the prime 2, where the transfer maps the higher Adams-Novikov elements in $ko_*(\mathbb{R}P^\infty)$ bijectively onto those in ko_*.

2.6 The *tmf* Homology of the Finite Skeleta of R

For completeness, we include the tmf-homology of the finite skeleta of R. These computations serve as starting points for the program of Minami to detect the 3-primary η_j family [27].

2.6.1 The Skeleta of R

Let $n = 12k + i$, for $0 < i \leq 12$. We wish to compute the *tmf*-homology of $R^{[n]}$.

Lemma 2.6.1. *There is a filtration of $H_*(R^{[12k+i]})$ such that the associated graded is*

$$Gr\big(H_*(R^{[12k+i]})\big) = \left(\bigoplus_{n=0}^{k-1} \Sigma^{12n} M\right) \oplus \Sigma^{12k} M_i,$$

where M_i is the subcomodule of M generated by all classes of degree at most i for $i < 12$, and M_{12} is M_9 plus a primitive class in dimension 12.

Proof. The required filtration is just the restriction of the filtration used in the proof of Lemma 2.4.1 to the subcomodule $H_*(R^{[12k+i]})$. □

The comodules M_i are the homology of $R^{[i]}$, and this splitting result and the follow theorem demonstrates that knowing their *tmf*-homology gives that of all finite skeleta. The proof of Theorem 2.4.4 shows the following

Theorem 2.6.2. *As a module over tmf_*,*

$$tmf_*(R^{[12k+i]}) = \mathbb{Z}_3\left[\frac{c_4}{3}\right]\{e_0, e_{12}, \ldots, e_{12(k-1)}\} \oplus \widetilde{M_i}e_{12k}/(c_6e_{12j} - 27e_{12(j+1)}),$$

where $\widetilde{M_i}$ is the tmf-homology of spectrum $R^{[i]}$.

The remainder of the section will be spent computing the modules $\widetilde{M_i}$. To save space, in what follows we use two indices: δ which ranges from 0 to 2 and ϵ which ranges from 0 to 1. When these appear, it means that all possible values of the index are actually present.

Proposition 2.6.3. *The spectra $R^{[1]}$, $R^{[2]}$, and $R^{[3]}$ are simply S^0. This implies that*

$$\widetilde{M_i} = tmf_*, \quad 1 \le i \le 3.$$

Lemma 2.6.4. *The spectrum $R^{[4]}$ is the cofiber of α_1. The tmf-homology of this is the extension of the module generated by $[\Delta^\epsilon e_0]$ and $[\alpha e_4]$ and subject to the relations*

$$\alpha[\alpha e_4] = \beta e_0,\ \alpha[\Delta e_0] = \beta^2[\alpha e_4],\ \alpha e_0 = \beta^3[\Delta^\epsilon e_0] = I[\alpha e_4] = \beta^4[\alpha e_4]$$

by the module

$$\mathbb{Z}_3[c_4, c_6, \Delta]\{[3e_4], [c_4e_4], [c_6e_4]\}.$$

The extension is determined by the two relations

$$c_4[3e_4] = 3[c_4e_4] \pm c_6e_0, \quad c_6[3e_4] = 3[c_6e_4] \pm c_4^2e_0.$$

Proof. Since the spectrum M_4 is the cone on α_1, we can use the long exact sequence in Ext to compute the Adams E_2 term (Figure 2-8).

As a module over the Adams E_2 term for tmf_*, this E_2 term is the extension of

$$\mathbb{F}_3[v_0, c_4, c_6, \Delta, \beta]\{e_0\}$$

by

$$\mathbb{F}_3[v_0, c_4, c_6, \Delta]\{[v_0e_4], [c_4e_4], [c_6e_4]\} \oplus \mathbb{F}_3[\Delta, \beta]\{[\alpha e_4]\},$$

subject to the relations

$$c_4[v_0e_4] = v_0[c_4e_4] \pm c_6e_0, \quad c_6[v_0e_4] = v_0[c_6e_4] \pm c_4^2e_0, \quad \alpha[\alpha e_4] = \beta,$$

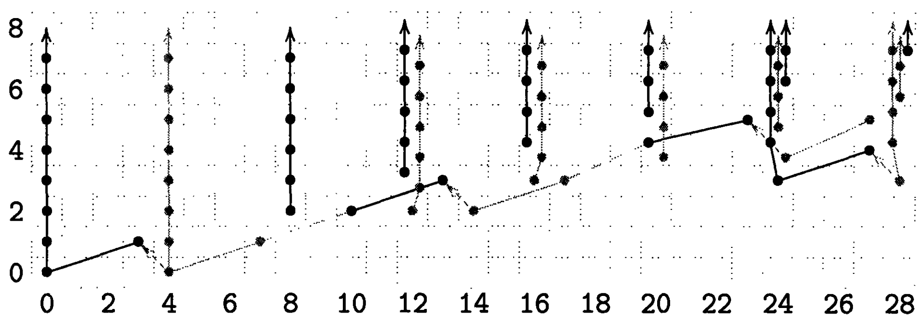

Figure 2-8: The Long Exact Sequence for $\mathrm{Ext}(M_4)$

and depicted in Figure 2-9.

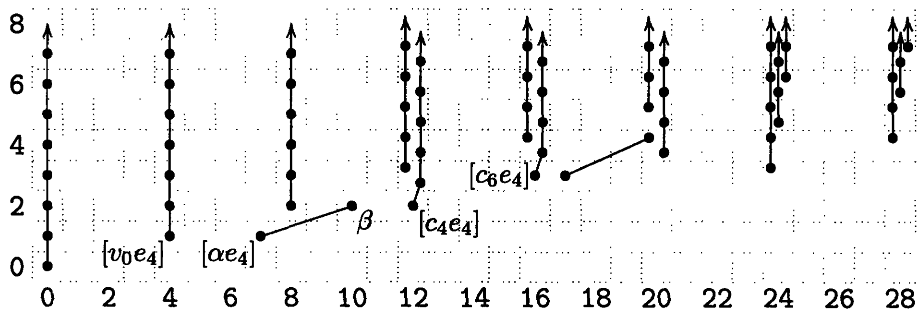

Figure 2-9: The Adams E_2 term for $tmf_*(R^{[4]})$

This Adams spectral sequence is a spectral module over the Adams spectral sequence for the *tmf*-homology of the sphere, and the two differentials in the Adams spectral sequence for the sphere,

$$d_2(\Delta) = \alpha\beta^2, \quad d_3([\alpha\Delta^2]) = \beta^5,$$

imply that Δe_0 and $\Delta^2 e_0$ are d_2 cycles and that the following differentials hold:

$$d_2(\Delta[\alpha e_4]) = \beta^3 e_0, \quad d_3(\alpha\Delta^2[\alpha e_4]) = \beta^5[\alpha e_4].$$

This last d_3 implies that in fact,

$$d_3(\Delta^2 e_0) = \beta^4[\alpha e_4],$$

using the relation involving α multiplication on $[\alpha e_4]$. □

Lemma 2.6.5. *The spectra $R^{[5]}$, $R^{[6]}$, and $R^{[7]}$ are the cofiber of the extension of α over the mod 3 Moore spectrum. The tmf-homology of these spectra, $\widetilde{M_i}$ is the tmf_* module generated by*

$$[\tfrac{c_4}{3}\Delta^\delta e_0],\ [\tfrac{c_6}{3}\Delta^\delta e_0],\ [\Delta^\epsilon e_0],\ [\alpha e_4],\ [\beta e_5],$$

and subject to the relations

$$\alpha[\beta e_5] = \beta[\tfrac{c_4}{3}e_0],\ \alpha[\alpha e_4] = \beta e_0,\ \alpha[\Delta e_0] = \beta^2[\alpha e_4],$$
$$(\alpha, \beta^3)e_0 = I([\alpha e_4], [\beta e_5]) = \beta^4[\alpha e_4] = 0.$$

Proof. In the long exact sequence in Ext induced by the inclusion of the 4-skeleton into $R^{[5]}$, the inclusion of the 5-cell kills the element $[v_0 e_4]$ (Figure 2-10).

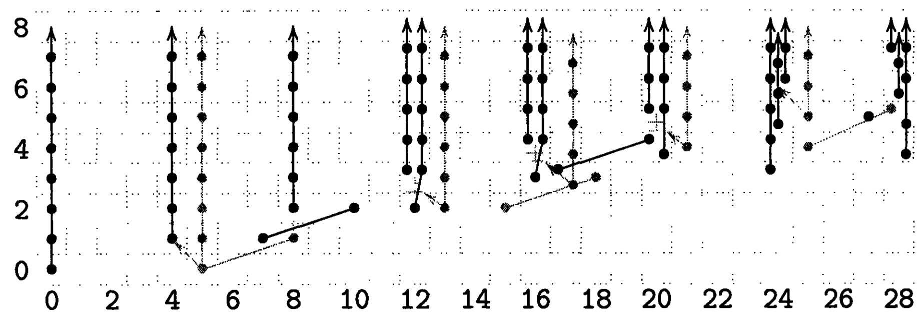

Figure 2-10: The Long Exact Sequence for $\mathrm{Ext}(M_5)$

The elements $[c_4 e_4]$ and $[c_6 e_4]$ survive, and the relations in the Ext term for the 4-skeleton ensure that in the Adams E_2 term for $\widetilde{M}_5$,

$$v_0[c_4e_4] = c_6e_0, \quad v_0[c_6e_4] = c_4^2e_0.$$

Moreover, since α and β multiplications on the class $[v_0e_4]$ are trivial, the classes $[\alpha e_5]$ and $[\beta e_5]$ survive to the Adams E_2 page (Figure 2-11).

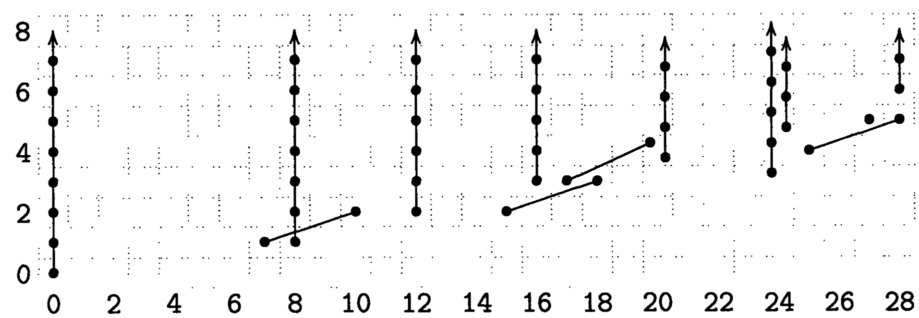

Figure 2-11: The Adams E_2 term for $tmf_*(R^{[5]})$

A computation in the bar complex establishes that

$$v_0[\alpha e_5] = c_4 e_0.$$

This shows that the Adams E_2 page, as a module over that for tmf_*, is

$$\mathbb{F}_3[v_0, c_4, c_6, \Delta, \beta]\{e_0, [\tfrac{c_4}{v_0}e_0], [\tfrac{c_6}{v_0}e_0], [\alpha e_4], [\beta e_5]\}$$
$$/\big(\alpha[\alpha e_4] - \beta e_0, \beta[\tfrac{c_4}{v_0}e_0] - \alpha[\beta e_5], \alpha e_0, I([\beta e_5], [\alpha e_4])\big)$$

The differentials again follow from those in the Adams spectral sequence of tmf_*. $\square$

At this point, the patterns of extensions and differentials repeats. This makes the final computations substantially easier.

Lemma 2.6.6. *The spectrum $R^{[8]}$ is the spectrum C from §2.2, where the middle cell is replaced by the mod 3 Moore spectrum. The module $\widetilde{M_8}$ sits in a short exact sequence*

$$0 \to tmf_*\{[\tfrac{c_4}{3}\Delta^\delta e_0], [\tfrac{c_6}{3}\Delta^\delta e_0], [\Delta^\delta e_0], [\beta e_5]\}/((\alpha,\beta)([\tfrac{c_4}{3}{}^\epsilon\Delta^\delta e_0], [\tfrac{c_6}{3}\Delta^\delta e_0]), I[\beta e_5])$$
$$\to \widetilde{M_8} \to \mathbb{Z}_3[c_4, c_6, \Delta]\{[3e_8], [c_4e_8], [c_6e_8]\} \to 0,$$

where the extension is determined by the two relations

$$c_4[3e_8] = 3[c_4e_8] \pm c_4[\tfrac{c_4}{3}e_0], \quad c_6[3e_8] = 3[c_6e_4] \pm c_4[\tfrac{c_6}{3}e_0].$$

Proof. The long exact sequence in Ext coming from the short exact sequence in homology induced by the inclusion of $R^{[5]}$ into $R^{[8]}$ is determined by the connecting homomorphism which takes e_8 to $[\alpha e_4]$ (Figure 2-12). The linearity of this map shows

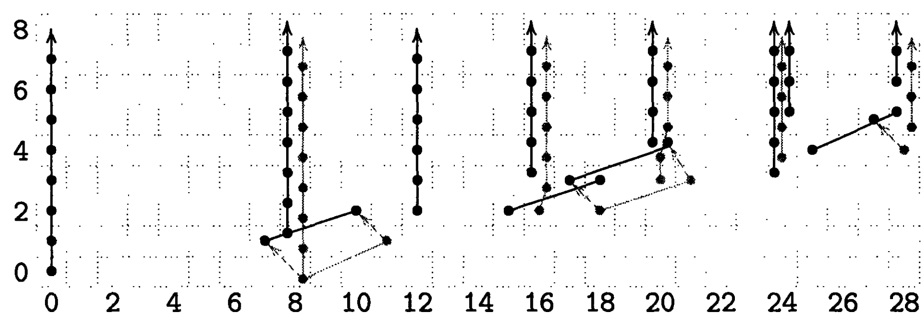

Figure 2-12: The Long Exact Sequence for $\mathrm{Ext}(M_8)$

that the Adams E_2 term for $\widetilde{M_8}$ (Figure 2-13)is an extension of

$$\mathbb{F}_3[v_0, c_4, c_6, \Delta]\{e_0, [\tfrac{c_4}{3}e_0], [\tfrac{c_6}{3}e_0]\} \oplus \mathbb{F}_3[\Delta, \beta] \otimes E(\alpha)\{[\beta e_5]\}$$

by

$$\mathbb{F}_3[v_0, c_4, c_6, \Delta]\{[v_0e_8], [c_4e_8], [c_6e_8]\},$$

subject to the extensions

$$c_4[v_0e_8] = v_0[c_4e_8] \pm c_4\tfrac{c_4}{3}e_0, \quad c_6[v_0e_8] = v_0[c_6e_8] \pm c_6\tfrac{c_4}{3}e_0.$$

