

Optimal Trading Strategies vs. a Statistical Adversary

by

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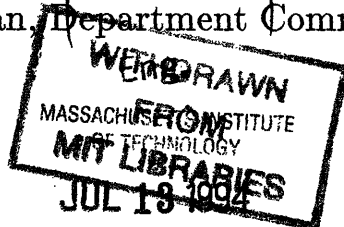
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Abstract

The *distributional approach* and *competitive analysis* have traditionally been used for the design and analysis of on-line algorithms. The former assumes a specific distribution on inputs, while the latter assumes inputs are chosen by an unrestricted adversary. This thesis employs the *statistical adversary* (recently proposed by Raghavan) to analyze and design on-line algorithms for *two-way currency trading*. The statistical adversary approach may be viewed as a hybrid of the distributional approach and competitive analysis. By statistical adversary, we mean an adversary that generates input sequences, where each sequence must satisfy certain general statistical properties. The on-line algorithms presented in this thesis have some very attractive properties. For instance, the algorithms are *money-making*; they are guaranteed to be profitable when the optimal off-line algorithm is profitable. Previous on-line algorithms although “competitive”, can lose money, even though the *optimal off-line* algorithm makes money. Against a weak statistical adversary, our methods yield an algorithm that outperforms the optimal off-line “buy-and-hold” strategy. Furthermore, it is guaranteed to make a substantial profit when it is known that the market is *active* and *stable* (i.e. there are fluctuations but the upward and downward fluctuations tend to balance each other). In fact, our algorithm even makes money when the market exhibits a slightly unfavorable trend.

Thesis Supervisor: Mauricio Karchmer

Title: Professor of Mathematics

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Chapter 1

Introduction

In many situations, we are forced to choose between different alternatives without knowledge of each alternative's future worth. Our choices must be made in an *on-line* manner. However, we often have partial information about the future. For example, we may know that with very high probability, certain statistical properties are satisfied. When making choices to maximize our future gain, it makes sense to incorporate such information into these decisions. Similarly, on-line algorithms that solve these types of problems can gain significantly by including some knowledge of the future. In the areas of finance, economics, and operations research, we find examples of this kind of decision making process (e.g. [20, 25, 29, 33]). In this thesis, we examine a two-way currency trading problem against a *statistical adversary* [26].

1.1 Techniques for analyzing on-line algorithms

The analysis of on-line algorithms has typically involved either *distributional analysis* or *competitive analysis*. In the former approach, the input is assumed to conform to a “natural” or “typical” probability distribution. Based upon this distribution, one seeks strategies with good average case performance. In the latter approach, the input is generated by an adversary. In this case, one seeks to design on-line algorithms which compare favorably against the optimal off-line algorithm.

In practice, the premise that a fixed distribution governs the nature of the input

is questionable. Moreover, even if the input in question follows a particular fixed (or stable) probability distribution, it is often difficult to identify or construct a mathematical model that accurately reflects the true distribution. For instance, a great deal of effort has been invested in an attempt to identify probability distributions of currency exchange rates, but there is still no evidence that such distributions exist. A wide variety of (sometimes conflicting) opinions concerning the existence and/or nature of such distributions can be found in [7, 10, 12, 21, 22, 23, 28, 32].

The competitive analysis approach first appeared in works on *bin packing* in the 1970's [14, 16, 17, 34] and then was explicitly formulated in the 1980's [4, 6, 18, 30]. This approach considers input sequences that are generated by an adversary and measures performance with respect to the *optimal off-line* algorithm using the same input sequence. Under this model, one avoids making the assumptions required by the distributional approach. Instead, the assumption is made that the input will be strictly adversarial. Specifically, the input sequence is chosen to minimize the algorithm's overall performance relative to the optimal off-line algorithm. Such a powerful adversary often does not reflect the nature of the input to many practical problems.

Indeed, a few researchers have recognized the need for some middle ground approach that lies somewhere between the pessimistic competitive analysis and the more optimistic distributional approach. They introduced a number of alternative approaches [3, 5, 19, 26, 35]. In this thesis we focus on Raghavan's *statistical adversary* [26]. Here, the underlying idea is to limit the power of the adversary in some way dependent on the particular problem. Namely, the adversary is required to generate input sequences satisfying certain (statistical) properties. For example, the adversary may be required to maintain certain bounds on the number of requests of a certain type, or to produce input sequences of which certain subsequences must satisfy particular constraints. The premise of this approach is that input sequences arising in reality exhibit and conform to certain long-term statistics. Thus, by exploiting such statistical constraints, the goal is to eliminate the possibility of extremely bad input sequences, which do not occur frequently in reality. The hope is to show that one can

place limited and realistic restrictions on an adversary which then allow the on-line algorithm to perform well.

An important issue regarding the statistical adversary is whether the on-line player should be allowed to make use of the statistical parameters associated with the adversary. On the one hand, it would be more elegant to seek on-line algorithms that perform optimally (or well) for all possible choices of the parameters and to use these parameters only for the analysis (i.e. to express the performance in terms of these parameters). On the other hand, we may allow our algorithms to use these parameters for the purpose of obtaining better performance. We refer to algorithms of the former type as *universal* algorithms, and algorithms of the latter type as *non-universal* algorithms. Certainly, universal algorithms are more desirable because they do not assume knowledge of the parameters. However, in the universal approach, the performances of two different algorithms are likely to depend differently upon the parameters. In order to decide which algorithm to use, one needs to judge, perhaps by conducting statistical tests, which algorithm will perform better under the particular circumstances. By doing so, one is transforming the two universal algorithms into a single non-universal algorithm. In addition, since a non-universal algorithm is tuned to the parameters, it can be designed to perform significantly better than the corresponding universal algorithm.

1.2 Two-way currency trading

In this thesis we use the framework of the statistical adversary to analyze the *two-way currency trading* game that is discussed in [11]. Specifically, we consider a discrete variant of this problem in which the on-line player begins with some money, say dollars, and is given an opportunity to invest in another currency, say yen, for some period of time. The player would like to maximize his returns during this time by taking advantage of fluctuations in the exchange rates by converting back and forth between dollars and yen. The player assumes that a statistical adversary is controlling the change in exchange rates. In the following chapters, we will analyze this problem

for both strong and weak adversaries.

The two-way currency trading problem was first analyzed using traditional competitive analysis. In [11], the authors present a competitive on-line strategy. However, though this strategy is competitive, it is possible to experience losses although the off-line strategy might be profitable. Before formally analyzing the problem, we give an overview of the foreign exchange market.

1.2.1 The foreign exchange market

Financial investors trade currencies as they would other commodities (e.g. stocks and bonds). In 1993, the *daily* worldwide volume of currency trading was around \$750 billion[27]. One might question whether this volume is due to the growth of international trade or active currency trading (speculation). The volume of international trade is less than 1% of the currency trading volume. Over 95% of the transactions are between financial firms and banks. Therefore, one must conclude that currency trading is not only an instrument of international trade, but has become a major source of economic activity and investment.

Before 1973, the foreign exchange market followed a fixed exchange rate scheme. With fixed exchange rates, the exchange rate was set by the participating governments. Over time, the exchange rate no longer reflected the actual market price for various currencies. As a result, currencies often had to be re-valued. In 1973, a floating exchange rate scheme was enacted. Now, the exchange rate is determined by market pressures rather than international agreements.

Today, the foreign exchange market is composed of financial firms and government banks. London, New York, and Tokyo form the major trading centers with additional smaller trading centers located throughout the world. Because of the overlap in time zones, the foreign exchange market is active 24 hours a day. Naturally, the volume of activity will vary depending on which trading centers are open. Unlike most other investments, there is no fee for performing a foreign exchange transaction. Instead, the cost of transactions is incorporated into the *bid-ask spread*. The bid-ask spread is the difference between the buy and sell exchange rates. Thus, financial firms or

banks can generate revenue by conducting transactions. In this thesis, transactions costs will refer to the bid-ask spread.

When investing in foreign exchange, there are two main steps involved: forecasting and trading. Forecasting is predicting how exchange rates will behave in the future. Given a forecast, the investor must decide how much to invest and when to invest. For this, he relies on his trading strategy. If forecasting were perfect, trading would be trivial. Unfortunately, forecasts are often wrong in direction, magnitude, and timing. Furthermore, the reliability of forecasts is often unknown. Trading strategies must be able to handle this uncertainty. Ideally, a trading strategy should be robust enough to handle faulty forecasts, but still aggressive enough to profit when given accurate forecasts.

Traditionally, both forecasting and trading have been done by human experts. The recent trend has been toward computerized trading. Specific algorithms are being used to both forecast and trade. Most of the emphasis has been on forecasting, while trading has received scant attention. In this thesis, we examine optimal trading strategies against a statistical adversary.

1.2.2 Forecasting

Forecasting techniques can be divided into two classes: fundamentals and technical analysis. Fundamentals rely on basic economic principles such purchasing-power parity and interest rate differentials. They tend to predict long term behavior. Technical analysis uses historical data to detect patterns and trends. These predictions are generally short-term and generally cannot be explained by economic principles. Both techniques have their merits and drawbacks. Fundamentals can give an investor an economic reason for the forecast, while technical analysis cannot. However, fundamentals sometimes only give a direction of change (either up or down), while technical analysis can give estimates of the magnitude. In addition, since fundamental are generally used for long term predictions, it may take years to accurately evaluate the performance of a fundamental model. On the other hand, short-term technical analysis models can be evaluated in weeks or months. Moreover, performance is

the overriding factor, when judging financial models. If a technical analysis model can consistently give accurate forecasts, few people will care about the basis of the underlying theory.

