Introductory Study of

Hypercomplex Number Systems and Their Applications in Geometry.

by

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Introduction and Summary

In the first chapter, the general hypercomplex number systems with n-units are discussed; some generalized theorems and new results are given. In the second chapter, the study concerns with the linearization of Riemann Space which idea is primarily the result of the present writer; connection between classical Riemann geometry and Einstein(1929) geometry is shown here. The geodesics defined with respect to the Linearized space are generally not the same as the ordinary geodesics deduced from the quadratic expression unless certain conditions are satisfied. In the third chapter, we study this linearized geometry at one particular point and consider various transformation properties. The fact that we have connected both the $+++-$ and the four-dimensional dilemma of the special theory of relativity with the theory of hypercomplex numbers is the most interesting point of this chapter. Geometrically, it is also shown that the space whose dimension is 4 occupies a peculiar position in linearized geometry in that it makes the dimensions of both spaces, actual and hypercomplex, equal. We have next established tentatively a transition rule between quantities in spin space and quantities in actual space. It is shown that this choice is not arbitrary and, guided by the equation of light cone in Einstein's relativity theory, we deduce Dirac equations in a different way. Throughout this part of the paper, it can be seen that the mass term has nothing to do with the fifth dimension, so-called, it is merely connected with the invariant interval. In chapter

4, the Dirac's equations receive~~d~~ a little attention and are discussed ~~[illegible]~~ in detail. In the last chapter, we study the connections of hypercomplex spaces at different points. It is shown that the solution of problems of two or more bodies depend on the negation of "possibility of displacement of the spin space".

Owing to the limit of space and time, the author regrets to say that may important points which could be and should be considerably developed have to be left out.

Note added by the author: As the paper ~~is~~ near finishing, the writer discovered a method by which the matrix representation of a hypercomplex system can in all future cases be easily found. The brief discussion, accompanied by an illustration, is given in the "appendix".

§ 1.

General Hypercomplex Number Systems

Just as the interest in related branches of Geometry was aroused by the advance of Einstein's Theory of Relativity, the study of Hypercomplex Number Systems has come into renewed attention by the Dirac's Theory of Linearization of the Relativitistic Quantum Equations of Electron. It seems desirable to treat the subject in a more logical and general way as, besides its intrinsic interest, the generalization of Dirac's Theory to the problems of two or more bodies essentially depend. Most results here deduced are merely generalizations of previous results obtained in the special case (of sedenions), though some new results are also worked out.

1. Generators

A set of independent numbers

$$E_\mu \qquad \mu = 1, 2, 3, \cdots\cdots n$$

will be called the generators of a Hypercomplex Number System of order n if they satisfy, besides the ordinary postulates e.g. associative, distributive laws, the following $\frac{1}{2}n(n+1)$ relations:

$$(1.1) \qquad E_{(\mu} E_{\nu)} = \delta_{\mu\nu} \qquad \text{where } E_{(\mu} E_{\nu)} = \frac{1}{2}(E_\mu E_\nu + E_\nu E_\mu).$$

The system is <u>closed</u> with elements E_μ , $E_\mu E_\nu$, $E_\mu E_\nu E_\sigma$ etc.,

which will be called the basic elements of the system. They are, together with Unity,

$$1 + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \cdots \quad + 1 = (1+1)^n = 2^n$$

in number.

Transformations of the type $\Lambda E \Lambda^{-1}$ (1.2)

where E is any member of the ~~grop~~ system and Λ is any arbitrary member of the system and where Λ^{-1} is defined as $\Lambda\Lambda^{-1} = 1$, are called canonical transformations. From the elementary theory of the groups, it is seen that the correspondence is one to one and that the relations (1) are kept invariant.

It is easily shown that all the basic elements (2^n in number) are linearly independent[1]; we have the result that every hypercomplex number system of order n has exactly 2^n linearly independent basic elements. It may be noted here that <u>every basic element</u>, possibly for a minus sign, <u>is its own Inverse</u>.

If we call the property $AB + BA = 0$

where A,B any elements of the system as 'anti-commutative' and the property that $AB = BA$

as 'commutative', then it can be verified that <u>every</u> basic element is either commutative or anticommutative with <u>any other</u> basic element of the system. This property still holds when these elements are under the canonical transformations (1.2).

2. Matrix Representation.

By a Theory in Algebra, every associative algebra is equivalent to a matrix algebra.[2] Therefore we can represent any element of the group by $(A^{\mu}{}_{\nu})$, the reason for the upper and lower indices will be given when we consider the geometrical representations of the system.

Now every matrix of m rows and columns has m^2 linearly independent basic elements. In order that it ~~can~~ may be <u>completely</u> represented by our hypercomplex numbers, we must restrict the system by the folowing condition i.e. $m^2 = 2^n$

But m is an integer, therefore n must be even; this proves

Theorem 1. No complete hypercomplex number systems of odd orders exist. That is , the independent generators can not be 1,3,5 etc., but may be 2,4,6 etc. This theorem, when properly interpreted, shows that why we can have only sedenions, quatenions but not, for instance, some system between them.

Theorem 11. There is one and only one element e in the system which anticommutes with every generator.[3)]

That there is one can be easily verified for this is $E_1 E_2 E_3 \cdots E_n$; it anticommutes with every generator. To prove the converse, that there is only one, we proceed as follows.

Lemma 1a. Every generator anticommutes with a basic element of even degree if this basic element includes this generator as a factor; it commutes ~~with~~ if this basic element does not include this generator as a factor.

Lemma 1b. Every generator commutes with a basic element of an odd degree if this basic element includes this generator as a factor; it anticommutes if this basic element does not include this generator as a factor.

A basic element e.g. $E_{n_1} E_{n_2} \cdots E_{n_f}$ $\quad n_1, n_2 \cdots n_f = 1, 2, 3 \cdots n,$

is called of odd degree if f is odd, even if f is even. The above Lemma asserts that

1 commutes with all E_ν

E_α antocommutes with all E_ν if ν not equal α ; otherwise commutes.

$E_\alpha E_\beta$ anticommutes with ~~all~~ E_ν if $\nu = \alpha$ or β ; commutes with all others.

$E_\alpha E_\beta E_\gamma$ commutes with E_ν if $\nu = \alpha, \beta$ or γ ; anticommutes with all others.

etc.

