Delzant-type classification of near-symplectic toric 4-manifolds

by

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Abstract

Delzant's theorem for symplectic toric manifolds says that there is a one-to-one correspondence between certain convex polytopes in $\mathbb{R}^n$ and symplectic toric $2n$-manifolds, realized by the image of the moment map. I present proofs of this theorem and the convexity theorem of Atiyah-Guillemin-Sternberg on which it relies. Then, I describe Honda's results on the local structure of near-symplectic 4-manifolds, and inspired by recent work of Gay-Symington, I describe a generalization of Delzant's theorem to near-symplectic toric 4-manifolds. One interesting feature of the generalization is the failure of convexity, which I discuss in detail.

Thesis Supervisor: Victor W. Guillemin
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Chapter 1

Background

1.1 Hamiltonian group actions

Let (M, ω) be a symplectic manifold, let G be a connected compact Lie group, and let $\varphi : G \times M \to M$ be a smooth Lie group action $(g, m) \mapsto g \cdot m = \varphi_g(m)$. φ is *symplectic* if $\varphi_g^* \omega = \omega, \forall g \in G$. For $\xi \in \mathfrak{g}$, we define the vector field X_ξ by $X_\xi(m) = \frac{d}{dt}|_{t=0}(exp(t\xi) \cdot m)$.

Let $H \in C^\infty(M)$. Define the vector field X_H by $\iota(X_H)\omega = dH$. X_H is called the *Hamiltonian vector field associated to* H. ω defines a Poisson structure on M by $\{H, F\} = \omega(X_H, X_F)$. We say the group action φ is *Hamiltonian* if there exists a map $\Phi : M \to \mathfrak{g}^*$ such that for all $\xi \in \mathfrak{g}$, $X_\xi = X_{\Phi^\xi}$, where $\Phi^\xi \in C^\infty(M)$ is defined by $\Phi^\xi(m) = \langle \xi, \Phi(m) \rangle$, and furthermore, the map $j : \mathfrak{g} \to C^\infty(M), \xi \mapsto \Phi^\xi$ is a Lie algebra homomorphism. In this case the action φ is automatically symplectic, because $\mathcal{L}_{X_\xi}\omega = d\iota(X_\xi)\omega = d \cdot d\Phi^\xi = 0$. We call Φ the *moment map* for φ.

The following two properties of the moment map will be fundamental.

Proposition 1.1.1 (Equivariance of the moment map).

$$Ad_g^* \Phi(g \cdot m) = \Phi(m) \tag{1.1.1}$$

Proof.

$$\begin{aligned}\langle \xi, Ad_g^* d\Phi(g\cdot m)\rangle &= \langle g\xi g^{-1}, d\Phi(g\cdot m)\rangle \\ &= d\varphi_g{}^* \circ \iota(X_{g\xi g^{-1}}(g\cdot m))\omega_{g\cdot m}\end{aligned}$$

Now,

$$\begin{aligned}X_{g\xi g^{-1}}(g\cdot m) &= \frac{d}{dt}|_{t=0} \exp(g(t\xi)g^{-1})\cdot(g\cdot m) \\ &= \frac{d}{dt}|_{t=0} g\exp(t\xi)\cdot m \\ &= d\varphi_g(X_\xi(m))\end{aligned}$$

So,

$$\begin{aligned}\langle \xi, Ad_g^* d\Phi(g\cdot m)\rangle &= \iota(X_\xi(m))(\varphi_g^*\omega)_m \\ &= \iota(X_\xi(m))\omega_m \\ &= \langle \xi, d\Phi(m)\rangle\end{aligned}$$

□

For the next property, we rewrite $X_\xi = X_{\Phi\xi}$ as the requirement that for all $v \in T_m M, \xi \in \mathfrak{g}$,

$$\langle d\Phi_m(v), \xi\rangle = \omega(X_\xi(m), v) \tag{1.1.2}$$

For $m \in M$, let G_m be the stabilizer group of m under the action φ, and let $\mathfrak{g}_m$ be its Lie algebra.

Proposition 1.1.2 ([6]). *The image of $d\Phi_m$ is $\mathfrak{g}_m^0$,the annihilator in $\mathfrak{g}^*$ of $\mathfrak{g}_m$.*

Proof. The symplectic form ω defines an isomorphism $T_m M \to T_m M^*$. Under this isomorphism, (1.1.2) shows that the map $d\Phi : T_m M \to \mathfrak{g}^*$ has the map $\xi \mapsto \iota(X_\xi)\omega_m, \mathfrak{g} \to T_m M^*$ as its transpose. The proposition then follows from linear algebra.

□

1.2 Equivariant Moser-Darboux theorems

In this section, we review detailed proofs of equivariant local and semi-local versions of the Darboux-Moser theorem, which we require to develop local canonical forms. The presentation is standard and follows [7].

Theorem 1.2.1 (Semi-local equivariant Darboux-Moser). *Let M be a manifold and let G be a compact connected Lie group acting on M. Let $X \subset M$ be a submanifold. Let ω_0 and ω_1 be two G-invariant symplectic forms on M such that $\omega_0 = \omega_1$ at X. Then there exists a neighborhood U of X and a G-equivariant diffeomorphism $f : U \to M$, fixing each point in X, such that $f^*\omega_1 = \omega_0$.*

Proof. Choose a G-invariant Riemannian metric ρ on M (e.g. by averaging over G, using compactness). Let $U_0 \subset NX$ be a tubular neighborhood of the zero section of the normal bundle to X such that $exp_\rho : U_0 \to U$ is a diffeomorphism, for U a tubular neighborhood of X in M. Define the map $\phi_t : U \to U$ by $\phi_t(u) = exp_\rho(t \cdot exp_\rho^{-1}(u))$, so $\phi_1 = id$, $\phi_0 : U \to X$ and ϕ is a deformation retraction. Define the vector field $\xi_t = \frac{d}{dt}\phi_t$.

Set $\omega_t = t\omega_1 + (1-t)\omega_0$. We would like to construct a flow f_t such that $f_t^*\omega_t = \omega_0$. Set $\eta_t = \frac{d}{dt} f_t$. Setting $\sigma = \omega_1 - \omega_0$, we have

$$\frac{d}{dt}(f_t^*\omega_t) = f_t^*(d\iota(\eta_t)\omega_t) + f_t^*(\sigma)$$

Now,

$$\sigma - \phi_0^*\sigma = \int_0^1 \frac{d}{dt}\phi_t^*\sigma = \int_0^1 \phi_t^*(d\iota(\xi_t)\sigma)dt$$

But $\phi_0^*\sigma = 0$ because $\omega_1 = \omega_0$ at X. So

$$\sigma = d\int_0^1 \phi_t^*(\iota(\xi_t)\sigma)dt$$

Choose η_t such that $\iota(\eta_t)\omega_t = -\int_0^1 \phi_t^*(\iota(\xi_t)\sigma)dt$, shrinking U_0 as necessary so that ω_t is nondegenerate on it. Note $\eta_t = 0$ along X. Integrating η_t, we obtain a flow f_t satisfying $f_t^*\omega_t = \omega_0$. Finally, f_t is G-invariant because η_t, ξ_t, and ω_i are, and ϕ_t is

G-equivariant.

□

Remark. The global Darboux-Moser theorem presupposes a smooth family of cohomologous symplectic forms $\omega_t, t \in [0,1]$. Here, the agreement on the submanifold X and the existence of a retraction of a tubular neighbourhood onto X means the cohomology condition is trivially satisfied for the family $\omega_t = t\omega_1 + (1-t)\omega_0$. The proof here uses a form of the Poincaré Lemma to construct the coboundary explicitly, and makes the G-equivariance explicit.

One consequence of Theorem 1.2.1 is that it allows us to linearize the G-action and the symplectic form simultaneously at a fixed point, as follows.

Corollary 1.2.2. *Let (M,ω) be a symplectic manifold on which G acts symplectically. Let $x \in M$ be a fixed point of the G-action φ, so that G acts on T_xM by $g \mapsto d\varphi_g$. Then there exist neighborhoods $0 \in U_o \subset T_xM$ and $x \in U \subset M$ and a G-equivariant diffeomorphism $h : U_o \to U$ such that $h^*\omega = \omega_x$.*

Proof. As before, let ρ be a G-invariant Riemannian metric on M, and let $0 \in U_0' \subset T_xM$ and $x \in U' \subset M$ be such that $exp_\rho : U_0' \to U'$ is a diffeomorphism. For any $v \in T_xM$, notice that

$$\exp_\rho(d\varphi_g \cdot sv) = g \cdot \exp_\rho(sv)$$

because, by G-invariance of ρ, both are geodesics tangent to v at x, so $\exp_\rho$ is G-equivariant. Set $\omega_0 = \omega_x$ and $\omega_1 = \exp_\rho^*\omega$ on U_0'. (Here G acts by $d\varphi$.) By Theorem 1.2.1, there's a G-equivariant diffeomorphism $f : U_0 \to U_0, f^*\omega_1 = \omega_0$, i.e. $f^* \exp_\rho^* \omega = \omega_0 = \omega_x$ on U_0, and setting $h = \exp_\rho \circ f : U_0 \to U$ have $h(g \cdot u) = g \cdot h(u)$ as desired.

□

We can similarly linearize the G-action and the symplectic form along G-invariant submanifolds:

Corollary 1.2.3. *Let $X \subset M$ be a G-invariant submanifold. Then after choosing a G-invariant metric ρ, $g \mapsto d\varphi_g$ defines a G-action that's a bundle-map on NX,* *the normal bundle. Then there exist neighborhoods $U_0 \subset NX$ of the zero section and $X \subset U \subset M$ and a G-equivariant diffeomorphism $h : U_0 \to U$ such that $h^*\omega = \omega_X$, where ω_X is any symplectic form on NX that agrees with ω at the zero section (with a certain embedding $NX \to T_X M$).*

Proof. Again, let ρ be a G-invariant Riemannian metric on M, and use it to identify NX with a subbundle of $T_X M$. Choose tubular neighborhoods $U_0' \subset NX$ of the zero section and $U' \subset M$ of X such that $\exp_\rho : U_0' \to U'$ is a diffeomorphism. By G-invariance, $d\varphi$ maps $NX \to NX$ and $TX \to TX$. As in Corollary 1.2.2, for $v \in N_x X$, we have

$$\exp_\rho(d\varphi_g \cdot sv) = g \cdot \exp_\rho(sv)$$

so $\exp_\rho$ is G-equivariant. Set $\omega_0 = \omega_X$ where the right-hand side is any symplectic form which agrees with $\exp_\rho^* \omega$ along the zero section. Setting $\omega_1 = \exp_\rho^* \omega$, we can apply Theorem 1.2.1 to conclude.

□

1.3 Local forms

The following standard fact shows that the linearized actions act as complex representations. Recall, given (M, ω), a compatible almost-complex structure is a fibre-preserving automorphism $J : TM \to TM$ such that $J^2 = -1$ and $(v, w) \mapsto \omega(v, Jw)$ defines a Riemannian metric on M.

Proposition 1.3.1. *Let (M, ω) be a symplectic manifold with a compact group G acting symplectically on it. Then M admits a G-invariant compatible almost-complex structure.*

Proof. Let ρ be a G-invariant Riemannian metric on M. Defining A by $\rho(u, v) = \omega(u, Av)$, $-A^2$ is symmetric and positive definite. Setting $P = \sqrt(-A^2)$ and $J = AP^{-1}$, J is an invariant compatible complex structure.

□

We can use Corollary 1.2.3 to describe the neighborhood of an orbit explicitly, as follows:

Lemma 1.3.2 (Equivariant slice theorem). *Let G be a compact Lie group acting (symplectically) on M, let $x \in M$, and let $G \cdot x$ be its orbit. A G-invariant neighborhood of the orbit is equivariantly diffeomorphic to a neighborhood of the zero section in the bundle $G \times_{G_x} W$, where G_x is the isotropy group of x and $W = T_xM/T_x(G \cdot x)$. Furthermore, the diffeomorphism can be chosen to be a symplectomorphism when W is identified with a particular embedding of $N(G \cdot x)$ in TM and the symplectic form on the bundle agrees with the pullback of ω along the zero section.*

Proof. As usual, choose a G-invariant metric on M and use it to identify $N(G \cdot x)$ with a subbundle of TM. $d\varphi$ is a G action which is a bundle-map on $N(G \cdot x)$, and for $v \in N(G \cdot x)$, $\exp(sd\varphi_g \cdot v) = g \cdot \exp(sv)$. Therefore, for a small neighborhood $0 \in U_0 \subset N_x(G \cdot x)$, $\exp(U_0)$ is G_x-invariant, and the map

$$G \times N_x(G \cdot x) \to M, (g, v) \mapsto g \cdot \exp(v) \tag{1.3.1}$$

descends to a map

$$G \times_{G_x} N_x(G \cdot x) \to M \tag{1.3.2}$$

which is easily seen to be a diffeomorphism around the zero section. The left hand side is also easily seen to be identical with the bundle $N(G \cdot x)$ and Corollary 1.2.3 gives the conclusion.

□

For an isotropic orbit of a Hamiltonian action, we can further describe the slice $W = T_xM/T_x(G \cdot x)$.

Lemma 1.3.3. *Let G be a compact Lie group which acts in a Hamiltonian manner on (M, ω), let $x \in M$, and let $G \cdot x$ be an isotropic orbit. Set $V = T_x(G \cdot x)^\omega/(T_x(G \cdot x) \cap T_x(G \cdot x)^\omega)$. Then G_x acts on V and we can identify $W = \mathfrak{g}_x^0 \times V$, where $\mathfrak{g}_x^0$ is the annihilator of $\mathfrak{g}_x$ in $\mathfrak{g}^*$. Furthermore, a neighborhood of $G \cdot x \subset M$ is equivariantly* *symplectomorphic to a neighborhood of zero in the bundle* $Y = G \times_{G_x} (\mathfrak{g}_x^0 \times V)$*. Here* Y *has the symplectic form which is the product of the one induced by the canonical form on* T^*G *for the first two factors and some other symplectic form on* V*.*

Proof. First, since G_x acts symplectically and preserves $T_x(G \cdot x)$, it also preserves $T_x(G \cdot x)^\omega$, so G_x acts on V by the linear isotropy action. This time, choose the invariant metric ρ to be induced by a compatible invariant almost-complex structure J, i.e.

$$\rho(v, w) = \omega(v, Jw) \tag{1.3.3}$$

Then V can be identified with a subspace of $N(G \cdot x)$ as follows:

$$\begin{aligned} T_x(G \cdot x)^\omega &= \{v \in T_xM | \omega(v, w) = 0, \forall w \in T_x(G \cdot x)\} \\ &= \{v \in T_xM | \rho(v, -Jw) = 0, \forall w \in T_x(G \cdot x)\} \end{aligned} \tag{1.3.4}$$

so $T_x(G \cdot x)^\omega = (JT_x(G \cdot x))^\perp = J(T_x(G \cdot x))^\perp$. Since $G \cdot x$ is isotropic, 1.3.3 shows that $J(T_x(G \cdot x)) \perp T_x(G \cdot x)$. So we can identify

$$V \cong J(T_x(G \cdot x))^\perp \cap T_x(G \cdot x)^\perp \tag{1.3.5}$$

and this gives the orthogonal splitting

$$T_xM \cong T_x(G \cdot x) \oplus J(T_x(G \cdot x)) \oplus V \tag{1.3.6}$$

which, grouping the first two terms together, is also a symplectic splitting, because

$$(T_x(G \cdot x) \oplus J(T_x(G \cdot x)))^\omega \cong V \tag{1.3.7}$$

Furthermore, for the moment map Φ, from equation (1.1.2) we have that

$$\ker d\Phi_x = T_x(G \cdot x)^\omega \tag{1.3.8}$$

By definition, $V \subset T_x(G \cdot x)^\omega$, and $T_x(G \cdot x) \subset T_x(G \cdot x)^\omega$ because $G \cdot x$ is isotropic.