In a survey of major financial trading firms[31], the majority of firms used both techniques. Although very different in nature, they can often be used in a complementary fashion. For example, fundamentals can be used to filter out salient currencies, after which, technical analysis can offer additional analysis and reinforce suspicions. In essence, the investor is given two different views of the future. One view based on economics, and the other based on trends and patterns. Fundamentals belongs to the study of economics and is beyond the scope of this thesis. Technical analysis provides algorithms for forecasting and, therefore, is in the same spirit of the trading strategies found in this thesis. The following is a brief summary of technical analysis.

In the 1960's and 1970's, technical analysis was viewed with great skepticism. The predominate economic theory was that markets were efficient. The market price is a reflection of all current information, and future prices are solely determined by future news. Since changes in news are random (or as close to random as anyone can tell), future price movements should follow a random walk. However, for markets to be perfectly efficient, some rather unrealistic assumptions must hold. First, it assumes all players are perfectly rational. Second, all players receive identical information simultaneously. Third, given the same information, all players calculate the same optimal strategy and execute concurrently. In reality, there are millions of different players, some of whom are more rational than others. Information arrives to different players at different times,¹ and, even given the same information, two rational players can come to different conclusions based upon their risk aversions and time horizons (short vs. long term).

Although markets are not perfectly efficient, there is no reason to believe prices should be predictable. The market could be so close to an efficient market that it is impossible to tell the difference (e.g. the market inefficiencies might cancel each

¹Fundamentalists believe the way to beat the market is to get better, more complete information faster than your competitors.

other out). A great deal of research has been put into the area of forecasting foreign exchange rates with inconclusive (often contradictory) results. Some researchers find evidence of predictability ([9],[15]), while others conclude the opposite([13],[24]). Some find evidence that daily rates are leptokurtic², while others find that in periods of high volatility, rates are contrarian³. Furthermore, there is evidence that volatility may or may not be clustered. Though there is no agreement on the exact form of the exchange rate, the general consensus is the exchange rate sometimes deviates substantially from the Gaussian random walk.

Many different techniques are used to forecast exchange rates. Some of them are quite simplistic, while others are very involved. The following is a list of current techniques in the literature. This list is certainly not exhaustive, and many other successful techniques may not even be published (for obvious reasons).

Trading Range Break Rule Sell when the price exceeds its last peak, and buy when it goes below its last trough.

Moving Averages Maintain both a short-term and a long-term average (e.g. three weeks and fifteen weeks). Sell when the short-term average goes above the long-term average, and buy when the short-term average dips below the long-term average. This strategy is very popular. By adjusting the lengths of the short and long term, one can create a wide variety of different strategies.

Linear regressions Create a time-series regression using the previous exchange rates as independent variables. After calculating the regression coefficients, predict the next exchange range and then buy or sell accordingly. This technique is very flexible. By using absolute values, the regression can predict volatility. In addition, seasonal adjustments and non-linear indicators can be added. Many analysts believe that non-linear statistics will greatly improve forecasting. However, completely non-linear techniques are not currently available.

²Prices are leptokurtic if they tend to move in the same direction. In other words, if prices are more likely to go up if they just went up, and more likely to go down, if they just went down

³Contrarian is the opposite of leptokurtic. Prices are more likely to move in the opposite direction.

Neural Nets Theoretically, neural nets can compute any function and have the ability to learn the data set. Unfortunately, their implementation is often difficult. Designing the exact structure of the net is highly subjective. Also, when training, the net can be trapped into local minimums. Nevertheless, many financial firms use them as a filter. Neural nets can choose which stocks or currencies are interesting, and then further analysis can be performed.

1.2.3 Trading

After given a forecast, the investor must now decide how to invest. One obvious strategy is the buy-and-hold strategy. Given a forecast that the price will increase, the investor will buy the currency and hold onto it. If the forecast is for the price to drop, the investor will not buy the currency⁴. In this thesis, we will compare optimal trading strategies to the buy-and-hold strategy in a statistical adversarial model. For forecasts that call for sudden short-term changes, the buy-and-hold strategy is very close to optimal. However, for forecasts that call for gradual long-term changes, the optimal strategy does exponentially better.

Other trading strategies include Merton's constant rebalanced portfolio[25] and Cover's universal portfolio[8]. With Merton's strategy, the investor chooses to invest a constant, fixed percentages of his wealth in each currency. For example, he may choose to invest 30% in dollars and 70% in yen. As the exchange rate varies, he will exchange currencies, so that the 30% - 70% ratio is maintained. In universal portfolios, the investor again splits his wealth among the different currencies. However, past behavior is used to determine what percentage invested in each currency. Initially, the investor splits his wealth equally among all currencies. As a particular currency begins to perform well, relative to other currencies, the percentage invested in that currency increases. Likewise, as a currency performs poorly, the percentage invested decreases.

⁴A more sophisticated investor might sell the currency short.

1.3 Summary of Results

We begin by considering a statistical adversary that is forced to generate sequences of exchange rates of a known length, say n , such that the optimal off-line return on these sequences is larger than a known quantity, Π . We call this adversary the (n, Π) -adversary.

- We present a general scheme that identifies optimal on-line strategies against statistical adversaries constrained by such (and similar) features. Against each particular adversary, this scheme yields the optimal on-line strategy in the form of a dynamic program in terms of the statistical parameters. For usage, one can then efficiently pre-compute the on-line algorithm for arbitrary choices of these parameters.
- We identify the optimal on-line strategy relative to the (n, Π) -adversary and show that it is *money-making* as long as $\Pi > 1$. Unfortunately, it is shown that this strategy can only guarantee a very small fraction of the optimal off-line profit.
- Next we consider stronger adversaries that do not provide the player with either the value of n or the value of Π . Against these adversaries, we show that it is not possible to design a money-making strategy.
- In contrast, we then consider a weaker adversary in which the player has knowledge of the overall movement of the exchange rates during the n time periods and a factor $\alpha > 1$ by which the exchange rate changes (either up or down) during each time period. We call this new model the *fixed fluctuation* model. We then identify and analyze the corresponding optimal on-line strategy. This strategy exhibits some striking properties.
 - Aside from being money-making, this strategy always outperforms the optimal off-line *buy-and-hold strategy*.
 - When the market is *stable* and *active* (i.e. there are fluctuations but the upward and downward fluctuations tend to balance each other with respect

to the trading period), this strategy yields exponential profits (in n), even if the market has a slight *unfavorable* trend. This is somewhat surprising. Intuitively, it is reasonable to believe that one should avoid any financial transactions during stable periods (in comparison, the buy-and-hold strategy will not make any profit in this situation).

Based on preliminary experimental results, it appears that the fixed fluctuation model may provide a practical approach to investing strategies.

The rest of this thesis is organized as follows. In Chapter 2 we define the problem, identify the optimal money-making strategy against the (n, Π) -adversary, and derive some of its properties. In Chapter 3 we introduce *the fixed fluctuation model* as a weaker adversary and find the optimal strategy. In Chapter 4, we discuss various implementation issues and give some preliminary results. Finally, Chapter 5 contains some concluding remarks.

Chapter 2

Trading Against the Statistical Adversary

2.1 Definitions and notation

We consider a discrete variant of the two-way trading problem in which the on-line player is given 1 dollar, and is given the job of maximizing his return over a time period of n days by exchanging money back and forth from/to yen. With these two currencies, there is an associated exchange rate sequence $E = e_1, e_2, \dots$, where e_i , the exchange rate for the i th day, equals the number of yen that can be purchased for one dollar on that day. The player is required to finish the game with all the money converted back to the initial currency. We assume that there are no transaction fees and that the player may trade arbitrary fractions of dollars/yen.

For any strategy S and finite exchange rate sequence E , let $R_S(E)$ denote the *return* of S when it begins the game with 1 dollar and exchanges money in accordance with E . Let OPT denote the optimal off-line two-way trading strategy. Notice that for a given exchange rate sequence, E , OPT will always convert all available dollars to yen at all exchange rates which are local maxima in E , and all available yen back to dollars at all exchange rates which are local minima, with the exception that the last transaction must be yen to dollars. We say that a trading strategy, S , is a *money-making* strategy if for any exchange rate sequence, E , with $R_{OPT}(E) > 1$,

then $R_S(E) > 1$, as well.

2.2 Optimal trading strategies against the statistical adversary

We can specify trading strategies in a “normal form” as follows. Every day, the player might wish to convert some dollars to yen and/or some yen to dollars. It should be clear that only one transaction (i.e. dollars to yen or yen to dollars) is sufficient. It is convenient to conceptualize every such transaction as if the strategy first converted *all* yen back to dollars and then converted a fraction, s , of the total available dollars to yen. Any such transaction can be specified by one number, $s \in [0, 1]$. In this way, we can specify the activity of any conversion strategy by the sequence s_1, s_2, \dots, s_n where for each day, i , s_i is the fraction of dollars that should be converted to yen, immediately after all available yen are converted to dollars. By the rules of our game, s_n must be zero.