The truth of the Lemma is then self-evident by actually multiplying all them out. We may now proceed to prove the second part of Theorem 11. The most general form of an element of the system is

$$T = C_0 + \sum_{\mu=1}^{n} C_\mu E_\mu + \sum_{\mu\neq\nu} C_{\mu\nu} E_\mu E_\nu + \sum_{\mu\neq\nu\neq\sigma} C_{\mu\nu\sigma} E_\mu E_\nu E_\sigma + \cdots$$

The condition that it anticommutes with all the generators is

$$T E_\alpha + E_\alpha T = 0$$

for all $\alpha = 1, 2, 3, \cdots n$. This condition, by using Lemma 1, ~~condition~~ leads to a system of linear relations among the <u>independent</u> basic elements which are impossible; hence every co-efficient occurred must vanish. This gives

$$C_0 = 0 \qquad C_\mu = 0 \qquad C_{\mu\nu} = 0 \qquad etc.,$$

except the last one which does not appear in any of the linear relations. This proves our theorem. By a similiar argument, we can easily prove the following

<u>Theorem 111. There is one and only one element in the system which commutes with every generator.</u> In fact this member is Unity which commutes with every member of the sustem. (This theorem is a hint to indicate that in solving the problems of two bodies in Dirac's theory, we <u>can not use</u> four row-and-columned matrices.)

If we denote the element $E_1 E_2 E_3 \cdots E_n$

by E_0 and normalized by a factor so that its square is Unity; then every member of the set

$$E_0, E_1, E_2, E_3, \cdots E_n$$

anticommutes with any other member of the same set and they all satisfy the realtions (1.1); they form a (n+1)-fold "normalized anticommutative set." The following theorem is easily proved:

Theorem 1V. <u>The multiplier of every member of the set by any other member of the set, normalization factor being here $\sqrt{-1}$, forms together with this member itself, another normalized anticommutative set.</u>

In this way, we can obtain a total of <u>n+2</u> (n+1)-fold "anticommutative sets." This theorem may be called the <u>generalized</u> '<u>coupling theorem</u>'[4] of Eddington since it was discovered by him in the case of sedenions.

Consider two sets of elements

$$A: \quad E_\mu E_0 \qquad \mu = 1, 2, 3, \cdots\cdots n/2$$

$$B: \quad E_{\mu'} \qquad \mu' = \frac{n}{2}+1, \frac{n}{2}+2, \cdots n.$$

It is seen that the following properties hold true:

1. Every memeber of the set A commutes with every member of the set B and conversely.

2. Every member of the set A anticommutes with every member of its own set; similiarly for the set B.

3. E_0 anticommutes with every member of either set and Unity commutes with every member of the either set.

They form <u>indeed</u> two sub-hypercomplex number systems each of order n/2. Conversely if we have two sets of hypercomplex numbers with these properties assigned to them, we can build from them a system with twice as many generators as the sub-system has.[5] That is, we can build a sedenion system by multiplication of two quatenions and we can build a ~~octonion~~ system of 256 units by the multiplication of two sedenions and so on. It may not be out of place here to remark that in solving problems of two bodies these properties we referred to are exactly what we require.[6]

3. Transformation Properties.

If we subject E_μ to an arbitrary linear homogeneous transformation, the relations (1) will in general not hold unless the transformation matrix T is orthogonal i.e. $t_{\alpha\alpha'}t_{\beta\alpha'} = \delta_{\alpha\beta}$ where $T = (t_{ij})$. These transformations will also be called the "canonical transformations" since they keep relations (1) invariant. Under a canonical transformation

$$E_0' = E_1' E_2' E_3' \cdots E_n'$$
$$= t_{1\alpha_1} t_{2\alpha_2} t_{3\alpha_3} \cdots t_{n\alpha_n} E_{\alpha_1} E_{\alpha_2} E_{\alpha_3} E_{\alpha_4} \cdots E_{\alpha_n}$$

which, by the help of Lemma 1 and the condition of orthogonality, observing that the terms on the right hand side all vanish except those with indices different, gives

$$E_0' = \sum \pm t_{1\alpha_1} t_{2\alpha_2} t_{3\alpha_3} \cdots t_{n\alpha_n} \; E_1 E_2 \cdots E_n = |T| \, E_0$$

that is, E_0 transforms like a density, hence

Theorem V, For all orthogonal transformations of the generators with determinant 1, E_0 remains invariant.

§ 2

Linearization of Riemannian Space.

The fundamental metric of a Riemannian Space of n dimensions is given by the quadratic form

(2.1) $$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \qquad \mu,\nu = 1,2,3,\cdots n$$

which may be written as

(2.2) $$\left(E_0\, d\sigma + \sum E_\alpha\, dx^\alpha\right)^2 = 0$$

where

(2.3) $$E_{(\alpha} E_{\beta)} = g_{\alpha\beta}$$

and where E_0 is such a quantity that it anticomutes with every E_α.[7)] This metric will be referred to as the CLASSICAL metric. It is evident that these E_α introduced are isomorphic with the hyper-complex number systems of order n introduced in the first chapter.

We shall now apply the principle of linearization and consider the expression *

(2.4) $$E_0\, d\sigma + \sum E_\alpha\, dx^\alpha = 0$$

as of more fundamental importance than (2.2). It is not necessary to enter the discussion at the present moment what this significance is until later we have understood the meaning of (2.4).

*When this idea occurred to the writer who has worked it out somewhat in detail he was unaware and was very glad to find out that the same idea has occured to Fock and Iwanenko: Über eine mögliche geometrische Deutung der relativistischen Quantentheorie Zt. für Phy. 54 p.798; dieselbe: Géometrié quantiqué linéaire ét déplacement parallele Com. Ren. 188 p. 1470; and Fock: Geometrisierung der Diracschen Theorie des Elecktrons Zt. für Phy. 57 p.261. However the treatment is ~~somewhat~~ different.

Define

(2.5) $$E^{\mu} = g^{\mu\nu} E_{\nu} \quad \text{where} \quad (g^{\mu\nu}) = (g_{\mu\nu})^{-1}$$

It follows that, since E_{ν} is covariant, E^{μ} is contravariant. It can be proved that [8]

(2.6) $$E^{(\mu} E^{\nu)} = g^{\mu\nu}$$

and

(2.7) $$E^{(\mu} E_{\nu)} \equiv E^{\mu} E_{\nu} + E_{\nu} E^{\mu} = \delta^{\mu}_{\nu} \quad \begin{matrix} = 0 \text{ if } \mu \neq \nu \\ = 1 \text{ if } \mu = \nu. \end{matrix}$$

The Geodesics of this Geometry will be defined as

(2.8) $$\delta \int E_0 \, d\sigma = 0$$

i.e.

