Simplicial and Cubical Complexes: Analogies and Differences

by

Gábor Hetyei

M.S., Eötvös Lóránd University, Budapest (1988)

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1994

Signature of Author ....
Department of Mathematics
April 28, 1994

Certified by ....
Richard P. Stanley
Professor of Mathematics
Thesis Supervisor

Accepted by ....
David Vogan
Chairman, Departmental Graduate Committee
Department of Mathematics

Simplicial and Cubical Complexes:

Analogies and Differences

by

Gábor Hetyei

Submitted to the Department of Mathematics
on April 28, 1994, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Abstract

The research summarized in this thesis consists essentially of two parts. In the first, we generalize a coloring theorem of Baxter about triangulations of the plane (originally used to prove combinatorially Brouwer's fixed point theorem in two dimensions) to arbitrary dimensions and to oriented simplicial and cubical pseudomanifolds. We show that in a certain sense no other generalizations may be found. Using our coloring theorems we develop a purely combinatorial approach to cubical homology. (This part is joint work with Richard Ehrenborg.)

In the second part, we investigate the properties of the Stanley ring of cubical complexes, a cubical analogue of the Stanley-Reisner ring of simplicial complexes. We compute its Hilbert-series in terms of the f-vector. We prove that by taking the initial ideal of the defining relations, with respect to the reverse lexicographic order, we obtain the defining relations of the Stanley-Reisner ring of the triangulation via "pulling the vertices" of the cubical complex. We show that the Stanley ring of a cubical complex is Cohen-Macaulay when the complex is shellable and its defining ideal is generated by homogeneous forms of degree two when the complex is also a subcomplex of the boundary complex of a convex cubical polytope. We present a cubical analogue of balanced Cohen-Macaulay simplicial complexes: the class of edge-orientable shellable cubical complexes. Using Stanley's results about balanced Cohen-Macaulay simplicial complexes and the degree two homogeneous generating system of the defining ideal, we obtain an infinite set of examples for a conjecture of Eisenbud, Green and Harris. This conjecture says that the h-vector of a polynomial ring in n variables modulo an ideal which has an n-element homogeneous system of parameters of degree two, is the f-vector of a simplicial complex.

Thesis Supervisor: Richard P. Stanley
Title: Professor of Mathematics

To My Parents

"Soli Deo gloria"

Acknowledgements

My first and foremost expression of gratitude goes to my advisor Richard Stanley for both his exemplary patience, and his good advice.

Richard Ehrenborg is not only the co-researcher of Chapter 2, but also one of my best friends. His help has been invaluable even in the most practical matters.

Laura Anderson helped me base my intuition for Chapter 1 on piecewise linear topology. The inspiration for Chapter 2 came from Dan Kleitman and Gian-Carlo Rota. I also appreciate the helpful advice received from Dan Kan, James R. Munkres and Richard P. Stanley. Dan Klain spent a lot of his precious time reading the preprint, and I am thankful for his invaluable suggestions. In Chapter 3 the starting point was given by Richard Stanley. He was also the one who drew my attention to the Eisenbud-Greene-Harris conjecture. I am grateful to Joe Harris and especially to David Eisenbud for several fruitful discussions. Steve Kleiman enlarged my knowledge of commutative algebra and Bernd Sturmfels gave me essential information about Gröbner bases. Brian Taylor supplied me with important facts about algebras with straightening laws.

Finally, I wish to thank Julio Bastida, who convinced me to apply to MIT, and Ervin Fried, without whom I would not be a mathematician.

Contents

Folly is an endless maze,
Tangled roots perplex her ways.
How many have fallen there!
They stumble all night over bones of the dead,
And feel they know not what but care,
And wish to lead others, when they should be led.
(William Blake: Songs of Innocence)

Introduction

In this thesis we investigate the analogies and differences between certain combinatorial, geometric and topological properties of cubical and simplicial complexes.

In the preliminary Chapter 1 we introduce the notion of *abstract polyhedral complexes*. These complexes are families of finite sets, generalizing the concepts of abstract simplicial and cubical complexes, and they are not necessarily representable as a complex of convex polyhedra in some Euclidean space. Only simplicial complexes are always identifiable with convex cell complexes, for cubical complexes we indicate the construction of a small counterexample. We show a way to weaken the definition of geometric realizability to require only having a coherent system of geometric realizations of the faces, such that all cubical complexes will be "representable" in this weaker sense.