Suppose that the on-line player knows Π , the return of the optimal off-line player for an n -day game. We now derive the optimal on-line strategy for any $n \geq 2$ number of days. To derive the optimal strategy we require the following observation. Consider Figure 2-1, which illustrates an exchange rate sequence of 10 days (yen per dollar). First, notice that the optimal (off-line) return is

$$\Pi = \left(\frac{e_2}{e_5}\right) \left(\frac{e_6}{e_7}\right) \left(\frac{e_8}{e_9}\right) = \left(\frac{e_2 e_3 e_4}{e_3 e_4 e_5}\right) \left(\frac{e_6}{e_7}\right) \left(\frac{e_8}{e_9}\right)$$

In general, it can be shown that for any n -day sequence,

$$\Pi = \prod_{i=1}^{n-1} \max\{1, e_i/e_{i+1}\}.$$

Any conversion strategy “realizes” its dollar profit only on downward runs of the exchange rate sequence. If Π is known at the beginning of the sequence, then on each

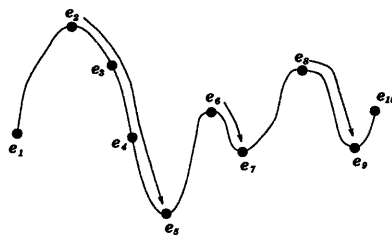


Figure 2-1: Realization of profit in a 10-day game

day, the on-line player can determine exactly what the optimal off-line profit would be for an $(n - 1)$ -day sequence starting on that day – that is, if the off-line player were to start a “new game” consisting of $n - 1$ days. We illustrate this by referring to Figure 2-1. Since on day 2 we are not in a downward run with respect to the previous day, the optimal off-line return must remain at Π if a new game is to begin at that moment. On the other hand, on day 3 (knowing only the first three rates) we know that the off-line player has just realized a factor e_2/e_3 of his total dollar return, so the optimal off-line return must be $\Pi' = \Pi/e_3$ if a new game were to be started that moment. In this way, the on-line player knows, after each exchange rate is revealed, exactly what the optimal off-line return would be if a new game were to be started that day.

This observation enables a dynamic programming derivation of the optimal on-line strategy using the Principle of Optimality [2]. In our context it can be stated as follows: the optimal on-line strategy has the property that, whatever the initial rate and the initial choice of how many dollars to trade, the remaining trades must constitute an optimal on-line strategy with regard to the state resulting from the first trade.

More formally, for $n \geq 2$, let $R_n(\Pi, e_1)$ denote the return of the optimal on-line strategy for an n -day game, given that the first exchange rate is e_1 and that the optimal off-line return for the entire period is Π . When exchange rates are chosen by an adversary who tries to minimize the optimal on-line return, we have

$$\begin{aligned}
R_n(\Pi, e_1) &= \max_{\substack{\text{fraction of dol-} \\ \text{lars } s_1 \text{ to trade}}} \min_{\substack{\text{2nd day} \\ \text{rate } e_2}} \left[\begin{array}{l} \text{total worth in dol-} \\ \text{lars after 2nd day} \\ \text{rate is revealed} \end{array} \right] \cdot R_{n-1}(\text{updated } \Pi, e_2) \\
&= \max_{0 \leq s_1 \leq 1} \min_{e_2 \geq \frac{e_1}{\Pi}} \left[\left(\frac{e_1}{e_2} - 1 \right) s_1 + 1 \right] \cdot R_{n-1} \left(\min \left\{ \Pi, \frac{\Pi e_2}{e_1} \right\}, e_2 \right) \quad (2.1)
\end{aligned}$$

where the lower bound, $\frac{e_1}{\Pi}$, on possible second day rates, e_2 , is due to the assumption that the total off-line return is Π . In addition, the following is clearly a boundary condition:

$$R_2(\Pi, \cdot) = \Pi. \quad (2.2)$$

Thus, (2.1) and (2.2) identify the optimal on-line strategy, which we denote by S^* . The first transaction that S^* performs is the purchase of yen with a fraction s_1^* of its dollars, where s_1^* is the quantity which maximizes the right hand side of (2.1).

Lemma 1 S^* is a money-making strategy.

Proof. We prove by induction on $n \geq 2$ that for any $\Pi > 1$ there exists a number $B_n(\Pi)$ such that $1 < B_n(\Pi) \leq R_n(\Pi, \cdot)$

Base case ($n = 2$): Then by (2.2) $R_2(\Pi, \cdot) = \Pi > 1$ and we take $B_2(\Pi) = R_2(\Pi, \cdot)$.

Induction step: By the induction hypothesis on $n - 1$ for any $\Pi > 1$ there exists a number $B_{n-1}(\Pi)$ with $1 < B_{n-1}(\Pi) \leq R_{n-1}(\Pi, \cdot)$. Let $\Pi > 1$ be given. Consider the following three cases:

- case (i) $e_1 = e_2$: Then $\min \{\Pi, \Pi e_2/e_1\} = \Pi$, and $R_n(\Pi, e_1) = R_{n-1}(\Pi, e_2)$ for all $0 \leq s_1 \leq 1$. Hence, in this case $b_1 \stackrel{\text{def}}{=} B_{n-1}(\Pi) > 1$ is a lower bound on $R_n(\Pi, \cdot)$.
- case (ii) $e_1 < e_2$: Since $e_2/e_1 > 1$, $\min \{\Pi, \Pi e_2/e_1\} = \Pi$. The adversary can choose e_2 arbitrarily high so that $\frac{e_1}{e_2} \rightarrow 0$ thus diminishing the return on the investment s_1 . In this case, for all $0 \leq s_1 \leq 1$, the on-line return is greater than but can be made arbitrarily close to $(1 - s_1)R_{n-1}(\Pi, e_2) \geq (1 - s_1)B_{n-1}(\Pi)$. Hence, for any choice of $s_1 < \frac{B_{n-1}(\Pi)-1}{B_{n-1}(\Pi)}$, $b_2 \stackrel{\text{def}}{=} (1 - s_1)B_{n-1}(\Pi) > 1$ is a lower bound on $R_n(\Pi, \cdot)$.

- case (iii) $e_1 > e_2$: Now, for every positive fraction s_1 , $(e_1/e_2 - 1)s_1 + 1 > 1$. If $\Pi e_2/e_1 > 1$ then by the induction hypothesis, there exists a number $B_{n-1}(\Pi e_2/e_1)$ with $1 < B_{n-1}(\Pi e_2/e_1) \leq R_{n-1}(\Pi e_2/e_1, e_2) > 1$. Otherwise $\Pi e_2/e_1 = 1$ and $R_{n-1}(\Pi e_2/e_1, e_2) = 1$. In either case, for each positive s_1 , $b_3 \stackrel{\text{def}}{=} (e_1/e_2 - 1)s_1 + 1 > 1$ is a lower bound on $R_n(\Pi, \cdot)$.

Thus, combining the conclusions of the above cases, for any choice of $s_1 < \frac{B_{n-1}(\Pi)-1}{B_{n-1}(\Pi)}$, $B_n(\Pi) \stackrel{\text{def}}{=} \min\{b_1, b_2, b_3\}$ is greater than one and $B_n(\Pi) \leq R_n(\Pi, \cdot)$, which completes the induction step. ■

The task of obtaining a closed form expression for (2.1) seems to be rather hard. The following lemma provides an upper bound on the return of S^* for any Π and n .

Lemma 2 For any $\Pi > 1$ and $n \geq 2$,

$$R_n(\Pi, \cdot) \leq \frac{1}{1 - (1 - \frac{1}{\Pi})^{n-1}}$$

Proof. Consider the following restricted version of the (n, Π) -adversary. On each day, i , $i \geq 2$, the restricted adversary has two options. either to decrease the exchange rate by a factor of Π (i.e. $e_i = \frac{e_{i-1}}{\Pi}$), or to increase the exchange rate by a very large factor so that the current dollar value of the previous investment, $s_{i-1} \frac{e_{i-1}}{e_i}$, is negligible. Notice that once the adversary chooses and acts the first option, there will be no more downward fluctuations since the optimal off-line player has realized a return of Π dollars. Hence, if this is the case, the game is over.

Denote by $\hat{R}_n(\Pi)$ the *optimal* on-line return against this restricted adversary. Clearly, $\hat{R}_2(\Pi) = \Pi$. Suppose the on-line player invested on the first trading day $0 \leq s \leq 1$ dollars. Then, if the adversary chooses the first option (i.e. $e_2 = \frac{e_1}{\Pi}$), the game is over and the on-line return is $s\Pi + 1 - s$. On the other hand, if the second option is chosen, then the dollar return on the first investment, s will be arbitrarily close to zero, and the total on-line return will be arbitrarily close to $(1 - s)\hat{R}_{n-1}(\Pi)$ as the player can still obtain an optimal return for the rest of his money.