(2.9) $$\delta \int E_{\nu} \, dx^{\nu} = 0$$

which gives, by easy calculation, the n linear partial differential equations

(2.10) $$A_{\mu\sigma} \, dx^{\sigma} = 0$$

where $A_{\mu\sigma}$ is defined as $$\frac{\partial E_{\mu}}{\partial x^{\sigma}} - \frac{\partial E_{\sigma}}{\partial x^{\mu}}$$

It can be proved that $A_{\mu\sigma}$ is covariant of the second rank. It can also be shown that the geodesics (2.10) defined in our Geometry do not, in general, coincide with the geodesics defined in the classical Riemann Geometry. In fact, we have, from (2.4)

(2.11) $$\frac{d}{d\sigma}\left(E_0 + E_{\nu} \frac{dx^{\nu}}{d\sigma}\right) = 0$$

and make the convention that

(2.12) $$\frac{d}{d\sigma}(E_0) = 0$$

which ~~will~~ may be justified later; considerations on substituting the equations of geodesics (2.10), we obtain

(2.13) $$\frac{d}{d\sigma}\left(E_{\nu} \frac{dx^{\nu}}{d\sigma}\right) = E_{\nu} \frac{d^2 x^{\nu}}{d\sigma^2} + \frac{dE_{\nu}}{d\sigma} \frac{dx^{\nu}}{d\sigma} = E_{\nu} \frac{d^2 x^{\nu}}{d\sigma^2} + \frac{\partial E_{\alpha}}{\partial x^{\sigma}} \frac{dx^{\alpha}}{d\sigma} \frac{dx^{\sigma}}{d\sigma} = 0$$

Multiplying by E^{μ} first in front and then in back and add, we have

$$(2.14) \quad \frac{d^2x^\nu}{d\sigma^2} + \left(E^\nu \frac{\partial E_\mu}{\partial x^\sigma}\right) \frac{dx^\mu}{d\sigma} \frac{dx^\sigma}{d\sigma} = 0 \quad \text{where } 2\left(E^\nu \frac{\partial E_\mu}{\partial x^\sigma}\right) \equiv E^\nu \frac{\partial E_\mu}{\partial x^\sigma} + \frac{\partial E_\mu}{\partial x^\sigma} E^\nu,$$

which may be written as

$$(2.15) \quad \frac{d^2x^\nu}{d\sigma^2} + \Gamma_{\mu\sigma}^{\cdot\cdot\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\sigma}{d\sigma} = 0 \quad \text{where } 2\Gamma_{\mu\sigma}^{\cdot\cdot\nu} \equiv \left(E^\nu \frac{\partial E_\mu}{\partial x^\sigma} + \frac{\partial E_\mu}{\partial x^\sigma} E^\nu\right).$$

In this summation only the symmetrical part of $\Gamma_{(\mu\sigma)}^{\cdot\cdot\nu}$ survives. We can represent $\Gamma_{\mu\sigma}^{\cdot\cdot\nu}$ in terms of the well-known Christoffel symbols and some antisymmetric functions.

$$(2.16) \quad 2\begin{bmatrix}\mu\nu\\ \sigma\end{bmatrix} = \left(\frac{\partial E_\mu}{\partial x^\nu} E_\sigma\right) + \left(E_\mu \frac{\partial E_\sigma}{\partial x^\nu}\right) + \left(\frac{\partial E_\nu}{\partial x^\mu} E_\sigma\right) + \left(\frac{\partial E_\sigma}{\partial x^\mu} E_\nu\right)$$
$$- \left(\frac{\partial E_\mu}{\partial x^\sigma} E_\nu\right) - \left(\frac{\partial E_\nu}{\partial x^\sigma} E_\mu\right) = 2\left\{\Gamma_{(\mu\nu),\sigma} + \Lambda_{\sigma\nu,\mu} + \Lambda_{\sigma\mu,\nu}\right\}$$

where Λ is defined as

$$(2.17) \quad \Lambda_{\alpha\beta,\gamma} = \frac{1}{2}\left(\Gamma_{\alpha\beta,\gamma} - \Gamma_{\beta\alpha,\gamma}\right)$$

Hence

$$(2.18) \quad \Gamma_{(\mu\nu),\sigma} = \begin{bmatrix}\mu\nu\\ \sigma\end{bmatrix} + \Lambda_{\nu\sigma,\mu} + \Lambda_{\mu\sigma,\nu}$$

and

$$(2.19) \quad \Gamma_{(\mu\nu)}^{\cdot\cdot\sigma} = \begin{Bmatrix}\mu\nu\\ \sigma\end{Bmatrix} + \Lambda_{\nu\sigma}^{\cdot\cdot\mu} + \Lambda_{\mu\sigma}^{\cdot\cdot\nu}.$$

Thus the geodesics in our geometry reduces to

$$(2.20) \quad \frac{d^2x^\alpha}{d\sigma^2} + \begin{Bmatrix}\mu\nu\\ \alpha\end{Bmatrix} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} = \left(\Lambda^{\alpha}_{\cdot\nu,\mu} + \Lambda^{\alpha}_{\cdot\mu,\nu}\right) \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}$$

These are not the same as the classical geodesics unless the right hand side vanishes. If the condition (2.12) were not imposed, we would get more complicated formulae.

In order to restrict ourselves only to the fundamental quantities (E_μ) and (E^μ) to define covariant differentiation, we proceed in the following manner of <u>geodesic</u> displacement.

Let a covariant vector be denoted by V_α then $V_\alpha \frac{dx^\alpha}{d\sigma}$ is, by definition, invariant, hence $\frac{d}{d\sigma}\left(V_\alpha \frac{dx^\alpha}{d\sigma}\right)$ is invariant along an absolutely defined curve. Geodesics are of such curves. On using the equations for geodesics (2.10), we obtain the result that

$$\left(\frac{\partial V_\mu}{\partial x^\tau} - \Gamma_{\mu\tau}^{\cdot\cdot\sigma} V_\sigma\right) \frac{dx^\mu}{d\sigma} \frac{dx^\tau}{d\sigma}$$

is invariant. Hence either

$$(2.21)\qquad \nabla_\tau V_\mu = \frac{\partial V_\mu}{\partial x^\tau} - \Gamma_{\mu\tau}^{\cdot\cdot\sigma} V_\sigma$$

or

$$(2.22)\qquad \nabla_\tau V_\mu = \frac{\partial V_\mu}{\partial x^\tau} - \Gamma_{\tau\mu}^{\cdot\cdot\sigma} V_\sigma$$

may be taken as the definition of covariant differentiation; the former is however preferable. That they are actually covariant tensors can be proved directly. From (2.21) and (2.22) we see that Λ defined by equation (2.17) are tensors of the third rank.