We define subdivisions in an abstract way and show a “natural triangulation” of polyhedral complexes. This triangulation will be the plausible generalization of the “triangulation via pulling the vertices” of a convex polytope.

Using this triangulation we prove that the apparently topological notion of shellability is in fact combinatorial, and does not depend on geometric representation.

In Chapter 2 we investigate generalizations of a coloring lemma by Baxter, which was originally used to prove Brouwer's fixed-point theorem in the plane. Baxter's lemma in dual form may be stated as follows. Given a triangulation of the sphere, and a coloring of the vertices with 4 colors, we denote by A_1 the number of triangles whose vertices are colored with 234 in clockwise order minus the number of triangles whose vertices are colored with 234 in counterclockwise order. We define A_2, A_3, A_4 similarly. Then we have

$$A_1 = -A_2 = A_3 = -A_4.$$

We generalize this theorem to triangulated and cubically subdivided manifolds of arbitrary dimension in several ways. From our cubical results we obtain a way to construct a cubical analogue of simplicial homology.

In Subsection 2.2.1 we apply techniques of algebraic topology to prove the generalized Baxter's Theorem for triangulations of compact orientable manifolds. To each coloring we associate a continuous function, and the result will essentially depend on the local character of the degree of a continuous function between compact orientable manifolds.

In Subsection 2.2.2 we show a combinatorial lemma for colorings of orientable pseudomanifolds of which the boundariless version seems to be only a special case of the generalized Baxter's Theorem, and which appeared as a corollary of a theorem on octahedral colorings by Ky Fan in [12]. However, an easy observation shows that this lemma implies not only the generalized Baxter's theorem directly, but a Master Theorem for simplicial colorings: a complete system of linear identities for colorings with an arbitrary number of colors. The path-construction idea that we use in the proof of Lemma 4 was first applied in Cohen's proof of Sperner's lemma in [7], and it occurs in [12] in a more complicated situation, but we use it as the sole device to obtain an explicit bijection, in a setting where no induction on dimension is needed.

In Subsection 2.2.3 we prove that the linear equations presented in Subsection 2.2.2 generate all linear relations among our generalized A_i numbers. Hence, in a certain sense we have all generalizations of Baxter's theorem to triangulations colored with an arbitrary number of colors.

In Subsection 2.2.4 we sketch how one can show the Master Theorem and the completeness of its equations, using simplicial homology. We also indicate the way to get a similar complete theory in all cases when the coloring object has zero n-th homology. In particular, Ky Fan's above mentioned octahedral result will follow as a corollary from this theory.

In Subsection 2.2.5 we show reductions of Sperner's lemma to our generalized Baxter's theorem and to Lemma 4. The first reduction sheds a light on the connection between Baxter's result (which was used to give a combinatorial proof of Brouwer's fixed-point theorem in the plane) and Sperner's lemma (which is often used for the same purpose in arbitrary dimensions). It also shows that Sperner's lemma depends on some facts of algebraic topology which imply Brouwer's fixed-point theorem directly. The second reduction offers a combinatorial proof of Sperner's lemma that does not use induction on dimension.

In Subsection 2.3.2 we prove the analogues of the results of Subsection 2.2.2 for orientable cubical pseudomanifolds. Again, we manage to avoid induction on dimension, but in order to do so, we need to make some geometric observations about the standard n-cube which have no or only trivial simplicial analogues. These facts about the Hamming-distance preserving functions that map along the edges of the standard n-cube, which are interesting by their own merit, are explained in Subsection 2.3.1.

In Subsection 2.3.3 we show that in the cubical case we also obtained all possible linear relations in Subsection 2.3.2. This subsection ends the survey on generalizations of Baxter's theorem.

In Section 2.4, we build up the theory of cubical homology. While doing so, we refer to only one of the key lemmas about cubical colorings. Thus, analogously to the simplicial case, all our cubical results will also have a homological proof. The cubical case is more "combinatorial" in the sense that we can easily define a notion of homotopy equivalence without abandoning the world of discrete cubical complexes. Thus we can show the vanishing of the positive degree homology groups of the standard n-cube purely combinatorially. The construction of cubical homology is in Subsection 2.4.1, the concept of homotopy is developed in Subsection 2.4.2.

In Chapter 3 we investigate the properties of the Stanley ring of cubical complexes. This ring is one of the possible cubical analogues of the Stanley-Reisner ring of simplicial complexes. Ironically, while in the simplicial case commutative algebra was instrumental in obtaining combinatorial inequalities, this time combinatorics seems to give some commutative algebraic insight.