The on-line player will choose s to maximize his total return, and the adversary, while knowing the choice of s , will choose one of his options to minimize the on-line return. Hence, the optimal on-line return can be made arbitrarily close to

$$\hat{R}_n(\Pi) = \max_{0 \leq s \leq 1} \min\{s\Pi + 1 - s, (1 - s)\hat{R}_{n-1}(\Pi)\}.$$

Both these quantities are linear functions of s . The first function is increasing and the second, decreasing. Hence, the on-line player must choose s satisfying

$$s\Pi + 1 - s = (1 - s)\hat{R}_{n-1}(\Pi). \quad (2.3)$$

Let s^* be the solution of (2.3). Thus, $\hat{R}_n(\Pi) = (1 - s^*)\hat{R}_{n-1}(\Pi)$. Solving for s^* we obtain

$$s^* = \frac{\hat{R}_{n-1}(\Pi) - \hat{R}_n(\Pi)}{\hat{R}_{n-1}(\Pi)}. \quad (2.4)$$

By substituting (2.4) into $\hat{R}_n(\Pi) = s^*\Pi + 1 - s^* = s^*(\Pi - 1) + 1$, and rearranging, we obtain

$$\frac{1}{\hat{R}_n(\Pi)} = \frac{1}{\Pi} + \frac{\Pi - 1}{\Pi} \cdot \frac{1}{\hat{R}_{n-1}(\Pi)}.$$

Thus,

$$\begin{aligned} R_2^{-1}(\Pi) &= \frac{1}{\Pi}; \\ R_n^{-1}(\Pi) &= \frac{1}{\Pi} + \frac{\Pi - 1}{\Pi} R_{n-1}^{-1}(\Pi). \end{aligned}$$

It is easy to see (e.g. by induction on n) that $R_n^{-1}(\Pi) = 1 - \left(1 - \frac{1}{\Pi}\right)^{n-1}$. Hence,

$$\hat{R}_n(\Pi) = \frac{1}{1 - \left(1 - \frac{1}{\Pi}\right)^{n-1}}. \quad (2.5)$$

Since $\hat{R}_n(\Pi)$ is the return against the restricted (n, Π) -adversary, it must be an upper bound for $R_n(\Pi, \cdot)$, the optimal return against the (unrestricted) (n, Π) -adversary. ■

Using the approximation $\left(1 - \frac{1}{\Pi}\right)^{n-1} = \left(1 - \frac{1}{\Pi}\right)^{\Pi \frac{n-1}{\Pi}} \approx e^{-\frac{n-1}{\Pi}}$, for large Π , we

obtain

$$\hat{R}_n(\Pi) \approx \frac{1}{1 - e^{-\frac{n-1}{\Pi}}}.$$

It is not hard to see that

- if $\Pi = \omega(n)$, then $e^{-\frac{n-1}{\Pi}} \approx 1 - \frac{n-1}{\Pi}$, and $\hat{R}_n(\Pi) \approx \frac{\Pi}{n-1}$;
- If $\Pi = \Theta(n)$, then $\hat{R}_n(\Pi) = \frac{1}{1-e^{-c}}$ where c is some positive constant;
- If $\Pi = o(n)$, then $\hat{R}_n(\Pi)$ approaches 1.

Hence, the optimal return against the (n, Π) -adversary can be a minuscule fraction of the optimal off-line return.

2.3 Games against stronger adversaries

One can think of several meaningful ways to strengthen the original adversary. Here we consider two stronger adversaries which correspond to the cases where the on-line player does not know Π or does not know n *a priori*. In either case we prove the nonexistence of a money-making strategy for non-degenerate strategies. A *non-degenerate strategy* is one that makes at least one non-zero transaction.

Lemma 3 *For any $n > 2$, and any non-degenerate on-line strategy S that only knows n in advance, there is an exchange rate sequence, $E = e_1, e_2, \dots, e_n$ for which $R_S(E) < 1$ and $R_{OPT}(E) > 1$, even if S also knows in advance that there is a positive off-line profit.*

Proof. Let $n = 3$. We show how the adversary can construct a sequence, $E = e_1, e_2, e_3$ for which $R_S(E) < 1$ and $R_{OPT}(E) > 1$. Let e_1 be any positive real. If S does not purchase any yen on the first day, then, since S is non-degenerate, it must buy some yen on the second day and the adversary can take $e_1 \gg e_2 < e_3$. Clearly, $R_{OPT}(E)$ can be made arbitrarily large and $R_S(E) < 1$. Therefore, assume that S trades $s_1 > 0$ dollars on the first day (with rate e_1). If $s_1 = 1$, the adversary can take

any $e_1 < e_3 < e_2$ with a clear loss to S and a return of e_2/e_3 to OPT. Thus, assume that $s_1 < 1$. Let δ be any positive real such that $\delta < s_1$. For any $0 < \varepsilon < \frac{s_1 - \delta}{1 - s_1}$ let

$$e_2 = \frac{s_1 e_1 (1 + \varepsilon)}{s_1 (1 + \varepsilon) - \varepsilon - \delta}; \quad (2.6)$$

$$e_3 = e_2 / (1 + \varepsilon). \quad (2.7)$$

First, notice that since $\varepsilon < \frac{s_1 - \delta}{1 - s_1}$, e_2 and e_3 are positive and hence, well-defined exchange rates. Also, it is easy to see that $e_2 > e_1$. Therefore, to perform optimally from this stage onward, S must convert the remaining dollars to yen on the second day and all yen back to dollars on the last day. Thus,

$$R_S(E) \leq \frac{s_1 e_1 + (1 - s_1) e_2}{e_3}. \quad (2.8)$$

Substituting (2.6) and (2.7) for e_2 and e_3 in (2.8) respectively, it is not hard to verify that $R_S(E) \leq 1 - \delta$. Clearly, $R_{\text{OPT}}(E) = e_2/e_3 = 1 + \varepsilon$.

It is easy to extend this exchange rate sequence to any length $n > 3$. ■

Lemma 4 *For any $\Pi > 1$, and any non-degenerate on-line strategy S that only knows Π in advance, there exists an exchange rate sequence, $E = e_1, e_2, \dots$ for which $R_S(E) < 1$ and $R_{\text{OPT}}(E) = \Pi$.*

Proof. Fix $\Delta > \Pi$. Consider the exchange rate sequence defined by: $e_1 = 1$, and $e_{i+1} = e_i \Delta$, $i \geq 1$. As S is a non-degenerate algorithm we can assume, without loss of generality, that it invests some dollars, t_1 , on the first trading day. Otherwise, the adversary can wait until S invests some amount and then start. Hence, we assume that $t_1 > 0$.

After investing t_1 on the first day, the exchange rate rises so that $e_2 = e_1 \Delta$. Then, S 's net worth in dollars is now

$$\frac{t_1 e_1}{\Delta e_1} + 1 - t_1 = \frac{t_1}{\Delta} + 1 - t_1,$$

and has incurred a loss of

$$1 - \left(\frac{t_1}{\Delta} + 1 - t_1 \right) = t_1 \left(1 - \frac{1}{\Delta} \right).$$

In order to recover from this initial loss, we claim that on each day $i \geq 2$, S must invest at least $\frac{t_1}{\Delta}$ dollars. Otherwise, the adversary will drop the exchange rate, by a factor Π , on the following day thus ending the game with S having a loss (i.e. $R_S(E) < 1$) and an optimal off-line return, Π . The claim is proven by induction on i . *Base case ($i = 2$):* Let t_2 be the dollars invested on the second day. In case of a drop by a factor of Π on the third day (i.e. to the rate $\frac{e_2}{\Pi}$), the gain will be $t_2(\Pi - 1)$. This gain must exceed the previous loss, so

$$t_2(\Pi - 1) > t_1 \left(1 - \frac{1}{\Delta} \right) \implies t_2 > \frac{t_1 \Delta - 1}{\Delta \Pi - 1} > \frac{t_1}{\Delta}.$$

Induction step: On each day where $e_i = e_{i-1}\Delta$, S incurs losses. Therefore, S 's wealth on day i is less than his wealth on day 2. In the case of a drop of Π on the $(i + 1)$ th day, the gain will be $t_i(\Pi - 1)$. Again, this gain must exceed the sum of all previous losses (which is at least as large as the first day's loss), so $t_i > \frac{t_1}{\Delta}$. Therefore, the induction step is complete.

It follows that once the on-line player has invested t dollars, he must invest a minimum amount of $\frac{t}{\Delta}$ on each of the remaining days. This is, of course, impossible. Once his money runs out, the adversary drops the exchange rate by a factor of Π , and the on-line player ends the game with a loss. ■

2.4 Games against weaker adversaries

Against the (n, Π) -adversary, the on-line player is forced to invest very small amounts on most days since the adversary can depreciate most daily investments by increasing the exchange rate arbitrarily. Theoretically, this can be done until the second to last day. Such exchange rate sequences are, of course, unrealistic. By imposing additional suitable constraints it is possible to reduce such threats. For example,

by requiring that all rates are drawn from the interval $[m, M]$ this threat can be substantially reduced. There are many other constraints that can be added. In fact, the possibilities are practically unlimited. We now specify a few.

- *maximum daily fluctuation ratio*: a number $\alpha > 1$ such that for every day i , the next day's rate, e_{i+1} , is in $[\frac{e_i}{\alpha}, e_i\alpha]$. Although we measure the time difference between two successive exchange rates by “days”, these time differences may be of any size (seconds, minutes, etc.), and, in fact, they need not be of a fixed size.
- *minimum and maximum bounds on exchange rates*: numbers, m and M , such that all exchange rates are within the interval $[m, M]$.
- *maximum run length*: a number ρ such that there is no monotone increasing (decreasing) subsequence of consecutive exchange rates of length longer than ρ .
- *number of extrema points*: a number k such that the number of minima and maxima in the exchange rate sequence is k .
- *statistical functions of exchange rate sequences*: “standard” statistical functions like mean and standard deviation may be considered.