The differentials are again determined by those of tmf_*. The only classes which support non-trivial α multiplication are multiples of $[\beta e_5]$, and here, the differentials

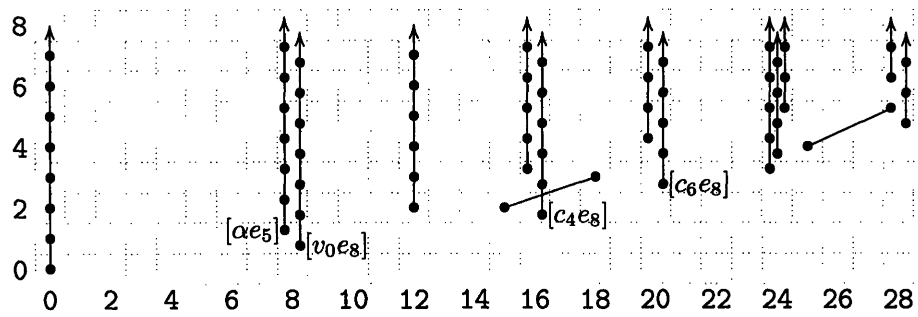

Figure 2-13: The Adams E_2 term for $tmf_*(R^{[8]})$

are the same as for $\widetilde{M}_5$:

$$d_2(\Delta^i[\beta e_5]) = i\alpha\beta^2\Delta^{i-1}[\beta e_5], \quad d_3([\alpha\Delta^2][\beta e_5]) = \beta^5[\beta e_5].$$

□

Lemma 2.6.7. *The spectra $R^{[9]}$, $R^{[10]}$, and $R^{[11]}$ are the cofiber of the map from $\Sigma^4 C(\alpha)$ to C which is multiplication by 3 on the 4 and 8 cells. The module M_9 can be expressed via the short exact sequence*

$$0 \to tmf_*\{[\alpha e_9]\} \to \widetilde{M}_9 \to \mathbb{Z}_3\left[\frac{c_4}{3}\right] e_0 \to 0,$$

where the only extension is given by

$$c_6 e_0 = 9[\alpha e_9].$$

Proof. The cofiber sequence coming from the inclusion of $R^{[8]}$ into $R^{[9]}$ induces a long exact sequence on Ext (Figure 2-14). The connecting homomorphism is

$$e_9 \mapsto [v_0 e_8] + [\tfrac{c_4}{3} e_0].$$

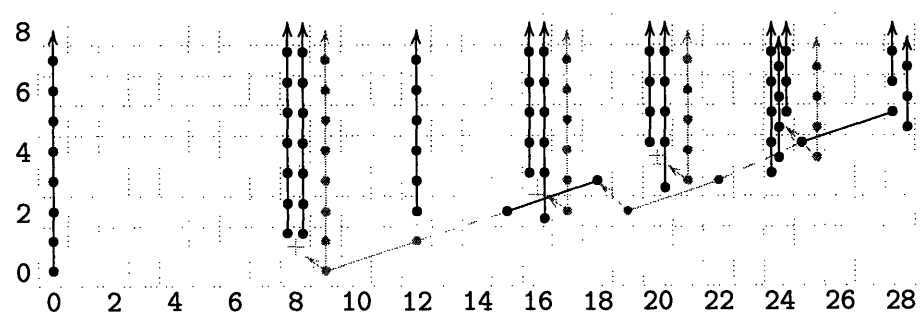

Figure 2-14: Long Exact Sequence for $\mathrm{Ext}(M_9)$

This is a map of modules over the Adams E_2 term for tmf_*, and just as before, the element $[\alpha e_9]$ is in the kernel of this map. This gives hidden extensions analogous to the ones for $\widetilde{M_4}$ and $\widetilde{M_5}$ in the Adams E_2 term for $\widetilde{M_9}$:

$$\alpha[\alpha e_9] = \beta e_5, \quad v_0[\alpha e_9] = [\tfrac{c_6}{3}e_0].$$

The c_4 and c_6 extensions coming from $[v_0 e_8]$ give two more extensions:

$$v_0[c_4 e_8] = c_4[\tfrac{c_4}{3}e_0], \quad v_0[c_6 e_8] = c_4[\tfrac{c_6}{3}e_0].$$

This establishes that the Adams E_2 term is given by the extension of

$$\mathbb{F}_3[v_0, c_4, c_6, \Delta]\{[\alpha e_9]\}$$

by

$$\mathbb{F}_3[v_0, c_4, c_6, \Delta]\{e_0, [\tfrac{c_4}{v_0}e_0], [\tfrac{c_4^2}{v_0^2}e_0]\},$$

where $c_6 e_0 = v_0^2[\alpha e_9]$ (Figure 2-15).

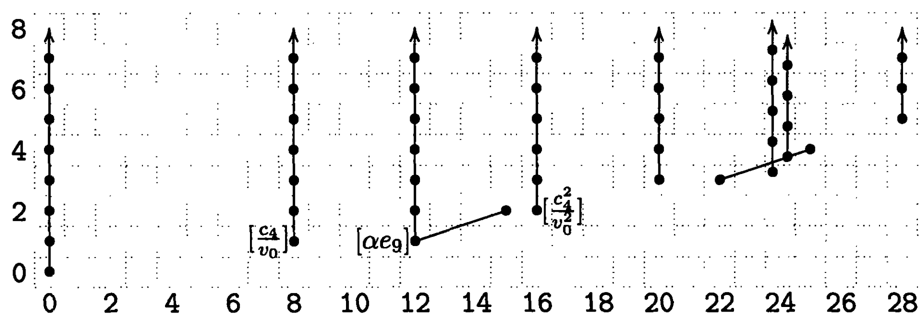

Figure 2-15: The Adams E_2 term for $tmf_*(R^{[9]})$

Just as before, the ordinary Adams differentials determine the differentials, recalling that $[\frac{c_6}{v_0^2}e_0] = [\alpha e_9]$:

$$d_2(\Delta^k[\tfrac{c_6}{v_0^2}e_0]) = k\alpha\beta^2\Delta^{k-1}[\tfrac{c_6}{v_0^2}e_0] = \beta^2[\beta e_5], d_3(\Delta^2[\beta e_5]) = \beta^5[\alpha e_9].$$

The Adams differentials here preserve the exact sequence, and this establishes the statement of the Lemma. □

Remark. For completeness, we note that if it were possible to include a 13-cell, attaching it to the 9-cell via α, then the attaching map in long exact sequence in *tmf* homology would take the copy of tmf_* coming from the 13-cell isomorphically onto the factor $tmf_*\{[\alpha e_9]\}$.

Proposition 2.6.8. *Since the twelve dimensional class is primitive in M_{12}, we conclude that as a tmf_*-module,*

$$\widetilde{M_{12}} = \widetilde{M_9} \oplus \Sigma^{12} tmf_*.$$

2.6.2 The Skeleta of P^∞

The analysis of the preceding section allows us to completely determine the structure of the groups $tmf_*(P^n)$. However, due to the complexity of the combinatorial problem, explicit demonstration of these groups in unenlightening. We instead present the following theorem concerning bounds on the orders of these groups.

Theorem 2.6.9. *If $n = 12k + i$, then 3^{3k+2} annihilates the torsion subgroup of $tmf_*(P^n)$. Moreover, if $i \geq 5$, then there are elements of order exactly 3^{3k+1}, and if $i \geq 9$, then there are elements of order exactly 3^{3k+2}.*

Proof. This is immediate with the consideration that the large torsion subgroups are generated by high powers of $\frac{c_6}{27}$. If we consider only a finite skeleton of P^∞, then we include only finitely many powers of this element. The largest such element occurs in dimension $12k$. If i is at least 5, then we have the element $\frac{c_4}{3}$ on this element. If i is at least 9, then we have the element $\frac{c_4^2}{9}$ on this element. These provide the elements of exact order. □

2.7 The Σ_3 Tate Homology of *tmf*

The analysis used to compute the *tmf* homology of R applies to compute the homotopy of

$$tmf^{t\Sigma_3} = \Sigma\left(tmf \wedge P^\infty\right)_{-\infty} = \Sigma \varprojlim \left(tmf \wedge P^\infty_{-n}\right).$$

A mod 3 form of James periodicity shows that as $\mathcal{A}(1)_*$-comodules,

$$H_*\left(P^\infty_{-12k+3}\right) = \Sigma^{-12k} H_*\left(P^\infty_3\right).$$

The Adams spectral sequence argument in §2.5 shows that the map

$$\pi_*\left(tmf \wedge P^\infty_{-12(k+1)+3}\right) \to \pi_*\left(tmf \wedge P^\infty_{-12k+3}\right)$$

is surjective on the G_* summand and zero on the $\widehat{tmf}_*$ summand. This implies that there are no $\lim^1$ terms coming from the inverse system of homotopy groups. Moreover, this is a system of tmf_*-modules, and considering the action of c_4 and c_6 in each of the modules in the inverse system allows us to conclude

Theorem 2.7.1. *The homotopy of the Σ_3 Tate spectrum of tmf is an indecomposable tmf_* module, and*

$$\pi_*(tmf^{t\Sigma_3}) = \mathbb{Z}_3\left[\frac{c_4}{3}, \left(\frac{c_6}{27}\right)^{\pm 1}\right]^\wedge_I,$$

where I is the ideal in $\pi_0(tmf^{t\Sigma_3})$ generated by elements of positive Adams filtration.

Chapter 3

The 5-local Homotopy of eo_4

3.1 Organization of the Chapter

In §3.2, we review the Gorbounov-Hopkins-Mahowald Hopf algebroid and the stacks associated with it. In §3.3, we apply the techniques from the previous section to compute the rational homotopy of eo_{p-1}. In §3.4, we state the theorem which the rest of the chapter will be spent proving. The middle sections of the paper compute the Adams-Novikov E_2 term for the homotopy of eo_4, loosely following Bauer's computation of the 3-local homotopy of *tmf* [5]. We introduce the Bockstein spectral sequences needed for computation in §3.5, and we carry out the prime independent computations. In §3.6, we restrict attention to the prime 5, competing the computations for eo_4. We try to present proofs that follow formally from Massey product considerations, and if we have not included proofs of any required lemmas, we also include proofs from the bar complex. Finally, in §3.7, we compute the Adams differentials.

3.2 The Geometric Model for EO_{p-1}

The success of the geometric model of elliptic curves for building a geometric model for EO_2 and for building a connective version eo_2 leads to a search for analogous models for primes bigger than 3.

Manin showed that the Jacobian of the Artin-Schreier curve over $\mathbb{F}_p$

$$y^{p-1} = x^p - x$$

admits a formal summand of height $p-1$ [25]. Since this is the first interesting height at the prime p, Hopkins, Mahowald, and Gorbounov used this fact to build a geometric model analogous to the story of elliptic curves and *tmf* at the prime 3, and they show that the formal completion of the Jacobians of the family of curves over $\mathbb{W}(\mathbb{F}_{p^{p-1}})$

$$y^{p-1} = x^p + a_1 x^{p-1} + \cdots + a_p, \quad x \mapsto x + r, \ (x, y) \mapsto (\lambda^{p-1} x, \lambda^p y) \qquad (3.1)$$

carries the Lubin-Tate universal deformation of the Honda formal group, together with an action of $\mathbb{Z}/p \rtimes \mathbb{Z}/(p-1)^2$, a maximal finite subgroup of $\mathbb{G}_{p-1}$ [14]. Such curves are non-singular if the discriminant Δ of the polynomial

$$x^p + a_1 x^{p-1} + \cdots + a_p$$

is a unit.

The scaling action on the Artin-Schreier family given by Equation 3.1 allows us to split off a graded Adams summand. This splitting is algebraically realized by considering Equation 3.1 as a homogeneous graded equation, where $|x| = 2(p-1)$, $|y| = 2p$, $|r| = 2(p-1)$ and $|a_i| = 2i(p-1)$, and the λ action fixes the graded pieces. The degree of the discriminant is $2p(p-1)^2$.

3.2.1 The Moduli Stacks Used

Lurie's derived algebraic geometry produces sheaves of E_∞ ring spectra over various moduli stacks associated to this family of curves. Since the global sections are all closely related, we briefly introduce the stacks involved. In all cases, stackification takes place in the flat topology. Since this is not the topology to which Lurie's machinery applies, we show that there are natural étale, affine covers. We first note that curves of the form given by Equation 3.1 are corepresented by the graded Hopf algebroid

$$(A, \Gamma) = (\mathbb{Z}_p[a_1, \ldots, a_p], A[r]).$$

The first stack considered was the stackification of the Hopf algebroid associated to corepresenting non-singular curves of the form given by Equation 3.1, completed at the maximal ideal I of the degree zero part. In other words, the stack we consider is

$$\mathcal{M}_{p-1}^{\wedge} = Stack\big(Proj(A[\Delta^{-1}]_I^{\wedge}), Proj(\Gamma[\Delta^{-1}]_I^{\wedge})\big).$$

This is essentially the stack first considered by Hopkins, Gorbounov, and Mahowald, as it singles out the height $p-1$ information, and the global sections of the sheaf associated to this stack is the $K(p-1)$-local spectrum EO_{p-1} described earlier.

Part of the power of Lurie's machinery is that we can weaken the conditions on our stack, looking not only at a formal neighborhood of the maximal ideal of the degree zero part but rather at the entire ring corepresenting non-singular curves of the form given by Equation 3.1. Better said, Lurie's machinery produces an appropriate sheaf of E_∞ ring spectra $\mathcal{O}_{p-1}$ over the stack

$$\mathcal{M}_{p-1} = Stack\big(Proj(A[\Delta^{-1}]), Proj(\Gamma[\Delta^{-1}])\big).$$

The global sections of this sheaf is an $H\mathbb{F}_p$ local spectrum denoted $\mathrm{eo}_{p-1}[\Delta^{-1}]$.

It is hoped that a connective version of this spectrum can be constructed. The stack we consider is the full weighted projective space given by

$$\mathcal{M}_{\mathrm{eo}_{p-1}} = Stack(Proj(A), Proj(\Gamma)).$$

While Lurie's machinery does not apply directly to this moduli problem, it seems likely that there does exist an appropriate sheaf on this moduli stack extending the sheaf $\mathcal{O}_{p-1}$. The global sections of this sheaf would be a spectrum local with respect to $K(1) \vee K(2) \vee \dots K(p-1)$.