We see that the classical Riemann Geometry corresponds to the case when all Λ vanish; on the other hand, we shall show that the covariant differentiation adopted in (2.21) is practically identical with the definition of Einstein in his 1929 papers. For if we write

$$(2.23)\qquad E^\mu = \sum_{1}^{n} {}_t h^\mu E^t$$ where E^t are hypercomplex numbers of order n,

and

$$(2.24)\qquad E_\mu = \sum_{1}^{n} {}^t h_\mu E^t$$

~~We thus~~ get

$$(2.25)\qquad E_\mu E_\nu = \sum_{1}^{n} {}^t h_\mu {}^t h_\nu$$

and similiar formulae for $E^\mu E^\nu$ and $E^\mu E_\nu$.

We obtain, thus

$$(2.26)\qquad \Gamma_{\mu\tau}^{\cdot\cdot\sigma} = \left(E^\sigma \frac{\partial E_\mu}{\partial x_\tau} \right) = \sum_{t=1}^{n} {}_t h^\sigma \frac{\partial\, {}^t h_\mu}{\partial x^\tau}$$

if we assume that

$$(2.27)\qquad \frac{\partial E^t}{\partial x_\tau} = 0 \qquad t = 1, 2, \cdots n \qquad \tau = 1, 2, \cdots n$$

Our Γ would then be the same as his Δ. The assumption (2.27) is equivalent to (see Chapter 3) the assumption ~~that~~ of distant parallelism of spin co-ordinates.

We could define analogous Riemann-Christoffel Tensors in this theory but it has little interest ~~at~~ for our immediate purpose and we shall not pursue the matter further.

§ 3.

Study of Hypercomplex Geometry at one point.

At one point, the $g_{\mu\nu}$ are constants; we can, therefore, by suitable choise of co-ordinate system, reduce the relations

$$E_{(\mu} E_{\nu)} = g_{\mu\nu}$$

to

(3.1) $$E_{(\mu} E_{\nu)} = \delta_{\mu\nu} \quad \text{as in § 1,}$$

which are the generators of a hypercomplex number system of order n considered in chapter 1. That is, at a point, the geometry (2.4) reduces to the form

(3.2) $$E_0 \, d\sigma + \sum E_\mu \, dx^\mu = 0$$ where E_μ are generators of a hypercomplex number system of order n.

By the considerations developed in chpater 1, it is evident that if we study the hypercomplex numbers we can study them in the "matrix-way". We thus associate at every point of space a matrix-space in the sence that every matrix--that is, every hypercomplex number--we consider is a tensor in that space. Owing to the fact that $E_\mu E_\nu$ also lies in this space and is a tensor of the same kind, we must consider them as co-contravariant tensors $E^{\alpha}_{\cdot\beta}$ of the second rank. The multiplication rule would then be

(3.3) $$(E_\mu E_\nu)^{\alpha}_{\cdot\beta} = E_\mu{}^{\alpha}_{\cdot\tau} \, E^{\tau}_{\cdot\beta} \, .$$

This "matrix-space" is introduced, at least up to the present, only in helping to describe the Geometry (3.2); it may be called a "Hilfsraum" or ~~"[illegible]"~~ "Auxiliary Space" if we please. But time to time we shall also use the name "spin space" or "hypercomplex space" when it is advisable to render the meaning more explicite.

By way of definition E_μ $\mu=1,\cdots n$ are covariant and E_0 invariant. It is to be noted that E_0 has eigenwerte 1 or -1 and that it has an equal number of 1 as -1 eigenwerte. The former is obvious since $E_0 E_0 = 1$ means that in applying E_0 twice successively to its eigenvektor, it restores its original length. The latter can be seen from the following consideration: $\Lambda E_0 \Lambda^{-1}$ has necessarily the same eigenwerte as E_0 (since the characteristic equation is not changed by canonical transforamtions). If we put Λ equal to any of the generators E_μ say, then

$$(3.4) \qquad E_\mu E_0 E_\mu^{-1} = E_\mu E_0 E_\mu = -E_\mu^2 E_0 = -E_0$$

that is $-E_0$ has the same eigenwerte as $+E_0$. This proves our assertion. Since it is invariant by definition, we can therefore subdivide the hypercomplex space into two invariant sub-spaces H_1, H_2 each of which is of dimension $m/2$ where m is defined as $m^2 = 2^n$. It can be easily proved that every generator is now reduced, consisting of components with one index in H_1 while the other index in H_2 and conversely; there is no component of any generator lies totally in H_1 or H_2. However we do not need these properties expilicitly in this paper, we shall not psuh the subject further. Reader who is interested in this part may have reference to Schouten'[1] Paper where special case of n = 4 is treated and many of them admit an easy generalization. We consider now the transforamtion properties in both spaces and their relations to each other.

a) Transformations of the co-ordinates in the hypercomplex space.

Let us denote the co-ordinate systems in the hypercomplex space by ζ_{α_i} $i=1,\cdots n$. By an arbitrary transformation of the co-ordinates $\zeta_{\alpha_i} \rightarrow \zeta_{b_i}$ $\quad d\zeta_{\alpha_i} = \sum_1^n \frac{\partial \zeta_{\alpha_i}}{\partial \zeta_{b_j}} d\zeta_{b_j} \quad i, j = 1, 2, \cdots n,$

hypercomplex numbers (mixed tensors of the second rank) would undergo the transformation,

$$(3.5) \qquad E_{\mu}{}^{b_i}{}_{\cdot b_j} = T^{b_i}{}_{\cdot \alpha_s} E_{\mu}{}^{\alpha_s}{}_{\cdot \alpha_t} T'^{\alpha_t}{}_{\cdot b_j}$$

where

$$(3.6) \qquad T^{b_i}{}_{\cdot \alpha_s} = \frac{\partial \zeta_{b_i}}{\partial \zeta_{\alpha_s}} ; \; T'^{\alpha_t}{}_{\cdot b_j} = \frac{\partial \zeta_{\alpha_t}}{\partial \zeta_{b_j}}$$

It is easily proved that $TT' = 1$ i.e. $T' = T^{-1}$

$$(3.7) \therefore \quad E \longrightarrow TET^{-1}$$

that is, they are undergoing a canonical transformation considered in chapter 1. E_0 transforms into

$$(3.8) \qquad E_0{}^{b_i}{}_{b_j} = T^{b_i}{}_{\cdot \alpha_s} E_0{}^{\alpha_s}{}_{\cdot \alpha_t} T'^{\alpha_t}{}_{\cdot b_j}$$

or, when the spin-space was sub-divided by the consideration above,

$$(3.9) \qquad = \sum_{\alpha_s = 1}^{m/2} T^{b_i}{}_{\cdot \alpha_s} T'^{\alpha_s}{}_{\cdot b_j} - \sum_{\alpha_t = \frac{m}{2}+1}^{m} T^{b_i}{}_{\cdot \alpha_t} T'^{\alpha_t}{}_{\cdot b_j}$$

which will be invariant and only then if the transformation matrix T is the reduced matrix i.e.

$$(3.10) \qquad T = \begin{Vmatrix} T_1 & 0 \\ 0 & T_2 \end{Vmatrix}$$

where T_1 and T_2 are both m/2 square matrices. That is, if the transformation in spin space is such that it keeps both the sub-spaces invariant, ~~the geometry~~ E_0 remains invariant. A transformation of the form $\begin{Vmatrix} 0 & T_1 \\ T_2 & 0 \end{Vmatrix}$ would only change $E_0 \rightarrow -E_0$ that is, interchanging of the two sub-spaces.

b) Transformations of the co-ordinates in the actual space.

