Orbispaces

by

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Abstract

In this thesis, I introduce a new definition for orbispaces based a notion of stratified fibration and prove it's equivalence with other existing definitions. I study the notion of orbispace structures on a given stratified space. I then set up two parallel theories of stratified fibrations, one for topological spaces, and one for simplicial sets.

Modulo a technical comparison between the two theories, I construct a classifying space for orbispace structures. Using a conjectural obstruction theory, I then prove that every compact orbispace is equivalent to the quotient of a compact space by the action of a compact Lie group.

Thesis Supervisor: Michael J. Hopkins
Title: Professor of Mathematics

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Contents

Chapter 1

Introduction

This thesis naturally splits in two parts.

In the first part, we introduce a new, easily understandable definition for orbispaces (the topological analogue of orbifolds). From our point of view, an orbispace is a pair of spaces $E \to X$, whose fibers are classifying spaces of finite groups. The local model, being the Borel construction $(Y \times EG)/G$ mapping to the topological quotient Y/G (also known as the coarse moduli space).

Other equivalent notions, such as stacks, topological groupoids or complexes of groups are well known. So in some sense, we are only add a new item to this list. In the first chapter of the thesis, we explain all these approaches in more detail, and provide constructions to go from one to the other. Later, in chapter 3, we prove the equivalence of our new definition at the level of 2-categories.

The map from $(Y \times EG)/G$ to Y/G is not a fibration since the homotopy type of the fibers varies according to the size of the stabilizer group. We introduce a new notion of stratified fibration, which is adapted to this situation. We then show that an orbispace $E \to X$ is a stratified fibration in our sense.

Finally, we explain how one might use our definition of orbispaces in various circumstances. More specifically, we treat of the question of bundle theory, group actions, sheaf theory, and a little bit of elementary algebraic topology.

In the second part of this thesis, we describe a theory of stratified simplicial sets. Though we do not prove this here, we expect that the theories of stratified spaces and stratified simplicial sets correspond and thus we freely translate results between the two worlds.

The goal of this second half is, among other things, to outline the proof of the global quotient conjecture. It states that every compact orbispace is a quotient of a compact space by a compact Lie group. On our way to proving this conjecture, we introduce a few new notions of independent interest:

We define a stratified simplicial set to be a simlicial set equipped with a map to the nerve of that poset. We then introduce a new conjectural model structure on that category. We give an explicit construction of a stratified classifying space $\underline{\mathsf{Orb}}$ for orbispace structures on a stratified space. The simplicial set $\underline{\mathsf{Orb}}$ is stratified by the poset of isomorphism classes of finite groups. It comes with a universal orbispace structure $E_{\underline{\mathsf{Orb}}} \to \underline{\mathsf{Orb}}$, and has the property that homotopy classes of stratified maps into $\underline{\mathsf{Orb}}$ are in bijective correspondence with isomorphism classes of orbispace structures.

Given a topological group K and a family of subgroups $\mathcal{F}$, we show that $B_{\mathcal{F}}K$, the well known classifying space for $\mathcal{F}$, is a stratified classifying space for the structure of "being of the form Y/K", where Y is a K-space with stabilizers in $\mathcal{F}$. Given a compact orbispace $E \to X$, we then use obstruction theory to show that the representing map $X \to \underline{\mathsf{Orb}}$ always lifts to a map $X \to B_{\mathcal{F}}K$, for some appropriate K and $\mathcal{F}$. In other words, every orbispace is a global quotient. The group K can be taken to be a large unitary group.

Towards the end, we explain an interesting connection between vector bundles on orbispaces and global quotients by Lie groups. In particular, we show the the excision propoerty for K-theory is equivalent to the global quotient problem.

We also study global quotient by more general topological groups. We introduce the notion of a group which is contractible with respect to a family of subgroups. Then we show that if K is contractible with respect to its finite subgroups, and if every finite groups occurs as a subgroup of K, then the categories of K-spaces and of orbipsaces are homotopy equivalent.

Chapter 2

Survey of existing definitions

In the next chapter, we will introduce a new definition for orbifolds and orbispaces. Before doing so, we survey the most important existing definitions and their interrelations.

The word "orbispace" can take different meanings, depending on which underlying category of "spaces" one works in, and which groups one allows. Morally, an orbispace is something that looks locally like the quotient of a space by a group. If we take "space" to mean manifold and "group" to mean finite group, this results in the usual notion of an orbifold [13][25]. If we interpret "space" to mean algebraic variety, we recover the notion of Deligne-Mumford stack [6][22]. If on the other hand we work with simplicial complexes, then we recover the notion of complex of groups of Bridson and Heafliger [4].

Remark 2.1 We should warn the reader that sometimes the definition of orbifolds is stated in a way that requires them to be "effective" (for example [32][34]). Namely, they have to be be modeled by the quotient of a manifold by the action of a finite group acting *faithfully*. We shall stay away from this requirement which we find unnatural for our setting.

Remark 2.2 In this chapter, we use topological spaces as our category of "spaces". But we only consider those spaces arising as geometric realizations of simplicial complexes. So for us, a space will always come along with a triangulation.

Remark 2.3 Unlike geometers, when topologists glue two spaces together (for example when attaching a cell) they do not identify *open* subsets of these spaces. To account for this, we will modify the usual notion of cover. For us, a cover will always mean a cover by *closed* subspaces[1]. For example the intervals $[0,1]$ and $[1,2]$ form a cover of $[0,2]$.

We now survey the most important definitions of orbispaces existing in the literature, and their connection to each other. Since the definitions listed below have their origin in some different category of "spaces", they been modified to fit the world of topology. We have tried to state them in a way that is as close as possible from the original setting, and hope that the reader will be able to adapt them back to the context (s)he is working in.

Definition 2.4 (topological groupoids) *A topological groupoid consists of two spaces $\mathcal{G}_0$ and $\mathcal{G}_1$ called objects and morphisms along with maps $s, t : \mathcal{G}_1 \to \mathcal{G}_0$ called source and target, a map $u : \mathcal{G}_0 \to \mathcal{G}_1$ called unit, and a map $\mu : \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1$ called multiplication (all these maps are compatible with the triangulations). These maps satisfy the usual axioms for groupoids: unit, associativity, existence of inverse. We use the symbol $\mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ to denote the groupoid $(\mathcal{G}_0, \mathcal{G}_1, s, t, u, \mu)$. Given a point $x \in \mathcal{G}_0$ the group $s^{-1}(x) \cap t^{-1}(x)$ is denoted $\mathrm{Aut}(x)$ and is called the automorphism group of the object x.*

An orbispace is a topological groupoid, where all objects $x \in \mathcal{G}_0$ have finite automorphism groups and where the image of u is a connected component of $\mathcal{G}_1$.

Definition 2.5 (stacks) *A stack is a functor F (in the sense of bicategories [2]) from the category of spaces to the category of groupoids. For any cover $\{V_i\}$ of a space T, the map*

$$F(T) \to \varprojlim \Big[\coprod F(V_{ijk}) \substack{\leftarrow\\ \leftarrow\\ \leftarrow} \coprod F(V_{ij}) \leftleftarrows \coprod F(V_i) \Big] \tag{2.1}$$

is an equivalence of groupoids. Here V_{ij} and V_{ijk} denote the double and triple inter-

[1]More precisely, our Grothendieck topology is generated by proper surjective maps

sections of V_i and the limit is taken in the bicategorical sense.

To a space X we associate the stack $Y(X)$ given by $Y(X)(T) := \mathrm{Hom}(T, X)$, where the set $\mathrm{Hom}(T, X)$ is viewed as a discrete groupoid. To a group G we associate a stack BG given by $BG(T) = \{G\text{-principal bundles on } T\}$.

A map of stacks $f : F \to F'$ is representable if for any point $p \in F(pt)$, the induced map $\mathrm{Aut}(p) \to \mathrm{Aut}(f(p))$ is injective. Let us call orbisimplex a stack of the form $Y(\Delta^n) \times BG$, where Δ^n denotes the n-simplex and G is a finite group.

An orbispace is a stack $F = \varinjlim F^{(n)}$, where each $F^{(n)}$ is obtained from the previous one by taking a pushout

$$\begin{array}{ccc} \coprod \partial\tau_i & \longrightarrow & \coprod \tau_i \\ {\scriptstyle \coprod \alpha_i}\downarrow & & \downarrow \\ F^{(n-1)} & \longrightarrow & F^{(n)}. \end{array} \tag{2.2}$$

Here the τ_i are orbisimplices $Y(\Delta^n) \times BG_i$, with boundaries $\partial\tau_i = Y(\partial\Delta^n) \times BG_i$. The attaching maps $\alpha_i : \partial\tau_i \to F^{(n-1)}$ are required to be representable.

Definition 2.6 (complexes of groups) *A complex of groups is a space X along with the following data. To each simplex σ_i of X, we associate a group G_i. To each incidence $\sigma_i \supset \sigma_j$ we associate a group homomorphism $\phi_{ij} : G_i \to G_j$. To each double incidence $\sigma_i \supset \sigma_j \supset \sigma_k$ we associate a group element $g_{ijk} \in G_k$ satisfying*

$$\phi_{ik} = Ad(g_{ijk})\, \phi_{jk}\, \phi_{ij}. \tag{2.3}$$

Finally, for each triple incidence $\sigma_i \supset \sigma_j \supset \sigma_k \supset \sigma_\ell$ the above elements must satisfy the cocycle condition

$$g_{ij\ell}\; g_{jk\ell} = g_{ik\ell}\; \phi_{k\ell}(g_{ijk}). \tag{2.4}$$

An orbispace is a complex of groups where the G_i are finite and the ϕ_{ij} are injective.

Definition 2.7 (charts) *An orbispace atlas on a space X is a cover $\{U_i\}$, closed under finite intersections, along with the following data. For each U_i, we are given a "branched covering" $\pi_i : \widetilde{U}_i \to U_i$, with an action $G_i \circlearrowright \widetilde{U}_i$ by a finite group G_i. The maps π_i are invariant under the action of G_i and induce homeomorphisms between*

$G_i\backslash\widetilde{U}_i$ and U_i. For each inclusion $U_i \subset U_j$, we are given injective group homomorphisms $\phi_{ij} : G_i \to G_j$ and ϕ_{ij}-equivariant maps $\alpha_{ij} : \widetilde{U}_i \to \widetilde{U}_j$ satisfying $\pi_i = \pi_j \circ \alpha_{ij}$. For any point $x \in \widetilde{U}_i$, the homomorphisms ϕ_{ij} induces an isomorphism between the G_i-stabilizer of x and the G_j-stabilizer of its image $\alpha_{ij}(x) \in \widetilde{U}_j$. For each double inclusion $U_i \subset U_j \subset U_k$, we are given group elements $g_{ijk} \in G_k$ satisfying

$$\alpha_{ik} = g_{ijk}\alpha_{jk}\alpha_{ij} \tag{2.5}$$

and the cocycle conditions (2.3) and (2.4).

An orbispace is a space X along with an orbispace atlas $(\{U_i\}, \{\widetilde{U}_i\}, \pi_i, \alpha_{ij}, \phi_{ij}, g_{ijk})$.

2.1 Comparison between definitions

We now explain how to go from any one of the above definitions to another. This will be done in the following order:

$$\begin{array}{ccc} \textbf{(groupoids)} & \longrightarrow & \textbf{(stacks)} \\ \uparrow & & \downarrow \\ \textbf{(charts)} & \longleftarrow & \textbf{(complexes of groups)}. \end{array}$$

2.1.1 From groupoids to stacks

To go from a topological groupoid $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ to a stack $F = F(\mathcal{G})$, first consider the prestack $\widetilde{F}$ given by $\widetilde{F}(T) = \big(\mathrm{Hom}(T, \mathcal{G}_1) \rightrightarrows \mathrm{Hom}(T, \mathcal{G}_0)\big)$, and then stackify $\widetilde{F}$ to get F. More precisely, this is done by letting

$$F(T) = \varinjlim_{\{V_i\}} \left[\varprojlim \left[\coprod \widetilde{F}(V_{ijk}) \mathrel{\substack{\leftarrow\\[-1ex]\leftarrow\\[-1ex]\leftarrow}} \coprod \widetilde{F}(V_{ij}) \leftleftarrows \coprod \widetilde{F}(V_i) \right] \right],$$

where the colimit is taken over all covers $\{V_i\}$ of T ordered by refinement, and the limit is as described in Definition 2.5.

Proposition 2.8 *The stack F is of the desired form.*

Proof. Let $\sigma \subset \mathcal{G}_0$ be an n-simplex. The assumption on the image of u implies that $s^{-1}(\sigma)$ is a disjoint union of n-simplices. To see that, we show that paths $\gamma : [0,1] \to \sigma$ satisfy the unique lifting property. Consider two lifts $\tilde{\gamma}_1, \tilde{\gamma}_2 : [0,1] \to s^{-1}(\sigma)$ with same initial point $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0)$. The path $\delta(t) := \tilde{\gamma}_1(t)\tilde{\gamma}_2(t)^{-1}$ satisfies $\delta(0) \in \mathrm{Im}(u)$. Since $\mathrm{Im}(u)$ is a connected component on $\mathcal{G}_1$, all of δ lands in $\mathrm{Im}(u)$ and therefore $\tilde{\gamma}_1 = \tilde{\gamma}_2$. The same argument shows that t maps n-simplices to n-simplices.

At this point, let us barycentrically subdivide $\mathcal{G}_0$ and $\mathcal{G}_1$. Now all simplices have an order on their vertices and the structure maps preserve that order.

An n-simplex $\sigma \subset \mathcal{G}_0$ determines a full subgroupoid $\underset{\sim}{\sigma} \subset \mathcal{G}$ given by $\underset{\sim}{\sigma}_0 = t(s^{-1}(\sigma))$ and $\underset{\sim}{\sigma}_1 = s^{-1}(\underset{\sim}{\sigma}_0)$. Since s and t map n-simplices to n-simplices and since all structure maps preserve a given order on the vertices of these simplices, $\underset{\sim}{\sigma}$ is just the direct product of a discrete groupoid $\underset{\sim}{G}$ with the simplex Δ^n. Clearly $\underset{\sim}{G}$ is connected, so it is equivalent to the groupoid $G \rightrightarrows pt$, where $G = \mathrm{Aut}(x)$ for some point $x \in \sigma$. Let $\partial\underset{\sim}{\sigma}$ be the groupoid made from of all the boundaries of all the n-simplices of $\underset{\sim}{\sigma}_0$ and $\underset{\sim}{\sigma}_1$.

The n-skeleton $\mathcal{G}^{(n)} = (\mathcal{G}_0^{(n)} \rightrightarrows \mathcal{G}_1^{(n)})$ is obtained from the $(n-1)$-skeleton by a pushout diagram :

$$\begin{array}{ccc} \coprod \partial\underset{\sim}{\sigma} & \longrightarrow & \coprod \underset{\sim}{\sigma} \\ {\scriptstyle \coprod \alpha}\downarrow & & \downarrow \\ \mathcal{G}^{(n-1)} & \longrightarrow & \mathcal{G}^{(n)}. \end{array} \tag{2.6}$$

Letting $F^{(n)}$ be the stack represented by $\mathcal{G}^{(n)}$, we get from (2.6) the desired pushout of stacks (2.2). Indeed, Δ^n represents $Y(\Delta^n)$, $\underset{\sim}{G} \simeq (G \rightrightarrows pt)$ represents BG, and therefore $\underset{\sim}{\sigma}$ represents $Y(\Delta^n) \times BG$ as desired.

Now we explain why the attaching maps $\alpha : \partial\underset{\sim}{\sigma} \to \mathcal{G}^{(n-1)}$ are representable. In the language of groupoids, we need to show that they induce monomorphisms on the automorphism groups of objects. This is actually quite trivial trivial since both $\partial\underset{\sim}{\sigma}$ and $\mathcal{G}^{(n-1)}$ are subgroupoids of $\mathcal{G}$ and α is just the inclusion. □

2.1.2 From stacks to complexes of groups

Given a stack F, let $X := \pi_0 F(pt)$ be the underlying space of our complex of groups. For each closed orbisimplex $\sigma_i \subset F$, denote by n_i its dimension and by G_i its isotropy group. That is, we have $\sigma_i \simeq Y(\Delta^{n_i}) \times BG_i$. These are the groups that decorate the simplices of X. For each σ_i we have

$$\sigma_i(\Delta^{n_i}) = \mathrm{Hom}(\Delta^{n_i}, \Delta^{n_i}) \times \{G_i\text{-principal bundles on } \Delta^{n_i}\}.$$

Let $p_i \in \sigma_i(\Delta^{n_i})$ be the object corresponding to $\mathrm{Id}_{\Delta^{n_i}} \times$ (trivial bundle). Note that $\mathrm{Aut}(p_i) \simeq G_i$. For each face σ_j of σ_i in F, we let σ_i^j be the corresponding "abstract face" of σ_i, namely $\sigma_i^j \simeq Y(\Delta^{n_j}) \times BG_i$. It maps to σ_j via the attaching map of σ_i. We let $p_i^j \in \sigma_j(\Delta^{n_j})$ be the image of p_i under the composite $\sigma_i(\Delta^{n_i}) \to \sigma_i(\Delta^{n_j}) \supset \sigma_i^j(\Delta^{n_j}) \to \sigma_j(\Delta^{n_j})$, where the first arrow is $\sigma_i(\Delta^{n_j} \hookrightarrow \Delta^{n_i})$ and the last one is the attaching map of σ_i restricted to σ_i^j. We note that $\sigma_i(\Delta^{n_j} \hookrightarrow \Delta^{n_i})(p_i)$ lands in the essential image of the full subgroupoid $\sigma_i^j(\Delta^{n_j})$ of $\sigma_i(\Delta^{n_j})$, so p_i^j is well defined up to unique isomorphism.

We let $[\sigma_i] := \pi_0\ \sigma_i(pt)$ denote the simplex of X corresponding to σ_i. For each incidence $[\sigma_i] \supset [\sigma_j]$, we note that p_i^j is isomorphic to p_j in $\sigma_j(\Delta^{n_j})$. We pick such an isomorphism $\varphi_{ij} : p_i^j \to p_j$ and let

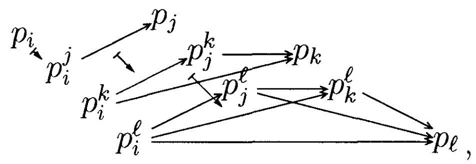
$$\phi_{ij} : G_i = \mathrm{Aut}(p_i) \longrightarrow \mathrm{Aut}(p_i^j) \xrightarrow{\varphi_{ij}(\ )\varphi_{ij}^{-1}} \mathrm{Aut}(p_j) = G_j.$$

These are the group homomorphisms that are part of the data of our complex of groups. They are injective because the attaching maps are representable.

A double incidence $[\sigma_i] \supset [\sigma_j] \supset [\sigma_k]$ leads to three morphisms $\varphi_{ij} : p_i^j \to p_j$, $\varphi_{jk} : p_j^k \to p_k$, and $\varphi_{ik} : p_i^k \to p_k$. Let $\varphi_{ij}^k : p_i^k \to p_j^k$ denote the restriction of φ_{ij} to Δ^{n_k}. We then define $g_{ijk} := \varphi_{ik}(\varphi_{ij}^k)^{-1}\varphi_{jk}^{-1} \in G_k = \mathrm{Aut}(p_k)$. These are the group elements that come in the definition of our complex of groups. It is straightforward to verify the condition $\phi_{ik} = Ad(g_{ijk})\,\phi_{jk}\,\phi_{ij}$ from the definition of ϕ_{ij} and g_{ijk}.

Now we need to show the cocycle identity for the g_{ijk}. A triple incidence $[\sigma_i] \supset$

$[\sigma_j] \supset [\sigma_k] \supset [\sigma_\ell]$ leads to a diagram



where the arrows "$\longrightarrow$" are the morphisms φ and the arrows "$\mapsto$" denote restriction functors between groupoids. Each triangle in the above tetrahedron corresponds to a g_{ijk}, and the cocycle relation is then clear:

$$\begin{aligned} g_{ij\ell}\, g_{jk\ell} &= \left[\varphi_{i\ell}(\varphi_{ij}^\ell)^{-1}\varphi_{j\ell}^{-1}\right]\left[\varphi_{j\ell}(\varphi_{jk}^\ell)^{-1}\varphi_{k\ell}^{-1}\right] \\ &= \left[(\varphi_{i\ell}\varphi_{ik}^\ell)^{-1}\varphi_{k\ell}^{-1}\right]\varphi_{k\ell}\left[\varphi_{ik}^\ell(\varphi_{ij}^\ell)^{-1}(\varphi_{jk}^\ell)^{-1}\right]\varphi_{k\ell}^{-1} = g_{ik\ell}\ \phi_{k\ell}(g_{ijk}). \end{aligned}$$

Note that we have mapped $g_{ijk} = \varphi_{ik}(\varphi_{ij}^k)^{-1}\varphi_{jk}^{-1} \in \mathrm{Aut}(p_k)$ to the corresponding element $\varphi_{ik}^\ell(\varphi_{ij}^\ell)^{-1}(\varphi_{jk}^\ell)^{-1}$ of $\mathrm{Aut}(p_k^\ell)$.

2.1.3 From complexes of groups to charts

Let X be a complex of groups. Given a simplex σ in X, we let U_σ be the union of all simplices τ of the barycentric subdivision of X such that $\sigma \cap \tau$ is the barycenter of σ. The intersection $U_\sigma \cap U_{\sigma'}$ is either empty or equal to $U_{\sigma''}$, where σ'' is the simplex whose vertices is the union of the vertices of σ and of σ'. So the U_σ for a cover of X which is closed under finite intersections.

For a simplex σ_i, let us use U_i instead of U_{σ_i}. Also, we let $G_i^j := \phi_{ij}(G_i)$. For each U_k, we define the corresponding $\widetilde{U}_k$ to be the quotient

$$\widetilde{U}_k := \Big(\bigcup_{\sigma_j \supset \sigma_k} (\sigma_j \cap U_k) \times G_k/G_j^k \Big) \Big/ \sim\,,$$

where the equivalence relation is generated by $(x, mG_i^k) \sim (x, m\, g_{ijk} G_j^k)$. The spaces $\widetilde{U}_j$ admit a left action of G_j given by $h \cdot (x, mG_i^j) = (x, hmG_i^j)$ and the map π_j : $(x, mG_i^j) \mapsto x$ induces a homeomorphism $G_j \backslash \widetilde{U}_j \simeq U_j$. The ϕ_{ij} and g_{ijk} required for Definition 2.7 are taken identical to those in Definition 2.6.

The α's are given by $\alpha_{jk}(x, mG_i^j) := (x, \phi_{jk}(m)g_{ijk}^{-1}G_i^k)$. To see that they are well defined we take two representatives (x, mG_i^k) and $(x, m\, g_{ijk}G_j^k)$ of the same point of $\widetilde{U}_k$ and check using (2.4) that their values agree:

$$\alpha_{k\ell}(x, mG_i^k) = \big(x, \phi_{k\ell}(m)g_{ik\ell}^{-1}G_i^\ell\big) \sim \big(x, \phi_{k\ell}(m)g_{ik\ell}^{-1}\, g_{ij\ell}G_j^\ell\big) \quad \text{and}$$
$$\alpha_{k\ell}(x, m\, g_{ijk}G_j^k) = \big(x, \phi_{k\ell}(m\, g_{ijk})g_{jk\ell}^{-1}G_j^\ell\big).$$

One then checks the ϕ_{jk}-equivariance:

$$\begin{aligned}
\alpha_{jk}\big(h \cdot (x, mG_i^j)\big) &= \alpha_{jk}\big(x, hmG_i^j\big) \\
&= (x, \phi_{jk}(hm)\, g_{ijk}^{-1}\, G_i^k) \\
&= \big(x, \phi_{jk}(h)\, \phi_{jk}(m)\, g_{ijk}^{-1}\, G_i^k\big) \\
&= \phi_{jk}(h) \cdot \big(\alpha_{jk}(x, mG_i^j)\big)
\end{aligned}$$

and the compatibility with the g_{ijk}:

$$\begin{aligned}
g_{jk\ell}\, \alpha_{k\ell}\, \alpha_{jk}(x, mG_i^j) &= \big(x, g_{jk\ell}\, \phi_{k\ell}(\phi_{jk}(m)g_{ijk}^{-1})g_{ik\ell}^{-1}G_i^\ell\big) \\
&= \big(x, g_{jk\ell}\, \phi_{k\ell}\, \phi_{jk}(m)\phi_{k\ell}(g_{ijk}^{-1})g_{ik\ell}^{-1}G_i^\ell\big) \\
&= \big(x, \phi_{j\ell}(m)g_{jk\ell}(g_{ik\ell}\, \phi_{k\ell}(g_{ijk}))^{-1}G_i^\ell\big) \\
&= \big(x, \phi_{j\ell}(m)g_{ij\ell}^{-1}G_i^\ell\big) \\
&= \alpha_{j\ell}(x, mG_i^j).
\end{aligned}$$

Finally, we need to check that ϕ_{ij} induces an isomorphism between the stabilizer of a point and that of its image under α_{ij}. Recall that G_i^j denotes $\phi_{ij}(G_i)$.

Lemma 2.9 *If x is in the interior of σ_k, then the stabilizer of $(x, G_k^\ell) \in \widetilde{U}_\ell$ under the action of G_ℓ is exactly G_k^ℓ.*

Proof. Let $\approx$ denote the relation given by $(x, mG_j^\ell) \approx (x, m'G_{j'}^\ell)$ if the element $(m\, g_{jk\ell})^{-1}\, m'\, g_{j'k\ell}$ belongs to G_k^ℓ, where k is chosen such that $x \in \sigma_k \setminus \partial\sigma_k$. We claim that $\approx$ is the equivalence relation generated by $\sim$. This implies the statement about the stabilizer of (x, G_k^ℓ) since $\approx$ doesn't identify distinct points of $\{x\} \times G_\ell/G_k^\ell$.

We first show that $\approx$ extends $\sim$. For two points $(x, mG_i^\ell) \sim (x, m\, g_{ij\ell} G_j^\ell)$. We check by (2.4) that $(m\, g_{ik\ell})^{-1} m\, g_{ij\ell}\, g_{jk\ell} \in G_k^\ell$ and therefore $(x, mG_i^\ell) \approx (x, m\, g_{ij\ell} G_j^\ell)$. Clearly $\approx$ is reflexive and symmetric, so we check transitivity. Consider the situation $(x, mG_j^\ell) \approx (x, m' G_{j'}^\ell) \approx (x, m'' G_{j''}^\ell)$. Since both group elements $(m\, g_{jk\ell})^{-1} m'\, g_{j'k\ell}$ and $(m'\, g_{j'k\ell})^{-1} m''\, g_{j''k\ell}$ lie in G_k^ℓ, the same holds for their product, and we deduce that $(x, mG_j^\ell) \approx (x, m'' G_{j''}^\ell)$. □

By the above Lemma, the G_j-stabilizer of a point $(x, mG_i^j) \in \widetilde{U}_j$ is exactly $m\, G_i^j\, m^{-1} \subset G_j$, provided that x sits in the interior of σ_i. The stabilizer of its image $\alpha_{jk}(x, mG_i^j) = (x, \phi_{jk}(m) g_{ijk}^{-1} G_i^k)$ is then equal to $\phi_{jk}(m) g_{ijk}^{-1} G_i^k\, g_{ijk}\, \phi_{jk}(m)^{-1}$. We now check using (2.3) that ϕ_{jk} induces isomorphisms between these two groups:

$$\begin{aligned}
\phi_{jk}\big(mG_i^j m^{-1}\big) &= \phi_{jk}\big(m\, \phi_{ij}(G_i) m^{-1}\big) \\
&= \phi_{jk}(m) \phi_{jk}(\phi_{ij}(G_i)) \phi_{jk}(m)^{-1} \\
&= \phi_{jk}(m) g_{ijk}^{-1}\, \phi_{ik}(G_i) g_{ijk}\, \phi_{jk}(m)^{-1} = \phi_{jk}(m) g_{ijk}^{-1} G_i^k\, g_{ijk}\, \phi_{jk}(m)^{-1}.
\end{aligned}$$

2.1.4 From charts to groupoids

Let X be an orbispace given by charts $U_i \simeq G_i \backslash \widetilde{U}_i$. We construct a topological groupoid $\mathcal{G}$ as follows. Its object space is given by $\mathcal{G}_0 = \coprod \widetilde{U}_i$. Its arrows are generated by elements $[h] : x \to h \cdot x$ and $[\alpha_{ij}] : x \to \alpha_{ij}(x)$ subject to the relations

$$[h][h'] = [hh'], \qquad [\alpha_{ij}][h] = [\phi_{ij}(h)][\alpha_{ij}], \quad \text{and} \quad [\alpha_{ik}] = [g_{ijk}][\alpha_{jk}][\alpha_{ij}]. \tag{2.7}$$

These generate a topological groupoid $\mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ whose orbit space $\mathcal{G}_0/\mathcal{G}_1$ is homeomorphic to X. We have implicitly used that $\alpha_{ij} h = \phi_{ij}(h) \alpha_{ij}$ and $\alpha_{ik} = g_{ijk} \alpha_{jk} \alpha_{ij}$ since otherwise, (2.7) would not be equations between morphisms sharing the same source and target.

We need to show that the automorphism groups of objects are finite and that the image of the unit map u is clopen of $\mathcal{G}_1$. Both these facts will be a direct consequence of the following proposition:

Proposition 2.10 *For $x, y \in \widetilde{U}_i \subset \mathcal{G}_0$, the only morphisms $x \to y$ are of the form*

$[h] : x \to h \cdot x = y$. *Moreover, two morphisms* $[h], [h'] : x \to y$ *are equal in* $\mathcal{G}_1$ *if and only if* $h = h'$ *in* G_i. *In other words*

$$s^{-1}(\widetilde{U}_i) \cap t^{-1}(\widetilde{U}_i) = G_i \times \widetilde{U}_i. \tag{2.8}$$

First we show that the automorphism groups are finite. Let $x \in \widetilde{U}_i$. By Proposition 2.10, all the self arrows $x \to x$ are given by elements of G_i. Since G_i is finite, we conclude that $\mathrm{Aut}(x)$ is finite. Now we explain why $Im(u)$ is clopen in $\mathcal{G}_1$. The image of u is the disjoint union of $Im(u|_{\widetilde{U}_i})$. By (2.8), each one is clopen in $s^{-1}(\widetilde{U}_i) \cap t^{-1}(\widetilde{U}_i)$. Since the $s^{-1}(\widetilde{U}_i) \cap t^{-1}(\widetilde{U}_i)$ are the intersections of two clopen subsets of $\mathcal{G}_1$, $Im(u|_{\widetilde{U}_i})$ is clopen in $\mathcal{G}_1$. We conclude that $Im(u)$ is clopen.

Proof of proposition 2.10. Let $\pi : \mathcal{G} \to X$ denote the projection. If $\pi(x) \neq \pi(y)$ there are no morphisms between x and y and so there is nothing to show. If $\pi(x) = \pi(y)$ then x and y lie in the same G_i-orbit, and so there exists an arrow $[h_0] : x \to y$, where $h_0 \in G_i$. By composing with $[h_0^{-1}]$, we can restrict ourselves to the case $x = y$.

Consider the inclusion ι of the stabilizer group G_x of x, into the groupoid $\mathcal{G}_x := \pi^{-1}(\pi(x)) \subset \mathcal{G}$. The first statement of our proposition claims that ι is full, and the second one the ι is faithful. So we are reduced to proving the following lemma:

Lemma 2.11 *The inclusion functor* $\iota : G_x \to \mathcal{G}_x$ *is an equivalence of groupoids.*

Proof. We construct a retraction $r : \mathcal{G}_x \to G_x$ and a natural transformation ν from $\imath \circ r$ to the identity. Recall that $x \in \widetilde{U}_i$. For each object y of $\mathcal{G}_x$ with $y \in \widetilde{U}_j$ we pick an object $z \in \widetilde{U}_k$, where $U_k = U_i \cap U_j$, and two group elements $a \in G_i$ and $b \in G_j$ such that $a\,x = \alpha_{ki}(z)$ and $y = b\,\alpha_{kj}(z)$. This data can be pictured as follows :

$$x \xrightarrow{[a]} \alpha_{ki}(z) \xleftarrow{[\alpha_{ki}]} z \xmapsto{[\alpha_{kj}]} \alpha_{kj}(z) \xrightarrow{[b]} y, \tag{2.9}$$

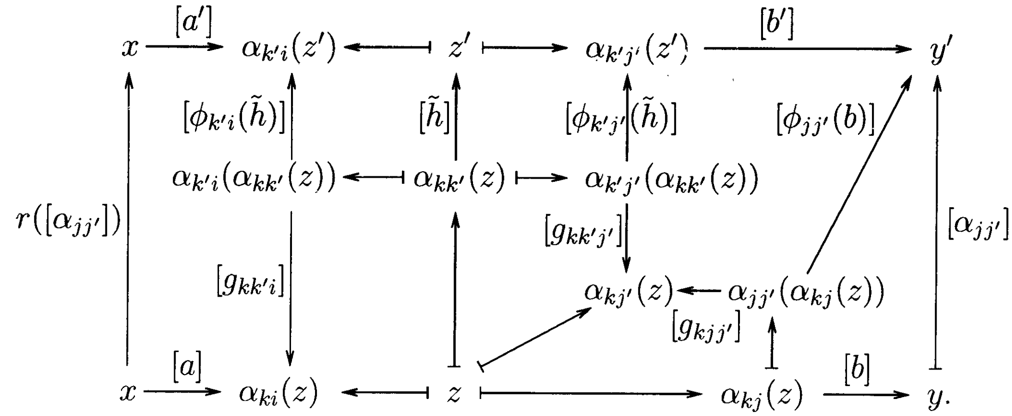
where we use the arrows "$\to$" and "$\mapsto$" to distinguish between the two kinds of generators for $\mathcal{G}_1$.

We now define $r : \mathcal{G}_x \to G_x$. At the level of objects, r sends everything to x. For the arrows of the form $[h] : y \to y' = h \cdot y$ we let z, a, b and z', a', b' be the choices corresponding to y and y' respectively. The arrow $r([h])$ is the only possible one that makes the following diagram commutative:

$$\begin{array}{ccccccccc}
x & \xrightarrow{[a']} & \alpha_{ki}(z') & \xleftarrow{[\alpha_{ki}]} & z' & \xrightarrow{[\alpha_{kj}]} & \alpha_{kj}(z') & \xrightarrow{[b']} & y' \\
\uparrow{\scriptstyle r([h])} & & \uparrow{\scriptstyle [\phi_{ki}(\tilde{h})]} & & \uparrow{\scriptstyle [\tilde{h}]} & & \uparrow{\scriptstyle [\phi_{kj}(\tilde{h})]} & & \uparrow{\scriptstyle [h]} \\
x & \xrightarrow[{[a]}]{} & \alpha_{ki}(z) & \xleftarrow[{[\alpha_{ki}]}]{} & z & \xrightarrow[{[\alpha_{kj}]}]{} & \alpha_{kj}(z) & \xrightarrow[{[b]}]{} & y.
\end{array} \tag{2.10}$$

More precisely, since ϕ_{kj} induced isomorphisms on stabilizers, there is a unique element $\tilde{h} \in G_k$ such that $\phi_{kj}(\tilde{h}) = b'^{-1} h\, b$. We then let $r([h]) := [a'^{-1}\phi_{ki}(\tilde{h})\, a]$.

To define $r([\alpha_{jj'}])$, we proceed similarly. Let z, a, b and z', a', b' be the choices corresponding to y and y'. The diagram that we want to make commutative now looks like this:

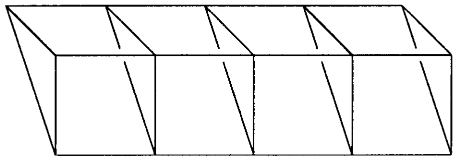


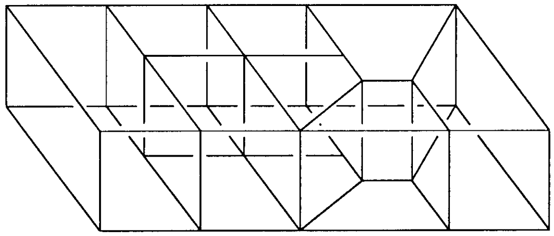
(2.11)

As before, there is a unique element $\tilde{h} \in G_{k'}$ satisfying $\phi_{k'j'}(\tilde{h}) = b'^{-1}\phi_{jj'}(b)\, g_{kjj'}^{-1} g_{kk'j'}$, and we let $r([\alpha_{jj'}]) = [a'^{-1}\phi_{k'i}(\tilde{h})\, g_{kk'i}^{-1} a]$.

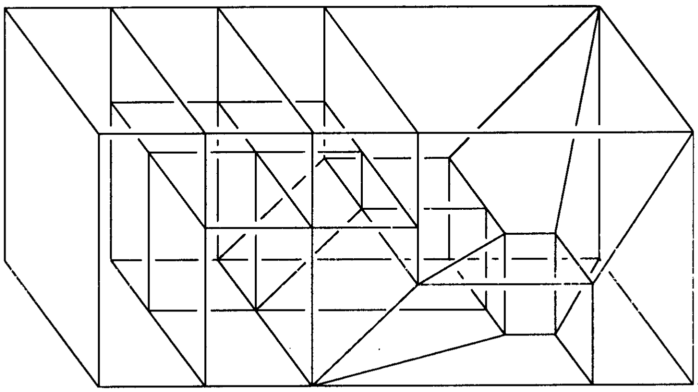
We now go back to (2.9) to define our natural transformation $\nu : \imath \circ r \to 1$. It is given by $\nu_y := [b][\alpha_{kj}][\alpha_{ki}]^{-1}[a] : x \to y$. Using (2.10) and (2.11), it is immediate to verify that ν is natural with respect to the morphisms $[h]$ and $[\alpha_{jj'}]$.

