Solving the Dirac Equation in a Two-Dimensional Spacetime Background with a Kink

by

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Abstract

In the following paper, I study the Dirac equation in curved spacetime and solve this equation in two-dimensional spacetime backgrounds discovered by Jackiw et al[4], [5]. I will first discuss flat spacetime and introduce the Dirac equation, which describes the relativistic wavefunctions of spin-$\frac{1}{2}$ particles. I will go on to discuss curved spacetime and introduce the Vierbein field, which will relate an arbitrary curved spacetime to the simpler flat spacetime. After examining various transformation properties in curved and flat spacetime, I will use the properties to postulate the Dirac equation in arbitrary curved coordinates. By checking the invariance of the Dirac action under both coordinate and Lorentz transformations, I will verify that the postulated Dirac action satisies the proper symmetries and properties that it should.

In order to solve the Dirac equation in the spacetime backgrounds found by Jackiw, I will need to examine the equation in the two- and three-dimensional cases. I will then reduce the three-dimensional Dirac equation to two dimensions to describe states independent of the third dimension, because the spacetime backgrounds were derived in an Ansatz which reduced the metric from three to two spacetime dimensions. I will then be able to solve the reduced Dirac equation in each of four spacetime backgrounds. The solutions to the first three spacetime backgrounds involve spherical Bessel functions. The last spacetime background has a kink, and its solution involves a hyperbolic cosine function enveloping an oscillatory factor.

Thesis Supervisor: Roman W. Jackiw
Title: Zacharias Professor of Physics

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Contents

Chapter 1

Flat Spacetime

In order to examine the wavefunction of spin-$\frac{1}{2}$ particles in the recently discovered kink spacetime background, I will first need to analyze the properties and invariances of flat and curved spacetime. I begin by analyzing the Dirac equation in flat spacetime.

Throughout this paper, I will use letters from the modern Roman alphabet $(a, b, c...)$ to denote flat spacetime indices, called Lorentz indices. We will start our analysis considering an arbitrary number of spacetime dimensions, which we denote D. We define our contravariant and covariant Lorentz vectors as

$$A^a = (A^0, A^1, ..., A^{D-1}) \tag{1.1}$$

$$A_a = (A_0, A_1, ..., A_{D-1}) = (A^0, -A^1, ..., -A^{D-1}) = \eta_{ab} A^b, \tag{1.2}$$

where η_{ab} is the flat spacetime Minkowski metric. Any repeated index is implicitly summed over. By this convention, repeated indices will always be one lower index and one upper index.

The first coordinate A^0, typically has some relation to time, while the other coordinates are related to the $(D-1)$ spatial dimensions. We've chosen the convention that the spatial coordinates change sign when raised of lowered, while the time coordinate's sign is conserved. According to this definition, η_{ab} must be the diagonal matrix with 1 in the upper left entry and -1's throughout the rest of the diagonal. We define η^{ab} to be its inverse tensor, so that we can now use η_{ab} to lower indices and

η^{ab} to raise indices. Since η_{ab} is its own inverse, both tensors have the same matrix representation. In the 4-dimensional spacetime we live in, the Minkowski metric is

$$\eta_{ab} = \eta^{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.3}$$

By defining upper and lower indices, it becomes much easier to work with relativistic quantities. We can now define $x^a = (ct, \vec{x})$ and $p^a = (\frac{E}{c}, \vec{p})$ to be the position and momentum Lorentz vectors, which include time and energy respectively. According to relativistic theory, x^a and p^a must now transform as Lorentz vectors. Any Lorentz vector must transform the same way under any Lorentz transformation, so that we get the same Lorentz vector in a different Lorentz frame. A Lorentz transformation is a linear transformation of the coordinates that preserves the inner product of any two Lorentz vectors, $A_a B^a$. This inner product is a scalar, so we say that scalars are invariant under Lorentz transformations[6].

We can express the transformation as a two-tensor $\Lambda^a{}_b$, and we find the condition on it so that the inner product is the same in any Lorentz frame.

$$A'^a = \Lambda^a{}_b A^b, \tag{1.4}$$

$$\Lambda^c{}_a \eta_{cd} \Lambda^d{}_b = \eta_{ab} \tag{1.5}$$

Lorentz transformations can be decomposed into rotations, boosts, and space or time reversal. Rotations refer to spatial rotations of our frame, and boosts transform to frames moving with a constant velocity with respect to the original frame. By taking the determinant of both sides of the above equation, we see that det $\Lambda^a{}_b$ must be ± 1. Space or time reversal arise from a discrete Lorentz transformation with determinant -1, while all other transformations can be broken down into a series of infinitesimal transformations. We can thus check the Lorentz invariance of any quantity under propery Lorentz transformations (det $+1$) by checking whether it's invariant under infinitesimal Lorentz transformations. I will use this later in the paper to test that the Dirac action that I find is Lorentz invariant.

For infinitesimal transformations, we let $\Lambda^a{}_b = \delta^a_b + \varepsilon^a{}_b$ where $\varepsilon^a{}_b \ll 1$, so that the coordinates are only changed by a small amount. Plugging this into the above constraint on $\Lambda^a{}_b$ and keeping only terms up to first order in $\varepsilon^a{}_b$, we get the constraint that ε_{ab} is antisymmetric.

$$\varepsilon_{ab} = -\varepsilon_{ba} \tag{1.6}$$

When analyzing the effect of an infinitesimal Lorentz transformation, a Lorentz vector A^a is transformed by $\delta A^a = \varepsilon^a{}_b A^b$, where ε_{ab} is a constant arbitrary antisymmetric infinitesimal parameter. Before continuing, I should also note that the derivative with respect to a contravariant vector transforms like a covariant vector and vice versa. I will thus use the compact notation,

$$\frac{\partial}{\partial x^a} \rightarrow \partial_a, \quad \frac{\partial}{\partial x_a} \rightarrow \partial^a. \tag{1.7}$$

I will also simplify notation by using units where $\hbar = c = 1$.

1.1 Dirac Equation in Flat Spacetime

In quantum mechanics, the momentum operator $\vec{p}$ is given in position space by $-i\nabla$. The Hamiltonian gives the energy of the system by the operation $i\frac{\partial}{\partial t}$. We see from this that it's natural to define the covariant momentum operator p_a to act this way on contravariant position space by the operation $i\partial_a$. This forces contravariant momentum to act on contravariant position space in the expected manner. We expect the relativistic equation $-p_a p^a + m^2 = 0$ to hold for valid quantum mechanical states, where m is the mass of the particle. If we apply this equation directly, we get the Klein-Gordon equation:

$$\partial_a \partial^a \psi + m^2 \psi = 0 \tag{1.8}$$

If we take ψ to be a single wavefunction of the coordinates x^a, it becomes impossible to define a probability density which is positive definite and transforms properly under Lorentz transformations. The probability density and probability current must form a contravariant vector in order for the probability interpretation of quantum mechanics to hold. The second derivative in the Klein-Gordon equation forces us to include a derivative in the contravariant vector, which allows for negative probability. In order the get a probability vector that isn't defined with a derivative, we need our original equation to contain only first derivatives. Dirac proposed such a linear differential equation:

$$i\gamma^a\partial_a\psi + m\psi = 0 \tag{1.9}$$

Rather than having ψ be a single wavefunction, it's now a vector of wavefunctions, called a spinor. I'll call column vectors like ψ spinors and the matrices that act on them bispinors. The γ^a are the gamma matrices, which act as a contravariant vector of matrices. These are constant matrices in any Lorentz frame. The equation is Lorentz invariant, since the inner product $\gamma^a\partial_a\psi$ is preserved. This is the Dirac equation in flat spacetime. If we act on the Dirac equation with $-i\gamma^b\partial_b + m$, the cross terms cancel and give

$$\gamma^b\gamma^a\partial_b\partial_a\psi + m^2\psi = 0. \tag{1.10}$$

We now impose the anti-commutation rule

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \tag{1.11}$$

on the gamma matrices, and we find that the Klein-Gordon equation holds for each component of the spinor. Our anti-commutation algebra of the gamma matrices imposes the relativistic equation $-p_a p^a + m^2 = 0$ on valid spinors.

Through linear algebra, any higher order linear differential equation can be expressed as a first order linear differential matrix equation. The Dirac equation did just that. By our anti-commutation rule, we see that $(\gamma^0)^{-1} = \gamma^0$, so we could also express the Dirac equation as

$$i\partial_0 \psi = -i\gamma^0 \gamma^j \partial_j \psi - m\gamma^0 \psi \tag{1.12}$$

where j is summed over just the spatial coordinates. Since the term on the left is the time derivative, this is the Dirac equivalent to the Schrödinger equation. Our Hamiltonian is

$$H = -i\gamma^0 \gamma^j \partial_j - m\gamma^0 = \gamma^0 \gamma^j p_j - m\gamma^0. \tag{1.13}$$

Since the energy of any spinor must be real, the Hamiltonian must be Hermitian. This means that γ^0 and $\gamma^0\gamma^j$ must be Hermitian. This gives the relation $\gamma^{a\dagger} = \gamma^0 \gamma^a \gamma^0$ for each gamma matrix. We define the Dirac adjoint for any bispinor matrix M or spinor ψ and express this relation as such:

$$\overline{M} \equiv \gamma^0 M^\dagger \gamma^0 \tag{1.14}$$

$$\overline{\psi} \equiv \psi^\dagger \gamma^0 \tag{1.15}$$

$$\overline{\gamma^a} = \gamma^a. \tag{1.16}$$

Note that $\overline{\psi}$ is a row vector, while ψ is a column vector. Since $\overline{\gamma^a} = \gamma^a$, we say that the gamma matrices are Dirac Hermitian. The anti-commutation and Hermiticity rules are the only constraints on the gamma matrices, so any set of matrices satisfying these constraints is a valid representation. We will always choose to work in an irreducible representation, so that the dimensions of the gamma matrices and spinor are minimized. Any two irreducible representations are related by a similarity transformation, so changing representations just mixes the components of the spinor. The physics is the same in any representation. The constraints can be used to show that the gamma matrices are traceless and must be of even dimension. In four spacetime dimensions the simplest gamma matrices are 4×4, while in two or three dimensions the gamma matrices are 2×2[1].