So we can identify

$$T_xM/\ker d\Phi_x \cong J(T_x(G\cdot x)) \tag{1.3.9}$$

So $d\Phi_x : J(T_x(G\cdot x)) \longrightarrow \mathfrak{g}^*$ is an isomorphism onto its image, and by Proposition 1.1.2, $d\Phi_x(T_xM) = \mathfrak{g}_x^0$. Summarizing, we have:

$$T_xM = (T_x(G\cdot x) \oplus \mathfrak{g}_x^0) \oplus V \tag{1.3.10}$$

where all direct sums are orthogonal splittings, and the second one is a symplectic splitting.

Finally, applying the equivariant slice theorem to this splitting we obtain an equivariant symplectomorphism to the bundle

$$Y \cong G \times_{G_x} (\mathfrak{g}_x^0 \times V) \tag{1.3.11}$$

where the G_x action on $(\mathfrak{g}_x^0 \times V)$ is the product of the co-adjoint action on $\mathfrak{g}_x^0$ (by equivariance of the moment map) and and the linear isotropy action on V. The natural symplectic form on this bundle is as described in the statement (we can take the constant form ω_x on the V factor), and by the identification via Φ it's of the form required by the equivariant slice theorem.

□

The remainder of this section will be devoted to analyzing the moment map in the above case. The calculation follows [9], Lemma 3.5.

To see the moment map on Y, it's convenient to construct Y via symplectic reduction of a space on which the moment map is easy to calculate. This is done as follows.

Recall, T^*G, like any cotangent bundle, has a canonical 1-form θ defined, for $p \in T^*G, v \in T_P(T^*G)$ by

$$\langle \theta_p, v\rangle = \langle p, d\pi_p v\rangle$$

where $\pi : T^*G \longrightarrow G$ is the projection, which induces a canonical symplectic form

$\omega_G = -d\theta_G$. Let $H \subset G$ be a subgroup, and let H act on G by the right action $h \cdot g \mapsto gh^{-1}$.

The actions of H and G on G (like any diffeomorphism) induce symplectomorphisms of (T^*G, ω_G):

Writing $(q,p) \in T^*G, p \in T_q^*G$, we define the action of G by

$$g \cdot (q,p) \mapsto (g \cdot q, (dg^{-1})^* p) \tag{1.3.12}$$

but

$$\begin{aligned} \langle \theta_p, v \rangle &= \langle p, d\pi_p v \rangle \\ &= \langle (dg^{-1})^* p, dg d\pi_p v \rangle \\ &= \langle g^* \theta_p, v \rangle \end{aligned} \tag{1.3.13}$$

Similarly, the action of H is

$$h \cdot (q,p) \mapsto (q \cdot h^{-1}, (dh_R)^* p) \tag{1.3.14}$$

but

$$\begin{aligned} \langle \theta_p, v \rangle &= \langle p, d\pi_p v \rangle \\ &= \langle (dh_R)^* p, dh_R^{-1} d\pi_p v \rangle \\ &= \langle h^* \theta_p, v \rangle \end{aligned} \tag{1.3.15}$$

Since both actions preserve θ_G, both preserve ω_G.

Since $g^*\theta_G = \theta_G$ we have, for $\xi \in \mathfrak{g}$:

$$0 = (L)_{X_\xi} \theta_G = \iota(X_\xi) d\theta_G + d\iota(X_\xi)\theta_G \tag{1.3.16}$$

so

$$\iota(X_\xi)\omega_G = d(\iota(X_\xi)\theta_G) \tag{1.3.17}$$

and similarly for the right H action, so both actions are Hamiltonian with moment map

$$\xi \mapsto \iota(X_\xi)\theta_G(p) = \langle p, d\pi X_\xi \rangle \tag{1.3.18}$$

In the case of the (left) G-action, we have, for $p \in T^*G, p = (g, v)$,

$$d\pi_p(X_\xi) = (dg_R)\xi \tag{1.3.19}$$

so the moment map is

$$p = (g, v) \mapsto (\xi \mapsto \langle v, (dg_R)\xi \rangle) \tag{1.3.20}$$

or

$$\hat{\Phi}_G^\xi = \langle (dg_R)^* v, \xi \rangle \tag{1.3.21}$$

Similarly, for the (right) H-action, we have

$$\hat{\Phi}_H^\xi = \langle -j^*(dg)^* v, \xi \rangle \tag{1.3.22}$$

where $j : \mathfrak{h} \to \mathfrak{g}$ is the inclusion.

Suppose H also acts in a Hamiltonian way on a symplectic vector space (V, ω_V), with moment map $\Phi_H^V : V \to \mathfrak{h}^*$. We can define two commuting Hamiltonian actions on $T^*G \times V$ by

$$g' \cdot (g, v) \xrightarrow{\varphi_G} (g'g, v) \tag{1.3.23a}$$

$$h \cdot (g, v) \xrightarrow{\varphi_H} (gh^{-1}, h \cdot v) \tag{1.3.23b}$$

In what follows, let (g, η, v) be coordinates on $T^*G \times V$, with $g \in G, \eta \in T_g^*G, v \in V$. Denoting the G and H -action's respective moment maps by $\Phi_G : T^*G \times V \to \mathfrak{g}$ and $\Phi_H : T^*G \times V \to \mathfrak{h}$, by the previous calculations we have

$$\Phi_G(g, \eta, v) = (dg_R)^* \eta \tag{1.3.24a}$$

$$\Phi_H(g, \eta, v) = -j^*(dg)^* \eta + \Phi_H^V(v) \tag{1.3.24b}$$

We would now like to consider the symplectic reduction of $T^*G \times V$ with respect to the H action, and show that the resulting space, $\Phi_H^{-1}(0)/H$, is our model bundle Y.

We can write the set $\Phi_H^{-1}(0)$ as

$$\Phi_H^{-1}(0) = \{(g, \eta, v) | \langle -j^*(dg)^*\eta + \Phi_H^V(v), \xi \rangle = 0, \forall \xi \in \mathfrak{h}\} \tag{1.3.25}$$

For each fixed pair (g, v), we can identify the fibre $\Phi_H^{-1}(0)|_{g \times T_g^*G \times v}$ with $\mathfrak{h}^0 \subset \mathfrak{g}^*$, the annihilator of $\mathfrak{h}$. First, by (1.3.25), the fibre's an affine subspace of T_g^*G. Let $A : \mathfrak{g} \to \mathfrak{h}$ be any H-equivariant projection (so $A \circ j = \mathrm{Id}$). The map $\mathfrak{k} \mapsto (dg^*)^{-1}(\mathfrak{k} + A^*\Phi_H^V(v))$ is then an isomorphism $\mathfrak{h}^0 \to \Phi_H^{-1}(0)|_{g \times T_g^*G \times v}$. Thus we've identified

$$G \times \mathfrak{h}^0 \times V \cong \Phi_H^{-1}(0) \subset T^*G \times V \tag{1.3.26}$$

via the map

$$(g, \mathfrak{k}, v) \mapsto (g, (dg^*)^{-1}(\mathfrak{k} + A^*\Phi_H^V(v)), v) \tag{1.3.27}$$

In these coordinates, we can rewrite (1.3.24a) as

$$\begin{aligned} \Phi_G(g, \mathfrak{k}, v) &= (dg_R)^*(dg^*)^{-1}(\mathfrak{k} + A^*\Phi_H^V(v)) \\ &= Ad^*(g)(\mathfrak{k} + A^*\Phi_H^V(v)) \end{aligned} \tag{1.3.28}$$

Since the G action commutes with the H action, it descends to the quotient space with the same moment map. Setting $H = G_x$, so $\mathfrak{h}^0 = \mathfrak{g}_x^0$, it's clear that the quotient $\Phi_H^{-1}(0)/H = G \times_{G_x} (\mathfrak{g}_x^0 \times V)$ as desired. We've now shown:

Lemma 1.3.4 ([9], Lemma 3.5). *For any G_x-equivariant projection $A : \mathfrak{g} \mapsto \mathfrak{g}_x^0$, there's a symplectic structure on Y such that Lemma 1.3.3 is true and the moment map on Y is given by $\Phi_Y([g, \eta, v]) = Ad^*(g)(\eta + A^*\Phi_V(v))$.*

Chapter 2

Hamiltonian torus actions

In this section, we apply the canonical local forms to the case where the group G is a torus T^n.

2.1 Convexity

Proposition 2.1.1. *Let $G = T^n$. Then* (1) *G-orbits are isotropic, and* (2) *Φ is constant on G orbits.*

Proof. The same calculation shows both. For $\eta, \xi \in \mathfrak{g}$:

$$\begin{aligned} \iota(X_\eta)(d\Phi^\xi) &= \omega(X_\eta, X_\xi) \\ &= \{\Phi^\eta, \Phi^\xi\} \\ &= \jmath([\eta, \xi]) = 0 \end{aligned}$$

(Claim (2) also follows directly by equivariance of Φ since $Ad^{T^n} = Id$.)

□

The following lemma is quoted without proof from elementary representation theory.

Lemma 2.1.2. *Let T^n act linearly and unitarily on $\mathbb{C}^m$. Then there exists an orthogonal decomposition $\mathbb{C}^m = \oplus_{k=1}^m V_{\lambda(k)}$ into one-dimensional T-invariant com-*

plex subspaces and linear maps $\lambda^{(i)} \in \mathfrak{t}^*, i = 1, ..., m$ *such that on* V_{λ^k}, T^n *acts by* $(e^{it_1}, ..., e^{it_n}) \cdot v = e^{i\sum_j \lambda_j^{(k)} t_j} v$.

The covectors $\lambda^{(k)}$ are called the *weights* of the representation.

Corollary 2.1.3 (Local convexity [6]). *Let* $x \in M$ *be a fixed point of a Hamiltonian* T^n *action with moment map* Φ*, and let* $p = \Phi(x)$*. Then there exist open neighborhoods* $x \in U \subset M$ *and* $p \in U' \subset \mathfrak{t}^*$ *such that* $\Phi(U) = U' \cap (p + S(\lambda^{(1)}, ..., \lambda^{(k)}))$*, where* $\lambda^{(i)}$ *are the weights of the isotropy representation* $g \mapsto dg$ *on* T_xM *and* $S(\lambda^{(1)}, ..., \lambda^{(k)}) = \{\sum_{i=1}^n s_i \lambda^{(i)}, s_i \geq 0\}$.

Proof. By choosing a compatible invariant almost-complex structure J on M and the induced invariant Riemannian metric ρ, we make T_xM into a complex vector space and obtain a unitary representation $g \mapsto dg$. Furthermore, any one-dimensional complex subspace is symplectic, since $\omega_x(v, Jv) = \rho(v, v) \neq 0$. Thus the T-invariant subspaces V_λ are symplectic, and they are pairwise symplectically orthogonal because they're J-invariant and ρ-orthogonal. We can therefore write $\omega_x = \sum_{i=1}^m dz_i \wedge d\bar{z}_i = \omega_0$, where z_i is a complex coordinate on $V_{\lambda^{(i)}}$, and this identifies $(T_xM, \omega) \cong (\mathbb{C}^m, \omega_0)$. The induced T^n action on $(\mathbb{C}^m, \omega_0)$ is described in Lemma 2.1.2 and its moment map is $z \mapsto \sum_i |z_i|^2 \lambda^{(i)}$, which can be checked easily. Finally, by the equivariant Darboux theorem, a neighborhood $0 \in \hat{U} \subset (\mathbb{C}^m, \omega_0)$ is equivariantly symplectomorphic to a neighborhood $x \in U \subset M$, so the image of the moment map is the same, up to translation.

□

Corollary 2.1.4 (Relative local convexity [6]). *Let* $x \in M$ *have orbit* $T \cdot x$ *and let* T_x *be the isotropy group of* x*. Let* $p = \Phi(x)$ *and let* $\lambda^{(1)}, ..., \lambda^{(k)}$ *be the weights of the isotropy representation on a slice* V *at* x*. Let* $\pi : \mathfrak{t}_x \to \mathfrak{t}$ *be the inclusion. Then there exist neighborhoods* $U \subset M$ *of* $T \cdot x$ *and* U' *of* p *such that* $\Phi(U) = U' \cap (p + S'(\lambda^{(1)}, ..., \lambda^{(k)}))$ *where* $S'(\lambda^{(1)}, ..., \lambda^{(k)}) = (\pi^*)^{-1} S(\lambda^{(1)}, ..., \lambda^{(k)})$.

Proof. By Lemma 1.3.4, around $T \cdot x$, we have $\Phi([g, \eta, v]) = Ad^*(g)(\eta + A^*\Phi_V(v))$, up to translation. Since $G = T, \mathrm{Ad} = \mathrm{Id}$, so $\Phi([g, \eta, v]) = p + \eta + A^*\Phi_V(v)$. By the previous corollary, $\Phi_V(V) = S(\lambda^{(1)}, ..., \lambda^{(k)}) \subset \mathfrak{t}_x^*$. Finally, for any projection A, the set $\{\eta + A^*S(\lambda^{(1)}, ..., \lambda^{(k)}) | \eta \in \mathfrak{t}_x^0\}$ is equal to $S'(\lambda^{(1)}, ..., \lambda^{(k)})$.

□

Recall, a function $f : M \to \mathbb{R}$ is Bott-Morse if each component of its critical set C_f is a submanifold of M, and for each $x \in C_f$, the Hessian $d^2 f_x$ is nondegenerate on $N_x C_f$. (The index of $d^2 f_x$ is constant on each component of C_f.) The following lemma is key:

Lemma 2.1.5 ([6]). *For each $\xi \in \mathfrak{g}$, Φ^ξ is Bott-Morse, and the indices and coindices of its critical manifolds are all even.*

Proof. From our canonical local form,

$$\Phi^\xi([g, \eta, v]) = \langle p, \xi \rangle + \langle \eta, \xi \rangle + \sum |z_i|^2 \langle A^* \lambda^{(i)}, \xi \rangle$$

Modulo some mess from the quotient, the result can be read off: for $x \in C_{\Phi^\xi}$ have $\xi \in \mathfrak{g}_x$, so η is free to vary in $\mathfrak{g}_x^0$. We also have there that $z_i = 0$ or z_i is free and $\langle A^* \lambda^{(i)}, \xi \rangle = 0$, so the critical sets are manifolds. The index is $2k$ where k is the number of i's such that $\langle A^* \lambda^{(i)}, \xi \rangle < 0$, and similarly for coindex, because each V_λ is 2-dimensional.