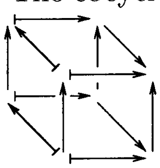
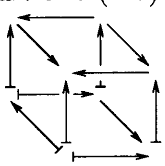
So far, we have not used the cocycle conditions (2.3) and (2.4). These are needed to verify that r is well defined, namely that $r([h])$ and $r([\alpha_{jj'}])$ satisfy the three relations (2.7). This is done by a careful diagram chase, where the diagrams look like this:

,

,

and

.

In all these cases, we start with the data that the rightmost face commutes, and slowly work our way to the left until we show that the leftmost face commutes. For each 3-cell, we use one of the properties of ϕ_{ij} and g_{ijk} in order to show that all the 2-faces commute. The cocycle conditions (2.3) and (2.4) are used when encountering cells of the form  and  respectively. □

Chapter 3

Orbispaces from the point of view of the Borel construction

We will now describe our new definition of orbispaces and relate it to the existing definitions discussed in Chapter 1. Parts of this chapter rely on the technology of Chapter 4.

Recall that the Borel construction of a G-space Y is the quotient $(Y \times EG)/G$ of the product of Y with a contractible space EG that has a free action of G. The Borel construction is also called the homotopy quotient of Y by G. An explicit model for EG is provided by the geometric realization of the simplicial space $\cdots G^3 \substack{\rightarrow\\\rightarrow\\\rightarrow} G^2 \rightrightarrows G$, where the face maps are the projections omitting a factor and the degeneracy maps repeat an entry. Using this particular model of EG, we get a model for the Borel construction as the geometric realization of the nerve of the groupoid $Y \times G \rightrightarrows Y$. Indeed, the product $Y \times EG$ is the realization of the nerve of the groupoid $Y \times G^2 \rightrightarrows Y \times G$. Since quotients commutes with geometric realizations therefore

$$(Y \times EG)/G = |Y \times G^2 \rightrightarrows Y \times G|/G = |Y \times G^2/G \rightrightarrows Y \times G/G| = |Y \times G \rightrightarrows Y|.$$

Note that $Y \times EG$ could have been replaced by any free G-space $\tilde{Y}$ admitting an equivariant map to Y which is an acyclic fibration. Hence forward, it is this more general construction that we shall call the Borel construction.

Given a topological group K and an action $Y \circlearrowleft K$ whose stabilizers are finite, we let $[Y/K]$ be the orbispace quotient (see example 3.2). The Borel construction $\tilde{Y}/G$ remembers a large amount of the orbispace homotopy type of $[Y/G]$. For example, the ordinary cohomology of $[Y/G]$ is nothing else than the cohomology of $\tilde{Y}/G$. However, the K-theory of $K^*[Y/G] = K^*_G(Y)$ is typically not isomorphic to $K^*(\tilde{Y}/G)$. It is instead the completion of $K^*[Y/G]$ at the ideal consisting of virtual vector bundles of dimension zero. However, we can keep track of the whole orbispace homotopy type if we remember the topological quotient Y/G. This leads to the following definition:

Definition 3.1 *An orbispace is an object in the following 2-category:*

An object is a map of spaces $p : E \to X$ which is locally isomorphic to the Borel construction of a finite group G acting on a space. In other words X has a cover $\{U_i\}$ by closed[1] *subspaces, where each restriction $p^{-1}(U_i) \to U_i$ is homeomorphic to the Borel construction $\tilde{Y}_i/G_i \to Y_i/G_i$ of some action $Y_i \circlearrowleft G_i$. The space E is called the total space and X the topological quotient.*

A morphism of orbispaces $(E, X) \to (E', X')$ is a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow{g} & X'. \end{array}$$

If $g = g'$, there may exist 2-morphisms $(E, X) \overset{\to}{\underset{\to}{\Downarrow}} (E', X')$ between maps (f, g) and (f', g'). A 2-morphism is a homotopy $h : E \times [0,1] \to E'$ such that $p' \circ h_t = g \circ p$ for all $t \in [0,1]$

$$\begin{array}{ccc} E & \overset{f}{\underset{f'}{\underset{\Downarrow h}{\rightrightarrows}}} & E' \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & X'\,. \end{array} \tag{3.1}$$

If $g \neq g'$, there are no 2-morphisms. Two 2-morphisms are considered to be the same if they are homotopic to each other relatively to the endpoints.

[1]If we used open covers instead, we would get an equivalent definition (see Remark 3.6).

Example 3.2 Let K be a topological group and Y a K-space with finite isotropy groups. Then $[Y/K] := \big((Y \times EK)/K \to Y/K\big)$ is an orbispace.

Note that a map $(f, g) : (E, X) \to (E', X')$ is entirely determined by $f : E \to E'$. So we will use $f : (E, X) \to (E', X')$ as a shorthand notation.

Remark 3.3 An equivalent way or recording the information is to have E foliated by the fibers of p. The leaf space of this foliation is X. We will often switch between these two equivalent ways of giving the data.

Given an orbispace $p : E \to X$, the fibers $p^{-1}(x)$ are all $K(\pi, 1)$'s, where the π are finite groups depending on the point $x \in X$. This induces a stratification of X by the isomorphism type of $\pi = \pi_1(p^{-1}(x))$.

Definition 3.4 *Let X be a space equipped with a stratification by the poset of isomorphism classes of finite groups (see Definition 4.4).*

An orbispaces structure on X is an orbispace $p : E \to X$ such that for every $x \in X$, the group $\pi_1(p^{-1}(x))$ is isomorphic to the group indexing the stratum of x.

Having the correct homotopy type of fibers is not quite enough to be an orbispace. For example, the map

$$\big(K(G, 1) \times [0, 1]\big) \Big/ (y, 1) \sim (y', 1) \longrightarrow [0, 1] \tag{3.2}$$

is not an orbispace (unless G is trivial). More generally, if M_f is the mapping cylinder of a fibration $f : K(G, 1) \twoheadrightarrow K(H, 1)$, then $M_f \to [0, 1]$ is an orbispace if and only if $\pi_1(f) : G \to H$ is injective.

The following theorem gives the exact conditions when $E \to X$ is an orbispace structure on X.

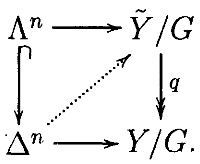
Theorem 3.5 *A map $p : E \to X$ is an orbispace if and only if the following conditions are satisfied:*

- *The map p is a stratified fibration in the sense of Definition 4.9.*

- *The fibers of p are $K(\pi, 1)$'s and their fundamental groups are all finite.*

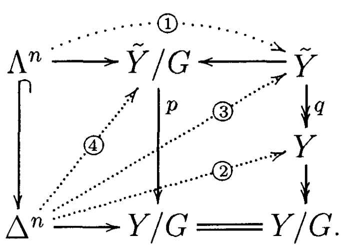
- *Let F_x and F_y be the fibers over points x, $y \in X$ and let $\gamma : [0,1] \to X$ be a directed*[2] *path from x to y. Then the corresponding map*[3] *$\nabla_\gamma : F_x \to F_y$ is injective on π_1.*

Proof. We first show that orbispaces satisfy the above conditions. All the conditions are local (for the first one, this is the content of Lemma 4.19), so we may assume that $(E, X) = (\tilde{Y}/G, Y/G)$, where $\tilde{Y}$, Y and G are as in Definition 3.1. We first check that $p : \tilde{Y}/G \to Y/G$ is a stratified fibration. Let $q : \tilde{Y} \to Y$ be the projection. Let $\Lambda^n \hookrightarrow \Delta^n$ be a generating directed cofibration and consider the lifting diagram



(3.3)

Since Λ^n is simply connected and $\tilde{Y}$ is a free G-space, we can lift $\Lambda^n \to \tilde{Y}/G$ to a map $\Lambda^n \to \tilde{Y}$. There is a lift $\Delta^n \to Y$ by Lemma 4.20. There is a lift $\Delta^n \to \tilde{Y}$ because q is a fibration. Finally we compose with the projection $\tilde{Y} \to \tilde{Y}/G$ to get our desired lift $\Delta^n \to \tilde{Y}/G$. This diagram chase is best visualized as follows:



This finishes the proof that p is a stratified fibration. Its fibers have the required homotopy type since

$$p^{-1}([y]) = p^{-1}(yG/G) = q^{-1}(yG)/G = q^{-1}(y)/\mathrm{Stab}_G(y)$$

[2]See Definition 4.22.
[3]See Lemma 4.23.

and $q^{-1}(y)$ is contractible.

We now show that ∇_γ is injective on π_1. By definition, ∇_γ is the composite $F_x \times \{1\} \hookrightarrow F_x \times [0,1] \xrightarrow{\ell} \tilde{Y}/G$ where ℓ is a lift

$$\begin{array}{ccccc} F_x \times \{0\} & & \xrightarrow{\quad \iota \quad} & & \tilde{Y}/G \\ \downarrow & & \overset{\ell}{\nearrow} & & \downarrow \\ F_x \times [0,1] & \longrightarrow & [0,1] & \xrightarrow{\gamma} & Y/G. \end{array}$$

Since ∇_γ is homotopic to ι, it's enough to show that $\iota_* : \pi_1(F_x) \to \pi_1(\tilde{Y}/G)$ is injective. Let $\hat{x} \in Y$ be a representative of x. The fiber $q^{-1}(\hat{x})$ is a universal cover for F_x. The map ι lifts to an inclusion $q^{-1}(\hat{x}) \hookrightarrow \tilde{Y}$, so ι_* in injective.

Now, let's assume that $p : E \to X$ satisfies the three conditions in the statement of the theorem. We want to show that it's an orbispace. Given a point $x \in X$, we need to find a neighborhood of U such that $p^{-1}(U) \to U$ is homeomorphic to a Borel construction $\tilde{Y}/G \to Y/G$.

Let U be a star-shaped closed neighborhood of x. By picking U small enough, we can make sure that for all points y in U, the straight path from y to x is directed. Let G be the fundamental group of $p^{-1}(U)$, let $\tilde{Y}$ be its universal cover, and let Y be the leaf space of $\tilde{Y}$ with respect to the foliation inherited from U. Since $p^{-1}(U)$ deformation retracts to F_x, we have $\pi_1(F_x) = \pi_1(p^{-1}(U)) = G$. The map $\pi_1(F_y) \to \pi_1(p^{-1}(U))$ is a monomorphism, so the preimages of F_y in $\tilde{Y}$ have contractible connected components. This proves that the fibers of the projection $q : \tilde{Y} \to Y$ are all contractible. Clearly $\tilde{Y} \circlearrowleft G$ is free and $(p^{-1}(U), U) \simeq (\tilde{Y}/G, Y/G)$.

We now show that $q : \tilde{Y} \to Y$ is a fibration. By Lemma 4.30, it's enough to show that q is a stratified fibration. Let $\Lambda^n \hookrightarrow \Delta^n$ a directed cofibration and consider the lifting problem

$$\begin{array}{ccc} \Lambda^n & \longrightarrow & \tilde{Y} \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & Y. \end{array}$$

Since p is a stratified fibration,there is a lift $\Delta^n \to p^{-1}(U)$. We can then lift it to $\tilde{Y}$

because Δ^n is simply connected. This finishes the verification that $p : E \to X$ in an orbispace. $\square$

Remark 3.6 For each point $x \in X$, we have constructed a closed neighborhood U_x such that $p^{-1}(U_x) \to U_x$ is homeomorphic to a Borel construction. The U_x form a closed cover of X as required for definition 3.1, but if we replaced U_x by their interiors, we would get an open cover. This shows that the notion of orbispace doesn't depend on the choice of open versus closed covers.

We will soon establish the equivalence of Definition 3.1 with the other definitions presented in Chapter 2. But before, we would like explain how to do various constructions one might be interested in.

3.1 Bundles and pullbacks

A bundle on an orbispace (E, X) is a bundle $P \to E$ along with a leaf-wise flat connection. Namely, for each path γ in a leaf of E, we are given an isomorphism $\nabla_\gamma : P_{\gamma(0)} \to P_{\gamma(1)}$. This isomorphism only depends on the homotopy class of γ and is compatible with composition of paths. The space of P is itself the total space of an orbispace. The topological quotient is given by $P/\sim$, where $x \sim y$ if there is a path γ with $\nabla_\gamma(x) = y$.

The pull-back of a bundle $P \to E$ along a map $f : (E, X) \to (E', X')$ is the usual pullback $f^*(P)$. The leaf-wise flat connection is given by

$$\nabla_\gamma : \big(f^*(P)\big)_x = P_{f(x)} \xrightarrow{\nabla_{f \circ \gamma}} P_{f(y)} = \big(f^*(P)\big)_y.$$

As an example, we explain how to build the tangent bundle of an orbifold.

Example 3.7 Let $p : E \to X$ be an orbifold. It is locally isomorphic to the Borel construction of a smooth action on a manifold. Recall that E comes foliated by the fibers of p. Given a point $x \in E$ we consider a small neighborhood U of x and the induced foliation on that neighborhood. If U is chosen conveniently, its leaf space $U/\sim$ is a manifold, and thus we may let T_x be the tangent space $T_{[x]}(U/\sim)$.

We first show that U may be chosen so that its leaf space is a manifold. By definition, E is locally of the form $\tilde{M}/G$ for some $\tilde{M}$ mapping to a manifold M. Pick a preimage $\hat{x} \in \tilde{M}$ of $x \in \tilde{M}/G$. The action of G on $\tilde{M}$ is proper and free, so we may pick a neighborhood V of $\hat{x}$ that doesn't intersect any of its translates by G. Let us also pick V so that its intersection with any leaf of $q : \tilde{M} \to M$ is connected. Clearly, the leaf space of V is then $q(V)$, which is a manifold. Since V does not intersect any of its translates, it is homeomorphic to its image U in M'/G. This gives us the desired neighborhood of x. If $U' \subset U$, is a smaller neighborhood of x, then $U'/\sim$ will be an open submanifold of $U/\sim$ and thus the construction doesn't depend on the choice of neighborhood.

We now construct the leaf-wise connection ∇. Clearly, it is enough to do it locally. So for each point $x \in E$ we need a neighborhood W in $p^{-1}(x)$, and for each point $x' \in W$ an isomorphism $T_x \simeq T_{x'}$. Given $x \in E$, we may pick W to be the intersection of U with the leaf of x, where U is as above. If x' is another point of W, then both T_x and $T_{x'}$ are given by $T_{[x]}(U/\sim)$, so they come with a preferred isomorphism between them: the identity.

Pulling back general maps of orbispaces is done in a slightly different way. The construction is similar to homotopy pullbacks, the only difference being that instead of arbitrary paths, one only uses those that stay in a fixed leaf. Consider the following diagram of orbispaces:

$$\begin{array}{ccccc} E' & \xrightarrow{f} & E & \xleftarrow{f'} & E'' \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p''} \\ X' & \xrightarrow{g} & X & \xleftarrow{g'} & X'' \end{array} \tag{3.4}$$

The pullback (E''', X''') is then given by

$$\begin{aligned} E''' := \big\{ (x, y, \gamma) \in E' \times E'' \times E^{[0,1]} \,\big| \\ f(x) = \gamma(0), f'(y) = \gamma(1),\ p \circ \gamma \text{ is constant} \big\} / \sim \end{aligned} \tag{3.5}$$

where $(x, y, \gamma) \sim (x, y, \gamma')$ if γ and γ' are leaf-wise homotopic relatively to their endpoints. Two points (x_0, y_0, γ_0) and (x_1, y_1, γ_1) are in the same leaf if there exists leaf-wise homotopy $(x_t, y_t, \gamma_t)_{t\in[0,1]}$ from one to the other. In (3.5), we could have omitted the operation of modding out by $\sim$. Indeed, the equivalence classes of $\sim$ are all contractible, so we would just get a different model for the pullback.

If either f or f' is a leaf-wise fibration, we can also use the usual pullback $E' \times_E E''$. The leaves are then the connected components of the pullbacks of leaves.

3.2 Group actions

Since orbispaces form a 2-category, there are two possibly different notions of actions. Let G be a topological group with multiplication $\mu : G^2 \to G$. A strict action of G on an orbispace (E, X) consists of actions on E and on X commuting with the projection. A weak action of G on (E, X) is a map $\nu : G \times (E, X) \to (E, X)$ and an associator 2-morphism $\alpha : \nu \circ (\mu \times 1) \to \nu \circ (1 \times \nu)$. The associator is required to make the following diagram commute

$$\begin{array}{ccc} \nu \circ (\mu^{(2)} \times 1) & \xrightarrow{\alpha\circ(1\times\mu\times 1)} & \nu \circ (1 \times \nu) \circ (1 \times \mu \times 1) \\ {\scriptstyle \alpha\circ(\mu\times 1\times 1)}\downarrow & & \downarrow{\scriptstyle \nu\circ(1\times\alpha)} \\ \nu \circ (\mu \times \nu) & \xrightarrow[\alpha\circ(1\times 1\times\nu)]{} & \nu \circ (1 \times \nu) \circ (1 \times 1 \times \nu), \end{array} \qquad (3.6)$$

where $\mu^{(2)} : G^3 \to G$ denotes the multiplication. Note that a weak action on (E, X) induces a usual (strict) action on X.

A map $f : (E, X) \to (E', X')$ between two orbispaces equipped with weak G-actions is said to be G-equivariant if we also have an intertwiner $\beta : f \circ \nu \to \nu' \circ (1 \times f)$

that makes the following diagram commute :

$$\begin{array}{ccc} f\circ\nu\circ(\mu\times 1) & \xrightarrow{f\circ\alpha} & f\circ\nu\circ(1\times\nu) \\ & & \downarrow{\scriptstyle \beta\circ(1\times\nu)} \\ \downarrow{\scriptstyle \beta\circ(\mu\times 1)} & & \nu'\circ(1\times f)\circ(1\times\nu) \\ & & \downarrow{\scriptstyle \nu'\circ(1\times\beta)} \\ \nu'\circ(\mu\times f) & \xrightarrow[\alpha'\circ(1\times 1\times f)]{} & \nu'\circ(1\times\nu')\circ(1\times 1\times f). \end{array} \tag{3.7}$$

Remark 3.8 The notion of weak group action is the natural specialization of the notion of A_∞-action to the world of 2-categories (as opposed to ∞-categories). We refer the reader to [23] and [3] for more details on the theory of A_∞-spaces and their actions. We could also consider weak group objects and their actions. These come with their own associator $\mu\circ(\mu\times 1)\to\mu\circ(1\times\mu)$, and so the diagram (3.6) would then be replaced by a pentagon diagram.

We now show that the notions of strict and weak actions are equivalent. This is a special case of the rectification procedure for strictifying algebras over operads [3, Thm. 4.49] [5, Sec. 18.3]. The corresponding result for stacks has appeared in [31].

Theorem 3.9 *Given a weak action (ν,α) of a topological group G on an orbispace (E,X), there exists an equivalent orbispace $(\widetilde{E},X)$ carrying a strict action $\tilde{\nu}$ of G. Moreover, the equivalence $f:(\widetilde{E},X)\to(E,X)$ is G-equivariant in the weak sense.*

Proof. Let WG be the topological monoid given as follows (see [3],[33]). Its elements consist of a collection of points $0=x_0\le x_1\le\ldots\le x_k$ on the positive real line satisfying $x_i-x_{i-1}\le 1$. Moreover, each point x_i is decorated by a group element $g_i\in G$. The points of WG are denoted $(x_0\ldots x_k;g_0\ldots g_k)$. If $x_i=x_{i+1}$, we also identify $(\ldots x_i,x_i\ldots;\ldots g_i,g_{i+1}\ldots)$ with $(\ldots x_i\ldots;\ldots g_ig_{i+1}\ldots)$. The multiplication is given by

$$(x_0\ldots x_k;g_0\ldots g_k)\cdot(x_0'\ldots x_\ell';g_0'\ldots g_\ell')=(x_0\ldots x_k,y_0\ldots y_\ell;g_0\ldots g_k,g_0'\ldots g_\ell'),$$

where $y_i=x_k+1+x_i'$. We have a natural homomorphism $p_G:WG\to G$ given by $(x_0 \dots x_k; g_0 \dots g_k) \mapsto g_0 \cdots g_k$ and a section $G \to WG$, $g \mapsto (0; g)$. This section is not a homomorphism. The map $WG \times [0,1] \to WG : ((x_0 \dots x_k; g_0 \dots g_k), t) \mapsto (tx_0 \dots tx_k; g_0 \dots g_k)$ is a deformation retraction on the image of G, hence p_G is a homotopy equivalence[4]. Since the fibers of p_G are contractible, (WG, G) is actually a (strict) monoid in the category of orbispaces.

Note that WG is a colimit $WG = \varinjlim WG^{(n)}$, where each $WG^{(n)}$ is obtained from $WG^{(n-1)}$ by gluing an "n-cell" $G^{n+1} \times [0,1]^n$ and freely generating a monoid from it. Combinatorially, $WG^{(n)}$ is the subspace of WG where at most n consecutive x_i's have distance < 1, and the "n-cell" that generates it consists of element $(x_0 \dots x_k; g_0 \dots g_k)$ with $k \leq n$.

Now, let us relate all this to our weak action of G on (E, X). We use the skeleta $WG^{(n)}$ to inductively define a strict action of WG on E. First, since $WG^{(0)}$ is free on G our map $\nu : G \times E \to E$ defines an action of $WG^{(0)}$. The associator map $\alpha : G^2 \times [0,1] \times E \to E$ is exactly what we need to extend this to an action of $WG^{(1)}$. Now, the identity (3.6) claims the existence of a fiber-wise homotopy between two maps $G^3 \times [0,1] \times E \to E$. These two maps can be assembled into a single map $G^3 \times \partial[0,1]^2 \times E \to E$, and then (3.6) claims that it extends to a map defined on the whole of $G^3 \times [0,1]^2 \times E$. Again, this is exactly what we need to extend our action to $WG^{(2)}$. Note that so far all our maps commute with the projections on $G \times X$ and on X, so we are actually on our way to defining a (strict) action of (WG, G) on (E, X) in the category of orbispaces.

To finish our construction, recall that all higher cells are of the form $G^{n+1} \times [0,1]^n$. Suppose that we have built the action of $WG^{(n-1)}$. That action provides a map $G^{n+1} \times \partial[0,1]^n \times E \to E$ and we wish to extend it over $G^{n+1} \times [0,1]^n \times E \to E$. Our

[4]In fact, WG is the canonical cofibrant replacement of G in the model category of topological monoids (assuming G is cofibrant as a space), and can be constructed as a monadic bar construction $WG = B(*, \text{Free monoid}, G)$.

extension problem looks like this:

$$\begin{array}{ccc} G^{n+1} \times \partial[0,1]^n \times E & \longrightarrow & E \\ \big\downarrow & \nearrow & \big\downarrow \\ G^{n+1} \times [0,1]^n \times E & \longrightarrow & X \end{array}$$

At this point, we note that each fiber over $G \times X$ of the inclusion $G^{n+1} \times \partial[0,1]^n \times E \hookrightarrow G^{n+1} \times [0,1]^n \times E$ is 2-connected and that the fibers of $E \to X$ are 1-truncated. So by obstruction theory (i.e. Theorem 4.29), our desired map $G^{n+1} \times [0,1]^n \times E \to E$ exists. This finishes our inductive construction of the strict action of (WG, G) on (E, X).

Now we define our orbispace $\widetilde{E}$ by

$$\widetilde{E} := (G \times EWG \times E)/WG,$$

where WG acts on $G \times EWG \times Y$ by $\tilde{g} \cdot (g, x, y) = (g\tilde{g}^{-1}, \tilde{g}x, \tilde{g}y)$. The strict action $G \circlearrowright \widetilde{E}$ is given by $h \cdot (g, x, y) = (hg, x, y)$. The space $\widetilde{E}$ comes with a G-equivariant map f to E given by $f(g, x, y) = gy$. Since $WG \simeq G$, the map $f : (\widetilde{E}, X) \to (E, X)$ is a fiber-wise homotopy equivalence, and therefore an equivalence of orbispaces.

To show that $f : (\widetilde{E}, X) \to (E, X)$ is G-equivariant in the weak sense, we introduce an auxiliary orbispace (E', X). It is given by $E' := (WG \times EWG \times E)/WG \simeq EWG \times E$ and admits strictly WG-equivariant equivalences

$$(\widetilde{E}, X) \xleftarrow{\sim} (E', X) \xrightarrow{\sim} (E, X).$$

Since strict WG-actions are the same thing as weak G-actions, one sees that these maps are weakly G-equivariant. Now we just need to check that the inverse of a weakly equivariant map is also weakly equivariant, and that the composite of two weakly equivariant maps is weakly equivariant. These routine verifications are left to the reader. □

3.3 Sheaf cohomology

A sheaf $\mathcal{F}$ on an orbispace (E, X) is a sheaf on X along with the following additional data. For each open $U \subset E$ and leaf-wise homotopy from $h : U \times [0,1] \to E$ from $h_0 = \mathrm{Id}_U$ to some map $h_1 : U \to V$ (not necessarily an inclusion) we are given a map $\mathcal{F}(h) : \mathcal{F}(V) \to \mathcal{F}(U)$. Here, by leaf-wise homotopy, we mean that for any point $y \in U$, the path $\{y\} \times [0,1]$ stays in a fixed leaf of E. The maps $\mathcal{F}(h)$ are compatible with restriction and composition of homotopies. Replacing $\mathcal{F}$ by its étale space $|\mathcal{F}| \to E$, this additional data is equivalent to a leaf-wise (flat) connection on $|\mathcal{F}|$, as considered in section 3.1. This notion of sheaf is equivalent to what is known as an étale sheaf (see [24] for a treatment of sheaves from the point of view of étale groupoids).

Let $\mathsf{Sh}(E, X)$ denote the category of sheaves on (E, X). Given a map $f : (E, X) \to (E', X')$, we have the classical operations on sheaves f^* and f_*. The pullback functor $f^* : \mathsf{Sh}(E', X') \to \mathsf{Sh}(E, X)$ is most easily defined by pulling back the étale space of the sheaf $|f^*\mathcal{F}| := f^*|\mathcal{F}|$.

The pushforward functor $f_* : \mathsf{Sh}(E, X) \to \mathsf{Sh}(E', X')$ agrees with the usual pushforward of sheaves if f is a leaf-wise fibration. Otherwise, it is defined by $f_*(\mathcal{F})(U) := \Gamma(P; p^*(\mathcal{F}))$, where P is the pullback (3.5) of $E \to E' \hookleftarrow U$ and $p : P \to X$ is the projection. We have the usual adjunction $f^* \dashv f_*$.

Given a sheaf of abelian groups $\mathcal{A}$ on (E, X), the sheaf cohomology $H^n((E, X); \mathcal{A})$ is the derived functor of global sections. The following result relates the sheaf cohomology of (E, X) to that of its total space E (see [26] and [7, chapt 6] for analogous results).

Theorem 3.10 *If (E, X) is an orbispace, $p : (E, E) \to (E, X)$ the canonical map, and $\mathcal{A}$ a sheaf of abelian groups on (E, X), then p induces a natural isomorphism of sheaf cohomology groups $H^*((E, X); \mathcal{A}) \simeq H^*((E, E); p^*\mathcal{A})$.*

Note that the sheaf cohomology of the orbispace $(E, E) = (\mathrm{Id} : E \to E)$ is the usual sheaf cohomology $H^*(E; \mathcal{A})$.

Proof. The map $p : (E,E) \to (E,X)$ may be made into a leaf-wise fibration by replacing (E,E) by (Z,E), where

$$Z := \{\gamma \in E^{[0,1]} \,|\, p \circ \gamma \text{ is constant}\}/\text{homotopy rel. endpoints.}$$

Since all the fibers of Z are contractible, (Z,E) is indeed equivalent to (E,E). Call $q : (Z,E) \to (E,X)$ the evaluation map $q(\gamma) := \gamma(1)$ and call $\pi : (E,X) \to pt$ the unique map to the point. We want to show that $H^n((E,E); f^*\mathcal{A}) = H^n((Z,E); q^*\mathcal{A}) := R^n(\pi q)_*(q^*\mathcal{A})$ is isomorphic to $H^n((E,X);\mathcal{A}) := R^n\pi_*\mathcal{A}$.

We examine the Grothendieck spectral sequence

$$R^n\pi_* R^m q_*(q^*\mathcal{A}) \Rightarrow R^{n+m}(\pi q)_*(q^*\mathcal{A}). \tag{3.8}$$

Let us consider a point $x \in E$ and look at the stalk of $R^m q_*(q^*\mathcal{A})$ at x. Call $i : \{x\} \to (E,X)$ the inclusion, $F \subset Z$ the fiber of q over x, with map $j : (F,F) \to (Z,E)$ and projection $\bar{q} : (F,F) \to \{x\}$. These all fit into the following commutative diagram:

$$\begin{array}{ccccc}
(E,E) & \xleftarrow{\sim} & (Z,E) & \xleftarrow{j} & (F,F) \\
 & \searrow^{p} & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \bar{q}} \\
pt & \xleftarrow{\pi} & (E,X) & \xleftarrow{i} & \{x\}
\end{array}$$

Note that a sheaf on (F,F) is really the same thing as a sheaf on F. And since F is a contractible the stalk at x of $R^m q_*(q^*\mathcal{A})$ can be computed:

$$\imath^* R^m q_*(q^*\mathcal{A}) = R^m \bar{q}_* \jmath^*(q^*\mathcal{A}) = H^m(F, \jmath^* q^*\mathcal{A}) = H^m(F, \bar{q}^*\imath^*\mathcal{A}) = \begin{cases} \imath^*\mathcal{A} & \text{if } \; m = 0 \\ 0 & \text{otherwise,} \end{cases}$$

where the first equality holds because localization is exact, and the last one holds because F is contractible and $\bar{q}^*\imath^*\mathcal{A}$ is a constant sheaf. So the Grothendieck spectral sequence (3.8) degenerates and thus

$$\begin{aligned} H^n((E,X),\mathcal{A}) =& R^n\pi_*\mathcal{A} = (R^n\pi_*)q_*q^*\mathcal{A} \\ =& R^n(\pi q)_*q^*\mathcal{A} = H^n((Z,E);q^*\mathcal{A}) = H^n((E,E),p^*\mathcal{A}). \end{aligned}$$

□

This proof can be interpreted in the following way. The map $(E,E) \to (E,X)$ is a fibration with contractible fibers, so the Leray spectral sequence $H^*(base; H^*(fiber)) \Rightarrow H^*(total\ space)$ degenerates and produces the required isomorphism.

3.4 Algebraic invariants

The homology, cohomology, homotopy groups of (E,X) are defined to be those of E. The universal cover of (E,X) is $(\widetilde{E}, \widetilde{E}/\sim)$, where $\widetilde{E}$ is the universal cover of E and $\widetilde{E}/\sim$ is its leaf space with respect to the foliation induced from E. A cover of (E,X) is again just a cover of E, and the usual correspondence between subgroups of $\pi_1(E,X)$ and covers of (E,X) carries over from spaces. A good treatment of the above subjects, both from the point of view of étale groupoids and from the points of view of the Borel construction, is available in Moerdijk's paper [25]. See [14] for an account of rational cohomology of orbispaces, and [4, pp 604-608] for the relationship between π_1 and covering spaces.

If X is compact, the K-theory of (E,X) is the Grothendieck group of orbi-vector bundles, as described in the section 3.1. If $(F,Y) \subset (E,X)$ is a pair (i.e. if $F = E|_Y$), a relative K-class is given by $\mathbb{Z}/2$-graded vector bundles $V = V_0 \ominus V_1$ on (E,X) and an isomorphism $f : V_0|_{(F,Y)} \to V_1|_{(F,Y)}$. The fact that this defines a cohomology theory is surprisingly tricky (see Proposition 6.11 for the proof of the excision axiom), and relies on the fact that all compact orbispaces are global quotients (Theorem 6.6). A similar attempt to define K-theory for Lie orbispaces would fail the excision axiom.

3.5 Equivalence of our definitions

This section shows the equivalence of Definition 3.1 with Definition 2.4 at the level of bicategories. To do this, we first need to complete Definition 2.4 and explain what the morphisms and the two morphisms are. This is quite tricky and can be done in two different ways.

The first approach is to take continuous functors $\mathcal{G} \to \mathcal{G}'$ as our morphisms and continuous natural transformations as our 2-morphisms. One then needs to formally invert (in a weak sense) a class of morphisms called weak equivalences (see [28], [29]). The second approach is to define a morphism $\mathcal{G} \to \mathcal{G}'$ via its "graph" $\mathcal{G}_0 \leftarrow \Gamma \to \mathcal{G}'_0$. The space Γ admits commuting actions of $\mathcal{G}_1$ and $\mathcal{G}'_1$ subject to certain conditions. A 2-morphism $\Gamma \to \Gamma'$ is then a map commuting with the actions of $\mathcal{G}_1$ and $\mathcal{G}'_1$ (see [16], [24], [27]). We will use the first approach.

3.5.1 The bicategory of topological groupoids

Let Orb be the 2-category given in Definition 3.1, and let Gpd denote the 2-category of topological groupoids satisfying the conditions of Definition 2.4, continuous functors and continuous natural transformations. We will show that Orb is equivalent to the bicategory of fractions $\mathsf{Gpd}[W^{-1}]$ introduced by Pronk [28], [29].

Definition 3.11 *A map of spaces $X \to X'$ is topologically surjective if there exists a closed cover[5] $\{U_i\}$ of X' for which each U_i admits a section $U_i \to X$.*

Definition 3.12 *A weak equivalence is a continuous functor $f : \mathcal{G} \to \mathcal{H}$ for which the map*

$$\mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \to \mathcal{H}_0 \tag{3.9}$$

[5] We work with closed covers, but the arguments are identical to those using open covers.

is topologically surjective (the functor is essentially surjective), and the diagram

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{f} & \mathcal{H}_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ \mathcal{G}_0 \times \mathcal{G}_0 & \xrightarrow{f\times f} & \mathcal{H}_0 \times \mathcal{H}_0. \end{array} \tag{3.10}$$

is a pullback (the functor is fully faithful). A weak equivalence is denoted by the symbol $\xrightarrow{\sim}$. *We let* $W \subset \mathsf{Gpd}_1$ *be the class of all weak equivalences.*

One way of constructing weak equivalences is by pulling back a groupoid $\mathcal{G}$ along a topologically surjective map $f : V \twoheadrightarrow \mathcal{G}_0$.

Lemma 3.13 *Let* $\mathcal{G}$ *be a groupoid and* $f : V \to \mathcal{G}_0$ *a topologically surjective map. Let* $f^*\mathcal{G}_1$ *be the space* $V \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} V$. *Then* $f^*\mathcal{G} := (f^*\mathcal{G}_1 \rightrightarrows V)$ *is a groupoid, and the projection functor* $f^*\mathcal{G} \to \mathcal{G}$ *is a weak equivalence.*

Proof. The multiplication $f^*\mathcal{G}_1 \times_V f^*\mathcal{G}_1 \to f^*\mathcal{G}_1$ is given by

$$V \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} V \times_V V \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} V = V \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} V \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} V$$
$$\to V \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} V \to V \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} V,$$

where the second map is the projection and the last map is the multiplication in $\mathcal{G}$. The groupoid axioms are easy to check. The map

$$\left(V \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} V\right) \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_0$$

is topologically surjective because all the maps in all the pullbacks are. The diagram

$$\begin{array}{ccc} V \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} V & \longrightarrow & \mathcal{G}_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ V \times V & \longrightarrow & \mathcal{G}_0 \times \mathcal{G}_0 \end{array}$$

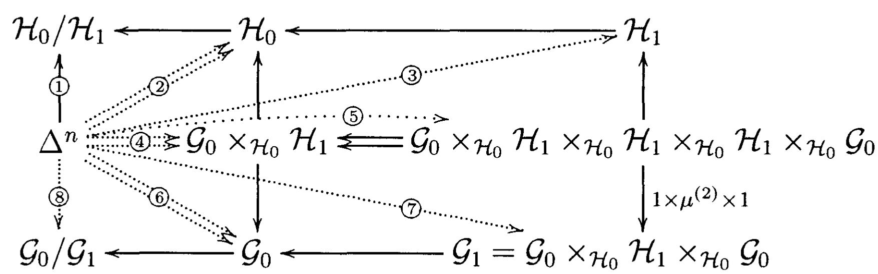
a pullback diagram. We have checked the two conditions in Definition 3.12, so the functor $f^*\mathcal{G} \to \mathcal{G}$ is indeed a weak equivalence. □

We also have the following well known result.

Lemma 3.14 *Let $f : \mathcal{G} \to \mathcal{H}$ be a weak equivalence. Then the corresponding map $\bar{f} : \mathcal{G}_0/\mathcal{G}_1 \to \mathcal{H}_0/\mathcal{H}_1$ is a homeomorphism.*

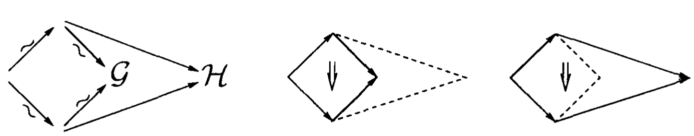
Proof. The two maps $\mathcal{H}_0 \to \mathcal{H}_0/\mathcal{H}_1$ and $\mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \to \mathcal{H}_0$ are topologically surjective. So we can triangulate $\mathcal{H}_0/\mathcal{H}_1$ such that every simplex $\sigma : \Delta^n \to \mathcal{H}_0/\mathcal{H}_1$ lifts to a map $\tilde{\sigma} : \Delta^n \to \mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1$. We define $\bar{f}^{-1}(\sigma(t))$ to be the image of $\tilde{\sigma}(t)$ in $\mathcal{G}_0/\mathcal{G}_1$. Assuming that $\bar{f}^{-1}$ is well defined, it is clear from the construction that $\bar{f}^{-1}(\bar{f}(x)) = x$ and $\bar{f}(\bar{f}^{-1}(y)) = y$.