The Dirac equation in four spacetime dimensions describes a spin-$\frac{1}{2}$ particle. The four components of the spinor can be made to be the spin-up and spin-down wavefunctions for the positive and negative energy solutions. The negative energy solutions create a duplicate of the expected positive energy spectrum, due to relativity giving us an equation for E^2 rather than for E. The ground state of a system of spin-$\frac{1}{2}$ particles is taken to have all of these negative energy states filled. An unfilled negative energy state can be viewed as having an anti-particle in this state. This means that the excitation of a particle from a negative to positive energy state corresponds to the creation of a particle and anti-particle. I'll only be working with single particle theory for a spin-1/2 particle, so the Dirac equation will properly describe it.

1.2 Dirac Action in Flat Spacetime

We postulate that the Dirac action in flat spacetime should be

$$S = \int d^D x \overline{\psi}(i\gamma^a \partial_a \psi + m\psi). \tag{1.17}$$

This integral extends over all spacetime. In order for this to be a valid action, it must produce the Dirac equation when we set $\delta S = 0$ under an arbitrary variation of the spinor $\delta\psi$. Since ψ is complex, we can vary its real and imaginary parts independently. This is equivalent to varying ψ and ψ^* independently. Since $\overline{\psi}$ is related to ψ^*, we can choose to vary ψ and $\overline{\psi}$ independently. Variation of $\overline{\psi}$ gives

$$\delta S = \int d^D x \delta\overline{\psi}(i\gamma^a \partial_a \psi + m\psi) = 0. \tag{1.18}$$

Since this holds for arbitrary $\overline{\psi}$, the Dirac equation must hold at all points in spacetime. Variation of ψ would give us the Dirac adjoint of the same equation, where we would integrate by parts and drop the boundary term since $\psi \rightarrow 0$ at infinity. This action does indeed reproduce the Dirac equation.

The Dirac action must also be Lorentz invariant. Since the action's a scalar, it must be the same in any Lorentz frame. I showed earlier that an infinitesimal Lorentz transformation is given by $\delta x^a = \varepsilon^a{}_b x^b$, where ε_{ab} is antisymmetric. The transformation also changes ψ by a similarity transformation S, $\psi \rightarrow S\psi$. The similarity transformation also transforms the gamma matrices, which must transform like Lorentz vectors. This means that S must satisfy

$$S^{-1}\gamma^a S = \Lambda^a{}_b\gamma^b. \tag{1.19}$$

Expanding in terms of infinitesimals, we find that $S = 1 + \frac{1}{2}\varepsilon_{ab}\Sigma^{ab}$ where

$$\Sigma^{ab} \equiv \frac{1}{4}[\gamma^a, \gamma^b]. \tag{1.20}$$

By Taylor expansion, we can expand scalar, vector, and spinor fields to see how they transform under Lorentz transformations.

$$\phi(x + \delta x) = \phi(x) + \delta x^a \partial_a \phi(x) \tag{1.21}$$

$$\delta\phi = \varepsilon^a{}_b x^b \phi = \frac{1}{2}\varepsilon_{ab}(x^a\partial^b - x^b\partial^a)\phi \equiv \frac{1}{2}\varepsilon_{ab}\delta^{ab}\phi \tag{1.22}$$

$$\delta^{ab}\phi = (x^a\partial^b - x^b\partial^a)\phi \tag{1.23}$$

Since ϕ must be invariant under any arbitrary antisymmetric ε_{ab}, any field that transforms this way is a Lorentz scalar. Similar expansions for vector and spin fields, along with the appropriate Lorentz or similarity transformation, give the transformations[1]

$$\delta^{ab}A_c = (x^a\partial^b - x^b\partial^a)A_c + \delta^a_c A^b - \delta^b_c A^a \tag{1.24}$$

$$\delta^{ab}A^c = (x^a\partial^b - x^b\partial^a)A^c + \eta^{ac}A^b - \eta^{bc}A^a \tag{1.25}$$

$$\delta^{ab}\psi = (x^a\partial^b - x^b\partial^a)\psi + \Sigma^{ab}\psi \tag{1.26}$$

$$\delta^{ab}\overline{\psi} = (x^a\partial^b - x^b\partial^a)\overline{\psi} - \overline{\psi}\Sigma^{ab}. \tag{1.27}$$

Any fields that transform in these ways are Lorentz vectors or spinors. A Lorentz tensor has an arbitrary number of Lorentz indices and transforms like a vector, except that we get the extra terms that we got from the vector transformation for each index.

[1]To get the last equation, I used the relation $\overline{\Sigma^{ab}} = -\Sigma^{ab}$.

For example, a rank-2 tensor transforms as

$$\delta^{ab} A^{cd} = (x^a \partial^b - x^b \partial^a) A^{cd} + \eta^{ac} A^{bd} - \eta^{bc} A^{ad} + \eta^{ad} A^{cb} - \eta^{bd} A^{ca}. \tag{1.28}$$

From these, we find that $\overline{\psi}\psi$ is a Lorentz scalar, and $\overline{\psi}\gamma^a\psi$ is a Lorentz vector. We can now define $\overline{\psi}\gamma^a\psi$ to be the Lorentz invariant probability vector and see that probability density is $\psi^\dagger\psi$, similar to the positive definite probability density defined nonrelativistically. The probability current also behaves the same way as its nonrelativistic counterpart.

$\overline{\psi}\gamma^a\partial_a\psi$ is also a Lorentz scalar, since the extra terms from the transformations of ψ and $\overline{\psi}$ cancel. To see this, we note that

$$[\gamma^c, \Sigma^{ab}] = \eta^{ac}\gamma^b - \eta^{bc}\gamma^a \tag{1.29}$$

follows from the anti-commutation properties of the gamma matrices. Since each term in the action is a Lorentz scalar, the total action itself is a Lorentz scalar. We have therefore defined a valid Dirac action for flat spacetime.

Chapter 2

Curved Spacetime

We will now generalize from flat to curved spacetime. I will start by switching from working with flat coordinates x^a to arbitrary curved coordinates x^μ. I will use Greek letters $(\mu, \nu, \rho ...)$ to denote curved spacetime indices. When I refer to spacetime indices, I will mean the Greek indices; when I refer to Lorentz indices, I will mean the Roman indices.

In these coordinates, the metric is no longer constant. Instead we use the spacetime metric $g_{\mu\nu}(x)$ to lower indices and its inverse $g^{\mu\nu}(x)$ to raise indices. We define a Vierbein field $e^a{}_\mu(x)$ to relate the metric to the flat spacetime metric η_{ab}. The Vierbein is defined such that

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu \tag{2.1}$$

at every spacetime point. We also define the inverse Vierbein field $E_a{}^\mu(x)$ at each point by requiring that

$$e^a{}_\mu E_b{}^\mu = \delta^a_b, \quad e^a{}_\mu E_a{}^\nu = \delta^\nu_\mu. \tag{2.2}$$

From these definitions, it follows that we can find the inverse metric from the inverse Vierbein.

$$g^{\mu\nu} = \eta^{ab} E_a{}^\mu E_b{}^\nu \tag{2.3}$$

We can now use g to raise to lower spacetime indices, η to raise of lower Lorentz indices, and the Vierbein and inverse Vierbein to change between spacetime and Lorentz indices. It is necessary to introduce this curved spacetime metric since spacetime is not flat in general.

2.1 Transformations in Curved Spacetime

We define a quantity in spacetime to be a vector or tensor in the same way as we did in flat spacetime. We can convert a spacetime tensor to a flat spacetime tensor by use of the Vierbien and insist that this new tensor transform as a tensor, as described in the previous section. For example, for A^μ to be a vector, $e^a{}_\mu A^\mu$ must transform as a Lorentz vector. These lead to transformation rules for infinitesimal transformations similar to those in flat spacetime.

$$\delta^{\alpha\beta} A_\mu = (x^\alpha \partial^\beta - x^\beta \partial^\alpha) A_\mu + \delta^\alpha_\mu A^\beta - \delta^\beta_\mu A^\alpha \tag{2.4}$$

$$\delta^{\alpha\beta} A^\mu = (x^\alpha \partial^\beta - x^\beta \partial^\alpha) A^\mu + g^{\alpha\mu} A^\beta - g^{\beta\mu} A^\alpha \tag{2.5}$$

2.1.1 Transformation of Spacetime Indices

We need to be careful to remember that the spacetime metric and Vierbein are coordinate dependent. Thus, an ordinary partial derivative of a tensor, such as $\partial_\alpha A^\mu$, does not transform as a tensor in curved spacetime, due to the derivatives of the metric. We define a new derivative operator D_α that transform as a tensor when acting on a tensor. We define

$$D_\alpha A^\mu \equiv \partial_\alpha A^\mu + \Gamma^\mu_{\alpha\beta} A^\beta \tag{2.6}$$

$$D_\alpha A_\mu \equiv \partial_\alpha A_\mu - \Gamma^\nu_{\alpha\mu} A_\nu \tag{2.7}$$

where $\Gamma^\alpha_{\mu\nu}$ is a connection field called the Christoffel coefficients. D_α acts on an arbitrary tensor by ∂_α plus a Christoffel term for each index. Since D_α on any tensor must transform as a tensor, we must be able to pull the spacetime metric through this derivative to raise or lower indices. It must be the case that

$$D_\alpha g_{\mu\nu} = 0, \quad D_\alpha g^{\mu\nu} = 0. \tag{2.8}$$

Since the spacetime metric is a tensor, we can expand these derivatives in terms of Christoffel symbols and the metric. From this, we find that $\Gamma^\alpha_{\mu\nu}$ is symmetric in μ and ν and can be found in terms of the metric and its derivative:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu}) \tag{2.9}$$

The Christoffel symbols vanish when we use a constant metric, so D_α reverts back to ∂_α in flat spacetime. Using this equation for the Christoffel coefficients, D_α acting on any tensor is now another tensor. We could also define D^α by using the metric to raise the index on the derivative. It's also worth noting that the antisymmetric combination $\partial_\alpha A_\mu - \partial_\mu A_\alpha$ is a tensor, since the Christoffel symbols cancel.