□

Lemma 2.1.6 ([6]). *For each $\xi \in \mathfrak{g}$, Φ^ξ has a unique connected component of local maxima.*

Proof. Let $C_1, ..., C_k$ be the connected critical manifolds of Φ^ξ consisting of local maxima, and let $C_{k+1}, ..., C_N$ be the remaining connected critical manifolds. $M = \amalg_{i=1}^N W_i$, where W_i is the stable manifold of C_i. Note $\dim(W_i) = \text{index}(C_i) + \dim(C_i)$. For $i = 1, ..., k$, $\dim(W_i) = \dim(M)$, so $W_i, i = 1, ..., k$ is open. For $i = k+1, ..., N$, codim $C_i \geq 2$. Therefore $M \setminus \cup_{i=k+1}^N W_i$ is connected, i.e. $\cup_{i=1}^k W_i$ is connected, so $k = 1$, and there is a unique connected component of local maxima.

□

Corollary 2.1.7 (Global convexity [6]). *$\Phi(M) \subset \mathfrak{t}^*$ is a convex polytope, specifically the convex hull of the image of the fixed points, $\Phi(M^T)$.*

Proof. Let $p \in \partial\Phi(M), x \in \Phi^{-1}(p)$. By Corollary 2.1.4, there exist neighborhoods $U \subset M$ of x and U' of p such that $\Phi(U) = U' \cap (p + S'(\lambda^{(1)}, ..., \lambda^{(k)}))$ where $\lambda^{(1)}, ..., \lambda^{(k)}$ are the weights of the isotropy representation on a slice V at x. We can choose $\xi \in \mathfrak{t}$ such that $\langle \cdot, \xi \rangle = 0$ on a boundary component of S' and $\langle \cdot, \xi \rangle < 0$ on S'. Then if $\langle \Phi(p), \xi \rangle = a$, $\langle \Phi(x), \xi \rangle \leq a$ for $x \in U$, i.e. a is a local maximum of Φ^ξ, so by Lemma 2.1.6, $\Phi^\xi \leq a$ on M. Repeating this argument for each face of S', we have $\Phi(M) \subset p + S'(\lambda^{(1)}, ..., \lambda^{(k)})$. Applying this argument to all boundary components of $\Phi(M)$, $\Phi(M)$ is convex. Finally, by the local canonical form, if $\Phi(x)$ is an extremal point of $\Phi(M)$, we must have $\mathfrak{g}_x^0 = \emptyset$, so x is a fixed point.

□

The following connectedness result requires a more involved Morse-theoretic argument, and will be quoted without proof.

Lemma 2.1.8 (Connectedness [1], [9]). *For every $a \in \mathfrak{t}^*$, the fiber $\Phi^{-1}(a)$ is connected.*

2.2 Delzant's theorem

In this section we consider the case of an effective Hamiltonian T^n action on M^{2n} (effective means that the action has trivial kernel.) In this case we say M is a *toric* $2n$-manifold. In what follows we'll often write $\Delta = \Phi(M)$.

Proposition 2.2.1 (Smoothness). *Let T^m act linearly on $(\mathbb{C}^n, \omega_0)$. By Lemma 2.1.2, we have $\mathbb{C}^n = \oplus_{k=1}^n V_{\lambda^{(k)}}$ such that on V_{λ^k}, T^m acts by $(e^{it_1}, ..., e^{it_n}) \cdot v = e^{i\sum_j \lambda_j^{(k)} t_j} v$, i.e. the action factors through a map $T^m \to^\Psi T^n, \exp(t) \mapsto \exp(\langle \lambda^{(1)}, t \rangle, ..., \langle \lambda^{(n)}, t \rangle)$. (So $\lambda^{(k)} \in \mathbb{Z}^m$.)Then if T^m acts effectively, $m \leq n$. If $m = n$, $\lambda^{(k)}$ are a $\mathbb{Z}$-basis of $\mathbb{Z}^m \cong \mathfrak{t}^*$.*

Proof. The map $T^m \to^\Psi T^n$ lifts to the linear map $\mathfrak{t}^m \to^\psi \mathfrak{t}^n$ given by the weights. If $m > n$, ψ and hence Ψ has nontrivial kernel, contradicting effectiveness. Similarly, if

$m = n$, ψ must have trivial kernel, i.e. be an isomorphism. In this case, if $\{\lambda^{(k)}\}$ is not a $\mathbb{Z}$-basis of $\mathbb{Z}^m$, then there exist lattice points in $\mathfrak{t}^n$ that are not the images of lattice points in $\mathfrak{t}^m$. Since ψ is onto, this means that Ψ has nontrivial kernel, contradicting effectiveness.

□

Corollary 2.2.2. *For an effective Hamiltonian T^n action in M^{2n}, the moment polytope $\Delta = \Phi(M)$ satisfies the following* Delzant conditions*: (1)* simplicity *- n edges meet at each vertex (2)* rationality *- each vertex is of the form $\{p + \sum t_i v_i, t_i \geq 0, v_i \in \mathfrak{t}^*\}$, such that v_i has integral entries (3)* smoothness *- at each vertex, $\{v_i\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.*

Proof. Apply Proposition 2.2.1 and the analysis in the proof of Corollary 2.1.3 to the fixed points of the action.

□

Proposition 2.2.3. *The map $M/T \to \Delta, m \mapsto \Phi(m)$ is a bijection.*

Proof. Let $x \in M, p = \Phi(x)$. Recall our local model for a neighborhood of $T \cdot x \subset M$, i.e. a neighborhood of the zero section in the bundle $T \times_{T_x} (\mathfrak{t}_x^0 \times V)$ with moment map $[g, \eta, v] \mapsto p + \eta + \sum |z_i|^2 A^* \lambda_{(i)}$, with z_i coordinates on V. By definition of A and $\mathfrak{t}_x^0$, image$(A^*) \cap \mathfrak{t}_x^0 = \emptyset$, so if the set $\{A^* \lambda_{(i)}\}$ is independent, then $\Phi : M/T \to \Delta$ is locally a bijection onto its image. Since A^* is injective, it's sufficient to show that the weights are independent. Let dim $T_x = k$. Then dim $V = 2k$. The action of T_x on V is also effective, because otherwise, by the local form the action of T wouldn't be effective. So by Proposition 2.2.1, $\lambda_{(i)}$ are independent. Therefore $\Phi : M/T \to \Delta$ is locally a bijection. To see that it's a global bijection, use Lemma 2.1.8 to see that each set $\Phi^{-1}(a)$ must be a single orbit.

□

Any polytope $\Delta \subset \mathfrak{t}^*$ satisfying the conditions of Corollary 2.2.2 is called a *Delzant polytope*. Delzant [3] proved that for any Delzant polytope Δ, there exists a unique symplectic toric manifold M such that $\Phi(M) = \Delta$. The construction for the existence proof can be found in essentially the same form in any of [3, 5, 9, 10], and will be skipped. The uniqueness proof I reproduce below is due to [9].

Theorem 2.2.4 (Delzant [3]). *Let M_1, M_2 be two compact, connected, symplectic toric 2n-manifolds with moment maps Φ_1, Φ_2 and moment polytopes Δ_1, Δ_2. If $\Delta_1 = \Delta_2$, then there exists a T^n-equivariant symplectomorphism $f : M_1 \longrightarrow M_2$ such that $\Phi_2 \circ f = \Phi_1$.*

The proof involves several intermediate results.

Proposition 2.2.5. *Let $\alpha \in \Delta$. Then a neighborhood of $\Phi^{-1}(\alpha)$ is determined by (Δ, α).*

Proof. By the canonical local form, we need to determine the subspace $\mathfrak{g}_x^0$ and the weights $\lambda^{(i)} \in \mathfrak{g}_x^*$. First, note that the subspace $\mathfrak{g}_x^0 \subset \mathfrak{g}^*$ is the subspace parallel to the affine face of Δ that α belongs to. Let $\mathfrak{g}_x^0$ have codimension k. By simplicity, each codimension k face F_k belongs to k codimension $k-1$ faces $\{F_{k-1}^i\}_{i=1}^k$. For each codimension $k-1$ face F_{k-1}^i, we can choose a covector $\alpha_{k,i} \in F_{k-1}^i \setminus F_k$, eg $\alpha_{k,i} \in F_{k-1}^i \cap F_k^\perp$. The covector $\alpha_{k,i}$ is the image $A^*\lambda^{(i)}$ for some projection A corresponding to the choice of $\alpha_{k,i}$, but this choice is irrelevant because any two such models are symplectomorphic.

□

Corollary 2.2.6. *Suppose $\Delta = \Delta_1 = \Delta_2$. Then for any $\alpha \in \Delta$, there exists a neighborhood U of α such that, for $M_{1U} = \Phi_1^{-1}(U)$ and $M_{2U} = \Phi_2^{-1}(U)$, there exists a T^n-equivariant symplectomorphism $f : M_{1U} \longrightarrow M_{2U}$ such that $\Phi_2 \circ f = \Phi_1$.*

Proof. By Proposition 2.2.5, both M_{1U} and M_{2U} are equivariantly symplectomorphic to the same canonical model.

□

To piece these local symplectomorphisms together, we follow [9] and use sheaf cohomology.

Let $\mathcal{U}$ be a cover of Δ with the property that each $U \in \mathcal{U}$ has the property in Corollary 2.2.6, and let $\mathcal{H}_U$ be the set of all moment-preserving T-equivariant symplectomorphisms $M_{1U} \longrightarrow M_{1U}$. Notice that this group is abelian: $\mathcal{H}_U$ acts on fibers, the fibres are G/G_x, and the only G-equivariant diffeomorphisms $G/G_x \longrightarrow G/G_x$ are multiplications by elements of G. Since G is commutative, $\mathcal{H}_U$ is commutative on fibres, so commutative. We can therefore use the groups $\mathcal{H}_U$ to define a sheaf of abelian groups and its sheaf cohomology $H^*(\Delta, \mathcal{H})$.

By Corollary 2.2.6, $\mathcal{U}$ defines a 1-cochain in the sheaf as follows. For each $U_i \in \mathcal{U}$ choose a T-equivariant moment preserving symplectomorphism $f_i : M_{1U_i} \longrightarrow M_{2U_i}$. For each pair $U_i, U_j \in \mathcal{U}$, set $h_{ij} = f_i^{-1} \circ f_j \in \mathcal{H}_{U_i \cap U_j}$. This is a cocycle: for $U_i, U_j, U_k \in \mathcal{U}, U_i \cap U_j \cap U_k \neq \emptyset$, $h_{ij} \circ h_{jk} \circ h_{ki} = f_i^{-1} \circ f_j \circ f_j^{-1} \circ f_k \circ f_k^{-1} \circ f_i = Id$, so h_{ij} defines a cohomology class in $H^1(\Delta, \mathcal{H})$.

Suppose h_{ij} is a coboundary, i.e. $h_{ij} = h_i \circ h_j^{-1}$, for $h_i, h_j \in \mathcal{H}_{U_i}$. Then $f_i^{-1} \circ f_j = h_i \circ h_j^{-1}$, or $f_j \circ h_j = f_i \circ h_i$ on $U_i \cap U_j$. Then the map $x \in M_{1U_i} \mapsto (f_i \circ h_i)(x)$ is a globally well-defined T-equivariant moment preserving symplectomorphism $M_1 \longrightarrow M_2$.

So, we will show that $H^1(\Delta, \mathcal{H}) = 0$ by showing that $H^k(\Delta, \mathcal{H}) = 0, \forall k > 0$.

Define auxiliary sheafs as follows:

Let $\underline{\ell \times \mathbb{R}}$ be the locally constant sheaf on Δ with values in the abelian group $\ell \times \mathbb{R}$, where ℓ is the integer lattice $\mathbb{Z}^n \subset \mathfrak{t}$.

For each $U \in \Delta$, let $\tilde{C^\infty}(U)$ be the set of smooth T-invariant functions on M_U, so $f \in \tilde{C^\infty}(U) \implies f = h \circ \Phi$ for some smooth function h on U. Call this sheaf C^∞.

Define a map $j : \underline{\ell \times \mathbb{R}} \longrightarrow C^\infty$ by $j(\xi, c)(x) = \langle \xi, \Phi(x) \rangle + c, \forall x \in M_U$.

Define a map $\Lambda : C^\infty \longrightarrow \text{Symp}(M_U)$ by $\Lambda(f)(x) = \exp(X_f)$ where X_f is the Hamiltonian vector field associated to f (this is the time-1 flow).

Proposition 2.2.7. $\Lambda : C^\infty \longrightarrow \mathcal{H}$, *i.e. the Hamiltonian flow preserves* Φ *and is* T*-equivariant.*

Proof. $\exp(X_f)$ preserves Φ: For $\xi \in \mathfrak{t}$,

$$\iota(X_f) d\Phi^\xi = \omega(X_f, X_\xi) = -\iota(X_\xi) df = 0$$

by T-invariance of f.

$\exp(X_f)$ is T-equivariant: Have $f(x) = f(t \cdot x) \implies df_x = df_{t\cdot x} \circ dt_x$. Since T acts symplectically, have

$$\iota(dt_x X_{f_x})\omega_{t\cdot x} \circ dt_x = \iota(X_{f_x})\omega_x = df_x = df_{t\cdot x} \circ dt_x$$

Cancelling dt_x, have

$$\iota(dt_x X_{f_x})\omega_{t\cdot x} = df_{t\cdot x}$$

i.e. $dt \circ X_f = X_f$, so X_f is T-equivariant, and its flow is also.

□

Lemma 2.2.8. *The sequence of sheaves* $0 \to \underline{\ell \times \mathbb{R}} \to^j C^\infty \to^\Lambda \mathcal{H}$ *is exact.*

Proof. j is injective because any open set in Δ suffices to determine an affine function.

$im(j) \subset ker(\Lambda)$ since, for any $(\xi, r) \in \ell \times \mathbb{R}$, $j((\xi, r))(x) = \Phi^\xi(x) + r$, so it's the moment for ξ. By definition, $\exp X_{\Phi^\xi}(x) = \exp(\xi) \cdot x = id \cdot x$ because $\xi \in \ell$.

$ker(\Lambda) \subset im(j)$: Let $f \subset \tilde{C^\infty}(U), \Lambda(f) = id, f = h \circ \Phi$. Since the flow of f is G-invariant and tangent to the orbits, at each point $x \in \Delta$, there exists $\xi_x \in \mathfrak{g}$ such that on $\Phi^{-1}(x)$, $X_f = X_{\xi_x}$, and locally ξ_x can be chosen to be continuous. On the interior of Δ, the G-action is free. So on int(Δ), $Id = \exp(X_f) = \exp(X_\xi) = \exp(\xi) \cdot x \implies \exp(\xi) = id \in G$, or that $\xi_x \in \ell$ for $x \in$ int(Δ). Since ℓ is discrete and ξ_x is continuous, we must have that ξ_x is locally constant on int(Δ), so also on $\partial\Delta$. So $df_x = d\Phi^\xi_x = d\langle\Phi(x), \xi\rangle \implies f(x) = \langle\Phi(x), \xi\rangle + r$, as claimed.

□

Will now show that $\Lambda : C^\infty \to \mathcal{H}$ is surjective.