We now show that $\bar{f}^{-1}$ is well defined. Suppose that we have two sections $\Delta^n \to \mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1$. The two maps from $\Delta^n \to \mathcal{H}_0$ agree in $\mathcal{H}_0/\mathcal{H}_1$ hence differ by a map to $\mathcal{H}_1$ (after refining the triangulation). Assemble the two maps to $\mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1$ and the map to $\mathcal{H}_1$ to a map $\Delta^n \to \mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0$. Compose with the multiplication map to get a map $\Delta^n \to \mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0$. By (3.10), $\mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0 = G_1$. We have a map $\Delta^n \to \mathcal{G}_1$ whose source and target are the two maps $\Delta^n \to \mathcal{G}_0$. Their projections in $\mathcal{G}_0/\mathcal{G}_1$ are therefore equal. This diagram chase is best visualized by the following picture:



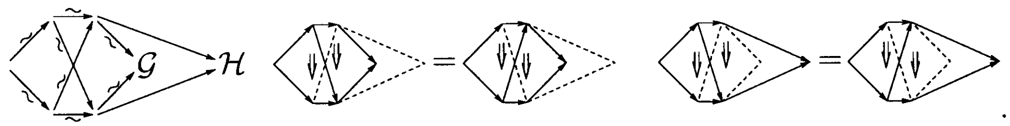
□

We now explain how the localized bicategory $\mathsf{Gpd}[W^{-1}]$ is constructed. We follow Pronk [28]. The objects of $\mathsf{Gpd}[W^{-1}]$ are those of Gpd. The morphisms of $\mathsf{Gpd}[W^{-1}]$ are diagrams of the form $\mathcal{G} \xleftarrow{\sim} \mathcal{K} \to \mathcal{H}$. Finally, the 2-morphisms of $\mathsf{Gpd}[W^{-1}]$ are equivalence classes of 2-diagrams



,

where two two such diagrams are equivalent if they fit into a bigger 2-diagram:



The various compositions in $\mathsf{Gpd}[W^{-1}]$ involve cumbersome diagrams which are detailed in [28].

3.5.2 The functor $\mathsf{Gpd}[W^{-1}] \to \mathsf{Orb}$

The bicategory $\mathsf{Gpd}[W^{-1}]$ has the following universal property shown in [28, section 3.3]. For any bicategory $\mathcal{D}$, the bicategory of functors from Gpd to $\mathcal{D}$ that send elements of W to equivalences is equivalent to the bicategory of functors from $\mathsf{Gpd}[W^{-1}]$ to $\mathcal{D}$. We now recall a result of Pronk [28], [29].

Theorem 3.15 *Let $\mathcal{C}$, $\mathcal{D}$ be bicategories and $W \subset \mathcal{C}_1$ a class of weak equivalences. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor which sends elements of W to equivalences. Then the corresponding functor $\tilde{F} : \mathcal{C}[W^{-1}] \to \mathcal{D}$ is an equivalence if and only if the following conditions hold:*

- *F is essentially surjective on objects.*
- *For every 1-morphism f in $\mathcal{D}$, there exists a $w \in W$ such that $Fg \overset{\sim}{\Rightarrow} f \circ Fw$ for some g in $\mathcal{C}_1$.*
- *F is fully faithful on 2-morphisms.* □

So, in order to show that Orb and $\mathsf{Gpd}[W^{-1}]$ are equivalent, we need to construct a (weak) functor $F : \mathsf{Gpd} \to \mathsf{Orb}$ and show that it satisfies the conditions of Theorem 3.15. A first approximation of $F(\mathcal{G})$ is given by $(|N\mathcal{G}|, \mathcal{G}_0/\mathcal{G}_1)$. This would be very convenient since F would then be a strict functor. But $|N\mathcal{G}| \to \mathcal{G}_0/\mathcal{G}_1$ is unfortunately not a stratified fibration, so we use Quillen's small object argument to replace it by a stratified fibration $|N\mathcal{G}|' \to \mathcal{G}_0/\mathcal{G}_1$. The category of regular CW-complexes doesn't have enough colimits so there are further technical details to make this argument work precisely. We then let

$$F(\mathcal{G}) := (|N\mathcal{G}|', \mathcal{G}_0/\mathcal{G}_1). \tag{3.11}$$

To show that $F(\mathcal{G}) \in \mathsf{Orb}$, we need to show that it satisfies the three conditions of Theorem 3.5. The map $p : |N\mathcal{G}|' \to \mathcal{G}_0/\mathcal{G}_1$ is a stratified fibration by construction and the fibers of p are homotopy equivalent to those of $\tilde{p} : |N\mathcal{G}| \to \mathcal{G}_0/\mathcal{G}_1$. So it's enough to check the second and third conditions on $\tilde{p}$. Let $y\mathcal{G}_0/\mathcal{G}_1$ be a point represented by $\tilde{y} \in \mathcal{G}_0$. The fiber $\tilde{p}^{-1}(y)$ is the realization of the full subgroupoid $\mathcal{G}_y \subset \mathcal{G}$ on the $\mathcal{G}_1$-orbit of $\tilde{y}$. Since $\mathrm{Aut}_{\mathcal{G}}(\tilde{y})$ is finite, we have

$$p^{-1}(y) \simeq \tilde{p}^{-1}(y) = |N\mathcal{G}_y| \simeq K\big(\mathrm{Aut}_{\mathcal{G}}(\tilde{y}), 1\big). \tag{3.12}$$

Now we show that the maps $\nabla_\gamma : p^{-1}(x) \to p^{-1}(y)$ are injective on π_1. Again, it's enough to check it on the corresponding map $\tilde{p}^{-1}(x) \to \tilde{p}^{-1}(y)$. We show this locally. Since $\mathcal{G}_0$ and $\mathcal{G}_1$ admit triangulations making all structure maps simplicial, there exists a neighborhood $y \in \bar{U} \subset \mathcal{G}_0/\mathcal{G}_1$ and a deformation retraction to $\bar{U} \searrow \{y\}$, covered by $U_0 \searrow (\mathcal{G}_y)_0$ and $U_1 \searrow (\mathcal{G}_y)_1$, where U_0 and U_1 are the preimages of $\bar{U}$ in $\mathcal{G}_0$ and $\mathcal{G}_1$ respectively. The groupoid $U := (U_1 \rightrightarrows U_0)$ is a full subgroupoid of $\mathcal{G}$, and we can assemble the above maps to a deformation retraction of groupoids $r : U \searrow \mathcal{G}_y$. Given a point $x \in \mathcal{G}_0/\mathcal{G}_1$, the map $\tilde{p}^{-1}(x) \to \tilde{p}^{-1}(y)$ is the realization of $r : \mathcal{G}_x \to \mathcal{G}_y$. So we need to show that r induces monomorphisms on the automorphism groups of objects. If this was not the case, we would have a non-identity $g \in (\mathcal{G}_x)_1$ whose image $r(g)$ is an identity. Since r is a deformation retraction, we would also get a path from g to $r(g)$. But this contradicts the fact that $Im(u)$ is a connected component of $\mathcal{G}_1$. We conclude that $\tilde{p}^{-1}(x) \to \tilde{p}^{-1}(y)$ is injective on π_1. This finishes the proof that $F(\mathcal{G}) \in \mathsf{Orb}$.

We now continue the definition of our functor F. A continuous functor $f : \mathcal{G} \to \mathcal{H}$ produces maps $(|Nf|, \bar{f}) : (|N\mathcal{G}|, \mathcal{G}_0/\mathcal{G}_1) \to (|N\mathcal{H}|, \mathcal{H}_0/\mathcal{H}_1)$. We compose it with the inclusion $|N\mathcal{H}| \hookrightarrow |N\mathcal{H}|'$. The inclusion $|N\mathcal{G}| \hookrightarrow |N\mathcal{G}|'$ is a directed cofibration with respect to the stratification inherited from $\mathcal{H}_0/\mathcal{H}_1$, so we can extend $|Nf|$ to

$$F(f) : (|N\mathcal{G}|', \mathcal{G}_0/\mathcal{G}_1) \to (|N\mathcal{H}|', \mathcal{H}_0/\mathcal{H}_1). \tag{3.13}$$

If f is an identity, pick $F(f)$ to be an identity.

A continuous natural transformation $h : f \Rightarrow g : \mathcal{G} \underset{\Downarrow}{\rightrightarrows} \mathcal{H}$ gives a simplicial homotopy $Nf \Rightarrow Ng$ and hence a homotopy $|Nf| \Rightarrow |Ng|$. The existence of h also implies that $\bar{f} = \bar{g}$, so we get a diagram

$$\begin{array}{ccc} |N\mathcal{G}| & \overset{|Nf|}{\underset{|Ng|}{\rightrightarrows}} \ \Downarrow_{|Nh|} & |N\mathcal{H}| \\ \downarrow & & \downarrow \\ \mathcal{G}_0/\mathcal{G}_1 & \xrightarrow{\bar{f}=\bar{g}} & \mathcal{H}_0/\mathcal{H}_1 . \end{array} \tag{3.14}$$

We then compose (3.14) with the inclusion $|N\mathcal{H}| \hookrightarrow |N\mathcal{H}|'$. Assembling (3.11) and (3.14), we get a map

$$F(f) \cup |Nh| \cup F(g) : \ |N\mathcal{G}|' \cup \big(|N\mathcal{G}| \times [0,1]\big) \cup |N\mathcal{G}|' \to |N\mathcal{H}|'. \tag{3.15}$$

The inclusion $|N\mathcal{G}|' \cup (|N\mathcal{G}| \times [0,1]) \cup |N\mathcal{G}|' \hookrightarrow |N\mathcal{G}|' \times [0,1]$ is a directed cofibration with respect to the stratification inherited from $\mathcal{H}_0/\mathcal{H}_1$ so we can extend (3.15) to the whole $|N\mathcal{G}|' \times [0,1]$. This is our 2-morphism $F(h) : F(f) \Rightarrow F(g)$.

To finish the construction of F, we still need 2-morphisms $\varphi_{f,g} : F(g) \circ F(f) \Rightarrow F(g \circ f)$ (recall that all 2-morphisms are invertible). If $f : \mathcal{G} \to \mathcal{H}$ and $g : \mathcal{H} \to \mathcal{K}$ are two composable 1-arrows, so we get a diagram

$$\begin{array}{ccccc} |N\mathcal{G}| & \xrightarrow{|Nf|} & |N\mathcal{H}| & \xrightarrow{|Ng|} & |N\mathcal{K}| \\ \downarrow & & \downarrow & & \downarrow \\ |N\mathcal{G}|' & \xrightarrow{F(f)} & |N\mathcal{H}|' & \xrightarrow{F(g)} & |N\mathcal{K}|' . \end{array} \tag{3.16}$$

Both maps $F(g) \circ F(f)$ and $F(g \circ f)$ extend $|N(g \circ f)|$ from $|N\mathcal{G}|$ to $|N\mathcal{G}|'$. By the same argument as before, the map

$$\big(F(g) \circ F(f)\big) \cup \big((|N(g \circ f)|) \circ \mathrm{pr}_1\big) \cup F(g \circ f) : |N\mathcal{G}|' \cup (|N\mathcal{G}| \times [0,1]) \cup |N\mathcal{G}|' \to |N\mathcal{K}|'$$

extends to $|N\mathcal{G}|' \times [0,1]$. This is our 2-morphism $\varphi_{f,g} : F(g) \circ F(f) \Rightarrow F(g \circ f)$.

To show that F is a functor, we still need to show that it preserves the various composition. The identities are sent to identities, so we don't have to worry about them.

Let $h_1 : f \Rightarrow g : \mathcal{G} \rightleftarrows \mathcal{H}$ and $h_2 : g \Rightarrow k : \mathcal{G} \rightleftarrows \mathcal{H}$ be vertically composable 2-morphisms, and let $\bullet$ denote vertical composition. The 2-morphisms $F(h_2) \bullet F(h_1)$ and $F(h_2 \bullet h_1)$ are represented by maps $|N\mathcal{G}|' \times [0,1] \to |N\mathcal{H}|'$. They agree on $(|N\mathcal{G}| \times [0,1]) \cup (|N\mathcal{G}|' \times \{0,1\})$, so we get a map

$$(|N\mathcal{G}| \times [0,1]^2) \cup (|N\mathcal{G}|' \times \partial[0,1]^2) \to |N\mathcal{H}|'. \tag{3.17}$$

The map (3.17) extends to $|N\mathcal{G}|' \times [0,1]^2$, which shows that $F(h_2) \bullet F(h_1)$ and $F(h_2 \bullet h_1)$ are homotopic relatively to the end points. This shows that $F(h_2) \bullet F(h_1) = F(h_2 \bullet h_1)$ on Orb.

Now suppose that $h_1 : f_1 \Rightarrow g_1 : \mathcal{G} \rightleftarrows \mathcal{H}$ and $h_2 : f_2 \Rightarrow g_2 : \mathcal{H} \rightleftarrows \mathcal{K}$ are horizontally composable. We need to show that

$$F(h_2 \circ h_1) \bullet \varphi_{f_1,f_2} = \varphi_{g_1,g_2} \bullet \big(F(h_2) \circ F(h_1)\big). \tag{3.18}$$

The two maps in (3.18) agree on the subspace $(|N\mathcal{G}| \times [0,1]) \cup (|N\mathcal{G}|' \times \{0,1\}) \subset |N\mathcal{G}|' \times [0,1]$. So, by the same argument as (3.17) they're equal as 2-morphisms of Orb. This finishes the construction of $F : \mathsf{Gpd} \to \mathsf{Orb}$.

Theorem 3.16 *Let* Orb *be the 2-category given in Definition 3.1, and* Gpd *be the 2-category of topological groupoids satisfying the conditions of Definition 2.4. Let* $W \subset \mathsf{Gpd}_1$ *be the weak equivalences, as given in Definition 3.12.*

Then the functor $F : \mathsf{Gpd} \to \mathsf{Orb}$ constructed above extends to an equivalence $\tilde{F} : \mathsf{Gpd}[W^{-1}] \to \mathsf{Orb}$.

Proof. Let $f : \mathcal{G} \xrightarrow{\sim} \mathcal{H}$ be a weak equivalence. We first explain why

$$\begin{array}{ccc} |N\mathcal{G}|' & \xrightarrow{F(f)} & |N\mathcal{H}|' \\ \downarrow p_{\mathcal{G}} & & \downarrow p_{\mathcal{H}} \\ \mathcal{G}_0/\mathcal{G}_1 & \xrightarrow{\bar{f}} & \mathcal{H}_0/\mathcal{H}_1 \end{array} \tag{3.19}$$

is an equivalence in Orb. By Lemma 3.14, $\bar{f}$ is a homeomorphism. Given a point $x \in \mathcal{G}_0/\mathcal{G}_1$ with preimage $\hat{x} \in \mathcal{G}_0$, its fiber $p_{\mathcal{G}}^{-1}(x)$ is a $K(\mathrm{Aut}_{\mathcal{G}}(\hat{x}), 1)$ by (3.12). The map $\mathrm{Aut}_{\mathcal{G}}(\hat{x}) \to \mathrm{Aut}_{\mathcal{H}}(f(\hat{x}))$ is an isomorphism by (3.10), so $F(f)$ is a homotopy equivalence on each fiber. By Theorem 4.21, the map $F(f)$ is has a homotopy inverse relatively to the projections to $\mathcal{H}_0/\mathcal{H}_1$. This is exactly what we wanted to show.

We have checked that $F : \mathsf{Gpd} \to \mathsf{Orb}$ sends weak equivalences to equivalences. So by the universal property of $\mathsf{Gpd}[W^{-1}]$, the functor F extends to a functor $\tilde{F} : \mathsf{Gpd}[W^{-1}] \to \mathsf{Orb}$. To show that $\tilde{F}$ is an equivalence, we verify the conditions of Theorem 3.15.

First, we show that F is essentially surjective. Let $p : E \to X$ be an orbispace and let $\{U_i\}$ be a closed cover such that $(p^{-1}(U_i), U_i) \simeq (\tilde{Y}_i/G_i, Y_i/G_i)$. Let $q_i : \tilde{Y}_i \to Y_i$ be the projections and let $s_i : Y_i \to \tilde{Y}_i$ be sections. Let $\mathcal{G}_0$ be the disjoint union of the Y_i's and $s : \mathcal{G}_0 \to E$ be induced by the s_i. We then let $\mathcal{G}$ be the topological groupoid with objects $\mathcal{G}_0$ and arrows given by

$$\mathrm{Hom}_{\mathcal{G}}(x, y) = \big\{\gamma \in E^{[0,1]} \,\big|\, s(x) = \gamma(0),\, s(y) = \gamma(1),\, p \circ \gamma \text{ is constant}\big\}/\!\sim, \tag{3.20}$$

where $\gamma \sim \gamma'$ if they are homotopic relatively to their endpoints, and within their p-fiber. In other words, $\mathcal{G}$ is the groupoid pulled back from fib-$\Pi_1(E)$ along the map $s : \mathcal{G}_0 \to E$, where fib-$\Pi_1(E)$ is the fiber-wise fundamental groupoid of E.

We now show that $\mathcal{G}$ satisfies the two conditions in Definition 2.4. The automorphism groups $\mathrm{Aut}_{\mathcal{G}}(x)$ are finite since they agree with the fundamental groups of the fibers of p. We now show that the subspace of identities is a union of connected components of $\mathcal{G}_1$. The space $\mathcal{G}_1$ is the disjoint union of the subspaces $\mathcal{G}_1(i,j) := \mu^{-1}(Y_i \times Y_j)$. Since $\mathcal{G}_1(i,j)$ intersects $Im(u)$ only when $i = j$, we can restrict our attention to the full subgroupoids $\mathcal{G}_1(i,i) \rightrightarrows Y_i$. These are isomorphic to the action groupoids $Y_i \times G_i \rightrightarrows Y_i$. The identities are $Y_i \times \{e\}$ which is indeed a connected component of $Y_i \times G_i$. we have shown that $\mathcal{G} \in \mathsf{Gpd}$.

We now build an equivalence $F(\mathcal{G}) \to (E, X)$. Since $\mathcal{G}$ is a subgroupoid of fib-$\Pi_1(E)$ we get a diagram

$$\begin{array}{ccccccc} |N\mathcal{G}|' & \hookleftarrow & |N\mathcal{G}| & \longrightarrow & \big|N\big(\text{fib-}\Pi_1(E)\big)\big| & \longleftarrow & E \\ & \searrow & \downarrow & & \downarrow & \swarrow & \\ & & \mathcal{G}_0/\mathcal{G}_1 & = & X, & & \end{array} \tag{3.21}$$

where all the top horizontal maps are fiber-wise homotopy equivalences. We first apply Theorem 4.21 to get a homotopy inverse $|N(\text{fib-}\Pi_1(E))| \dashrightarrow E$, and then extend it to $|N\mathcal{G}|'$ using Theorem 4.29. The resulting map $(|N\mathcal{G}|', \mathcal{G}_0/\mathcal{G}_1) \dashrightarrow (E, X)$ is then an equivalence by Theorem 4.21. This finishes the proof that F is essentially surjective.

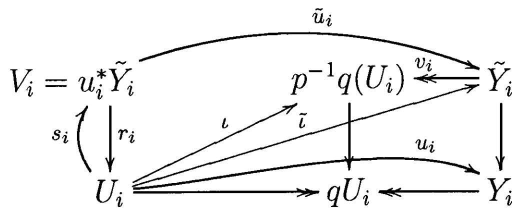
We now verify the second condition of Theorem 3.15. Let $\mathcal{G}, \mathcal{H}$ be topological groupoids, and let $f : F(\mathcal{G}) \to F(\mathcal{H})$ be an orbispace morphism. We need to find a groupoid $\tilde{\mathcal{G}} \xrightarrow{\sim} \mathcal{G}$ and a continuous functor $\tilde{f} : \tilde{\mathcal{G}} \to \mathcal{H}$ making the diagram

$$F(\tilde{\mathcal{G}}) \underset{\sim}{\xrightarrow{w}} F(\mathcal{G}) \xrightarrow{f} F(\mathcal{H}), \qquad F(\tilde{f}) : F(\tilde{\mathcal{G}}) \to F(\mathcal{H}) \tag{3.22}$$

commute up to a 2-morphism.

Let $\iota : \mathcal{H}_0 \hookrightarrow |N\mathcal{H}|'$ be the inclusion, and $q : \mathcal{H}_0 \to \mathcal{H}_0/\mathcal{H}_1$ and $p : |N\mathcal{H}|' \to \mathcal{H}_0/\mathcal{H}_1$ be the projections. We claim that $|N\mathcal{H}|'$ has a closed cover $\{V_i\}$ that fiber-wise retracts into the image of ι. Let $\{U_i\}$ be a cover of $\mathcal{H}_0$ which is fine enough so that $p^{-1}(q(U_i)) \to q(U_i)$ are homeomorphic to Borel constructions $\tilde{Y}_i/G_i \to Y_i/G_i$. By picking the cover fine enough, we can also make sure that all the U_i and $\tilde{Y}_i$ are simply connected. Let $v_i : \tilde{Y}_i \to \tilde{Y}_i/G_i = p^{-1}(q(U_i))$ be the projection. Since $\tilde{Y}_i$ is the universal cover of $p^{-1}(q(U_i))$, the map $\iota : U_i \to p^{-1}(q(U_i))$ lifts to a map $\tilde{\iota} : U_i \to \tilde{Y}_i$.

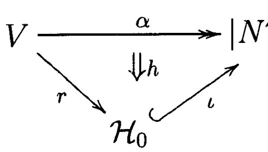
Let u_i denote the composite of $\tilde{\iota}$ with the projection to Y_i. Let V_i be the pullback $u_i^*\tilde{Y}_i$ with map $\tilde{u}_i : V_i \to \tilde{Y}_i$ and projection $r_i : V_i \to U_i$. Since r_i has contractible fibers, we can pick a section $s_i : U_i \to V_i$.



The map $s_i \circ r_i$ is fiber-wise homotopic to the identity on V_i, so $v_i \circ \tilde{u}_i$ is fiber-wise homotopic to

$$v_i \circ \tilde{u}_i \circ s_i \circ r_i = v_i \circ \tilde{\iota} \circ r_i = \iota \circ r_i.$$

Let V denote the disjoint union of the V_i's and let $r : V \to \mathcal{H}_0$ and $\alpha : V \to |N\mathcal{H}|'$ be the maps induced from the r_i and $v_i \circ \tilde{u}_i$ respectively. Since each $v_i \circ \tilde{u}_i$ is topologically surjective, so is α. We have constructed a cover $V \to |N\mathcal{H}|'$ that fiber-wise retracts into the image of ι:



(3.23)

Let $\alpha^*\mathcal{G}_0$ be the pullback of $\mathcal{G}_0 \hookrightarrow |N\mathcal{G}|' \xrightarrow{f} |N\mathcal{H}|' \twoheadleftarrow V$, and let $\tilde{\mathcal{G}}$ be groupoid pulled back from $\mathcal{G}$ along the map $\alpha^*\mathcal{G}_0 \to \mathcal{G}_0$. By Lemma 3.13, the projection $\tilde{\mathcal{G}} \to \mathcal{G}$ is a weak equivalence.

We now build a functor $\tilde{f} : \tilde{\mathcal{G}} \to \mathcal{H}$. On the objects, it is given by $r \circ (\alpha^* f) : \alpha^*\mathcal{G}_0 \to V \to \mathcal{H}_0$. An arrow in $\tilde{\mathcal{G}}$ consists of an arrow $g \in \mathcal{G}_1$, and two points $v_0, v_1 \in V$ such that $f(s(g)) = \alpha(v_0)$ and $f(t(g)) = \alpha(v_1)$. Let $f(g)$ denote the image of the path $\{g\} \times \Delta^1 \subset |N\mathcal{G}|'$ under the map f. The composition $h(v_0)^{-1} \cdot f(g) \cdot h(v_1)$ is then a path from $\iota(r(v_0))$ to $\iota(r(v_1))$, where h is the homotopy in (3.23). This path lies entirely in a fiber of $|N\mathcal{H}|'$. Since $\mathcal{H}$ is a full subgroupoid of fib-$\Pi_1|N\mathcal{H}|'$, the path $h(v_0)^{-1} \cdot f(g) \cdot h(v_1)$ gives us a morphism in from $r(v_0)$ to $r(v_1)$. This is what we define $\tilde{f}(g, v_0, v_1)$ to be.

To check that $\tilde{f} : \tilde{\mathcal{G}} \to \mathcal{H}$ is a functor, we pick two composable arrows (g_1, v_0, v_1) and (g_2, v_1, v_2) in $\tilde{\mathcal{G}}$. The two paths

$$\begin{aligned} &\tilde{f}(g_1, v_0, v_1)\tilde{f}(g_2, v_1, v_2) = h(v_0)^{-1} \cdot f(g_1) \cdot h(v_1) \cdot h(v_1)^{-1} \cdot f(g_2) \cdot h(v_2) \quad \text{and} \\ &\tilde{f}\big((g_1, v_0, v_1)(g_2, v_1, v_2)\big) = \tilde{f}(g_1 g_2, v_0, v_2) = h(v_0)^{-1} \cdot f(g_1 g_2) \cdot h(v_2) \end{aligned} \tag{3.24}$$

are clearly homotopic, and so they represent the same element in $\mathcal{H}_1$. This finishes the construction of $\tilde{f}$.

We have constructed all the maps in (3.22). Now we need to provide the 2-morphism $H : f \circ w \Rightarrow F(\tilde{f})$. On $\tilde{\mathcal{G}}_0 \subset |N\tilde{\mathcal{G}}|'$, it is given by

$$H : f \circ w = \alpha \circ (\alpha^* f) \overset{h}{\Rightarrow} \iota \circ r \circ (\alpha^* f) = \iota \circ \tilde{f} = F(\tilde{f}). \tag{3.25}$$

To extend H to the one skeleton $|N\tilde{\mathcal{G}}|^{(1)}$, we consider the lifting problem

$$\begin{array}{ccc} \big(\tilde{\mathcal{G}}_0 \times [0,1]\big) \cup \big(|N\tilde{\mathcal{G}}|^{(1)} \times \{0,1\}\big) & \xrightarrow{h \cup (f \circ w) \cup F(\tilde{f})} & |N\mathcal{H}|' \\ \big\downarrow & \nearrow & \big\downarrow p \\ |N\tilde{\mathcal{G}}|^{(1)} \times [0,1] & \longrightarrow & \mathcal{H}_0/\mathcal{H}_1 . \end{array} \tag{3.26}$$

Since the fibers of p are $K(\pi, 1)$'s the only obstructions come from the 2-cells. The boundary of the 2-cells are given by paths $f \circ w(g, v_0, v_1) = f(g)$, $\tilde{f}(g, v_0, v_1)$, $h(v_0)$ and $h(v_1)$, where $(g, v_0, v_1) \in \tilde{\mathcal{G}}_1$ is some arrow. But the two paths $\tilde{f}(g, v_0, v_1)$ and $h(v_0)^{-1} \cdot f(g) \cdot h(v_1)$ are homotopic by construction. So there are no obstructions. Similarly, there are no obstructions to extending H to the whole $|N\mathcal{H}|'$. This finishes the construction of the 2-arrow $H : f \circ w \Rightarrow F(\tilde{f})$.

Now we check the last condition of Theorem 3.15, namely that F is fully faithful on 2-morphisms. Let $\mathcal{G}$, $\mathcal{H}$ be topological groupoids and $f, g : \mathcal{G} \to \mathcal{H}$ be continuous functors. We want to construct an inverse F^{-1} to the natural map

$$F : \text{2-Hom}_{\mathsf{Gpd}}(f, g) \to \text{2-Hom}_{\mathsf{Orb}}\big(F(f), F(g)\big). \tag{3.27}$$

Given a 2-morphism $h : F(f) \Rightarrow F(g)$ and an object $x \in \mathcal{G}_0$, we let $F^{-1}(h) : f(x) \to g(x)$ be the arrow corresponding to the path $h(x)$ in $|N\mathcal{H}|'$. This is an inverse to (3.27), and thus proves that F is fully faithful on 2-morphisms.

We have checked the three conditions in Theorem 3.15, so $\tilde{F} : \mathsf{Gpd}[W^{-1}] \to \mathsf{Orb}$ is an equivalence of bicategories. □

Chapter 4

Stratified fibrations

4.1 Triangulations

One of our main working tools, will be triangulations. So we will work in the subcategory of spaces which are regular CW-complexes (see [10] for basic background). These are the spaces arising as geometric realization of simplicial sets. It is also convenient to restrict the class of morphisms. So we make the following working definition:

Definition 4.1 *The objects of the category* spaces *are topological spaces arising as geometric realizations of simplicial sets. The morphisms are the continuous maps which are semi-algebraic of each simplex of the source i.e. the graph of the map is defined (locally) by finitely many algebraic equalities and inequalities.*

We now define what we mean by a triangulation of a space. Since our definition varies slightly from the usual one, we use a different terminology.

Definition 4.2 *Let X be an object of* spaces. *An oriented triangulation of X is a simplicial set Y and an isomorphism between X and $|Y|$.*

We have the following convenient lemma about cofibrations.

Lemma 4.3 *Let $i : A \hookrightarrow B$ be a cofibration. Then there exist spaces C, D and maps $f : C \to A$, $g : C \to D$ such that $B \simeq A \cup_f (C \times [0,1]) \cup_g E$. More precisely, we get* *a commutative diagram*

$$\begin{array}{ccc} A & \hookrightarrow & A \cup_f (C \times [0,1]) \cup_g D \\ \| & & \downarrow \simeq \\ A & \xhookrightarrow{\quad i \quad} & B \end{array} \tag{4.1}$$

where the top arrow is the obvious inclusion. Moreover, if i is a weak equivalence, then g is a weak equivalence.

Proof. Triangulate B is a way compatible with A and let D be the union of all simplices that do not intersect A. The map $t : A \sqcup D \to [0,1]$ sending A to 0 and D to 1 extends by linearity to a map on the whole of B. Let $C := t^{-1}(1/2)$.

An n-simplex of B has $k+1$ vertices in A and $\ell + 1$ vertices in A for some k, ℓ satisfying $k + \ell + 1 = n$. The intersection of that simplex with C is then isomorphic to $\Delta^k \times \Delta^\ell$. We then assemble the maps $\Delta^k \times \Delta^\ell \twoheadrightarrow \Delta^k \hookrightarrow \Delta^n$ and $\Delta^k \times \Delta^\ell \twoheadrightarrow \Delta^\ell \hookrightarrow \Delta^n$ to maps $f : C \to A$ and $g : C \to D$. Since $\Delta^n = \Delta^k * \Delta^\ell$, we get a map $\Delta^k \times \Delta^\ell \times [0,1] \twoheadrightarrow \Delta^n$ which is a homeomorphism over the interior of $[0,1]$. Assembling these maps over all the simplices of B, produces a map $C \times [0,1] \to B$. This provides the isomorphism $A \cup_f (C \times [0,1]) \cup_g D \xrightarrow{\sim} B$.

If i is a weak equivalence, then so is the bottom arrow in the following diagram.

$$\begin{array}{ccc} C \times \{1\} & \xrightarrow{\quad g \quad} & D \\ \downarrow & & \downarrow \\ A \cup_f (C \times [0,1]) & \xrightarrow{\sim} & A \cup_f (C \times [0,1]) \cup_g D \end{array} \tag{4.2}$$

Since (4.2) is a homotopy pushout diagram, g is also a weak equivalence. □

4.2 Stratifications

Definition 4.4 *Let $\mathbb{J}$ be a poset and X a topological space. A stratification of X by $\mathbb{J}$ is an upper semi-continuous function $s : X \to \mathbb{J}$ which takes finitely many values on each compact subspace. It defines a partition of X into strata $X_j := s^{-1}(j)$. We* *also introduce the notations*

$$X_{\leq j} := s^{-1}\{i \in \mathbb{J} | i \leq j\} \qquad X_{<j} := s^{-1}\{i \in \mathbb{J} | i < j\}$$
$$X_{\geq j} := s^{-1}\{i \in \mathbb{J} | i \geq j\} \qquad X_{>j} := s^{-1}\{i \in \mathbb{J} | i > j\}.$$

The subsets $X_{\leq j}$ and $X_{<j}$ are open while $X_{\geq j}$ and $X_{>j}$ are closed.

Let (X, s) and (X', s') be two $\mathbb{J}$-stratified spaces. A continuous map $f : X \to X'$ is stratified if it satisfies $s' \circ f = s$, or equivalently if it sends X_j to X'_j.

Definition 4.5 *Let (X, s) be a stratified space. An oriented triangulation $\mathcal{T}$ is compatible with the stratification if given any simplex $\sigma : \Delta^n \to X$ of $\mathcal{T}$, the composite $s \circ \sigma$ is constant on the subsets*

$$\big\{(t_0, \ldots, t_n) \in \Delta^n \,\big|\, t_i = 0 \text{ for } i < j,\ t_j \neq 0\big\} \subset \Delta^n.$$

On compact spaces, the existence of compatible oriented triangulations is an easy corollary of the existence of (usual) triangulation.

Lemma 4.6 *Given a compact stratified space X, there exists a compatible oriented triangulation.*

Proof. By [17], there exists a (usual) triangulation $\mathcal{T}$ of X such that each stratum is a union of open simplices of $\mathcal{T}$. The barycentric subdivision of $\mathcal{T}$ is then a compatible oriented triangulation. □

Let $\mathbb{J}^{op}$ be the opposite poset to $\mathbb{J}$, with elements denoted j^{op}. To each $\mathbb{J}$-stratification $s : X \to \mathbb{J}$ there is an associated $\mathbb{J}^{op}$-stratification $s^{op} : X \to \mathbb{J}^{op}$, defined up to some appropriate notion of weak equivalence.

Definition 4.7 *Let X be an $\mathbb{J}$-stratified space and $\mathcal{T}$ a compatible oriented triangulation. The opposite stratification $s^{op} : X \to \mathbb{J}^{op}$ is given by*

$$s^{op}(x) := \Big[\max_{y \in \Delta^n} s(\sigma(y))\Big]^{op} = \min_{y \in \Delta^n} s(\sigma(y))^{op}, \tag{4.3}$$

where $\sigma : \Delta^n \to X$ is the smallest non-degenerate simplex of $\mathcal{T}$ in the image of which

x lies. The space X, equipped with this new stratification will be denoted X^{op}. We also define $X_j^{op} := X_{j^{op}}^{op}$, $X_{\geq j}^{op} := X_{\leq j^{op}}^{op}$ and similarly for $X_{>j}^{op}$, $X_{\leq j}^{op}$ and $X_{<j}^{op}$.

We illustrate here an example of a stratified space X and its opposite X^{op}:

X $\qquad\qquad$ X^{op}

The poset for this example is $\mathbb{J} =$ (*"white"* $<$ *"black"* $<$ *"stripes"*).

Note that in general, the strata X_j^{op} are homotopy equivalent to X_j, but the closure relations are reversed. We also have

$$X_{\leq j}^{op} \simeq X_{\leq j}, \qquad X_{<j}^{op} \simeq X_{<j}, \qquad X_{\geq j}^{op} \simeq X_{\geq j}, \quad \text{and} \quad X_{>j}^{op} \simeq X_{>j}. \tag{4.4}$$

4.3 Directed cofibrations

Before introducing stratified fibrations, we recall the classical notion of (Serre) fibrations. The symbols $\twoheadrightarrow$ and $\hookrightarrow$ will be used to denote fibrations and cofibrations respectively.

Let Δ^n be the topological n-simplex and $\Lambda^n := Cone(\partial\Delta^{n-1})$ be the n-horn. A map $E \to X$ is a fibration if for any maps $\Delta^n \to X$ and $\Lambda^n \to E$ making the diagram

$$\begin{array}{ccc} \Lambda^n & \longrightarrow & E \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta^n & \longrightarrow & X \end{array} \tag{4.5}$$

commute, there exists a lift $\Delta^n \to E$ extending the given map on Λ^n. The inclusions $\Lambda^n \hookrightarrow \Delta^n$ are called the generating acyclic cofibrations.

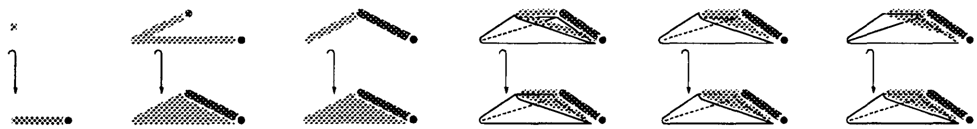
Just like Serre fibrations, we define stratified fibrations by a lifting property. The maps which play the role of acyclic cofibrations are called directed cofibrations. We begin by defining a set of generating directed cofibrations. We then define stratified fibrations to be the maps satisfying the right lifting property with respect to the generating directed cofibrations, and directed cofibrations as those satisfying the left lifting property with respect to stratified fibrations.

Definition 4.8 *Let Δ^n be the n-simplex and $\Lambda^{n,i} \subset \Delta^n$ be the union of all the facets containing the ith vertex. Let $s : \Delta^n \to \mathbb{J}$ be a stratification which is constant on the subsets*

$$\{(t_0, \ldots, t_n) \in \Delta^n \,|\, t_i = 0 \text{ for } i < j,\ t_j \neq 0\} \tag{4.6}$$

and $s' := s|_{\Lambda^{n,i}}$. Then $(\Lambda^{n,i}, s') \hookrightarrow (\Delta^n, s)$ is a generating directed cofibration if $i < n$.

We illustrate the generating directed cofibrations for $n = 1, 2, 3$:



(4.7)

Note that $(\Lambda^{n,n}, s') \hookrightarrow (\Delta^n, s)$ is sometimes a generating directed cofibration. It is one if and only if s is constant on the edge linking the nth and $n-1$st vertex of Δ^n. Indeed, the linear map flipping that edge is then a stratified homeomorphism between the pairs $(\Delta^n, \Lambda^{n,n})$ and $(\Delta^n, \Lambda^{n,n-1})$.