2.1.2 Transformation of Lorentz Indices

We now turn our attention to tensors that have both spacetime and Lorentz indices. Such a tensor must transform like a Lorentz tensor in its Lorentz indices, and it must transform as a spacetime tensor in its spacetime indices. D_α acting on this tensor would transform like a tensor in its spacetime indices, but it wouldn't transform as a tensor in its Lorentz indices. We define a new type of derivative $\mathcal{D}_\alpha$ which does yield a tensor when it acts on a tensor. Starting with D_α acting on the tensor, we add another connection field, called the spin connection one-form $\omega_\mu{}^a{}_b$, for each Lorentz index. This connection field acts on the Lorentz indices in the same way that the Christoffel coefficients act on the spacetime indices. For example, the derivative of the field V^a_μ is

$$\mathcal{D}_\alpha V^a_\mu \equiv D_\alpha V^a_\mu + \omega_\alpha{}^a{}_b V^b_\mu \tag{2.10}$$

$$\mathcal{D}_\alpha V_{a\mu} \equiv D_\alpha V^{a\mu} + \omega_{\alpha ab} V^b_\mu. \tag{2.11}$$

Since $\mathcal{D}_\alpha$ acting on any tensor is a tensor, we must be able to move the Vierbien through this derivative, so that we can raise, lower, and change between spacetime and Lorentz indices. This follows from the same logic that told us we must be able to pull the spacetime metric through the D_α derivative. This tells us that[2]

$$\mathcal{D}_\alpha e^a{}_\mu = 0, \quad \mathcal{D}_\alpha E_a{}^\mu = 0. \tag{2.12}$$

We can use these relations to calculate the spin connection from the Vierbien and its derivatives. Each of the two relations yields a separate equivalent equation for the spin connection:

$$\omega_\mu{}^a{}_b = -E_b{}^\mu D_\alpha e^a{}_\mu \tag{2.13}$$

$$\omega_\mu{}^b{}_a = -e^a{}_\mu D_\alpha E_b{}^\mu \tag{2.14}$$

The right sides of these equations add to zero, since they give the derivative of δ^a_b. Thus, $\omega_{\mu ab}$ must be antisymmetric. Plugging in the definition of D_α, we get the spin connection. This definition of the spin connection allows $\mathcal{D}_\alpha$ acting on any tensor to be a tensor.

$$\omega_\mu{}^a{}_b = e^a{}_\alpha(\partial_\mu E_b{}^\alpha + \Gamma^\alpha_{\mu\nu} E_b{}^\nu) \tag{2.15}$$

$$= E_b{}^\nu(-\partial_\mu e^a{}_\nu + \Gamma^\alpha_{\mu\nu} e^a{}_\alpha) \tag{2.16}$$

$$\omega_{\mu ab} = -\omega_{\mu ba} \tag{2.17}$$

2.1.3 Transformation of Spinors

We next turn our attention to spinors with an arbitrary number of spacetime and Lorentz indices. To be invariant, this quantity must transform as a spinor as well as transforming like a tensor in its spacetime and Lorentz indices. If we act $\mathcal{D}_\alpha$ on such a spinor, the spacetime and Lorentz indices transform properly, but it doesn't transform as a spinor. We define a new derivative ∇_α that adds another connection field Γ_μ for the spinor. For consistancy, we also define its action on adjoint spinors and bispinor matrices accordingly. Again, Ψ represents a spinor and M represents an arbitrary bispinor matrix, such as the Dirac matrices. These can have any number of indices, which are transformed accordingly by $\mathcal{D}_\alpha$.

$$\nabla_\alpha \Psi^{a\mu} \equiv \mathcal{D}_\alpha \Psi^{a\mu} + \Gamma_\mu \Psi^{a\mu} \tag{2.18}$$

$$\nabla_\alpha M^{a\mu} \equiv \mathcal{D}_\alpha M^{a\mu} + [\Gamma_\mu, M^{a\mu}] \tag{2.19}$$

$$\nabla_\alpha \overline{\Psi^{a\mu}} \equiv \mathcal{D}_\alpha \overline{\Psi^{a\mu}} - \overline{\Psi^{a\mu}} \Gamma_\mu \tag{2.20}$$

Since we require that ∇_α acting on any spinor must transform as a spinor, it turns out that we must be able to pull any Dirac matrix through this derivative. This means that the derivative of the Dirac matrices must vanish, which allows us the find what this connection field must be[2].

$$\nabla_\alpha \gamma^a = 0 \tag{2.21}$$

$$\Gamma_\mu = \frac{1}{2}\omega_{\mu ab}\Sigma^{ab} \tag{2.22}$$

We again make use of Eq. (1.29) in this derivation. Using this connection field, the derivative ∇_α of any spinor now transforms as a spinor.

2.2 Dirac Action in Curved Spacetime

Using this invariant derivative along with the curved spacetime metric, we can now convert the Dirac equation into curved spacetime. We postulate the curved spacetime Dirac action to be

$$S_D = \int d^D x \sqrt{-g}\, \overline{\psi}(i\gamma^a E_a{}^\mu(\partial_\mu \psi + \frac{1}{2}\omega_{\mu bc}\Sigma^{bc}\psi) + m\psi). \tag{2.23}$$

The inverse Vierbein is introduced into the equation to convert the flat gamma matrices into curved spacetime, since the Vierbein relates curved spacetime to flat spacetime. The extra $\frac{1}{2}\omega_{\mu ab}\Sigma^{ab}$ term is added to make the derivative of ψ invariant under coordinate transformations.

Finally, we introduced the factor $\sqrt{-g}$ to the integral, where g is the determinant of the spacetime metric with lower indices. This term acts as a Jacobian under the integral and is needed because the curved spacetime coordinates are arbitrary. $\sqrt{-g}$ can also be expressed as e, where e is the determinant of the Vierbein. This is easily seen by taking the determinant of both sides of Eq. (2.1) and realizing that $\det \eta = -1$. $\sqrt{-g}$ therefore acts as a Jacobian which allows us to switch from integrating over flat spacetime to integrating over our curved spacetime coordinates.

We vary this action in the same manner as before, noting that independent variation with respect to the real and imaginary parts of ψ is equivalent to independent variation of ψ and $\overline{\psi}$. Varying the action with respect to $\delta\overline{\psi}$ gives us the Dirac equation in curved spacetime:

$$i\gamma^a E_a{}^\mu(\partial_\mu \psi + \frac{1}{2}\omega_{\mu bc}\Sigma^{bc}\psi) + m\psi = 0 \tag{2.24}$$

We could also vary with respect to $\delta\psi$ to get the Dirac adjoint of the Dirac equation. The sign of the extra $\frac{1}{2}\omega_{\mu ab}\Sigma^{ab}$ term works out, since taking the adjoint changes the sign twice, due to the minus signs from the complex conjugate of i and from $\overline{\Sigma^{ab}} = -\Sigma^{ab}$. Also, when we integrate by parts, we get a term of the form $\partial_\mu(\sqrt{-g}E_a{}^\mu)$. Through a bit of algebra, we can show that

$$\partial_\mu(\sqrt{-g}E_a{}^\mu) = \sqrt{-g}E_b{}^\mu \omega_{\mu a}{}^b. \tag{2.25}$$

From this, the proper adjoint Dirac equation emerges. Thus, the adjoint Dirac equation is consistent with the Dirac equation.

$$iE_a{}^\mu(-\partial_\mu\overline{\psi} + \frac{1}{2}\omega_{\mu bc}\overline{\psi}\Sigma^{bc})\gamma^a + m\overline{\psi} = 0 \tag{2.26}$$

In order for this Dirac action to be valid, it must be a scalar. This means that it's invariant under both spacetime coordinate transformations and Lorentz transformations. We will examine first its transformation properties under arbitrary coordinate transformations, called diffeomorphisms, and then its transformations properties under infinitesimal Lorentz transformations.

2.2.1 Transformation under Diffeomorphisms

When transforming under a diffeomorphism, we will change from our current spacetime coordinates x^μ to new spacetime coordinates which we'll denote with a prime, x'^ν. The primed coordinates are arbitrary functions of all the unprimed coordinates, such that there exists a one-to-one correspondence between coordinates. We can therefore also express the unprimed coordinates as functions of the primed coordinates, by taking the inverse. The spinor, and thus the adjoint spinor as well, is not transformed by a diffeomorphism, so $\psi'(x') = \psi(x)$. Thus must be so, since the spinor cannot depend on our choice of coordinates. The Vierbein must be transformed as

$$e'^a{}_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} e^a{}_\nu(x), \quad E'_a{}^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} E_a{}^\nu(x) \tag{2.27}$$

so that the primed Vierbein exchange Lorentz indices with the primed spacetime indices. $\frac{\partial x'^\mu}{\partial x^\nu}$ is the Jacobian matrix for the transformation. From this, we find that the primed spacetime metric is given by the unprimed metric along with two sets of this Jacobian matrix.

The differential $d^D x$ in the integral is transformed by the absolute value of the Jacobian, which is the determinant of the Jacobian matrix. The factor $\sqrt{-g} = e$ is transformed by the inverse Jacobian:

$$d^D x' = |\det \frac{\partial x'^\mu}{\partial x^\nu}| d^D x \tag{2.28}$$

$$\sqrt{-g'(x')} = e'(x') = \det e'^a{}_\mu(x') = \det(\frac{\partial x^\nu}{\partial x'^\mu} e'^a{}_\nu(x')) = \sqrt{-g(x)} \det \frac{\partial x^\nu}{\partial x'^\mu} \tag{2.29}$$

Since the Jacobian and inverse Jacobian cancel, the quantity $d^D x\sqrt{-g}$ remains invariant. This property is what makes the $\sqrt{-g}$ necessary in the integral. However, we have the absolute value of the Jacobian, so they only cancel when the Jacobian is positive. A negative Jacobian corresponds to either a space reflection or time reversal. Thus, the action will change sign under this type of a reversal, but the Dirac equation will stay the same. This means that the action will be invariant under proper coordinate transformations.

By the chain rule, we get a Jacobian matrix in the expression for the primed coordinates each time a quantity has a partial derivative with respect to the primed coordinates.

$$\partial'_\mu \psi' = \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \psi \tag{2.30}$$

Since the Jacobian matrix from the primed to unprimed coordinates is the inverse of the Jacobian matrix from the unprimed to primed coordinates,

$$\frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x'^\alpha}{\partial x^\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x'^\nu} = \delta^\mu_\nu . \tag{2.31}$$

From these properties, we find that ${E_a}^\mu \partial_\mu \psi$ is invariant.