Choose $\alpha \in U \subset \Delta$ simply-connected such that M_α is a deformation retract of M_U (this is possible by our local model.) Let $f : M_U \to M_U \in \mathcal{H}(U)$. By G-equivariance, can write $f(p) = \gamma(\Phi(p)) \cdot p$ where $\gamma : U \to G$ is smooth. Since $\pi_1(U) = 0$, can lift γ to a map $\tilde{\gamma} : U \to \mathfrak{g}$ such that $\gamma = \exp\tilde{\gamma}$. So have $f(p) = \gamma(\Phi(p)) \cdot p = (\exp(\tilde{\gamma}(\Phi(p)))) \cdot p = \exp(X_{\tilde{\gamma}(\Phi(p))}(p))$, i.e. f is the time-1 flow of the vector field $p \mapsto X_{\tilde{\gamma}(\Phi(p))}$. We would like to show that $Y = X_{\tilde{\gamma}\circ\Phi}$ is a Hamiltonian vector field.

Set $U_0 = U \cap \text{int}(\Delta)$. As we've seen, M_{U_0} is a principal G-bundle.

Proposition 2.2.9. *$\iota(Y)\omega|_{M_{U_0}}$ is a basic form on this bundle.*

Proof. We need to show that (1) $\iota(Y)\omega$ is G-invariant, and (2) that for all vectors v tangent to the fibre, $\iota(v)\iota(Y)\omega = 0$.

(1): Let $v \in T_xM$. Then

$$\begin{aligned} g^*(\iota(Y)\omega)_x(v) &= (\iota(Y)\omega)_{g\cdot x}(dg_x v) \\ &= \iota(X_{\tilde{\gamma}\circ\Phi(g\cdot x)_{g\cdot x}})\omega_{g\cdot x}(dg_x v) \\ &= \iota(X_{\tilde{\gamma}\circ\Phi(x)_{g\cdot x}})\omega_{g\cdot x}(dg_x v) \\ &= \iota(dg_x X_{\tilde{\gamma}\circ\Phi(x)_x})\omega_{g\cdot x}(dg_x v) \\ &= \iota(X_{\tilde{\gamma}\circ\Phi(x)_x})\omega_x(v) = \iota(Y)\omega_x(v) \end{aligned}$$

(2) is true because Y is tangent to the fibres and the fibres are isotropic.

□

Since it's basic, we have $\iota(Y)\omega = \Phi^*\nu$ for some 1-form ν on U_0. Write $f_t = \exp(tY)$.

Corollary 2.2.10. $f_t^*(\iota(Y)\omega) = \iota(Y)\omega$

Proof. Since M_{U_0} is dense in M_U, it suffices to check there. By the above, we have there that $f_t^*\iota(Y)\omega = f_t^*\Phi^*\nu = (\Phi \circ f_t)^*\nu = \Phi^*\nu = \iota(Y)\omega$.

□

Corollary 2.2.11. *$\iota(Y)\omega$ is exact.*

Proof. $\frac{d}{dt} f_t^*\omega = f_t^*\mathcal{L}_Y\omega = f_t^* d\iota(Y)\omega = df_t^*\iota(Y)\omega = d\iota(Y)\omega$. Integrating from 0 to 1, have $0 = f_1^*\omega - f_0^*\omega = d\iota(Y)\omega$, since $f_0 = id$ and $f_1 = f$ is a symplectomorphism. Therefore, have shown that $\iota(Y)\omega$ is closed, so defines a cohomology class.

To see exactness of $\iota(Y)\omega$, we need to show that its cohomology class is zero. Recall that we chose U such that M_U is a deformation retract of M_α, so the inclusion $j : M_\alpha \to M_U$ induces an isomorphism in cohomology $j^* : M_u \to M_\alpha$. Since Y is tangent to M_α, and M_α is isotropic, $j^*(\iota(Y)\omega) = \iota(Y)j^*\omega = 0$, so $\iota(Y)\omega$ is exact.

□

Corollary 2.2.12. *The map* $\Lambda : C^\infty \to \mathcal{H}$ *is surjective.*

Proof. Using the notation of the above corollaries, $\iota(Y)\omega = dh$ for some $h \in C^\infty(M_U)$. Since $\iota(Y)\omega$ is G-invariant, can choose h to be G-invariant (e.g. by averaging). Then f is the time-1 Hamiltonian flow of the G-invariant function h.

□

Proof of Theorem 2.2.4. By the above, we have a short exact sequence of sheaves of abelian groups $0 \to \underline{\ell \times \mathbb{R}} \to C^\infty \to \mathcal{H} \to 0$, inducing a long exact sequence in cohomology. C^∞ is "flabby", so $H^i(\Delta, C^\infty) = 0, \forall i > 0$. Δ is contractible, so $H^i(\Delta, \underline{\ell \times \mathbb{R}}) = 0, \forall i > 0$. The long exact sequence then gives $H^1(\Delta, \mathcal{H}) = 0$, which completes the argument.

□

Chapter 3

Canonical forms for near-symplectic 4-manifolds

A *near-symplectic structure* on a compact 4-manifold M is a closed 2-form ω which is self-dual and harmonic with respect to some metric ρ_ω, and is transverse to the zero section of the bundle $\wedge_2^{+,\rho}$ of self-dual 2-forms. By transversality, the zero set Z_ω of ω is a 1-manifold, i.e. a disjoint union of circles C_i. We call Z_ω the *vanishing locus.* Since $\omega_p \wedge \omega_p = \omega_p \wedge *\omega_p = 0$ iff $\omega_p = 0$, ω is symplectic on $M \setminus Z_\omega$ (the *symplectic locus*). It's a result due to Honda that if $b_2^+(M) > 0$, for generic pairs (ρ, ω) with ω ρ-self-dual and harmonic, ω is transverse, i.e. (M, ω) is near-symplectic.

Remark. Auroux et. al. ([2]) give an equivalent definition of a near-symplectic structure that's independent of a Riemannian metric and show that a metric with respect to which the form is self-dual can always be constructed. Their analysis is similar to that in our Appendix.

In [8], Honda proves that near each component C_i of Z_ω, there is a neighborhood that is symplectomorphic to one of two canonical models $(S^1 \times D^3, \omega_A)$ and $(S^1 \times D^3, \omega_B)$. I'll present his argument in this chapter. The main tool in the proof is a Darboux-Moser type theorem for near-symplectic structures which will be extended to the equivariant case in Chapter 4. The theorem relies on the existence of a canonical splitting of the normal bundle NC, which I'll describe first.

3.1 Normal bundle splittings and standard forms

Assume M is oriented; then so is a neighborhood of C, $N(C)$, and so is the normal bundle to C, NC. $\pi_1(BSO(3)) = 0$, so this bundle is trivial and we can choose a ρ_ω-orthonormal frame for NC along C. Exponentiating the frame with respect to some metric (ρ_ω works, but so does any other one, e.g. a G-invariant one in the presence of a G-action), we obtain a diffeomorphism $S^1 \times D^3 \to^\psi N(C)$. Let (θ, x_1, x_2, x_3) be such coordinates on $S^1 \times D^3$. Then in the chart ψ, the tangent vectors $\{\frac{\partial}{\partial\theta}, \frac{\partial}{\partial x_i}\}$ form an oriented ρ_ω-orthonormal basis at all points $(\theta, 0)$.

Since $\omega(\theta, 0) = 0$, we can Taylor-expand ω in the coordinates ψ to write

$$\begin{aligned}\omega = {} & L_1(\theta,x)(d\theta dx_1 + dx_2 dx_3)\\ & + L_2(\theta,x)(d\theta dx_2 + dx_3 dx_1)\\ & + L_3(\theta,x)(d\theta dx_3 + dx_1 dx_2) + Q\end{aligned} \tag{3.1.1}$$

where $L_i(\theta,x) = \sum_{i=1}^3 L_{ij}(\theta)x_j$ are linear in x and Q is quadratic or higher in x. Note that this particular form holds since ω is ρ_ω-self-dual with $\{\frac{\partial}{\partial\theta}, \frac{\partial}{\partial x_i}\}$ an oriented ρ_ω-orthonormal basis at all points $(\theta, 0)$.

Using $d\omega = 0$, calculating using the above expression, and equating 0th order terms, we obtain

$$\frac{\partial L_1}{\partial x_2} - \frac{\partial L_2}{\partial x_1} = 0, \frac{\partial L_1}{\partial x_3} - \frac{\partial L_3}{\partial x_1} = 0, \frac{\partial L_2}{\partial x_3} - \frac{\partial L_3}{\partial x_2} = 0 \tag{3.1.2a}$$

$$\frac{\partial L_1}{\partial x_1} + \frac{\partial L_2}{\partial x_2} + \frac{\partial L_3}{\partial x_3} = 0, \tag{3.1.2b}$$

Since $\frac{\partial L_i}{\partial x_j} = L_{ij}(\theta)$, this shows that the matrix $\{L_{ij}\}_{i,j=1}^3$ is traceless and symmetric. Thus L_{ij} is diagonalizable.

Proposition 3.1.1. *For ω transverse to the zero section of $\wedge^+_{\rho_\omega}$, L_{ij} has full rank.*

Proof. The fibre of $\wedge^+_{\rho_\omega}$ at a point $x \in M$ has dimension 3, and since $\omega = 0$ along C, the image of $\partial\omega|_{N_xC}$ must span the fibre. L_{ij} is this derivative.

□

By the Proposition, L_{ij} has no zero eigenvalues. Thus it must have two positive eigenvalues and one negative eigenvalue (or vice versa).

I will now describe these eigenspaces in an invariant way as subspaces of the normal bundle NC using only the metric ρ_ω.

Let $A = \sum a_i \frac{\partial}{\partial x_i}, B = \sum b_i \frac{\partial}{\partial x_i} \in N_{(\theta,0)}C$ be two tangent vectors in the above coordinates. Then

$$\begin{aligned} \sum_{i,j} a_j b_i L_{ij} &= \sum_i b_i (\sum_j a_j \iota(\frac{\partial}{\partial x_j}) d[\iota(\frac{\partial}{\partial x_k})\iota(\frac{\partial}{\partial x_\ell})\omega]) \\ &= \iota(A) \sum_i b_i d[\iota(\frac{\partial}{\partial x_k})\iota(\frac{\partial}{\partial x_\ell})\omega] \end{aligned} \tag{3.1.3}$$

where $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_\ell}\}$ is an oriented basis of $N_{(\theta,0)}C$.

Proposition 3.1.2. *Let $V \cong \mathbb{R}^3$ be a vector space with the standard inner product and orientation, and let ω be a skew-symmetric bilinear form on V. Then the map $v \in V \mapsto^q \omega(v', v'')$, where $\{v, v', v''\}$ is an oriented orthogonal basis of V and $|v'| = |v''| = |v|^{1/2}$, is well-defined and linear.*

Proof. q is well-defined because any orientation-preserving orthogonal transformation $(v', v'') \mapsto (\tilde{v}', \tilde{v}'')$ leaves ω invariant. One can check that if (x, y, z) are standard coordinates on $\mathbb{R}^3$, $\omega = adx \wedge dy + bdy \wedge dz + cdx \wedge dz$, and $v = (x_0, y_0, z_0)$, then $q(v) = az_0 + bx_0 - cy_0$.

□

Corollary 3.1.3. *Given ρ_ω, there is a natural orthogonal splitting of the normal bundle NC into a 2-dimensional subbundle and a line bundle.*

Proof. By the previous proposition we can define a bilinear form $H : N_pC \times N_pC \to \mathbb{R}$ by

$$H(v, w) = \iota(v)(d(q(\tilde{w}))_p \tag{3.1.4}$$

where q is as in the proposition and $\tilde{w}$ is any extension of w to a vector field near p. (H is independent of the choice of $\tilde{w}$ by vanishing of ω at C.) By (3.1.3), the associated map $\tilde{H} : N_pC \to N_pC$ obtained via ρ_ω is represented by the matrix L_{ij}, and so induces an orthogonal splitting of the normal bundle NC into a 2-dimensional subbundle and a line bundle which are the spans of the positive and negative eigenspaces.

□

Since $NC \cong S^1 \times D^3$ is trivializable, we can classify such splittings by maps $S^1 \to \mathbb{RP}^2$. Up to homotopy, these are classified by $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$. We can distinguish these splittings by whether the line bundle is orientable or not.

Proposition 3.1.4 ([8]). *There are model near-symplectic structures on $S^1 \times D^3$ with vanishing locus $x = 0$ and self-dual with respect to the flat metric that induce both types of splittings.*

Proof. Representatives ω_A and ω_B are defined as follows.

The oriented splitting: On $S^1 \times D^3$, set

$$\begin{aligned} \omega_A &= x_1(d\theta dx_1 + dx_2 dx_3) \\ &\quad + x_2(d\theta dx_2 + dx_3 dx_1) \\ &\quad - 2x_3(d\theta dx_3 + dx_1 dx_2) \end{aligned} \tag{3.1.5}$$

here $L_{ij}(\theta) = \mathrm{diag}(1, 1, -2)$ with fixed positive and negative eigenspaces.

The unoriented splitting: Set $\Omega = \omega_A$ on $[0, 2\pi] \times D^3$. Then glue $\{2\pi\} \times D^3 \to^{\phi} \{0\} \times D^3$ by $\theta \mapsto \theta - 2\pi$, $x_1 \mapsto x_1$, $x_2 \mapsto -x_2$, $x_3 \mapsto -x_3$. Then $\phi^*\Omega = \Omega$ so Ω induces a form ω_B on the quotient.

□

3.2 Contact boundaries and Reeb flow

Proposition 3.2.1 ([8]). *Both models $(S^1 \times D^3, \omega_A)$ and $(S^1 \times D^3, \omega_B)$ admit compatible contact structures on their boundaries $S^1 \times S^2$.*

Proof. For (A), consider the 1-form

$$\lambda = -\frac{1}{2}(x_1^2 + x_2^2 - 2x_3^2)d\theta + x_2 x_3 dx_1 - x_1 x_3 dx_2 \tag{3.2.1}$$

We have $d\lambda = \omega_A$. Let $i : S^1 \times S^2 \to S^1 \times D^3$ be the inclusion. Then since $\sum_i x_i dx_i = 0$ on $T(S^1 \times S^2)$, $i^*(\lambda \wedge d\lambda) \neq 0$ iff $\lambda \wedge d\lambda \wedge \sum_i x_i dx_i \neq 0$ near $S^1 \times S^2$. But

$$\lambda \wedge d\lambda \wedge \sum_i x_i dx_i = -(\frac{1}{2}(x_1^2 + x_2^2)(x_1^2 + x_2^2 + 2x_3^2) + 2x_3^4) d\theta dx_1 dx_2 dx_3 \tag{3.2.2}$$

so is nonzero where required.