We now introduce a new notion of stratified fibration. It is much stronger than the stratified fibrations of Huges [20] and Friedman [9].

Definition 4.9 *A stratified map $p : E \to X$ is a stratified fibration if it satisfies the right lifting property with respect to the generating directed cofibrations. In other words, for every generating directed cofibration $\Lambda^n \hookrightarrow \Delta^n$ and every commutative diagram (4.5) of stratified maps, there has to exist a lift $\Delta^n \to E$ making both triangles commute.*

A stratified map $A \to B$ is a directed cofibration if it satisfies the left lifting property with respect to stratified fibrations. In other words, for every diagram

$$\begin{array}{ccc} A & \longrightarrow & E \\ \big\downarrow & \nearrow & \big\downarrow \\ B & \longrightarrow & X, \end{array} \tag{4.8}$$

where $E \to X$ is a stratified fibration, there exists a lift $B \to E$.

To be able to work with the above definitions, we need a better understanding of directed cofibrations. The following Lemma is classical from the theory of model categories [19, section 2.1] [11, section I.4].

Lemma 4.10 *a. If $A \to B$ is a directed cofibration and $A \to C$ is an arbitrary stratified map, then $C \to B \cup_A C$ is a directed cofibration.*

b. If $A_i \to A_{i+1}$ are directed cofibrations, then $A_0 \to \varinjlim A_i$ is a directed cofibration. A similar statement holds when i ranges over some arbitrary ordinal, but then we should also insist that $A_\beta = \varinjlim_{\alpha<\beta} A_\alpha$ whenever β is a limit ordinal.

c. Let $A \to B$ be a directed cofibration and let $A' \to B'$ be a retract of it. In other words, suppose that we have a commuting diagram

$$\begin{array}{ccccc} A' & \longrightarrow & A & \longrightarrow & A' \\ \downarrow & & \downarrow & & \downarrow \\ B' & \longrightarrow & B & \longrightarrow & B', \end{array} \tag{4.9}$$

where the horizontal composites are identities. Then $A' \to B'$ is a directed cofibration.

Proof. *a.* Given a stratified fibration $E \to X$, and a commutative square $C \to E$, $B \cup_A C \to X$, we first find a lift $B \to E$

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & E \\ \downarrow & & \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & B \cup_A C & \longrightarrow & X. \end{array} \tag{4.10}$$

Then we assemble the maps $B \to E$ and $C \to E$ into a map $B \cup_A C \to E$.

b. Given a stratified fibration $E \to X$ and a commutative square $A_1 \to E$,

$\varinjlim A_i \to X$, we inductively build partial lifts $A_i \to E$

$$\begin{array}{ccc} A_i & \longrightarrow & E \\ \Big\downarrow & \nearrow & \Big\downarrow \\ A_{i+1} & \longrightarrow & X. \end{array} \tag{4.11}$$

These maps assemble to the desired map $\varinjlim A_i \to E$.

c. Given a stratified fibration $E \to X$ and a commutative square $A' \to E$, $B' \to X$, we first find a lift $B \to E$

$$\begin{array}{ccccc} A & \rightleftarrows & A' & \longrightarrow & E \\ \Big\downarrow & & \Big\downarrow & \nearrow & \Big\downarrow \\ B & \rightleftarrows & B' & \longrightarrow & X. \end{array} \tag{4.12}$$

Composing the map $B' \to B$ with the lift $B \to E$ gives a solution to our problem. □

Lemma 4.10 also has a converse [19, Corollary 2.1.15].

Lemma 4.11 *The class of maps that can be obtained, starting from the class of generating directed cofibration, and using the constructions of Lemma 4.10 is exactly the class of directed cofibrations.* □

The following Lemma provides some basic examples of directed cofibrations.

Lemma 4.12 *Let (X, A) be a pair of spaces and let $s : X \times [0,1] \to \mathbb{J}$ be a stratification such that $s|_{\{x\}\times[0,1)}$ is constant for all $x \in X$. Then the inclusion*

$$\big(X \times \{0\}\big) \cup \big(A \times [0,1]\big) \hookrightarrow X \times [0,1] \tag{4.13}$$

is a directed cofibration.

Proof. Let $\mathcal{T}$ be an oriented triangulation of X which is compatible with the subspace A and with the stratification of $X \times [0,1]$. Let $S_n \subset X \times [0,1]$ be the subspaces given by

$$S_n := (X \times \{0\}) \cup \big((A \cup X^{(n)}) \times [0,1]\big), \tag{4.14}$$

where $X^{(n)}$ denotes the n-skeleton of $\mathcal{T}$. It is enough by Lemma 4.10.b to show that each inclusion $S_{n-1} \hookrightarrow S_n$ is a directed cofibration. We can write it as a pushout

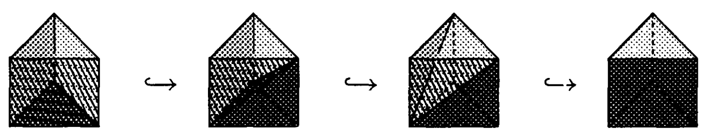
$$\begin{array}{ccc} \coprod\Big((\Delta^n \times \{0\}) \cup (\partial\Delta^n \times [0,1])\Big) & \longrightarrow & S_{n-1} \\ \big\downarrow & & \big\downarrow \\ \coprod(\Delta^n \times [0,1]) & \longrightarrow & S_n, \end{array} \tag{4.15}$$

where the coproducts run over the set of n-simplices of X. So by Lemma 4.10.a we are reduced to the case $X = \Delta^k$ and $A = \partial\Delta^k$.

We wish to show that $(\Delta^n \times \{0\}) \cup (\partial\Delta^n \times [0,1]) \hookrightarrow \Delta^n \times [0,1]$ is a directed cofibration. Using the standard triangulation of $\Delta^n \times [0,1]$, we find sequence of spaces

$$\big(\Delta^n \times \{0\}\big) \cup \big(\partial\Delta^n \times [0,1]\big) = Y_0 \hookrightarrow Y_1 \hookrightarrow \ldots \hookrightarrow Y_{n+1} = \Delta^n \times [0,1]$$

illustrated below for $n = 2$

$Y_0 =$  $= Y_3$.

The space Y_i contains the i first $(n+1)$-simplices of $\Delta^n \times [0,1]$ and we have a sequence of pushout diagrams

$$\begin{array}{ccc} \Lambda^{n+1,n-i} & \longrightarrow & Y_i \\ \big\downarrow & & \big\downarrow \\ \Delta^{n+1} & \longrightarrow & Y_{i+1}. \end{array} \tag{4.16}$$

Each inclusion $\Lambda^{n+1,n-i} \hookrightarrow \Delta^{n+1}$ is a generating directed cofibration. Therefore by Lemma 4.10.ab, so is $Y_{i-1} \to Y_i$ and so is $(\Delta^n \times \{0\}) \cup (\partial\Delta^n \times [0,1]) \hookrightarrow \Delta^n \times [0,1]$. □

Corollary 4.13 *If $X \times [0,1]$ is stratified with two strata $X \times [0,1)$ and $X \times \{1\}$, then the inclusion $X \times \{0\} \hookrightarrow X \times [0,1]$ is a directed cofibration.*

Lemma 4.14 *Let $\iota : A \hookrightarrow B$ be an inclusion which is also a homotopy equivalence.*

Give both A and B a constant stratification. Then ι is a directed cofibration.

Proof. Let $r : B \to A$ be a deformation retraction, and $h : B \times [0,1] \to B$ be the homotopy between $\iota \circ r$ and Id_B. Let $r' : B \times \{0\} \cup A \times [0,1] \to A$ be the map which is r on B and projection on $A \times [0,1]$, and let C be the pushout of the diagram

$$\begin{array}{ccc} B \times \{0\} \cup A \times [0,1] & \xrightarrow{r'} & A \\ \big\downarrow{\scriptstyle \iota'} & & \big\downarrow \\ B \times [0,1] & \longrightarrow & C. \end{array} \tag{4.17}$$

The inclusion $B \times \{0\} \cup A \times [0,1]$ is a directed cofibration by Lemma 4.12, and so is ι' by Lemma 4.10.*a*.

Now we note that, since h is constant on A, it factor through a map $h' : C \to B$. Let $i_1 : B \times \{1\} \to C$ denote the inclusion. Then the diagram

$$\begin{array}{ccccc} A & = & A & = & A \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ B & \xrightarrow{i_1} & C & \xrightarrow{h} & B \end{array} \tag{4.18}$$

expresses ι as a retract of ι'. So ι is a directed cofibration by Lemma 4.10.*c*. ☐

Theorem 4.15 *Let (B, s) be a $\mathbb{J}$-stratified space and $A \subset B$ a subspace. Suppose that the image of s has no infinite descending chains (for example B is compact). Then the following are equivalent:*

1. *$A_{\leq j} \to B_{\leq j}$ is a homotopy equivalence for all $j \in \mathbb{J}$.*
2. *The map*

$$\bigcup_{j \in S} A_{\leq j} \to \bigcup_{j \in S} B_{\leq j} \tag{4.19}$$

   is a homotopy equivalence for all subsets $S \subset \mathbb{J}$.

3. *$A \hookrightarrow B$ is a directed cofibration.*

Proof. Since $\mathbb{J}$ doesn't have infinite descending chains, we can extend the partial order to a well order $\mathbb{J}'$. Let $\iota : \mathbb{J} \to \mathbb{J}'$ denote the identity.

The implication 2. $\Rightarrow$ 1. is trivial so we show 1. $\Rightarrow$ 2. Assume that 1. holds. We show that (4.19) is a homotopy equivalence by induction on the first element $\alpha \in \mathbb{J}'$ which is not in $\iota(S)$. If α is a limit ordinal, then

$$\bigcup_{j \in S} A_{\leq j} = \varinjlim_{\beta < \alpha} \bigcup_{\substack{j \in S, \\ \iota(j) \leq \beta}} A_{\leq j} \simeq \varinjlim_{\beta < \alpha} \bigcup_{\substack{j \in S, \\ \iota(j) \leq \beta}} B_{\leq j} = \bigcup_{j \in S} B_{\leq j},$$

where the middle equivalence holds by the inductions hypothesis. If $\alpha = \beta + 1$, we let $k := \iota^{-1}(\beta)$ and write

$$\bigcup_{j \in S} A_{\leq j} = \Big(\bigcup_{\substack{j \in S \\ j \neq k}} A_{\leq j}\Big) \cup_{\left(\bigcup_{\substack{j \in S, \\ j < k}} A_{\leq j}\right)} A_{\leq k} \simeq \Big(\bigcup_{\substack{j \in S \\ j \neq k}} B_{\leq j}\Big) \cup_{\left(\bigcup_{\substack{j \in S, \\ j < k}} B_{\leq j}\right)} B_{\leq k} = \bigcup_{j \in S} B_{\leq j}$$

where the middle equivalence holds by the inductions hypothesis and because $A_{\leq k} \simeq B_{\leq k}$.

We now show 2. $\Rightarrow$ 3. Let

$$A_\alpha := \bigcup_{j :\, \iota(j) < \alpha} A^{op}_{\leq j} \qquad \text{and} \qquad B_\alpha := \bigcup_{j :\, \iota(j) < \alpha} B^{op}_{\leq j}.$$

By (4.4) and (4.19), the cofibration $A_\alpha \hookrightarrow B_\alpha$ is a homotopy equivalence. Consider the spaces $C_\alpha := A \cup B_\alpha$. We have a sequence of inclusions

$$A = C_0 \hookrightarrow C_1 \hookrightarrow \ldots \to \varinjlim C_j = B, \tag{4.20}$$

so by Lemma 4.10.b, it's enough to show that each inclusion $C_\alpha \hookrightarrow C_{\alpha+1}$ is a directed cofibration. The horizontal arrows in

$$\begin{array}{ccc} A_\alpha & \overset{\sim}{\hookrightarrow} & B_\alpha \\ \big\downarrow & & \big\downarrow \\ A_{\alpha+1} & \overset{\sim}{\hookrightarrow} & B_{\alpha+1} \end{array} \tag{4.21}$$

are homotopy equivalences, so that diagram is homotopy cocartesian. The inclusion

$A_{\alpha+1} \cup_{A_\alpha} B_\alpha \hookrightarrow B_{\alpha+1}$ is therefore a homotopy equivalence. Using Lemma 4.3 we write $B_{\alpha+1}$ as a homotopy pushout

$$B_{\alpha+1} = \left(A_{\alpha+1} \cup_{A_\alpha} B_\alpha\right) \cup_f \left(D \times [0,1]\right) \cup_g E, \tag{4.22}$$

where $g : D \to E$ is a homotopy equivalence. Since $C_\alpha = A \cup_{A_{\alpha+1}} \left(A_{\alpha+1} \cup_{A_\alpha} B_\alpha\right)$ and $C_{\alpha+1} = A \cup_{A_{\alpha+1}} B_{\alpha+1}$, we also get

$$C_{\alpha+1} = C_\alpha \cup_f \left(D \times [0,1]\right) \cup_g E. \tag{4.23}$$

The stratification is $s \circ f \circ \mathrm{pr}_1$ on $D \times (0,1)$ and j on E. Now consider the following sequence of inclusions illustrated in (4.26):

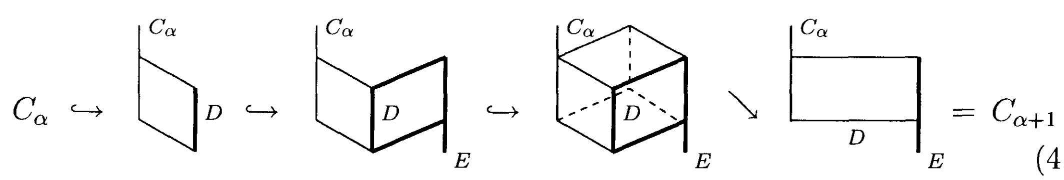
$$\begin{aligned} C_\alpha &\hookrightarrow C_\alpha \cup_f (D \times [0,1]) \\ &\hookrightarrow C_\alpha \cup_f \left(D \times ([0,1] \times \{0\} \cup \{1\} \times [0,1])\right) \cup_g E \\ &\hookrightarrow C_\alpha \cup_f (D \times [0,1]^2) \cup_g E. \end{aligned} \tag{4.24}$$

The stratification is $s \circ f \circ \mathrm{pr}_1$ on $D \times [0,1) \times [0,1]$ and is j on $D \times \{1\} \times [0,1]$ and on E.

The three inclusions in (4.24) are directed cofibrations. The first one is by Lemma 4.12. The second one is because $D \times \{1\} \subset D \times [1,2] \cup_g E$ is a homotopy equivalence, and so we can apply Lemmas 4.14, and 4.10.*a*. To see that the third one is, we apply Lemma 4.12 to $D \times ([0,1] \times \{0\} \cup \{1\} \times [0,1]) \hookrightarrow D \times [0,1]^2$ and finish by Lemma 4.10.*a*. By Lemma 4.10.*b* the composite

$$C_\alpha \hookrightarrow C_\alpha \cup_f \left(D \times [0,1]^2\right) \cup_g E \tag{4.25}$$

is also a directed cofibration.



(4.26)

We now observe that $C_\alpha \hookrightarrow C_{\alpha+1} = C_\alpha \cup_f (D \times [0,1]) \cup_g E$ is a retract of (4.25) by using the diagonal map $\Delta : [0,1] \to [0,1]^2$ and the projection $\mathrm{pr}_1 : [0,1]^2 \to [0,1]$. So by Lemma 4.10.*c*, we have that $C_\alpha \hookrightarrow C_{\alpha+1}$ is a directed cofibration. We now go back to (4.20) and apply Lemma 4.10.*b*. This finishes the proof that $A \hookrightarrow B$ is a directed cofibration.

We now show 3. $\Rightarrow$ 2. Suppose that $A \to B$ is a directed cofibration, we want to show that (4.19) is a homotopy equivalence. By Lemma 4.11, it is enough to check it for generating cofibrations, and to check that the constructions of Lemma 4.10 preserve that property. The latter is straightforward, so we concentrate on the former.

Let $\Lambda^{n,i} \hookrightarrow \Delta^n$ be a generating directed cofibration. The sets (4.19) are either empty or of the form

$$\left\{t \in \Lambda^{n,i} \,\middle|\, \sum_{j=0}^{r} t_j \neq 0\right\} \quad \text{and} \quad \left\{t \in \Delta^{n} \,\middle|\, \sum_{j=0}^{r} t_j \neq 0\right\} \tag{4.27}$$

for some appropriate r depending on S. Let $f_i : \Delta^n \to \Delta^n$ be the projection

$$f_i : (t_0, t_1, \dots, t_n) \mapsto (t_0, \dots, t_i + t_n, \dots, t_{n-1}, 0). \tag{4.28}$$

The straight line homotopy between Id and f_i is a deformation retraction from the spaces (4.27) onto the facet $t_n = 0$. This shows that they are both contractible, and in particular that they are homotopy equivalent. □

Here are some more examples of directed cofibrations (which include the generating one):

Example 4.16 Let Z be a space and $a \leq b : Z \to \mathbb{R}_{\geq 0}$ be two upper semi-continuous functions. Let

$$A := \{(z,t) \in Z \times \mathbb{R}_{\geq 0} \,|\, t \leq a(z)\} \quad \text{and} \quad B := \{(z,t) \in Z \times \mathbb{R}_{\geq 0} \,|\, t \leq b(z)\}. \tag{4.29}$$

Let $s : B \to \mathbb{J}$ be a stratification such that $s|_{\{z\}\times[0,b(z)]}$ is increasing for all $z \in Z$. Then the inclusion $A \hookrightarrow B$ is a directed cofibration.

Proof. The fibers of the projections $A_{\leq j} \to (Z \times \{0\})_{\leq j}$ and $B_{\leq j} \to (Z \times \{0\})_{\leq j}$ are intervals, so we have homotopy equivalences $A_{\leq j} \simeq (Z \times \{0\})_{\leq j} \simeq B_{\leq j}$. The inclusion $A \hookrightarrow B$ satisfies the first condition of Theorem 4.15, and is therefore a directed cofibration. □

The following corollary of Theorem 4.15 will be used frequently in future proofs:

Lemma 4.17 *Let $p : E \to X$ be a stratified fibration and consider the lifting problem*

$$\begin{array}{ccc} S^k & \xrightarrow{\alpha} & E \\ \big\downarrow & \overset{\ell}{\nearrow} & \big\downarrow \\ D^{k+1} & \xrightarrow{\beta} & X. \end{array} \tag{4.30}$$

Suppose that we have a solution $\tilde{\ell}$ to a similar lifting problem

$$\begin{array}{ccc} S^k & \xrightarrow{\alpha} & E \\ \big\downarrow & \overset{\tilde{\ell}}{\nearrow} & \big\downarrow \\ D^{k+1} & \xrightarrow{\tilde{\beta}} & X, \end{array} \tag{4.31}$$

and that $\tilde{\beta}$ factors as $\tilde{\beta} = \beta \circ b$. Suppose moreover that $b|_{S^k}$ is the identity and that the fibers of $b : D^{k+1} \to D^{k+1}$ are all contractible. Then our original lifting problem (4.30) admits a solution.

Proof. Let C be the mapping cylinder of b and $\iota : D^{k+1} \to C$ be the inclusion of the codomain. By theorem 4.15, the inclusion $D^k \cup (\partial D^k \times [0,1]) \hookrightarrow C$ is a directed cofibration. It follows that

$$\begin{array}{ccccccc} & & D^k \cup \big(\partial D^k \times [0,1]\big) & \longrightarrow & D^k & \xrightarrow{\tilde{\ell}} & E \\ & & \big\downarrow & & \overset{h}{\nearrow} & & \big\downarrow \\ D^{k+1} & \xrightarrow{\iota} & C & \longrightarrow & D^{k+1} & \xrightarrow{\beta} & X. \end{array} \tag{4.32}$$

admits a solution $h : C \to E$. Now letting $\ell := h \circ \iota$ produces the solution to (4.30). □

4.4 Stratified fibrations

The statements dual to Lemma 4.10 hold by the same formal arguments.

Lemma 4.18 *1. The pullback of a stratified fibration along a stratified map is a stratified fibration.*

2. The composite of stratified fibrations is a stratified fibration.

3. A retract of a stratified fibration is a stratified fibration.

The property of being a stratified fibration is a local property with respect to closed covers.

Lemma 4.19 *Let $p : E \to X$ be a map and $\{V_\alpha\}$ a closed cover of X. If all the restrictions $p|_{V_\alpha} : E|_{V_\alpha} \to V_\alpha$ are stratified fibrations, then so is p.*

Proof. Let $a, b : Z \to \mathbb{R}_{\geq 0}$ and $\iota : A \hookrightarrow B$ be as in (4.29). The generating acyclic cofibrations are of that form, it's enough to show that

$$\begin{array}{ccc} A & \longrightarrow & E \\ {\scriptstyle \iota}\big\downarrow & \nearrow & \big\downarrow \\ B & \xrightarrow{f} & X \end{array} \tag{4.33}$$

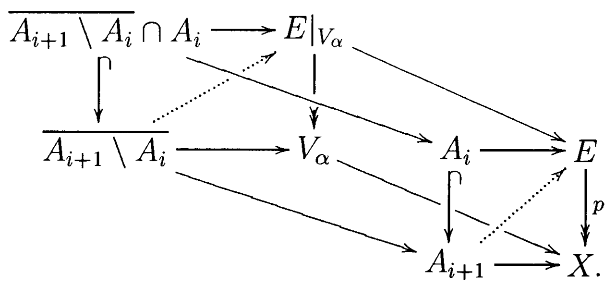
has a lift. Triangulate B so that each simplex in that triangulation maps to a given V_α and so that the projection $\mathrm{pr} : B \to Z$ is simplicial.

We can build B from A by successively adding each simplex of that triangulation. More precisely, we can write ι as a sequence

$$A = A_0 \hookrightarrow A_1 \hookrightarrow \ldots \hookrightarrow A_r = B \tag{4.34}$$

where each A_i is of the form $\{(z, t) \in Z \times \mathbb{R}_{\geq 0} \,|\, t \leq a_i(z)\}$ for some appropriate functions $a_i : Z \to \mathbb{R}_{\geq 0}$. Each layer $A_{i+1} \setminus A_i = \{(z, t) \,|\, a_i(z) < t \leq a_{i+1}(z)\}$ maps to some V_α, and so does it closure $\overline{A_{i+1} \setminus A_i}$. The inclusion $\overline{A_{i+1} \setminus A_i} \cap A_i \hookrightarrow \overline{A_{i+1} \setminus A_i}$ is of the form (4.29), and is therefore a directed cofibration.

We now build lifts $A_i \to E$ inductively on i. Suppose that we have $A_i \to E$. In order to extend it to A_{i+1}, we find some V_α containing $f(\overline{A_{i+1} \setminus A_i})$ and write down the following commutative diagram



The lift $\overline{A_{i+1} \setminus A_i} \to E|_{V_\alpha}$ exists since $p|_{V_\alpha}$ is a stratified fibration. Assembling it with the existing map $A_i \to E$ produces the desired lift $A_{i+1} \to E$. □

Ramified covers provide the first interesting examples of stratified fibrations.

Lemma 4.20 *A map of stratified spaces $E \to X$, which is a covering when restricted to each stratum X_j, is a stratified fibration.*

Proof. Let $\Lambda^{n,i} \hookrightarrow \Delta^n$ be a generating directed cofibration and consider the lifting problems

$$\begin{array}{ccc} \Lambda^{n,i} & \xrightarrow{f} & E \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & X. \end{array} \tag{4.35}$$

Without loss of generality, we can assume that $\mathbb{J} = \{0, 1, \ldots, n\}$.

Since we're not in the case $i = n = 1$, both $(\Lambda^{n,i})_0$ and $(\Delta^n)_0$ are contractible. The map $E_0 \to X_0$ is a cover, so we have a (unique) lift

$$\begin{array}{ccc} (\Lambda^{n,i})_0 & \xrightarrow{f} & E_0 \\ \downarrow & \overset{\ell}{\nearrow} & \downarrow \\ (\Delta^n)_0 & \longrightarrow & X_0. \end{array} \tag{4.36}$$

Taking the closure of the graph of ℓ produces a map $\bar{f} : \overline{(\Delta^n)_0} = \Delta^n \to E$.

We now show that $\bar{f}$ agrees with f on $\Lambda^{n,i}$. Clearly, $\bar{f} = f$ on $\overline{(\Lambda^{n,i})_0}$. If $i = 0$, that's all of $\Lambda^{n,i}$. Otherwise $\overline{(\Lambda^{n,i})_0} = (\Lambda^{n,i})_0 \cup d^0(\Lambda^{n-1,i-1})$, where $d^0 : \Delta^{n-1} \to \Delta^n$ is the 0th coface map. Since we're not in the case $i = n = 2$, both $(d^0(\Lambda^{n-1,i-1}))_1$ and $(d^0(\Delta^{n-1}))_1$ are contractible. The map $E_1 \to X_1$ is a cover, so the diagram

$$\begin{array}{ccc} \left(d^0(\Lambda^{n-1,i-1})\right)_1 & \xrightarrow{f=\bar{f}} & E_1 \\ \big\downarrow & \overset{\exists!}{\nearrow} & \big\downarrow \\ \left(d^0(\Delta^n)\right)_1 & \longrightarrow & X_1. \end{array} \tag{4.37}$$

has a unique lift. Both f and $\bar{f}$ are solutions of (4.37), so they agree on $(d^0(\Delta^n))_1$. This shows that $f = \bar{f}$ on $\overline{(\Lambda^{n,i})_0} \cup (d^0(\Delta^n))_1$. Since $\overline{(\Lambda^{n,i})_0} \cup (d^0(\Delta^n))_1$ is dense in $\Lambda^{n,i}$, $\bar{f}$ agrees with f on $\Lambda^{n,i}$. □

Theorem 4.21 (Whitehead's Theorem) *Let*

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & X & \end{array} \tag{4.38}$$

be a commuting diagram where p_1 is a stratified fibration. Assume that f induces homotopy equivalences on all fibers. Then f has a right homotopy inverse g relatively to X. Namely $p_1 \circ g = p_2$ and $f \circ g$ is homotopic to Id_{E_2} by a homotopy $h : E_2 \times [0,1] \to E_2$ satisfying $p_2 \circ g = p_2 \circ \mathrm{pr}_1$.

If p_2 is also a stratified fibration, then E_1 and E_2 are homotopy equivalent as spaces over X.

Proof. Let $\mathcal{T}$ be an oriented triangulation of E_2 compatible with the stratification. We build the $g : E_2 \to E_1$ by induction on the skeleta $E_2^{(k)}$ of the above triangulation.

Assume that g has been defined on $E_2^{(k-1)}$, along with the corresponding homotopy $h : f \circ g \Rightarrow \mathrm{Id}$. To extend it to $E_2^{(k)}$, we do it on each k-simplex individually. Let $i_0, i_1 : \Delta^k \to \Delta^k \times [0,1]$ be the two inclusions, and r the retraction of Δ^k onto its last vertex e_k. Let $\sigma : \Delta^k \to E_2$ be a simplex, and let F_1, F_2 be the fibers of p_1, p_2 over the point $p_2(\sigma(e_k)) \in X$. We want to define $g \circ \sigma : \Delta^k \to E_1$ in a way compatible with the existing map $g \circ \sigma|_{\partial\Delta^k}$.

We pick a lift m of

$$\begin{array}{ccccc} \partial\Delta^k & \xrightarrow{\sigma} & E_2^{(k-1)} & \xrightarrow{g} & E_1 \\ {\scriptstyle i_0}\big\downarrow & & & {\scriptstyle m}\nearrow & \big\downarrow{\scriptstyle p_1} \\ \partial\Delta^k \times [0,1] & \xrightarrow{r} & \Delta^k & \xrightarrow{p_2 \circ \sigma} & X. \end{array} \tag{4.39}$$

The element $\alpha := f \circ m \circ i_1 : \partial\Delta^k \to E_2$ represents an element of $\pi_{k-1}(F_2)$. The three maps σ, $h \circ \sigma|_{\partial\Delta^k}$ and $f \circ m$ assemble to a disk $\eta : D^k \to E_2$ bounding α. This shows that α is nullhomotopic in E_2.

The class α is also nullhomotopic in F_2. Let $j_0, j_1 : D^k \to D^k \times [0,1]$ be the inclusions, $r' : D^k \times [0,1] \to D^k$ be a retraction of D^k onto its basepoint, and n be a lift of

$$\begin{array}{ccccc} (D^k \times \{0\}) \cup (S^{k-1} \times [0,1]) & \xrightarrow{\mathrm{pr}_1} & D^k & \xrightarrow{\eta} & E_2 \\ \big\downarrow{\scriptstyle j_0} & & & {\scriptstyle n}\nearrow & \big\downarrow{\scriptstyle p_2} \\ D^k \times [0,1] & \xrightarrow{r'} & D^k & \xrightarrow{p_2 \circ \eta} & X. \end{array} \tag{4.40}$$

Then $n \circ j_1$ produces such a nullhomotopy.

Since $\alpha = f \circ m \circ i_1$ is zero in $\pi_{k-1}(F_2)$, we know by assumption that $m \circ i_1$ is zero in $\pi_{k-1}(F_1)$. Let $\beta : D^k \to\subset E_1$ be a disk bounding it, and make sure that $f \circ \beta$ is homotopic in F_2 to the map $n \circ j_1 : D^k \to E_2$ provided by (4.40).

The maps m and β assemble to a disk $\tau : D^k \to E_1$ bounding $g \circ \sigma|_{\partial\Delta^k}$. This disk can not be used to define $g \circ \sigma$ because its composite with p_1 does not agree with $p_2 \circ \sigma$. However, we are in a situation where we can apply Lemma 4.17. The map τ plays the role of ℓ in (4.30), and the map $b : D^k \to \sigma$ is assembled from r and from the constant map at e_k. So we get our map $g \circ \sigma : \Delta^k \to E_1$ making the diagram (4.38) commute. This extends g to each k-simplex σ, and thus to the hole of $E_2^{(k)}$.

We now extend the homotopy h. Again, we extend it to each k-simplex σ individually. Consider the map $n : D^k \times [0,1] \to E_2$ of (4.40), the homotopy between $n \circ j_1 : D^k \to E_2$ and $f \circ \beta$, and the composite of f with the map $C \to E_1$ constructed in the proof of Lemma 4.17. These three maps assemble to a homotopy

$D^k \times [0,1] \to E_2$ between σ and $g \circ f \circ \sigma$. This homotopy satisfies the assumptions of Lemma 4.17, so we get a new homotopy $h \circ (\sigma \times [0,1]) : D^k \times [0,1] \to E_2$, compatible with the projection to X. We have extended h to each k-simplex σ, thus finishing the inductive construction of g and h.

Now assume that p_2 is also a stratified fibration. By applying the first part of the theorem to the map g, we learn that it also has a right homotopy inverse. The map f had a right homotopy inverse, which itself has a right homotopy inverse. So, by a well known argument, we deduce that f has a two sided homotopy inverse. In other words, f is a homotopy equivalence. □

Here's an alternate proof of Theorem 4.21 which uses Theorem 4.29 (and a bit of non-abelian cohomology).

Proof. Let C be the mapping cone of f and r its obvious map to X. The obstructions to the existence of a lift

$$\begin{array}{ccc} E_1 & = & E_1 \\ \downarrow & \overset{\ell}{\nearrow} & \downarrow p \\ C & \xrightarrow{r} & X \end{array} \qquad (4.41)$$

lie in the relative cohomology group $H^*(C, E_1; \mathcal{F})$, where $\mathcal{F}$ denotes the cosheaf $\pi_*(\textit{Fiber of } p)$. This relative cohomology group can be computed by the Leray-Serre spectral sequence

$$H^*\big(X; H^*(\textit{Fiber of } r,\ \textit{Fiber of } p; \mathcal{F})\big) \Rightarrow H^*(C, E_1; \mathcal{F}). \qquad (4.42)$$

Since $\mathcal{F}$ is constant on each fiber $r^{-1}(x)$, and since the inclusions $p^{-1}(x) \to r^{-1}(x)$ are homotopy equivalences, we have $H^*(r^{-1}(x), p^{-1}(x); \mathcal{F}) = 0$ for all $x \in X$. The spectral sequence (4.42) is identically zero, and so are the obstruction groups $H^*(C, E_1; \mathcal{F})$. Composing the solution ℓ of (4.41) with the inclusion $E_2 \to C$ produces a right homotopy inverse to f. □

4.5 The fundamental category

If $E \to X$ is a Serre fibration, then any path in the base X induces a homotopy class of homotopy equivalences between the fibers over the endpoints. More precisely, we get a functor from the fundamental groupoid of X with values in the homotopy category of spaces. Something similar happens for stratified fibrations.

Given a stratified space (X, s), let us say that a path $\gamma : [0,1] \to X$ is directed if $s \circ \gamma : [0,1] \to \mathbb{J}$ is (weakly) increasing. The composition of directed paths is directed, so we get:

Definition 4.22 *The fundamental category $\Pi_1(X)$ of a stratified space X has an object for each point $x \in X$ and a morphism $x \to y$ for each directed path γ from x to y. Two paths from x to y are identified in $\Pi_1(X)$ if they are homotopic through directed paths. The composition of morphisms is given by concatenation of paths.*

Two objects are isomorphic in $\Pi_1(X)$ if and only if they are in the same connected component of a the same stratum X_i.

Lemma 4.23 *Let $p : E \to X$ be a stratified fibration. Then the assignment $x \mapsto p^{-1}(x)$ extends to a functor ∇ from $\Pi_1(X)$ to the homotopy category of spaces.*

Proof. Let x, y be points in X and let γ be a path from x to y. Let F_x and F_y denote the fibers over x and y respectively. To build $\nabla_\gamma : F_x \to F_y$, we consider the lifting problem

$$\begin{array}{ccccc} F_x \times \{0\} & \longrightarrow & & & E \\ \downarrow & & \overset{\ell}{\nearrow} & & \downarrow \\ F_x \times [0,1] & \xrightarrow{\mathrm{pr}_2} & [0,1] & \xrightarrow{\gamma} & X, \end{array} \tag{4.43}$$

where the top map is the inclusion. We define ∇_γ, to be the composite $F_x \times \{1\} \hookrightarrow F_x \times [0,1] \to E$.

We now show that ∇_γ only depends on the homotopy class of γ. Let $\gamma' : [0,1] \to X$ be another path, and let $h : [0,1]^2 \to X$ be a homotopy between γ and γ'.

Precomposing the solution to

$$\begin{array}{ccc} F_x \times (\{0\} \times [0,1] \cup [0,1] \times \{0,1\}) & \xrightarrow{\ell \cup \ell'} & E \\ \downarrow & \nearrow & \downarrow \\ F_x \times [0,1]^2 & \xrightarrow{\mathrm{pr}_2} [0,1]^2 \xrightarrow{h} & X \end{array} \tag{4.44}$$

with the inclusion of $F_x \times \{1\} \times [0,1]$ provides the desired homotopy between ∇_γ and $\nabla_{\gamma'}$. This also shows that the lift ℓ is well defined up to homotopy. □

Corollary 4.24 *Let $E \to X$ be a stratified fibration. Then the homotopy type of the fibers is constant on each connected component of each stratum of X.*

Lemma 4.25 *Let X be a stratified space and $\mathcal{T}$ a compatible oriented triangulation. Let X^{op} be the opposite of X defined with respect to $\mathcal{T}$ (see Definition 4.7). Then we have an equivalence of categories between $\Pi_1(X^{op})$ and $\Pi_1(X)^{op}$.*

Proof. Given a point $x \in X$, let $\sigma : \Delta^n \to X$ be the smallest non-degenerate simplex of $\mathcal{T}$ in the image of which x lies. Let $e_0 \in \Delta^n$ be the 0th vertex. Since $\mathcal{T}$ is compatible with the stratification, x is in the same stratum as $\sigma(e_0)$. This shows that for every point $x \in X$, there exists a vertex of $\mathcal{T}$ which is in the same connected component of the same stratum.

Two points in the same connected component of the same stratum are isomorphic as objects of $\Pi_1(X)$. Therefore the full subcategory $\Pi'_1(X)$ whose objects are the vertices of $\mathcal{T}$ is equivalent to $\Pi_1(X)$.

Any directed path between objects of $\Pi'_1(X)$ can be homotoped to a path that follows the edges of the triangulation. Similarly, any homotopy can be homotoped to one that remains in the 2-skeleton on X. So we get the following presentation of $\Pi'_1(X)$. We have a generator for each edge of $\mathcal{T}$. If an edge has a constant stratification, then we also have an inverse to the above generator. Each 2-simplex of $\mathcal{T}$ gives a relation.

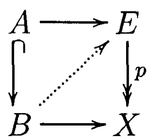
By the same argument, $\Pi_1(X^{op})$ is equivalent to a category $\Pi'_1(X^{op})$ on the same set of objects as $\Pi'_1(X)$, and with the opposite presentation. We conclude that

$\Pi'_1(X^{op}) = \Pi'_1(X)^{op}$ and therefore $\Pi_1(X^{op}) \simeq \Pi_1(X)^{op}$. □

4.6 Obstructions to lifting

This section sets up an obstruction theory for lifting maps across stratified fibrations.

If $p : E \to X$ is a usual fibration, the obstructions to finding a lift



live in the relative cohomology groups $H^*(B, A; \pi_*(F))$, where F denotes the fiber of p. It might happen that $\pi_1(B)$ acts non-trivially on $\pi_*(F)$. In that case, the above groups should be understood as the cohomology of a locally constant sheaf on B.

If $E \to X$ is a stratified fibration, then $\pi_*(F)$ still makes sense, but it will fail in general to be locally constant. It will instead be a constructible cosheaf.