It is possible to incorporate any (subset) of the above constraints in (2.1) to yield an optimal on-line strategy against the corresponding, more constrained adversary. For some of these adversaries, we simply have to replace the bounding interval for possible choices of e_2 , which was originally $[\frac{e_1}{\Pi}, \infty)$. Clearly, if we add more constraints, the resulting strategy must be money-making. Intuitively, by the inclusion of additional “effective” constraints, one should obtain strategies with superior performance. The appeal of this scheme is that the users of our strategies may choose their own set of statistical features and obtain optimal on-line performance against an adversary that reflects “financial nature” according to their own beliefs.

Chapter 3

The Fixed Fluctuation Model

3.1 The adversary

Using our scheme, we now derive and analyze the optimal strategy against a weak adversary that is restricted by a constraint which is, in a sense, a hybrid of the (n, Π) -constraint and the *maximum daily fluctuation ratio*. This constraint will reduce the unrealistic threats that can be made by the (n, Π) -adversary. The parameters of the new constraint are (α, m, n) where α represents a fixed ratio between any two successive exchange rates, m denotes the number of downward changes, and n is the total number of changes. Since each downward change in the exchange rate corresponds to a realization of dollar profit, we know that for each exchange rate sequence conforming with (α, m, n) , the optimal off-line profit is α^m . Notice that n in this constraint measures the total number of the α -changes whereas in the (n, Π) -adversary, n is the length of the exchange rate sequence. The corresponding adversary is called the (α, m, n) -adversary.

In practice, daily fluctuation ratios are *variable*. Thus, to approximate fixed fluctuations the player must dynamically “scale” the time axes by waiting until the exchange rate fluctuates by ratios close to α and then act as if one “day” has elapsed. In essence, time axes is being scaled to measure volatility rather than actual physical time. Instead of having decision points every fixed unit of physical time, decision points only occur after the exchange rates has moved some fixed amount. Thus, in pe-

periods of low volatility, time is expanded by having very few decision points. Similarly, time is compressed during high volatility. A major advantage of fixed fluctuations is that it considerably simplifies the analysis. In the next section, we give an exact solution for the optimal strategy. As a by-product, the user may choose α to filter out insignificant, “noisy” fluctuations (e.g. very small fluctuations must be avoided when transaction fees are introduced) and to control the number of transactions per unit of time.

We assume that the on-line player knows (α, m, n) . Notice that this constraint strictly subsumes the knowledge of (n, Π) in the previous constraint since $\Pi = \alpha^m$. However, against the (α, m, n) -adversary the player is given additional valuable knowledge. For sequences conforming with (α, m, n) , we expect the exchange rate to change at a rate of $\alpha^{2\frac{m}{n}-1}$ (i.e. the forecast). Hence, even knowledge of the ratio $\frac{m}{n}$ may be extremely valuable as it represents the trend during the period in question. Given the knowledge of a particular trend (either downward or upward) one can use standard techniques (via the use of *future contracts*) to guarantee the profit of the buy-and-hold strategy. Hence, of particular interest is the case $m = \frac{1}{2}n$ in which exactly half of the changes are upward and half the changes are downward. If this is the case, we say that the exchange rate is *stable* and *active*.

3.2 The optimal strategy

Let $R_\alpha(m, n)$ be the optimal on-line return with parameters α , m , and n . When the on-line player invests s , his return is either $(\alpha s + 1 - s)R_\alpha(m - 1, n - 1)$ or $(\frac{s}{\alpha} + 1 - s)R_\alpha(m, n - 1)$, which correspond to a downward change and an upward change respectively. The adversary will choose the minimum of these two values. Hence, the following recurrence identifies the optimal on-line strategy which we call S^{**} .

$$\begin{aligned} R_\alpha(0, n) &= 1, \\ R_\alpha(n, n) &= \alpha^n, \end{aligned}$$

$$R_\alpha(m, n) = \max_{0 \leq s \leq 1} \min \left\{ (\alpha s + 1 - s) \cdot R_\alpha(m-1, n-1), \left(\frac{s}{\alpha} + 1 - s\right) \cdot R_\alpha(m, n-1) \right\} \quad (3.1)$$

We now derive some interesting properties of the strategy S^{**} . First, notice that the left operand of the “min” in (3.1) is increasing with s while the right operand is decreasing with s . Hence, the optimal strategy sets s so that $(\alpha s + 1 - s) \cdot R_\alpha(m-1, n-1) = (\frac{s}{\alpha} + 1 - s) \cdot R_\alpha(m, n-1)$. Solving for s ,

$$s = \frac{R_\alpha(m, n-1) - R_\alpha(m-1, n-1)}{(\alpha - 1) \cdot R_\alpha(m-1, n-1) - (\frac{1}{\alpha} - 1) \cdot R_\alpha(m, n-1)}.$$

Substituting for s , we obtain

$$\begin{aligned} R_\alpha(m, n) &= \left(\frac{(\alpha - 1) \cdot (R_\alpha(m, n-1) - R_\alpha(m-1, n-1))}{(\alpha - 1) \cdot R_\alpha(m-1, n-1) + \frac{\alpha-1}{\alpha} \cdot R_\alpha(m, n-1)} + 1 \right) \cdot R_\alpha(m-1, n-1) \\ &= \frac{\frac{\alpha+1}{\alpha} R_\alpha(m, n-1) \cdot R_\alpha(m-1, n-1)}{R_\alpha(m-1, n-1) + \frac{1}{\alpha} R_\alpha(m, n-1)}. \end{aligned}$$

Setting $R_\alpha^{-1}(m, n) \stackrel{\text{def}}{=} \frac{1}{R_\alpha(m, n)}$ and inverting both sides,

$$R_\alpha^{-1}(m, n) = \frac{\alpha}{\alpha + 1} R_\alpha^{-1}(m, n-1) + \frac{1}{\alpha + 1} R_\alpha^{-1}(m-1, n-1). \quad (3.2)$$

Set $\beta \stackrel{\text{def}}{=} \frac{1}{\alpha+1}$, and let $B(k; n, p) \stackrel{\text{def}}{=} \sum_{i=0}^k \binom{n}{i} p^i (1-p)^{n-i}$, the partial binomial sum. The following lemma provides a solution to (3.2).

Lemma 5 $R_\alpha^{-1}(cn, n) = B(n(1-c) - 1; n-1, 1-\beta) + \alpha^{n(1-2c)} B(cn-1; n-1, 1-\beta)$

Proof. Recall the initial conditions of S^{**} (3.1). For all n ,

$$\begin{aligned} R_\alpha^{-1}(0, n) &= 1; \\ R_\alpha^{-1}(n, n) &= \alpha^{-n}. \end{aligned}$$

Intuitively, $R_\alpha^{-1}(m, n)$ has no meaning for $m > n$ or $m < 0$. We now extend $R_\alpha^{-1}(m, n)$ to these cases, while still satisfying both the recurrence and initial conditions.

Let $R_\alpha^{-1}(m, n) = 1, m < 0$ and $R_\alpha^{-1}(m, n) = \alpha^{(n-2m)}, m > n$. Note that for $n = m$, $\alpha^{(n-2m)} = \alpha^{-n}$, so the two conditions combine to $R_\alpha^{-1}(m, n) = \alpha^{(n-2m)}, m \geq n$.

Claim: The extended $R_\alpha^{-1}(m, n)$ satisfies the recurrence and the initial conditions.

Proof of claim: By induction on n . For the base case, $n = 1$, we have $R_\alpha^{-1}(m, 1) = 1$ for $m \leq 0$, and $R_\alpha^{-1}(m, 1) = \alpha^{(1-2m)}$ for $m > 0$. The initial conditions $R_\alpha^{-1}(0, 1) = 1$ and $R_\alpha^{-1}(1, 1) = \alpha^{-1}$ are satisfied. We assume the induction hypothesis for $n - 1$, and prove it for n .

1. For $m \leq 0$,

$$\begin{aligned} R_\alpha^{-1}(m, n) &= \frac{\alpha}{\alpha + 1} R_\alpha^{-1}(m, n - 1) + \frac{1}{\alpha + 1} R_\alpha^{-1}(m - 1, n - 1) \\ &= \frac{\alpha}{\alpha + 1} \cdot 1 + \frac{1}{\alpha + 1} \cdot 1 \\ &= 1. \end{aligned}$$

2. For $m \geq n$,

$$\begin{aligned} R_\alpha^{-1}(m, n) &= \frac{\alpha}{\alpha + 1} R_\alpha^{-1}(m, n - 1) + \frac{1}{\alpha + 1} R_\alpha^{-1}(m - 1, n - 1) \\ &= \frac{\alpha}{\alpha + 1} \alpha^{n-1-2m} + \frac{1}{\alpha + 1} \cdot \alpha^{n-1-2m+2} \\ &= \frac{1}{\alpha + 1} (\alpha^{n-2m} + \alpha^{n-2m+1}) \\ &= \alpha^{n-2m}. \end{aligned}$$

■

Consider the directed graph in Figure 3-1 . Each node is labeled (x, y) with the “root” labeled (m, n) . The value stored at each node is $R_\alpha^{-1}(x, y)$. For node (x, y) , x corresponds to the vertical height in the grid. “Leaf” nodes have height 1, and the “root” has height n . y labels the left-to-right diagonals. The rightmost diagonal is m , the diagonal immediately below is $m - 1$, etc. The left most diagonal (a single node) is labeled $m - n + 1$.