For case (B), the proof is the same after gluing. □

To conclude this section, I reproduce Honda's description of the Reeb vector fields on $N(C)$. Note that the flat metric on $S^1 \times D^3$ is given by $\rho(x, y) = \frac{1}{c}\omega_A(x, Jy)$, where $J = -\frac{1}{c}A$, A is the matrix representation of ω_A, i.e.

$$A = \begin{pmatrix} 0 & x_1 & x_2 & -2x_3 \\ -x_1 & 0 & -2x_3 & -x_2 \\ -x_2 & 2x_3 & 0 & x_1 \\ 2x_3 & x_2 & -x_1 & 0 \end{pmatrix} \tag{3.2.3}$$

and $c = \sqrt{x_1^2 + x_2^2 + 4x_3^2}$. By compatibility, requiring the Reeb vector field X to be in $\ker(i^* d\lambda)$ is equivalent to requiring it to be in the image under J of the ρ-normal to $S^1 \times S^2$, i.e. up to scalars,

$$X = J(\sum_i x_i \frac{\partial}{\partial x_i}) = \frac{-1}{\sqrt{x_1^2 + x_2^2 + 4x_3^2}}((x_1^2 + x_2^2 - 2x_3^2)\frac{\partial}{\partial \theta} - 3x_2 x_3 \frac{\partial}{\partial x_1} + 3x_1 x_3 \frac{\partial}{\partial x_2}) \tag{3.2.4}$$

Normalizing by $\lambda(X) = 1$ gives

$$X = \frac{1}{f}((x_1^2 + x_2^2 - 2x_3^2)\frac{\partial}{\partial \theta} - 3x_2 x_3 \frac{\partial}{\partial x_1} + 3x_1 x_3 \frac{\partial}{\partial x_2}) \tag{3.2.5}$$

where $f = -\frac{1}{2}[(x_1^2 + x_2^2)(x_1^2 + x_2^2 + 2x_3^2) + 4x_3^4]$.

Setting $r^2 = x_1^2 + x_2^2, \beta = \arctan(x_2/x_1)$, we can rewrite the vector field as

$$X = \frac{1}{f}((r^2 - 2x_3^2)\frac{\partial}{\partial \theta} + 3x_3 \frac{\partial}{\partial \beta}) \tag{3.2.6}$$

where $f = -\frac{1}{2}[(r^2)(r^2 + 2x_3^2) + 4x_3^4]$.

Thus the flow preserves the x_3 coordinate and r^2, and rotates in the (x_1, x_2)-plane and along S^1. On $S^1 \times S^2$, we have $r^2 + x_3^2 = k$ (usually $k = 1$, but can take any $k \neq 0$), so the flow can be written in the form

$$\begin{aligned} x_1(t) &= \sqrt{k - x_3^2}\cos(R_1(x_3)t) \\ x_2(t) &= \sqrt{k - x_3^2}\sin(R_1(x_3)t) \\ x_3(t) &= x_3(0) \\ \theta(t) &= R_2(x_3)t + c \end{aligned} \tag{3.2.7}$$

where R_i are functions of x_3. Specifically, since $-2f = k^2 + 3x_3^4$,

$$\begin{aligned} R_1 &= -2\frac{3x_3}{k^2 + 3x_3^4} \\ R_2 &= -2\frac{r^2 - 2x_3^2}{k^2 + 3x_3^4} \end{aligned} \tag{3.2.8}$$

We can now consider the closed orbits of the Reeb flow. Note that for $x_3 = 0$, $R_1 = 0$, and the closed orbit is of the form $(x_1, x_2, 0) =$ constant, i.e. flow along the θ direction. Similarly, for $r = 0$, the flow is along the θ direction. For $r^2 - 2x_3^2 = 0$, $R_2 = 0$, so the closed orbit is of the form $(x_3, r, \theta) =$ constant, i.e. flow along the β direction. The other closed orbits occur when $R_1/R_2 \in \mathbb{Q}$.

Remark. Note that the cases $x_3 = 0$ and $r = 0$ correspond respectively to the stable and unstable gradient directions in the Morse-Bott theory for the function $x_1^2 + x_2^2 - 2x_3^2 = r^2 - 2x_3^2$, which is the moment for the S^1 action given by rotation in θ and the numerator in R_2. The significance of the numerator in R_1 is unclear.

Remark. Honda ([8]) also proves that the contact structures induced on the boundaries $S^1 \times S^2$ by the contact forms λ_A and λ_B are both overtwisted and distinct, but I'll skip the proof because I won't make use of it later.

3.3 Honda-Moser theorems

Let $\{\omega_t\}, t \in [0,1]$ be a smooth family of self-dual harmonic 2-forms with respect to metrics ρ_t, transverse to the zero sections of their respective bundles of self-dual forms, such that the number of components of Z_{ω_t} is constant so that we can identify all Z_{ω_t} via isotopy. For simplicity, assume $Z_\omega = C$ is constant. Assume further that (i) $[\omega_t] \in H^2(M;\mathbb{R})$ is constant, and (ii) $[\omega_t] \in H^2(M,C;\mathbb{R})$ is constant.

Theorem 3.3.1 (Global Honda-Moser [8]). *Under the above assumptions, there exists a 1-parameter family f_t of C^0-homeomorphisms of M, smooth away from C and fixing C, such that $f_t^*\omega_t = \omega_0$.*

As in the proof of the Moser-Darboux theorem, the requirement $f_t^*\omega_t = \omega_0$, $f_0 =$ Id, implies that $f_t^*(d\iota(X_t)\omega_t) + f_t^*(\frac{d\omega_t}{dt}) = 0$, where $X_t = \frac{df_t}{dt}$. The proof thus reduces to choosing a 1-form η_t satisfying $d\eta = \frac{d\omega_t}{dt}$ such that the equation $\iota(X_t)\omega_t = -\eta_t$ defines a vector field X_t which is sufficiently continuous and zero along Z_ω. The complication in the near-symplectic case is that ω_t is degenerate along Z_ω so can't be smoothly inverted. To deal with this complication, we will choose η very carefully.

Lemma 3.3.2. *There exists a smooth family of 1-forms $\tilde{\eta}_t$ such that $\frac{d\omega_t}{dt} = d\tilde{\eta}$ and $i^*\tilde{\eta}_t$ is exact, where $i : C \to M$ is the inclusion.*

Proof. Recall the relative cohomology exact sequence:

$$H^1(M;\mathbb{R}) \to^{i^*} H^1(C;\mathbb{R}) \to^{\delta} H^2(M,C;\mathbb{R}) \to H^2(M;\mathbb{R})$$

In deRham cohomology, for $[\alpha] \in H^1(C,\mathbb{R})$, $\delta[\alpha] = [d\tilde{\alpha}]$ where $\tilde{\alpha}$ is any extension of α to a 1-form on M.

By assumption (i), there exists a smooth family of 1-forms $\hat{\eta}_t$ such that $\frac{d\omega_t}{dt} = d\hat{\eta}_t$. $\hat{\eta}_t$ is clearly an extension of $i^*\hat{\eta}_t$ to M, and by condition (ii), $[d\hat{\eta}_t] = 0 \in H^2(M,C;\mathbb{R})$. Therefore, by exactness, there exists $[\beta_t] \in H^1(M;\mathbb{R})$ such that $[i^*\hat{\eta}_t] = [i^*\beta_t]$. Setting $\tilde{\eta}_t = \hat{\eta}_t - \beta_t$ gives the required 1-form.

□

Lemma 3.3.3. *There exists a smooth family of 1-forms* η_t *such that* $\frac{d\omega_t}{dt} = d\eta_t$ *and* $\eta_t = 0$ *at* Z_ω up to second order.

Proof. Let $\tilde{\eta}_t$ be as in the previous lemma and let f_t be a smooth family of functions such that $i^*\tilde{\eta}_t = df_t$. Choose coordinates on $N(C) \cong S^1 \times D^3$ via exponentiating a ρ_ω-orthonormal frame along C as above. Set

$$f_t(\theta, x_1, x_2, x_3) = f_t(\theta) + \sum_i \tilde{\eta}_i(\theta, 0)x_i + \frac{1}{2}\sum_{i,j} \frac{\partial \tilde{\eta}_i}{\partial x_j}(\theta, 0)x_i x_j$$

where $\tilde{\eta} = \tilde{\eta}_\theta d\theta + \sum_i \tilde{\eta}_i dx_i$ on N(C). Then

$$\begin{aligned} df_t(\theta, x_1, x_2, x_3) = & \frac{\partial f_t}{\partial \theta}(\theta)d\theta + \sum_i \frac{\partial \tilde{\eta}_i}{\partial \theta}(\theta, 0)x_i d\theta \\ & + \sum_i \tilde{\eta}_i(\theta, 0)dx_i + \frac{1}{2}\sum_{i,j} \frac{\partial \tilde{\eta}_i}{\partial x_j}(\theta, 0)(x_i dx_j + x_j dx_i) \\ & + O(x^2) \end{aligned}$$

Notice that

$$\begin{aligned} \frac{\partial f}{\partial \theta}(\theta) =& \tilde{\eta}_\theta(\theta, 0) \\ d\tilde{\eta}_t(\theta, 0) =& \frac{d\omega_t}{dt}(\theta, 0) = 0 \\ \Longrightarrow & \frac{\partial \tilde{\eta}_\theta}{\partial x_i}(\theta, 0) = \frac{\partial \tilde{\eta}_i}{\partial \theta}(\theta, 0) \\ & \frac{\partial \tilde{\eta}_i}{\partial x_j}(\theta, 0) = \frac{\partial \tilde{\eta}_j}{\partial x_i}(\theta, 0) \end{aligned}$$

Substituting, we obtain

$$\begin{aligned} df_t(\theta, x_1, x_2, x_3) = & (\tilde{\eta}_\theta(\theta, 0) + \sum_i \frac{\partial \tilde{\eta}_\theta}{\partial x_i}(\theta, 0)x_i)d\theta \\ & + \sum_i (\tilde{\eta}_i(\theta, 0) + \sum_j (\frac{\partial \tilde{\eta}_i}{\partial x_j}(\theta, 0)x_j))dx_i \\ & + O(x^2) \end{aligned} \tag{3.3.1}$$

Setting $\eta_t = \tilde{\eta}_t - df_t$ (after damping f_t to zero away from C by a cutoff function, so that it extends by zero to all of M) we obtain the required 1-form.

□

Proof of theorem 3.3.1 . Define the vector field X_t by $\iota(X_t)\omega = -\eta_t$, where η_t is as in the previous lemma. As a matrix, we've seen that ω_t corresponds to

$$A(\theta, x) = \begin{pmatrix} 0 & L_1 & L_2 & L_3 \\ -L_1 & 0 & L_3 & -L_2 \\ -L_2 & -L_3 & 0 & L_1 \\ -L_3 & L_2 & -L_1 & 0 \end{pmatrix} (\theta, x) + Q(\theta, x)$$

where Q is quadratic or higher in x, and this is in the coordinates coming from the ρ_ω-orthonormal frame along C. Setting $X_t = a_t \frac{\partial}{\partial \theta} + \sum_i a_i \frac{\partial}{\partial x_i}$, we have $(a_\theta, a_1, a_2, a_3)A = -(\eta_\theta, \eta_1, \eta_2, \eta_3)$, or

$$\begin{aligned}(a_\theta, a_1, a_2, a_3) &= -(\eta_\theta, \eta_1, \eta_2, \eta_3)A^{-1} \\ &= (\eta_\theta, \eta_1, \eta_2, \eta_3)\frac{1}{L_1^2 + L_2^2 + L_3^2} \begin{pmatrix} 0 & L_1 & L_2 & L_3 \\ -L_1 & 0 & L_3 & -L_2 \\ -L_2 & -L_3 & 0 & L_1 \\ -L_3 & L_2 & -L_1 & 0 \end{pmatrix} + Q'\end{aligned}$$

where Q' is second order or higher in x.

Notice that by nondegeneracy of L_{ij}, $L_1^2 + L_2^2 + L_3^2 \neq 0$ unless $x = 0$. Given this nondegeneracy, the expression above has leading term of order 1 in x. Thus $|X_t| < k|x|$ near C, X_t is smooth elsewhere, and the flow fixes C.

□

Theorem 3.3.4 (Local Honda-Moser [8]). *Let (M, ω) be a near-symplectic manifold. Then near each component C_i of Z_ω, there is a neighborhood that is symplectomorphic to one of the two models $(S^1 \times D^3, \pm\omega_A)$ or $(S^1 \times D^3, \pm\omega_B)$.*

Proof. Assume that ω is such that NC splits in the oriented manner. The proof for the unoriented case is similar (working in the nonreduced space of the unoriented model). We can choose coordinates via an ρ_ω-orthonormal frame such that $\{\frac{\partial}{\partial x_i}\}_{i=1,2}$

span the two dimensional subbundle along C and $\{\frac{\partial}{\partial x_3}\}$ spans the one-dimensional subbundle. Then in these coordinates,

$$\begin{aligned}\omega = &(L_{11}(\theta)x_1 + L_{12}(\theta)x_2)(d\theta dx_1 + dx_2 dx_3)\\ &+ (L_{21}(\theta)x_1 + L_{22}(\theta)x_2)(d\theta dx_2 + dx_3 dx_1)\\ &+ L_{33}(\theta)x_3(d\theta dx_3 + dx_1 dx_2) + Q\end{aligned}$$

where $(L_{ij})_{i,j=1,2}$ is positive-definite and $L_{33} < 0$ (for the opposite case, change signs in ω_A below to match). Using these ω-adapted coordinates, write

$$\begin{aligned}\omega_A = &x_1(d\theta dx_1 + dx_2 dx_3)\\ &+ x_2(d\theta dx_2 + dx_3 dx_1)\\ &- 2x_3(d\theta dx_3 + dx_1 dx_2)\end{aligned}$$

Now, set $\omega_t = t\omega + (1-t)\omega_A$. We can still define $L_{ij}(t)$ as before, only we do so in this fixed coordinate system. Since we defined ω_A so that its eigenspaces correspond to those of ω, $L_{ij}(t)$ is nondegenerate for all $t \in [0,1]$ and has the same form as in the global theorem. Finally, on a tubular neighborhood $N(C)$, the cohomological conditions of the global theorem are trivially satisfied, so the proof of the global theorem carries through otherwise unmodified.

□

Chapter 4

Near-symplectic toric 4-manifolds

I now consider the case in which T^2 acts effectively on a near-symplectic 4-manifold. This chapter is inspired by the work of Gay-Symington ([4]) but takes a different approach. In particular, I make the simplifying assumption throughout that there is a global Hamiltonian T^2 action rather than only a locally toric structure. I use the existence of a metric as in Condition 4.1.1 to show that near each component of Z_ω the toric structure is of a certain standard form.

Definition 4.0.1 ([4]). A smooth T^2 action on a near-symplectic manifold (M, ω) is *Hamiltonian* if there exists a smooth map $\Phi : M \to \mathfrak{t}^*$ such that Φ is a moment map for the action on $M \setminus Z_\omega$. A near symplectic 4-manifold is *toric* if it has an effective Hamiltonian T^2-action.

From this definition, it's clear that the local analysis of the moment map in Chapters 1 and 2, including the local canonical forms, local convexity, and Delzant conditions, carry through unchanged on $M \setminus Z_\omega$.

4.1 Equivariant Honda-Moser theorems

To obtain a canonical local form for the T^2 action near Z_ω, it's convenient to extend the results of Chapter 3 to be equivariant. It is easy to extend Lemmas 3.3.2, 3.3.3 and Theorem 3.3.1 to the equivariant setting, as follows.

Lemma 4.1.1. *In Lemmas 3.3.2, 3.3.3 and Theorem 3.3.1 we can choose the one-forms $\tilde{\eta}$ and η to be G-invariant and the diffeomorphism f to be G-equivariant, for any connected compact Lie group G that acts symplectically on (M, ω_t).*

Proof. Since averaging over G preserves cohomology classes and ω_t is invariant, $\tilde{\eta}$ may be averaged at the end of the proof of Lemma 3.3.2 to obtain an invariant form satisfying all the conditions. Similarly, since ω_t is invariant and G is compact and fixes Z_ω, η can be averaged at the end of the proof of Lemma 3.3.3 to obtain an invariant form as required, preserving the estimate on its vanishing. Finally, since η_t and ω_t are G-invariant, so is the vector field X_t in the proof of Theorem 3.3.1, and its flow f is G-equivariant. □

It is less easy to extend Theorem 3.3.4 in a useful way, making use of a T^2 action. For this purpose I state the following condition explicitly:

Condition 4.1.1. Assume there exists a Riemannian metric ρ such that ω is self-dual and transverse with respect to ρ *and ρ is T^2-invariant.*

Remark. This condition is always satisfied for any near-symplectic ω. A proof of this fact due to D. Auroux is explained in the Appendix.