From the arguments of chapter 1, it is seen that the only transformations which keeps the relations (1.1) invariant are those that are orthogonal. In that case $E_0 \rightarrow -E_0$ if the determinant is -1 and E_0 remains invariant if the determinant is 1. (It is to be noted that the general transformations which keep the quadratic expression of (3.2) invariant is orthogonal and that relativity transformations in the case of $n = 4$ are restricted to those with determinant $+1$.

A particular case of the transformations in the hypercomplex case is that when T is in the form $e^{i\phi}\,1$ where ϕ may have <u>any</u> value. This transformation does not affect any of the hypercomplex numbers; the corresponding transformation in the actual space can only be the identity transformation. This case would have no interest if we deal only with the mixed tensors in the hypercomplex space but which would play a rôle, an important rôle indeed if we accept Weyl's idea, if we consider not only the mixed tensors but also vectors in the hypercomplex space. This ~~gives~~ would give rise to the conception of "pseudo-vector".

One can easily convince oneself that the transforamtions considered above in both spaces are the <u>most general possible</u> transformations that keep (1.1) invariant. We tabulate here the correspondences between the two spaces:

a) Trans. in Matrix Space		b) Trans. in Space
General Transf. leaves two subspaces invariant.	$\leftarrow$	Orthogonal Transf. of determinant 1.
General Transf. leaves two subspaces interchanged.	$\leftarrow$	Orthogonal Transf. of determinant -1.
General transf. of the form	$\rightarrow$	Identity Transf.

A case of particular interest is that when the dimensions of both spaces are the same that is when $m = n$. This has one and only <u>one solution that</u> is when $m = n = 4$*.

* It is admittedly true that in this case an intimate relation between the two spaces may exist but I don't think this consideration leads to the identification of the two spaces (as Eddington did[9]). That would only lead to the confusion of terminology and lose their

real geometrical significances.

Thus the space of the type (3·2) of four dimensions occupies a peculiar position in the hypercomplex geometry. We shall therefore consider this more in detail.

The hypercomplex number system of order 4 has been investigated more or less thoroughly by various writers.[10)] It receives the special name of "'Dirac's numbers" or sedenions as one generally calls it. Its matrix representation was first discovered by Dirac. The treatment of Eddington is particularly elegant and enables one to make easy generalizations when n is any number. We shall adopt his method here.

He starts with three 4-point matices grouped according to (12,34), (13,24) and (14,23): (all element +1)

$$(3\cdot11)\quad S_\alpha=\begin{Vmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{Vmatrix}\qquad S_\beta=\begin{Vmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{Vmatrix}\qquad S_\gamma=\begin{Vmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{Vmatrix}$$

and introduces a fourth matrix S_δ, the identity matrix. Further three diagonal matrices with elements + 1 or -1 are introduced (their spur is zero)

$$(3\cdot12)\quad D_\alpha=\begin{Vmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{Vmatrix}\qquad D_\beta=\begin{Vmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{Vmatrix}\qquad D_\gamma=\begin{Vmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{Vmatrix}$$

and $D_\delta=S_\delta=1$.

Then the following properties are easily proved: 1, Each matrix commutes with the matrix of the same group; 2, S_a commutes with D_b if $a=b$, otherwise they anticommute; (Unity Matrix of course commutative) 3, the product $S_a D_b$ is of the type S_a of the four-point matrices but they may contain -1 as element or elements; and 4, the 16 products $S_a D_b$ are linearly independent. With the help of them, he was able to find the anti-commutative sets; they are

$$
(3{\cdot}13)\quad
\begin{array}{llllll}
1) & S_\alpha & D_\beta & S_\gamma D_\gamma & iS_\gamma D_\beta & iS_\alpha D_\gamma \\
2) & S_\alpha & D_\gamma & S_\beta D_\beta & iS_\beta D_\gamma & iS_\alpha D_\beta \\
3) & S_\beta & D_\alpha & S_\gamma D_\gamma & iS_\gamma D_\alpha & iS_\beta D_\gamma \\
4) & S_\beta & D_\gamma & S_\alpha D_\alpha & iS_\alpha D_\gamma & iS_\beta D_\alpha \\
5) & S_\gamma & D_\alpha & S_\beta D_\beta & iS_\beta D_\alpha & iS_\gamma D_\beta \\
6) & S_\gamma & D_\beta & S_\alpha D_\alpha & iS_\alpha D_\beta & iS_\gamma D_\alpha
\end{array}
$$

when any one of the sets is found, others can be obtained by the ~~coup~~ "coupling theorem" of chapter 1. Two important properties should be noted: 1, " ~~No more than~~ Only three real matrices are in the set" and 2, the real ones are symmetrical and the imaginary ones are antisymmetrical about the diagonal, that is Hermitian. The property 1 does not survive under arbitrary transformations 10) but ~~it~~ ~~one~~ we can easily convince ourselves that no matter what the transformations may be the statement that "no more than three real matrices can be found such that they all satisfy the relations (1.1)" is always true. The property 2 does not survive except under unitary transformations or, if real transformations only are considered, under orthogonal transformations. The property "no more than three matrices can be found such that they all satisfy the relations (1.1)" was first noticed by Eddington. What is the significance of this property when applied to geometry (3.2)? This means that, if the geometry (3.2) is to be considered as real, one of the co-ordinates must be pure imaginary. This property when coupled with the theorem 1 of chapter 1 gives the most remarkable and beautiful result that Einstein's invariant interval in the special theory of relativity is the only possible real one when ~~have~~ we have optional choice of dimensions lying in 3, 4 or 5 and that one dimension of which must be imaginary. This is the consequence of our linearization of Geometry. Eddington has remarked that, though from an entirely different consideration as here presented,

"

" .... the matrix theory offers an <u>explanation</u> why one of the dimensions of our world differs from the other three. We have traced it down to the fact that not more than three real four-point matrices can satisfy simultaneously $E_\mu^2=1$ $E_\mu E_\nu = 0$."[11] Thus the linearization theory, whether in our present form or in the ~~conception~~ consideration of Eddington <u>indeed</u> <u>explains</u> the $+ + + -$ mystery of the special theory of relativity. The space of dimensions 3 or 5 has already been ruled out by Theorem 1 of chapter 1.