In Section 3.1 we show how to reduce greatly the number of relations defining the Stanley ring $K[\square]$ for all cubical complexes, and we compute the Hilbert-series of $K[\square]$. We observe that this Hilbert-series is identical with the Hilbert-series of the Stanley-Reisner ring associated to the triangulation of the cubical complex via pulling the vertices.

In Section 3.2 we use the observed coincidence of Hilbert-series to establish a connection between the Stanley ring of a cubical complex and the Stanley-Reisner ring of its triangulations via pulling the vertices: We show that the face ideal of the triangulation via pulling the vertices is the initial ideal with respect to the reverse lexicographic order of the face ideal of our cubical complex. This result is analogous Sturmfels' result on initial ideals of toric ideals but the techniques involved in the proof are substantially different. It would be a challenging task to find a common generalization of the two statements.

In Section 3.3 we take a closer look at shellable cubical complexes and their Stanley ring. Using an idea of Hochster we establish the Cohen-Macaulay property of the rings. At the combinatorial level, we prove that the edge-graph of shellable cubical complexes is bipartite.

Section 3.4 contains the hardest theorem of this chapter. We show that in the case of shellable subcomplexes of the boundary complex of a convex cubical polytope, the Stanley-ring may be defined by homogeneous relations of degree two.

In Section 3.5 we introduce the notion of edge-orientable cubical complexes, which turn out to be a cubical analogue of completely balanced simplicial complexes. Not only their Stanley ring contains an explicitly constructible linear system of parameters, but they also have a completely balanced triangulation.

Using almost all previous results of Chapter 3 we construct an infinite number of examples to a commutative algebraic conjecture of Eisenbud, Greene and Harris in Section 3.6. According to this conjecture the h-vector of a polynomial ring in n variables modulo an ideal which has an n-element homogeneous system of parameters of degree two, is the f-vector of a simplicial complex. Taking the face ideal of the boundary complex of any edge-orientable convex cubical polytope, and factoring out by a linear system of parameters we obtain an example to the conjecture. The fact the conjecture holds for our examples is not trivial: it follows from Stanley's result about the h-vectors of completely balanced Cohen-Macaulay simplicial complexes. Our examples are the first nontrivial ones: the only example given by Eisenbud, Greene and Harris was any polynomial ring with the ideal generated by the squares of the variables. Interestingly, from our results we cannot know whether we have still examples or also some counter-examples if we drop the condition of edge-orientability: this observation hints a concrete way to attack the Eisenbud-Greene-Harris conjecture.

Chapter 1

Preliminaries on polyhedral complexes

1.1 Abstract polyhedral complexes and geometric realizations

Definition 1 *An* (abstract) polyhedral complex $\mathcal{C}$ *is a family of finite sets (called faces) on a vertex set* V *satisfying the following conditions.*

(i) We have $\{v\} \in \mathcal{C}$ *for every* $v \in V$.

(ii) For every $\sigma \in \mathcal{C}$ *we have either* $\sigma = \emptyset$ *or there is an injective map* $\phi_\sigma : \sigma \longrightarrow \mathbb{R}^n$ *for some* $n = n(\sigma)$ *such that* $\phi_\sigma(\sigma)$ *is the vertex set of the convex polytope* $\text{conv}(\phi_\sigma(\sigma))$, *and the faces contained in* σ *are exactly the inverse images under* ϕ_σ *of the vertex sets of the faces of* $\text{conv}(\phi_\sigma(\sigma))$. *We call* ϕ_σ *a* geometric realization of σ.

(iii) If $\sigma, \tau \in \mathcal{C}$ *then* $\sigma \cap \tau \in \mathcal{C}$.

In particular, a polyhedral complex $\mathcal{C}$ is called *simplicial* or *cubical* respectively, if every face has a geometric realization as a simplex or as a cube respectively. We will denote cubical complexes by the symbol $\square$ and simplicial complexes by the symbol $\triangle$.

For every face σ we call the dimension of the polytope associated to σ the *dimension of* σ. Maximal faces are called *facets*, their facets are *subfacets*. The one-dimensional faces are also called *edges* and two vertices $u, v \in V$ are called *adjacent* if $\{u, v\}$ is an edge. The dimension of the facets is the dimension of the polyhedral complex. We denote the number of i-dimensional faces of a d-dimensional polyhedral complex by f_i and call $(f_{-1}, f_0, \ldots, f_d)$ the *f-vector* of the polyhedral complex.