Definition 4.26 *A constructible sheaf is a contravariant functor from the fundamental category* $\Pi_1(X)$ *to the category of abelian groups. A constructible cosheaf on a stratified space* X *is a covariant functor from* $\Pi_1(X)$ *to abelian groups.*

By proposition 4.25, a constructible cosheaf $\mathcal{A}$ on X is equivalent to a constructible sheaf $\mathcal{A}^{op}$ on X^{op}. Dually, a constructible sheaf on X is equivalent to a constructible cosheaf on X^{op}.

Example 4.27 Let $p : E \to X$ be a stratified fibration. Suppose that all the fibers $p^{-1}(x)$ are connected and have trivial π_1-action on π_k. Then composing the functor ∇ given by Lemma 4.23 with the k-th homotopy group functor produces a constructible cosheaf. We denote it by $\pi_k(F)$, the letter F standing for "a fiber".

There is a well known notion of cohomology of a space with coefficients in a sheaf $\mathcal{F}$. Similarly, we can take homology with coefficients in a cosheaf $\mathcal{A}$. In our setting, the easiest ways to define them is to introduce the simplicial chain and cochain complexes

$C_*(X;\mathcal{A})$ and $C^*(X;\mathcal{F})$ given by

$$C_k = \bigoplus_{\substack{k\text{-simplices} \\ \sigma}} \mathcal{A}(\sigma) \qquad \text{and} \qquad C^k = \prod_{\substack{k\text{-simplices} \\ \sigma}} \mathcal{F}(\sigma). \tag{4.45}$$

Here $\mathcal{A}(\sigma)$ and $\mathcal{F}(\sigma)$ denote the values of $\mathcal{A}$ and $\mathcal{F}$ at the 0th vertex of σ (or equivalently, at any point in the interior of σ). The differential in C_* and C^* is the usual alternating sum of (co)face maps.

We define the cohomology of a constructible cosheaf by using the opposite constructible sheaf.

Definition 4.28 *Given a constructible cosheaf $\mathcal{A}$ on a space X, we define its cohomology to be*

$$H^*(X;\mathcal{A}) := H^*(X;\mathcal{A}^{op}).$$

Similarly, for a pair (X,Y), we let $H^*(X,Y;\mathcal{A}) := H^*(X,Y;\mathcal{A}^{op})$.

Given a simplex σ of X with corresponding simplex σ^{op} of X^{op}, the first vertex of σ is the last vertex of σ^{op}, and vice versa. So the complex $C^*(X;\mathcal{A})$ computing cosheaf cohomology now looks like

$$C^*(X;\mathcal{A}) = \prod_{\substack{k\text{-simplices} \\ \sigma}} \mathcal{A}(x_\sigma), \tag{4.46}$$

where x_σ denotes the last vertex of σ. Given a cochain $c \in C^n(X;\mathcal{A})$, with components $c(\sigma) \in \mathcal{A}(x_\sigma)$, its differential is then given by

$$\delta c(\sigma) = \sum_{i=0}^{n-1} (-1)^i c(d_i\sigma) + (-1)^n \mathcal{A}(\gamma) c(d_n\sigma), \tag{4.47}$$

where d_i are the usual face maps, γ is the edge from the $(n-1)$st to the nth vertex of σ, and $\mathcal{A}(\gamma)$ is the map from the value of $\mathcal{A}$ at the $(n-1)$st vertex to the value of $\mathcal{A}$ at the nth vertex. The relative cochain complex $C^*(X,Y;\mathcal{A}) \subset C^*(X,\mathcal{A})$ is the subcomplex of cochains vanishing on the simplices of Y.

We can now state the main theorem of this section:

Theorem 4.29 *Let $E \to X$ be a stratified fibration, whose fibers are connected and have trivial action of π_1 on π_*. Let $A \hookrightarrow B$ be a cofibration of stratified spaces and let $A \to E$ and $B \to X$ be stratified maps making the following diagram commute:*

$$\begin{array}{ccc} A & \longrightarrow & E \\ \big\downarrow & \nearrow & \big\downarrow \\ B & \longrightarrow & X. \end{array} \tag{4.48}$$

Then the obstructions to finding a lift $\ell : B \to E$ making both triangles commute live in the cosheaf cohomology groups $H^{k+1}(B, A; \pi_k(F))$.

Proof. We build the lift $B \to E$ by induction on the skeleta of B.

Suppose that we have a lift $\ell : A \cup B^{(k)} \to E$. Such a lift defines a cocycle $c = c_\ell \in C^{k+1}(B, A; \pi_k(F))$ as follows. Let $i_0, i_1 : \Delta^{k+1} \to \Delta^{k+1} \times [0,1]$ be the two inclusions, and r the retraction to the last vertex of Δ^{k+1}. Then given a simplex $\sigma : \Delta^{k+1} \to B$, we pick a lift m of

$$\begin{array}{ccccc} \partial\Delta^{k+1} & \xrightarrow{\sigma} & X^{(k)} & \xrightarrow{\ell} & E \\ {\scriptstyle i_0}\big\downarrow & & & {\scriptstyle m}\nearrow & \big\downarrow \\ \partial\Delta^{k+1} \times [0,1] & \xrightarrow{r} & \Delta^{k+1} & \xrightarrow{\sigma} & X \end{array} \tag{4.49}$$

and let $c(\sigma)$ be the map $m \circ i_1 : \partial\Delta^{k+1} \to \Delta^{k+1} \times [0,1] \to F \subset E$, where $F = p^{-1}(x_\sigma)$ is the fiber over the last vertex of σ.

The values of c are well defined elements of $\pi_k(F)$. Suppose that we have two solutions m and m' of (4.49), giving two elements $c(\sigma), c(\sigma)' \in \pi_k(F)$. We can then find a lift n of

$$\begin{array}{ccccc} \partial\Delta^{k+1} \times \big(\{0\} \times [0,1] \cup [0,1] \times \{0,1\}\big) & & \xrightarrow{(\ell\times[0,1]) \cup m \cup m'} & & E \\ \big\downarrow & & {\scriptstyle n}\nearrow & & \big\downarrow \\ \partial\Delta^{k+1} \times [0,1]^2 & \xrightarrow{1\times \mathrm{pr}_1} & \partial\Delta^{k+1} \times [0,1] & \xrightarrow{\sigma\circ r} & X. \end{array} \tag{4.50}$$

The map $n \circ (i_1 \times 1) : \partial\Delta^{k+1} \times [0,1] \to \partial\Delta^{k+1} \times [0,1]^2 \to F \subset E$ then provides a homotopy between $c(\sigma)$ and $c(\sigma)'$, thus proving their equality in $\pi_k(F)$.

The lift ℓ extends to $B^{(k+1)}$ if and only if $c = 0$. Indeed, if ℓ extends, we can use that extension to produce a solution of (4.49). The resulting map $c(\sigma) : \partial\Delta^{k+1} \to F$ will then be constant, hence trivial in $\pi_k(F)$. Conversely, suppose that $c = 0$ in $\pi_k(F)$. The homotopy m defining c and the disk bounding c assemble to a disk D^{k+1} bounding $\ell \circ \sigma|_{\partial\Delta^{k+1}}$. That disk satisfies the hypothesis of Lemma 4.17 so we get our desired lift.

We now show that c is closed and therefore defines an element in $H^{k+1}(B, A; \pi_k(F))$. Let $i_0, i_1 : \Delta^{k+2} \to \Delta^{k+2} \times [0,1]$ be the two inclusions, and r the retraction to the last vertex of Δ^{k+2}. Given a simplex $\tau : \Delta^{k+2} \to B$, we let s be a lift of

$$\begin{array}{ccccc} sk_k(\Delta^{k+2}) & \xrightarrow{\tau} & X^{(k)} & \xrightarrow{\ell} & E \\ {\scriptstyle i_0}\big\downarrow & & & {\scriptstyle s}\nearrow & \big\downarrow \\ sk_k(\Delta^{k+2}) \times [0,1] & \xrightarrow{r} & \Delta^{k+2} & \xrightarrow{\tau} & X, \end{array} \tag{4.51}$$

and consider the map $f := s \circ i_1 : \mathrm{sk}_k(\Delta^{k+2}) \to \mathrm{sk}_k(\Delta^{k+2}) \times [0,1] \to F \subset E$. Precomposing f with the coface maps $d^i : \partial\Delta^{k+1} \to \mathrm{sk}_k(\Delta^{k+2})$ produces the $k+3$ terms in (4.47). For $i < k+2$, the map $f \circ d^i$ agrees with the definition of $c(d_i\tau)$. To see that the remaining map $f \circ d^{k+2}$ represents $\mathcal{A}(\gamma)c(d_{k+2}\tau)$, we use a diagram similar to (4.50). The top map is instead assembled from the the restriction to $d^{k+2}(\partial\Delta k + 1)$ of the map $s \circ i_1$ given in (4.51), and from the composite of the homotopies used to define c and $\mathcal{A}(\gamma)$. The $k+3$ summands in (4.47) assemble to a map $sk_k(\Delta^{k+2}) \to F$. Therefore $\delta c(\tau)$ is zero in $\pi_k(F)$.

Assuming now that $[c] = [c_\ell] = 0 \in H^{k+1}(B, A; \pi_k(F))$ we produce a new lift $\ell' : B^{(k)} \to E$ which agrees with ℓ on $B^{(k-1)}$. The new obstruction cocycle $c_{\ell'}$ then vanishes identically and ℓ' thus extends to the $(k+1)$-skeleton. To build ℓ', we write $c = \delta b$ for some $b \in C^k(B, A; \pi_k(F))$ and geometrically subtract b from ℓ. More concretely, given a simplex $\rho : \Delta^k \to B$, we assemble $\ell \circ \rho$ and $-b(\rho)$ to a map $D^k \to E$ bounding $\ell \circ \rho|_{\partial\Delta^k}$. We then apply Lemma 4.17 to produce the new lift ℓ'.

At last, we verify that $c_{\ell'} = 0$. Let $\sigma : \Delta^{k+1}B$ be a simplex. It is enough by Lemma 4.17 to build an appropriate disk D^{k+1} bounding $\ell' \circ \sigma|_{\partial\Delta^{k+1}}$. Such a disk can be obtained by assembling the homotopy m of (4.49), the $k+2$ mapping cylinders used in the construction of ℓ' for each facet of σ (see proof of Lemma 4.17), the homotopy used in the definition of $\mathcal{A}(\gamma)c_\ell(d_{k+1}\sigma)$ for the cosheaf $\mathcal{A} = \pi_k(F)$ (see Example 4.27 and Lemma 4.23), and the homotopy $S^k \times [0,1] \to F$ proving the equality $c_\ell(\sigma) = \delta b(\sigma)$ in $\pi_k(F)$. □

A startling consequence of Theorem 4.29 is that the property of being a stratified fibration is in some sense independent of the stratification of the base.

Proposition 4.30 *Let (X, s) be a $\mathbb{J}'$-stratified space, $f : \mathbb{J}' \to \mathbb{J}$ a map of posets, and $s := f \circ s'$ the corresponding $\mathbb{J}$-stratification. Let $L_f : \Pi_1(X, s') \to \Pi_1(X, s)$ be the map on fundamental categories induced by the identity $X \to X$. Let $p : E \to X$ be a stratified fibration with respect to s'.*

If the functor $\nabla_{s'} : \Pi_1(X, s') \to Ho(\mathsf{spaces})$ factors through L_f then $E \to X$ is also a stratified fibration with respect to s.

Proof. Let $\iota : A \hookrightarrow B$ be a cofibration, and consider our usual lifting problem

$$\begin{array}{ccc} A & \longrightarrow & E \\ {\scriptstyle\iota}\downarrow & & \downarrow \\ B & \xrightarrow{\beta} & X. \end{array}$$

By Theorem 4.29, the obstructions to finding a lift $B \to E$ lie in $H^{k+1}(B, A; \pi_k(F))$. Suppose that ι is directed, with respect to $s \circ beta$, we want to show that the obstruction groups vanish. By the long exact sequence in cohomology, it's enough to show that the restriction maps $H^*(B; \pi_k(F)) \to H^*(A; \pi_k(F))$ are isomorphisms. Without loss of generality, we can assume that B is compact, and therefore that $\mathbb{J} = \{0, 1, \ldots, k\}$.

Let $\mathcal{F}$ be the sheaf $(\pi_k(F))^{op}$ used to define $H^*(-; \pi_k(F))$. We show by induction on $j \in \mathbb{J}$, that

$$H^*(B^{op}_{\leq j}; \mathcal{F}) \to H^*(A^{op}_{\leq j}; \mathcal{F}) \tag{4.52}$$

are isomorphisms. So let's assume that (4.52) is an isomorphism. In order to show that the same thing holds for $j+1$, it's enough by the five lemma to show that

$$\widetilde{H}^*(B^{op}_{\leq j+1}/B^{op}_{\leq j'};\mathcal{F}) \to \widetilde{H}^*(A^{op}_{\leq j+1}/A^{op}_{\leq j};\mathcal{F}) \tag{4.53}$$

is an isomorphism. Now $A^{op}_{\leq j+1}/A^{op}_{\leq j} \hookrightarrow B^{op}_{\leq j+1}/B^{op}_{\leq j}$ is a homotopy equivalence because ι was assumed to be a directed cofibration with respect to s. And the sheaf $\mathcal{F}$ is locally constant on $B^{op}_{\leq j+1} \setminus B^{op}_{\leq j}$ because of the factorization of $\nabla_{s'}$. It follows that (4.53) are isomorphisms. The maps (4.52) are also isomorphisms, the obstruction groups vanish, and so our lift exists. □

4.7 Stratified simplicial sets

Unfortunately, applying simple categorical constructions to $\mathbb{J}$-stratified spaces quickly produces pathological spaces. For example, the terminal object is the set $\mathbb{J}$ equipped with the topology generated by the subsets $\{j \in \mathbb{J} \,|\, j \leq j_0\}$. In particular, we have not been able to find a convenient model category structure on $\mathbb{J}$-stratified spaces. Because of this problem, we introduce an analogue of stratifications in the world of simplicial sets. A stratification of a topological space X is some kind of map from X to $\mathbb{J}$. Analogously, we let:

Definition 4.31 *Let $\mathbb{J}$ be a poset and $N\mathbb{J}$ its nerve. An stratification of a simplicial set X by $\mathbb{J}$ is a map $s : X \to N\mathbb{J}$.*

Given $j \in \mathbb{J}$, we also introduce the notations

$$\begin{array}{lll} X_j := s^{-1}(j) & X_{\leq j} := s^{-1}(\mathbb{J}_{\leq j}) & X_{<j} := s^{-1}(\mathbb{J}_{<j}) \\ & X_{\geq j} := s^{-1}(\mathbb{J}_{\geq j}) & X_{>j} := s^{-1}(\mathbb{J}_{>j}) \end{array} \tag{4.54}$$

where $\mathbb{J}_{\leq j}$, $\mathbb{J}_{<j}$, $\mathbb{J}_{\geq j}$, and $\mathbb{J}_{>j}$ are the sub-posets of elements smaller or equal, smaller, greater or equal, and greater than j respectively.

As for topological spaces and simplicial sets, there is an adjunction between the category of $\mathbb{J}$-stratified spaces and $\mathbb{J}$-stratified simplicial sets. Given a stratified simplicial set X, its geometric realization $|X|$ is stratified via the composite

$$|X| \xrightarrow{|s|} |N\mathbb{J}| \to \mathbb{J}, \tag{4.55}$$

where the second map is given by $(j_0 \leq \ldots \leq j_k; t_0, \ldots, t_k) \mapsto \min_{r:t_r \neq 0} j_r$.

Geometric realization has a right adjoint $\mathrm{Sing}_{\mathbb{J}}$ given by

$$\mathrm{Sing}_{\mathbb{J}}(X)_k = \coprod_{s:\Delta[k] \to N\mathbb{J}} \mathrm{Map}_{\mathbb{J}}\Big(\big|(\Delta[k], s)\big|, X\Big).$$

The map $\mathrm{Sing}_{\mathbb{J}}(X) \to N\mathbb{J}$ is given by $(f : |(\Delta[k], s)| \to X) \mapsto s \in \mathrm{Hom}(\Delta[k], N\mathbb{J}) = (N\mathbb{J})_k$.

We now define our model structure on the category of $\mathbb{J}$-stratified simplicial sets. We use the criterion, originally due to Kan, for building (cofibrantly generated) model structures [18, section 11.3], [19, chapter 2]. Recall that the i-horn $\Lambda[n, i] \subset \Delta[n]$ is the sub-simplicial set of $\Delta[n]$ generated by all proper faces containing the ith vertex.

Definition 4.32 *Let $\mathbb{J}$ be a poset and $\mathsf{sSet} \downarrow N\mathbb{J}$ be the category of simplicial sets over $N\mathbb{J}$. The stratified model structure on $\mathsf{sSet} \downarrow N\mathbb{J}$ is given by:*

- *The weak equivalences are a map $X \to Y$ that induces weak equivalences of simplicial sets $X_{\leq j} \to Y_{\leq j}$ for all $j \in \mathbb{J}$.*
- *The generating cofibrations are the maps $\partial\Delta[n] \to \Delta[n]$, where the stratification $s : \Delta[n] \to N\mathbb{J}$ is arbitrary.*
- *The generating acyclic cofibrations are the maps $\Lambda[n, i] \subset \Delta[n]$, where $i < n$ or s assigns the same value to the nth and $n-1$st vertices of $\Delta[n]$.*

Conjecture 4.33 *Assuming that $\mathbb{J}$ doesn't have infinite descending chains, then definition 4.32 defines a model category structure on $\mathsf{sSet} \downarrow N\mathbb{J}$.*

The fibrations in this conjectured model category are what we call stratified fibrations of simplicial sets. Using this model category, there exists an analog of Theorem 4.29 for lists across stratified fibrations.

We expect that the geometric realization of a stratified fibration of simplicial sets is a stratified fibration of spaces.

4.8 Geometric realization

In [30], Quillen showed that the geometric realization of a Kan fibration is a Serre fibration. We beleive that the corresponding fact also holds for stratified fibrations (Theorem 4.38).

We have been able to prove this fact only assuming the following technical result:

Conjecture 4.34 *Let $p : E \to X$ be a map of stratified topological spaces and $j \in \mathbb{J}$ an element such that $E_k = X_k = \emptyset$ for all elements $k \in \mathbb{J}$ which are not comparable with j.*

Suppose that p can be written as

$$\begin{array}{c} E \\ \Big\downarrow{\scriptstyle p} \\ X \end{array} = pushout \left(\begin{array}{ccccc} E_{\geq j} & \longleftarrow & \widetilde{E} & \hookrightarrow & \overline{E} \\ \Big\downarrow{\scriptstyle \hat{p}} & & \Big\downarrow{\scriptstyle \tilde{p}} & & \Big\downarrow{\scriptstyle \bar{p}} \\ X_{\geq j} & \longleftarrow & \widetilde{X} & \hookrightarrow & \overline{X} \end{array} \right) \tag{4.56}$$

If the three maps $\hat{p}$, $\overline{p}$ and $\widetilde{E} \to E_{\geq j} \times_{X_{\geq j}} \widetilde{X}$ are stratified fibrations, then so is p.

The archetype for a Serre fibration is the projection map $F \times X \to X$. Any Serre fibration is locally a retract of one of that form. The following Lemma provides a similar archetype for stratified fibrations.

Lemma 4.35 *Let X be be a $\mathbb{J}$-stratified space and let $(\{F_j\}, \{r_{jj'} : F_j \to F_{j'}\}_{j<j'})$ be a diagram of spaces indexed by $\mathbb{J}$. Suppose moreover that all the $r_{jj'}$ are Serre*

fibrations. Let E be the space

$$E = \coprod_{j} F_j \times X_{\geq j} \Big/ \sim , \tag{4.57}$$

where $(v, x) \sim (r_{jj'}(v), x)$ for all $x \in X_{\geq j'}$ and $v \in F_j$. Then the projection $p : E \to X$ is a stratified fibration.

Proof. Since stratified fibration is a local property, we may assume that $\mathbb{J}$ is finite. We prove the lemma by induction on the size of $\mathbb{J}$. We assume that it holds for $|\mathbb{J}| = j - 1$ and want to show that it holds for $|\mathbb{J}| = j$. Without loss of generality, we can assume that $\mathbb{J} = \{1, \ldots, j\}$.

The map $p : E \to X$ is a pushout of

$$\begin{array}{ccccc} F_j \times X_j & \twoheadleftarrow & F_{j-1} \times X_j & \hookrightarrow & E' \\ \downarrow & & \downarrow & & \downarrow \\ X_j & = & X_j & \hookrightarrow & X \end{array}$$

where

$$E' = \coprod_{j' < j} F_{j'} \times X_{\geq j'} \Big/ \sim . \tag{4.58}$$

The map $E' \to X$ is a stratified fibrations by induction. The conditions of Conjecture 4.34 are satisfied and so p is a stratified fibration. □

In order to apply Lemma 4.35 to the question of geometric realization, we show the following technical lemma.

Lemma 4.36 *Let $p : E \to \Delta[n]$ be a stratified fibration. Given a face $\sigma : \Delta[k] \hookrightarrow \Delta[n]$ we let $\Gamma(\sigma; p)$ denote the simplicial set of sections of $\sigma^* p$. We then have a homeomorphism*

$$|E| = \coprod_{\sigma : \Delta[k] \hookrightarrow \Delta[n]} |\Gamma(\sigma; p)| \times \Delta^k \Big/ (s|_{d_i(\sigma)}, x) \sim (s, d^i(x)), \tag{4.59}$$

where $d_i : \Delta[n]_k \to \Delta[n]_{k-1}$ and $d^i : \Delta^{k-1} \to \Delta^k$ are the face and coface maps.

Proof. Given a face $\sigma : \Delta[k] \to \Delta[n]$, we build a map $f_\sigma : |\Gamma(\sigma; p)| \times \Delta^k \to |E|$. Let $((\tau, t), y) \in |\Gamma(\sigma; p)| \times \Delta^k$ be a point, where $\tau \in \Gamma(\sigma; p)_r$, $t \in \Delta^r$, and $y \in \Delta^k$. The adjoint of $\tau : \Delta[r] \to \Gamma(\sigma; p)$ is a map

$$\tilde{\tau} : \Delta[r] \times \Delta[k] \to \sigma^* E.$$

which commutes with the projections to $\Delta[k]$.

For suitable map $\kappa : \Delta[m] \to \Delta[k]$, there exists an embedding $\iota : \Delta[r] \times \Delta[k] \hookrightarrow \Delta[m]$ making the diagram

$$\begin{array}{ccc} \Delta[r] \times \Delta[k] & \xrightarrow{\tilde{\tau}} & \sigma^* E \\ {\scriptstyle \iota}\downarrow & \overset{\nu}{\nearrow} & \downarrow{\scriptstyle \sigma^* p} \\ \Delta[m] & \xrightarrow{\kappa} & \Delta[k] \end{array} \tag{4.60}$$

commute (one can always pick $m = (r + 1)(k + 1) - 1$). The fibers of $|\kappa|$ and $|\kappa \circ \iota|$ are all contractible, so ι is a directed cofibration. Therefore (4.60) admits a lift ν. Let $\bar{\nu} : \Delta[m] \to E$ be the composite of ν with the inclusion $\sigma^* E \hookrightarrow E$. The adjoint of ι then fits into the following commutative triangle:

$$\begin{array}{ccc} & & \Gamma(\sigma; \kappa) \\ & \overset{\tilde{\iota}}{\nearrow} & \downarrow{\scriptstyle \nu^*} \\ \Delta[r] & \xrightarrow{\tau} & \Gamma(\sigma; p). \end{array} \tag{4.61}$$

Let $x := |\tilde{\iota}|(y)$. The preimage under κ of the ith vertex of $\Delta[k]$ is isomorphic to some sub-simplex $\Delta[\ell_i] \subset \Delta[m]$. The simplicial set of sections of κ is isomorphic to the product

$$\Gamma(\sigma; \kappa) = \Delta[\ell_0] \times \Delta[\ell_1] \times \ldots \times \Delta[\ell_k], \tag{4.62}$$

so we can view x as a point in $\prod \Delta^{\ell_i}$. The simplex Δ^m can be identified with the join $\Delta^{\ell_0} * \ldots * \Delta^{\ell_k}$ so we get a corresponding projection map

$$\varphi : \left(\Delta^{\ell_0} \times \Delta^{\ell_1} \times \ldots \times \Delta^{\ell_k}\right) \times \Delta^k \twoheadrightarrow \Delta^m. \tag{4.63}$$

At this point, we let $z := \varphi(x, y) \in \Delta^m$ and define $f_\sigma((\tau, t), y) := (\bar{\nu}, z)$.

We now show that f_σ is well defined. Let us replace ι by $\iota' := d^i \circ \iota : \Delta[r] \times \Delta[k] \to \Delta[m+1]$. Then we have $d_i(\nu') = \nu$ and therefore also $d_i(\bar{\nu}') = \bar{\nu}$. Composing with $d^i : \Delta[m] \to \Delta[m+1]$ induces a map $\Gamma(\sigma, \kappa) \to \Gamma(\sigma, \kappa')$ which, under the isomorphism (4.62) becomes

$$\begin{aligned} 1\times \ldots \times d^j \times \ldots \times 1 : \Delta[\ell_0] \times \ \ldots \ \times \Delta[\ell_a] \times \ \ldots \ \times \Delta[\ell_k] \\ \to \Delta[\ell_0] \times \ldots \times \Delta[\ell_a + 1] \times \ldots \times \Delta[\ell_k] \end{aligned}$$

for some appropriate j and a. We have a commutative diagram

$$\begin{array}{ccc} (\Delta^{\ell_0} \times \ldots \times \Delta^{\ell_a} \times \ldots \times \Delta^{\ell_k}) & \xrightarrow{\varphi} & \Delta^m \\ \downarrow {\scriptstyle 1\times\ldots\times d^j\times\ldots\times 1} & & \downarrow {\scriptstyle d^i} \\ (\Delta^{\ell_0} \times \ldots \times \Delta^{\ell_a+1} \times \ldots \times \Delta^{\ell_k}) & \xrightarrow{\varphi'} & \Delta^m \end{array}$$

and we can now compute

$$\begin{aligned} (\bar{\nu}', z') = (\bar{\nu}', \varphi'(x', y)) = (\bar{\nu}', \varphi'(|\tilde{\iota}'|(t), y)) = \Big(\bar{\nu}', \varphi'\big(|(1\times \ldots \times d^j \times \ldots \times 1) \circ \tilde{\iota}|(t), y\big)\Big) \\ = (\bar{\nu}', d^i\varphi(|\tilde{\iota}|(t), y)) = (\bar{\nu}', d^i\varphi(x, y)) = (\bar{\nu}', d^i(z)) = (d_i(\bar{\nu}'), z) = (\bar{\nu}, z). \end{aligned}$$

This shows us that $f_\sigma((\tau, t), y)$ doesn't depend on the choice of $\iota : \Delta[r] \times \Delta[k] \to \Delta[m]$.

We now assemble all the f_σ to get a map f from the RHS of (4.59) to $|E|$. Suppose that we have two equivalent points $((\tau, t), y) \in |\Gamma(\sigma; p)| \times \Delta^k$ and $((\tau', t'), y') \in |\Gamma(\sigma'; p)| \times \Delta^{k-1}$ where $\sigma' = d_i(\sigma)$. We can assume that $t = t'$, $y = d^i(y')$ and $\tau' = \tau \circ res$ where $res : \Gamma(\sigma, p) \to \Gamma(\sigma'; p)$ is the restriction map. The map $\tilde{\tau}' : \Delta[r] \times \Delta[k-1] \to \sigma'^* E$ is then the pullback of $\tilde{\tau}$ along the map $d^i : \Delta[k-1] \to \Delta[k]$. We may also take ι', κ', ν' to be the corresponding pullbacks of ι, κ, ν. We then have $\bar{\nu}' = d_I \bar{\nu}$, where d_I is some appropriate composition of face maps and $\tilde{\iota}' = res \circ \tilde{\iota}$, where $res : \Gamma(\sigma, \kappa) \to \Gamma(d_i(\sigma), \kappa')$ is the restriction. Under the isomorphism (4.62), this restriction map become the projection

$$\mathrm{pr}_i : \Delta[\ell_0] \times \ldots \times \Delta[\ell_i] \times \ldots \times \Delta[\ell_k] \to \Delta[\ell_0] \times \ldots \times \widehat{\Delta[\ell_i]} \times \ldots \times \Delta[\ell_k].$$

It satisfies the equation $\varphi(t, d^i(x)) = d^I\big(\varphi'(\mathrm{pr}_i(t), x)\big)$. Putting all this together, we see that

$$\begin{aligned} f_{\sigma'}((\tau',t),y') &= (\bar{\nu}', z') = \big(\bar{\nu}', \varphi'(x',y')\big) = \big(\bar{\nu}', \varphi'(|\tilde{\iota}'|(t), y')\big) \\ &= \big(d_I\bar{\nu}, \varphi'(|res \circ \tilde{\iota}|(t), y')\big) = \big(\bar{\nu}, d^I\varphi'(|\mathrm{pr}_i \circ \tilde{\iota}|(t), y')\big) \\ &= \big(\bar{\nu}, \varphi(|\tilde{\iota}|(t), d^i(y'))\big) = \big(\bar{\nu}, \varphi(|\tilde{\iota}|(t), y)\big) = \big(\bar{\nu}, \varphi(x,y)\big) = (\bar{\nu}, z) = f_\sigma((\tau,t),y). \end{aligned}$$

This finishes the proof that f : [RHS of (4.59)] $\longrightarrow |E|$ is well defined.

Finally, we show that f is an isomorphism. We do this by constructing an inverse f^{-1}. Let $(\bar{\nu}, z) \in |E|$ be a point given by $\bar{\nu} \in E_m$ and $z \in \Delta^m$. We assume furthermore that $\bar{\nu}$ is non-degenerate and that $z \in \mathring{\Delta}^m$. Let σ be the smallest sub-simplex of $\Delta[n]$ such that $p \circ \bar{\nu}$ factorizes as

$$\begin{array}{ccc} \Delta[m] & \xrightarrow{\bar{\nu}} & E \\ {\scriptstyle\kappa}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \Delta[k] & \xrightarrow{\sigma} & \Delta[n]. \end{array} \tag{4.64}$$

The map (4.63) is an isomorphism over $\mathring{\Delta}^k$, so we can define $(x,y) := \varphi^{-1}(z)$.

Now we consider the standard triangulation of $\Delta^{\ell_0} \times \ldots \times \Delta^{\ell_k}$. Let $\alpha : \Delta^r \to \Delta^{\ell_0} \times \ldots \times \Delta^{\ell_k}$ be a simplex in the image of which x lies and $t := \alpha^{-1}(x) \in \Delta^r$. We let τ be the composite

$$\tau : \Delta[r] \xrightarrow{\alpha} \Delta[\ell_0] \times \ldots \times \Delta[\ell_k] = \Gamma(\sigma; \kappa) \hookrightarrow \Gamma(\sigma; p). \tag{4.65}$$

The image of $(\bar{\nu}, z)$ under the map f^{-1} is then declared to be $((\tau,t),y)$. This shows that f is an isomorphism and thus finishes the proof of the lemma. □

Lemma 4.36 and Lemma 4.35 together imply

Lemma 4.37 *Let $s : \Delta[n] \to N\mathbb{J}$ be a stratification and $|s| : \Delta^n \to \mathbb{J}$ the induced stratification on the geometric realization.*

Let $p : E \to \Delta[n]$ be a stratified fibration with respect to s. Then its geometric realization $|p| : |E| \to \Delta^n$ is a stratified fibration with respect to $|s|$.

Proof. Let $\tilde{s}$ be the stratification of Δ^n by it's poset of faces. We first show that $|p|$

is a stratified fibration with the respect $\tilde{s}$.

By Lemma 4.36, $|p|$ is of the form (4.57) where the F_j's are replaced by $|\Gamma(\sigma; E)|$. In order to apply Conjecture 4.34, we need to show that the restriction maps $|\Gamma(\sigma; E)| \to |\Gamma(\sigma'; E)|$ are Serre fibrations. We show that the corresponding maps of simplicial sets

$$r : \Gamma(\sigma; E) \to \Gamma(\sigma'; E) \tag{4.66}$$

are Kan fibrations. Indeed, a lift

$$\begin{array}{ccc} A & \longrightarrow & \Gamma(\sigma; E) \\ {\scriptstyle \iota}\big\downarrow & \nearrow & \big\downarrow \\ B & \longrightarrow & \Gamma(\sigma'; E) \end{array} \tag{4.67}$$

is equivalent to a lift

$$\begin{array}{ccc} (A \times \sigma) \cup (B \times \sigma') & \longrightarrow & E \\ \big\downarrow & \nearrow & \big\downarrow \\ B \times \sigma & \longrightarrow & \Delta[n]. \end{array} \tag{4.68}$$

If ι is an acyclic cofibration, then $(A \times \sigma) \cup (B \times \sigma') \hookrightarrow B \times \sigma$ is a directed cofibration. The diagram (4.68) has a lift, and so does (4.67). We conclude that (4.66) is a Kan fibration. So by Conjecture 4.34, $|p|$ is a stratified fibration with the respect to $\tilde{s}$.

In order to show that $|p|$ is a stratified fibration with respect to $|s|$, we appeal to Lemma 4.30. We need to show that the restriction (4.66) are homotopy equivalences whenever $|\sigma|$ and $|\sigma'|$ are in the same stratum.

So we go back to (4.67), but now $A \hookrightarrow B$ is an arbitrary cofibration. Again, $(A \times \sigma) \cup (B \times \sigma') \hookrightarrow B \times \sigma$ is a directed cofibration, so (4.68) and (4.67) have lifts. We conclude that (4.66) is an acyclic Kan fibration. $\square$

We can now prove the main theorem of this section (modulo Conjecture 4.34).

Theorem 4.38 *If $p : E \to X$ is a stratified fibration of simplicial sets, then its geometric realization $|p| : |E| \to |X|$ is a stratified fibration in the sense of definition 4.9.*

Proof. The space $|X|$ is the covered by its skeleta $|X^{(i)}|$. So by Lemma 4.19, it's enough to show the result when X is finite dimensional. Let us assume by induction that it holds when X has dimension $< n$.

Let X be a simplicial set of dimension n. The $(n-1)$-skeleton, along with the images of the n-simplices form a closed cover of $|X|$. So by Lemma 4.19, it's enough to show that the restrictions of $|p|$ to these various subspaces are all stratified fibrations.

The map $\left|E|_{X^{(n-1)}}\right| \to |X^{(n)}|$ is a stratified fibration by induction, so we concentrate on the other case. Let $\sigma : \Delta[n] \to X$ be a simplex and K its image in X. To show that $\left|E|_K\right| \to |K|$ is a stratified fibration, we write it as a pushout of

$$\begin{array}{ccccc} \left|E|_Z\right| & \longleftarrow & \left|E|_{\partial\Delta[n]}\right| & \hookrightarrow & \left|E|_{\Delta[n]}\right| \\ \downarrow & & \downarrow & & \downarrow \\ |Z| & \longleftarrow & \partial\Delta^n & \hookrightarrow & \Delta^n, \end{array} \tag{4.69}$$

where Z is the image of $\partial\Delta[n]$ in X. The leftmost vertical map is a stratified fibration by the induction hypothesis. The rightmost map is a stratified fibration by Lemma 4.37. Therefore (4.69) satisfies the hypothesis of Conjecture 4.34 and $\left|E|_K\right| \to |K|$ is a stratified fibration. This finishes the inductive step. □

4.9 Equivariant stratified fibrations

In the presence of groups actions, one can develop a similar theory of stratified fibrations. Let K be a topological group, and $\mathcal{F}$ a family of subgroups (i.e. a set of subgroups closed under subconjugacy). We work in the category $(K, \mathcal{F})$-spaces of K-spaces with stabilizers in $\mathcal{F}$. A stratification of an K-space X is a K-invariant upper semi-continuous function $s : X \to \mathbb{J}$.

Definition 4.39 *An equivariant generating directed cofibration is an inclusion of the form*

$$K/G \times \Lambda^n \hookrightarrow K/G \times \Delta^n, \tag{4.70}$$

where $\Lambda^n \hookrightarrow \Delta^n$ is a non-equivariant directed cofibration (see Definition 4.8), and

$G \in \mathcal{F}$.

An equivariant stratified fibration is a map satisfying the right lifting property with respect to the set of equivariant generating directed cofibrations. An equivariant directed cofibration is a map satisfying the left lifting property with respect to the class of equivariant stratified fibrations.

The characterization of directed cofibrations given in Theorem 4.15 has an immediate generalization to the equivariant situation.

Theorem 4.40 *Let (B, s) be an equivariant $\mathbb{J}$-stratified space and $A \subset B$ a subspace. Suppose that the image of s has no infinite descending chains (for example B is compact). Then the following are equivalent:*

- $A^G_{\leq j} \to B^G_{\leq j}$ *is a homotopy equivalence for all $j \in \mathbb{J}$ and all $G \in \mathcal{F}$.*
- $A \hookrightarrow B$ *is an equivariant directed cofibration.*

Proof. The proof of Theorem 4.15 goes through word for word. □

In other words, a map $A \to B$ in $(K, \mathcal{F})$-spaces is an equivariant directed cofibration if and only if the corresponding maps on fixed points $p : A^G \to B^G$ are directed cofibrations. Similarly, we get a characterization of equivariant stratified fibrations in terms of non-equivariant ones.

Theorem 4.41 *Let $p : E \to X$ be an equivariant stratified map. Then p is an equivariant stratified fibrations if and only if the corresponding maps on fixed points $p : E^G \to X^G$ are stratified fibrations.*

Proof. The lifting problems

$$\begin{array}{ccc} K/G \times \Lambda^n & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ K/G \times \Delta^n & \longrightarrow & X \end{array} \quad \text{and} \quad \begin{array}{ccc} \Lambda^n & \longrightarrow & E^G \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & X^G \end{array}$$

are equivalent. □

Chapter 5

Stratified classifying spaces

Given a poset $\mathbb{J}$ and a functor $F : \mathbb{J} \to Ho(\mathsf{spaces})$, a natural problem is to classify stratified fibrations $E \to X$ such that ∇ factors as

$$\begin{array}{ccc} \Pi_1(X) & \xrightarrow{\quad\nabla\quad} & Ho(\mathsf{spaces}) \\ & \searrow \quad \nearrow_{F} & \\ & \mathbb{J} & \end{array} \tag{5.1}$$

In other words, we ask for the classification question given specified homotopy types of fibers and homotopy classes of the gluing maps.