So far the geometry represented by (3.2) has not received interpretation; that is, it connects quantities in the hypercomplex space on the one hand and the quantities in the actual space on the other hand. We do not know how to <u>work</u> with them unless some convention is made; that is, some sort of transition rule by which a quantity in either space is translated into a quantity of the other space. The folowing is a tentative discussion of this process.

Geometry (3.2) may be written as

$$E_0 + \sum E_\mu \frac{dx^\mu}{d\sigma} = 0 \tag{3.14}$$

which, when multiplied by an arbitrary factor, invariant, k, becomes

$$kE_0 + \sum E_\mu k\frac{dx^\mu}{d\sigma} = 0 \tag{3.15}$$

We then observe that in the old relativity theory when $ds = 0$ that is $d\sigma = 0$ it gives the track of light wave-quanta. We shall naturally expect that when we put $d\sigma \to 0$ in (3.15) we should get the equation of motion of light wave ψ say. How this is to be brought out? This can be done by the following process; it is extremely unlikely that any other process will do. We write

$$k\frac{dx^\mu}{d\sigma} \to \frac{\partial \psi}{\partial x^\mu} \qquad k1 \to k\psi \tag{3.16}$$

as the transition rule, then it follows at once that equation (3.15) becomes, when $k \to 0$

$$(3.17) \quad \left(\sum E_\mu \frac{\partial}{\partial X^\mu}\right)\psi = 0$$

which, as can be verified, represents the quanta-mechanical equation of the motion of electro-magnetic waves. We now extend this ~~case~~ to the case <u>when ds</u> is <u>not equal</u> to <u>zero</u>, (Remember that the geometry in the classical sence here is Euclidean.) therefore k can not be zero. Equation (3.15) becomes

$$(3.18) \quad \left(\sum E_\mu \frac{\partial}{\partial X^\mu} + kE_0\right)\psi = 0$$

We now inquire next what is the significance of k? It is invariant and equals zero when ds equal zero. It does not equal to zero when ds does not equal to zero. We know that the <u>only</u> invariant satisfies these condition is the so-called "<u>proper mass</u>." Therefore k must be proportional to the invariant mass associated with wave function ψ. It equals to it save a numerical factor. The above equation then becomes

$$(3.19) \quad \left(\sum E_\mu \frac{\partial}{\partial X^\mu} + mE_0\right)\psi = 0$$

where a numerical factor is ommitted by suitable choice of the units. This equation is the "Linearized Wave Equation" of the motion of ψ with which is associated <u>something</u> whose ds is $\neq$ zero in the <u>classical</u> sence. This equation was first discovered by Dirac[11] and has been associated the name of "Dirac's Equations" with it. We shall discuss it a little ~~more~~ in detail in the next chapter.

§ 4.

Brief Discussion of Dirac's Equations.

In our derivation of the Dirac's equations, we were guided by the equation of light cone in Einstein's special theory of relativity. Dirac shows that, because the necessity of the requirement of the General Transfomration Theory and the requirement of the Theory of Relativity, it is almost forced upon him that the wave equation must be linearized in the $\frac{\partial}{\partial x^{\alpha}}$'s. This eqaation gives, when an electro-magnetic field is present, not only the ordinary wave terms but also the corrections which were experimentally verified and were attributed to the spin of the electron. The assumption of spin has created many insurmountable difficulties which we shall not discuss here.[12] Not only this, the Dirac's equations settle once for all the time-worn controversies regarding the "relativity fine structure".* We now know that the "spin" has its origin in the Geometry itself; this is evident from the discussions of Eddington[13] ~~(Proc. Roy. Soc.)~~ that this spin term comes into geometry before any conception of eletron or wave has been made. To discuss these more in detail would ~~trespass~~ be out of place here; we shall however consider some simple properties of a monochromatic wave.

* Milikan and Bowen: Phil. Mag. 49, 923.

Let the wave be represented by

$$(4.1)\qquad \psi = \bar{v}\, e^{i(px + qy + rz + Wt)} \qquad x_1 = x \quad x_2 = y \quad x_3 = z \quad x_4 = it$$

where $\bar{v}$ is a vector in the "spin space" and p, q..'s are constants; if we substitute this vlaue in (3.19) and multiplying the equation left-hand-sidely by iE_0 , the equations bécomes

$$(4.2)\quad \left(pE_1 + qE_2 + rE_3 + \frac{W_0}{i}E_4 + mi1\right)\bar{v} = 0.$$

(with suitable change of notations).

Write

$$I = pE_1 + qE_2 + rE_3 + \frac{W}{i}E_4 + mi1$$

and

$$J = pE_1 + qE_2 + rE_3 - \frac{W}{i}E_4 + mi1$$

When we assume $E's$ to be Hermitian ~~conjugates~~ as we have seen, we obtain

$$I^* = pE_1 + qE_2 + rE_3 - \frac{W}{i}E_4 - mi1$$

$$J^* = pE_1 + qE_2 + rE_3 + \frac{W}{i}E_4 - mi1$$

where I^*, J^* are the Hermitian conjugates of I, J respectively.

$$\therefore IJ^* = J^*I = I^*J = I^{-} = p^2+q^2+r^2-W^2+m^2$$

From $I\bar{v} = 0$ ~~froms~~ follows that $J^*I\bar{v} = 0$

Therefore $\bar{v} = 0$ if $(p^2+q^2+r^2-W^2+m^2) \neq 0$ (4.3)

But if this condition is satisfied (it can be easily verifìed that this is the relativity energy momentum equation.) then solutions for which $\bar{v} \neq 0$ may be found. Let us dénote the rank of I by a ; then equation $I\bar{v}=0$ has $4-a$ linearly independent solutions. If the rank of J^* is b then since $IJ^* = 0$ $\therefore a+b \leq 4$ and since $I - J^* = 2mi1$ $\therefore a+b \geq 4$; hence $a+b=4$. But if we set $m \to -m$ then

$$I \longrightarrow J^* \qquad \therefore\ a = b = 4$$

Hence, the number of linearly independent solutions of (4.2) is two. This conclusion was due to Neumann.[15] (It can be seen that in a sence these monochromatic waves are so polarized such as to make the "spin" possibility.) The condition (4.3) shows that if $W \to -W$ it is also satisfied; this would lead to nothing new in the classical theory. But in Dirac's theory, when $W \to -W$, $I\bar{v} = 0 \to J^*\bar{v} = 0$! This gives two other solutions with negative energy! (This is the origin of Dirac's recent theory of Proton and Electron.[14]) We thus obtain four wave functions $\psi_1^+ \psi_2^+ \psi_1^- \psi_2^-$; since $I J^* = 0$, they are perpendicular to each other. They thus determine two ~~per~~ mutually perpendicular planes in the spin space. Every vector can be split into two components: one in the plane determined by ψ_1^+, ψ_2^+, the other in the plane determined by $\psi_1^- \psi_2^-$. The two waves are respectively

$$e^{i(px+qy+rz+Wt)} \qquad e^{i(px+qy+rz-Wt)}$$

the latter corresponds to electron with negative energy. ~~to the inversed time direction;~~

§ 5.