This only leaves the $\frac{1}{2}\gamma^a {E_a}^\mu \omega_{\mu bc} \Sigma^{bc}$ term. Since the gamma matrices are invariant, it's left to find the transformation of ${E_a}^\mu \omega_{\mu bc}$ under diffeomorphisms. Since $\omega_{\mu bc}$ can be decomposed into terms involving the Vierbein and its derivative, calculation of $\omega'_{\mu bc}$ in terms of the unprimed frame leads to a large mess of Jacobian matrices and their derivatives. Through much algebra, I managed to find an expression for $E'_a{}^\mu \omega'_{\mu bc}$. This quantity is not invariant, but the extra terms from the transformation are symmetric in b and c. Since Σ^{bc} is antisymmetric, $\gamma^a {E_a}^\mu \omega_{\mu bc} \Sigma^{bc}$ turns out to be invariant due to the cancelling of these terms.

We've now seen that each part of the action is invariant under diffeomorphisms, so the total Dirac action must be invariant under arbitrary coordinate transformations. If it also turns out to be invariant under Lorentz transformations of the flat spacetime coordinates, then this will be a valid action for us to use.

2.2.2 Transformation under Infinitesimal Lorentz Transformations

We now examine the transformation properties of the action under infinitesimal local Lorentz transformations. These transform just the Lorentz indices. Since our curved coordinates are independent of our flat spacetime coordinates, the transformations have no effect on the curved coordinate dependence. Our infinitesimal antisymmetric parameter $\varepsilon_{ab}(x)$ now becomes coordinate dependent, since the infinitesimal Lorentz transformation is local at each spacetime coordinate. For tensors with an arbitrary number of spacetime and Lorentz indices, only the Lorentz indices are transformed. For each Lorentz index, the transformation is

$$\delta A^a = \varepsilon^{ab}(x)\delta A_b, \quad \delta A_a = \varepsilon_{ab}(x)\delta A^b. \tag{2.32}$$

For an arbitrary tensor, one of these terms arises for each Lorentz index. Since the Vierbein has one Lorentz index, one such term appears when we vary it. To vary the spin connection, we plug Eq. (2.15) into our variation and find the variation properties of ${\omega_\mu}^a{}_b$.

$$\delta {\omega_\mu}^a{}_b = {\varepsilon^a}_c {\omega_\mu}^c{}_b + {\varepsilon_b}^c {\omega_\mu}^a{}_c + \partial_\mu {\varepsilon_b}^a \tag{2.33}$$

The spinor and adjoint spinor also transform under the infinitesimal Lorentz transformations in the same way that we saw them transform in Eq. (1.26).

$$\delta\psi = \frac{1}{2}\varepsilon_{ab}\Sigma^{ab}\psi, \quad \delta\overline{\psi} = -\frac{1}{2}\varepsilon_{ab}\overline{\psi}\Sigma^{ab} \tag{2.34}$$

The Dirac matrices transform only in their Lorentz index. We used their transformation property previously to find the transformations of ψ and $\overline{\psi}$.

Since we now know how each component of the Dirac action transforms, we can now vary the action. The $\sqrt{-g}$ factor is invariant, since it depends on only on the spacetime coordinates. $\delta(\overline{\psi}\psi) = 0$ since the variations of ψ and $\overline{\psi}$ carry opposite signs and cancel. This means that the mass term is invariant. $\gamma^a {E_a}^\mu$ is also invariant, since the variation of the two factors cancel due to the antisymmetry of ε_{ab}. When we vary $\overline{\psi}\gamma^a {E_a}^\mu \partial_\mu \psi$, we now only get terms from the variation of $\partial_\mu\psi$ and $\overline{\psi}$. The variation of $\partial_\mu\psi$ yields the spinor variation along with a term arising from acting the derivative of ε_{ab}. The spinor and adjoint spinor variations cancel as they did for the mass term, leaving only

$$\delta(\overline{\psi}\gamma^a E_a{}^\mu \partial_\mu \psi) = \frac{1}{2}\partial_\mu \varepsilon_{bc} E_a{}^\mu \overline{\psi}\gamma^a \Sigma^{bc}\psi. \tag{2.35}$$

Variation of $\frac{1}{2}\overline{\psi}\gamma^a E_a{}^\mu \omega_{\mu bc}\Sigma^{bc}\psi$ also sees the cancellation of the terms from the variation of ψ and $\overline{\psi}$, leaving only the variation from $\omega_{\mu bc}\Sigma^{bc}$. The two terms from the variation of Σ^{bc} cancel two of the terms from the variation of $\omega_{\mu bc}$. The leaves only the term from the variation of $\omega_{\mu bc}$ involving the derivative of ε_{ab}.

$$\delta(\frac{1}{2}\overline{\psi}\gamma^a E_a{}^\mu \omega_{\mu bc}\Sigma^{bc}\psi) = \frac{1}{2}\partial_\mu \varepsilon_{cb} E_a{}^\mu \overline{\psi}\gamma^a \Sigma^{bc}\psi \tag{2.36}$$

By antisymmetry of ε_{ab}, these two last remaining terms cancel. The variation of the Dirac action therefore vanishes under arbitrary infinitesimal local Loretz transformations. Since the action transforms as a scalar under both Lorentz and coordinate transformations, we have defined a valid action for the Dirac equation generalized into curved spacetime.

Chapter 3

Dimensional Reduction

Since I will be solving the Dirac equation in two spacetime dimensions, I will simplify the Dirac equation for the two dimensional case. The formulas for the spacetime metric with kinks discovered by Jackiw et al[4], [5] were found using an Ansatz which reduced from three to two dimensions, so I will also simplify the equation for the three dimensional case. I will then dimensionally reduce from three to two dimensions by use of the Kaluza-Klein Ansatz. To avoid confusion, I will use Greek and Roman letter from the beginning of the alphabet $(\alpha, \beta, ...; a, b, ...)$ to denote spacetime and Lorentz indices respectively in two dimensions. I will use Greek and Roman letter from the middle of the alphabet $(\mu, \nu, ...; i, j, ...)$ to denote spacetime and Lorentz indices respectively in three dimensions.

3.1 Two Dimensional Dirac Equation

Two dimensional spacetime consists of one spatial dimension and a dimension corresponding to time. I will start by introducing the totally antisymmetric ε_{ab} in two dimensions. This quantity does not transform as a tensor, so I will not raise and lower indices as such. For convenience, ε^{ab} will be defined to be equal to ε_{ab}. We define ε_{ab} such that $\varepsilon_{01} = 1$ and ε_{ab} changes sign under the exchange of the indices, so that it vanishes when an index is repeated.

The spin connection field $\omega_{\alpha ab}$ has only one unique Lorentz component $\omega_{\alpha 01}$, since it's antisymmetric. We therefore define a connection field with no Lorentz indices and can express $\omega_{\alpha ab}$ in terms of this field.

$$\omega_\alpha \equiv \omega_{\alpha 01} \tag{3.1}$$

$$\omega_{\alpha ab} = \varepsilon_{ab}\omega_\alpha \tag{3.2}$$

We define a new bispinor matrix γ^5 to be the product of the gamma matrices. We will not raise or lower this index, since the index is never summed over. The 5 is used as the index since it was originally defined in four spacetime dimensions and was used along with the gamma matrices as a fifth Dirac matrix. Due to the anticommutation relations of the gamma matrices, $\Sigma^{ab} = \frac{1}{2}\gamma^a\gamma^b$ when $a \neq b$ and vanishes when $a = b$. We can therefore express Σ^{ab} in terms of γ^5.

$$\gamma^5 \equiv \gamma^0\gamma^1 \tag{3.3}$$

$$\Sigma^{ab} = \frac{1}{2}\varepsilon^{ab}\gamma^5 \tag{3.4}$$

We can now express the Dirac action and equation in terms of ω_α and γ^5. To see this, note that

$$\varepsilon^{ab}\varepsilon_{ac} = \delta^b_c \tag{3.5}$$

$$\varepsilon^{ab}\varepsilon_{ab} = 2. \tag{3.6}$$

The Dirac action and Dirac equation in two dimensions now simplify to:

$$S_2 = \int d^2x\sqrt{-g}\ \overline{\psi}(i\gamma^a E_a{}^\alpha(\partial_\alpha\psi + \frac{1}{2}\omega_\alpha\gamma^5\psi) + m\psi) \tag{3.7}$$

$$i\gamma^a E_a{}^\alpha(\partial_\alpha\psi + \frac{1}{2}\omega_\alpha\gamma^5\psi) + m\psi = 0 \tag{3.8}$$

Since the gamma matrices are 2×2 in two or three dimensions, the math is simplest when we use Pauli matrices for our representation. The Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.9}$$

The Pauli matrices satisfy the anticommutation relation $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ and are Hermitian. We can choose γ^0 to be one Pauli matrix and γ^1 to be $\pm i$ times another Pauli matrix. The gamma matrices will thus satisfy the necessary anticommutation relations and be Dirac Hermitian. Choosing our gamma matrices as such, γ^5 turns out to be plus or minus the last Pauli matrix.

3.2 Three Dimensional Dirac Equation

Three dimensional spacetime includes two spatial dimensions and one dimension of time. Similar to its two dimensional counterpart, we now introduce the totally antisymmetric ε_{ijk}. This is defined such that $\varepsilon_{012} = 1$ and ε_{ijk} changes sign under the the exchange of any two indices. This implies that any terms with repeated indices vanish, and ε_{ijk} remains the same under a cyclic exchange of indices.

Since the spin connection field is antisymmetric, it only has three unique Lorentz components. We can therefore use ε_{ijk} to define a connection field with only one Lorentz index. We can express the full spin connection field in terms of this new reduced field.

$$\omega_\mu{}^k \equiv \frac{1}{2}\varepsilon^{ijk}\omega_{\mu ij} \tag{3.10}$$

$$\omega_{\mu ij} = \varepsilon_{ijk}\omega_\mu{}^k \tag{3.11}$$

To see that these equations are consistent, we note that

$$\varepsilon^{ijk}\varepsilon_{imn} = \delta^j_m\delta^k_n - \delta^j_n\delta^k_m \tag{3.12}$$

$$\varepsilon^{ijk}\varepsilon_{ijm} = 2\delta^k_m. \tag{3.13}$$

Again, we define γ^5 to be the product of the gamma matrices. Since we now take the product of three gamma matrices, our expression for Σ^{ij} changes. Using the anticommutation rules for the gamma matrices, we can express Σ^{ij} simply in terms of the gamma matrices and γ^5.

$$\gamma^5 \equiv \gamma^0\gamma^1\gamma^2 \tag{3.14}$$

$$\Sigma^{ij} = \frac{1}{2}\varepsilon^{ijk}\gamma_k\gamma^5. \tag{3.15}$$

We can now put these expressions into the Dirac action and equation. We therefore simplify the Dirac action and Dirac equation in three dimensions to:

$$S_3 = \int d^3x\sqrt{-g}\ \overline{\psi}(i\gamma^j E_j{}^\mu(\partial_\mu\psi + \frac{1}{2}\omega_{\mu k}\gamma^k\gamma^5\psi) + m\psi) \tag{3.16}$$

$$i\gamma^j E_j{}^\mu(\partial_\mu\psi + \frac{1}{2}\omega_{\mu k}\gamma^k\gamma^5\psi) + m\psi = 0 \tag{3.17}$$

As in two dimensions, it's simplest to choose the gamma matrices using the Pauli matrices. Choosing γ^0 to be one Pauli matrix and γ^1 and γ^2 to be $\pm i$ times the other two Pauli matrices, our gamma matrices satify the necessary anticommutation rules and are Dirac Hermitian. In any 2×2 representation, we see that γ^5 is just $\pm i$ times the identity. This will simplify our calculations.