Given a set $X \subset V$ we will denote by $\mathcal{C}|_X$ the polyhedral complex $\{\sigma \in \mathcal{C} : \sigma \subseteq X\}$.

Definition 2 *We call a map* $\phi : V \longrightarrow \mathbb{R}^n$ *a* geometric realization of the polyhedral complex $\mathcal{C}$ in $\mathbb{R}^n$ *if for each* $\sigma \in \mathcal{C} \setminus \{\emptyset\}$ *the map* $\phi|_\sigma$ *is a geometric realization of the face* σ*, and for all* $\sigma, \tau \in \mathcal{C}$ *we have* $\operatorname{conv}(\phi(\sigma)) \cap \operatorname{conv}(\phi(\tau)) = \operatorname{conv}(\phi(\sigma \cap \tau))$.

Remarks

1. Given a polyhedral complex $\mathcal{C}$, a geometric realization ϕ_σ of a face $\sigma \in \mathcal{C} \setminus \{\emptyset\}$ provides also a geometric realization of the subcomplex $\mathcal{C}|_\sigma$.

2. For simplicial complexes we may replace condition (ii) in Definition 1 with the requirement that the subset of every face has to be a face.

It is a well-known fact that every simplicial complex has a geometric realization. (See for example [23, p. 110].) Abstract polyhedral complexes in general, however, may have no geometric realization. (An example of such a polyhedral complex is the cubical complex with three squares F_1, F_2, F_3 incident in such a way that they form a Möbius strip. The proof of the fact that this complex has no geometric realization is implicit in the proof of Theorem 15.) When $\mathcal{C}$ has a geometric realization ϕ, then the system $\{\operatorname{conv}(\phi(\sigma)) : \sigma \in \mathcal{C} \setminus \{\emptyset\}\} \cup \{\emptyset\}$ is a *polyhedral complex* or *convex complex*, as defined on p. 39 of [13], p. 126 of [3], or p. 60 of [17].

Even though cubical complexes may have no geometric realization, their faces have a *coherent system* of geometric realizations in the following sense.

Definition 3 *Let $\mathcal{C}$ be a polyhedral complex and let us choose a geometric realization ϕ_σ for every $\sigma \in \mathcal{C} \setminus \{\emptyset\}$. We call the system of geometric realizations $\Phi := \{\phi_\sigma : \sigma \in \mathcal{C} \setminus \{\emptyset\}\}$ a* weak geometric realization *of $\mathcal{C}$. We consider two weak geometric realizations $\Phi = \{\phi_\sigma : \sigma \in \mathcal{C} \setminus \{\emptyset\}\}$ and $\Psi = \{\psi_\sigma : \sigma \in \mathcal{C} \setminus \{\emptyset\}\}$* affinely equivalent *or* congruent *respectively if the polytope $\text{conv}(\phi_\sigma(\sigma))$ is affinely equivalent resp. congruent to $\text{conv}(\psi_\sigma(\sigma))$ for all $\sigma \in \mathcal{C} \setminus \{\emptyset\}$.*

We call the weak geometric realization $\Phi = \{\phi_\sigma : \sigma \in \mathcal{C} \setminus \{\emptyset\}\}$ affinely coherent *or* congruently coherent *respectively if for every pair of nonempty faces $\tau \subseteq \sigma$ the polytopes $\text{conv}(\phi_\sigma(\tau))$ and $\text{conv}(\phi_\tau(\tau))$ are affinely equivalent or congruent respectively.*

The following proposition is a straightforward consequence of the definitions, and of the fact that any two simplices of the same dimension are affinely equivalent.

Proposition 1 *For every geometric realization ϕ of a polyhedral complex $\mathcal{C}$ the set*

$$\Phi := \{ \phi|_\sigma : \sigma \in \mathcal{C} \setminus \{\emptyset\}\}$$

is a congruently coherent weak geometric realization.

Every weak geometric realization Φ of a simplicial complex $\triangle$ is affinely coherent, and any two weak geometric realizations of $\triangle$ are affine equivalent.

Definition 4 *We call a weak geometric realization Φ* principal *if it may be written as*

$$\Phi = \{ \phi|_\sigma : \sigma \in \mathcal{C} \setminus \{\emptyset\}\}$$

for some geometric realization ϕ.