Connection of Hypercomplex Spaces at Different Points.

1. Homogeneous and Inhomogeneous Space Manifold.

In chapter 3, we have discussed the hypercomplex geometry at a single point of the space manifold which is assummed to be Euclidean (in classical sence) at this point. If there are two or more points in the manifold, we may, as we have done, associate a hypercomplex space with each point in question. How we are to connect them? We consider two separate cases: 1) The hypercomplex space at the point B may be obtained by some process of displacement from the hypercomplex space at the point A and 2) It is not possible to do so; that is, the two spaces are fundamentlly distinct and we can not obtain the space at B from that at A by any of processes employed in 1. If the hypercomplex space at <u>every</u> point B of the space manifold <u>can</u> be obtained by some displacement from a certain point A, the space will be called a "<u>homogeneous manifold</u>". In an "inhomogeneus" field, we may however displace the hypercomplex space at A to B by some process but then this displaced space A', say, cannot be made identical with B by <u>any</u> process. Since they are distinct, no conneetion is possible between them*, hence the following commutative lawS muat hold

$$E_\mu' E_\nu = E_\nu E_\mu'$$

where E_ν' is any hypercomplex number belonging to A' and E_μ any hypercomplex number belonging to B. We can, as seen from the end

*Except possibly the Pauli Exclusion Principle in the solution of wave equations.

of the section 2 of chapter 1, associate at every point a hypercomplex space of 2m dimensions if there are only two distiicnt hypercomplex spaces in the entire field and that each is of dimension $m = 2^{n/2}$ and consider then all the hypercomplex spaces in the manifold can be obtained by the displacement method. For more than two, the process is analogeous. Hence: Any inhomogeneous spaze can be made homogeneous by increasing the dimensions of the associated hypercomplex spaces. If there are an infinite number of hypercomplex spaces, all of which are distinct, assocaited with an equal number of points of the field, we must, in order to make the space homogeneous that is the possibility of a displacement, assocaiate at every point of the manifold a matrix space of infinite number of dimensions (matrices appeared would then be of infinite number of rows and columns.) The justification that whether they may be chosen as Hermitian must be sought for ~~in~~ from a deeper investigation and will not be discussed here.

The significance of the above considerations lies in the fact that for the problems of two or more bodies, we must, in order to make the possibility of displacement, increase the dimension of the associated hypercomplex space. The physical interpretation of distinct hypercomplex ~~sapces~~ spaces is, when applied to wave equations, the spin associated with one electron (a point in space) is essentially distiicnt from the spin assocoiated with a different electron (at other point of the space). Thus although the classical geometrical theory has no counter part for the treatment of the problems of two or more bodies, the spin geometry has! We can not increase the dimensions of the actual space but we can increase the dimensions of the auxiliary space as much as we like without leading to any logical inconsistency. The so-called "interaction" would either appear as geometrical property in the composition of two spin spaces or as a result of the Exclusion Principle of Pauli ~~has~~ which has so far no geometrical interpretation. It is extremely likely that both play important rôles and it is conjectured that the Pauli Exclusion Principle may have its geometrical significance in the process of "composition". To ~~discuss~~ investigate more fully this subject would be outside of the scope of this thesis but I hope I shall return to this subject sooner or later. Now we shall briefly consider the theory of linear displacement of a spin quantity.

2. Displacement of spin spaces.

The method by which a hypercomplex space frame can be displced to an arbitrary but infinitesimally nearby point is called the method of pseudo- parallel displacement. It is called parallel in analogous to the case when the manifold is Euclidian. Neglecting quantities of higher orders, the displacement is in general of the form

$$\delta e^{\nu} = f(e^{\alpha})\, \delta x^{\beta}$$

where e^{ν} is some vector (contravariant) ~~of~~ in the spin space. For covariant vectors, similiar formula is obtained. In general we are interested ~~in~~ only in the so-called linear displacements for which the displacement formula is of the form

$$\delta e^{\nu} = T^{\nu}_{\alpha\beta}\, e^{\alpha}\, \delta x^{\beta}$$

where the $T^{\nu}_{\alpha\beta}$ are entirely arbitrary with well defined modes of transformations. If the space is homogeneous, that is the hyperspace at one point can be obtained by the displacement of a hyper-complex space at another point, we can make the convention that

$$\nabla E = \delta E + \partial E = 0$$

together by suitable chosing of co-ordinates we can make

$$\partial E = 0$$

If however the vector is pseudo-vector, we can write

$$\delta e^{\nu} = \Gamma^{\nu}_{\alpha\beta}\, e^{\alpha} dx^{\beta} + \varphi_{\beta}\, e^{\nu} dx^{\beta}$$ where φ's are parameters with well-defined mode of transformations.

In general, the displacement of vectors in the spin space has no connection at all with the displacement of a vector in actual space, but if the above conventions were adopted and with suitable assumptions, it can be shown that they are connected. This result is mainly due to Schouten*

* I must admit that it was mainly his lectures at Massachusetts Institute of Technology during the winter semester of 1930-1 that inspired me to write this paper. His work on this subject will appear in the coming issue of Journal of Mathematics and Physics.

We see that the electric-magnetic terms nearly come in automatically if we can accept the idea that these φ's are actually the electric magnetic potentials derived macroscopically from the formulae

$$\varphi_{\beta} = \iiint \left\{ \frac{\rho \frac{dx^{\beta}}{dt}}{r} \right\}_{t-r} dV \qquad \beta = 1, 2, 3, 4,$$

and if we replace ordinary differentiations in the wave equations (3·19) by the covariant differentiations. However, as the first part of this idea is hard to be accepted unless further investigation can prove that, we must leave this as rather a mathematical speculation.

§ 6.

Conclusion.