In analogy with classical topology, we want a classifying space B such that homotopy classes of stratified maps $X \to B$ are in bijection with fiber-wise homotopy equivalence classes of stratified fibrations $E \to X$ making (5.1) commute.

Instead of developing the general theory, we deal directly with our example of interest: orbispaces structures.

5.1 Orbispace structures

In this section, $\mathbb{J}$ denotes the poset of isomorphism classes of finite groups. The order relation is given by $[H] < [G]$ if there exists a monomorphism $H \to G$. Recall from Definition 3.4 and Theorem 3.5 that an orbispace structure on a $\mathbb{J}$-stratified space X is a stratified fibration $E \to X$. The fibers F_v over points v in the G-stratum are

$K(G,1)$'s, and the maps $\nabla_\gamma : F_v \to F_w$ are injective on fundamental groups.

Definition 5.1 *Let $\mathcal{J}$ be a collection of representatives for each isomorphism class of finite groups. The* classifying space for orbispace structures $\underline{\mathsf{Orb}}$ *is the simplicial set whose n-simplices are given by:*

- *for each vertex of $\Delta[n]$, a group $G_i \in \mathcal{J}$,*
- *for each edge of $\Delta[n]$, a monomorphism $\phi_{ij} : G_i \to G_j$,*
- *for each 2-face of $\Delta[n]$, an element $g_{ijk} \in G_k$ satisfying*

$$\phi_{ik} = Ad(g_{ijk})\, \phi_{jk}\, \phi_{ij}, \tag{5.2}$$

- *for each 3-face of $\Delta[n]$ these group elements satisfy the cocycle condition*

$$g_{ij\ell}\, g_{jk\ell} = g_{ik\ell}\, \phi_{k\ell}(g_{ijk}). \tag{5.3}$$

We fix $\phi_{ii} = 1$ and $g_{iij} = g_{ijj} = 1$.

The stratification $\underline{\mathsf{Orb}} \to N\mathbb{J}$ is given by forgetting the ϕ_{ij} and the g_{ijk}. There is a universal orbispace structure $E_{\underline{\mathsf{Orb}}} \to \underline{\mathsf{Orb}}$. An n-simplex of $E_{\underline{\mathsf{Orb}}}$ is given by:

- *for each vertex of $\Delta[n]$, a group $G_i \in \mathcal{J}$,*
- *for each edge of $\Delta[n]$, a monomorphism $\phi_{ij} : G_i \to G_j$ and an element $x_{ij} \in G_j$,*
- *for each 2-face of $\Delta[n]$, an element $g_{ijk} \in G_k$ satisfying (5.2) and*

$$x_{ik} = g_{ijk}\, \phi_{jk}(x_{ij})\, x_{jk}, \tag{5.4}$$

- *for each 3-face of $\Delta[n]$, the g_{ijk} satisfy (5.3).*

We fix $x_{ii} = 1$. The map $E_{\underline{\mathsf{Orb}}} \to \underline{\mathsf{Orb}}$ is given by forgetting the x_{ij}.

The fact that $|E_{\underline{\mathsf{Orb}}}| \to |\underline{\mathsf{Orb}}|$ is an orbispace is not entirely obvious. We need to check that $E_{\underline{\mathsf{Orb}}} \to \underline{\mathsf{Orb}}$ is a stratified fibration and that its fibers have the correct homotopy type. This is done in the following lemma.

Lemma 5.2 *The map $|E_{\underline{\mathsf{Orb}}}| \to |\underline{\mathsf{Orb}}|$ is a stratified fibration. The fiber F_v over a point v in the G-stratum is a $K(G,1)$. The maps $\nabla : F_v \to F_{v'}$ induce injective homomorphisms on π_1.*

Proof. It's enough by Theorem 4.38 to check that $E_{\underline{\mathsf{Orb}}} \to \underline{\mathsf{Orb}}$ is a stratified fibration of simplicial sets. Let $\Lambda[n,j] \hookrightarrow \Delta[n]$ be an acyclic cofibration and let

$$\begin{array}{ccc} \Lambda[n,j] & \longrightarrow & E_{\underline{\mathsf{Orb}}} \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta[n] & \longrightarrow & \underline{\mathsf{Orb}} \end{array} \tag{5.5}$$

be our usual lifting problem. So we are given groups G_i on the vertices of $\Delta[n]$, homomorphisms ϕ_{ij} on the edges, and elements g_{ijk} on the 2-faces, satisfying (5.2) and (5.3). We are also given elements x_{ij} on the edges of $\Lambda[n,j]$, and we want to extend them to the rest of the edges while respecting the relation (5.4). We do this by a case by case study.

If $n = 1$, then $x_{01} \in G_1$ can be taken arbitrarily.

If $n = 2$ and $j = 0$, we solve (5.4) for x_{12} in terms of x_{01} and x_{02}. If $n = 2$ and $j = 1$, we solve (5.4) for x_{02} is terms of x_{01} and x_{12}. If $n = 2$, $j = 2$ and $G_1 = G_2$ then ϕ_{12} is invertible and we can also solve (5.4) for x_{01} in terms of x_{02} and x_{12}.

If $n = 3$, then all the x_{ij} are already given to us, but we still need to check the condition (5.4) on the 2-face which is not in $\Lambda[n,j]$. If $n = 3$, $j = 0$ we compute

$$\begin{aligned} x_{13} = \phi_{13}(x_{01})^{-1} g_{013}^{-1}\, x_{03} &= g_{123}\, \phi_{23}(\phi_{12}(x_{01}))^{-1} g_{123}^{-1}\, g_{013}^{-1}\, g_{023}\, \phi_{23}(x_{02})\, x_{23} \\ &= g_{123}\, \phi_{23}(x_{01})^{-1} \phi_{23}(g_{012})^{-1} \phi_{23}(x_{02})\, x_{23} = g_{123}\, \phi_{23}(x_{12})\, x_{23}\,. \end{aligned}$$

If $n = 3$, and $j = 1$ or 2 we can write down

$$g_{013}\, \phi_{13}(x_{01})\, x_{13} = g_{013}\, g_{123}\, \phi_{23}(\phi_{12}(x_{01}))g_{123}^{-1}\, g_{123}\, \phi_{23}(x_{12})\, x_{23}$$
$$g_{013}g_{123}\phi_{23}(g_{012}^{-1}\, x_{02})\, x_{23} = g_{023}\, \phi_{23}(x_{02})\, x_{23}\,.$$

If $j = 1$, we start from x_{03}=LHS and conclude that x_{03}=RHS. If $j = 2$, we start from x_{03}=RHS and conclude that x_{03}=LHS.

If $n = 3$, $j = 3$ and $G_2 = G_3$, then ϕ_{23} is invertible and we can compute

$$x_{02} = \phi_{23}^{-1}\big(g_{023}^{-1}\, x_{03}\, x_{23}^{-1}\big) = \phi_{23}^{-1}\big(g_{023}^{-1}\, g_{013}\, \phi_{13}(x_{01})\, x_{13}\, x_{23}^{-1}\big)$$
$$= \phi_{23}^{-1}\big(g_{023}^{-1}\, g_{013}\, g_{123}\, \phi_{23}(\phi_{12}(x_{01}))g_{123}^{-1}\, g_{123}\, \phi_{23}(x_{12})\big) = g_{012}\, \phi_{12}(x_{01})\, x_{12}$$

If $n \geq 4$ all the data needed for a map $\Delta[n] \to E_{\underline{\mathsf{Orb}}}$ is already present in the map $\Lambda[n, j] \to E_{\underline{\mathsf{Orb}}}$. This finishes the verification that $E_{\underline{\mathsf{Orb}}} \to \underline{\mathsf{Orb}}$ is a stratified fibration.

We now check that the fibers F_v have the correct homotopy type. Since the homotopy type of fibers doesn't depend on the particular point in a stratum, we can assume that v is a vertex of $|\underline{\mathsf{Orb}}|$. So F_v is the geometric realization of the corresponding fiber of $E_{\underline{\mathsf{Orb}}} \to \underline{\mathsf{Orb}}$. An n-simplex in that fiber is an n-simplex in $E_{\underline{\mathsf{Orb}}}$ where $G_i = G$, $\phi_{ij} = 1$ and $g_{ijk} = 1$ for all i, j, k. The x_{ij}'s satisfy (5.4), which now reads $x_{ik} = x_{ij}x_{kj}$, so the fiber is isomorphic to the standard model of $K(G, 1)$.

Finally, given a path γ from v_0 to v_1, we check that $\nabla_\gamma : F_{v_0} \to F_{v_1}$ induces an injective homomorphism of fundamental groups. As before, we assume that v_0 and v_1 are vertices of $\underline{\mathsf{Orb}}$. Since ∇_γ only depends on the homotopy class γ, we may also assume that γ is an edge $\{G_0, G_1, \phi\}$ of $\underline{\mathsf{Orb}}$.

Recall that ∇_γ is defined using a lift ℓ of the diagram (4.43). Such a lift

$$\ell : K(G_0, 1) \times \Delta[1] \to E_{\underline{\mathsf{Orb}}} \tag{5.6}$$

can be written explicitly. An n-simplex σ of $K(G_0, 1) \times \Delta[1]$ consists of elements $g_{ij} \in G_0$ and numbers $\varepsilon_i \in \{0, 1\}$ satisfying $g_{ik} = g_{ij}\, g_{jk}$ and $\varepsilon_i \leq \varepsilon_j$ for $i \leq j$, where $i \leq j$ range over the set of vertices of $\Delta[n]$. Its image $\ell(\sigma)$ assigns the group G_{ε_i} to the vertex i, the element $g_{ij} \in G_0$ to the edges ij with $\varepsilon_j = 0$ and $\phi(g_{ij}) \in G_1$ to the edges ij with $\varepsilon_j = 1$.

The map ∇_γ was defined by Precomposing ℓ with the inclusion of $K(G_0, 1) \times \{1\}$. In our case, we get the map $K(G_0, 1) \to K(G_1, 1)$ induced by ϕ. So ∇_γ induces ϕ on fundamental groups, which is injective by definition of $\underline{\mathsf{Orb}}$. □

If a map $X \to |\underline{\mathsf{Orb}}|$ is simplicial with respect to some triangulation $\mathcal{T}$ of X, then we get a group for every vertex a monomorphism for every edge and a group element for each triangle. All these subject to the cocycle conditions (5.2) and (5.3). Inversely, such a collection of data determines a simplicial map $X \to |\underline{\mathsf{Orb}}|$.

Definition 5.3 *Let X be a $\mathbb{J}$-stratified space and $\mathcal{T}$ an oriented triangulation of X. Recall the set $\mathcal{J}$ from Definition 5.1.*

An $\underline{\mathsf{Orb}}$-valued cocycle of X consists of a group $G_v \in \mathcal{J}$ for every vertex v, a monomorphism $\phi_{vw} : G_v \to G_w$ for every edge vw, and an element $g_{uvw} \in G_w$ for each triangle uvw, all subject to the relations (5.2) and (5.3). Moreover, if v belongs to the G-stratum of X, we require that $G_v \simeq G$.

An $E_{\underline{\mathsf{Orb}}}$-valued cocycle is an $\underline{\mathsf{Orb}}$-valued cocycle along with elements $x_{vw} \in G_w$ for each edge vw, subject (5.4).

Two cocycles c, c' on X are equivalent if $c \sqcup c'$ extends to a cocycle on $X \times [0, 1]$.

Given a stratified map $X \to |\underline{\mathsf{Orb}}|$, the pullback of $|E_{\underline{\mathsf{Orb}}}|$ to X induces an orbispace structure on X. This provides a bijection between homotopy classes of maps into $|\underline{\mathsf{Orb}}|$ and orbispace structures on X. In other words, $|\underline{\mathsf{Orb}}|$ is a classifying space for orbispace structures.

Theorem 5.4 *Let $\mathbb{J}$ be the poset of isomorphism classes of finite sets. Given a $\mathbb{J}$-stratified space X, the assignment $f \mapsto f^*|E_{\underline{\mathsf{Orb}}}|$ provides a bijection*

$$\left\{ \begin{array}{c} \textit{Homotopy classes of} \\ \mathbb{J}\textit{-stratified maps } X \to |\underline{\mathsf{Orb}}|. \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Isomorphism classes of} \\ \textit{orbispace structures on } X. \end{array} \right\} \quad (5.7)$$

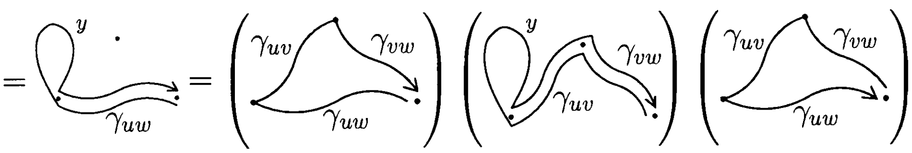
Proof. We first show that the orbifold structure $f^*|E_{\underline{\mathsf{Orb}}}|$ only depends on the homotopy class of f. Let f' be another map and $h : X \times [0, 1] \to |\underline{\mathsf{Orb}}|$ a homotopy between

f and f'. Composing the lift of

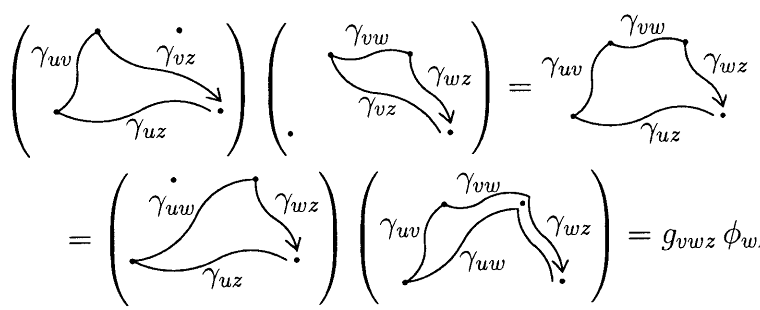
$$\begin{array}{ccc} f^*|E_{\underline{\mathsf{Orb}}}| & \hookrightarrow & h^*|E_{\underline{\mathsf{Orb}}}| \\ {\scriptstyle i_0}\downarrow & \nearrow & \downarrow \\ f^*|E_{\underline{\mathsf{Orb}}}| \times [0,1] & \longrightarrow & X \times [0,1] \end{array} \tag{5.8}$$

with the inclusion $i_1 : f^*|E_{\underline{\mathsf{Orb}}}| \hookrightarrow f^*|E_{\underline{\mathsf{Orb}}}| \times [0,1]$ produces a map $f^*|E_{\underline{\mathsf{Orb}}}| \to f'^*|E_{\underline{\mathsf{Orb}}}|$. Another application of the homotopy lifting property shows that $f^*|E_{\underline{\mathsf{Orb}}}|$ and $f'^*|E_{\underline{\mathsf{Orb}}}|$ are actually homotopy equivalent as spaces over X (one could also use Theorem 4.21).

Given an orbispace structure $E \to X$, we now construct a map $f : X \to |\underline{\mathsf{Orb}}|$ such that $E \simeq f^*|E_{\underline{\mathsf{Orb}}}|$. Let $\mathcal{T}$ be an oriented triangulation of X. For every vertex v, we pick a point b_v in the fiber F_v and an isomorphism $\alpha_v : \pi_1(F_v, b_v) \xrightarrow{\sim} G_v$ for an appropriate $G_v \in \mathcal{J}$. For each edge vw, we pick a path γ_{uv} over that edge, linking b_v to b_w. Such an edge induces a group homomorphism $\tilde{\phi}_{vw} : \pi_1(F_v, b_v) \to \pi_1(F_{w}, b_w)$ by sending a loop $y \in \pi_1(F_v, b_v)$ to $\gamma_{vw}^{-1} \cdot y \cdot \gamma_{vw}$ and then pushing it into F_w. Let $\phi_{vw} := \alpha_w \circ \tilde{\phi}_{vw} \circ \alpha_v^{-1}$.For each triangle uvw of X, the three paths γ_{uw}, γ_{uv} and γ_{vw} assemble to a loop $S^1 \to E$. Pushing it into F_w produces an element of $\pi_1(F_w, b_w)$. We let $g_{uvw} \in G_w$ be the image of that loop under α_w. One now checks that $c = \{G_v, \phi_{vw}, g_{uvw}\}$ is an $\underline{\mathsf{Orb}}$-valued cocycle on X i.e. a map $f : X \to |\underline{\mathsf{Orb}}|$:

$\phi_{uw}(y) =$ 

$$= g_{uvw}\, \phi_{vw}(\phi_{uv}(y)) g_{uvw}^{-1} = \big(Ad(g_{uvw})\phi_{vw}\phi_{uv}\big)(y).$$

$g_{uvw}\, g_{vwz} =$ 

$= g_{vwz}\, \phi_{wz}(g_{uvw})$.

Suppose that we make some other choices $\{\mathcal{T}', b'_{v'}, \alpha'_{v'}, \gamma'_{v'w'}\}$. These give a cocycle c' and a corresponding map f'. Let $\mathcal{T}''$ be a triangulation of $X \times [0,1]$ that restricts to $\mathcal{T}$ on $X \times \{0\}$ and to $\mathcal{T}'$ on $X \times \{1\}$. Extend $\{\gamma_{vw}, \gamma'_{v'w'}\}$ to all the edges of $\mathcal{T}''$. This produces a cocycle c'' on $X \times [0,1]$ extending $c \sqcup c'$, in other words a homotopy between f and f'.

We now show that f doesn't change when we replace E by another space E' which is homotopy equivalent over X. Let $e : E \to E'$ be such an equivalence. One then checks that $b'_v := e(b_v)$, $\alpha'_v := \alpha_v \circ \pi_1(e|_{F_v})^{-1}$, and $\gamma'_{vw} := e \circ \gamma_{vw}$ induce the same cocycle as $b_v, \alpha_v, \gamma_{vw}$. This finishes the proof that $f : X \to |\underline{\mathsf{Orb}}|$ is well defined up to homotopy.

We now show that $f^*|E_{\underline{\mathsf{Orb}}}| \to X$ is equivalent to the original orbispace $E \to X$. A simplex of $f^*|E_{\underline{\mathsf{Orb}}}|$ is the same thing as a simplex $\sigma \subset X$ along with data $\{x_{vw}\}$ extending the $\underline{\mathsf{Orb}}$-valued cocycle $c|_\sigma$ to an $E_{\underline{\mathsf{Orb}}}$-valued cocycle. We use the notation $[\sigma, \{x_{vw}\}]$ to denote such a simplex.

We now describe an equivalence $e : f^*|E_{\underline{\mathsf{Orb}}}| \to E$. It sends the 0-simplex of $K(G_v, 1)$ to x_v. Given an edge $[vw, x]$ of $f^*|E_{\underline{\mathsf{Orb}}}|$, consider the path $\gamma_{vw} \cdot \lambda_{vw}$, where $\lambda_{vw} = e(\alpha_w^{-1}(x))$. Using Lemma 4.17, that path can be rectified to a path ε_{vw} over the edge vw. We then let $e(vw, x) := \varepsilon_{vw}$.

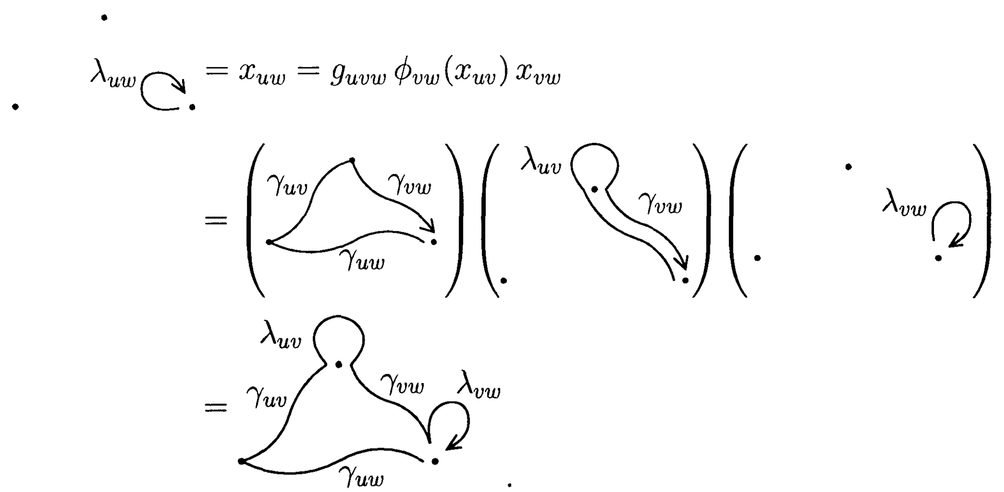
So far, we have constructed the map e on $\mathrm{sk}_1(f^*|E_{\underline{\mathsf{Orb}}}|)$. We now use obstruction theory to extend it to the whole $f^*|E_{\underline{\mathsf{Orb}}}|$. By Theorem 4.29, the obstructions to finding a lift

$$\begin{array}{ccc} \mathrm{sk}_1(f^*|E_{\underline{\mathsf{Orb}}}|) & \xrightarrow{e} & E \\ \big\downarrow & \nearrow & \big\downarrow \\ f^*|E_{\underline{\mathsf{Orb}}}| & \longrightarrow & X \end{array} \tag{5.9}$$

lie in $H^{k+1}\big(f^*|E_{\underline{\mathsf{Orb}}}|, \mathrm{sk}_1(f^*|E_{\underline{\mathsf{Orb}}}|); \pi_k F\big)$. Since all the fibers are $K(\pi,1)$'s, we only need to check that the obstruction in

$$H^2\big(f^*|E_{\underline{\mathsf{Orb}}}|\,, \mathrm{sk}_1(f^*|E_{\underline{\mathsf{Orb}}}|); \pi_1 F\big) = \prod_{\substack{\text{2-simplices} \\ [uvw, \{x_{uv}, x_{uw}, x_{vw}\}]}} G_w \tag{5.10}$$

vanishes. So, for each 2-simplex $[uvw, \{x_{uv}, x_{uw}, x_{vw}\}]$ of $f^*|E_{\underline{\mathsf{Orb}}}|$, we need to check that the loop composed of ε_{uv}, ε_{uw}, and ε_{vw} represents the trivial element in $\pi_1(F_w)$. Equivalently, we show that the loops λ_{uw} and $\gamma_{uw}^{-1} \cdot \gamma_{uv} \cdot \lambda_{uv} \cdot \gamma_{vw} \cdot \lambda_{vw}$ are homotopic:



(5.11)

In order to finish the proof, we need to show that the map $f' : X \to |\underline{\mathsf{Orb}}|$ associated to $f^*|E_{\underline{\mathsf{Orb}}}|$ is homotopic to f. By naturality, it's enough to check this when $X = |\underline{\mathsf{Orb}}|$ and $f = \mathrm{Id}$. In that case, the triangulation of $|\underline{\mathsf{Orb}}|$ may be taken to be the standard one. The points x_v may be taken to be the 0-simplices of $|E_{\underline{\mathsf{Orb}}}|$, we can pick the α_v to be the identity, and we can choose the edges of $|E_{\underline{\mathsf{Orb}}}|$ with $x_{01} = 1$ for our paths γ_{vw}. It is then easy to check that the corresponding $\underline{\mathsf{Orb}}$-valued cocycle on $|\underline{\mathsf{Orb}}|$ gives the identity map $\mathrm{Id} : |\underline{\mathsf{Orb}}| \to |\underline{\mathsf{Orb}}|$. □

The orbispace $|E_{\underline{\mathsf{Orb}}}| \to |\underline{\mathsf{Orb}}|$ has the following other universal property. It is a terminal object up to homotopy in the category of orbispaces and representable maps.

Theorem 5.5 *Let $E \to X$ be an orbispace. Then the space of representable orbispace maps $(E, X) \to (|E_{\underline{\mathsf{Orb}}}|, |\underline{\mathsf{Orb}}|)$ is weakly contractible.*

Proof. To show that $M := \mathrm{Map}\big((E, X), (|E_{\underline{\mathsf{Orb}}}|, |\underline{\mathsf{Orb}}|)\big)$ is weakly contractible, it's enough that any map $S^n \to M$ is nullhomotopic. But a map $S^n \to M$ is the same thing as a map $S^n \times (E, X) \to (|E_{\underline{\mathsf{Orb}}}|, |\underline{\mathsf{Orb}}|)$. So, by letting $S^n \times (E, X)$ play the role of (E, X), it's enough to show that any two maps $(f, g), (f', g') : (E, X) \to (|E_{\underline{\mathsf{Orb}}}|, |\underline{\mathsf{Orb}}|)$ are homotopic. Without loss of generality, we may assume that g is a map classifying

$E \to X$.

Let $\mathcal{T}$ be a triangulation of X such that the image $g'(\sigma)$ of every simplex $\sigma \in \mathcal{T}$ lands in a simplex of $|\underline{\mathsf{Orb}}|$. For each vertex $v \in X$, we consider the simplex $\tau_v \subset |\underline{\mathsf{Orb}}|$ in the interior of which $g'(v)$ lies. By sending v to the 0th vertex of τ, and extending linearly we obtain a new map $g'' : X \to |\underline{\mathsf{Orb}}|$. Moreover, the straight line homotopy between g' and g'' preserves the stratification of $|\underline{\mathsf{Orb}}|$. So, by the homotopy lifting property, it lifts to a homotopy from f' to some other map $E \to |E_{\underline{\mathsf{Orb}}}|$. This reduces us to the case when $g' : X \to |\underline{\mathsf{Orb}}|$ is compatible with the triangulations.

Now, fix $g : X \to |\underline{\mathsf{Orb}}|$ to be the particular map constructed in the proof of Theorem 5.4 using $\mathcal{T}$. At this point, we may also replace (E, X) by the equivalent orbispace $(f^*|E_{\underline{\mathsf{Orb}}}|, X)$. Finally, using a fiber-wise homotopy, the map $f^*|E_{\underline{\mathsf{Orb}}}| \to |E_{\underline{\mathsf{Orb}}}|$ can be replaced by a map respecting the two triangulations. We have equipped all our spaces with triangulations and reduced to the case when all the maps respect the triangulations. Therefore, we can assume that E and X were simplicial sets to begin with.

We are reduced to the following situation. We have a simplicial set X and a map $g : X \to \underline{\mathsf{Orb}}$. We let $f : g^*E_{\underline{\mathsf{Orb}}} \to E_{\underline{\mathsf{Orb}}}$ be the canonical map. We then have another pair of maps (f', g') commuting with the projections, and we want a diagram homotopy between the two diagrams

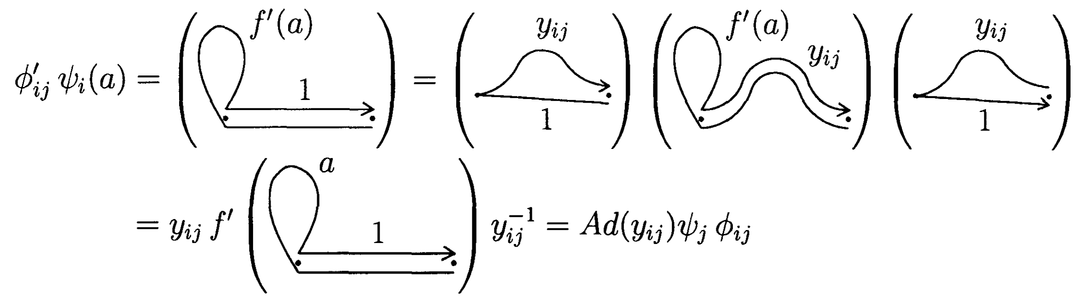
$$\begin{array}{ccc} g^*E_{\underline{\mathsf{Orb}}} & \xrightarrow{f'} & E_{\underline{\mathsf{Orb}}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{g'} & \underline{\mathsf{Orb}} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} g^*E_{\underline{\mathsf{Orb}}} & \xrightarrow{f} & E_{\underline{\mathsf{Orb}}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & \underline{\mathsf{Orb}}. \end{array} \tag{5.12}$$

We begin by writing down a homotopy $h : X \times \Delta[1] \to E_{\underline{\mathsf{Orb}}}$ between g and g'. For each n-simplex $\sigma \in X$, we write down the restriction of h to the corresponding sub-simplicial set $\Delta[n] \times \Delta[1] \subset X \times \Delta[1]$. Namely, we write down an $\underline{\mathsf{Orb}}$-valued cocycle on $\Delta[n] \times \Delta[1]$. Let $\{G_i, \phi_{ij}, g_{ijk}\}) := g(\sigma)$ and $\{G'_i, \phi'_{ij}, g'_{ijk}\}) := g'(\sigma)$. Let $F_i = K(G_i, 1) \subset E_{\underline{\mathsf{Orb}}}$ be the fiber over the ith vertex of $g(\sigma)$, and $F'_i = K(G'_i, 1)$ the fiber over the ith vertex of $g'(\sigma)$. Let $\psi_i : G_i \to G'_i$ be the homomorphism induced by $f' : F_i \to F'_i$. Note that ψ_i is injective because f' was assumed to be representable.

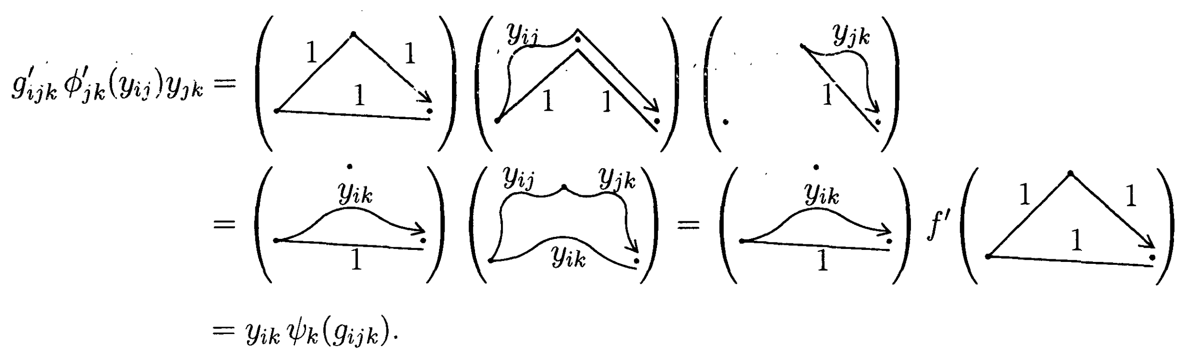
Finally, let $y_{ij} \in G'_j$ be such that $\{G'_i, G'_j, \phi'_{ij}, y_{ij}\}$ is the image of $\{G_i, G_j, \phi_{ij}, 1\}$ under f'. The y_{ij} satisfy the conditions

$$\phi'_{ij}\,\psi_i = Ad(y_{ij})\,\psi_j\,\phi_{ij} \qquad \text{and} \qquad g'_{ijk}\,\phi'_{jk}(y_{ij})\,y_{jk} = y_{ik}\,\psi_k(g_{ijk}), \tag{5.13}$$

which are best verified pictorially:



and



We now write the cocycle on $\Delta[n] \times \Delta[1]$ explicitly. We use lower indices for the $\Delta[n]$ coordinate and upper indices for the $\Delta[1]$ coordinate:

$$\begin{array}{lllll} G_i^0 = G_i & G_i^1 = G'_i & \phi_{ij}^{00} = \phi_{ij} & \phi_{ij}^{01} = \phi'_{ij}\,\psi_i & \phi_{ij}^{11} = \phi'_{ij} \\ g_{ijk}^{000} = g_{ijk} & g_{ijk}^{001} = y_{ik}\,\psi_k(g_{ijk})y_{jk}^{-1} & g_{ijk}^{011} = g'_{ijk} & g_{ijk}^{111} = g'_{ijk}. & \end{array} \tag{5.14}$$

We then check the cocycle conditions (5.2) and (5.3) using (5.13):

$$\begin{aligned}\phi^{01}_{ik} = \phi'_{ik}\,\psi_i &= Ad(y_{ik})\psi_k\,\phi_{ik} = Ad\big(g^{001}_{ijk}\,y_{jk}\,\psi_k(g^{-1}_{ijk})\big)\psi_k\,\phi_{ik}\\ &= Ad(g^{001}_{ijk})Ad(y_{jk})\psi_k Ad(g^{-1}_{ijk})\phi_{ik}\\ &= Ad(g^{001}_{ijk})\phi'_{jk}\,\psi_j\,\phi^{-1}_{jk}\,\phi_{jk}\,\phi_{ij} = Ad(g^{001}_{ijk})\phi'_{jk}\,\psi_j\,\phi_{ij} = Ad(g^{001}_{ijk})\phi^{01}_{jk}\,\phi^{00}_{ij}\end{aligned}$$

$$\begin{aligned}\phi^{01}_{ik} = \phi'_{ik}\,\psi_i &= Ad(y_{ik})\psi_k\,\phi_{ik} = Ad(g^{011}_{ijk})Ad({g'_{ijk}}^{-1})Ad(y_{ik})\psi_k\,\phi_{ik}\\ &= Ad(g^{011}_{ijk})Ad({g'_{ijk}}^{-1})\phi'_{ik}\,\psi_i = Ad(g^{011}_{ijk})\phi'_{jk}\,\phi'_{ij}\,\psi_i = Ad(g^{011}_{ijk})\phi^{11}_{jk}\,\phi^{01}_{ij}\end{aligned}$$

$$\begin{aligned}g^{001}_{ijk}\,g^{001}_{jk\ell} &= y_{i\ell}\,\psi_\ell(g_{ij\ell})\,y^{-1}_{j\ell}\,y_{j\ell}\,\psi_\ell(g_{jk\ell})\,y^{-1}_{k\ell} = y_{i\ell}\,\psi_\ell(g_{ik\ell}\,\phi_{k\ell}(g_{ijk}))y^{-1}_{k\ell}\\ &= y_{i\ell}\,\psi_\ell(g_{ik\ell})y^{-1}_{k\ell}Ad(y_{k\ell})\big(\psi_\ell(\phi_{k\ell}(g_{ijk}))\big)\\ &= y_{i\ell}\,\psi_\ell(g_{ik\ell})y^{-1}_{k\ell}\,\phi'_{k\ell}(\psi_k(g_{ijk})) = g^{001}_{ik\ell}\,\phi^{01}_{k\ell}(g^{000}_{ijk})\end{aligned}$$

$$\begin{aligned}g^{001}_{ij\ell}\,g^{011}_{jk\ell} &= y_{i\ell}\,\psi_\ell(g_{ij\ell})y^{-1}_{j\ell}\,g'_{jk\ell} = y_{i\ell}\,\psi_\ell(g_{ij\ell})\psi_\ell(g_{jk\ell})y^{-1}_{k\ell}\,\phi'_{k\ell}(y_{jk})^{-1}\\ &= y_{i\ell}\,\psi_\ell(g_{ik\ell})\psi_\ell(\phi_{k\ell}(g_{ijk}))y^{-1}_{k\ell}\,\phi'_{k\ell}(y^{-1}_{jk})\\ &= g'_{ik\ell}\,\phi'_{k\ell}(y_{ik})Ad(y_{k\ell})\big(\psi_\ell(\phi_{k\ell}(g_{ijk}))\big)\phi'_{k\ell}(y^{-1}_{jk})\\ &= g'_{ik\ell}\,\phi'_{k\ell}\big(y_{ik}\,\psi_k(g_{ijk})y^{-1}_{jk}\big) = g^{011}_{ik\ell}\,\phi^{11}_{k\ell}(g^{001}_{ijk})\end{aligned}$$

$$g^{011}_{ij\ell}\,g^{111}_{jk\ell} = g'_{ij\ell}\,g'_{jk\ell} = g'_{ik\ell}\,\phi'_{k\ell}(g'_{ijk}) = g^{001}_{ik\ell}\,\phi^{11}_{k\ell}(g^{001}_{ijk}).$$

Assembling the cocycles (5.14) over the simplices of X produces an $\underline{\mathsf{Orb}}$-valued cocycle on $X \times \Delta[1]$. This is our simplicial homotopy between g and g'.

To build the corresponding homotopy between f and f', we proceed similarly, but with $(g^*E_{\underline{\mathsf{Orb}}}) \times \Delta[1]$ instead of $X \times \Delta[1]$. For each simplex in $g^*E_{\underline{\mathsf{Orb}}}$, we write an $E_{\underline{\mathsf{Orb}}}$-valued cocycle on $\Delta[n] \times \Delta[1]$. It is given by (5.14) and

$$x^{00}_{ij} = x_{ij} \qquad x^{01}_{ij} = x'_{ij} \qquad x^{11}_{ij} = x'_{ij} \tag{5.15}$$

where x_{ij} and x'_{ij} are provided by the image of our simplex under f and f' respectively. They satisfy $x'_{ij} = y_{ij}\psi_j(x_{ij})$ which allows us to check that (5.15) satisfy the cocycle condition (5.4):

$$
\begin{aligned}
x_{ik}^{01} &= x'_{ik} = y_{ik}\psi_k(x_{ik}) = y_{ik}\psi_k\big(g_{ijk}\phi_{jk}(x_{ij})x_{jk}\big) \\
&= y_{ik}\psi_k(g_{ijk})y_{jk}^{-1}y_{jk}\psi_k((\phi_{jk}(x_{ij}))\psi_k(x_{jk}) \\
&= g_{ijk}^{001}y_{jk}\psi_k(\phi_{jk}(x_{ij}))y_{jk}^{-1}x'_{jk} = g_{ijk}^{001}\phi'_{jk}(\psi_j(x_{ij}))x'_{jk} = g_{ijk}^{001}\phi_{jk}^{01}(x_{ij}^{00})x_{jk}^{01}
\end{aligned}
$$

$$
x_{ik}^{01} = x'_{ik} = g'_{ijk}\phi'_{jk}(x'_{ij})x'_{jk} = g_{ijk}^{011}\phi_{jk}^{11}(x_{ij}^{01})x_{jk}^{11}.
$$

Assembling all the cocycles over the simplices of $g^*E_{\underline{\mathsf{Orb}}}$ produces an $E_{\underline{\mathsf{Orb}}}$-valued cocycle on $(g^*E_{\underline{\mathsf{Orb}}}) \times \Delta[1]$. This is our simplicial homotopy between f and f'. We have constructed a commutative diagram

$$
\begin{array}{ccc}
(g^*E_{\underline{\mathsf{Orb}}}) \times \Delta[1] & \xrightarrow{f'} & E_{\underline{\mathsf{Orb}}} \\
\downarrow & & \downarrow \\
X \times \Delta[1] & \xrightarrow{g'} & \underline{\mathsf{Orb}},
\end{array}
$$

providing a homotopy between the the two diagrams (5.12). This finishes the proof that any two representable maps into $(|E_{\underline{\mathsf{Orb}}}|, |\underline{\mathsf{Orb}}|)$ are homotopic. $\square$

5.2 Quotient structures

Given a topological group K and an action $K \circlearrowleft Y$ with finite stabilizers, we can form the quotient orbispace $(EK \times Y)/K \to Y/K$. We want to know which orbispaces $E \to X$ are of the above form. As a first step, we classify the ways a stratified space X can be written as Y/K.