Now we are in the position to define a *standard congruently coherent weak geometric realization* for every cubical complex $\square$.

Definition 5 *The* geometric standard n-cube *is the convex polytope*

$$[0,1]^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \, : \, 0 \leq x_i \leq 1\}.$$

We define the (abstract) standard n-cube $\square^n$ *to be the vertex set of* $[0,1]^n$ *together with the inherited face-structure on the vertices. We call any* 2^n*-element set with an isomorphic face-structure an n-dimensional cube. We call a geometric realization* $\phi : \square^n \longrightarrow \mathbb{R}^n$ standard *if we have* $\phi(V(\square^n)) = \{0,1\}^n$.

For any cubical complex $\square$ *we call a weak geometric realization* $\Phi = \{\phi_\sigma \, : \, \sigma \in \square\}$ standard, *if* $\mathrm{conv}(\phi_\sigma(\sigma))$ *is congruent to a standard cube for every* $\sigma \in \square \setminus \{\emptyset\}$.

Given the fact that the faces of standard cubes are standard cubes, up to congruence equivalence there is one and only one standard weak geometric realization of a cubical complex $\square$.

1.2 Geometric, abstract and natural subdivisions

Next we define polyhedral subdivisions of polyhedral complexes in both geometric and abstract setting.

Definition 6 *Let* C *be a polyhedral complex on the vertex set* V *and* $\phi : V \longrightarrow \mathbb{R}^n$ *a geometric realization of* C*. A polyhedral complex* C' *on the vertex set* $V' \supseteq V$*, together with a map* $\phi' : V' \to \mathbb{R}^n$ *is a* geometric subdivision *of* (C, ϕ) *if* ϕ' *has the following properties.*

(1) $\phi = \phi'|_V$ *holds.*

(2) We have $\bigcup_{\sigma \in \mathcal{C} \setminus \{\emptyset\}} \mathrm{conv}(\phi(\sigma)) = \bigcup_{\sigma' \in \mathcal{C}' \setminus \{\emptyset\}} \mathrm{conv}(\phi'(\sigma'))$.

(3) For every nonempty face $\sigma' \in \mathcal{C}'$ *the polyhedron* $\mathrm{conv}(\phi'(\sigma'))$ *is contained in* $\mathrm{conv}(\phi(\sigma))$ *for some* $\sigma \in \mathcal{C}$.

When $\mathcal{C}'$ *is a simplicial complex then we call the subdivision* $(\mathcal{C}', \phi')$ *a* geometric triangulation *of* $(\mathcal{C}, \phi)$.

Remark Note that at the light of condition (3), we may replace condition (2) by

(2') $\bigcup_{\sigma \in \mathcal{C} \setminus \{\emptyset\}} \mathrm{conv}(\phi(\sigma)) \subseteq \bigcup_{\sigma' \in \mathcal{C}' \setminus \{\emptyset\}} \mathrm{conv}(\phi'(\sigma'))$.

Given a geometric subdivision $(\mathcal{C}', \phi')$ of a geometric realization $(\mathcal{C}, \phi)$ we may define the *carrier map* $S : \mathcal{C}' \longrightarrow \mathcal{C}$ by $S(\sigma')$ being the smallest face $\sigma \in \mathcal{C}$ satisfying

$$\mathrm{conv}(\phi'(\sigma')) \subseteq \mathrm{conv}(\phi(\sigma)).$$

Then for every $\sigma \in \mathcal{C}$ the family of sets $S^{-1}(\mathcal{C}|_{\sigma})$ is a *full subcomplex* of $\mathcal{C}'$. Full subcomplexes are defined as follows.

Definition 7 *A subcomplex* $\mathcal{D}'$ *of a polyhedral complex* $\mathcal{D}$ *is* full *if every* $\sigma \in \mathcal{D}$ *having all vertices in* $\mathcal{D}'$ *belongs to* $\mathcal{D}'$. *Formally*

$$\forall v \in \sigma \, (\{v\} \in \mathcal{D}') \Rightarrow \sigma \in \mathcal{D}.$$

In fact, if $\phi'(v')$ belongs to $\mathrm{conv}(\phi(\sigma))$ for every $v' \in \sigma'$, then $\mathrm{conv}(\phi'(\sigma'))$ is contained in $\mathrm{conv}(\phi(\sigma))$, and so we have $S(\sigma') \subseteq \sigma$, i.e., $\sigma' \in S^{-1}(\mathcal{C}|_{\sigma})$.