In this paper, the writer attempts to introduce a new field which is wide open. It is evident from the considerations of chapter 2 and 3 that the generalization of Dirac's equations to a Riemann space is not so easy as one might expect. The deductions of the Dirac's equations in chapter ~~3~~ 3 based mainly on the ~~diea~~ idea of light cone and the ds in the theory of relativity. One cannot expect to get the generalization by simply replacing $\frac{\partial}{\partial x^\beta}{}^*$ by $\nabla_\beta{}^*$ as many authors do but one must be guided by the Riemann ds and the equations of the geodesics which an uncharged "classical" particle is expected to take in the general theory of relativity. In ~~chapter~~ chapter 5, we have presented some new features which, the author hopes, may ultimately lead to the solution of the problem of two bodies and to such questions ~~of~~ as the geometrical significance of the modern <u>matrix</u> quantum theory and that why the space should be of a certain nature.

In conclusion, I wish to thank Prof. D. J. Struik of the Department of Mathematics of Massachusetts Institute of Technology for his kind interest and encouragement.

Notes

1) that is to say, no linear relations can exist among them.

2) cf. e.g. Dickson

3) The first part of this theorem has been noticed by Eddington in the case of sedenions. cf. Eddington: Symmetrical Treatment of Wave Equations Pro. Roy. Soc.A Vol. 1~~17~~21 p. 524-542

4) Eddington: loc. cit.

5) We can easily see by multiplication.

6) cf. Eddington: Interaction of Electric Charges Proc. Roy. Soc. A Vol. 126 p. 696

7) In fact E_0 may be taken as the E_0 of the hypercomplex number system of order n.

8) for instance

$$E^{(\mu}E^{\nu)} = g^{\mu\alpha}g^{\nu\beta}E_{(\alpha}E_{\beta)} = g^{\mu\alpha}g^{\nu\beta}g_{\alpha\beta} = g^{\mu\nu}$$

9)

10) They only survive under real transformations of the spin co-ordinates

11)

12)

13)

14) Dirac: Thepry of Electrons and Protons Proc. Roy. Soc. A Vol. 126 p. 360

15) Neumann: Einige Bemerkungen zür Diracshen Theorie Zt. Phy. 48 p.868

16) The theoretical discussion of this method can be carried and ~~are~~ is based on the properties of "<u>composite</u> <u>matrix</u>" but I shall reserve this for a later paper.

17) By actual Multiplication or otherwise.

APPENDIX

On the method of finding the Matrix Representation of a Hypercomplex Number Systems.[16)]

If the matrix representation of a hypercomplex number system of nth order is known, we can find the matrix representation of a hypercomplex number system of 2nth order by the following method.

Let us denote the matrix representations of the given hypercomplex number system by A_1 , A_2, $A_3 \cdots$(they are m row and columned) and let us denote the matrix whose elements are (of $\underline{m^2}$ rows and columneds) $\alpha_{st} \equiv a_{il} u_{jm}$ where $s=(ij)$ $t=(lm)$, ordered (where u_{jm} is the element of the unit matrix) according to 11,12,13,....,21,22,23,.....33,32,33,...... etc., by A'_1 A'_2 A'_3 ... and let us denote the matrix whose elements are $\beta_{st} \equiv u_{il} a_{jm}$ where $s=(ij), t=(lm)$ ordered as above by A'' A''_2 A'' ..... .
They form thus <u>two</u> <u>sets</u> of matrices

i) A'_1 A'_2 A'_3 $\cdots$

ii) A''_1 A''_2 A''_3 $\cdots$

<u>and have m^2 rows and columneds</u>. We can prove the following properties:[17)]
1, every element of the group (i) commutes with every element of the group ii; and 2, every element of one group anticommutes with every other element of its own group. They are <u>indeed</u> the two sub-hypercomplex sets considered in the chapter 1. From them, we can easily build up all the generators of the desired hypercomplex system of order 2n.

As an example, we can illustrate by requiring to find the sedenion system (n=4) from a quaterion system (n=2) whose matrix representation are known to be

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We have $A' = \{\alpha_{(ij)\,(lm)}\} = \alpha_{st}$

$$A' = \left\| \begin{matrix} \alpha_{(11)(11)} & \alpha_{(12)(11)} & \alpha_{(21)}\alpha_{(11)} & \alpha_{(22)(11)} \\ \alpha_{(11)(12)} & \alpha_{(12)(12)} & \alpha_{(21)}\alpha_{(12)} & \alpha_{(22)(12)} \\ \alpha_{(11)(21)} & \alpha_{(12)(21)} & \alpha_{(21)(21)} & \alpha_{(22)(21)} \\ \alpha_{(11)(22)} & \alpha_{(12)(22)} & \alpha_{(21)(22)} & \alpha_{(22)(22)} \end{matrix} \right\|^{*} = \left\| \begin{matrix} a_{11} & 0 & a_{21} & 0 \\ 0 & a_{11} & 0 & a_{21} \\ a_{12} & 0 & a_{22} & 0 \\ 0 & a_{12} & 0 & a_{22} \end{matrix} \right\|$$

since $u_{ij} = 0$ if $i \neq j$

Therefore

$$A_1' = \left\| \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix} \right\| \qquad A_2' = \left\| \begin{matrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{matrix} \right\| \qquad A_3' = \left\| \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix} \right\|$$

and $A'' = \left\| \begin{matrix} u_{11}a_{11} & u_{11}a_{21} & u_{21}a_{11} & u_{21}a_{21} \\ u_{11}a_{12} & u_{11}a_{22} & u_{21}a_{12} & u_{21}a_{22} \\ u_{12}a_{11} & u_{12}a_{21} & u_{22}a_{11} & u_{22}a_{21} \\ u_{12}a_{12} & u_{12}a_{22} & u_{22}a_{12} & u_{22}a_{21} \end{matrix} \right\| = \left\| \begin{matrix} a_{11} & a_{21} & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & a_{21} \\ 0 & 0 & a_{12} & a_{22} \end{matrix} \right\|$

Therefore

$$A_1'' = \left\| \begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{matrix} \right\| \qquad A_2'' = \left\| \begin{matrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{matrix} \right\| \qquad A_3'' = \left\| \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix} \right\|$$

These are exactly the ρ's and σ's in Dirac's paper who has derived them probably from experimantally. From these matrices, we can easily build the generators of the actual sedenion systam by the help of chapter 1.

We can easily continue this process to build the 16 point-matices which are required in dealing the probhem of two bodies and its hypercomplex number system is of order 2n and higher hyper-complex number systems.

* We can supplement by an equation here as

$$
{}^{*}= \left\| \begin{matrix} a_{11}u_{11} & a_{11}u_{21} & a_{21}u_{11} & a_{21}u_{21} \\ a_{11}u_{12} & a_{11}u_{22} & a_{21}u_{12} & a_{21}u_{22} \\ a_{12}u_{11} & a_{12}u_{21} & a_{22}u_{11} & a_{22}u_{21} \\ a_{12}u_{12} & a_{12}u_{22} & a_{22}u_{12} & a_{22}u_{21} \end{matrix} \right\|
$$