3.3 Dirac Action in Hermitian Form

Recall that the gamma matrices were taken to be Dirac Hermitian to make our Hamiltonian Hermitian. This forces our energy and momentum eigenvalues to be real, so we say that the Dirac equation is Hermitian. The Hermiticity of the Dirac equation is apparent in the fact that variation of the action with respect to ψ or $\overline{\psi}$ give Dirac adjoints of the same equation. This can only be the case when the Dirac action is Hermitian. However, the action treats ψ or $\overline{\psi}$ quite differently, so it's not immediately apparent that this action is Hermitian. Since the action is a scalar, saying that it is Hermitian is the same as saying that it is real. I will now put this action in explicitly Hermitian form.

The Dirac adjoint of a scalar is just its complex conjugate. To decompose the action into a Hermitian and anti-Hermitian part, we add and subtract the adjoint. This is the same as decomposing it into real and imaginary parts.

$$S = \frac{1}{2}(S + \overline{S}) + \frac{1}{2}(S - \overline{S}) \tag{3.18}$$

We will now see that the anti-Hermitian part $S - \overline{S}$ vanishes. The Vierbein and spacetime metric are taken to be real. When we subtract the adjoint action from the action, the mass term cancels since $\overline{\psi}\psi$ is Dirac Hermitian. The remaining terms take the form

$$\frac{1}{2}(S - \overline{S}) = \int d^Dx \frac{i}{2}\sqrt{-g}E_a{}^\mu(\overline{\psi}\gamma^a\partial_\mu\psi + \partial_\mu\overline{\psi}\gamma^a\psi + \frac{1}{2}\omega_{\mu bc}\overline{\psi}\,[\gamma^a, \Sigma^{bc}]\,\psi). \tag{3.19}$$

Using Eq. (1.29) and Eq. (2.25), we can rearrange this term into a total derivative under the integral:

$$\frac{1}{2}(S - \overline{S}) = \int d^Dx\partial_\mu(\frac{i}{2}\sqrt{-g}\;\overline{\psi}\gamma^a E_a{}^\mu\psi) \tag{3.20}$$

Since we integrate a total derivative over all space, we're left with only the boundary terms, which vanishes at infinity. The anti-Hermitian part of the action vanishes, so we see that the action is indeed Hermitian.

We can therefore express the action as explicitly Hermitian.

$$S_D = \frac{1}{2}(S_D + \overline{S_D}) = \int d^Dx\sqrt{-g}[\frac{i}{2}E_a{}^\mu(\overline{\psi}\gamma^a\partial_\mu\psi - \partial_\mu\overline{\psi}\gamma^a\psi) + \frac{i}{4}E_a{}^\mu\omega_{\mu bc}\overline{\psi}\{\gamma^a, \Sigma^{bc}\}\psi + m\overline{\psi}\psi] \tag{3.21}$$

Since this expression treats ψ and $\overline{\psi}$ the same, it is explicitly Hermitian.

The anticommutator $\{\gamma^a, \Sigma^{bc}\}$ vanishes when any two of the three Lorentz indices repeat, due to the anticommutation rules. In two dimensions, at least two of the three indices must always repeat, so the spinor connection field vanishes completely in the Hermitian form. In three dimensions, the non-vanishing terms arise only when we use all three different gamma matrices, so $\{\gamma^a, \Sigma^{bc}\}$ will be proportional to γ^5. The two- and three-dimensional actions in Hermitian form are

$$S_2 = \int d^2x\sqrt{-g}[\frac{i}{2}E_a{}^\alpha(\overline{\psi}\gamma^a\partial_\alpha\psi - \partial_\alpha\overline{\psi}\gamma^a\psi) + m\overline{\psi}\psi] \tag{3.22}$$

$$S_3 = \int d^3x\sqrt{-g}[\frac{i}{2}E_j{}^\mu(\overline{\psi}\gamma^j\partial_\mu\psi - \partial_\mu\overline{\psi}\gamma^j\psi) + \frac{i}{2}E_j{}^\mu\omega_{\mu j}\overline{\psi}\gamma^5\psi + m\overline{\psi}\psi]. \tag{3.23}$$

Since terms cancel, the two- and three-dimensional actions take this simpler Hermitian form.

When we vary the Hermitian action with respect to $\delta\overline{\psi}$, we need to integrate by parts due to the derivative of $\overline{\psi}$. After using Eq. (2.25), we get equations of motion of the form

$$iE_a{}^\mu\gamma^a\partial_\mu\psi + \frac{i}{2}E_a{}^\mu\omega_\mu{}^{ab}\gamma_b\psi + \frac{i}{4}E_a{}^\mu\omega_{\mu bc}\{\gamma^a, \Sigma^{bc}\}\psi + m\psi = 0 \tag{3.24}$$

$$iE_a{}^\alpha\gamma^a\partial_\alpha\psi - \frac{i}{2}\varepsilon^{ab}E_a{}^\alpha\omega_\alpha\gamma_b\psi + m\psi = 0 \tag{3.25}$$

$$iE_j{}^\mu\gamma^j\partial_\mu\psi - \frac{i}{2}\varepsilon^{ijk}E_i{}^\mu\omega_{\mu j}\gamma_k\psi + \frac{i}{2}E_j{}^\mu\omega_{\mu j}\gamma^5\psi + m\psi = 0. \tag{3.26}$$

An extra term emerges in each due to the derivative of the Vierbein and $\sqrt{-g}$ from the integration by parts. Through a bit of algebra, we can see that these equations are equivalent to the Dirac equations we derived before. This verifies that our Hermitian form of the action is equivalent to our original action.

3.4 Reduction from Three to Two Dimensions

We now want to see what happens to our Dirac equation when we reduce from an arbitrary three-dimensional metric by restricting motion in one coordinate. To reduce the Dirac equation from three to two dimensions, I will make use of the Kaluza-Klein Ansatz. We take the three-dimensional spacetime metric to be in the form

$$g_{\mu\nu} = \begin{pmatrix} g_{\alpha\beta} - A_\alpha A_\beta & -A_\alpha \\ -A_\beta & -1 \end{pmatrix}. \tag{3.27}$$

The first row and column represent the first two spacetime dimensions of the μ and ν index respectively, where $g_{\alpha\beta}$ is a reduced two-dimensional spacetime metric. α and β run over 0 and 1. The second row and column represent the third dimension, which we plan to reduce. These correspond to μ or ν equal to 2. The two-dimensional field A_α is a connection field between the two dimensions in which we're interested and the extra third dimension. Once we've restricted ourselves to two dimensions, the particle will still feel the effects of this field on the remaining dimensions.

Given $g_{\mu\nu}$ in this form, we can take the Vierbein $e^j{}_\mu$ to be

$$e^j{}_\mu = \begin{pmatrix} e^a{}_\alpha & 0 \\ A_\alpha & 1 \end{pmatrix}. \tag{3.28}$$

The first column represents the two-dimensional spacetime coordinates $\mu = \alpha$, and the second column represents the extra spacetime dimension $\mu = 2$. The first row represents the two-dimensional Lorentz dimensions $j = a$, and the second row represents the extra Lorentz dimension $j = 2$. $e^a{}_\alpha$ is the reduced two-dimensional Vierbein which produces the two-dimensional spacetime metric $g_{\alpha\beta}$. Plugging $e^j{}_\mu$ into Eq. (2.1), we verify that this Vierbein produces the desired metric $g_{\mu\nu}$.

By imposing Eq. (2.2), we can find the inverse Vierbein.

$$E_j{}^\mu = \begin{pmatrix} E_a{}^\alpha & -E_a{}^\beta A_\beta \\ 0 & 1 \end{pmatrix} \tag{3.29}$$

We use the same convention as before, where the columns represent the spacetime index and the rows represent the Lorentz index. $E_a{}^\alpha$ is the inverse Vierbein of our two-dimensional metric. We can also construct the inverse metric from the inverse Vierbein.

$$g^{\mu\nu} = \begin{pmatrix} g^{\alpha\beta} & -g^{\alpha\delta} A_\delta \\ -g^{\gamma\beta} A_\gamma & g^{\gamma\delta} A_\gamma A_\delta - 1 \end{pmatrix}. \tag{3.30}$$

The rows represent the first index, and the columns represent the second index.

Now that we have the metric and Vierbein in this form, we use them to calculate the three-dimensional spin connection field $\omega_\mu{}^j$. Plugging our two-dimensional Vierbein and A_α into Eq. (2.15), we get expressions for each component of $\omega_\mu{}^j$ in terms of the two-dimensional Vierbein $e^a{}_\alpha$, the connection field A_α, and their derivatives. To simply notation, we define the field strength of the connection field.

$$f_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{3.31}$$

Since our field strength $f_{\alpha\beta}$ is antisymmetric and runs over two indices, it has only one independent component. We can therefore define a field strength for the connection field with no indices. We choose to scale the field strength by a factor of $\sqrt{-g}$, using the two-dimensional metric, to simplify our expressions.

$$f \equiv \frac{1}{\sqrt{-g}} f_{01} \tag{3.32}$$

$$f_{\alpha\beta} = \sqrt{-g}\varepsilon_{ab} f \tag{3.33}$$

Using these definitions, I worked through the algebra to get each component of $\omega_\mu{}^j$ in terms of the two-dimensional spin connection and Vierbein along with the connection field A_α and its field strength[1]. After a bit of algebra, we find that the three-dimensional spin connection is

$$\omega_\mu{}^j = \begin{pmatrix} -\frac{1}{2} e^a{}_\alpha f & \omega_\alpha + \frac{1}{2} A_\alpha f \\ 0 & \frac{1}{2} f \end{pmatrix}. \tag{3.34}$$

The rows represent the first spacetime index, and the columns represent the second Lorentz index. Notice that the two-dimensional spin connection with no Lorentz indices ω_α appears in our expression. This will make it possible to relate our three-dimensional Dirac equation to its reduced two-dimensional form.