Let $\mathcal{M}$ be one of any of the three stacks described above, and let B be the corresponding coöbject ring. The natural map

$$Proj(B) \to \mathcal{M}$$

is flat but not étale, since a polynomial ring on one generator is not a finitely generated module. However, in the case for smooth curves, we can make a faithfully flat base change to give an étale cover.

Proposition 3.2.1. *Let $\bar{B}$ be the quotient $B/(a_p)$, and let $\bar{\Gamma} = \bar{B} \otimes_B \Gamma \otimes_B \bar{B}$. Then the map*

$$Proj(\bar{B}) \to \mathcal{M}$$

is an étale cover.

Proof. We prove the result by showing that there is a faithfully flat extension of B such that any curve of the form (3.1) can be translated to one which has $a_p = 0$. However, this is readily done. Let

$$\hat{B} = B[t]/(t^p + a_1 t^{p-1} + \cdots + a_p).$$

The extension $B \to \hat{B}$ is faithfully flat. However, given any curve of the form (3.1), we can transform it into one represented by $\bar{B}$ (suitably extended) by applying the transformation $x \mapsto x + t$. This shows that the stackification of the Hopf algebroid $(\bar{B}, \bar{\Gamma})$ is $\mathcal{M}$.

The proof that this is an étale cover is almost the same. We again need only show that $\bar{B} \to \bar{\Gamma}$ is étale. This, however, again follows from the invertibility of the discriminant, since Δ is divisible by a_{p-1}, implying that the discriminant of the polynomial

$$r^p + \cdots + a_{p-1} r$$

never vanishes modulo any maximal ideal of $\bar{B}$. □

All computations are done over this affine cover. We moreover assume that there is a suitable extension of this cover to $\mathcal{M}_{\mathrm{eo}_{p-1}}$ which has the same form (though here étaleness is more difficult to show). Lurie's machinery ensures that the E_2 term of the Adams-Novikov spectral sequence for the homotopy of the global sections of our sheaf of E_∞ ring spectra over $\mathcal{M}$ is

$$\mathrm{Ext}_{(\bar{B},\bar{\Gamma})}(\bar{B}, \bar{B}).$$

With all of this in place, we could provide a definition of eo_{p-1}. The Adams-Novikov spectral sequence for the sheaf over $\mathcal{M}_{\mathrm{eo}_{p-1}}$ would show that the negative homotopy groups are concentrated in dimensions at most $p(p^2 - 2)$ and that the product of negative dimensional elements with positive dimensional elements is never a non-zero element of non-negative dimension. This would allow us to safely take the connective cover of the global sections, producing the spectrum eo_{p-1}, and because the positive and negative dimensional elements do not interact, we can deduce that the Adams-Novikov spectral sequence for eo_{p-1} has E_2 term

$$\mathrm{Ext}_{(\bar{A},\bar{\Gamma})}(\bar{A},\bar{A}).$$

3.3 Rational Computations

Because they will prove useful for later computations, we list formulas for the right unit in the Hopf algebroid $(\bar{A},\bar{\Gamma})$. At an arbitrary prime, we have

$$\eta_R(a_i) = \sum_{j=0}^{i} \binom{p-j}{i-j} a_j r^{i-j}.$$

At the prime 5, this gives

$$\begin{aligned}
\eta_R(a_1) &= a_1 + 5r,\\
\eta_R(a_2) &= a_2 + 4a_1 r + 10r^2,\\
\eta_R(a_3) &= a_3 + 3a_2 r + 6a_1 r^2 + 10r^3,\\
\eta_R(a_4) &= a_4 + 2a_3 r + 3a_2 r^2 + 4a_1 r^3 + 5r^4,\\
\eta_R(a_5) &= a_5 + a_4 r + a_3 r^2 + a_2 r^3 + a_1 r^4 + r^5.
\end{aligned}$$

3.3.1 Rational Information

The rational case is substantially easier to compute.

Lemma 3.3.1. *There are classes c_i of degree $2i(p-1)$ in A such that*

$$H^*(A\otimes\mathbb{Q},\Gamma\otimes\mathbb{Q}) = H^0(A\otimes\mathbb{Q},\Gamma\otimes\mathbb{Q}) = \mathbb{Q}[c_2,\dots,c_p].$$

Proof. Since p is a unit, we can transform Equation 3.1 into one of the form

$$y^{p-1} = x^p + a_2 x^{p-2} + \cdots + a_p$$

by applying the morphism $x \mapsto x - \frac{a_1}{p}$. There are no translations in x which preserve this form of the curve, so we conclude that rationally, $\mathcal{M}_{\mathrm{eo}_{p-1}}$ is affine. In the language of Hopf algebroids, we conclude that $(A\otimes\mathbb{Q},\Gamma\otimes\mathbb{Q})$ is equivalent to the trivial Hopf algebroid $A = \Gamma = \mathbb{Q}[c_2',\dots,c_p']$, where $c_i' = \eta_R(a_i)$ evaluated at our choice of r. However, the trivial Hopf algebroid has no higher cohomology, and H^0 is just A. This in particular shows the first equality.

The second follows quickly from algebraic manipulations. The denominators of the elements c_i' are powers of p, so we can multiply by a sufficiently high power of p

to get new generators that actually lie in A:

$$c_i = \sum_{j=0}^{i} \binom{p-j}{i-j} (-1)^j a_j a_1^{i-j} p^j.$$

□

At the prime 5, we have the elements c_i have the following form:

$$\begin{aligned}
c_2 &= -2a_1^2 + 5a_2 \\
c_3 &= 4a_1^3 - 15a_1a_2 + 25a_3 \\
c_4 &= -3a_1^4 + 15a_1^2a_2 - 50a_1a_3 + 125a_4 \\
c_5 &= 4a_1^5 - 25a_1^3a_2 + 125a_1^2a_3 - 625a_1a_4 + 3125a_5
\end{aligned}$$

3.4 Statement of the Main Result

Recall from §3.2 that we have generators c_i that rationally are polynomial generators. In our p-local context, this means their products can be written as some power of p times a sum of integral generators. To find the generators of $H^0(A,\Gamma)$, we have to add these and the obvious relations. The proof of the following theorem will be one of the goals for the rest of the chapter:

Theorem 3.4.1. *As an algebra over* $\mathbb{Z}_{(5)}$,

$$H^0(A,\Gamma) = \mathbb{Z}_{(5)}[c_2, c_3, \Delta_i, \Delta'_{15}, \Delta'_{18}, \Delta]/(rels),$$

where i ranges from 4 to 22, where the degree of Δ_i is $8i$, and where the expressions of these elements in terms of the elements a_i and their relations are induced by the formulas from Table 3.1, together with the natural inclusion of $H^0(A,\Gamma)$ into A.

This ring has a distinguished ideal:

$$\mathfrak{m} = (5, c_2, c_3, \Delta_i, \Delta'_j).$$

The ring $H^0(A,\Gamma)$ is the zero line of the Adams-Novikov E_2 term, and it is easier to compute the full E_2 term and then read off the zero line. The remainder of the chapter does just that.

3.5 Preliminary, Prime Independent Remarks

We will compute the Adams-Novikov spectral sequence via a sequence of Bockstein spectral sequences. It is clear from the formulation of the right units that the chain of ideals

$$I_0 = (p) \subset I_1 = (p, a_1) \subset \cdots \subset I_{p-1} = (p, a_1 \ldots, a_{p-1})$$

is invariant. The quotients $(A/I_k, \Gamma/I_k)$ are therefore Hopf algebroids, and we can compute using a Miller-Novikov style algebraic Bockstein spectral sequence. If we filter by powers of these invariant ideals, we get spectral sequences of the form

$$H^*(A/I_k, \Gamma/I_k) \otimes \mathbb{Z}_{(p)}[a_{k-1}] \Rightarrow H^*(A/I_{k-1}, \Gamma/I_{k-1}).$$

This is a trigraded spectral sequence of algebras. If the degree of a homogeneous element x is written (s, t, u), where s is the cohomological degree, t is the internal dimension, and u is the Bockstein degree, then the degree of $d_r(x)$ is $(s+1, t, u+r)$.

The first three Bockstein spectral sequences are the same for all primes.

3.5.1 Computation of $H^*(A/I_{p-1}, \Gamma/I_{p-1})$

The Hopf algebroid $(A/I_{p-1}, \Gamma/I_{p-1})$ is the Hopf algebra $(\mathbb{F}_p, \mathbb{F}_p[r]/r^p)$. The cohomology of this is $\mathbb{F}_p[b] \otimes E(a)$, where $|a| = \big(1, 2(p-1)\big)$, $|b| = \big(2, 2p(p-1)\big)$, and in the cohomology of the bar complex,

$$a = [r], \quad b = \langle \underbrace{a, \dots, a}_{p} \rangle = \left[\sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{j} r^{p-i} | r^i \right].$$

3.5.2 Computation of $H^*(A/I_{p-2}, \Gamma/I_{p-2})$

We run the Bockstein spectral sequence for adding in a_{p-1}. The E_1 term is a polynomial algebra on elements a, b, and a_{p-1} of tridegrees

$$|a| = \big(1, 2(p-1), 0\big), \ |b| = \big(2, 2p(p-1), 0\big), \ |a_{p-1}| = \big(0, 2(p-1)^2, 1\big).$$

For dimension reasons, all of these are permanent cycles, so the spectral sequence collapses.

3.5.3 Computation of $H^*(A/I_{p-3}, \Gamma/I_{p-3})$

The E_1 term of this Bockstein spectral sequence is a polynomial algebra on the elements from the previous part, together with a_{p-2}. The tridegrees of the elements a and b are not changed, while the rest are:

$$|a_{p-1}| = \big(0, 2(p-1)^2, 0\big), \ |a_{p-2}| = \big(0, 2(p-1)(p-2), 1\big).$$

It is also clear that a, b, and a_{p-2} are all permanent cycles which do not bound. The formulation of the right unit shows that

$$d_1(a_{p-1}) = 2aa_{p-2}.$$

This leaves us the following algebra for the E_2 page:

$$\mathbb{F}_p[b, a_{p-1}^p, a_{p-2}] \otimes E(a)/aa_{p-2}\{1, x_1, \dots, x_{(p-1)}\}/(ax_k, a_{p-2}x_k),$$

where the x_k has tridegree $(1, 2(1+k(p-1))(p-1), 0)$ and is represented by aa_{p-1}^k.

Proposition 3.5.1. *All of the x_k with the exception of x_{p-1} are non-bounding permanent cycles. We also have $d_{p-1}(x_{p-1}) = a_{p-2}^{p-1}b$.*

Proof. For dimension reasons, the only possible non-trivial differentials on x_k are of the form $x_k \mapsto ba_{p-2}^n$. We therefore have the following dimension computation on the internal degree:

$$2(p-1)(1+k(p-1)) = 2(p-1)(p+n(p-2)) \Rightarrow (k-1)(p-1) = n(p-2).$$

This has a unique solution in our range: $k = p-1$, $n = p-1$.

For the prime 5, we can also show easily the second part via direct computation in the bar complex:

$$a_4^4 r + 4a_4^3 a_3 r^2 + 3a_4^2 a_3^2 r^3 + 3a_4 a_3^3 r^4 \mapsto a_3^4(r^4|r + 2r^3|r^2 + 2r^2|r^3 + r|r^4) = a_3^4 b.$$

For all primes, this result follows from Lemma 6.2.1:

$$d_{p-1}(x_{p-1}) \doteq \langle a, \underbrace{d_1(a_{p-1}), \ldots, d_1(a_{p-1})}_{p-1} \rangle = \langle \underbrace{a, \ldots, a}_{p} \rangle a_{p-2}^{p-1} = ba_{p-2}^{p-1}.$$

□

This gives the following E_3 term, which, for dimension reasons, is also the E_∞ term:

$$\mathbb{F}_p[b, a_{p-2}, a_{p-1}^p] \otimes E(a)/(aa_{p-2}, a_{p-2}^{p-1}b)\{1, x_1, \ldots, x_{p-2}\}/ax_k = a_{p-2}x_k).$$

There are also the following Massey product relations:

$$\langle x_k, a, a_{p-2} \rangle = x_{k+1} = \langle a_{p-2}^{k+1}, \underbrace{a, \ldots, a}_{k+2} \rangle.$$

These in turn give multiplicative extensions between the elements x_i:

$$x_i x_j = \begin{cases} a_{p-2}^{p-2}b & i+j = p-2 \\ 0 & \text{otherwise} \end{cases},$$

where $x_0 = a$.

The element a_{p-1}^p is a distinguished permanent cycle that we will call Δ.

We can represent this E_∞ term as a picture for the prime 5 (Figure 3-1), with $t/8$ given by the horizontal axis and s given by the vertical one. This picture is repeated polynomially in Δ, represented by a box, and b, so we will only list the first part.

In the picture, a solid line of positive slope is multiplication by a, one of slope zero is multiplication by a_3, and the dotted lines are Massey products $\langle a_3, a, \cdot \rangle$. The case of the general prime is similar, except that the horizontal axis would be indexed as $t/2(p-1)$, and each row above the zeroth would have $p-1$ solid dots.

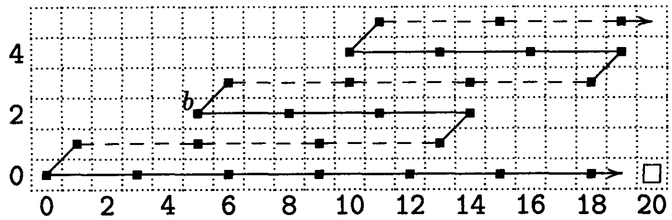

Figure 3-1: $H^*(A/I_{p-3}, \Gamma/I_{p-3})$

3.6 Computation at the Prime 5

From this point on, we will restrict our attention to the prime 5. In this case, we can find explicit representatives of the elements x_1, x_2, and x_3.

$$x_1 = a_4 r + a_3 r^2,\ x_2 = a_4^2 r + 2a_3 a_4 r^2 + 3a_3^2 r^3,\ x_3 = a_4^3 r + 3a_3 a_4^2 r^2 - a_3^2 a_4 r^3 - 3a_3^3 r^4.$$

3.6.1 Computation of $H^*(A/I_1, \Gamma/I_1)$

The computation here starts largely as before. The elements a, b, a_2, x_1, and Δ are all permanent cycles, for dimension reasons. The element x_1 is now represented as $a_4 r + a_3 r^2 + a_2 r^3$. However, beyond this the patterns of differentials becomes more complicated.