For a node (x, y) , its left child is $(x - 1, y - 1)$ and its right child is $(x, y - 1)$. To

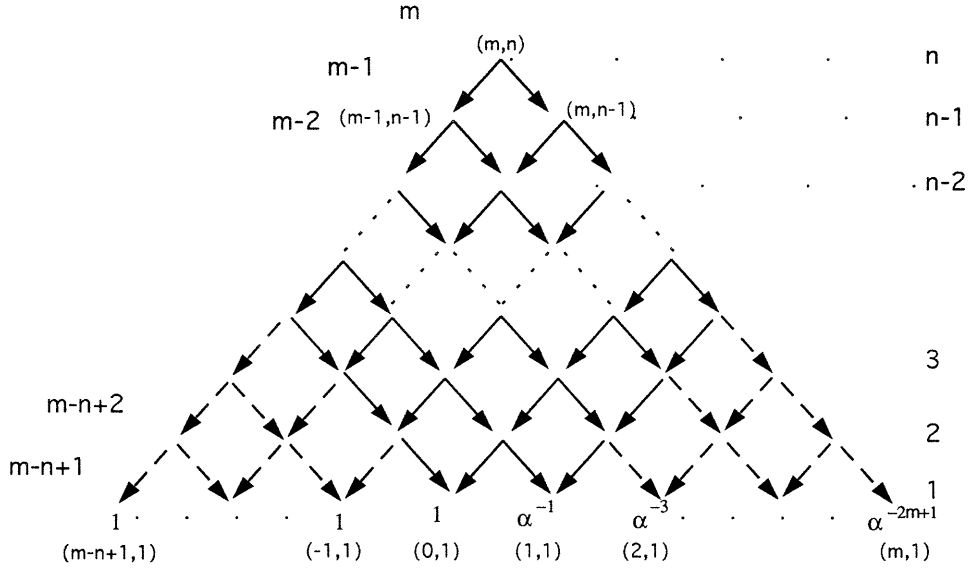


Figure 3-1: Directed graph showing the expansion of the recurrence

compute $R_\alpha^{-1}(x, y)$ from its children, we add $\frac{1}{\alpha+1}R_\alpha^{-1}(x-1, y-1)$ (the contribution of the left child) to $\frac{\alpha}{\alpha+1}R_\alpha^{-1}(x, y-1)$ (the contribution of the right child) (i.e. $R_\alpha^{-1}(x, y) = \frac{1}{\alpha+1}R_\alpha^{-1}(x-1, y-1) + \frac{\alpha}{\alpha+1}R_\alpha^{-1}(x, y-1)$). Thus, we can consider each left branch to be weighed by $\frac{1}{\alpha+1}$ and each right branch by $\frac{\alpha}{\alpha+1}$.

If we expand the recurrence $n-1$ times, we obtain an expression in $R_\alpha^{-1}(m, 1), R_\alpha^{-1}(m-1, 1), R_\alpha^{-1}(m-2, 1), \dots, R_\alpha^{-1}(m-n+1, 1)$. The number of times $R_\alpha^{-1}(m-k, 1)$ occurs is exactly the number of paths from (m, n) to $(m-k, 1)$, which is $\binom{n-1}{k}$. In addition, each term is weighed by $\frac{1}{\alpha+1}$ for each left branch and $\frac{\alpha}{\alpha+1}$ for each right branch. Each path to $(m-k, 1)$ has the same number of left and right moves, so the weight of each path is identical. Therefore,

$$\begin{aligned}
 R_\alpha^{-1}(m, n) &= \sum_{\text{leaf nodes}} [R_\alpha^{-1}(x, y)] \cdot [\text{Number of paths}] \cdot [\text{Weight of path}] \\
 &= \sum_{i=0}^{n-1} R_\alpha^{-1}(m-n+1+i, 1) \binom{n-1}{i} \left(\frac{\alpha}{\alpha+1}\right)^i \left(\frac{1}{\alpha+1}\right)^{(n-1-i)}
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{n-m-1} 1 \cdot \binom{n-1}{i} \left(\frac{\alpha}{\alpha+1}\right)^i \left(\frac{1}{\alpha+1}\right)^{(n-1-i)} \\
&\quad + \sum_{i=n-m}^{n-1} \alpha^{(2n-2m-2i-1)} \binom{n-1}{i} \left(\frac{\alpha}{\alpha+1}\right)^i \left(\frac{1}{\alpha+1}\right)^{(n-1-i)}.
\end{aligned}$$

In the second sum, we substitute j for $n-1-i$,

$$\begin{aligned}
R_\alpha^{-1}(m, n) &= \sum_{i=0}^{n-m-1} \binom{n-1}{i} \left(\frac{\alpha}{\alpha+1}\right)^i \left(\frac{1}{\alpha+1}\right)^{(n-1-i)} \\
&\quad + \sum_{j=0}^{m-1} \alpha^{(2j-2m+1)} \binom{n-1}{j} \left(\frac{\alpha}{\alpha+1}\right)^{(n-1-j)} \left(\frac{1}{\alpha+1}\right)^j \\
&= \sum_{i=0}^{n-m-1} \binom{n-1}{i} \left(\frac{\alpha}{\alpha+1}\right)^i \left(\frac{1}{\alpha+1}\right)^{(n-1-i)} \\
&\quad + \alpha^{(n-2m)} \sum_{j=0}^{m-1} \binom{n-1}{j} \left(\frac{\alpha}{\alpha+1}\right)^j \left(\frac{1}{\alpha+1}\right)^{(n-1-j)}.
\end{aligned}$$

■

Using the result of Lemma 5, the next lemma characterizes the performance of S^{**} .

Lemma 6 *For $m = cn$ with $c \in (0, 1)$, the following asymptotic relations hold.*

- If $0 \leq c \leq \beta$, then $R_\alpha(cn, n) \rightarrow 1$;
- If $\beta < c \leq \frac{1}{2}$, then $R_\alpha(cn, n) \rightarrow e^{\Omega(n)}$;
- If $\frac{1}{2} < c < 1 - \beta$, then $R_\alpha(cn, n) \rightarrow \alpha^{n(2c-1)} e^{\Omega(n)}$;
- If $1 - \beta \leq c \leq 1$, then $R_\alpha(cn, n) \rightarrow \alpha^{n(2c-1)}$.

Proof. Recall that

$$\begin{aligned}
B(k; n, p) &= \sum_{i=0}^k \binom{n}{k} p^i (1-p)^{n-i}; \\
R_\alpha^{-1}(m, n) &= B((1-c)n-1; n-1, 1-\beta) + \alpha^{(1-2c)n} B(cn-1; n-1, 1-\beta).
\end{aligned}$$

For the sake of brevity, define

$$\begin{aligned} B_1 &\stackrel{\text{def}}{=} B((1-c)n-1; n-1, 1-\beta) \\ B_2 &\stackrel{\text{def}}{=} B(cn-1; n-1, 1-\beta) \end{aligned}$$

To compute the upper bounds on B_1 and B_2 , we make use of the following Chernoff bound [1]:

Theorem 1 *Let X_1, \dots, X_n be n mutually independent random variables with*

$$\begin{aligned} \Pr[X_i = 1] &= p; \\ \Pr[X_i = 0] &= 1 - p. \end{aligned}$$

Let $X = X_1 + \dots + X_n$. Then for $a > 0$,

$$\Pr[X < pn - a] < e^{-a^2/2pn}.$$

$B(k; n, p)$ is simply the probability that at most k successes occur in a series of n Bernoulli trials with success probability p . We can use this Chernoff bound to bound $B_1 + \alpha^{n(1-2c)}B_2$. We provide bounds based on the value of c : By the theorem above, when $c > \beta$,

$$\begin{aligned} B_1 &= B(n(1-c)-1; n-1, 1-\beta) \\ &< e^{-(n-1)^2(c-\beta)^2/2(1-\beta)(n-1)} \\ &= e^{-(n-1)(c-\beta)^2/2(1-\beta)} \\ &= e^{-\Omega(n)}. \end{aligned}$$

When $c < \beta$,

$$B_1 = B(nc-1; n-1, \beta)$$

$$\begin{aligned}
&> 1 - e^{-(n-1)^2(\beta-c)^2/2\beta(n-1)} \\
&= 1 - e^{-(n-1)(\beta-c)^2/2\beta} \\
&= 1 - e^{-\Omega(n)};
\end{aligned}$$

When $c < 1 - \beta$,

$$\begin{aligned}
B_2 &= B(nc - 1; n - 1, 1 - \beta) \\
&< e^{-(n-1)^2(1-\beta-c)^2/2(1-\beta)(n-1)} \\
&= e^{-(n-1)(1-\beta-c)^2/2(1-\beta)} \\
&= e^{-\Omega(n)}.
\end{aligned}$$

When $c > 1 - \beta$,

$$\begin{aligned}
B_2 &= 1 - B((1 - c)n - 1; n - 1, \beta) \\
&> 1 - e^{-(n-1)^2(c-(1-\beta))^2/2\beta(n-1)} \\
&= 1 - e^{-(n-1)(c-(1-\beta))^2/2\beta} \\
&= 1 - e^{-\Omega(n)}.
\end{aligned}$$

We will need tighter bounds than the Chernoff bounds can provide in some of the case analysis below. The following theorem provides the necessary bounds. The theorem can be found in [1]:

Theorem 2 For any constants $1 \geq p > c \geq 0$,

$$\begin{aligned}
B(cn; n, p) &= \sum_{i=0}^{cn} \binom{n}{i} p^i (1-p)^{n-i} \\
&= 2^{n(H(c)+o(1))} p^{cn} (1-p)^{(1-c)n} \\
&= 2^{o(n)} \left(\frac{p}{c}\right)^{cn} \left(\frac{1-p}{1-c}\right)^{(1-c)n},
\end{aligned}$$

where $H(c) = -c \log c - (1 - c) \log(1 - c)$ is the entropy function.