Definition 5.6 *Let $\mathbb{J}$ be the poset of isomorphism classes of finite sets, and X a $\mathbb{J}$-stratified space.*

A K-quotient structure, on X consists of a K-space P with finite stabilizers, and a stratified homeomorphism $P/K \xrightarrow{\sim} X$, where the stratification on P/K is by the stabilizer group. We also require that $P \to X$ be a K-equivariant stratified fibration, where X is given the trivial action.

Like for orbispace structures, K-quotient structures have a classifying space.

Definition 5.7 *Let K be a topological group, and let $\mathcal{J}_K$ be a set of representatives for the conjugacy classes of finite subgroups of K. The classifying space for quotient structures $\underline{B}K$ is the simplicial space whose n-simplices are given by:*

- *for each vertex of $\Delta[n]$, a group $G_i \in \mathcal{J}_K$,*
- *for each edge of $\Delta[n]$, an element $k_{ij} \in K$ satisfying*

$$k_{ij}\, G_i\, k_{ij}^{-1} \subseteq G_j, \tag{5.16}$$

- *for each 2-face of $\Delta[n]$, these group elements satisfy the cocycle condition*

$$k_{ik}\, k_{ij}^{-1} k_{jk}^{-1} \in G_k \tag{5.17}$$

We fix $k_{ii} = 1$. The stratification $\underline{B}K \to N\mathbb{J}$ is given by sending the groups G_i to their isomorphism class and forgetting the rest of the data.

There is a universal K-quotient structure $\underline{E}K \to \underline{B}K$. An n-simplex of $\underline{E}K$ is given by:

- *for each vertex of $\Delta[n]$, a group $G_i \in \mathcal{J}_K$ and an element $y_iG_i \in K/G_i$.*
- *for each edge of $\Delta[n]$, an element $k_{ij} \in K$ satisfying (5.16) and*

$$y_iG_ik_{ij}^{-1} \subseteq y_jG_j \tag{5.18}$$

- *for each 2-face of $\Delta[n]$, the k_{ij} satisfy (5.17)*

The action $K \circlearrowleft \underline{E}K$ is induced by the action on the K/G_i's.

For convenience of notation, we write $(\underline{B}K)_n$ as a disjoint union of spaces $(\underline{B}K)_n^{\{G_i\}}$, according to the values of the G_i's. We now check that the quotient $|\underline{E}K|/K$ is homeomorphic to $|\underline{B}K|$ and the stratification by stabilizers agrees with the stratification coming from $\underline{B}K \to N\mathbb{J}$.

Lemma 5.8 *Let $p : \underline{E}K \to \underline{B}K$ be the canonical projection, and let $x \in |\underline{B}K|$ be a point in the G-stratum. Then the fiber $|p|^{-1}(x)$ is K-equivariantly isomorphic to K/G. Moreover, letting $f : (\underline{B}K)_n^{\{G_i\}} \to |\underline{B}K|$ be the natural map, we can be write the pullback of $|\underline{E}K|$ explicitly:*

$$f^*|\underline{E}K| = \coprod_{i=0}^{n} (K/G_i) \times (\underline{B}K)_n^{\{G_i\}} \times \Delta^i \Big/ \sim, \tag{5.19}$$

where

$$(y_i G_i; a, t) \sim (y_j k_{ij}(a)^{-1} G_j; a, t) \tag{5.20}$$

and Δ^i is identified with the last i-face of Δ^n.

Proof. Write $x = (\sigma, t)$ for some non-degenerate $\sigma \in |\underline{B}K|$ and some $t \in \mathring{\Delta}^n$. Let $\{G_i, k_{ij}, G_{ijk}\}$ be the data corresponding to σ. The fact that x lies in the G-stratum means that $G_0 = G$.

A simplex of $\underline{E}K$ whose image in $\underline{B}K$ is degenerate is necessarily itself degenerate. This means that $|p|^{-1}(x)$ is isomorphic to $(p_n)^{-1}(\sigma)$, where $p_n : (\underline{E}K)_n \to (\underline{B}K)_n$ is the map induced by p on the spaces on n-simplices.

So we need to show that $(p_n)^{-1}(\sigma) \simeq K/G$. A point in $(p_n)^{-1}(\sigma)$ is a collection $\{y_i G_i \in K/G_i\}$ satisfying (5.18). But (5.18) determines all the $y_i G_i$'s in terms of $y_0 G_0$. So the map $(p_n)^{-1}(\sigma) \to K/G = K/G_0$ given by $\{y_i G_i\} \mapsto y_0 G_0$ is injective.

Suppose now that we are given $y_0 G_0 \in K/G_0$. Then letting

$$y_i G_i := y_0 k_{0i}^{-1} G_i \tag{5.21}$$

produces a collection satisfying (5.18):

$$\begin{aligned} y_i G_i k_{ij}^{-1} &= y_0 k_{0i}^{-1} G_i k_{ij}^{-1} = y_0 k_{0i}^{-1} k_{ij}^{-1} k_{ij} G_i k_{ij}^{-1} \\ &\subseteq y_0 k_{0i}^{-1} k_{ij}^{-1} G_j = y_0 k_{0j}^{-1} k_{0j} k_{0i}^{-1} k_{ij}^{-1} G_j = y_0 k_{0j}^{-1} G_j = y_j G_j. \end{aligned} \tag{5.22}$$

This shows that $\{y_i G_i\} \mapsto y_0 G_0$ is a bijection. We have identified the fiber of p_n, hence the fiber $|p|$.

If $t \in \Delta^n$ lies in some face Δ^I, $I = \{i_0 \dots i_s\} \subset \{0 \dots n\}$, then its fiber $\{y_i G_i\}_{i \in \{0 \dots n\}} \simeq K/G_0$ gets identified with $\{y_i G_i\}_{i \in I} \simeq K/G_{i_0}$. As we saw in (5.21), this is exactly relation (5.20). □

We now finish checking that $|\underline{E}K| \to |\underline{B}K|$ is a K-quotient structure.

Lemma 5.9 *Let $|\underline{B}K|$ be given the trivial K-action. Then the map $|\underline{E}K| \to |\underline{B}K|$ is a K-equivariant stratified fibration.*

Proof. By Theorem , we need to check that the map of H-fixed points $|\underline{E}K|^H \to |\underline{B}K|^H = |\underline{B}K|$ is a stratified fibration for all subgroups $H < K$. The space $|\underline{B}K|$ is covered by its skeleta, so by Lemma 4.19, it's enough to show that $|\underline{E}K|^H_n \to |\underline{B}K|_n$ are stratified fibrations.

We assume by induction that $|\underline{E}K|^H_{n-1} \to |\underline{B}K|_{n-1}$ is a stratified fibration. The map $|\underline{E}K|^H_n \to |\underline{B}K|_n$ can be written as the pushout of

$$\begin{array}{ccccc} |\underline{E}K|^H_{n-1} & \longleftarrow & \alpha^*|\underline{E}K|^H & \hookrightarrow & \beta^*|\underline{E}K|^H \\ \downarrow & & \downarrow & & \downarrow \\ |\underline{B}K|_{n-1} & \longleftarrow & (\underline{B}K)_n \times \partial\Delta^n & \overset{\iota}{\hookrightarrow} & (\underline{B}K)_n \times \Delta^n, \end{array} \tag{5.23}$$

where $\beta : (\underline{B}K)_n \times \Delta^n \to |\underline{B}K|$ is the canonical map and $\alpha = \beta \circ \iota$. So by Conjecture 4.34, it's enough to show that $\beta^*|\underline{E}K|^H \to (\underline{B}K)_n \times \Delta^n$ is a stratified fibration.

But $(\underline{B}K)_n$ is the disjoint union of the $(\underline{B}K)_n^{\{G_i\}}$. Write $f : (\underline{B}K)_n^{\{G_i\}} \times \Delta^n$ for the canonical map and recall the expression (5.19) for $f^*|\underline{E}K|$. Taking H-fixed points, we get

$$f^*|\underline{E}K|^H = \coprod_{i=0}^{n} (K/G_i)^H \times (\underline{B}K)_n^{\{G_i\}} \times \Delta^i \Big/ \sim, \tag{5.24}$$

where

$$(y_i G_i; a, t) \sim (y_i k_{ij}(a)^{-1} G_j; a, t). \tag{5.25}$$

and Δ^i is identified with the last i-face of Δ^n. Let $G_i' := k_{0i}^{-1} G_i k_{0i}$. If we apply the change of coordinates $(y_i G_i; a, t) \mapsto (y_i k_{0i}(a) G_i'; a, t)$ to (5.24), the relation (5.25) becomes $(y_i G_i'; a, t) \sim (y_i G_j'; a, t)$. So we can apply Lemma 4.35 to our situation. The projections $(K/G_i)^H \to (K/G_j)^H$ are covering spaces, in particular fibrations, so $f^*|\underline{E}K|^H \to (\underline{B}K)_n^{\{G_i\}} \times \Delta^n$ is a stratified fibration. □

As was the case with Orb, it's useful to introduce a notion of cocycle corresponding $\underline{B}K$.

Definition 5.10 *Let X be a $\mathbb{J}$-stratified space and $\mathcal{T}$ an oriented triangulation. Recall the set $\mathcal{J}_K$ from definition 5.7.*

A $\underline{B}K$-valued cocycle on X consists of the following data. To each vertex v, we associate a group in $G_v \in \mathcal{J}_K$. For each simplex $\sigma : \Delta^n \to X$ of $\mathcal{T}$, and for each $0 \leq i, j \leq n$, we have a function $k_{ij} = k_{ij}^\sigma : \Delta^n \to K$. These functions satisfy the conditions (5.16) and (5.17) where the groups G_i are the ones associated to the vertices of σ.

Moreover, if $\iota : \Delta^k \to \Delta^n$ is the inclusion of a face, and if $\tau = \sigma \circ \iota$, we require that

$$k_{ij}^\tau = k_{\iota(i),\iota(j)}^\sigma \circ \iota. \tag{5.26}$$

Two cocycles c, c' are equivalent if $c \sqcup c'$ extends to a cocycle on $X \times [0,1]$.

A $\underline{B}K$-valued cocycle c on X produces a map $f : X \to |\underline{B}K|$ as follows. Let $\sigma : \Delta^n \to X$ be a simplex of $\mathcal{T}$, and let $\{G_i\}$ be the groups associated to the vertices of σ. The function f is then given by

$$f : \sigma(t) \mapsto (\{G_i, k_{ij}^\sigma(t)\}, t) \in |\underline{B}K|. \tag{5.27}$$

It is well defined because (5.26).

Given X an a $\mathbb{J}$-stratified map $X \to |\underline{B}K|$, we can pull back $|\underline{E}K|$ to get a quotient structure on X. This provides a bijection between homotopy classes of maps $X \to |\underline{B}K|$ and quotient structures on X.

Theorem 5.11 *Let $\mathbb{J}$ be the poset of isomorphism classes of finite groups, and K a topological group. Then, for any $\mathbb{J}$-stratified space X, the assignment $f \mapsto f^*|\underline{E}K|$*

provides a bijection

$$\left\{ \begin{array}{c} \textit{Homotopy classes of} \\ \mathbb{J}\textit{-stratified maps } X \to |\underline{B}K|. \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Isomorphism classes of} \\ K\textit{-quotient structures on } X. \end{array} \right\} \quad (5.28)$$

Proof. We first note that $f^*|\underline{E}K|$ only depends on the homotopy class of f. This is done as in (5.8), using Lemma 5.9. We also note that a K-equivariant homotopy equivalence $P \xrightarrow{\sim} P'$ between two K-quotient structures is necessarily a homeomorphism.

Given a K-quotient structure $p : P/K \xrightarrow{\sim} X$, we build a map $f : X \to |\underline{B}K|$ such that $P \simeq f^*|\underline{E}K|$. We do this by constructing a $\underline{B}K$-valued cocycle on X with respect to some oriented triangulation $\mathcal{T}$.

First, for every vertex v, we pick a point $b_v \in P_v := p^{-1}(v)$ whose stabilizer is in $\mathcal{J}_K$. The groups for our $\underline{B}K$-valued cocycle are then given by

$$G_v = \mathrm{Stab}_K(b_v). \quad (5.29)$$

Given a simplex $\sigma : \Delta^n \to X$, we let G_i be the group associated to its ith vertex. We use G_i^σ when we want to stress the dependence on σ. We also refer to the subset

$$\{(t_0, \ldots, t_n) \in \Delta^n \,|\, t_i = 0 \text{ for } i < j, t_j \neq 0\} \subset \Delta^n \quad (5.30)$$

as the j-stratum of Δ^n, and use the notation $\Delta^n_{\leq j}$, $\Delta^n_{\geq j}$ as in Definition 4.4.

We now produce a sequence of K-equivariant quotients of σ^*P

$$\sigma^*P = Q_0 \overset{q_1}{\twoheadrightarrow} Q_1 \overset{q_2}{\twoheadrightarrow} Q_2 \cdots \overset{q_n}{\twoheadrightarrow} Q_n. \quad (5.31)$$

They come with maps to $p_i : Q_i \to \Delta^n$ satisfying $p_{i-1} = p_i \circ q_i$. The q_i are isomorphisms over $\Delta^n_{\geq i}$, and the isotropy groups of Q_i are conjugate to G_i over $\Delta^n_{\leq i}$. We use Q_i^σ when we want to stress the dependence on σ. If $\iota : \Delta^k \to \Delta^n$ is the inclusion of a face, and $\tau = \sigma \circ \iota$, we also want inclusions $Q_i^\tau \to Q_{\iota(i)}^\sigma$ commuting with the q_i, and such that

$$\begin{array}{ccc} Q_i^\tau & \longrightarrow & Q_{\iota(i)}^\sigma \\ \big\downarrow & & \big\downarrow \\ \Delta^k & \xrightarrow{\iota} & \Delta^n \end{array} \tag{5.32}$$

is a pullback diagram.

We construct the Q_i^σ by induction on $\dim(\sigma)$ and on i. If $\dim(\sigma) = 0$ or $i = 0$ we don't have to do anything. So let's assume that we have Q_j^τ for all j if $\dim(\tau) < n$ and for $j < i$ if $\dim(\tau) = n$. Let $\sigma : \Delta^n \to X$ be an n-simplex and $i > 0$. By (5.32), Q_i is determined on the ith horn $\Lambda^{n,i}$. Let $r : \Delta^n \to \Lambda^{n,i}$ be a deformation retraction sending $\Delta^n \setminus \Lambda^{n,i}$ into $\Delta^n_{\leq i}$. By the homotopy lifting property, r is covered by a K-equivariant deformation retraction $\tilde{r} : Q_{i-1} \to Q_{i-1}|_{\Lambda^{n,i}}$. We define Q_i by identifying $x, y \in Q_{i-1}$ if $\tilde{r}(x)$ and $\tilde{r}(y)$ have the same image in $Q_i|_{\Lambda^{n,i}}$.

We now construct sections $s_i = s_i^\sigma$ of Q_i^σ, compatible with (5.32), and such that

$$\mathrm{Stab}_K\big(s_i(t)\big) = G_i \qquad \text{for} \quad t \in \Delta^n_{\leq i}. \tag{5.33}$$

On 0-simplices v, the value of s_0 is given by b_v. Assume now by induction that we have s_i^τ for all simplices τ of dimension smaller that n. Let σ be an n-simplex. The compatibility with (5.32) forces the value of s_i on the i-horn. We then extend it to Δ^n by the homotopy extension property.

Finally, we construct the functions $k_{ij} = k_{ij}^\sigma : \Delta^n \to K$ needed for our cocycle. We make sure that they satisfy

$$k_{ij}(t)\, q_{ij}\big(s_i(t)\big) = s_j(t), \tag{5.34}$$

where q_{ij} stands for for $q_j q_{j-1} \cdots q_{i+1}$. Note that (5.34) specifies $k_{ij}(t)$ up to left multiplication by an element of $\mathrm{Stab}_K(s_j(t))$. That stabilizer group is generically equal to G_j, so we only have a global ambiguity by G_j. If $\dim(\sigma) = 1$, we pick a value for k_{01} at the 0th vertex, subject to (5.34). This resolves the ambiguity, thus defining uniquely k_{01}. For arbitrary σ, the value of k_{ij} is given by (5.26) on some contractible subset of Δ^n. Again, the ambiguity is resolved, and so we have defined k_{ij}.

We now check that $\{G_i, k_{ij}\}$ satisfy the cocycle conditions (5.16) and (5.17). For the first one, we recall that G_i stabilizes s_i. It therefore also stabilizes $q_{ij}(s_i)$. Since G_j is the generic stabilizer of s_j, (5.16) holds generically, and by continuity it holds everywhere. The second condition holds because

$$\begin{aligned} s_k = k_{ik}\, q_{ik}\, s_i = k_{ik}\, q_{jk}\, q_{ij}\, s_i &= k_{ik}\, k_{ij}^{-1} k_{jk}^{-1} k_{jk}\, q_{jk}\, k_{ij}\, q_{ij}\, s_i \\ &= k_{ik}\, k_{ij}^{-1} k_{jk}^{-1} k_{jk}\, q_{jk}\, s_j = k_{ik}\, k_{ij}^{-1} k_{jk}^{-1} s_k \end{aligned}$$

and $\mathrm{Stab}_K(s_k) = G_k$ on a dense piece of Δ^n. This finishes the verification that $c := \{G_i, k_{ij}\}$ is a $\underline{B}K$-valued cocycle on X. Let $f : X \to |\underline{B}K|$ be the corresponding map.

Suppose now that we make some other choices $\{\mathcal{T}', b'_v, Q'_i, s'_i, k'_{ij}\}$. These give a cocycle c' and a corresponding map f'. We want to show that f and f' are homotopic. Let $\mathcal{T}''$ be a triangulation of $X \times [0,1]$ that restricts to $\mathcal{T}$ on $X \times \{0\}$ and to $\mathcal{T}'$ on $X \times \{1\}$. We can extend $\{Q_i, Q'_i, s_i, s'_i, k_{ij}, k'_{ij}\}$ to data $\{Q''_i, s''_i, k''_{ij}\}$ on the simplices of $\mathcal{T}''$ satisfying (5.31), (5.32), (5.34). This produces a cocycle c'' on $X \times [0,1]$ extending $c \sqcup c'$ i.e. a homotopy between f and f'. This finishes the proof that f is well defined up to homotopy.

We now show that $f^*|\underline{E}K|$ is isomorphic to the original K-quotient structure P. Let $x = \sigma(t)$ be a point of X, where $\sigma : \Delta^n \to X$ is a simplex of our triangulation $\mathcal{T}$, and $t \in \mathring{\Delta}^n$. We need to produce a K-equivariant map between its fiber P_x in P and the corresponding fiber $(f^*|\underline{E}K|)_x = |\underline{E}K|_{f(x)}$. Recall from (5.27) that $f(x)$ is given by $(\{G_i, k_{ij}(t)\}, t)$. Its fiber in $|\underline{E}K|$ is then given by

$$|\underline{E}K|_{f(x)} = \left\{ (y_i G_i) \in \prod^{n} K/G_i \,\middle|\, y_i G_i k_{ij}^{-1}(t) \subseteq y_j G_j \right\}, \tag{5.35}$$

and is isomorphic to K/G_0 by Lemma 5.8. Now consider the fibers $Q_i(t) := p_i^{-1}(t)$, where p_i are as in (5.31). Let $\psi_i : Q_i(t) \to K/G_i$ be the unique K-map sending $s_i(t)$ to the coset G_i. We can now write the equivalence on each fiber as

$$y \in P_x \;\mapsto\; \big(\psi_i\, q_{0i}(y)\big) \in |\underline{E}K|_{f(x)}\,, \tag{5.36}$$

where $q_{0i} = q_i \dots q_1 : P_x = Q_0(t) \to Q_i(t)$ is the projection. To check that the conditions in (5.35) are satisfied by the $\psi_i\big(q_{0i}(y)\big)$, we write $y = a\, s_0(t)$ for some $a \in K$ and compute

$$\psi_i\big(q_{0i}(y)\big) = \psi_i\big(a\, q_{0i}(s_0)\big) = \psi_i\big(a\, k_{0i}^{-1} k_{0i}\, q_{0i}(s_0)\big) = \psi_i(a\, k_{0i}^{-1} s_i) = a\, k_{0i}^{-1} G_i\,,$$

$$\text{so} \qquad \psi_i\big(q_{0i}(y)\big) k_{ij}^{-1} = a\, k_{0i}^{-1} G_i k_{ij}^{-1} \subseteq a\, k_{0i}^{-1} k_{ij}^{-1} G_j = a\, k_{0j}^{-1} G_j = \psi_j\big(q_{0j}(y)\big).$$

This finishes the proof that $f^*|\underline{E}K|$ is isomorphic to P.

Finally, suppose we start with a map $f : X \to |\underline{B}K|$. We need to show that the map f' associated to $P = f^*|\underline{E}K|$ is homotopic to f. By naturality, we may assume that $X = |\underline{B}K|$ and $f = \mathrm{Id}$. Triangulate each of the spaces $(\underline{B}K)_n \times \Delta^n$. Make sure that the projections $(\underline{B}K)_n \times \Delta^n \to \Delta^n$ and the attaching maps $(\underline{B}K)_n \times \partial\Delta^n \to |\underline{B}K|_{n-1}$ respect the triangulations. This induces a triangulation $\mathcal{T}$ of $|\underline{B}K|$, which refines the skeleton filtration. Note that the set of vertices necessarily agrees with $|\underline{B}K|_0$. Given $v = G \in \mathcal{J}_K$, we let b_v be the identity coset in $P_v = |\underline{E}K|_v = K/G$. Now, we need to choose quotients Q_i of $Q_0 = \sigma^*|\underline{E}K|$ as in (5.31). By Lemma 5.8, Q_0 can be written as

$$Q_0 = \coprod_{j=0}^{n} K/G_j \times \Delta^n_{\geq j} \Big/ \sim\,, \tag{5.37}$$

where $(y_i G_i, t) \sim (y_i k_{ij}^{-1}(t) G_j, t)$ and G_j are the groups associated to the vertices of σ, and k_{ij} are inherited from $\underline{B}K$. We then let

$$Q_i := \coprod_{j=i}^{n} K/G_j \times \Delta^n_{\geq i} \Big/ \sim\,. \tag{5.38}$$

Pick $s_i(t)$ to be the identity coset in K/G_i. Finally, the functions k_{ij} can be taken equal the ones which were already given to us from the beginning. They satisfy (5.34) since

$$k_{ij} q_{ij} s_i = k_{ij} q_{ij} G_i = k_{ij} k_{ij}^{-1} G_j = G_j = s_j. \tag{5.39}$$

One then checks that the corresponding map $f : |\underline{B}K| \to |\underline{B}K|$ is the identity. $\square$

The K-space $|\underline{E}K|$ actually has an other name. It is the classifying space for the family of finite subgroups of K. (See [Lück] for a survey of the subject.) This is the content of the following proposition:

Proposition 5.12 *Let H be a finite subgroup of K. Then the fixed point set $|\underline{E}K|^H$ is contractible.*

Proof. The simplicial space $\underline{E}K$ is isomorphic to the nerve of a category $\mathcal{C}$. The objects of $\mathcal{C}$ are the orbits K/G for $G \in \mathcal{J}_K$, equipped with a choice of base point. The morphisms are the K-equivariant, base points preserving maps between them. Taking H-fixed points means that we restrict the set of objects to those where the base point is H-fixed. So $(\underline{E}K)^H$ is the nerve of a full subcategory $\mathcal{C}' \subset \mathcal{C}$. An object of $\mathcal{C}'$ can also be viewed as an object of $\mathcal{C}$ equipped with a map from $(K/H, H) \in \mathcal{C}$. But $\mathcal{C}'$ has an initial object, so its nerve is contractible:

$$|\underline{E}K|^H = |(\underline{E}K)^H| = |\mathcal{C}'| \simeq *.$$

□

Given a K-quotient structure $P \to X$, we can form the corresponding orbispace $(P \times EK)/K \to X$. This operation is represented by a stratified map $|\underline{B}K| \to |\underline{\mathsf{Orb}}|$ which we can write explicitly:

Proposition 5.13 *Let K be a topological group, and recall the sets $\mathcal{J}$ and $\mathcal{J}_K$ from Definitions 5.1 and 5.7. Let $\kappa : \mathcal{J}_K \to \mathcal{J}$ be the map sending a group $G \in \mathcal{J}_K$ to the unique group in $\mathcal{J}$ to which it is isomorphic. Let $\beta_G : G \to \kappa(G)$ be an isomorphism.*

The orbispace $(|\underline{E}K| \times EK)/K \to |\underline{B}K|$ is then represented by (the realization of) the map

$$\begin{aligned} \Psi : \quad \underline{B}K \quad &\to \quad \underline{\mathsf{Orb}} \\ \{G_i, k_{ij}\} &\mapsto \{\kappa(G_i),\ \beta_j Ad(k_{ij})\beta_i^{-1},\ \beta_k(k_{ik}k_{ij}^{-1}k_{jk}^{-1})\}, \end{aligned} \tag{5.40}$$

where β_i is a shorthand notation for β_{G_i}.

Proof. Triangulate the spaces $(\underline{B}K)_n \times \Delta^n$ and make sure that the projections $(\underline{B}K)_n \times \Delta^n \to \Delta^n$ and the attaching maps $(\underline{B}K)_n \times \partial\Delta^n \to |\underline{B}K|_{n-1}$ respect the triangulations. This induces a triangulation $\mathcal{T}$ of $|\underline{B}K|$ refining the skeleton filtration. The vertices of $\mathcal{T}$ agree with $|\underline{B}K|_0 = \mathcal{J}_k$.

Let $\sigma : \Delta^n \to |\underline{B}K|$ be a simplex of that triangulation. By Lemma 5.8, we can write $\sigma^*|\underline{E}K|$ as

$$\sigma^*|\underline{E}K| = \coprod_{i=0}^{n} K/G_i \times \Delta^i \Big/ \sim,$$

where $(y_i k_{ij}(a)G_i; a, t) \sim (y_j G_j; a, t)$ and Δ^i is identified with the last i-face of Δ^n. Therefore, we can also write

$$(\sigma^*|\underline{E}K| \times EK)/K = \coprod_{i=0}^{n} EK/G_i \times \Delta^i \Big/ \sim, \tag{5.41}$$

where $(z_i k_{ij}(a)G_i; a, t) \sim (z_j G_j; a, t)$.

We now make the choices $\{b_i, \alpha_i, \gamma_{ij}\}$ used in the proof of Theorem 5.4. Pick some base point $e \in EK$. Over the ith vertex of Δ^n, the fiber of (5.41) is EK/G_i. We choose our base point $b_i := eG_i$. We have $\pi_i(EK/G_i) = G_i$ and we pick the isomorphism $\alpha_i := \beta_i$. For each i, j we pick some path $\delta_{ij} : [0,1] \to EK$ from e to ek_{ij}. The paths γ_{ij} are given by

$$\gamma(x) = \begin{cases} \delta_{ij}(x)G_i & \text{if } i < 1, \\ ek_{ij}G_i = eG_j & \text{if } i = 1, \end{cases}$$

where we have omitted the Δ^n coordinate.

We now check that

$$\phi_{ij} = \beta_j\, Ad(k_{ij})\beta_i^{-1} \quad \text{and} \quad g_{ijk} = \beta_k(k_{ik}k_{ij}^{-1}k_{jk}^{-1}). \tag{5.42}$$

The isomorphisms β_i don't play any role in the argument, so we omit them from the notation and identify G_i with $\kappa(G_i)$. Let $y \in G_i$ be an element represented by a path $z : [0,1] \to EK/G_i$. It admits a lift $\tilde{z} : [0,1] \to EK$ going from e to ey^{-1}.

The element $\phi_{ij}(y)$ is represented by the path $\varepsilon := \bar{\delta}_{ij}^{-1} \cdot zk_{ij}^{-1}G_j \cdot \bar{\delta}_{ij}$, where $\bar{\delta}_{ij}$ is the projection of δ_{ij} in EK/G_j. The lift of ϵ in EK is given by $\tilde{\varepsilon} = \delta_{ij}^{-1}k_{ij}^{-1} \cdot \tilde{z}k_{ij}^{-1} \cdot \bar{\delta}_{ij}yk_{ij}^{-1}$. We check that $\tilde{\varepsilon}(0) = e$ and $\tilde{\varepsilon}(1) = ek_{ij}y^{-1}k_{ij}^{-1}$ which gives us (5.42.a). Now, the element g_{ijk} is represented by the path $\eta := \bar{\delta}_{ik}^{-1} \cdot \bar{\delta}_{ij} \cdot \bar{\delta}_{jk}$. Its lift in EK is given by $\tilde{\eta} = \delta_{ik}^{-1}k_{ik}^{-1} \cdot \delta_{ij}k_{ik}^{-1} \cdot \delta_{jk}k_{ij}k_{ik}^{-1}$. We check that $\tilde{\eta}(0) = e$ and $\tilde{\eta}(1) = ek_{jk}k_{ij}k_{ik}^{-1}$ which gives us (5.42.b).

We have checked the formulas (5.40) on every individual simplex of $\mathcal{T}$. The choices can made in a compatible way over all the simplices of $\mathcal{T}$, so we have proven the result. □

If we are given a collection of subgroups of K, we can do the following useful variation on Definition 5.7.

Definition 5.14 *Let K be a topological group and $\mathcal{F}$ a collection of subgroups (i.e. a set of subgroups, closed under conjugation). Let $\mathcal{J}_{\mathcal{F}}$ be a set of representatives of the conjugacy classes in $\mathcal{F}$.*

Then one defines $B_{\mathcal{F}}K$ by replacing $\mathcal{J}_K$ by $\mathcal{J}_{\mathcal{F}}$ everywhere in Definition 5.7. Similarly, one defines $E_{\mathcal{F}}K \to B_{\mathcal{F}}K$.

One then has the following straightforward generalizations of Theorems 5.11.

Theorem 5.15 *Let $\mathbb{J}$ be the poset of isomorphism classes of finite groups, K a topological group and $\mathcal{F}$ a collection of subgroups. Then, for any $\mathbb{J}$-stratified space X, the assignment $f \mapsto f^*|E_{\mathcal{F}}K|$ provides a bijection*

$$\left\{ \begin{array}{c} \textit{Homotopy classes of} \\ \mathbb{J}\textit{-stratified maps} \\ X \to |B_{\mathcal{F}}K|. \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Isomorphism classes of} \\ K\textit{-quotient structures on } X \\ \textit{with stabilizers in } \mathcal{F}. \end{array} \right\} \quad (5.43)$$

□

Assume now that $\mathcal{F}$ is a family of subgroup (i.e. if it's closed under conjugation and under taking subgroups). Then then $|E_{\mathcal{F}}K|$ is a classifying space for $\mathcal{F}$:

Proposition 5.16 *The space $|E_{\mathcal{F}}K|^H$ is contractible for $H \in \mathcal{F}$ and empty otherwise.*

The proof is identical to that of Proposition 5.12. □

Chapter 6

Global quotients

The purpose of this chapter is to show that every orbispace (E, X) where X is compact is a global quotient by a compact Lie group K. Namely, there exists a K-space Y such that (E, X) is isomorphic to $[Y/K] := ((Y \times EK)/K, Y/K)$. It turns out that it's easier to answer the question if we relax the condition that K be a Lie group. In that case, we can prove the result even when X is non-compact.

6.1 A convenient group

Let U be the inductive limit of unitary groups

$$U := \varinjlim U(n!) \tag{6.1}$$

where the inclusions $U(n!) \hookrightarrow U((n+1)!)$ are given by $A \mapsto A \otimes \mathrm{Id}_{n+1}$ (See [12], [1], [15] for other occurrences of this group.) By Bott periodicity $\pi_i(U(n)) = 0$ for even i and $\mathbb{Z}$ for odd i, as long as $i < 2n$. The inclusion $U(n!) \hookrightarrow U((n+1)!)$ induces multiplication by $n+1$ on π_i so we conclude that

$$\pi_i(U) = \varinjlim \pi_i\big(U(n!)\big) = \begin{cases} 0 & \text{if } i \text{ is even} \\ \mathbb{Q} & \text{if } i \text{ is odd.} \end{cases} \tag{6.2}$$

The group U can also be defined as a colimit of all the $U(n)$'s where the colimit is instead indexed by the lattice $\mathbb{N}$, ordered by divisibility.

Let (E, X) be an orbispace represented by a map $f : X \to \underline{\mathsf{Orb}}$. It is of the form $[Y/U]$ if and only if there is a lift up to homotopy

$$\begin{array}{ccc} & & |\underline{B}U| \\ & \nearrow & \downarrow |\Psi| \\ X & \xrightarrow{f} & |\underline{\mathsf{Orb}}| \end{array} , \qquad (6.3)$$

where Ψ is the map described in Proposition 5.13.

For technical reasons, it will be useful to replace $|\underline{B}U|$ by another space mapping to it. Let $\mathcal{F}$ be the family of finite subgroups of U which are embedded by their regular representation $\lambda : G \to U(n) \hookrightarrow U$. We have

Lemma 6.1 *For any group $G \in \mathcal{F}$, the map $N_U(G) \to \mathrm{Aut}(G)$ is surjective.*

Proof. Let $\lambda : G \to U(n) \hookrightarrow U$ denote the regular representation. The action of $\mathrm{Aut}(G)$ on G induces a permutation representation $\rho : \mathrm{Aut}(G) \to U(n) \hookrightarrow U$ which normalizes $\lambda(G)$. Clearly, ρ is a section of the map $N_U(G) \to \mathrm{Aut}(G)$ so that map is surjective. □

Let $B_{\mathcal{F}}U$ be as in Definition 5.14 and let $\mathrm{Sing}(B_{\mathcal{F}}U)$ be the simplicial set obtained by applying level-wise the singular functor, and then taking the diagonal (i.e. the "geometric realization" functor from bisimplicial sets to simplicial sets). Note that a map into $\mathrm{Sing}(B_{\mathcal{F}}U)$ is the same thing as a $B_{\mathcal{F}}U$-valued cocycle as described in Definition 5.10.

Lemma 6.2 *Let $\tilde{\Psi}$ be the composite*

$$\tilde{\Psi} : \mathrm{Sing}(B_{\mathcal{F}}U) \hookrightarrow \mathrm{Sing}(\underline{B}U) \xrightarrow{\mathrm{Sing}\Psi} \mathrm{Sing}(\underline{\mathsf{Orb}}) = \underline{\mathsf{Orb}}, \qquad (6.4)$$

and let $\tilde{\Psi}^{op} : \mathrm{Sing}(B_{\mathcal{F}}U)^{op} \to \underline{\mathsf{Orb}}^{op}$ be the map between opposite simplicial sets. Then $\tilde{\Psi}^{op}$ is a stratified fibration.

Proof. To show that $\tilde{\Psi}^{op}$ is a stratified fibration, we replace the usual lifting diagram by its opposite

$$\begin{array}{ccc} \Lambda[n,j]^{op} & \longrightarrow & \mathrm{Sing}(B_{\mathcal{F}}U) \\ \downarrow & & \downarrow \tilde{\Psi} \\ \Delta[n]^{op} & \longrightarrow & \underline{\mathsf{Orb}}. \end{array} \tag{6.5}$$

Since $\Delta[n]^{op} = \Delta[n]$ and $\Lambda[n,j]^{op} = \Lambda[n, n-j]$, we can replace (6.5) by

$$\begin{array}{ccc} \Lambda[n,j] & \longrightarrow & \mathrm{Sing}(B_{\mathcal{F}}U) \\ \downarrow & & \downarrow \tilde{\Psi} \\ \Delta[n] & \longrightarrow & \underline{\mathsf{Orb}}, \end{array} \tag{6.6}$$

where $\Lambda[n,j] \hookrightarrow \Delta[n]$ now satisfies the opposite condition to that in Definition 4.32. Namely, $j > 0$ <u>or</u> the 0th and 1st vertices of $\Delta[n]$ are in the same stratum.

To check (6.6), we do a case by case analysis, similarly to the proof of Lemma 5.2. We have an $\underline{\mathsf{Orb}}$-valued cocycle $c = \{G_i, \phi_{ij}, g_{ijk}\}$ on $\Delta[n]$, a $B_{\mathcal{F}}U$-valued cocycle $c' = \{H_i, k_{ij}\}$ on $\Lambda[n,j]$, and they satisfy $\tilde{\Psi}(c') = c|_{\Lambda[n,j]}$. Namely

$$G_i = \kappa(H_i) \qquad \phi_{ij} = \beta_j Ad(k_{ij})\beta_i^{-1} \qquad g_{ijk} = \beta_k(k_{ik}k_{ij}^{-1}k_{jk}^{-1}), \tag{6.7}$$

where κ and β_i are as in (5.40). We want extend c' to the whole $\Delta[n]$ while preserving (6.7).