For clarity, we will rely on pictures of the E_r terms to describe the initial situations and tell us which elements could support a differential. In these Bockstein spectral sequences, the d_r-differential of any element must be divisible by a_2^r (more generally, by the new element to the r^{th} power). If we make the convention that a solid horizontal line means multiplication by the new, Bockstein element and an open circle means a polynomial algebra on this element, then we see that the possible targets of a d_r differential are open circles preceded horizontally by r solid lines. If we additionally make the convention that circles with dots in them are the non-Bockstein multiplicative generators, then the differentials are totally determined by their values on these elements. These conventions will allow us to immediately see which elements could support a differential.

The d_1 Differential

We have a single differential coming immediately from the bar complex:

$$d_1(a_3) = 3a_2 a.$$

If we extend this by multiplicativity, using the fact that $a_3 a = 0$, we see that all elements of the form a_3^k are d_1-cycles. To see if there are any other differentials, we first look at the picture (Figure 3-2), in which dashed horizontal lines are a_3 multiplications.

From this, we see the last possible d_1 differential:

Proposition 3.6.1. *We have* $d_1(x_3) \doteq a_2 a_3^2 b$.

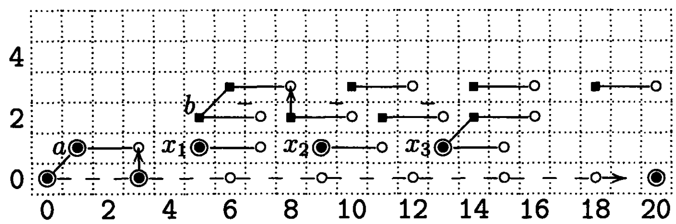

Figure 3-2: E_1 page for $H^*(A/I_1, \Gamma/I_1)$

Proof. The element x_3 can be written as

$$x_3 = \langle a_3^3, a, a, a, a \rangle.$$

From this it follows from a simplification of May's work on Massey products, as presented in [31] that

$$d_1(x_3) = \langle d_1(a_3^3), a, a, a, a \rangle = \langle -a_2 a_3^2 a, a, a, a, a \rangle = -a_2 a_3^2 \langle a, a, a, a, a \rangle = -a_2 a_3^2 b.$$

□

The d_2 Differential

From the picture of the E_2 page (Figure 3-3), we see immediately that the only elements that can support a d_2 differential are a_3^3 and x_2.

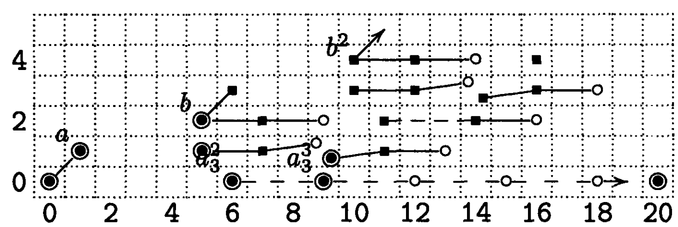

Figure 3-3: The E_2 page for $H^*(A/I_1, \Gamma/I_1)$

Proposition 3.6.2. $d_2(a_3^3) = -a_2^2 x_1$ *and* $d_2(x_2) = -a_2^2 b$.

Proof. Again, we have Massey product proofs. The element x_2 is the Massey product

$$x_2 = \langle a_3^2, a, a, a \rangle.$$

This means, by Proposition 6.2.5, that

$$d_2(x_2) \doteq \langle d_1(a_3), d_1(a_3), a, a, a \rangle = a_2^2 b.$$

In the bar complex, we have

$$a_3^3 + 3a_2 a_3 a_4 \mapsto -a_2^2 (a_2 r^3 + a_3 r^2 + a_4 r) = -a_2^2 x_1.$$

For the second differential, we appeal to the bar complex:

$$a_4^2r + 2a_3a_4r^2 + 3a_3^2r^3 + 2a_2a_4r^3 + 3a_2a_3r^4 \mapsto -a_2^2b.$$

$\square$

The d_3 Differential and Beyond

Given the sparsity of the spectral sequence above the filtration 0 line (Figure 3-4), it is clear that it now collapses.

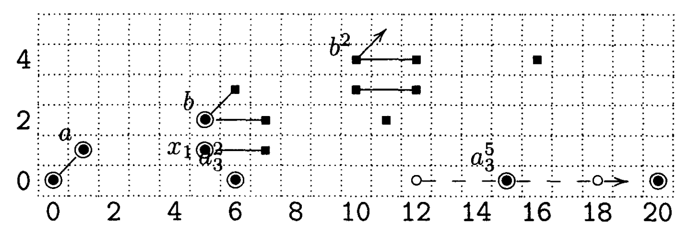

Figure 3-4: $H^*(A/I_1, \Gamma/I_1)$

For computational reasons, we give here some of the full names for some of the elements listed above. The elements a and b have their usual bar representatives, while

$$\begin{aligned}
x_1 &= a_4r + a_3r^2 + a_2r^3 \\
[a_3^2] &= a_3^2 + 2a_2a_4 \\
[a_3^5] &= a_3^5 + 2a_2^3a_3^3 + a_2^4a_3a_4 \\
\Delta &= a_4^5 - 2a_3^4a_4^2 - a_2a_3^2a_4^3 + 2a_2^2a_4^4 + a_2^3a_3^2a_4^2 + a_2^4a_4^3
\end{aligned}$$

With these elements, we can also compute the structure of H^* as a ring:

Proposition 3.6.3. *We have the multiplicative extension* $2a[a_3^2] = a_2x_1$, *and the full algebra of* $H^*(A/I_1, \Gamma/I_1)$ *is*

$$\mathbb{F}_5\left[a, b, x_1, a_2, [a_3^2], [a_3^5], \Delta\right]/\Big((a, x_1)^2,\ a(a_2, [a_3^2], [a_3^5]),\ a_2^2(b, x_1),$$
$$[a_3^2]^5 - [a_3^5]^2 = a_2^3[a_3^2]^4 + a_2^6[a_3^2]^3 + 2a_2^5\Delta,\ 2a[a_3]^2 - a_2x_1\Big)$$

Proof. The algebra structure will follow from the first part by direct computation. The first part follows from noting that the difference of these two elements is the bar differential of a_3a_4. $\square$

3.6.2 Computing $H^*(A/I_0, \Gamma/I_0)$

Because things are so spread out, this is actually easier to compute than the previous term. We start with the observation that, for dimension reasons, a, b, Δ, and x_1 are all permanent cycles. The bar representative of x_1 is $-r^5$.

The d_1 Differential

We first note the differential coming immediately from the bar complex:

$$d_1(a_2) = -a_1 a.$$

To continue, we use the picture of E_1 (Figure 3-5), marking this differential. We will use similar notation as before, but here solid lines with represent a_1 multiplications while dashed lines will represent a_2 multiplication. To further simplify the picture, we use a circled star to indicate a polynomial algebra on both a_1 and a_2.

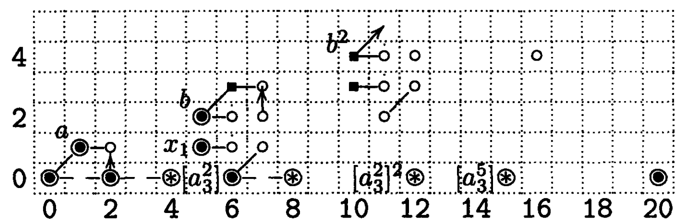

Figure 3-5: E_1 page for $H^*(A/I_0, \Gamma/I_0)$

The picture suggests to us another differential.

Proposition 3.6.4. *We have* $d_1([a_3^2]) = 3a_1x_1$.

Proof. The element a_3^2 can be realized as $\langle a_3, a, a_2\rangle$ or $\langle a_2, a, a_2, a\rangle$. Taking d_1 on this as on Massey products, we get

$$d_1(a_3^2) = \langle a_3, a, a_1, a\rangle = a_1x_1,$$

or

$$d_1(a_3^2) = \langle a_2, a, d_1(a_2), a\rangle = \langle a_2, a, a_1a, a\rangle = a_1x_1.$$

Similarly, from the bar complex, we have that $[a_3^2]$ is represented in the bar complex by $a_3^2 + 2a_2a_4$. We also have

$$a_3^2 \mapsto 6a_2a_3r + 2a_1a_3r^2 + a_1a_2r^3 + a_2^2r^2 + a_1^2r^4,$$

while

$$2a_2a_4 \mapsto -a_2a_3r + a_2^2r^2 + 2a_1a_2r^3 - 2a_1a_4r + a_1a_3r^2 + 2a_1^2r^4.$$

Adding these gives the result. □

The d_2 Differential and the E_∞ Page

At this point, our spectral sequence is again very sparse (Figure 3-6).

We again see that we can have but a single coherent differential.

Proposition 3.6.5. *We have* $d_2(a_2x_1) = a_1^2b$.

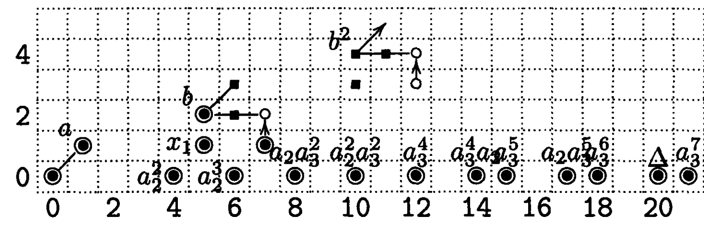

Figure 3-6: E_2 Page of $H^*(A/I_0, \Gamma/I_0)$

Proof. On this page, $x_1 = \langle a_2, a, a, a\rangle$, so $a_2x_1 = \langle a_2^2, a, a, a\rangle$. Proposition 6.2.5 shows that

$$d_2(\langle a_2^2, a, a, a\rangle) = \langle d_1(a_2), d_1(a_2), a, a, a\rangle = a_1^2 b.$$

For the bar version, we start by computing the bar differential on $a_2x_1 = a_2^2r^3 + a_2a_3r^2 + a_2a_4r$:

$$\begin{aligned} a_2x_1 \mapsto 2a_2a_3r|r + 4a_1a_4r|r + 3a_2^2r^2|r + 3a_1a_3r^2|r + a_1a_2r^3|r + a_1^2r^4|r \\ + 3a_2^2r|r^2 + 4a_1a_3r|r^2 + 3a_1a_2r^2|r^2 - a_1^2r^3|r^2 - 2a_2a_3r|r \\ + 3a_1a_2r|r^3 + a_1^2r^2|r^3 - 3a_2^2r^2|r - 3a_2^2r|r^2. \end{aligned}$$

If we add to this $-a_1a_2r^4 + a_1a_3r^3 + 2a_1a_4r^2$, then a little algebra shows us that the bar differential of this is exactly a_1^2b. □

It is clear that no further differentials are possible, so the spectral sequence collapses here.

3.6.3 $H^*(A, \Gamma)$

Everything we have done so far has led us to compute what happens when we add in the number 5. There is already an obvious differential given by $a_1 \mapsto 5r$. Additionally, x_1 has survived this long because it has represented r^5 which, mod 5, is a cycle since r is. Now the binomial theorem tells us exactly what it will hit:

$$r^5 \mapsto 5r^4|r + 10r^3|r^2 + 10r^2|r^3 + 5r|r^4.$$

In other words, $d_1(x_1) = 5b$. This gives us all of the differentials for dimension reasons, as we immediately see (Figure 3-7).

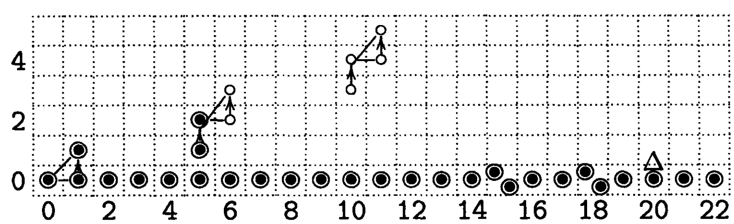

Figure 3-7: E_1 Page for $H^*(A, \Gamma)$

Since there are no more possible differentials, we conclude that $E_2 = E_\infty$. Additionally, we know the leading terms of the generators of H^0, since this is exactly what the Bockstein spectral sequences have been computing for us.

Corollary 3.6.6. *As an algebra, $H^0(A, \Gamma)$ is as described in Theorem 3.4.1*

Proof. The Bockstein spectral sequences demonstrated that the classes given are the algebra generators. The relations are simple consequences of algebraic manipulations, so these are also immediate. □

Putting everything we have seen so far together allows us to show the following theorem.

Theorem 3.6.7. *As an algebra,*

$$H^*(A, \Gamma) = H^0[a, b]/\Big(a^2, \mathfrak{m}(a, b)\Big).$$

Proof. The only surprise relation is $\mathfrak{m}(a, b)$, and this follows from the earlier fact that terms dominated in $(a_1, a_2, a_3)(a, b)$ were zero by the time we reached this last page. □

3.7 Adams Differentials and the 5-local Homotopy of eo_4

In this section, we compute the Adams' differentials for the homotopy of eo_4. Since the unit $S^0 \to \mathrm{eo}_4$ takes the elements $\alpha_1, \beta_1 \in \pi_*(S^0)$ to the classes $a, b \in \pi_*(\mathrm{eo}_4)$, and since we have the Toda relation that $\alpha_1\beta_1^p = 0$, we must conclude:

Theorem 3.7.1. *We have $d_9(\Delta) = ab^4$.*

We can see hidden multiplicative extensions by considering the Massey product representatives of the "left-over" classes $[a\Delta]$, $[a\Delta^2]$, and $[a\Delta^3]$.

Proposition 3.7.2. *We have*

$$\begin{aligned} [a\Delta] &= \langle \iota, ab^4, a \rangle \\ [a\Delta^2] &= \langle \iota, ab^4, ab^4, a \rangle \\ [a\Delta^3] &= \langle \iota, ab^4, ab^4, ab^4, a \rangle. \end{aligned}$$

Additionally, we have a hidden multiplicative extension

$$a[a\Delta^3] = b^{13}.$$

Proof. The first relations are immediate from the form of d_9. The hidden extension follows by "shuffling" in the a and then "shuffling" out the b^4 terms. □

Theorem 3.7.3. *We have $d_{33}([a\Delta^4]) \doteq b^{17}$.*

Proof. Proposition 6.2.4 gives

$$d_{33}(a\Delta^4) = \langle a, d_9(\Delta), d_9(\Delta), d_9(\Delta), d_9(\Delta)\rangle = b^{17}.$$

□

The spectral sequence collapses at this point, as there are not enough things in higher filtration to be the target of any further differentials.