Using the above bounds we can now derive bounds on $B_1 + \alpha^{n(1-2c)} B_2$ for all values

of $0 \leq c \leq 1$.

$0 \leq c \leq \beta$: In this case, $B_1 \rightarrow 1$ as n becomes large. Because the entire sum is at most 1, and $\alpha^{n(1-2c)}B_2$ is positive, $B_1 + \alpha^{n(1-2c)}B_2 \rightarrow 1$. Therefore, $R_\alpha(m, n) \rightarrow 1$.

$\beta < c \leq 1/2$: Here B_1 is exponentially small. We wish to show the same for $\alpha^{n(1-2c)}B_2$.

Here we will need the tight bound from Theorem 2.

$$\begin{aligned}
\alpha^{n(1-2c)}B_2 &= O(\alpha^{n(1-2c)}B(cn; n, 1 - \beta)) \\
&= 2^{o(n)}\alpha^{(1-2c)n} \left(\frac{1-\beta}{c}\right)^{cn} \left(\frac{\beta}{1-c}\right)^{(1-c)n} \\
&= 2^{o(n)} \left(\frac{1-\beta}{\alpha c}\right)^{cn} \left(\frac{\alpha\beta}{1-c}\right)^{(1-c)n} \\
&= 2^{o(n)} \left(\frac{\beta}{c}\right)^{cn} \left(\frac{1-\beta}{1-c}\right)^{(1-c)n}.
\end{aligned}$$

To determine that this function is exponentially small, we need only show that $V(c, \beta) = \left(\frac{\beta}{c}\right)^c \left(\frac{1-\beta}{1-c}\right)^{1-c} < 1$. First note that $V(\beta, \beta) = 1$. To complete the proof, we show that $V(c, \beta)$ is strictly decreasing as c increases beyond β . To do this we show that the derivative with respect to c of $\ln(V(c, \beta))$ is negative for these values of c .

$$\begin{aligned}
\ln'(V(c, \beta)) &= \ln \beta - 1 - \ln c - \ln(1 - \beta) + \ln(1 - c) + 1 \\
&= \ln \beta - \ln c - \ln(1 - \beta) + \ln(1 - c).
\end{aligned}$$

Now, $\ln \beta - \ln c - \ln(1 - \beta) + \ln(1 - c) < 0 \iff \frac{\beta}{c} \frac{1-c}{1-\beta} < 1$. But $\frac{\beta}{c} < 1$ and $\frac{1-c}{1-\beta} < 1$ for $\beta < c \leq 1/2$. So $B_1 + \alpha^{n(1-2c)}B_2 \rightarrow e^{-\Omega(n)}$.

Therefore, $R_\alpha(m, n) \rightarrow e^{\Omega(n)}$.

$1/2 < c < 1 - \beta$: In this region, B_2 is still exponentially small, so $\alpha^{n(1-2c)}B_2$ is $\alpha^{n(1-2c)}e^{-\Omega(n)}$.

We need only show that B_1 takes the same form. Consider $\alpha^{n(2c-1)}B_1$. We make a substitution of variables to show that this is exponentially small. Consider $d = 1 - c$. Then $\alpha^{n(2c-1)}B_1 = \alpha^{n(1-2d)}B((1-d)-1, n-1, 1-\beta)$ for $\beta < d < 1/2$. This is precisely the function analyzed in the previous case, which we showed

to be exponentially small. Thus, $B_1 + \alpha^{n(1-2c)} B_2 \rightarrow \alpha^{n(1-2c)} e^{-\Omega(n)}$. Therefore, $R_\alpha(m, n) \rightarrow \alpha^{n(2c-1)} e^{\Omega(n)}$.

$c \geq 1 - \beta$: B_1 is exponentially small. B_2 is moving exponentially close to 1, so $B_1 + \alpha^{n(1-2c)} B_2 \rightarrow \alpha^{n(1-2c)}$. Therefore, $R_\alpha(m, n) \rightarrow \alpha^{n(2c-1)}$.

■

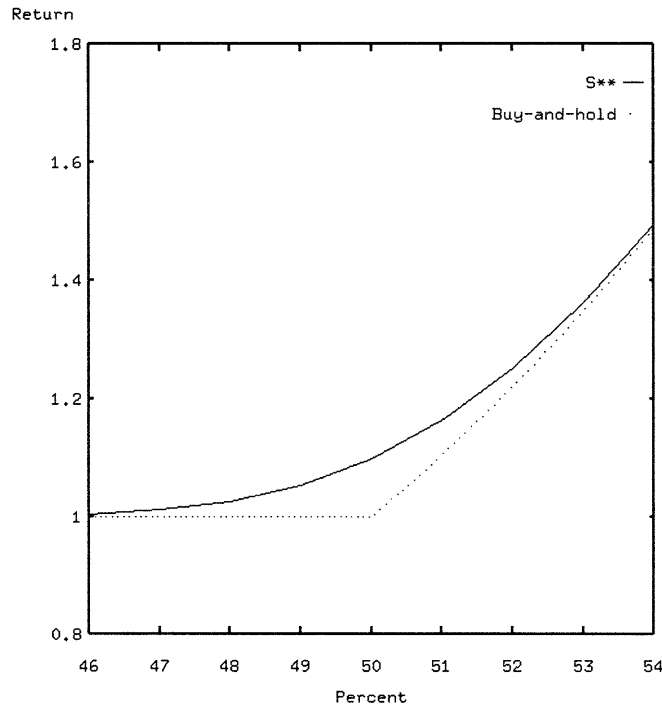


Figure 3-2: S^{**} vs. buy-and-hold as c varies ($\alpha = 1.01$ and $n = 500$)

The interpretation of Lemma 6 is quite surprising. Consider the behavior of the optimal buy-and-hold strategy. Buy-and-hold will invest all of its capital when $c > \frac{1}{2}$. On the other hand, when $c \leq \frac{1}{2}$ it will avoid any transaction. Hence, the return is 1 for $c \leq \frac{1}{2}$, and $\alpha^{n(2c-1)}$ for $c > \frac{1}{2}$. In the case where $0 \leq c \leq \beta$ or $1 - \beta \leq c \leq 1$, S^{**} asymptotically performs the same as buy-and-hold. However, for $\beta < c < 1 - \beta$, S^{**} performs exponentially better. In particular, for $c = \frac{1}{2}$, buy-and-hold will return 1, while S^{**} yields exponential return. Note that in the case $\beta < c < \frac{1}{2}$, the market is moving unfavorably yet the return is exponential in n . The relative advantage of S^{**} over buy-and-hold is greatest when the market is perfectly stable (i.e. $c = \frac{1}{2}$). This fact is illustrated in the graph of Figure 3-2.

Chapter 4

Implementation and Experimental Results

4.1 Implementation

Before running S^{**} on real data, one must set the parameters n , m , and α . The value of α may be chosen to capture “significant” changes in the exchange rate sequence. (e.g. one may choose sufficiently large α to filter out “insignificant” fluctuations). For a particular choice of n and α , the on-line player may choose a value for m according to the his beliefs, forecasts (and risk aversion). In any case, it would be unrealistic to assume that one knows the exact value of m . Let m^* be the actual number of profitable changes among the n changes.

In Figures 4-1 and 4-2 we plot the return of S^{**} as a function of m . At the point where $m = m^*$, S^{**} obtains a maximum. On the one hand, if S^{**} underestimates m^* , then S^{**} invests conservatively, since it “believes” that the number of remaining positive changes will be small. As m approaches zero, the return approaches 1, which is analogous to not trading at all. On the other hand, if S^{**} overestimates m^* , S^{**} invests more “aggressively” as it expects the exchange rate to be favorable. As m approaches n , the return approaches α^{2m^*-n} . This case is analogous to investing all the money on the first trading day and converting it back on the last trading day (buy-and-hold). In both cases (overestimating and underestimating), we see exponential

convergence to the limit cases.