If $n = 1$ and $j = 1$ we let $H_0 \in \mathcal{J}_{\mathcal{F}}$ be the unique group isomorphic to G_0. The group $H_0' := \beta_1^{-1}\phi_{ij}\beta_0(H_0)$ belongs to $\mathcal{F}$ because it's a subgroup of H_1. By Lemma 6.1, any isomorphism between groups of $\mathcal{F}$ can be obtained by conjugating by an appropriate element of U. So we may pick $k_{01} \in U$ such that $Ad(k_{01})|_{H_0} = \beta_1^{-1}\phi_{ij}\beta_0$. This defines a $B_{\mathcal{F}}U$-valued cocycle by viewing k_{01} as a constant function on Δ^1.

If $n = 1$, $j = 0$ and ϕ_{01} is invertible, we can run the same argument as above with ϕ_{01}^{-1} playing the role of ϕ_{01}.

If $n = 2$, $j = 2$, we extend k_{02} and k_{12} to the whole Δ^2 while preserving (6.7.b).

We then solve (6.7.c) for k_{01}. We now check (6.7.b) for k_{01}:

$$\begin{aligned}\beta_1 Ad(k_{01})\beta_0^{-1} &= \beta_1 Ad\big(k_{12}^{-1}\,\beta_2^{-1}(g_{012}^{-1})\,k_{02}\big)\beta_0^{-1} = \beta_1 Ad(k_{12}^{-1})\,\beta_2^{-1}Ad(g_{012}^{-1})\,\beta_2\,Ad(k_{02})\beta_0^{-1}\\ &= \beta_1 Ad(k_{12}^{-1})\,\beta_2^{-1}Ad(g_{012}^{-1})\,\phi_{02} = \beta_1 Ad(k_{12}^{-1})\,\beta_2^{-1}\phi_{12}\,\phi_{01}\\ &= \beta_1 Ad(k_{12}^{-1})\,\beta_2^{-1}\phi_{12}\,\beta_1\beta_1^{-1}\phi_{01} = \beta_1 Ad(k_{12}^{-1})Ad(k_{12})\,\beta_1^{-1}\phi_{01} = \phi_{01}.\end{aligned} \tag{6.8}$$

If $n = 2$, $j = 1$, we extend k_{01} and k_{12} to Δ^2, preserving (6.7.b). We solve (6.7.c) for k_{02} and check (6.7.b):

$$\begin{aligned}\beta_2\,Ad(k_{02})\beta_0^{-1} &= \beta_2 Ad\big(\beta_2^{-1}(g_{012})k_{12}\,k_{01}\big)\beta_0^{-1} = \beta_2\,\beta_2^{-1}Ad(g_{012})\,\beta_2\,Ad(k_{12})Ad(k_{01})\beta_0^{-1}\\ &= Ad(g_{012})\,\phi_{12}\,\beta_1 Ad(k_{01})\,\beta_0^{-1} = Ad(g_{012})\,\phi_{12}\,\phi_{01} = \phi_{02}.\end{aligned}$$

If $n = 2$, $j = 0$, and ϕ_{01} is invertible, we extend k_{01}, k_{02} to Δ^2, preserving (6.7.b), solve (6.7.c) for k_{12} and check (6.7.b):

$$\begin{aligned}\beta_2\,Ad(k_{12})\beta_1^{-1} &= \beta_2 Ad\big(\beta_2^{-1}(g_{012}^{-1})k_{02}k_{01}^{-1}\big)\beta_1^{-1} = \beta_2\,\beta_2^{-1}Ad(g_{012}^{-1})\beta_2\,Ad(k_{02})Ad(k_{01}^{-1})\beta_1^{-1}\\ &= Ad(g_{012}^{-1})\,\phi_{02}\,\beta_0\,Ad(k_{01}^{-1})\beta_1^{-1} = Ad(g_{012}^{-1})\,\phi_{02}\,\phi_{01}^{-1} = \phi_{12}.\end{aligned}$$

If $n \geq 3$, we extend k_{in} to the whole Δ^n while preserving (6.7.b) and let $k_{ij} := k_{jn}^{-1}\beta_n^{-1}(g_{ijn}^{-1})k_{in}$. They satisfy (6.7.b) by the same computation as (6.8), so we just check (6.7.c):

$$\begin{aligned}\beta_k\big(k_{ik}\,k_{ij}^{-1}k_{jk}^{-1}\big) &= \beta_k\big(k_{kn}^{-1}\beta_n^{-1}(g_{ikn}^{-1})k_{in}\,k_{in}^{-1}\beta_n^{-1}(g_{ijn})k_{jn}\,k_{jn}^{-1}\beta_n^{-1}(g_{jkn})k_{kn}\big)\\ &= \beta_k\big(k_{kn}^{-1}\,\beta_n^{-1}(g_{ikn}^{-1}\,g_{ijn}\,g_{jkn})\,k_{kn}\big) = \beta_k\big(k_{kn}^{-1}\,\beta_n^{-1}(\phi_{kn}(g_{ijk}))\,k_{kn}\big)\\ &= Ad(k_{kn}^{-1})\,\beta_n^{-1}\,\phi_{kn}(g_{ijk}) = g_{ijk}.\end{aligned}$$

This finishes the proof that $\tilde{\Psi}^{op}$ is a stratified fibration. □

6.2 Orbispaces are global quotients

We begin by a couple of technical lemmas. Let $\mathcal{C}$ be the category whose objects are $\mathcal{J}$ (see Definition 5.1) and whose morphisms are injective homomorphisms modulo conjugacy in the target $\mathrm{Hom}_{\mathcal{C}}(G,H) := \mathrm{Mono}(G,H)/H$.

Lemma 6.3 *Let ν be the projection*

$$\nu : \underline{\mathsf{Orb}} \to N\mathcal{C} \tag{6.9}$$

given by $\{G_i, \phi_{ij}, g_{ijk}\} \mapsto \{G_i, [\phi_{ij}]\}$ and let $x \in |N\mathcal{C}|$ be a point. Then the fiber $|\nu|^{-1}(x)$ is a $K(A,2)$'s for some finite group A.

Proof. We first show that ν is a stratified fibration. Once more, we write down the lifting problem

$$\begin{array}{ccc} \Lambda[n,j] & \longrightarrow & \underline{\mathsf{Orb}} \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] & \longrightarrow & N\mathcal{C}. \end{array} \tag{6.10}$$

So we are given groups G_i and morphisms $\psi_{ij} \in \mathrm{Hom}_{\mathcal{C}}(G_i, G_j)$, satisfying $\psi_{02} = \psi_{12}\,\psi_{01}$. We also have a compatible $\underline{\mathsf{Orb}}$-valued cocycle $\{G_i, \phi_{ij}, g_{ijk}\}$ on $\Lambda[n,j]$. We want to extend it to the whole $\Delta[n]$ preserving the relation $\psi_{ij} = [\phi_{ij}]$. We do it case by case:

If $n = 1$, we can take ϕ_{01} to be any representative of ψ_{01}.

If $n = 2$, we pick a representative $\phi_{k\ell}$ of $\psi_{k\ell}$, where $k\ell$ is the edge not in $\Lambda[n,j]$. Since $[\phi_{02}] = [\phi_{12}][\phi_{01}]$, we can pick an element g_{012} such that $\phi_{02} = Ad(g_{012})\phi_{12}\,\phi_{01}$.

If $n = 3$, $j \neq 3$ we can solve (5.3) for the missing g_{ijk}.

If $n = j$, $j = 3$ and ϕ_{23} is invertible, we can solve (5.3) for g_{012}.

This finishes the proof that ν is a stratified fibration. By Theorem 4.38, $|\nu|$ is a stratified fibration of spaces so by Corollary 4.24, it then enough to identify the homotopy type of the fibers of $|\nu|$ over the vertices of $|\underline{\mathsf{Orb}}|$.

Let $G \in \mathcal{J}$ be a group and $x \in N\mathcal{C}$ be the corresponding vertex. The fiber $\nu^{-1}(x)$ is the the simplicial set given by

$$\left(\nu^{-1}(x)\right)_n = \left\{\{G_i, \phi_{ij}, g_{ijk}\}_{i,j,k=0..n} \,\middle|\, G_i = G,\ [\phi_{ij}] = 1\right\}. \tag{6.11}$$

It can easily be checked that (6.11) is a fibrant simplicial set and that it's only non-trivial homotopy group is $\pi_2 = Z(G)$. □

Lemma 6.4 *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $f : \mathcal{C} \to \mathcal{D}$ be a functor such that for every $X \in Ob(\mathcal{D})$, the category $f^{-1}(X)$ is a groupoid. Let $x = (\sigma, y) \in |N\mathcal{D}|$ be a point, where $\sigma \in (N\mathcal{D})_n$ is a non-degenerate and $y \in \mathring{\Delta}^n$ and let σ^*f denote the pullback*

$$\begin{array}{ccc} \sigma^*N\mathcal{C} & \longrightarrow & N\mathcal{C} \\ {\scriptstyle \sigma^*f}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta[n] & \xrightarrow{\sigma} & N\mathcal{D}. \end{array}$$

Then $|f|^{-1}(x)$ is homeomorphic to the realization of the simplicial set of section of σ^*f.

Proof. Even if $f : N\mathcal{C} \to N\mathcal{D}$ is not a stratified fibration, the argument of Lemma 4.36 goes through. Indeed, the only place where that assumption is used, is in order to find a lift of (4.60). In our case, we have to solve

$$\begin{array}{ccc} \Delta[r] \times \Delta[n] & \longrightarrow & \sigma^*N\mathcal{C} \\ \downarrow & \nearrow & \downarrow{\scriptstyle \sigma^*f} \\ \Delta[(r+1)(k+1)-1] & \longrightarrow & \Delta[n]. \end{array} \tag{6.12}$$

The lift $\Delta[(r+1)(k+1)-1] \to \sigma^*N\mathcal{C}$ exists (and is unique) because of our assumption on the fibers of f. □

Lemma 6.5 *Let Gr be the category of finite groups and monomorphisms and $\mathcal{C}$ be the category of finite groups and monomorphisms modulo conjugacy in the target* $\mathrm{Hom}_{\mathcal{C}}(G, H) := \mathrm{Hom}_{Gr}(G, H)/H$. *Let $p : Gr \to \mathcal{C}$ be the projection functor. Given a diagram $\alpha = (G_0 \xrightarrow{j_1} G_1 \xrightarrow{j_2} \ldots \xrightarrow{j_n} G_n)$ in Gr and its image $\bar{\alpha} := p(\alpha)$ in $\mathcal{C}$, we let q*

and $\bar{q}$ *be the maps*

$$q : \mathrm{Aut}(\alpha) \to \mathrm{Aut}_{Gr}(G_0) \qquad \text{and} \qquad \bar{q} : \mathrm{Aut}(\bar{\alpha}) \to \mathrm{Aut}_{\mathcal{C}}(G_0). \tag{6.13}$$

For $\phi \in \mathrm{Aut}_{Gr}(G_0)$ *we then have*

$$\phi \in Im(q) \;\Leftrightarrow\; p(\phi) \in Im(\bar{q}). \tag{6.14}$$

Proof. Clearly $\phi \in Im(q) \Rightarrow p(\phi) \in Im(\bar{q})$, so we assume that $p(\phi) \in Im(\bar{q})$. Let $\{\psi_i \in \mathrm{Aut}_{\mathcal{C}}(G_i)\}$ an automorphism of $\bar{\alpha}$ such that $\psi_0 = p(\phi)$ and let $\tilde{\psi}_i$ be lifts of ψ_i in Gr. Make sure that $\tilde{\psi}_0 = \phi$. Since $\psi_i\, p(j_i) = p(j_i)\, \psi_{i-1}$, there exist elements $g_i \in G_i$ such that $\tilde{\psi}_i\, j_i = Ad(g_i) j_i\, \tilde{\psi}_{i-1}$. This is best described by the diagram

$$\begin{array}{ccccccc}
G_0 & \xrightarrow{j_1} & G_1 & \xrightarrow{j_2} & G_2 - & \cdots & \to G_n \\
\downarrow \tilde{\psi}_0 & g_1 & \downarrow \tilde{\psi}_1 & g_2 & \downarrow \tilde{\psi}_2 & \cdots & \downarrow \tilde{\psi}_n \\
G_0 & \xrightarrow[j_1]{} & G_1 & \xrightarrow[j_2]{} & G_2 - & \cdots & \to G_n.
\end{array}$$

Let $h_i \in G_i$ be the elements inductively defined by $h_0 = 1$ and $h_i = j_i(h_{i-1})\, g_i^{-1}$. The automorphisms $\phi_i := Ad(h_i)\, \tilde{\psi}_i \in \mathrm{Aut}(G_i)$ satisfy

$$\begin{aligned}
\phi_i\, j_i = Ad(h_i)\, \tilde{\psi}_i\, j_i &= Ad\big(j_i(h_{i-1}) g_i^{-1}\big)\, Ad(g_i)\, j_i\, \tilde{\psi}_{i-1} \\
&= Ad(j_i(h_{i-1}))\, j_i\, \tilde{\psi}_{i-1} = j_i\, Ad(h_{i-1})\, \tilde{\psi}_{i-1} = j_i\, \phi_{i-1},
\end{aligned}$$

which shows that $\{\phi_i\}$ is an element of $\mathrm{Aut}(\bar{\alpha})$. Since $\phi_0 = \phi$, this also shows that $\phi \in Im(q)$. □

We can now state the main theorem of this section.

Theorem 6.6 *Let* $U = \varinjlim U(n!)$ *be the group given in (6.1) and let* (E, X) *be an orbispace. Then there exists a* U*-space* Y *such that* (E, X) *is equivalent to* $[Y/U]$.

Proof. By Theorem 5.4 it is enough to solve the problem for $(E, X) = (|E_{\underline{\mathsf{Orb}}}|, |\underline{\mathsf{Orb}}|)$. Let

$$\Psi : \underline{B}U \to \underline{\mathsf{Orb}} \tag{6.15}$$

be the map (5.40) representing the functor $Y \mapsto [Y/U]$. By Theorem 5.11, finding a U-space Y such that $(E, X) \simeq [Y/U]$ is the same thing as finding a homotopy section of $|\Psi|$.

Let $\mathcal{F}$ be the family of finite subgroups which are embedded via a multiple of their regular representation, and let $\mathrm{Sing}(B_{\mathcal{F}}U)$ be as in Lemma 6.2. Since $|\mathrm{Sing}(B_{\mathcal{F}}U)|$ maps to $|\underline{B}U|$, it's enough to find a section of

$$\tilde{\Psi} : \mathrm{Sing}(B_{\mathcal{F}}U) \longrightarrow \underline{\mathsf{Orb}}. \tag{6.16}$$

Having a section of $\tilde{\Psi}$ is the same thing as having a section of $\tilde{\Psi}^{op}$. Lemma 6.2 checks that this map is a stratified fibration of simplicial sets and by a simplicial set analog of Theorem 4.29, the obstructions to the existence of a section of (6.16) lie in the cohomology groups $H^{k+1}(\underline{\mathsf{Orb}}, \pi_k(F))$.

So we need to understand the homotopy type of the fibers of $\tilde{\Psi}$. Taking opposites, and applying the singular functor doesn't change the homotopy type, so these fibers are homotopy equivalent to the fibers of

$$\Psi_{\mathcal{F}} : B_{\mathcal{F}}U \longrightarrow \underline{\mathsf{Orb}}. \tag{6.17}$$

Given a group $G \in \mathcal{J}$, let F denote the fiber of $\Psi_{\mathcal{F}}$ over the corresponding vertex of $|\underline{\mathsf{Orb}}|$. It's the simplicial set given by

$$F_n = \left\{ \{H_i, k_{ij}\}_{i,j=0..n} \middle| \, \kappa(H_i) = G,\, \beta_j Ad(k_{ij})\beta_i^{-1} = \mathrm{Id}_G,\, \beta_k(k_{ik}k_{ij}^{-1}k_{jk}^{-1}) = 1 \right\}. \tag{6.18}$$

There is the unique $H \in \mathcal{F}$ isomorphic to G, so all the H_i are equal to H. The k_{ij} centralize H and satisfy $k_{ik}k_{ij}^{-1}k_{jk}^{-1} = 1$, so F isomorphic to $B(Z_U(H))$.

Now, given a representation $\rho : H \to U(n)$, is well known that $Z_{U(n)}(H)$ is isomorphic to a product $U(n_1) \times \cdots \times U(n_r)$, where the n_i are the multiplicities of the irreducible representations ρ_i occurring in ρ. Taking the limit to infinity, we get that

$Z_U(H) \simeq U^r$. We have therefore identified the fibers of $\Psi_{\mathcal{F}}$ as

$$F = B\big(Z_U(H)\big) \simeq B(U^r). \tag{6.19}$$

Combining (6.19) with (6.2), we also get that $\pi_k(F) = \mathbb{Q}^r$ for even k and 0 otherwise. In our case, we also know that ρ is the regular representation of H so we see that $\pi_{\text{even}}(F)$ is the free $\mathbb{Q}$-vector space on the set of irreps of H. Using character theory, this can also be rewritten as

$$\pi_{\text{odd}}(F) = 0, \qquad\qquad \pi_{\text{even}}(F) = \mathbb{Q}[H]^H, \tag{6.20}$$

where H acts on $\mathbb{Q}[H]$ by conjugation. Given a monomorphism $H \to H'$ the corresponding map on fibers is induced by the inclusion $Z_U(H') \to Z_U(H)$. The homomorphisms $\pi_{\text{even}}(F') \to \pi_{\text{even}}(F)$ is then the pullback $\mathbb{Q}[H']^{H'} \to \mathbb{Q}[H]^H$.

In order to finish the proof, we now need to compute the obstructions groups $H^*(\underline{\mathsf{Orb}}; \pi_{\text{even}}(F))$, and show that they all vanish. Let $\mathcal{A}$ be the sheaf $\pi_{\text{even}}(F)^{op}$, used to define the above cohomology group (see Definition 4.28 and Example 4.27). More concretely, if $x = (\sigma, t) \in |\underline{\mathsf{Orb}}|$ is a point, where $\sigma = \{G_i, \phi_{ij}, g_{ijk}\}$ is an n-simplex and $t \in \overset{\circ}{\Delta}{}^n$, we can write down the stalk of $\mathcal{A}$ at x as

$$\mathcal{A}_x = \mathbb{Q}[G_0]^{G_0}.$$

Let $\mathcal{C}$ be the category whose objects are $\mathcal{J}$ (see Definition 5.1) and whose morphisms are injective homomorphisms modulo conjugacy in the target. We compute $H^{k+1}(\underline{\mathsf{Orb}}, \mathcal{A})$ using the Leray Serre spectral sequence

$$H^p\big(|N\mathcal{C}|;\ H^q(\textit{fibers of } |\nu|; \mathcal{A})\big) \quad \Rightarrow \quad H^{p+q}(|\underline{\mathsf{Orb}}|; \mathcal{A}) \tag{6.21}$$

associated to the map (6.9). The sheaf $\mathcal{A}$ is constant along the fibers of $|\nu|$ and it's stalks are rational. So by Lemma 6.3, the spectral sequence (6.21) collapses and we get

$$H^*(|N\mathcal{C}|;\mathcal{A}) \;=\; H^*(|\underline{\mathsf{Orb}}|;\mathcal{A}). \tag{6.22}$$

Now we compute (6.22) using the Leray Serre spectral sequence

$$H^p\big(|N\mathbb{J}|;\; H^q(\textit{fibers of } |\mu|;\mathcal{A})\big) \quad\Rightarrow\quad H^{p+q}(|N\mathcal{C}|;\mathcal{A}) \tag{6.23}$$

associated to the map $\mu : N\mathcal{C} \to N\mathbb{J}$ (as usual, $\mathbb{J}$ is the poset of isomorphism classes of finite groups).

Given a point $x \in |N\mathbb{J}|$, in the interior of some simplex $G_0 < \ldots < G_n$ we let $\mathcal{G}_x$ be the groupoid whose objects are the diagrams $G_0 \to \ldots \to G_n$ in $\mathcal{C}$. By Lemma 6.4, we know that $|\mu|^{-1}(x) = |N\mathcal{G}_x|$. In particular, the connected components of $|\mu|^{-1}(x)$ are in bijection with the isomorphism classes of diagrams $G_0 \to \ldots \to G_n$. Since the objects of $\mathcal{G}_x$ have finite isomorphism groups, and since finite groups have trivial rational cohomology, we get that

$$H^*(\mu^{-1}(x),\mathcal{A}) = H^0(\mu^{-1}(x),\mathcal{A}) = \bigoplus_{\substack{\text{isomorphism}\\ \text{classes of diagrams in } \mathcal{C}\\ \alpha=(G_0\to\ldots\to G_n)}} \big(\mathbb{Q}[G_0]^{G_0}\big)^{\mathrm{Aut}(\alpha)}. \tag{6.24}$$

So the spectral sequence (6.23) collapses, and we get

$$H^*(|\underline{\mathsf{Orb}}|;\mathcal{A}) \;=\; H^*(|N\mathcal{C}|;\mathcal{A}) \;=\; H^*(|N\mathbb{J}|,\mathcal{B})\,, \tag{6.25}$$

where $\mathcal{B}$ is the sheaf whose stalks are given by (6.24). By Lemma 6.5, we can rewrite these stalks as

$$\mathcal{B}_x = \bigoplus_{\substack{\text{isomorphism}\\ \text{classes of diagrams in } \mathsf{Gr}\\ \alpha=(G_0\to\ldots\to G_n)}} \mathbb{Q}[G_0]^{\mathrm{Aut}(\alpha)}, \tag{6.26}$$

where Gr is the category of finite groups and monomorphisms.

Let Z be simplicial set given by

$$Z_k = \Big\{\text{iso. classes of } \big(g \in G_0 \hookrightarrow G_1 \hookrightarrow \ldots \hookrightarrow G_k\big)\Big\}, \tag{6.27}$$

and let $Z(n)$ be the connected component of Z where g that order n. To each k-simplex $\sigma = [g \in G_0 \to \ldots \to G_k] \in Z_k(n)$, we can associate a $(k+1)$-simplex $\sigma_+ := [g \in \langle g \rangle \to G_0 \to \ldots \to G_k] \in Z_{k+1}(n)$ satisfying $d_0(\sigma_+) = \sigma$ and $d_{i+1}(\sigma_+) = d_i(\sigma)$. These simplices assemble to a map

$$Cone\big(Z(n)\big) \to Z(n), \tag{6.28}$$

thus providing a homotopy between $\mathrm{Id}_{Z(n)}$ and the constant map at $[1 \in \mathbb{Z}/n\mathbb{Z}] \in Z(n)$. In particular, this shows that $Z(n)$ is contractible. We have shown that Z is a disjoint union of contractible connected components, and in particular that

$$H^*(Z;\mathbb{Q}) = 0 \qquad \text{for} \quad * > 0. \tag{6.29}$$

Now let's try to compute $H^*(Z;\mathbb{Q})$ using the Leray Serre spectral sequence

$$H^p\big(|N\mathbb{J}|;\, H^q(\textit{fibers of } |\xi|;\mathbb{Q})\big) \quad \Rightarrow \quad H^{p+q}(|Z|;\mathbb{Q}). \tag{6.30}$$

associated to the projection $\xi : Z \to N\mathbb{J}$. Once again, we let $x \in |N\mathbb{J}|$ be a point in the interior of some simplex $G_0 < \ldots < G_n$, and we try to understand the fiber $|\xi|^{-1}(x)$. Clearly this fibers is discrete, so we get

$$H^*(|\xi|^{-1}(x),\mathbb{Q}) = H^0(|\xi|^{-1}(x),\mathbb{Q}) = \bigoplus_{\substack{\text{iso. classes of}\\ g\in G_0\hookrightarrow\ldots\hookrightarrow G_n}} \mathbb{Q}. \tag{6.31}$$

We now observe that the right hand sides of (6.26) and (6.31) are equal to each other. So we can assemble (6.25) and the collapsing spectral sequence (6.30) to get

$$H^*(|\underline{\mathsf{Orb}}|;\mathcal{A}) \;=\; H^*(|N\mathbb{J}|,\mathcal{B}) \;=\; H^*(Z,\mathbb{Q}) = 0 \qquad \text{for} \quad * > 0. \tag{6.32}$$

We have shown that are obstruction groups are all zero, which finishes the proof. □

As a corollary to our theorem, we get that every compact orbispace is a global quotient by the action of a compact Lie group.

Corollary 6.7 *Let (E, X) be an orbispace such that X is compact. Then there exists a compact Lie group K and a K-space Y such that $(E, X) \simeq [Y/K]$.*

Proof. Let $f : X \to |\underline{\mathsf{Orb}}|$ be the map classifying (E, X). By Theorem 6.6, there exists a lift (up to homotopy)

$$\begin{array}{ccc} & & |\underline{B}U| \\ & \overset{\tilde{f}}{\nearrow} & \downarrow \\ X & \xrightarrow{f} & |\underline{\mathsf{Orb}}| \end{array} .$$

Since X is compact and $|\underline{B}U| = \varinjlim |\underline{B}U(n!)|$, the map $\tilde{f}$ factors at some finite stage. The resulting map $X \to |\underline{B}U(n!)|$ classifies a $U(n!)$-space Y such that $(E, X) \simeq [Y/U(n!)]$. □

If (E, X) is a compact orbifold, then the $U(n!)$-space Y such that $(E, X) \simeq [Y/U(n!)]$ is automatically a manifold. So we have also proven that every compact orbifold is the quotient of a compact manifold by a compact Lie group.

6.3 Enough vector bundles

There is an interesting connection between global quotients by compact Lie groups and (finite dimensional) vector bundles on orbispaces.

Definition 6.8 *An orbispace $p : E \to X$ has enough vector bundles if for every subspace $X' \subset X$ and every vector bundle V on $(E', X') := (p^{-1}(X), X)$, there exists a vector bundle W on (E, X) and an embedding $V \hookrightarrow W|_{(E', X')}$.*

Theorem 6.9 *Let $p : E \to X$ be a compact orbispace (i.e. X is compact). Then the following are equivalent:*

1. *(E, X) is a global quotient by a compact Lie group.*

2. *(E, X) has enough vector bundles.*

3. *There exists a vector bundle W on (E, X) such that for every point $y \in E$, the action of $\pi_1(F)$ on W_y is faithful. Here $F = p^{-1}(p(y))$ stands for the fiber of y.*

Proof. We first show 1. $\Rightarrow$ 2. Let $Y \circlearrowleft K$ be such that $(E, X) \simeq [Y/K]$, and let $Y' \subset Y$ be the K-invariant subspace corresponding to $X' \subset X$. Let V be a vector bundle on (E, X) and let $\tilde{V}$ be the corresponding K-equivariant vector bundles on Y. It is well known that any equivariant vector bundle $\tilde{V}$ on a compact space Y' embeds in one of the form $Y' \times M$, where M is a representation of K. Let W be the vector bundle on (E, X) corresponding to $Y \times M \to Y$. Since $\tilde{V}$ embeds in $Y \times M|_{Y'}$, the bundle V embeds in $W_{(E',X')}$, as desired.

We show 2. $\Rightarrow$ 3. Suppose that (E, X) has enough vector bundles, and let $\{U_i\}$ be a finite cover of X such that $(p^{-1}(U_i), U_i) \simeq [Y_i/G_i]$. Let M_i be faithful representations of G_i, and let V_i be the vector bundles on $(p^{-1}(U_i), U_i)$ corresponding to $Y_i \times M_i \to Y_i$. Since M_i is faithful, the isotropy groups of $(p^{-1}(U_i), U_i)$ act faithfully on the fibers of V_i. Let W_i be vector bundles on (E, X) such that $V_i \hookrightarrow W_i|_{(E',X')}$, and let $W := \bigoplus W_i$. Clearly, the isotropy groups of (E, X) act faithfully on the fibers of W.

We now show 3. $\Rightarrow$ 1. Let $(P, P/\sim)$ be the total space of the frame bundle of W. It has no isotropy groups, and so it's equivalent to the space $Y := P/\sim$. The fibers of $P \to Y$ are contractible and P has a free action of the orthogonal group $O(n)$. So $E = P/O(n)$ is a Borel construction for $Y \circlearrowleft O(n)$. Similarly $X = Y/O(n)$. We have identified (E, X) with the global quotient $[Y/O(n)]$, which finishes the proof. □

Corollary 6.10 *Compact orbispaces have enough vector bundles.*

Proof. By corollary 6.7, all orbispaces are global quotients by compact Lie groups. □

As a consequence of Corollary 6.10, we can prove excision for K-theory.

Proposition 6.11 *Let $p : E \to X$ be a compact orbispace and let $(F, A) := (p^{-1}(A), A)$ and $(E', X') := (p^{-1}(X'), X')$ be two sub-orbispaces and let (F', A') be the intersection of (F, A) and (E', X'). Let $V' = V'_0 \ominus V'_1$ be a $\mathbb{Z}/2$-graded vector bundle on (E', X') and $f' : V'_0|_{(F',A')} \to V'_1|_{(F',A')}$ be an isomorphism.*

Then there exists a $\mathbb{Z}/2$-graded vector bundle V on (E, X) and an isomorphism $f : V_0|_{(F,A)} \to V_1|_{(F,A)}$ such that $(V, f)|_{(E',X')}$ represents the same class as (V', f') is *the relative K-theory group $K^0((E',X'),(F',A'))$. More precisely $(V,f)|_{(E',X')}$ is the sum of (V',f') and $(Z\ominus Z, \mathrm{Id}_Z|_{(F',A')})$, for some vector bundle Z on (E',X').*

Proof. Let $V' = V_0' \ominus V_1'$ and f be as above. By Theorem 6.9, we can find a vector bundle V_0 on (E,X) such that $V_0' \hookrightarrow V_0|_{(E',X')}$. Let Z be the orthogonal complement of V_0' in $V_0|_{(E',X')}$. We build V_1 by gluing $V_0|_{(F,A)}$ and $V_1' \oplus Z$ along the map

$$V_0|_{(F',A')} \simeq (V_0' \oplus Z)|_{(F',A')} \xrightarrow{f\oplus 1} .(V_1' \oplus Z)|_{(F',A')}.$$

We then let f be the natural map between $V_0|_{(F,A)}$ and $V_1|_{(F,A)}$. It is clear that (V,f) has the required properties. □

Remark 6.12 Theorem 6.9 still holds for Lie orbispaces (with identical proof), but Corollary 6.10 is not true any more. So we cannot use finite dimensional vector bundles in order to define K-theory of Lie orbispaces. One should instead use bundles of $\mathbb{Z}/2$-graded Hilbert spaces equipped with odd self-adjoint Fredholm operators.

6.4 Contractible groups

Recall the group $U = \varinjlim U(n!)$ from (6.1) and the family $\mathcal{F}$ of subgroups embedded via their regular representation. There are two main properties of U and $\mathcal{F}$ used in the proof of Theorem 6.6. The first one is that for any $G, H \in \mathcal{F}$ and any monomorphism $G \to H$ there exists an element $k \in U$ such that $Ad(k)|_G = f$ (see Lemma 6.1). The second one is that the centralizers $Z_U(H)$ are rational spaces for all $H \in \mathcal{F}$ (see equations (6.19) and (6.20)).

The idea is that, from the point of finite groups, these centralizers behave as if they were contractible. This motivates the following definition.

Definition 6.13 *A topological group K is contractible with respect to a family $\mathcal{F}$ if for every groups $G, H \in \mathcal{F}$ and every monomorphism $f : G \to H$, the space*

$$\{k \in K \mid Ad(k)|_G = f\} \tag{6.33}$$

is contractible.

As one might expect, it is easy to build contractible groups:

Proposition 6.14 *Let K be a topological group and $\mathcal{F}$ a family of subgroups. Then there exists a group K' containing K, which is contractible with respect to the family generated by $\mathcal{F}$.*

Proof. Each time we find a non-trivial map from a sphere into one of the spaces (6.33), we add a cell to kill it. We then freely generate a groups, modulo the relation that $Ad(k)|_G = f$ for all points k in that cell. This process terminates by the small object argument. □

Another example of contractible group is the unitary group of an infinite dimensional Hilbert space:

Example 6.15 Let $\mathcal{H}$ be a countably infinite dimensional Hilbert space. Let $U(\mathcal{H})$ be it group of unitary automorphisms, and let $\mathcal{F}$ be the family of finite (or compact Lie) subgroup G such that each irrep of G appears infinitely many times in $\mathcal{H}$. Then $U(\mathcal{H})$ is contractible with respect to $\mathcal{F}$.

Indeed, let $f : G \to H$ be a monomorphism between elements of $\mathcal{F}$, and consider the space (6.33). Since f is injective, the inclusion $G \hookrightarrow U(\mathcal{H})$ and the map f are equivalent representations of G, hence (6.33) in non-empty. The space (6.33) carries a simply transitive action of the centralizer $Z_{U(\mathcal{H})}(G) \simeq (U(\mathcal{H}))^r$. By Kuiper's theorem [21], the group $U(\mathcal{H})$ is contractible. Therefore $Z_{U(\mathcal{H})}(G)$ is contractible, and so is the space (6.33).

If we replace U by a group which contains all finite groups, and which is contractible with respect to them, them the proof of Theorem 6.6 goes through. As in Lemma 6.2, the map

$$|\mathrm{Sing}\underline{B}K| \to |\underline{\mathsf{Orb}}| \tag{6.34}$$

is a stratified fibration, and its fibers are $B(Z_K(H))$ as in (6.19). By (6.33), these fibers are contractible, and so the map (6.34) is a stratified homotopy equivalence.

Since $|\mathrm{Sing}\underline{B}K|$ maps to $|\underline{B}K|$ and $|\underline{B}K|$ maps to $|\underline{\mathsf{Orb}}|$, we also get that

$$|\underline{B}K| \simeq |\underline{\mathsf{Orb}}| \tag{6.35}$$

as stratified spaces. This suggests the following improvement of Theorem 6.6.

Theorem 6.16 *Let K be a group and let $\mathcal{F}$ be the family of its finite subgroups. Suppose that every finite group is isomorphic to an element of $\mathcal{F}$, and that K is contractible with respect to $\mathcal{F}$.*

Then the natural functor from $(K, \mathcal{F})$-spaces to the category of orbispaces and representable maps (i.e. injective on stabilizer groups) is an equivalence of topologically enriched categories.

Proof. Let $\mathcal{O}_K$ denote the category of orbits K/G for $G \in \mathcal{F}$ and K-equivariant maps between them. Let $\mathcal{O}_K$-spaces be the category of continuous contravariant functors $\mathcal{O}_K \to \mathsf{spaces}$. By Elmendorf's theorem [8], the categories of $(K, \mathcal{F})$-spaces and $\mathcal{O}_K$-spaces are topologically equivalent. The functor K-$\mathsf{spaces} \to \mathcal{O}_K$-$\mathsf{spaces}$ is given by

$$Y \mapsto \big(K/G \mapsto \mathrm{Map}_K(K/G, Y)\big) \tag{6.36}$$

and its homotopy inverse is the bar construction

$$F \mapsto B(F, \mathcal{O}_K, \iota) \tag{6.37}$$

where ι is the inclusion $\mathcal{O}_K \hookrightarrow K$-$\mathsf{spaces}$.

Let repr be the category of orbispaces and representable maps, and consider the the full subcategory of "orbipoints" $BG := (K(G,1), pt)$. Taking the standard simplicial model for BG and using the realization of the simplicial mapping space instead of all continuous maps, we obtain an equivalent subcategory $\mathcal{B}$. Its morphisms are given by

$$\mathrm{Hom}_{\mathcal{B}}(BG, BH) = \big(\mathrm{Mono}(G, H) \times EH\big)/H. \tag{6.38}$$

Let $\mathcal{B}$-spaces be the category of continuous contravariant functors $\mathcal{B} \to \mathsf{spaces}$. One has two functors similar to (6.36) and (6.37) given by

$$(E,X) \mapsto \big(BG \mapsto \mathrm{Map}_{\mathsf{repr}}(BG,(E,X))\big) \tag{6.39}$$

and

$$F \mapsto B(F,\mathcal{B},\iota), \tag{6.40}$$

where ι is the inclusion $\mathcal{B} \hookrightarrow \mathsf{repr}$. It is important that we only have monomorphisms in (6.38), otherwise the right hand side of (6.40) would fail the third condition of Theorem 3.5, and therefore wouldn't be an orbispace[1]. The proof of Elmendorf's theorem applies, and we get that the functors (6.39) and (6.40) are homotopy inverses.

In order to show that K-spaces and repr are equivalent, it's enough to show that $\mathcal{O}_K$-spaces and $\mathcal{B}$-spaces are equivalent. To see that, we compare the categories $\mathcal{O}_K$ and $\mathcal{B}$. The hom-spaces in $\mathcal{O}_K$ are given by

$$\mathrm{Hom}_{\mathcal{O}_K}(K/G,K/H) = \big\{k \in K \,\big|\, Ad(k)G \subset H\big\}/H. \tag{6.41}$$

By (6.33), the space $\{k \in K | Ad(k)G \subset H\}$ is homotopy equivalent to the set of monomorphisms from G to H. So (6.41) and (6.38) are homotopy equivalent.

We have shown that $\mathcal{O}_K$ and $\mathcal{B}$ are homotopy equivalent categories. Therefore, the same holds for $\mathcal{O}_K$-spaces and $\mathcal{B}$-spaces, and also for K-spaces and repr. □

Remark 6.17 The statement in Theorem 6.16 still holds if we replace finite groups by any class of topological groups that admits a set of representatives.

[1]We should also replace right hand side of (6.40) by a stratified fibration.

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