Using our reduced $\omega_\mu{}^j$, the spinor connection term in the three-dimensional Dirac

[1]To get the factors of $\sqrt{-g}$ correct, we note that $\frac{1}{\sqrt{-g}} = E_0{}^0 E_1{}^1 - E_0{}^1 E_1{}^0$.

equation gets reduced to

$$\frac{1}{2}i\gamma^j E_j{}^\mu \omega_{\mu k}\gamma^k \gamma_3^5 \psi = \frac{1}{2}i\gamma^a E_a{}^\alpha \omega_\alpha \gamma_2^5 \psi - \frac{1}{4}if\gamma_3^5 \psi. \tag{3.35}$$

I put subscripts on the γ^5 matrices here to avoid confusion between the ones that I defined in two and three dimensions[2]. The first term is the two-dimensional spinor connection term. Remarkably, only the field strength and not the full A_α appears in our three-dimensional spinor connection term. Since γ_3^5 is $\pm i$ in our representation, the only additional term in our spinor connection reduces to $\pm\frac{1}{4}f\psi$.

Since we restrict motion to be in only two-dimensions, our spinor will be independent of x^2. Since $\partial_2\psi = 0$, expanding the three-dimensional Dirac equation yields

$$i\gamma^a E_a{}^\alpha(\partial_\alpha\psi + \frac{1}{2}\omega_\alpha\gamma^5\psi) \pm \frac{1}{4}f\psi + m\psi = 0. \tag{3.36}$$

Now that we've reduced to two-dimensions, γ^5 will always be taken to be one defined in two dimensions.

The three-dimensional Dirac equation has reduced to the two-dimensional Dirac equation with an extra term which encompasses the effect that the third dimension has on the two dimensions in which we're working. It turns out that only the field strength f of our dimensional connection appears in our equation of motion. We will choose two of the Pauli matrices for γ^0 and γ^1. Since we can still choose $\pm i$ times the last Pauli matrix for γ^2, we're choosing either plus or minus for our field strength term. We can now solve this reduced two-dimensional equation using the spacetime background discovered by Jackiw et al.

[2] Note that $\gamma_3^5 = \gamma^2\gamma_2^5$.

Chapter 4

Solving the Two-Dimensional Dirac Equation

Now that we have found and verified the two-dimensional Dirac equation that we plan to solve, we turn our attention to the spacetime backgrounds discovered by Jackiw et al[4], [5]. They also reduced the three-dimensional spacetime metric to two dimesions by use of the Kaluza-Klein Ansatz, defining the field strength f in the same manner. This allowed them to solve for the following possible two-dimensional spacetime metrics:

$$\text{Case 1:}\quad f(t,x)=0, \qquad C>0, \quad g_{\alpha\beta}=\frac{2}{Ct^2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{4.1}$$

$$\text{Case 2:}\quad f(t,x)=0, \qquad C<0, \quad g_{\alpha\beta}=\frac{2}{|C|x^2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{4.2}$$

$$\text{Case 3:}\quad f(t,x)=\pm\sqrt{C}, \qquad C>0, \quad g_{\alpha\beta}=\frac{1}{Cx^2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{4.3}$$

$$\text{Case 4:}\quad f(t,x)=\sqrt{C}\tanh\frac{\sqrt{C}}{2}x, \quad C>0, \quad g_{\alpha\beta}=\begin{pmatrix}1/\cosh^4\frac{\sqrt{C}}{2}x & 0\\ 0 & -1\end{pmatrix} \tag{4.4}$$

In this matrix notation, the rows will always represent the first index and the columns will represent the second index. Since our two dimensions are time and space, we now use the convention $(x^0, x^1) \to (t, x)$. Our two-dimensional metrics are all in diagonal form, so our Dirac equation will simplify quite a bit. Since each equation depends on only one of the two coordinates, we will be able to use separation of variable techniques to reduce from a partial differential equation of two variables to a differential equation of one variable.

Notice that the first two cases are similar, with the role of time and space reversed. In both cases, the two-dimensional curvature of spacetime is the constant C, which is positive in the first and negative in the second case. These cases are the spacetime solutions which are symmetric in the third dimension, since $f = 0$. The third case represents the solutions that break this symmetry, where the plus or minus indicates that the symmetry of the third dimension can be broken in either direction. This solution has negative constant two-dimensional spacetime curvature $-2C$.

The last case interpolates between the plus and minus solutions from the third case. Since $\tanh x$ approaches $+1$ as x goes to positive infinity and -1 as x goes to negative infinity, we see that $f(x)$ behaves in these two limits as the positive and negative third case solutions. This is the kink solution, since it interpolates between the symmetry breaking solutions in both limits of x. The different behavior at both infinities makes it a kink[5].

Notice that our reduced Dirac equation depends on the field strength f but not on the field A_α. This means that we have a gauge invariance of this field. In this case, the gauge invariance would be a change in A_α by a full derivative,

$$\delta A_\alpha = \partial_\alpha \lambda, \tag{4.5}$$

where λ is any function of the coordinates. Thus, the solutions found will work for any three-dimensional spacetime metric with A_α transformed by this gauge invariance. Using Eq. (3.32), we can use any A_α field that has field strength f. Note that the field strength also depends on the metric we choose, due to its $\sqrt{-g}$ dependence. For the first two cases, where $f = 0$, we choose to use $A_\alpha = 0$ for convenience. In the other two cases, we choose

$$\text{Case 3:} \qquad A_\alpha = (\mp\frac{1}{\sqrt{C}x} \qquad 0) \tag{4.6}$$

$$\text{Case 4:} \qquad A_\alpha = (1/\cosh^2\frac{\sqrt{C}}{2}x \qquad 0). \tag{4.7}$$

4.1 Dirac Equation for Each Spacetime

For each case, we must choose an appropriate Vierbein, so that we can relate the our curved spacetime metric to the Minkowski metric. Each of our metrics is diagonal with the upper left component positive and the lower right component negative. We can therefore choose the Vierbein to be diagonal with the diagonal components being the square roots of the absolute values of the diagonal components of our spacetime metric. Checking Eq. (2.1), we see that this choice of Vierbein produces the correct metric.

Since our Vierbein and metric are both diagonal, our inverse Vierbein and inverse metric are also diagonal, with their components being the inverses of the components of the vierbeine and metric respectively. We find the Vierbein, inverse Vierbein and inverse metric in each case:

$$\text{Case 1:}\ \ e^a{}_\alpha = \sqrt{\frac{2}{C}\frac{1}{t}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad E_a{}^\alpha = \sqrt{\frac{C}{2}t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ g^{\alpha\beta} = \frac{Ct^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.8}$$

$$\text{Case 2:}\ \ e^a{}_\alpha = \sqrt{\frac{2}{|C|}\frac{1}{x}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad E_a{}^\alpha = \sqrt{\frac{|C|}{2}x}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ g^{\alpha\beta} = \frac{|C|x^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.9}$$

$$\text{Case 3:}\ \ e^a{}_\alpha = \frac{1}{\sqrt{C}x}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad E_a{}^\alpha = \sqrt{C}x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ g^{\alpha\beta} = Cx^2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.10}$$

$$\text{Case 4:}\ \ e^a{}_\alpha = \begin{pmatrix} 1/\cosh^2\frac{\sqrt{C}}{2}x & 0 \\ 0 & 1 \end{pmatrix}, \ E_a{}^\alpha = \begin{pmatrix} \cosh^2\frac{\sqrt{C}}{2}x & 0 \\ 0 & 1 \end{pmatrix}, \ g^{\alpha\beta} = \begin{pmatrix} \cosh^4\frac{\sqrt{C}}{2}x & 0 \\ 0 & -1 \end{pmatrix} \tag{4.11}$$

The first three cases all take similar forms, so we'll find that their solutions take similar forms. I next calculated the Christenoff coefficients for each case, using the metric and inverse metric in Eq. (2.9). Since the metrics are diagonal, I was able to use Kronecker deltas in each case to show that only a few of the Chistenoff coefficients didn't vanish. I next used Eq. (2.15) to find the spin connection form. Expressed as the two-dimensional spin connection without Lorentz indices, I found the spin connections to be

$$\text{Case 1:} \qquad \omega_\alpha = (0 \qquad -\frac{1}{t}) \tag{4.12}$$

$$\text{Case 2:} \qquad \omega_\alpha = (-\frac{1}{x} \qquad 0) \tag{4.13}$$

$$\text{Case 3:} \qquad \omega_\alpha = (-\frac{1}{x} \qquad 0) \tag{4.14}$$

$$\text{Case 4:} \qquad \omega_\alpha = (-\sqrt{C}\text{sech}^2\frac{\sqrt{C}}{2}x\tanh\frac{\sqrt{C}}{2}x \qquad 0). \tag{4.15}$$

Only the spin connection, inverse Vierbein, and field strength appear in the Dirac equation. Since the inverse Vierbein is diagonal and the spin connection has only one nonvanishing component, our Dirac equation simplifies quite a bit in each case. We find that the equations we must solve for the spinors now become

$$\text{Case 1:} \quad i\frac{\sqrt{C}}{2}(t\gamma^0\frac{\partial\psi}{\partial t} + t\gamma^1\frac{\partial\psi}{\partial x} - \frac{1}{2}\gamma^0\psi) + m\psi = 0 \tag{4.16}$$

$$\text{Case 2:} \quad i\frac{\sqrt{|C|}}{2}(x\gamma^0\frac{\partial\psi}{\partial t} + x\gamma^1\frac{\partial\psi}{\partial x} - \frac{1}{2}\gamma^1\psi) + m\psi = 0 \tag{4.17}$$

$$\text{Case 3:} \quad i\sqrt{C}(x\gamma^0\frac{\partial\psi}{\partial t} + x\gamma^1\frac{\partial\psi}{\partial x} - \frac{1}{2}\gamma^1\psi) + (m \pm \frac{\sqrt{C}}{4})\psi = 0 \tag{4.18}$$

$$\text{Case 4:} \quad i(\cosh^2\frac{\sqrt{C}}{2}x\gamma^0\frac{\partial\psi}{\partial t} + \gamma^1\frac{\partial\psi}{\partial x} - \frac{\sqrt{C}}{2}\tanh\frac{\sqrt{C}}{2}x\gamma^1\psi) + (m \pm \frac{\sqrt{C}}{4}\tanh\frac{\sqrt{C}}{2})\psi = 0. \tag{4.19}$$

Since we're in two dimensions, the spinors have two components and are functions of both t and x. Our differential equations each involve only one of the two variables, so we can use a separation of variables technique. The last three partial differential depend only on on x, so we will decompose an arbitrary spinor solution Ψ into solutions of the form

$$\Psi(t,x) = e^{-iEt}\psi(x) \tag{4.20}$$

where ψ depends only on x.