3.8 Formulas Relating the classes Δ_i

Δ_4	$\frac{1}{25}\left(4c_4 + 3c_2^2\right)$
Δ_5	$\frac{1}{25}\left(2c_5 + c_2c_3\right)$
Δ_6	$\frac{1}{125}\left(4c_3^2 - 8c_2c_4 + 2c_2^3\right)$
Δ_7	$\frac{1}{5}\left(c_3\Delta_4 - 2c_2\Delta_5\right)$
Δ_8	$\frac{1}{5^5}\left(-3c_2c_3^2 + 9c_2^2c_4 - 4c_4^2 + 3c_3c_5\right)$
Δ_9	$\frac{1}{5^5}\left(9c_3^3 + 32c_2c_3c_4 - 9c_2^2c_5 + 4c_4c_5\right)$
Δ_{10}	$\frac{1}{400}\left(4\Delta_5^2 + 2c_2\Delta_4^2 - 15\Delta_4\Delta_6\right)$
Δ_{11}	$\frac{1}{20}\left(3\Delta_5\Delta_6 - 2\Delta_4\Delta_7\right)$
Δ_{12}	$\frac{1}{5^8}\left(54c_3^4 - 279c_2c_3^2c_4 + 216c_2^2c_4^2 - 224c_4^3 + 81c_2^2c_3c_5 + 144c_3c_4c_5 - 27c_2c_5^2\right)$
Δ_{13}	$\frac{1}{15}\left(\Delta_4\Delta_9 - 4\Delta_5\Delta_8\right)$
Δ_{14}	$\frac{1}{50}\left(4c_4\Delta_{10} - 6c_3\Delta_{11} + 15\Delta_6\Delta_8 - 15\Delta_4\Delta_{10} + 15c_2\Delta_{12}\right)$
Δ'_{15}	$\frac{1}{5}\left(\Delta_5\Delta_{10} - 2\Delta_4\Delta_{11}\right)$
Δ_{15}	$\frac{1}{5^{10}}\big(162c_3^5 - 80c_2^3c_3^3 + 360c_2^4c_3c_4 + 160c_3c_4^3 + 2520c_2^2c_3c_4^2$ $-216c_2^5c_5 + 105c_2^2c_3^2c_5 - 900c_2^3c_4c_5 - 270c_3^2c_4c_5$ $+720c_2c_4^2c_5 - 105c_2c_3c_5^2 - 1215c_2c_3^3c_4 + 26c_5^3\big)$
Δ_{16}	$\frac{1}{50}\left(-8\Delta_5\Delta_{11} - 2c_2\Delta_4\Delta_{10} + 15\Delta_6\Delta_{10} - 30c_2\Delta_{14}\right)$
Δ_{17}	$\frac{1}{25}\left(-3c_3\Delta_{14} - 2c_2\Delta'_{15} + 20\Delta_8\Delta_9\right)$
Δ_{18}	$\frac{1}{5}\left(2\Delta_5\Delta_{13} - \Delta_4^2\Delta_{10} + \Delta_4\Delta_6\Delta_8\right)$
$\Delta_{18'}$	$\frac{1}{25}\left(2\Delta_9^2 + 8c_2\Delta_8^2 - 19\Delta_8\Delta_{10}\right)$
Δ_{19}	$\frac{1}{5}\left(8\Delta_8\Delta_{11} - \Delta_9\Delta_{10}\right)$
Δ_{21}	$\frac{1}{5}\left(2c_3\Delta'_{18} + 12c_2\Delta_{19} - 30\Delta_{10}\Delta_{11} + 15\Delta_9\Delta_{12}\right)$
Δ_{22}	$\frac{1}{5}\left(\Delta_9\Delta_{13} + 2\Delta_6\Delta_{16} + 3\Delta_5\Delta_{17} + \Delta_4\Delta_{18} + \Delta_7\Delta_{15}\right)$
Δ	$\frac{1}{5^{15}}\big(-100c_2^3c_3^2c_4^2 - 135c_3^4c_4^2 + 400c_2^4c_4^3 + 720c_2c_3^2c_4^3$ $-640c_2^2c_4^4 + 256c_4^5 + 80c_2^3c_3^3c_5 + 108c_3^5c_5 - 360c_2^4c_3c_4c_5$ $-630c_2c_3^3c_4c_5 + 560c_2^2c_3c_4^2c_5 - 320c_3c_4^3c_5 + 108c_2^5c_5^2$ $+165c_2^2c_3^2c_5^2 - 180c_2^3c_4c_5^2 + 90c_3^2c_4c_5^2 + 80c_2c_4^2c_5^2$ $-30c_2c_3c_5^3 + c_5^4\big)$

Table 3.1: Generators and Basic Relations for $H^0(A, \Gamma)$

Chapter 4

The eo_{p-1} Hopf Algebra

4.1 Introduction

With the understanding of eo_{p-1} developed in the previous chapter, we can turn attention to generalizing many of the results of Chapter 2. In this chapter, we introduce our conjectures, coupling them with provable statements in the non-connective cases. The machinery needed will be developed in § 4.3, and in § 4.4, we sketch out the results analogous to Theorem 2.5.1. We round out the chapter by working $K(p-1)$-locally, producing in § 4.5 a new Hopf algebra that computes the EO_{p-1} homology of a space. We also indicate how to compute the EO_{p-1} homology of $B\Sigma_p$, using this tool.

4.2 A New Spectrum

We begin by noting that the Gorbounov-Hopkins-Mahowald curves come equipped with an involution ι which on points looks like $(x, y) \mapsto (x, -y)$. If we consider the moduli problem of a GHM curve together with a fixed point of the involution ι, then we get a moduli stack $\mathcal{M}_{p-1}(\iota)$. This moduli stack has a forgetful map to $\mathcal{M}_{p-1}$ given by forgetting the fixed point.

A fixed point of the involution is equivalent to the data of a GHM curve together with a root of the right hand side. By using the morphism $x \mapsto x + r$, we can force this fixed point to be $(0, 0)$. This means that when we pull back the cover of $\mathcal{M}_{p-1}$ given by the GHM Hopf algebroid to $\mathcal{M}_{p-1}(\iota)$, we get the trivial Hopf algebroid

$$(A, \Gamma) = (\mathbb{Z}_p[a_1, \ldots, a_{p-1}][\Delta^{-1}], \mathbb{Z}_p[a_1, \ldots, a_{p-1}][\Delta^{-1}]).$$

Proposition 4.2.1. *The map $\mathcal{M}_{p-1}(\iota) \to \mathcal{M}_{p-1}$ is étale.*

Proof. The proof is similar to that of Proposition 3.2.1. The stack $\mathcal{M}_{p-1}$ is the stackification of the Hopf algebroid

$$(A, \Gamma) = (\mathbb{Z}_p[a_1, \ldots, a_{p-1}][\Delta^{-1}], A[r]/r^p + a_1 r^{p-1} + \cdots + a_{p-1} r).$$

The stack $\mathcal{M}_{p-1}(\iota)$ is the stackification of the Hopf algebroid (A, A), and the forgetful map is given by the map that sends $r \in \Gamma$ to $0 \in A$. This map is étale since the discriminant is invertible, making the polynomial

$$r^p + \cdots + a_{p-1}r$$

non-singular. □

The sheaf of E_∞ ring spectra produced by Lurie's machine is a sheaf in the étale topology. Evaluating it on $\mathcal{M}_{p-1}(\iota)$ produces an E_∞ ring spectrum denoted $\mathrm{eo}_{p-1}(\iota)[\Delta^{-1}]$. Since the moduli stack has a cover by the trivial Hopf algebroid, we conclude that the Adams Novikov spectral sequence for the homotopy of $\mathrm{eo}_{p-1}(\iota)[\Delta^{-1}]$ collapses, and

$$\pi_*(\mathrm{eo}_{p-1}(\iota)[\Delta^{-1}]) = \mathbb{Z}_p[a_1, \ldots, a_{p-1}][\Delta^{-1}].$$

We can actually make a slightly better statement. Let C_p be the p-cell complex

$$S^0 \cup_{\alpha_1} e^{2(p-1)} \cup_{\alpha_1} \cdots \cup_{\alpha_1} e^{2(p-1)^2}.$$

Proposition 4.2.2. *As ring spectra,*

$$\mathrm{eo}_{p-1}(\iota)[\Delta^{-1}] = \mathrm{eo}_{p-1}[\Delta^{-1}] \wedge C_p.$$

Proof. This statement is analogous to the ones for $p = 2$, $KU = KO \wedge C(\eta)$, and $p = 3$. The proof is identical. We first consider the unit map from the sphere into $\mathrm{eo}_{p-1}(\iota)[\Delta^{-1}]$. Since α_1 and its Massey powers are trivial in $\pi_*(\mathrm{eo}_{p-1}(\iota)[\Delta^{-1}])$, we conclude that the unit map extends over C_p. If we then smash this with $\mathrm{eo}_{p-1}[\Delta^{-1}]$ and compose with the action of $\mathrm{eo}_{p-1}[\Delta^{-1}]$ on $\mathrm{eo}_{p-1}(\iota)[\Delta^{-1}]$, then we get a map

$$\mathrm{eo}_{p-1}[\Delta^{-1}] \wedge C_p \to \mathrm{eo}_{p-1}(\iota)[\Delta^{-1}].$$

However, the element $r \in \Gamma$ detects α_1, so algebraically, the result of smashing with C_p is the addition of the truncated polynomial algebra on r to A. This shows that the map given is actually an isomorphism in π_*, making it an equivalence. □

Remark. We believe that this result may also be shown $K(p-1)$-locally using a homotopy fixed point spectral sequence argument. The spectrum $\mathrm{EO}_{p-1}(\iota)$ is the homotopy fixed points of E_n with respect to the $\mu_{(p-1)^2}$ part of the finite subgroup used to define EO_{p-1}. The spectrum EO_{p-1} could then be reconstructed by taking the homotopy fixed points with respect to $\mathbb{Z}/p$. The equivalence in the previous proposition amounts to showing that

$$\mathrm{EO}_{p-1}(\iota) \wedge C_p \cong \mathrm{EO}_{p-1}(\iota)[\mathbb{Z}/p],$$

just as with KU, KO, and the cone on η.

4.3 Hopes for eo_{p-1}

Conjecture 4.3.1. *All of the preceding propositions for* $\mathrm{eo}_{p-1}[\Delta^{-1}]$ *extend over the full weighted projective space, giving spectra* eo_{p-1} *and* $\mathrm{eo}_{p-1}(\iota)$*. These spectra satisfy the analogous relation*

$$\mathrm{eo}_{p-1} \wedge C_p \cong \mathrm{eo}_{p-1}(\iota).$$

Conjecture 4.3.2. *As a Hopf algebra,*

$$\mathcal{A}_{\mathrm{eo}_{p-1}*} := \pi_*(H\mathbb{Z}/p \wedge_{\mathrm{eo}_{p-1}} H\mathbb{Z}/p) = \mathcal{A}(1)_* \otimes E(\bar{a}_2, \dots, \bar{a}_{p-1}),$$

where again $\mathcal{A}(1)_*$ *is dual to the subalgebra generated by* β *and* $\mathcal{P}^1$*, and where* $|\bar{a}_i| = 2i(p-1)+1$*. The elements in* $\mathcal{A}(1)$ *again have their usual coproducts, while*

$$\psi(\bar{a}_j) = \sum_{k=0}^{j} \tfrac{1}{k!}\xi_1^k \otimes \bar{a}_{j-k} + \bar{a}_j \otimes 1,$$

where $\bar{a}_1 = \tau_1$ *and* $\bar{a}_0 = \tau_0$*.*

Before we can prove this, we need a proposition about algebras in the category of modules over a structured ring spectrum.

Proposition 4.3.3. *Let* $R \to S$ *be a map of* E_2 *ring spectra. If* M *is an* E_2 S*-algebra, and* N *is an* E_2 M*-algebra, then we have a push-out of* E_2 *algebras:*

$$\begin{array}{ccc} M \wedge_R S & \longrightarrow & M \wedge_R M\,. \\ \downarrow & & \downarrow \\ N & \longrightarrow & N \wedge_S M \end{array}$$

Proof. This is analogous to the statement in commutative rings that

$$Tor_{M\otimes_R S}(N, M \otimes_R M) \cong Tor_S(N, M).$$

The proof is actually identical, using the fact that we can "cancel" terms out of smashing over a ring spectrum. □

The push-out in commutative ring spectra induces an isomorphism

$$Tor_{M_*^R S}(N_*, M_*^R M) \xrightarrow{\cong} N_*^S M.$$

If $M = N$ is the quotient of S by a regular ideal, then we can can actually identify many of the terms, since $M_*^S M$ is just an exterior algebra on generators corresponding to the generators of the ideal. Moreover, if every module is flat of M_*, then the push-out square induces a short exact sequence of Hopf algebras

$$0 \to M_*^R S \to M_*^R M \to M_*^S M \to 0.$$

To prove Conjecture 4.3.2, let $R = \mathrm{eo}_{p-1}$, $S = \mathrm{eo}_{p-1}(\iota)$, and $M = H\mathbb{F}_p$. Every module over M_* is flat, just as before.

Conjecture 4.3.4. *The homotopy ring of $H\mathbb{F}_p \wedge_{\mathrm{eo}_{p-1}} \mathrm{eo}_{p-1}(\iota)$ corepresents the automorphism group of the "additive" Gorbounov-Mahowald curve*

$$y^{p-1} = x^p.$$

In other words,

$$H\mathbb{F}_{p_*}^{\mathrm{eo}_{p-1}}(\mathrm{eo}_{p-1}(\iota)) = \mathbb{F}_p[\xi_1]/\xi_1^p$$

as a primitively generated Hopf algebra

The spectrum $H\mathbb{F}_p \wedge_{\mathrm{eo}_{p-1}} \mathrm{eo}_{p-1}(\iota)$ represents the automorphisms of the additive point in the relative moduli stack $(\mathcal{M}_{\mathrm{eo}_{p-1}}, \mathcal{M}_{\mathrm{eo}_{p-1}(\iota)})$. The homotopy groups of this then carves out the truncated polynomial part indicated, by a simple computation involving quotients of Hopf algebroids.

The last computational piece we will need is the cohomology of $\mathcal{A}(1)$ at primes bigger than 2. To best describe it, we need a small bit of notation for Poincaré duality algebras. If A and B are connected, graded Poincaré duality algebras with top class in the same dimension and augmentation ideals I_A and I_B respectively, then we define a new connected Poincaré duality algebra $A \odot B$ by taking its augmentation ideal to be $I_A \oplus I_B$ modulo the relation that the top class in I_A is the top class of I_B.