The canonical form ω_A on $S^1 \times D^3$ can be given a Hamiltonian T^2 action which satisfies the above condition with the flat metric:

Example 4.1.1 (The "standard fold"). Let $\omega_A = x(d\alpha dx + dydz) + y(d\alpha dy + dzdx) - 2z(d\alpha dz + dxdy)$ on $S^1 \times D^3 \cong (\alpha, x, y, z)$. Setting $\theta = \arctan(y/x), r^2 = x^2 + y^2$, we have $\omega_A = -2z(d\alpha dz + rdrd\theta) - rdrd\alpha + r^2 d\theta dz$. Then

1. ω_A is invariant under the T^2 action $(t_1, t_2) \cdot (\alpha, r, \theta, z) = (\alpha + t_1, r, \theta + t_2, z)$
2. A moment map for the action is $(\alpha, r, \theta, z) \mapsto^{\Phi_0} (z^2 - \frac{1}{2}r^2, zr^2)$
3. ω_A is self-dual and transverse with respect to the flat metric.
4. The T^2 action preserves the flat metric.

Proof of 2. Setting $p_1 = z^2 - \frac{1}{2}r^2, q_1 = \alpha, p_2 = zr^2, q_2 = \theta$, we calculate $\omega_A = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$, as desired. □

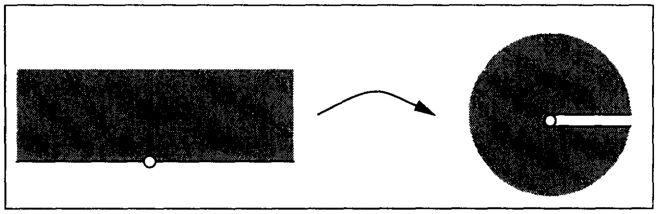

Figure 4-1: A fold.

It is worth pointing out several features of this model in detail. First, the orbit space $B = M/T^2$ can be identified with the half-plane $H = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 \geq 0\}$ via the map $(\alpha, r, \theta, z) \mapsto (z, r^2)$. Under this identification, $\partial B = \{(x_1, x_2) \in H | x_2 = 0\}$. Let $\pi : M \longrightarrow B$ be the quotient map, and let $S^1_1 \times S^1_2 \cong T^2$ be the standard splitting of T^2 into two circle subgroups. Since $r = 0$ there, each point in $\pi^{-1}(\partial B)$ has stabilizer S^1_2, and lies on a circle orbit generated by S^1_1. On $\mathrm{int}(B)$, each point has preimage a full T^2 orbit with trivial stabilizer.

Let $\mathfrak{t}_1 \oplus \mathfrak{t}_2$ be the splitting of the Lie algebra $\mathfrak{t}$ induced by the group splitting $T^2 \cong S^1_1 \times S^1_2$. It induces a splitting $\mathfrak{t}^* \cong \mathfrak{t}^*_1 \oplus \mathfrak{t}^*_2$. The moment map Φ_0 descends to a map $\phi_0 : B \longrightarrow \mathfrak{t}^*$, given in the coordinates (x_1, x_2) on H and $\mathfrak{t}^*_1 \oplus \mathfrak{t}^*_2$ on $\mathfrak{t}^*$ by $(x_1, x_2) \mapsto^{\phi_0} (x_1^2 - \frac{1}{2}x_2, x_1 x_2)$. This map is, in the terms of Gay-Symington ([4]), a *fold*, that is, it satisfies the following properties:

0. $\phi_0 : H \longrightarrow \mathbb{R}^2$ is smooth.

1. $\phi_0(0,0) = (0,0)$.

2. $\phi_{0H\setminus\{(0,0)\}}$ is an immersion.

3. ϕ_0 maps both $\{(x_1, 0) | x_1 > 0\}$ and $\{(x_2, 0) | x_2 < 0\}$ diffeomorphically onto $\{(p_1, 0) | p_1 > 0\}$.

4. ϕ_0 maps $\{(x_1, x_2) | x_2 > 0\}$ diffeomorphically onto $\mathbb{R}^2 \setminus \{(p_1, 0) | p_1 > 0\}$.

A familiar example of a fold is the complex map $z \mapsto z^2$, restricted to $H \subset \mathbb{C}$. A fold is illustrated in Figure 4-1, with the double p_1-axis drawn as two parallel lines for the purpose of illustration.

Here (p_1, p_2) are coordinates on $\mathfrak{t}^*$. The above description of a fold means that ϕ_0 maps ∂B to the positive p_1-axis. Along the positive p_1-axis, each point x has two pre-images under ϕ_0, which lie in ∂B, i.e. two orbits map to x and they are both circle orbits. The origin $(p_1, p_2) = (0,0)$ has a unique pre-image, also a circle orbit in ∂B. Notice that the double image along the positive p_1-axis lies in $\mathfrak{t}_1^* \times \{0\}$, i.e. in $\mathfrak{t}_2^0 \subset \mathfrak{t}^*$, the annihilator of $\mathfrak{t}_2$, the infinitesimal generator of the the stabilizer group of $\pi^{-1}(\partial B)$.

Finally, for a point not on the positive p_1-axis, the pre-image under ϕ_0 is a unique point in int(B) corresponding to a free T^2 orbit, and the set $\Phi_0^{-1}(\mathfrak{t}^* \setminus \{(p_1, 0) | p_1 \geq 0\})$ is equivariantly symplectomorphic to the manifold $(\mathfrak{t}^* \setminus \{(p_1, 0) | p_1 \geq 0\}) \times T^2$ with symplectic form $dp_1 \wedge dq_1 + dp_2 \wedge dq_2$, where q_1 is a coordinate on S_1^1 and q_2 is a coordinate on S_2^1, as shown in the example. The existence of these standard coordinates away from the fold is what permits the easy construction of near-symplectic toric 4-manifolds by patching together local models as in Sections 4.3 and 4.4.

Remark (Contact structure and Reeb flows). We remark that the contact form λ_A defined in Chapter 3 can be written in the above notation as

$$\lambda_A = -\frac{1}{2}(r^2 - 2z^2)d\alpha - zr^2 d\theta \tag{4.1.1}$$

so it's T^2 invariant, and its coefficients are the components of the moment map Φ_0. Indeed, this is just saying $\lambda_A = p_1 dq_1 + p_2 dq_2$, where $\omega_A = d\lambda_A = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ as above. Similarly, the Reeb vector field X can be written as

$$X = \frac{1}{f}((r^2 - 2z^2)\frac{\partial}{\partial \alpha} + 3z\frac{\partial}{\partial \theta}) \tag{4.1.2}$$

where $f = -\frac{1}{2}[(r^2)(r^2 + 2z^2) + 4z^4]$, so it's T^2 invariant and tangent to the fibres, and hence moment-preserving.

As we will see in Theorem 4.1.3, a consequence of Condition 4.1.1 is that near any component C of Z_ω, the manifold is equivariantly symplectomorphic to the above "standard fold".

Proposition 4.1.2. *Assuming Condition 4.1.1, the ρ-normal bundle NC and the splitting from Corollary 3.1.3 are T^2 invariant.*

Proof. The invariance of NC follows since T^2 preserves C and ρ is T^2-invariant.

To see that the splitting is preserved, we want $H_{g\cdot p}(dg \cdot v_p, dg \cdot w_p) = H_p(v_p, w_p)$, for all $p \in C, g \in T^2$, and H defined as in Corollary 3.1.3.

$$H(dg \cdot v_p, dg \cdot w_p) = \iota(dg \cdot v_p) dq(\widetilde{dg \cdot w_p})$$

Now, using the notation from Corollary 3.1.3,

$$\begin{aligned} d(q(\widetilde{dg \cdot w})) &= d[\omega((\widetilde{dg \cdot w})', (\widetilde{dg \cdot w})'')] \\ &= d[\omega(dg \cdot \widetilde{w}', dg \cdot \widetilde{w''})] \end{aligned}$$

because T^2 acts orthogonally and the quantity is independent of the extensions of the vector fields. So

$$\begin{aligned} \iota(dg \cdot v) d(q(\widetilde{dg \cdot w})) &= \iota(dg \cdot v) d[\omega(dg \cdot \widetilde{w}', dg \cdot \widetilde{w''})]_{g \cdot p} \\ &= \iota(v) d[\omega(\widetilde{w}', \widetilde{w''})]_p = H(v, w) \end{aligned}$$

where the second equality is by T^2 invariance of ω and the chain rule. ☐

We are now ready to extend Theorem 3.3.4 to the case with a T^2 action.

Theorem 4.1.3. *Assuming Condition 4.1.1, any component C of Z_ω has a neighborhood that's equivariantly symplectomorphic, up to an integral reparametrization of T^2, to the model in Example 4.1.1.*

Proof. Fix a metric ρ as in Condition 4.1.1. Choose $x \in C$. Consider G_x, the stabilizer group of x. We first show that G_x is a circle subgroup of $G = T^2$ and that the splitting of the normal bundle must be the oriented one. Note that because T^2 acts symplectically, it must map C to C.

If $G_x = G$, then G acts linearly on T_xM. Since ρ is G-invariant, G preserves the splitting of NC and maps C to C, G acts on T_xM as a subgroup of $O(1) \times O(1) \times O(2)$. By connectedness it must act as a subgroup of $SO(2)$, but this violates effectiveness, as there is no faithful representation $T^2 \to SO(2)$.

Since $G_x \neq G$ and is closed, it must have dimension 0 or 1. If it had dimension 0, the orbit G/G_x would have dimension 2, but $G/G_x \subset C$, which is 1-dimensional.

Thus G_x has dimension 1 and C is the orbit $G \cdot x$.

Choose an orthonormal basis for $\{V_1, V_2, V_3\}$ of N_xC such that $\{V_1, V_2\}$ span the 2-dimensional sub-space of the splitting of N_xC and V_3 spans the 1-dimensional subspace. Exponentiating these vectors by ρ, we obtain a slice for the G-orbit C at x. Since G preserves the splitting of NC, G_x acts on N_xC as a subgroup of $O(1) \times O(2)$, and by effectiveness and the equivariant slice theorem, this representation is faithful, i.e. $G_x \subset O(1) \times O(2)$ as a subgroup. By connectedness of T^2 and disconnectedness of $O(1)$, G_x acts nontrivially on the 1-dimensional subspace iff the line bundle is nonorientable, i.e. we are in the unoriented splitting. However, we must also have that G_x preserves the orientation of N_xC since it preserves the orientation of C and acts symplectically. The only 1-dimensional abelian subgroup of $(O(1) \times O(2)) \cap SO(3)$ is $1 \times SO(2)$, so G_x is a circle subgroup of T^2 and the splitting must be the oriented splitting.

For any closed circle subgroup G_x of T^2, we can choose a complement circle subgroup H such that T^2 splits as $T^2 = G_x \times H$. A consequence of effectiveness is that in these coordinates, the generators of the Lie algebras $\{\mathfrak{g}_x, \mathfrak{h}\}$ correspond to the image under some $A \in \mathrm{GL}(2, \mathbb{Z})$ of the standard basis $\{\mathfrak{t}_1, \mathfrak{t}_2\}$ of $\mathfrak{t}$. (The proof by "factoring" the action is the same as in Proposition 2.2.1.)

Since the splitting is oriented, the vector V_3 extends uniquely as a unit trivialization of the line bundle along C. We can use the complement subgroup H to transport the vectors $\{V_1, V_2\}$ along C to obtain an orthonormal frame $\{V_1, V_2, V_3\}$ for the bundle N_xC such that $\{V_1, V_2\}$ span the 2-dimensional sub-bundle of the splitting of NC and V_3 spans the line-bundle. (Note that the transportation of the vector V_3 along C by H agrees with the unique extension above.) Exponentiating this frame with respect to ρ gives coordinates $\{\theta, x_1, x_2, x_3\}$ on a neighbourhood of C such that along $S^1 \times \{0\}$, $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$ span the 2 dimensional sub-bundle and $\frac{\partial}{\partial x_3}$ spans the line bundle.

Thus, in the coordinates on $N(C)$ induced by the frame and the coordinates on T^2 induced by a splitting $T^2 = H \times G_x$, T^2 acts as $(t_1, t_2) \cdot (\theta, r, \alpha, x_3) \mapsto (\theta + t_1, r, \alpha + t_2, x_3)$, where $r^2 = x_1^2 + x_2^2, \alpha = \arctan(x_2/x_1)$. The rest of the argument of Theorem 3.3.4 goes through unmodified, using the equivariant version of the local Honda-Moser theorem as described above, applied to the given form ω and the form ω_A constructed in the above coordinates. □

Given the reparametrization of T^2 via the splitting needed in the last theorem, the following fact describes how the moment map changes.

Proposition 4.1.4. *If $\Phi :\to \mathfrak{t}^*$ is a moment map for a Hamiltonian T^2 action σ, and $\mu : p \mapsto Ap + b \in Aff(2, \mathbb{Z})$ (i.e. $A \in GL(2, \mathbb{Z}), b \in \mathbb{R}^2$), then $\mu \circ \Phi$ is a moment map for the torus action $\sigma'(t, x) = \sigma(A^{-T}t, x)$.*

(Note that both actions have the same orbits.)

Thus the moment map near a component C of Z_ω has image $(A^{-T} \circ \Phi_0)(S^1 \times D^3) + b$, where $A \in GL(2, \mathbb{Z})$ is the integral transformation corresponding to the splitting used in the symplectomorphism to the standard fold, Φ_0 is the moment map for the standard fold, and $b \in \mathbb{R}^2$. Note that by definition, the image of the p_1 axis under A^{-T} is independent of the choice of splitting (since it corresponds to $\mathfrak{g}_x^0$), so there is no ambiguity in the location of the fold.

4.2 Failure of convexity

Given the above, we have a completely canonical description of a neighborhood of any of the T^2 orbits. It is natural to ask to what extent the theorems of the previous chapters apply. In this section we consider convexity alone.

First, local convexity still holds away from Z_ω. In some sense local convexity also holds near Z_ω: the image of the moment map on a nice tubular neighborhood of a component C is a convex subset of of $\mathbb{R}^2$. In another sense, though, it fails, in that convex sets in the standard coordinates (z, r^2) of the orbit space are not mapped to convex sets by the moment map Φ_0.

To see how global convexity fails, it's interesting to see how the Morse-Bott theory of the moments in the near-symplectic case differs from that in the symplectic case of Chapter 2.