E must be real, so that Ψ does not blow up at early or late times. We can interpret E as the energy of the particle, since they are the eigenvalues of the equation

$$E\Psi(t,x) = i\frac{\partial\Psi}{\partial t} \tag{4.21}$$

The Hamiltonian is therefore the operator acting on the spinor once we've isolated $i\frac{\partial\Psi}{\partial t}$ on one side of the equation.

We now try to find the spinors of the form $e^{-iEt}\psi(x)$ which satify the Dirac equation. When we plug this into the Dirac equation, the t partial derivative replaces the $i\frac{\partial}{\partial t}$ term by E. We've now reduced the partial differential equation to a differential equation of just x. Solving this equation gives us two linearly independent eigenspinors $\psi(x)$ for each valid energy, since we're solving a two-component first order differential equation. An arbitrary solution $\Psi(t,x)$ is then given by integrating over these linearly independent solutions with arbitrary coefficients for each energy.

Since the first differential equation involves t, it turns out to be simpler in this case to decompose $\Psi(t,x)$ into solutions of the form $e^{-ipx}\psi(t)$, where p is interpreted as the momentum in the x-direction. From this, we get a similar differential equation for $\psi(t)$ involving only t.

4.2 Solving in the Symmetric Spacetime

Let's first analyze the first two cases, where the spacetime is symmetric in the third dimension since $f=0$. We start with case 2, since this one is more similar to the last two cases, involving a differential equation for x. After converting this to a differential equation of just x and rearranging terms, we get the equation:

$$E\psi = -i\gamma^5\frac{\partial\psi}{\partial x} + \frac{i}{2x}\gamma^5\psi - \frac{1}{x}\sqrt{\frac{2}{|C|}}m\gamma^0\psi \tag{4.22}$$

4.2.1 Massless Case

Let us first solve this in the case of a massless particle, where $m=0$. Since only the matrix γ^5 appears, we can completely separate the two components of the spinor by choosing a representation where γ^5 is diagonal. Since we can take γ^5 to be any Pauli matrix, we choose the diagonal one σ_3. We denote the two components of ψ as u and v, so we let

$$\psi \equiv \begin{pmatrix} u \\ v \end{pmatrix}. \tag{4.23}$$

The differential equation now separates into first order differential equations of u and v. We denote the derivative of u as u' and the derivative of v as v'.

$$u' = (iE + \frac{1}{2x})u \tag{4.24}$$

$$v' = (-iE + \frac{1}{2x})v \tag{4.25}$$

These equations are easily solved by noting we can put these in the form $\frac{d}{dx}(\ln u)$ and integrating both sides. We find that this massless Dirac equation is solved by a spinor of the form

$$\psi = x^{1/2} \begin{pmatrix} Ae^{iEx} \\ Be^{-iEx} \end{pmatrix} \tag{4.26}$$

where A and B are arbitrary constants.

4.2.2 Including the Mass Term

Let's now see how our solution is effected by the mass term. First, let's rescale our mass as $M = \sqrt{\frac{2}{|C|}}m$ to simplify notation. Both γ^5 and γ^0 appear in our equation now, and we can't choose both to be diagonal. We're going to have mixing between u and v no matter which Pauli matrices we pick. However, we can make the equations real by choosing $\gamma^5 = \sigma_2$ and $\gamma^0 = \sigma_1$. By choosing our equations to be real, we will get real solutions for the spinor. This will make the math a bit easier. We can now express the component coupled differential equations as

$$Eu = -v' - (M - \frac{1}{2})\frac{1}{x}v \tag{4.27}$$

$$Ev = u' - (M + \frac{1}{2})\frac{1}{x}u. \tag{4.28}$$

By multiplying both sides by some function of x, we can express the right sides of the equations as total derivatives. In this case, we get

$$Ex^{-\frac{1}{2}+M}u = -\frac{d}{dx}(x^{-\frac{1}{2}+M}v) \tag{4.29}$$

$$Ex^{-\frac{1}{2}-M}v = \frac{d}{dx}(x^{-\frac{1}{2}-M}u). \tag{4.30}$$

Since we have an $x^{-\frac{1}{2}}$ factor in front of both u and v, we find it convenient to absorb this factor into the variables to simplify our differential equations. We define

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix} \equiv x^{\frac{1}{2}} \begin{pmatrix} g \\ h \end{pmatrix}. \tag{4.31}$$

Our coupled differential equations now become

$$Eg = -h' - \frac{M}{x}h \tag{4.32}$$
$$Eh = g' - \frac{M}{x}g. \tag{4.33}$$

We can differentiate either equation and solve for the uncoupled second order differential equations. We find these to be of the form

$$g'' + (E^2 - \frac{M(M-1)}{x^2})g = 0 \tag{4.34}$$
$$h'' + (E^2 - \frac{M(M+1)}{x^2})h = 0. \tag{4.35}$$

We recognize these as the differential equations for the spherical Bessel functions. These are the same equations that must be satisfied by the radial wavefunction of a nonrelativistic free particle with angular momentum quantum number M[3]. The difference here is the the mass term now acts as our centrifugal term, instead of the angular momentum.

The solutions to these second order equations are of the form

$$g(x) = x[Aj_{M-1}(Ex) + Bn_{M-1}(Ex)] \tag{4.36}$$
$$h(x) = x[Cj_M(Ex) + Dn_M(Ex)]. \tag{4.37}$$

The functions j_M are the spherical Bessel functions of the first kind, and n_M are the spherical Bessel functions of the second kind. For integer M, these are defined as

$$j_M(x) \equiv (-x)^M (\frac{1}{x}\frac{d}{dx})^M \frac{\sin x}{x} \tag{4.38}$$
$$n_M(x) \equiv -(-x)^M (\frac{1}{x}\frac{d}{dx})^M \frac{\cos x}{x}. \tag{4.39}$$

The j_M's have the property that they go to zero and small x, while the n_M's go to infinity at small x. Their asymptotic behavior is

$$j_M(x) \rightarrow \frac{x^M}{(2M+1)!!} \text{ as } x \rightarrow 0, \qquad j_M(x) \rightarrow \frac{1}{x}\sin(x - \frac{M\pi}{2}) \text{ as } x \rightarrow \infty \tag{4.40}$$

$$n_M(x) \rightarrow -\frac{(2M-1)!!}{x^{M+1}} \text{ as } x \rightarrow 0, \quad n_M(x) \rightarrow \frac{1}{x}\cos(x - \frac{M\pi}{2}) \text{ as } x \rightarrow \infty. \tag{4.41}$$

Using the coupling relation between the components, we find that our spinor must be of the form

$$\psi = Ax^{3/2} \begin{pmatrix} j_{M-1}(Ex) \\ -j_M(Ex) \end{pmatrix} + Bx^{3/2} \begin{pmatrix} n_{M-1}(Ex) \\ -n_M(Ex) \end{pmatrix} \tag{4.42}$$

where A and B are arbitray coefficients. Notice that the first spinor goes to zero as $x^{3/2+M}$ for small x, while the second spinor blows up as $x^{3/2-M}$ at small x. The total spinor oscillates as $x^{1/2}\sin(Ex + \phi)$ with some phase shift ϕ for large x. The factor of $x^{1/2}$ in the solution which isn't in the solution for the nonrelativistic free radial wavefunction of a particle comes from the addition of the spinor connection term in our Dirac equation.

If the mass is quantized in units of $\sqrt{\frac{|C|}{2}}$, then we get the ordinary Bessel functions described above. However, for an arbitrary mass we must use fractional Bessel functions. The asymptotic behavior will still have the same dependence on M. Notice that $-M$ produces the same value of $M(M-1)$ as does $M-1$. We could therefore view the second type of Bessel functions as extensions of the first type into negative orders, by identifying n_{M-1} with j_{-M}. We see that the asymptotic dependence on M remains the same with this change of names.

For a massless particle, we use the zeroth spherical Bessel functions and those of order -1. We've identified the order -1 with the order 0 Bessel functions of the other type, so we get spinors proportional to $x^{1/2}\sin Ex$ and $x^{1/2}\cos Ex$. These are the real and imaginary combinations of the independent spinors found for the massless equation. Our massive solution is therefore consistent.

4.2.3 Time Dependent Symmetric Case

Let's look now at case 1. The first difference that we see here is that our differential equations are now dependent on t instead of x, and we replace the energy with the momentum p in the x-direction. The Dirac equation becomes

$$p\psi = -i\gamma^5 \frac{\partial \psi}{\partial t} + \frac{i}{2t}\gamma^5 \psi - \frac{M}{t}\gamma^1 \psi \tag{4.43}$$

where we scale the mass term to $M = \sqrt{\frac{2}{C}}m$.

In the massless case, we get the same equation as in case 2, with x and t exchanging places. Our solutions are again of the form $t^{1/2}\sin(pt + \phi)$ with some phase ϕ.

When we have a massive particle, we run into a problem. If we try to make the equation real, then γ^5 and $i\gamma^1$ must both be chosen to be σ_2. Since these must be different Pauli matrices, there's no way to make the entire equation real. Instead, let's choose $\gamma^5 = \sigma_2$ and $\gamma^1 = i\sigma_1$. The equations are now the same as in case 2 except that M is replaced by $-iM$.