Proposition 4.3.5. *The algebra* $\mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_p, \mathbb{F}_p)$ *is*

$$\mathbb{F}_p[v_0, \beta, v_1^p] \otimes \bigodot_{i=1}^{\left\lfloor \frac{p-1}{2} \right\rfloor} E(\alpha_i, \alpha_{p-i})/(v_0\alpha_i = 0, \alpha_i\alpha_{p-i} = v_0^{p-2}\beta),$$

where $|\alpha_i| = 2i(p-1) - 1$, $|\beta| = 2p(p-1) - 2$, and $|v_1^p| = 2p(p-1)$. The Adams filtrations of the elements α_i are i, while that of β is 2 and that of v_1^p is p.

Indicative Sketch of Conjecture 4.3.2. Proposition 4.3.3 gives a short exact sequence of Hopf algebras

$$0 \to \mathbb{F}_p[\xi_1]/\xi_1^p \to H\mathbb{F}_{p_*}^{\mathrm{eo}_{p-1}} H\mathbb{F}_p \to E(\tau_0, \tau_1, \bar{a}_2, \dots, \bar{a}_{p-2}) \to 0.$$

The computation of the coproducts is exactly as before. We can filter the Hopf algebra so that it becomes primitively generated extension of $\mathcal{A}(1)_*$. If we compute Ext over the associated graded, then we get

$$\mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_p, \mathbb{F}_p)[\bar{c}_2, \dots, \bar{c}_{p-1}],$$

where $\bar{c}_i = [\bar{a}_i]$. The degrees of the elements $\bar{c}_i$ are all smaller than the degree of β, so the possible targets of algebraic or Adams differentials are all greatly restricted by degree. In fact, since $|\bar{c}_i| = 2i(p-1)$, for degrees less than that of β, $\mathrm{Ext}_{\mathcal{A}}$ is zero except in topological degrees congruent to -1 or 0 modulo $2(p-1)$.

We complete the computation of the coproducts by singling out particular elements in Ext over the associated graded. The elements α_2 and $\bar{c}_i\alpha_1$ for $i < p-1$ are all present in Ext, and for degree reasons, if they were to survive both the algebraic and the Adams spectral sequences, the would give rise to p-torsion elements. However, we know from the work of Hopkins and Miller that the only p-torsion element in the range we consider is the element α_1. This implies that all of these elements must be killed. They cannot support any differentials for degree reasons, and since there are no elements of Adams filtration 0 in the relevant ranges, they can only be targeted by algebraic differentials. The only element in the appropriate dimension to kill $\bar{c}_i\alpha_1$ is $\bar{c}_{i+1}$, and this proves the result. □

To facilitate understanding, we include at the end of the chapter series of charts that show how the above argument plays out for the prime 5.

4.4 The eo_{p-1} homology of $B\Sigma_p$

Assuming Proposition 4.3.2, we can reprove most of the results true for the prime 3. If we again consider the cofiber R of the transfer map $B\Sigma_p \to S^0$, then there is an analogue to Lemma 2.4.1

Proposition 4.4.1. *There is a filtration of $H_*(R)$ such that the associated graded is*

$$Gr\big(H_*(R)\big) = \bigoplus_{k=0}^{\infty} \Sigma^{2p(p-1)k} M.$$

The same argument that showed that $\mathrm{Ext}_{\mathcal{A}}$ of this was torsion free works at other primes, so we see that $\mathrm{Ext}_{\mathcal{A}_p}\big(\mathbb{F}_p, H_*(R)\big)$ is an evenly generated polynomial algebra.

Conjecture 4.4.2. *As an eo_{p-1*} module,*

$$\mathrm{eo}_{p-1*}(R) = \mathbb{Z}_p\left[\frac{c_2}{p}, \dots, \frac{c_{p-2}}{p^{p-3}}, \frac{c_{p-1}}{p^{p-2}}, \frac{c_p}{p^p}\right]$$

The fractional multiples of the generators will be justified in § 4.5.

We moreover conjecture that the eo_{p-1} image of the transfer map again contains all of the higher Adams-Novikov filtration elements, since these are generated by α and β, and these elements will again not be present in $\mathrm{eo}_{p-1*}(R)$.

4.5 The EO_{p-1} homology of $B\Sigma_p$

While the previous sections contain only conjectures, if we consider the $K(p-1)$-local version, we can actual make honest statements. We first need a small number theoretic lemma.

Lemma 4.5.1. *If k is an integer, then $(k-1)^2$ divides $k^{k-1}-1$.*

Proof. It is obvious that $k-1$ divides $k^{k-1}-1$, leaving a quotient of $k^{k-2}+\cdots+1$. We can express the polynomial $x^{k-2}+\cdots+1$ in terms of $x-1$, and we get

$$(x-1)^{k-2}+\cdots+(k-1).$$

If we evaluate at k, then we get the result of the lemma. □

Let m denote the quotient of $k^{k-1}-1$ by $(k-1)^2$, and let $\widehat{K}(p-1)$ denote the $\mathcal{A}_\infty$ extension of $K(p-1)$ obtained by adjoining an m^{th} root of v_{p-1} [1]. This spectrum is a module over EO_{p-1}, where the module structure is determined by taking the quotient of the E_∞ ring spectrum $\mathrm{EO}_{p-1}(\iota)$ by the regular ideal $(p,\ldots,a_{p-2})$. This result, together with Proposition 4.3.3, proves the following theorem.

Theorem 4.5.2. *The homotopy of $\widehat{K}(p-1)\wedge_{\mathrm{EO}_{p-1}}\widehat{K}(p-1)$ is the Hopf algebra over $\widehat{K}(p-1)_*$*

$$\mathcal{A}_{\mathrm{EO}_{p-1}*}=\widehat{K}(p-1)_*[\xi_1]/\xi_1^p\otimes E(\tau_0,\tau_1,\bar{a}_2,\ldots,\bar{a}_{p-2}),$$

where ξ_1, τ_0, and τ_1 have their usual coproducts, and the coproducts on the elements $\bar{a}_i$ are those of Conjecture 4.3.2.

The Adams spectral sequence based on $\widehat{K}(p-1)$, as a module over EO_{p-1}, converges to the homotopy of the $\widehat{K}(p-1)$ nilpotent completion of $\mathrm{EO}_{p-1}\wedge X$. If X is the sphere S^0, then the Adams cosimplicial resolution of $\mathrm{EO}_{p-1}\wedge X$ converges to EO_{p-1}, since EO_{p-1}, being $K(p-1)$-local, is already $\widehat{K}(p-1)$-local.

This theorem allows us to immediately prove a result analogous to Theorem 2.4.4 for $\mathrm{EO}_{p-1*}(R)$.

Theorem 4.5.3. *As a module over EO_{p-1*},*

$$EO_{p-1*}(R)=\mathbb{Z}_p\left[\frac{c_2}{p},\ldots,\frac{c_{p-2}}{p^{p-3}},\frac{c_{p-1}}{p^{p-2}},\frac{c_p}{p^p}\right][\Delta^{-1}]_I^\wedge,$$

where I is the maximal ideal of $\pi_0 EO_{p-1}$.

Proof. The proof is exactly the same as for Theorem 2.4.4. The classes c_i arise from various v_0 multiples of the classes arising from $\bar{a}_i$, with the exception of c_p which corresponds to v_1^p. The earlier statements about the filtration of $H_*(R)$ apply equally well, giving this result. □

Working through the example of $X=S^0$ provides an important Adams differential. The class represented by $\bar{a}_{p-2}$ is not a cycle, but the class $[\bar{a}_{p-2}]\alpha_1$ is.

Proposition 4.5.4. *There is a d_2 differential of the form*

$$d_2([\sqrt[m]{v_{p-1}}])=[\bar{a}_{p-2}]\alpha_1.$$

The differentials originating on the root of v_{p-1} are artifacts of the algebraic differentials in the Cartan-Eilenberg spectral sequence for $\mathrm{Ext}_{\mathcal{A}_{\mathrm{eo}_{p-1}*}}$. This class would be an algebraic cycle, but for degree reasons, it now is an Adams d_2. Since this is a spectral sequence of algebras, we know that $[\sqrt[p]{v_{p-1}}]^p$ is a d_2 cycle. This is the class Δ.

There are again two purely topological differentials.

Proposition 4.5.5. *There is a d_{2p-1} differential of the form*

$$d_{2p-1}(\Delta) = \alpha_1\beta_1^{p-1}.$$

This forces a $d_{2(p-1)^2+1}$ differential of the form

$$d_{2(p-1)^2+1}(\alpha_1\Delta^{p-1}) = \beta^{(p-1)^2+1}.$$

4.6 Charts for Computing Ext at 5

To preclude clutter, we introduce the elements $\bar{c}_i$ one at a time. We begin with $\mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_5, \mathbb{F}_5)$ (Figure 4.6). The boxed and arrowed object in position $(40, 5)$ represents a polynomial algebra on v_1^5. The entire picture is repeated starting in this position, and this is what the box represents.

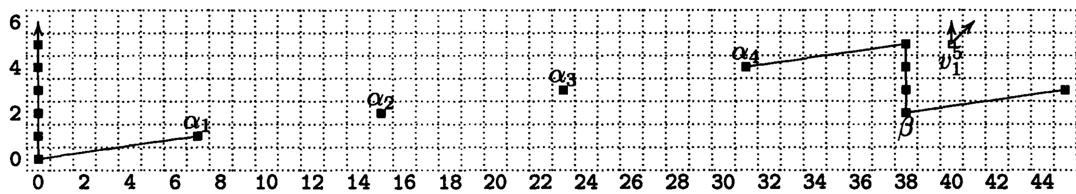

Figure 4-1: The Ring $\mathrm{Ext}_{\mathcal{A}(1)_*}(\mathbb{F}_5, \mathbb{F}_5)$

If we add $\bar{a}_2$, we see that there is a single differential

$$d_1(\bar{c}_2) = \alpha_2.$$

This gives a number of other differentials, including

$$d_1(\alpha_3\bar{c}_2) = v_0^3\beta, d_2(v_0\bar{c}_2^2) = \alpha_4, d_2(\alpha_1\bar{c}_2^2) = v_0^2\beta.$$

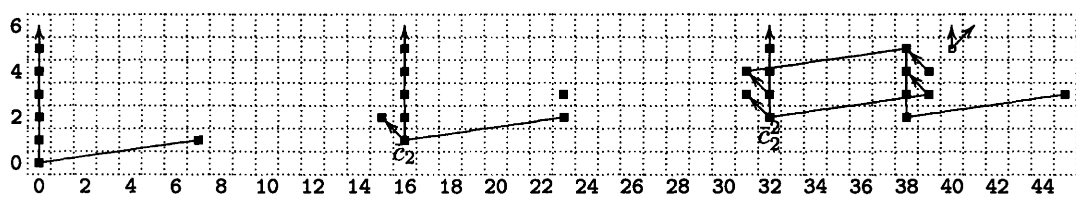

Figure 4-2: The Spectral Sequence for $\mathrm{Ext}_{\mathcal{A}(1)_*\otimes E(\bar{a}_2)}(\mathbb{F}_5, \mathbb{F}_5)$

Massey product considerations demonstrate an extension between $\alpha_1\bar{c}_2$ and α_3. This helps resolve the effects of adding in $\bar{a}_3$.

Massey product considerations again show an extension between $\alpha_1\bar{c}_3$ and $v_0\beta$. This helps complete understanding of the effects of adding in $\bar{a}_4$.

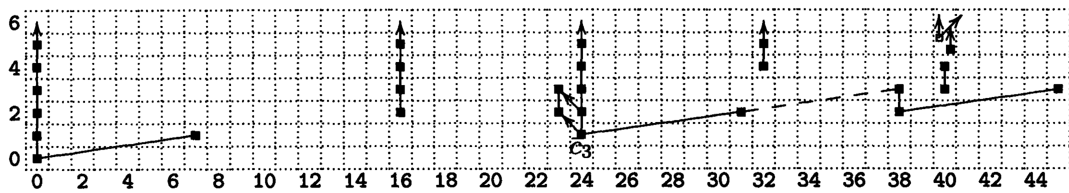

Figure 4-3: The Spectral Sequence for $\mathrm{Ext}_{\mathcal{A}(1)_*\otimes E(\bar{a}_2,\bar{a}_3)}(\mathbb{F}_5,\mathbb{F}_5)$

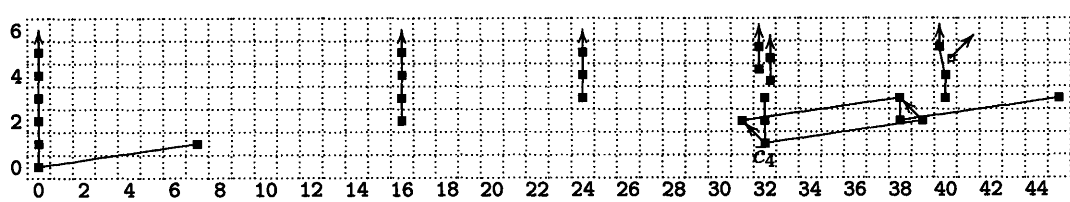

Figure 4-4: The Spectral Sequence for $\mathrm{Ext}_{\mathcal{A}(1)_*\otimes E(\bar{a}_2,\bar{a}_3,\bar{a}_4)}(\mathbb{F}_5,\mathbb{F}_5)$

Chapter 5

Cohomology of $\mathbb{Z}/p^k$ with Applications to Higher K-Theory

5.1 Introduction

The previous chapters have sought to improve the understanding and computability of relatively well-known tools. While the zero line of the homotopy of EO_{p-1} was not known, all of the higher filtration elements were understood, and this allowed a substantial bit of work. For heights beyond $p-1$ at p, almost nothing is known. This chapter establishes some of the pieces needed to complete the analogous computations.

5.2 The Structure of $S(k\overline{\rho}_{p-1})$

Let $\overline{\rho}_{p-1}$ denote the quotient of the regular representation ρ_p of $\mathbb{Z}/p$ by the obvious trivial summand. The module we consider is the symmetric algebra $S_{\mathbb{Z}}(k\overline{\rho}_{p-1})$.

We begin by recalling an unpublished result of Hopkins and Miller.