Let $\xi = (a,b) \in \mathfrak{t}$. For the standard fold, the moment Φ^ξ is given by

$$\Phi^\xi(\alpha, x, y, z) = a(z^2 - \frac{1}{2}(x^2+y^2)) + b(z(x^2+y^2))$$

and (suppressing the α direction by symmetry) its exterior derivative is

$$d\Phi^\xi = (2bz - a)xdx + (2bz - a)ydy + (2az + b(x^2+y^2))dz$$

For $(a,b) \neq (0,0)$, the critical set is described by two cases:

$$\text{crit}(\Phi^\xi) = \begin{cases} \{x = y = z = 0\} & a \neq 0 \\ \{x = y = 0\} & a = 0 \end{cases}$$

So in both cases, the critical set is a manifold. In these coordinates, the Hessian of Φ^ξ is

$$H\Phi^\xi = \begin{pmatrix} 2bz - a & 0 & 2bx \\ 0 & 2bz - a & 2by \\ 2bx & 2by & 2a \end{pmatrix}$$

For the case $a \neq 0$, along the critical set $\{x = y = z = 0\}$ (i.e. on C) we have

$$H\Phi^\xi = \begin{pmatrix} -a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 2a \end{pmatrix} \qquad (4.2.1)$$

which is non-degenerate along the normal bundle (which is all of D^3), so Φ^ξ is Morse-Bott. For the case $a = 0$, along the critical set $\{x = y = 0\}$ (i.e. on the line bundle) we have

$$H\Phi^\xi = \begin{pmatrix} 2bz & 0 & 0 \\ 0 & 2bz & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (4.2.2)$$

In this case the Hessian is non-degenerate in the normal directions (here being the x-y plane) everywhere except at $z = 0$, and as z crosses zero the stable manifold become unstable, so Φ^ξ is not Morse-Bott. Note that the z-axis maps under Φ to the folded double p_1 axis, and that the change in sign corresponds to the inside normal to the image changing from down to up pointing as z passes zero.

Recall, a neighborhood of each component of Z_ω is equivariantly symplectomorphic to the standard fold, up to an integral affine transformation of the moment map and torus. So in general, the moments Φ^ξ fail to be Morse-Bott exactly for the finite set of ξ for which $\xi \in \mathrm{span}(A\frac{\partial}{\partial q_2})$, where q_2 is the standard coordinate on T^2 and and $A \in \mathrm{GL}(2, \mathbb{Z})$ is the integral transformation of T^2 corresponding to the choice of splitting of T^2 to give the symplectomorphism to the standard fold near one of the components of Z_ω. Note however that this vector is, by construction of the symplectomorphism, always in the Lie algebra $\mathfrak{g}_x$ of the isotropy group G_x, for $x \in C$ a component of Z_ω, so is in fact invariantly defined.

The failure of Φ^ξ to be Morse-Bott for this small set of ξ might not be so bad if the other Φ^ξ behaved nicely. However, equation 4.2.1, shows that, depending on the sign of a, some of the components C of Z_ω will have stable manifolds of dimension 3, i.e. codimension 1. The absence of such stable manifolds is exactly what was required to show uniqueness of local maxima. In effect, the stable manifolds of the components of Z_ω form hypersurfaces which separate the stable manifolds of the local maxima.

We can draw this hypersurface easily in the case $b = 0$, using the flat metric on $S^1 \times D^3$. Then the gradient is (suppressing the α direction by symmetry)

$$\nabla\Phi^\xi = (-ax, -ay, 2az) \tag{4.2.3}$$

so the stable manifold of $S^1 \times \{0\}$ is the set $\{z = 0\}$. This manifold is T^2 invariant, so can be described by its image under Φ_0, which is the *negative* p_1-axis. This means that the separating hypersurface divides the manifold in a way that discounts local maxima/minima due to "going around a fold". This is illustrated in Figure 4-2. Choosing $b \neq 0$ curves the line up or down but maintains its tangency to the p_1-axis at the origin.

For examples of near-symplectic toric 4-manifolds whose moment maps have non-

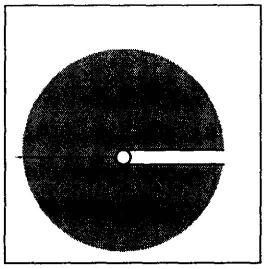

Figure 4-2: The separating hypersurface for $b = 0$, shown as the dotted line.

convex images, see the construction in 4.3.2 and the final section.

4.3 Classification theorems

In this section I would like to extend the classification Delzant's theorem provides to near-symplectic manifolds. The proof of uniqueness will follow with very little modification from that of Lerman-Tolman [9] presented in Chapter 2. The existence proof will be a consequence of the existence of the local models developed so far, similar to the patching argument of Gay-Symington ([4]), rather than the standard global existence proof for the Delzant theorem.

Because of possible overlaps in the image of the moment map Φ, due both to folds and to the failure of global convexity, for the purpose of this section the analogue of the Delzant polytope Δ is the following abstract object:

Definition 4.3.1. A *folded Delzant polygon* is a triple (B, F, ϕ), where B is a surface with corners, $F \subset (\partial B \setminus \{\text{corners}\})$ a discrete set, and $\phi : B \to \mathbb{R}^2$ a map that's a fold near F, an immersion on $B \setminus F$, takes edges to line segments with rational slopes, and whose image satisfies the smoothness property of the Delzant theorem near the corners.

Remark. Here the "smoothness property of the Delzant theorem" is a strict version that requires the images of the corners to be the standard corner up to *orientation-preserving* integral affine transformations; this implies local convexity at the corners.

Remark. By "a fold near F", we mean that there exist coordinates on B near F such that there ϕ is of the form $A \circ \phi_0 + b$, where $A \in \mathrm{GL}(2, \mathbb{Z}), b \in \mathbb{R}^2$, and ϕ_0 is the standard fold. In the paragraph below, the surface B will be the quotient M/T.

For some examples of folded Delzant polygons, see Figures 4-3 and 4-8, which show the images $\phi(B) \subset \mathbb{R}^2$. Note that the immersions ϕ may fail to be 1-to-1 even away from the folds, and that the images may have an arbitrary number of "holes".

4.3.1 Uniqueness

By the canonical forms developed up to now, for any near-symplectic toric 4-manifold (M, ω) satisfying Condition 4.1.1 the orbit space M/T^2, the vanishing locus Z_ω, and the map ϕ defined by $\Phi = \phi \circ \pi$ where $\pi : M \to M/T^2$ is the quotient, define a folded Delzant polygon $(M/T^2, \pi(Z_\omega), \phi)$.

We can state the uniqueness theorem, an analogue of Theorem 2.2.4, as follows.

Theorem 4.3.1. *Let $(M_1, \omega_1), (M_2, \omega_2)$ be two compact, connected, near-symplectic toric 4-manifolds satisfying Condition 4.1.1 with moment maps Φ_1, Φ_2. Let $\pi_i : M_i \to B_i$ be the quotients to the orbit spaces, let $F_i = \pi_i(Z_{\omega_i})$ be the images of the vanishing locii, and define $\phi_i : B_i \to \mathbb{R}^2$ by $\Phi_i = \phi_i \circ \pi_i$. If there is a diffeomorphism $\psi : (B_1, F_1) \to (B_2, F_2)$ such that $\phi_2 \circ \psi = \phi_1$, then there exists a T^2-equivariant symplectomorphism $\Psi : M_1 \to M_2$ such that $\Phi_2 \circ \Psi = \Phi_1$.*

Proof. The proof is an adaptation of propositions 2.2.5 to 2.2.12 as follows. Propositions 2.2.5 and 2.2.6 are true at a point $x \in F$ by the model for the standard fold, and still true elsewhere.

To check Proposition 2.2.7, we need to show that the map Λ is well-defined, i.e. that the vector field X_f can be defined over F. Write $\omega_A = -2z(d\alpha dz + rdrd\theta) - rdrd\alpha + r^2 d\theta dz$. Let $p_1 = z^2 - \frac{1}{2}r^2$, $p_2 = zr^2$ be coordinates on the base. Then a general 1-form ν on the base can be written $\nu = adp_1 + bdp_2$. Solving $\iota(X)\omega_A = \pi^*(\nu)$, we obtain $X = -a\frac{\partial}{\partial \alpha} + b\frac{\partial}{\partial \theta}$. For $f \in C^\infty(B)$, $\nu = df$, have $a = \frac{\partial f}{\partial p_1}, b = \frac{\partial f}{\partial p_2}$.

Here there is a technicality to worry about: if f is just some smooth function on the base in the coordinates (z, r^2), there's no guarantee that $a = \frac{\partial f}{\partial p_1}, b = \frac{\partial f}{\partial p_2}$ are well defined, or even bounded, at F. However, if we require that f is a smooth function on $\mathfrak{t}^*$ near $\phi(F)$, they are well-defined. So this fact requires a redefinition of the sheaf $\tilde{C^\infty}$ near Z_ω.

Given the above, since $\frac{\partial}{\partial\theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}$, X_f extends over F and there it is of the form $X_f = -a\frac{\partial}{\partial\alpha}$. Given this, the rest of the proposition generalizes by continuity.

The remaining propositions showing exactness of the sequence of sheaves go through unchanged.

The final step is to use the long exact sequence in sheaf cohomology to show that $H^1(B, \mathcal{H}) = 0$. It is no longer necessarily the case that B is contractible, or even simply-connected. However, the relevant portion of the long exact sequence is:

$$H^1(B, C^\infty) \to H^1(B, \mathcal{H}) \to H^2(B, \ell \times \mathbb{R})$$

The left hand term is still zero since C^∞ is flabby. I claim that $H^2(B, \ell \times \mathbb{R}) = 0$: Since B is a surface with non-empty boundary, it's homotopy-equivalent to a 1-complex, so it can be covered by contractible sets $\{U_\alpha\}$ such that no three intersect and each double intersection is contractible. This means that the Čech cohomology $\check{H}^2(B, \ell \times \mathbb{R})$ of this cover is trivially zero, and the corresponding sheaf cohomology is as well (see, eg., R. O. Wells, *Differential Analysis on Complex Manifolds*, p.64). □

4.3.2 Existence

The proof in this section follows ideas of Gay-Symington [4] and Symington [11].

Theorem 4.3.2. *Let (B, F, ϕ) be a folded Delzant polygon. Then there exists a near-symplectic toric 4-manifold M with moment map $\Phi = \phi \circ \pi$, orbit space B, and vanishing locus $\pi^{-1}(F)$, where π is the quotient by the T^2 action.*

Proof. Consider the 4-manifold with boundary given by $\tilde{M} = B \times T^2$. Set $\tilde{\Phi} = \phi \circ \tilde{\pi}$, where $\tilde{\pi}$ is projection on the first factor. Define a 2-form on $\tilde{M}$ by $\tilde{\omega} = \tilde{\Phi}^*(dp_1) \wedge dq_1 + \tilde{\Phi}^*(dp_2) \wedge dq_2$, where (q_1, q_2) are standard coordinates on T^2 and (p_1, p_2) are standard coordinates on $\mathbb{R}^2$. Note that on $\pi^{-1}(\text{int}(B)) = \text{int}(\tilde{M})$, $\tilde{\omega}$ is symplectic, and the natural T^2 action given by multiplication in the second factor is Hamiltonian with moment map $\tilde{\Phi}$.

Now, define another manifold $M = \tilde{M}/\sim$ which is constructed by collapsing the fibres above the boundary ∂B as follows:

1. For $x \in \partial B \setminus (F \cup \{\text{corners}\})$, collapse the T^2 fibre by taking the quotient by the S^1 subgroup generated by the 1-dimensional subspace of $\mathfrak{t}$ whose annihilator is parallel to the image of $d\phi_x$.
2. For $x \in F$, collapse the T^2 fibre by taking the quotient by the S^1 subgroup generated by the 1-dimensional subspace of $\mathfrak{t}$ whose annihilator is parallel to the image under ϕ of the edge that x lies on.
3. For $x \in \{\text{corners}\}$, collapse the entire T^2 fibre.

The models developed in Chapter 2 guarantee that steps 1 and 3 can be done preserving the manifold structure, T^2 action, moment map, and symplectic form, such that the form on the quotient is symplectic there. The model developed at the beginning of this chapter guarantees that step 2 can be done preserving the manifold structure, T^2 action, moment map, and symplectic form, such that the symplectic form on the quotient vanishes over F. □

4.4 Example(s)

In this section I analyze the near symplectic manifold corresponding to an example folded Delzant polygon in detail. Consider the folded Delzant polygon illustrated in Figure 4-3. Here we show the image $\phi(B) \subset \mathfrak{t}^*$, and label the vertices with their coordinates. B is four-sided polygon with a single fold, indicated by the circle in the figure. We draw two parallel lines to indicated the edges in the fold, though they really overlap.

We can assemble the near-symplectic manifold given by Section 4.3.2 from a series of models, as follows. First, consider the space $\mathbb{C}P^2 = (\mathbb{C}^3 \setminus \{0\})/\sim$, where $\sim$ is multiplication by non-zero scalars, with the T^2 action induced from the T^2 action on

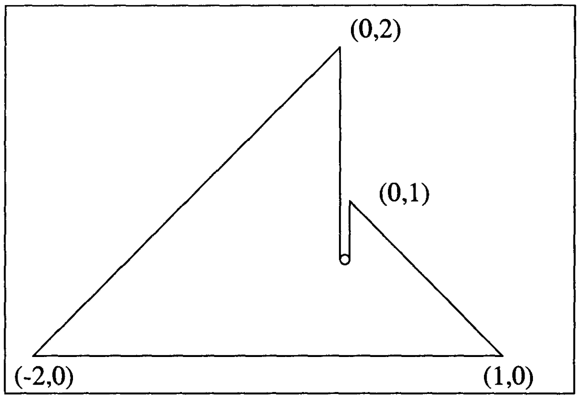

Figure 4-3: The example, $\mathbb{C}P^2 \# \mathbb{C}P^2$

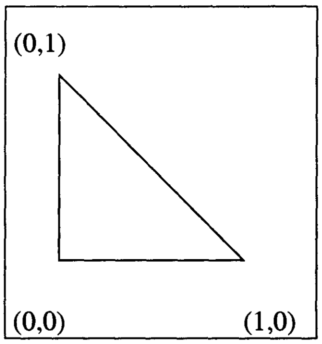

Figure 4-4: The image $\phi(\mathbb{C}P^2) \subset \mathbb{R}^2$

$\mathbb{C}^3$ given by $(t_1, t_2) \cdot (z_1, z_2, z_3) = (e^{it_1} z_1, e^{it_2} z_2, z_3)$. In homogeneous coordinates where $z_3 \neq 0$, define the moment map $\phi : \mathbb{C}P^2 \to \mathbb{R}^2$ to be

$$\phi([z_1, z_2, 1]) = (\frac{|z_1|^2}{1 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{1 + |z_1|^2 + |z_2|^2}) \tag{4.4.1}$$

Note that this moment map has image the right triangle shown in Figure 4-4, with vertices at $\{(0,1), (1,0), (0,0)\}$, and in the angle coordinates induced by the torus action, the collapsing of the T^2 fibres over the edges is as described in Section 4.3.2.