To get the exact solutions to these, we would have to define Bessel functions of imaginary order. From checking the asymptotic behavior, I found that these imaginary order Bessel functions must also have the same M dependence in the asymptotic limit. As $t \to \infty$, the spinor will oscillate as $t^{1/2}\sin(pt + \phi)$ with some phase shift. As $t \to 0$, the spinor behaves as $t^{1/2 \pm iM}$, which can be expressed as $t^{1/2}\sin(M \ln x + \phi)$ with some phase shift.

4.3 Solving in the Symmetry Breaking Spacetime

Let's now examine case 3, the spacetime which breaks the symmetry in the third dimension. We will again scale the mass term, this time as $M = \frac{m}{\sqrt{C}}$. The Dirac equation now becomes

$$E\psi = -i\gamma^5 \frac{\partial \psi}{\partial x} + \frac{i}{2x}\gamma^5 \psi - (M \pm \frac{1}{4})\frac{1}{x}\gamma^0 \psi \tag{4.44}$$

where the plus or minus indicates the symmetry breaking in either direction.

Notice that this is the same equation as in case 2 except that the mass term is shifted by $\frac{1}{4}$. We therefore use the same convention as before, $\gamma^5 = \sigma_2$ and $\gamma^0 = \sigma_1$. Our solutions are now the case 2 solutions with the mass term shifted.

$$\psi = Ax^{3/2} \begin{pmatrix} j_{M-1\pm 1/4}(Ex) \\ -j_{M\pm 1/4}(Ex) \end{pmatrix} + Bx^{3/2} \begin{pmatrix} n_{M-1\pm 1/4}(Ex) \\ -n_{M\pm 1/4}(Ex) \end{pmatrix} \tag{4.45}$$

The massless solution now involves fractional Bessel functions, of order $\frac{1}{4}$ and $-\frac{3}{4}$. We've identified the $\frac{1}{4}$ order Bessel functions of one kind with the $\frac{3}{4}$ order Bessel functions of the other kind. We've also similarly identified the $-\frac{1}{4}$ and $-\frac{3}{4}$ orders, so we're forced to use Bessel functions of negative fractional order.

Notice that when we switch between the plus and minus symmetry breaking solutions, the two independent spinor solutions are switched into each other. We therefore get the same spinor solutions for both massless symmetry breaking solutions. This finding is consistent, since there's nothing special about either direction in the third dimension that would make the symmetry breaking solutions differ.

The two massless spinor solutions go as $x^{3/4}$ and $x^{5/4}$ for small x. These spinors also go as $x^{1/2}\sin(Ex+\phi)$ with some phase shift for large x.

4.4 Solving in the Kink Spacetime

Only the last case is left to analyze. This is the case of the kink spacetime. After separation of variables, the Dirac equation becomes

$$E\cosh^2\frac{\sqrt{C}}{2}x\psi = -i\gamma^5\frac{\partial\psi}{\partial x} + i\frac{\sqrt{C}}{2}\tanh\frac{\sqrt{C}}{2}x\gamma^5\psi - (m \pm \frac{\sqrt{C}}{4}\tanh\frac{\sqrt{C}}{2}x)\gamma^0\psi. \tag{4.46}$$

I will only be analyzing the massless solution. We will again choose the representation which makes our equations real, so $\gamma^5 = \sigma_2$ and $\gamma^0 = \sigma_1$. The representation will also be chosen so that we use the plus sign before the field strength. Defining u and v to be the components of the spinor as before, we get the coupled differential equations

$$E \cosh^2 \frac{\sqrt{C}}{2}x \; u = -v' + \frac{\sqrt{C}}{4} \tanh \frac{\sqrt{C}}{2}x \; v \tag{4.47}$$

$$E \cosh^2 \frac{\sqrt{C}}{2}x \; v = u' - \frac{3\sqrt{C}}{4} \tanh \frac{\sqrt{C}}{2}x \; u. \tag{4.48}$$

The spinor connection and field strength terms combined. If we had used the minus sign for the field strength, the components of the spinor would exchange roles. I next multiplied both sides of each equation by quantities which makes the right sides of each equation total derivatives. This gives us

$$E(\cosh \frac{\sqrt{C}}{2}x)^{3/2} u = -\frac{d}{dx}[(\cosh \frac{\sqrt{C}}{2}x)^{-1/2} v] \tag{4.49}$$

$$E(\cosh \frac{\sqrt{C}}{2}x)^{1/2} v = \frac{d}{dx}[(\cosh \frac{\sqrt{C}}{2}x)^{-3/2} u]. \tag{4.50}$$

I next removed the fractional powers of the hyperbolic cosine by defining new functions g and h:

$$u \equiv (\cosh \frac{\sqrt{C}}{2}x)^{3/2} g, \qquad v \equiv (\cosh \frac{\sqrt{C}}{2}x)^{1/2} h \tag{4.51}$$

This removes the fractional exponents and simplifies our equations.

$$E \cosh^3 \frac{\sqrt{C}}{2}x \; g = -h' \tag{4.52}$$

$$E \cosh \frac{\sqrt{C}}{2}x \; h = g'. \tag{4.53}$$

By differentiating either equation, we can solve for the second order uncoupled differential equations. I will express these in dimensionless form, by making them functions of $z \equiv \frac{\sqrt{C}}{2}x$ and scaling the energy as $E \equiv \frac{\sqrt{C}}{2}\varepsilon$. The second order equations are

$$g'' - \tanh z \; g' + \varepsilon^2 \cosh^4 z \; g = 0 \tag{4.54}$$

$$h'' - 3 \tanh z \; h' + \varepsilon^2 \cosh^4 z \; h = 0. \tag{4.55}$$

I was unable to discover any analytic soluiton to these equations in general. However, the zero energy solutions are trivial from this point. Since their derivatives are zero, g and h must be constant. This gives a zero energy spinor solution of

$$\psi \equiv \begin{pmatrix} A(\cosh \frac{\sqrt{C}}{2}x)^{3/2} \\ B(\cosh \frac{\sqrt{C}}{2}x)^{1/2} \end{pmatrix}. \tag{4.56}$$

Near $x = 0$, $\cosh x$ is constant to first order expansion, so our zero energy spinor will be constant to first order here. $\cosh x$ has its minimum here and increases exponentially in either direction, so our spinor will rapidly increase in magnitude as we stray from this minimum. The asymptotic behavior of $\cosh x$ is

$$\cosh x \longrightarrow \frac{1}{2}e^{-x} \text{ as } x \rightarrow -\infty, \quad \cosh x \longrightarrow \frac{1}{2}e^{x} \text{ as } x \rightarrow +\infty. \tag{4.57}$$

The first spinor solution therefore goes as $e^{3\sqrt{C}|x|/4}$ at the infinities, while the second goes as $e^{\sqrt{C}|x|/4}$.

Let's consider the effect of the energy term now. Near $z = 0$, $\tanh z$ goes as z. This term can be dropped for small z. $\cosh z$ is constant to first order in z, so our equations become

$$g'' + \varepsilon^2 g = 0 \tag{4.58}$$

$$h'' + \varepsilon^2 h = 0. \tag{4.59}$$

for small z. These are harmonic oscillator differential equations, so the spinors will goes as $\sin(Ex+\phi)$ for small x. This means that we don't have any type of singularity to 0, as we did in the previous cases due to the behavior of the metric at 0.

We now check the behavior of the solutions as $z \rightarrow \pm\infty$. Due to the behavior of $\cosh z$ and $\tanh z$ at infinity, our equations take the form

$$x \rightarrow +\infty : \qquad g'' - g' + (\frac{\varepsilon}{4})^2 e^{4z} g = 0, \quad h'' - 3h' + (\frac{\varepsilon}{4})^2 e^{4z} h = 0 \tag{4.60}$$

$$x \rightarrow -\infty : \qquad g'' + g' + (\frac{\varepsilon}{4})^2 e^{-4z} g = 0, \quad h'' + 3h' + (\frac{\varepsilon}{4})^2 e^{-4z} h = 0. \tag{4.61}$$

From this, we see that at large z, the second derivative of g or h must be large enough in magnitude to cancel e^{4z} times the same function. In order for this to hold, the second derivative must dominate over the first derivative. We will therefore get the same asymptotic behavior for g and h. We find the behavior at infinity to be

$$x \to +\infty : \qquad g \sim \sin(\frac{\varepsilon}{8}e^{2z} + \phi), \quad h \sim \sin(\frac{\varepsilon}{8}e^{2z} + \phi) \tag{4.62}$$

$$x \to -\infty : \qquad g \sim \sin(\frac{\varepsilon}{8}e^{-2z} + \phi), \quad h \sim \sin(\frac{\varepsilon}{8}e^{-2z} + \phi). \tag{4.63}$$

with some phase shift.

The spinors oscillate at the infinities with a frequency on the order of $\frac{E}{4}e^{\sqrt{C}|x|}$. The effect of the energy term on the spinor is to add this oscillation factor, while the $(\cosh\frac{\sqrt{C}}{2}x)^{3/2}$ and $(\cosh\frac{\sqrt{C}}{2}x)^{1/2}$ factors for the two components continue the envelope this oscillation. We've seen that the frequency of oscillation increases exponentially at both infinities, while the frequency is on the order of E near $x = 0$.

The total asymptotic behavior of the spinors can be expressed as

$$x \to +\infty : \qquad \psi \sim \begin{pmatrix} A(\cosh\frac{\sqrt{C}}{2}x)^{3/2}\sin(\frac{E}{4\sqrt{C}}e^{\sqrt{C}x} + \phi_1) \\ B(\cosh\frac{\sqrt{C}}{2}x)^{1/2}\sin(\frac{E}{4\sqrt{C}}e^{\sqrt{C}x} + \phi_2) \end{pmatrix} \tag{4.64}$$

$$x \to -\infty : \qquad \psi \sim \begin{pmatrix} C(\cosh\frac{\sqrt{C}}{2}x)^{3/2}\sin(\frac{E}{4\sqrt{C}}e^{-\sqrt{C}x} + \phi_3) \\ D(\cosh\frac{\sqrt{C}}{2}x)^{1/2}\sin(\frac{E}{4\sqrt{C}}e^{-\sqrt{C}x} + \phi_4) \end{pmatrix}. \tag{4.65}$$

We've now analyzed the behavior of a massless spin-$\frac{1}{2}$ particle in the kink spacetime. A particle with some mass would also have a similar behavior at infinity, since the other terms in the Dirac equation would dominate over the mass term. We've examined the properties of the spinor in each of the four spacetime backgrounds. These solutions should give us some insight into the properties of these spacetimes.

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