Proposition 5.2.1. *As a $\mathbb{Z}/p$-module,*

$$S_{\mathbb{Z}}(\overline{\rho}_{p-1}) = S_{\mathbb{Z}}(\mathbb{A})\left\{\mathbb{1}, \overline{\rho}_{p-1}\right\} \oplus \textit{free},$$

where $\mathbb{A}$ and $\mathbb{1}$ are one dimensional trivial representations.

The number of free summands can also be computed, using a dimension count.

Proposition 5.2.2. *There are*

$$\left\lfloor \frac{1}{p}\binom{p+i-2}{i} \right\rfloor$$

permutation summands in $S^i(\overline{\rho}_{p-1})$.

From Proposition 5.2.1 and the simple recollection that the tensor product of a free module with any other module is again free, we conclude the following lemma.

Lemma 5.2.3. *Modulo free summands, as a $\mathbb{Z}/p$-module, if k is odd, then*

$$S(k\overline{\rho}_{p-1}) = S(\mathbb{A}_1, \dots, \mathbb{A}_p) \otimes \left(\mathbb{1} \oplus \binom{k}{1}\overline{\rho}_{p-1} \oplus \binom{k}{2}\mathbb{1} \oplus \cdots \oplus \binom{k}{1}\mathbb{1} \oplus \overline{\rho}_{p-1}\right).$$

If k is even, then

$$S(k\overline{\rho}_{p-1}) = S(\mathbb{A}_1, \dots, \mathbb{A}_p) \otimes \left(\mathbb{1} \oplus \binom{k}{1}\overline{\rho}_{p-1} \oplus \binom{k}{2}\mathbb{1} \oplus \cdots \oplus \binom{k}{1}\overline{\rho}_{p-1} \oplus \mathbb{1}\right).$$

Proof. This follows from the proposition immediately, using the binomial theorem and the fact that the symmetric algebra functor is exponential. The identifications of the tensor powers of $\overline{\rho}_{p-1}$ is a classical result. □

5.2.1 Computation of the Tate Cohomology

From Lemma 5.2.3, we can immediately compute the Tate cohomology of $\mathbb{Z}/p$ with coefficients in $S(k\overline{\rho}_{p-1})$.

Lemma 5.2.4.

$$\widehat{H}\big(\mathbb{Z}/p; S(k\overline{\rho}_{p-1})\big) = \mathbb{F}_p[x_2^{\pm 1}] \otimes \mathbb{F}_p[\Delta_1, \dots, \Delta_k] \otimes E(\alpha_1, \dots, \alpha_k),$$

where the generators α_i are in $\widehat{H}^1$ and correspond to the generators of $\widehat{H}^1(k\overline{\rho}_{p-1})$ in the decomposition in Lemma 5.2.3. The generators Δ_i are in $\widehat{H}^0$ and correspond to the trivial summands of the same name.

5.2.2 The Higher Cohomology of $\mathbb{Z}/p$

The computations already done essentially give this result. In dimensions greater than 0, the Tate cohomology coincides with the ordinary cohomology.

To concisely express the higher cohomology, we need some notation. Let I denote a subset of the set $\{1, \dots, k\}$, and let

$$\alpha_I = \prod_{i \in I} \alpha_i, \quad \|I\| = \left\lfloor \frac{|I|}{2} \right\rfloor.$$

With this notation, modulo the free summands ignored previously, we can complete the computation.

Lemma 5.2.5. *As an algebra, the higher cohomology is given by*

$$H^*\big(\mathbb{Z}/p; S(k\overline{\rho}_{p-1})\big) = \mathbb{F}_p[x_2] \otimes \mathbb{Z}_p[\Delta_1, \dots, \Delta_k] \otimes \bigotimes_I E\left(\frac{\alpha_I}{x_2^{\|I\|}}\right),$$

modulo the obvious relations involving the expressions α_I.

5.2.3 Concrete Example with H^0 Information

There is essentially only one example which can be worked out in full, and this carries interesting topological information. If we let $p = 3$ and $k = 1$, then we can essentially reconstruct the Hopkins-Miller result about the Adams-Novikov E_2 term for the homotopy of tmf. The module $\overline{\rho}_2$ can be identified with $\mathbb{Z}_3\{x, y\}$, and if $\langle g \rangle = \mathbb{Z}/3$, then

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{g} \begin{pmatrix} y \\ -x - y \end{pmatrix}.$$

One can readily compute the Poincaré series for the ring of invariants, using the following observations:

1. If $n = 3k$, then $S^n(\overline{\rho}_2)$ has a trivial summand, and if $n = 3k + 1$, then $S^n(\overline{\rho}_2)$ has a summand of $\overline{\rho}_2$.
2. If $n = 3k + j + 2$, where $0 \leq j < 3$, then $S^n(\overline{\rho}_2)$ has $k + 1$ summands of the regular representation ρ_3.

The first observation is essentially a restatement of Proposition 5.2.1, while the second follows from this by a dimension count. Together, these give the Poincaré series for the ring of invariants:

$$p_{H^0}(t) = \frac{1}{1 - t^3} + \frac{t^2 + t^3 + t^4}{(1 - t^3)^2} = \frac{1 - t^6}{(1 - t^2)(1 - t^3)^2},$$

where the first summand comes from the trivial factors and the second comes from the 3 types of regular representations. Direct computation allows us to find three invariant elements a_2 in $S^2(\overline{\rho}_2)$, b_3 in the permutation summand of $S^3(\overline{\rho}_2)$, and Δ_3 in the trivial summand. With these, it is easy to prove the following proposition.

Proposition 5.2.6.

$$H^0(\mathbb{Z}/3; S(\overline{\rho}_2)) = \mathbb{Z}_3[a_2, b_3, \Delta_3]/4a_2^3 - b_3^2 = 27\Delta^2.$$

In the topological setting, these are all graded objects, and there is an action of group of order 4. The group action sends Δ to $-\Delta$, and the elements degrees are 4 times their subscripts. When we pass to the invariants under this final group action, we can fully recover the Adams-Novikov E_2 term for the homotopy of tmf.

5.3 Applications to Higher Real K-Theory

The homotopy groups of $EO_n(G)$ are computed using the homotopy fixed point spectral sequence, the E_2 term of which is $H^*(G; E_{n*})$, and a theorem of Hewett shows that if $p^k(p-1)$ divides n and p^{k+1} does not, then the largest p-subgroup of $\mathbb{G}_n$ is $\mathbb{Z}/p^{k+1}$ [15].

The structure of E_{n*} as a G module is quite complicated for subgroups G for which p divides $|G|$, and regrettably, this is also the most interesting case, as these subgroups have higher cohomology that is closer to that of $\mathbb{G}_n$. However, if we restrict attention to $n = k(p-1)$ for $k \leq p$, then the computations of § 5.2 provide a starting point for the computations of these group cohomology computations. From Hewett's result, it is clear that the results for $k < p$ have a different flavor than those for $k = p$, and we handle them separately.

5.3.1 Height $k(p-1)$ for $k < p$

Devinatz and Hopkins compute a recursive formula for the action of $\mathbb{G}_n$ on E_{n*}. The formula can be recast as showing that there is a filtration of E_{n*} such that the associated graded is simply $S(k\overline{\rho}_{p-1})^{\wedge}_{I}[\mathbb{A}^{-1}]$, where I is a particular ideal which sits in the free summand of the symmetric group and where $\mathbb{A}$ is the product of the trivial one dimensional representations. This gives a spectral sequence of the form

$$H^*\big(\mathbb{Z}/p; S(k\overline{\rho}_{p-1})^{\wedge}_{I}[\mathbb{A}^{-1}]\big) \Rightarrow H^*(\mathbb{Z}/p; E_{k(p-1)*}). \tag{5.1}$$

Since I lies in the free summands, it does not affect the higher cohomology in any way. Similarly, $\mathbb{A}$ is a trivial summand, so the result of formally inverting it is simply to invert the class Δ in the cohomology. With these observations, however, the higher cohomology of E_1 term of Spectral Sequence 5.1 is exactly the result of Lemma 5.2.5 with the product of the classes Δ_i inverted. It remains only to compute the algebraic differentials and any differentials in the homotopy fixed point spectral sequence.

5.3.2 Height $p(p-1)$

Here the computations of Devinatz and Hopkins show that there is a filtration of E_{n*} such that the associated graded is $S\big(\overline{\rho}_{p(p-1)}\big)^{\wedge}_{I}[\Delta^{-1}]$, where I is a particular ideal in the symmetric algebra and Δ is a distinguished class corresponding essentially to the norm of the invertible class u. This gives a spectral sequence

$$H^*\big(\mathbb{Z}/p^2; S\big(\overline{\rho}_{p(p-1)}\big)^{\wedge}_{I}[\mathbb{A}^{-1}]\big) \Rightarrow H^*(\mathbb{Z}/p^2; E_{n*}). \tag{5.2}$$

To compute the E_1 term of this spectral sequence, we use the Hochschild-Serre Spectral Sequence based on the short exact sequence

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0.$$

This is a spectral sequence of the form

$$H^*\Big(\mathbb{Z}/p; H^*\big(\mathbb{Z}/p; S(\overline{\rho}_{p(p-1)})^{\wedge}_{I}[\mathbb{A}^{-1}]\big)\Big) \Rightarrow H^*\Big(\mathbb{Z}/p^2; S\big(\overline{\rho}_{p(p-1)}\big)^{\wedge}_{I}[\mathbb{A}^{-1}]\Big). \tag{5.3}$$

This spectral sequence is quite complicated, starting with the computation of the E_1 term.

We begin by recalling that the restriction of the representation $\overline{\rho}_{p(p-1)}$ to the subgroup $\mathbb{Z}/p$ is $p\overline{\rho}_{p-1}$. The action of the quotient $\mathbb{Z}/p$ on $H^*(\mathbb{Z}/p; p\overline{\rho}_{p-1})$ is readily determined to be the regular representation.

5.4 Recent Work and Indications of Future Developments

Lemma 5.4.1. *Let $n = p^k f(p-1)$ with $p \not| f$. If $g \in \mathbb{G}_n$ has order p, then, possibly after extending scalars to a larger residue field, there exists an element $\sigma \in \mathbb{G}_n$ such that $\sigma^{p^k} = g$. In other words, after passing to the Witt vectors of the algebraic closure of $\mathbb{F}_p$, every subgroup of $\mathbb{G}_n$ isomorphic to $\mathbb{Z}/p$ extends to a subgroup isomorphic to $\mathbb{Z}/p^{k+1}$.*

Proof. This is essentially a consequence of the Noether Theorem about automorphisms of division algebras over $\mathbb{Q}_p$.

We first recall the definition of $\mathbb{G}_n$. This is the group of units of the maximal order of the division algebra $\mathbb{D}_n$ over $\mathbb{Q}_p$ with Hasse invariant $\frac{1}{n}$. One of the properties of this division algebra is that it contains all extension fields of $\mathbb{Q}_p$ of degrees dividing n. In particular, it contains the ramified extension field of $\mathbb{Q}_p$ given by adjoining the p^{th} root of one g. We can moreover form the field extension $\mathbb{Q}_p[g][\sigma]$, where σ is a p^kth root of g. Since the degree of this extension is $p^k(p-1)$, this extension is a subring of $\mathbb{D}_n$. It is moreover a subring of the ring of integers, since g was. This implies that there is a p^kth root of g in $\mathbb{G}_n$, as was required. The extension of scalars ensures that the previous inclusion can still be satisfied. □

To demonstrate the effectiveness of this lemma, we need to recall the full form of Devinatz and Hopkins result about the action of the Morava stabilizer group on E_{n*}.

Proposition 5.4.2. *There is a filtration of $E_{p^k(p-1)*}$ such that the associated graded is a localization of a completion of the symmetric algebra on $\overline{\rho}_{p^k(p-1)}$. The spectral sequence of this filtration is of the form*

$$H^*(\mathbb{Z}/p^{k+1}; S(\overline{\rho}_{p^k(p-1)})^{\wedge}_I[\Delta^{-1}]) \Rightarrow H^*(\mathbb{Z}/p^{k+1}; E_{p^k(p-1)*}). \tag{5.4}$$

Theorem 5.4.3. *If there is an element a in E_* such that $(1-g)(a) = p \cdot \text{unit}$ and such that $|N(u)|$ divides $|a|$, where*

$$N(u) = \prod_{\theta \in \mathbb{Z}/p^{k+1}} \theta(u),$$

then possibly after extending scalars, Spectral Sequence (5.4) collapses at the E_1 term.

Proof. To prove this, we must produce a new invertible element v in degree 2 whose trace under the action of $\mathbb{Z}/p^k$ is 0 and which is not the norm of any other element.

Building v is quite easy. Let m be the quotient of $|a|$ by $|N(u)|$, and let

$$v = \frac{\frac{1-g}{p}(a)}{N(u)^m}.$$

The conditions on a ensure that this has the right degree and that this is well defined. Moreover, this is a unit in degree 2, meaning that modulo the maximal ideal $\mathfrak{m}$ in

$\pi_0(E_n)$, $v \equiv u$. The Devinatz-Hopkins result shows

$$M = (u, uu_1, \ldots, uu_{n-1}) \mod p, \mathfrak{m}^2$$

is a copy of the Dieudonné module. In particular, this is generated as a $\mathbb{Z}/p^k$ module by u. The equivalence of u and v modulo $\mathfrak{m}$ implies that v also generates M. However, since v is a traceless element, the structure of the Dieudonné module ensures that the $\mathbb{Z}/p^k$ submodule of $\pi_{-2}(E_n)$ generated by v is isomorphic to M itself. This gives the collapse of Spectral Sequence (5.4), since it shows that the associated graded of $\pi_*(E_n)$ built by Devinatz and Hopkins is equivariantly isomorphic to $\pi_*(E_n)$. □

5.4.1 Ravenel's Work and Hopes for Elements

Recent work of Ravenel might produce such an element of $\pi_*(E_n)$ [29, 30]. Ravenel produces two families of deformations of the Artin-Schreier curve

$$y^e = x^p - x,$$

where $n = f(p-1)$, and $e = p^f - 1$.

The first family is corepresented by the Lubin-Tate ring and the formal completion of the Jacobian has a one dimensional summand isomorphic to the universal deformation of the Honda group. This family suffers from the draw-back that there is no obvious action of $\mathbb{Z}/p$ on the curves. Ravenel remedies this problem by increasing the number of curves considered, enlarging the moduli stack to include a larger family. He shows up to first order that the formal completion of the Jacobian again has a summand isomorphic to the universal deformation. Moreover, this stack has an obvious action by $\mathbb{Z}/p$ (in fact, multiple copies of $\mathbb{Z}/p$). Regrettably, the natural étale cover of this stack is by a ring whose Krull dimension is larger than that of $\pi_* E_n$. Since Krull dimension is invariant under passing to the invariants under a group action, this implies that Ravenel's larger moduli stack is not the appropriate moduli stack for building EO_n.