We calculate, for $j, k \in \{1, 2\}, j \neq k$, (p_1, p_2) coordinates on $\mathbb{R}^2$,

$$\phi^*(dp_j) = \frac{1}{(|z_1|^2 + |z_2|^2 + 1)^2}((|z_k|^2 + 1)(z_j d\bar{z}_j + \bar{z}_j dz_j) - |z_j|^2(z_k d\bar{z}_k + \bar{z}_k dz_k)) \tag{4.4.2}$$

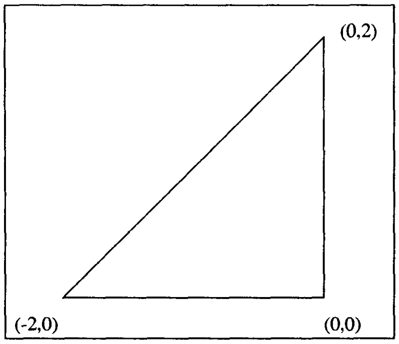

Figure 4-5: The image $\tilde{\phi}(\mathbb{C}P^2) \subset \mathbb{R}^2$

Setting $z_j = r_j e^{2\pi i \theta_j}$, we have

$$d\theta_j = -\frac{i}{2}\frac{\bar{z}_j dz_j - z_j d\bar{z}_j}{|z_j|^2} \tag{4.4.3}$$

So the symplectic form corresponding to this moment map is

$$\begin{aligned} d\theta_1 \wedge \phi^*(dp_1) + d\theta_2 \wedge \phi^*(dp_2) =& \frac{-i}{(|z_1|^2+|z_2|^2+1)^2}[(|z_2|+1)dz_1 \wedge d\bar{z_1} + (|z_1|^2+1)dz_2 \wedge d\bar{z_2}] \\ &+ \frac{-i}{(|z_1|^2+|z_2|^2+1)^2}[z_1\bar{z_2}d\bar{z_1} \wedge dz_2 - \bar{z_1}z_2 dz_1 \wedge d\bar{z_2}] \end{aligned} \tag{4.4.4}$$

This is the Fubini-Study form ω_{FS} on $\mathbb{C}P^2$, so we don't have to check that it extends symplectically to $\cup_{j=1}^3\{z_j = 0\}$.

Similarly, consider the space $\mathbb{C}P^2$ with the T^2 action induced by the action on $\mathbb{C}^3$ given by $(t_1, t_2) \cdot (z_1, z_2, z_3) = (e^{-it_1}z_1, e^{it2}z_2, z_3)$, and moment map

$$\tilde{\phi}([z_1, z_2, 1]) = (\frac{-2|z_1|^2}{1+|z_1|^2+|z_2|^2}, \frac{2|z_2|^2}{1+|z_1|^2+|z_2|^2}) \tag{4.4.5}$$

Note that this moment map has image the right triangle shown in Figure 4-5, with vertices at $\{(0,2), (-2,0), (0,0)\}$.

The new T^2 action gives a new angular coordinate $\tilde{\theta_1} = -\theta_1$, where $z_1 = r_1 e^{2\pi i \theta_1}$. Combined with the new moment map $\tilde{\phi}$, the sign changes cancel and this gives the induced symplectic form

$$d\tilde{\theta_1} \wedge \tilde{\phi}^*(dp_1) + d\theta_2 \wedge \tilde{\phi}^*(dp_2) = 2\omega_{FS} \tag{4.4.6}$$

which is twice the Fubini-Study form, so again it extends symplectically to $\cup_{j=1}^{3}\{z_j = 0\}$.

The last model we need is the space $\mathbb{C} \times \mathbb{R} \times S^1$. Let (r, θ) be polar coordinates on $\mathbb{C}$, x a coordinate on $\mathbb{R}$, and α a coordinate on S^1. Let T^2 act by $(t_1, t_2) \cdot (r, \theta, x, \alpha) = (r, \theta + t_2, x, \alpha + t_1)$. The moment map $\phi(r, \theta, x, \alpha) = (x, r^2)$ induces the symplectic form $d\theta_1 \wedge \phi^*(dp_1) + d\theta_2 \wedge \phi^*(dp_2) = d\alpha \wedge dx + d\theta \wedge 2rdr$ on $\mathbb{C} \times \mathbb{R} \times S^1$.

We construct the example space as follows. We remove from $(\mathbb{C}P^2, \omega_{FS})$ the ball corresponding to $\phi_1 + \phi_2 < 5/8$, i.e. set

$$M_A = (\mathbb{C}P^2, \omega_{FS}) \setminus \{\frac{|z_1|^2}{1 + |z_1|^2 + |z_2|^2} + \frac{|z_2|^2}{1 + |z_1|^2 + |z_2|^2} < 5/8\} \tag{4.4.7}$$

Similarly, remove a ball from $(\mathbb{C}P^2, 2\omega_{FS})$ by setting

$$M_B = (\mathbb{C}P^2, 2\omega_{FS}) \setminus \{\frac{2|z_1|^2}{1 + |z_1|^2 + |z_2|^2} + \frac{2|z_2|^2}{1 + |z_1|^2 + |z_2|^2} < 5/8\} \tag{4.4.8}$$

We restrict the model $\mathbb{C} \times \mathbb{R} \times S^1$ to the set $r^2 < 3/8$, $r^2 + x < 3/4$, $r^2 - x < 3/4$ to obtain the manifold M_C.

Finally, we rotate the standard fold by ninety degrees, shift up by $1/2$, and restrict to the set $p_2 > 2/8, p_1 + p_2 < 3/4$, $-p_1 + p_2 < 3/4$ to obtain the manifold M_D.

The last step is identifying the four manifolds over the strips indicated in Figure 4-6. Solid lines indicate edges over which the T^2-fibres are partially or completely (at corners) collapsed, while dashed lines indicate the open sets where patching will take place. We patch M_A, M_C, and M_D on $5/8 < p_1 + p_2 < 3/4$, M_B, M_C, and M_D on $5/8 < -p_1 + p_2 < 3/4$, and M_C and M_D on $1/4 < p_2 < 3/8$, via the coordinates $(p_1, \theta_1, p_2, \theta_2)$ in which all four manifolds have symplectic form $d\theta_1 \wedge dp_1 + d\theta_2 \wedge dp_2$. Since all the models are symplectic, the fibres are collapsed the same way over the edges, and the identification is a symplectomorphism on the open dense set where the

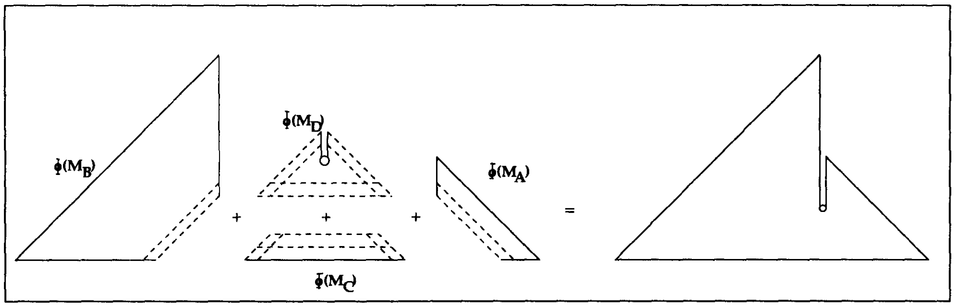

Figure 4-6: Patching the models

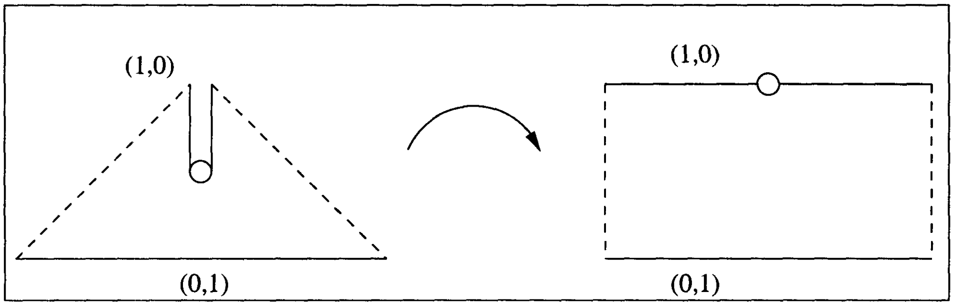

Figure 4-7: The patch, $S^3 \times [0,1]$, with collapsing directions labeled

fibres are T^2, the identification is a symplectomorphism everywhere.

We claim that the resulting manifold is $\mathbb{C}P^2 \# \mathbb{C}P^2$: We've connected two copies of $\mathbb{C}P^2 \setminus \{\text{ball}\}$ using a patch whose image under the moment map is given on the left side of Figure 4-7. (Here each edge is labelled with the generator of the subgroup which is collapsed in the fibre.) The patch is diffeomorphic to a trival T^2-fibration over $[0,1] \times [0,1]$, on which we collapse the fibres over $[0,1] \times \{0\}$ along the first S^1 factor and we collapse the fibres over $[0,1] \times \{1\}$ along the second S^1 factor, as shown on the right of Figure 4-7. This fibration, finally, is diffeomorphic to $S^3 \times [0,1]$, as can be seen by considering the model $S^3 \times [0,1] = \{(z_1, z_2, x) \in \mathbb{C}^2 \times \mathbb{R} \,|\, |z_1|^2 + |z_2|^2 = 1, x \in [0,1]\}$ with the T^2-action $(t_1, t_2) \cdot (z_1, z_2, x) = (e^{2\pi i t_1} z_1, e^{2\pi i t_2} z_2, x)$ and the projection $(z_1, z_2, x) \mapsto (x, |z_1|^2)$. The patching corresponds to that for the connected sum.

The construction in Section 4.3.2 guarantees the existence of near-symplectic toric 4-manifolds of a wide variety which may be less familiar. Some folded Delzant polygons giving rise to such manifolds are given in Figure 4-8.

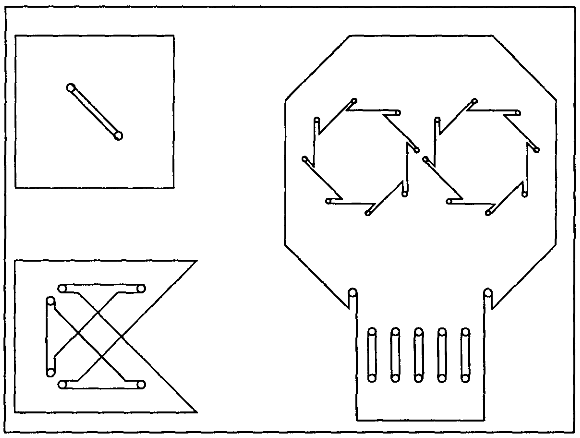

Figure 4-8: Assorted folded Delzant polygons. The left two come from [4].

Appendix A

Construction of invariant metrics

Given a near-symplectic G-invariant form ω, the following argument, explained to me by Denis Auroux and based on the non-equivariant argument in [2], p.63, guarantees the existence of a G-invariant Riemannian metric ρ_ω with respect to which ω is self-dual. Note that transverse vanishing is independent of the metric, so this guarantees that Condition 4.1.1 is always satisfied.

Consider a vector space $V \cong \mathbb{R}^4$ with a fixed postive-definite inner product ρ_0 and a given orientation. The Hodge-$*$ operator corresponding to ρ_0 and the Hodge inner-product on $\wedge^2 V$, $\langle \eta, \nu \rangle = \eta \wedge *\nu$, induce an orthogonal splitting of $\wedge^2 V = \wedge^2_{+,0} \oplus \wedge^2_{-,0}$ into the ρ_0-self-dual and anti-self-dual 2-forms. For any other positive-definite inner product ρ_i on V, its corresponding space of self-dual 2-forms $\wedge^2_{+,i}$ is a 3-plane in $\wedge^2 V$ on which the wedge-product restricts to a positive-definite bilinear form. Any such 3-plane can be written uniquely as the graph $P = \{\alpha + L_i(\alpha), \alpha \in \Lambda^2_{+,0}\}$ of a linear map $L_i : \wedge^2_{+,0} \to \wedge^2_{-,0}$ with operator norm less than 1. Conversely, any such linear map L_i defines a positive definite inner product ρ_i on V up to scaling, by specifying its space of self-dual 2-forms. Note that this space of maps is convex.

On the 4-manifold M with a G-action and a G-invariant near-symplectic form ω, choose any G-invariant Riemannian metric ρ_0. This induces a G-invariant splitting of the bundle of 2-forms $\wedge^2(T^*M) = \wedge^2_{+,0}(T^*M) \oplus \wedge^2_{-,0}(T^*M)$. Because ω is near-symplectic, it's self-dual with respect to some other (non-invariant) Riemannian metric ρ_1. By the discussion above, the bundle of ρ_1's self-dual forms is the graph of a section $L_1 \in \mathrm{Hom}(\wedge^2_{+,0}(T^*M), \wedge^2_{-,0}(T^*M))$ with pointwise operator norm less than 1, and ω is in this graph. Since the splitting $\wedge^2(T^*M) = \wedge^2_{+,0}(T^*M) \oplus \wedge^2_{-,0}(T^*M)$ is G-invariant, we can average L_1 over G, using the convexity above, to obtain a G-equivariant section $\tilde{L_1} \in \mathrm{Hom}(\wedge^2_{+,0}(T^*M), \wedge^2_{-,0}(T^*M))$, which still has pointwise operator norm less than 1, and has G-invariant graph. Since ω is G-invariant, ω is still in this graph. The graph of $\tilde{L_1}$ defines a conformal class of metrics having it as their bundle of self-dual 2-forms. Take one such metric $\tilde{\rho}_1$. Since $\tilde{L_1}$ has G-invariant graph, averaging $\tilde{\rho}_1$ over G preserves the conformal class and produces a G-invariant Riemannian metric ρ_ω on M having ω as a self-dual 2-form.

Appendix B

Hamiltonian S^1 actions

One obvious direction for further work is to see whether a generalization of Karshon's classification of 4-manifolds with Hamiltonian S^1-actions is true in the near-symplectic case, and in particular whether her result that "isolated fixed points implies toric variety" is true.

Honda's local model ω_B for the unoriented splitting provides a local counterexample to the toric claim, in that it has a Hamiltonian S^1 action but no T^2 action. I do not know if there is a compact example containing an unoriented splitting, or if the presence of unoriented splittings is the only obstruction. In any case, the analysis using the equivariant Honda-Moser theorems used in Chapter 4 provides a first step to generalizing Karshon's results, by describing local models for an S^1 action in a neighbourhood of Z_ω. In particular, I claim the following:

Let $x \in C$, a component of Z_ω.

(1) If x has trivial stabilizer, then a neighbourhood of C is equivariantly symplectomorphic to the standard fold, with Hamiltonian S^1 action given by the moment $\Phi_1 = z^2 - \frac{1}{2}r^2$.

(2) If x has S^1 stabilizer, then a neighbourhood of C is equivariantly symplectomorphic to the standard fold, with Hamiltonian S^1 action given by the moment $\Phi_2 = zr^2$.

(3) If x has stabilizer $\mathbb{Z}_k \subset S^1$, and the splitting is oriented, a neighbourhood of C is equivariantly symplectomorphic to the standard fold with Hamiltonian S^1 action

given by the moment $k\Phi_1 + \Phi_2$.

(4) If the splitting is unoriented, x must have stabilizer $\mathbb{Z}_2$, and a neighborhood of C is equivariantly symplectomorphic to the unoriented model ω_B with S^1 action given by rotation in α (i.e. along C) and moment map $z^2 - \frac{1}{2}r^2$, which is well-defined on the quotient.

The proofs are similar to the proof of Theorem 4.1.3. Given these local models, their local Morse theory is as described in Section 4.2. This is very different from the symplectic case as analyzed by Karshon and will require a different analysis.

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