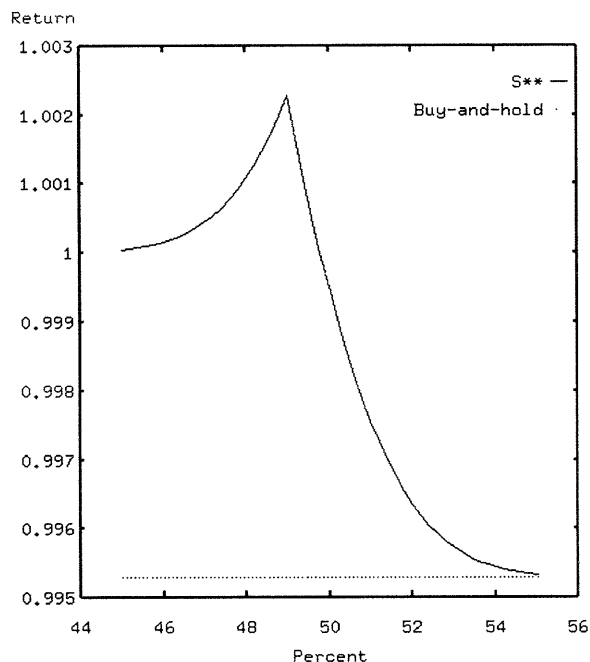


Figure 4-1: Returns of S^{**} as a function of m/n , $m^* < \frac{1}{2}$

The graph in Figure 4-1 illustrates the behavior when $m^* < \frac{1}{2}n$. In this case, the off-line buy-and-hold strategy does not invest and receives a return of 1. S^{**} always exceeds the buy-and-hold return when it underestimates the value of m^* . However, if S^{**} overestimates by too much, it may yield a return less than 1. Therefore, if S^{**} expects $m^* < \frac{1}{2}n$, then it is safe to underestimate.

A similar phenomenon is shown in Figure 4-2 for $m^* > \frac{1}{2}n$. The off-line buy-and-hold strategy will buy in the initial period and sell in the final period. Its return will be α^{2m^*-n} . If we incorrectly overestimate m^* , we will always exceed the buy-and-hold return. However, if we underestimate, then our return may be less than the off-line buy-and-hold return.

Based upon these graphs, it would appear that we need very accurate predictions to be successful. If we incorrectly estimate m^* , we can obtain returns that are worse than the off-line buy-and-hold. However, consider what it means for m^* to be different than m , where m is our estimate. Then, after n days, the exchange rate will differ from our expectation by a factor of $\alpha^{(m-m^*)}$. It is no surprise that if we expe-

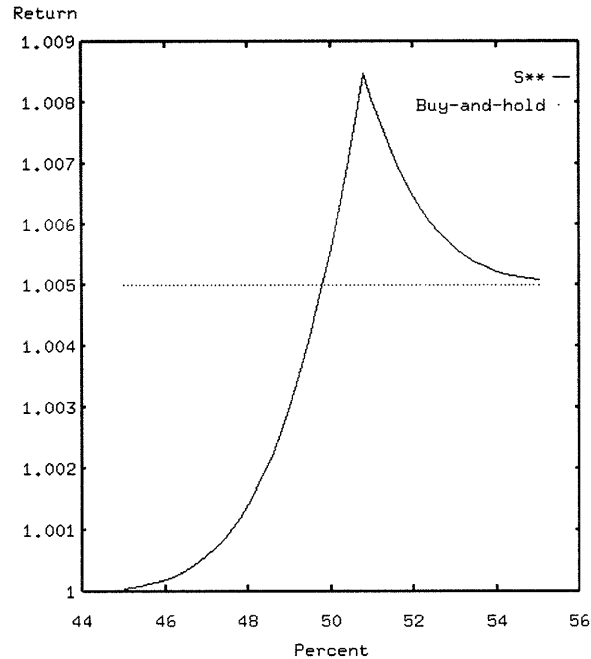


Figure 4-2: Returns of S^{**} as a function of m/n , $m^* > \frac{1}{2}$

rience an unanticipated exponential change in the exchange rate, then the algorithm will perform poorly. Fortunately, actual exchange rate sequences rarely exhibit this behavior. In fact, the simple strategy where we assume $m = \frac{1}{2}n$ performs fairly well on small samples of real data.

4.2 Experimental Results

S^{**} was tested on historical intra-day data for both US dollars vs. Japanese Yen and US dollars vs. German Marks. The intra-day data consisted of two sets. One set consisted of the exchange rate at every tick¹ (denoted by Set A). Set A contained exchange rates from October 10, 1993, to November 9, 1993. The other set contained the exchange rate every six minutes for the year 1992 (denoted by Set B). For both samples, the choice of α seemed natural. With almost every change, the exchange rate changed by a factor five points.² Thus, α was set to $1 + \frac{5}{\text{initial exchange rate}}$. Since five points is small compared to the exchange rate, additive changes of ± 5 points

¹A tick occurs every time the exchange rate changes, usually between 10-120 seconds.

²A point is the smallest unit used to measure exchange rates.

approximate multiplicative changes of α . Decision points were then inserted every time the rate changed by at least five points (approximately α).

In addition, the size of n has to be chosen. The total number of decision points in Set A is over 20,000 for US dollars vs. Deutschmarks. It is computationally unfeasible to set n this high. Thus, n was varied between 100 and 1000, and the entire set was broken into a series of games of size n . The returns from each trial of size n was then multiplied together to get the return for the entire sequence. To be completely fair, m was naively set to $\frac{n}{2}$ for all games.

We then ran S^{**} on the exchange rate sequences and received mixed results. The following table summarizes four different trial runs (with $n = 100$).

Sample	Return	Return with transaction costs
DM - Set A	2.23	1.33
JY - Set A	1.67	1.34
DM - Set B	1.04	0.73
JY - Set B	1.11	0.89

DM corresponds to the exchange rate of US dollars vs. German Deutschmarks, and JY to US dollars vs. Japanese yen. The first column gives the return assuming no transactions costs. The second column is the return with a 0.02% cost per transaction.³ Recall that in formulating S^{**} , we assumed no transactions costs. Thus, in the simulations, the transactions are chosen assuming no transaction costs, and then the transaction costs are subtracted from the return.

In Set A, S^{**} performed extremely (almost absurdly) well. Even with transaction costs, returns of 33% and 34% were obtained in *only one month*. However, we should note that S^{**} trades every tick (which is unrealistic), and the results are for only one exchange rate sequence. Unfortunately, we have been unable to obtain further tick data. In Set B, S^{**} performs marginally (4% and 11% returns) under no transaction costs and poorly with a .02% transaction costs (significant losses of 27% and 11%).

³0.02% is valid for large-scale transactions.

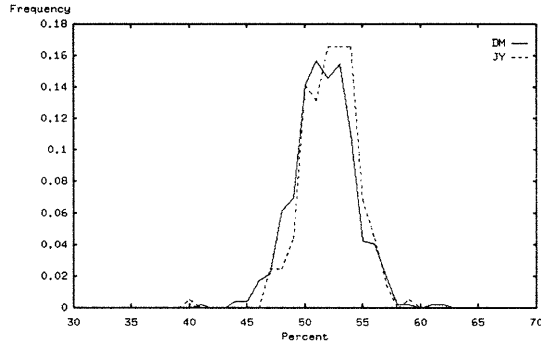


Figure 4-3: Distribution of c in Set A

These results are in stark contrast to the tick data. By examining whether the assumption $c = \frac{1}{2}$ holds, we can identify the underlying reasons for this difference.

Figures 4-3 and 4-4 show a histogram of c for the individual games⁴. In Set A, c is clustered near 50 percent, while in Set B, c varies widely. Since the tick data closely satisfies the assumption $m = \frac{1}{2}n$, S^{**} performed extremely well. However, on Set B, the assumption no longer holds, and S^{**} performed poorly.

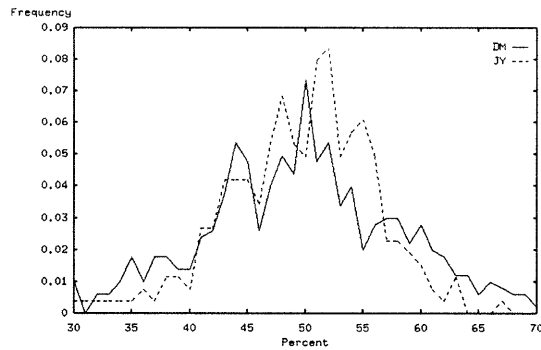


Figure 4-4: Distribution of c in Set B

⁴Recall that to facilitate the computation, the exchange rate sequence is broken into segments with $n = 100$.

Chapter 5

Conclusions

In this thesis, we examined the two-way currency trading problem. By using a statistical adversary, we struck a middle ground between the distributional approach and competitive analysis. For a wide variety of strong and weak adversaries, the optimal on-line strategy can be calculated via a simple dynamic program. For on-line algorithms that are *money-making*, the (n, Π) -adversary is a maximal adversary. Unfortunately, against this adversary, the on-line player is guaranteed only a minuscule fraction of the off-line profit.

In the fixed fluctuation model, we derived an exact form for the optimal strategy and calculated its asymptotic behavior. The optimal strategy always outperforms the optimal off-line “buy-and-hold” strategy, and in *active* and *stable* markets, the performance is exponentially better. Even in a slightly unfavorable market, the optimal strategy can achieve exponential return. The experimental results on actual data were mixed. When the data closely satisfied the model’s assumptions, the returns obtained were phenomenally high, even with transaction costs. However, if the data deviated from the assumptions, the optimal strategy became unprofitable.

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