However, it is hoped that the map from the corepresenting ring for Ravenel's family of curves to the Lubin-Tate ring is $\mathbb{Z}/p$ equivariant. It is easy to check that there is a distinguished element a in the corepresenting ring which transforms as

$$a \mapsto a + pr,$$

where r is a generator of the comorphism ring. It is also hoped that the element a (which behaves like v_f) maps to a non-zero element in $\pi_* E_n$. This element would satisfy all of the properties required for Theorem 5.4.3.

Chapter 6

A Computational Lemma for Differentials in Spectral Sequences

6.1 Introduction

6.1.1 Organization

In §6.2, we prove the key result that, subject to certain hypotheses, makes everything work out,

$$d_2(c \cdot a \cdot b) = \langle d_1(c), a, d_1(b) \rangle.$$

The remainder of the section establishes variants of this in a sequence of propositions. In §6.3, we use the main Lemma and its variants to reëstablish some classical results and demonstrate other simple applications.

6.1.2 Conventions

All of our algebras will be filtered differential graded algebras. If a is a homogeneous element of our algebra, then $|a|$ will denote its degree, and $\overline{a}$ will denote $(-1)^{|a|}a$. Moreover, all spectral sequences we consider are the spectral sequence associated to the given given filtration.

6.2 Higher Differentials out of Lower Ones

6.2.1 Main Result

Lemma 6.2.1. *Let a, b, and c be elements of $\mathcal{A}$ such $a \in F_0\mathcal{A}$, $b, c \in F_1\mathcal{A}$, and in $Gr(A)$, $d_1(b) \neq 0 \neq d_1(c)$ and*

$$a \cdot d_1(b) = d_1(c) \cdot a = 0.$$

Then we have

$$d_2(c \cdot a \cdot b) \in -(-1)^{|a|+|c|}\langle d_1(c), a, d_1(b) \rangle.$$

Proof. Ignoring the filtrations of the elements involved, the element $c \cdot a \cdot b$ visibly bounds one of the cycle representing the Massey product, since we can just apply the Leibnitz rule. The subtlety is incorporating the filtrations to allow us to apply this to spectral sequences.

The condition $a \cdot d_1(b) = 0$ implies that there is an element $x \in F_0\mathcal{A}$ such that

$$d(x) = d_0(x) = a \cdot d_1(b).$$

We similarly conclude that there is an element $y \in F_0\mathcal{A}$ such that

$$d(y) = d_0(y) = d_1(c) \cdot a.$$

The Leibnitz rule ensures that $c \cdot a \cdot b$ is a d_1 cycle. This means that we can find an element in $F_1\mathcal{A}$ such that the boundary of $c \cdot a \cdot b$ plus this element lands in filtration 0. The element is easy to find, however, given the bounding elements named above:

$$(-1)^{|a|} c \cdot x + y \cdot b.$$

For filtration reasons, the d_2 differential on $c \cdot a \cdot b$ is determined by taking the ordinary differential on

$$c \cdot a \cdot b - \big((-1)^{|a|} c \cdot x + y \cdot b\big).$$

This gives

$$-(-1)^{|a|} d_1(c) \cdot x - \overline{y} \cdot d_1(b) = -(-1)^{|a|+|c|}\big(y \cdot d_1(b) - \overline{d_1(c)} \cdot x\big).$$

However, this last term is obviously a representative of the Massey product in question.

It should also be noted that any two choices of x and y differ by a cycle. This change is perpetuated through the proof, giving a different representative of the Massey product. Conversely, any representative of the Massey product allows us to determine new choices for x and y, so we can conclude that in fact every element in the Massey product is the boundary of a representative of $c \cdot a \cdot b$. ☐

6.2.2 Variants of the Lemma

This lemma generalizes a great many ways. We can first consider strings of longer length.

When the algebra is commutative, we can generalize to strings of longer length.

Lemma 6.2.2. *Let $a \in F_0\mathcal{A}$ and $b \in F_1\mathcal{A}$. If for all $i < k$, $\langle a, \underbrace{d_1(b), \dots, d_1(b)}_{i}\rangle = 0$, with no indeterminacy then*

$$d_k(ab^n) \in (-1)^{(|a|-1)(k+1)} \frac{n!}{(n-k)!} \langle a, \underbrace{d_1(b), \dots, d_1(b)}_{k}\rangle b^{n-k}.$$

Proof. For $i < k$, let $x_i \in F_0\mathcal{A}$ be such that

$$d(x_i) = \langle a, \underbrace{d_1(b), \ldots, d_1(b)}_{i} \rangle.$$

With this notation, we note that for $j \leq k$,

$$\langle a, \underbrace{d_1(b), \ldots, d_1(b)}_{j} \rangle = x_{j-1} d_1(b).$$

If $|a|$ is odd, then $|x_i|$ is odd for all i, meaning that $\overline{x_i} = -x_i$. If $|a|$ is even, then again, so is $|x_i|$. In what follows, for ease of notation, we assume that $|a|$ is odd. If this is not the case, then a sign is introduced at every stage, producing the alternating signs shown in the statement of the lemma.

Now the proof follows by induction, with the base case being clear. Assume that

$$d_{m-1}(ab^n) = \frac{n!}{(n-m+1)!} \langle a, \underbrace{d_1(b), \ldots, d_1(b)}_{m-1} \rangle b^{n-m+1}.$$

The assumptions on the vanishing of these Massey products allows us to complete ab^n to a d_{m-1} cycle by noting that

$$d_{m-1}\left(ab^n - \frac{n!}{(n-m+1)!} x_{m-1} b^{n-m+1}\right) = 0.$$

The differential d_m is then given by

$$-\frac{n!}{(n-m+1)!}(n-m+1)\overline{x_{m-1}} d_1(b) b^{n-m} = \frac{n!}{(n-m)!}\left(x_{m-1} d_1(b)\right) b^{n-m}.$$

Recalling that $x_{m-1} d_1(b)$ is another name for the desired Massey product completes the proof. □

We can also formulate a form that has applications to Serre type spectral sequences, and the proof is exactly analogous.

Proposition 6.2.3. *Let $c \in F_s\mathcal{A}$, $a \in F_0\mathcal{A}$, and $b \in F_t\mathcal{A}$ be such that $d(c) \in F_0\mathcal{A}$, $d(a) = 0$, and $d(b) \in F_0\mathcal{A}$. Then if the analogous hypotheses of the previous lemma are satisfied,*

$$d_{s+t}(c \cdot a \cdot b) \in \langle d_s(c), a, d_t(b) \rangle,$$

where the Massey product is again viewed as occurring on the E_1 page.

If the algebras in question are bigraded algebras, then we can take the internal grading into consideration if the differential includes it. This type of example occurs in the Serre and Adams spectral sequences. We assume that the internal differential has degree -1.

Proposition 6.2.4. *Let $c \in F_s\mathcal{A}$, $a \in F_0\mathcal{A}$, and $b \in F_t\mathcal{A}$ be such that $d(c) \in F_0\mathcal{A}$, $d(a) = 0$, and $d(b) \in F_0\mathcal{A}$. Then if the analogous hypotheses of the previous lemma are satisfied,*

$$d_{s+t-1}(c \cdot a \cdot b) \in \langle d_s(c), a, d_t(b) \rangle,$$

where the Massey product is again viewed as occurring on the E_2 page.

6.2.3 A Massey Product Lemma for Massey Products

We can further generalize Lemma 6.2.1 by considering differentials on higher products. We begin with a simple form that can be readily proved.

Proposition 6.2.5. *If $a \in F_0\mathcal{A}$ and $b \in F_1\mathcal{A}$, a^2 and ab are zero in homology, and $\langle a, a, d_1(b) \rangle = 0$ in E_1, then*

$$d_2(\langle a, a, b^2 \rangle) = \langle a, a, d_1(b), d_1(b) \rangle.$$

6.3 Applications

6.3.1 Kraines' Results on Massey Powers

Proposition 6.2.4 allows for a quick proof of Kraines' results linking iterated self products with Steenrod operations at an odd prime [20].

Corollary 6.3.1. *If $x \in H^{2k+1}(X; \mathbb{F}_p)$, then*

$$\beta\mathcal{P}^k x \in \langle \underbrace{x, \ldots, x}_{p} \rangle.$$

Proof. We show this via universal example, using the Serre spectral sequence for the fibration

$$F = K(\mathbb{F}_p, 2k) \to EK(\mathbb{F}_p, 2k+1) \to B = K(\mathbb{F}_p, 2k+1).$$

The element $i_{2k} \in H^{2k}(F)$ transgresses to the element $i_{2k+1} \in H^{2k+1}(B)$.

The element $i_{2k+1} \cdot i_{2k}^{p-1}$ is a d_{2k+1} cycle, and the Kudo transgression theorem shows that this element transgresses to $\beta\mathcal{P}^k i_{2k+1}$. However, Propositions 6.2.4 and 6.2.2 show that we then have

$$\beta\mathcal{P}^k i_{2k+1} = \langle \underbrace{i_{2k+1}, \ldots, i_{2k+1}}_{p} \rangle.$$

□

Remark. Kraines shows a slightly stronger result, defining iterated Massey powers of an element. In this situation, we can modify the proof of Lemma 6.2.1 to reproduce his actual equality.

We can also prove an analogous statement for the Dyer-Lashof algebra, using $Q(S^n)$ and the path-space fibration $Q(S^{n-1}) \to * \to Q(S^n)$.

Corollary 6.3.2. *If X is an infinite loop space and if $x \in H_{2n+1}(X; \mathbb{F}_p)$, then*

$$\beta Q^n x = \langle \underbrace{x, \ldots, x}_{p} \rangle.$$

Proof. The proof is again via the Serre spectral sequence, using the example of $Q(S^{2n+1})$. The required result follows from simply equating the Massey product consequence of Proposition 6.2.4 with the consequence of the Kudo transgression theorem. □

Bibliography

[1] Vigleik Angeltveit, *Private correspondence.*

[2] ———, *Non-commutative ring spectra*, Ph.D. thesis, MIT, 2006.

[3] Greg Arone and Mark Mahowald, *The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres*, Invent. Math. **135** (1999), no. 3, 743–788.

[4] Andrew Baker and Andrej Lazarev, *On the Adams spectral sequence for R-modules*, http://hopf.math.purdue.edu/Baker-Lazarev/Rmod-ASS.pdf.

[5] Tilman Bauer, *Computation of the homotopy of the spectrum tmf*, arXiv:math.AT/0311328.

[6] Mark Behrens, *A modular description of the $K(2)$-local sphere at the prime 3*, Topology **45** (2006), no. 2, 343–402.

[7] Mark Behrens and Satya Pemmaraju, *On the existence of the self map v_2^9 on the Smith-Toda complex $V(1)$ at the prime 3*, Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, Contemp. Math., vol. 346, pp. 9–49.

[8] Donald M. Davis and Mark Mahowald, *Ext over the subalgebra A_2 of the Steenrod algebra for stunted projective spaces*, Current trends in algebraic topology, Part 1 (London, Ont., 1981), CMS Conf. Proc., vol. 2, pp. 297–342.

[9] ———, *v_1- and v_2-periodicity in stable homotopy theory*, Amer. J. Math. **103** (1981), no. 4, 615–659.

[10] ———, *The nonrealizability of the quotient $A//A_2$ of the steenrod algebra*, American Journal of Mathematics **104** (1982).

[11] ———, *The spectrum $(P\wedge bo)_{-\infty}$*, Math. Proc. Cambridge Philos. Soc. **96** (1984), no. 1, 85–93.

[12] Ethan S. Devinatz and Michael J. Hopkins, *The action of the Morava stabilizer group on the Lubin-Tate moduli space of lifts*, Amer. J. Math. **117** (1995), no. 3, 669–710.

[13] P. Goerss, H.-W. Henn, M. Mahowald, and C. Rezk, *A resolution of the $K(2)$-local sphere at the prime 3*, Ann. of Math. (2) **162** (2005), no. 2, 777–822.

[14] V. Gorbounov and M. Mahowald, *Formal completion of the Jacobians of plane curves and higher real K-theories*, J. Pure Appl. Algebra **145** (2000), no. 3, 293–308.

[15] Thomas Hewett, *Finite subgroups of division algebras over local fields*, J. Algebra **173** (1995), no. 3, 518–548.

[16] Michael J. Hopkins, *$K(1)$-local E_∞ ring spectra.*

[17] Michael J. Hopkins and Mark Mahowald, *Construction of some self maps with v_2^{32} periodicity.*

[18] Michael J. Hopkins and Haynes Miller, *E_n has a unique E_∞ structure*, Yet unpublished.

[19] Michael J. Hopkins and Haynes R. Miller, *Elliptic curves and stable homotopy I*, Difficult to find.

[20] David Kraines, *Massey higher products*, Trans. Amer. Math. Soc. **124** (1966), 431–449.

[21] M. Mahowald and R. James Milgram, *Operations which detect Sq^4 in connective K-theory and their applications*, Quart. J. Math. Oxford Ser. (2) **27** (1976), no. 108, 415–432.

[22] Mark Mahowald, *A new infinite family in ${}_2\pi_*{}^s$*, Topology **16** (1977), no. 3, 249–256.

[23] ———, *The image of J in the EHP sequence*, Ann. of Math. (2) **116** (1982), no. 1, 65–112.

[24] Mark Mahowald and Charles Rezk, *Brown-Comenetz duality and the Adams spectral sequence*, Amer. J. Math. **121** (1999), no. 6, 1153–1177.

[25] Ju. I. Manin, *Theory of commutative formal groups over fields of finite characteristic*, Uspehi Mat. Nauk **18** (1963), no. 6 (114), 3–90.

[26] John W. Milnor and John C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264.

[27] Norihiko Minami, *On the odd-primary Adams 2-line elements*, Topology Appl. **101** (2000), no. 3, 231–255.

[28] Lee S. Nave, *On the non-existence of Smith-Toda complexes*, 1998, http://hopf.math.purdue.edu.

[29] Douglas Ravenel, *Toward Higher Chromatic Analogs of Elliptic Cohomology*, In progress.

[30] ———, *Toward Higher Chromatic Analogs of Elliptic Cohomology II*, In progress.

[31] Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press Inc., Orlando, FL, 1986.