These observations motivate the following definition.

Definition 8 *Let* $\mathcal{C}$ *be a polyhedral complex on the vertex set* V *and* $\Phi = \{\phi_\sigma : \sigma \in \mathcal{C} \setminus \{\emptyset\}\}$ *a weak geometric realization of* $\mathcal{C}$. *Let* $\mathcal{C}'$ *be another polyhedral complex on the* *vertex set* $V' \supseteq V$ *and* $\Phi' = \{\phi'_{\sigma'} : \sigma \in \mathcal{C}' \setminus \{\emptyset\}\}$ *a weak geometric realization of* $\mathcal{C}'$. *We say that* $(\mathcal{C}', \Phi')$ *together with the* carrier map $S : \mathcal{C}' \longrightarrow \mathcal{C}$ *is a* subdivision *of* $(\mathcal{C}, \Phi)$, *if they satisfy the following.*

(A) $S^{-1}(\{\emptyset\}) = \{\emptyset\}$ *holds.*

(B) For every $v \in V$ *we have* $S(\{v\}) = \{v\}$.

(C) For every face $\sigma \in \mathcal{C} \setminus \{\emptyset\}$ *the family of sets* $S^{-1}(\mathcal{C}|_{\sigma})$ *is a full subcomplex of* $\mathcal{C}'$, *which has a geometric realization* ψ_σ *with the following properties.*

  (i) $\phi'_{\sigma'} = \psi_\sigma|_{\sigma'}$ *holds for all* $\sigma' \in S^{-1}(\sigma)$.

  (ii) We have $\psi_\sigma(v') \in \text{conv}(\phi_\sigma(S(v')))$ *for all vertex* v' *of* $S^{-1}(\mathcal{C}|_{\sigma})$.

  (iii) $(S^{-1}(\mathcal{C}|_{\sigma}), \psi_\sigma)$ *is a geometric subdivision of* $(\mathcal{C}|_{\sigma}, \phi_\sigma)$. *(In particular, the vertex set of* $S^{-1}(\mathcal{C}|_{\sigma})$ *contains* σ.*)*

Lemma 1 *Let* $(\mathcal{C}', \Phi')$ *and* $S : \mathcal{C}' \longrightarrow \mathcal{C}$ *be a subdivision of* $(\mathcal{C}, \Phi)$. *Then* S *has the following properties.*

1. S *is monotone.*

2. *For every* $\tau' \in S^{-1}(\mathcal{C}|_{\sigma})$ *there is a* $\sigma' \in \mathcal{C}'$ *such that* $\tau' \subseteq \sigma'$ *and* $\sigma = S(\sigma')$. *As a consequence,* S *is surjective.*

3. *For every* $\sigma' \in \mathcal{C}'$ *we have* $\sigma' \cap V \subseteq S(\sigma')$.

Proof:

1. First we show that S has to be monotone. Assume we are given $\sigma', \tau' \in \mathcal{C}'$ such that $\sigma' \subseteq \tau'$. We may assume $\tau' \neq \emptyset$, because otherwise we have $\sigma' = \tau' = \emptyset$ and $S(\sigma') = S(\tau') = \emptyset$. Obviously, $\tau' \in S^{-1}(\mathcal{C}|_{S(\tau')})$, and so $\sigma' \in S^{-1}(\mathcal{C}|_{S(\tau')})$, because by definition the family of sets $S^{-1}(\mathcal{C}|_{S(\tau')})$ has to be a subcomplex. Thus we have $S(\sigma') \subseteq S(\tau')$.

1.5. Before going on with the proof of our lemma, let us observe that using the monotonity of S we may generalize property (ii) of Definition 8 to the following.

(ii') $\text{conv}(\psi_\sigma(\sigma')) \subseteq \text{conv}(\phi_\sigma(S(\sigma')))$ holds for all $\sigma' \in S^{-1}(\mathcal{C}|_\sigma)$.

In fact, by (ii), every vertex v' of a face $\sigma' \in S^{-1}(\mathcal{C}|_\sigma)$ satisfies

$$\psi_\sigma(v') \in \text{conv}(\phi_\sigma(S(v'))),$$

and so by the monotonity of S we have $S(v') \subseteq S(\sigma')$, implying

$$\psi_\sigma(v') \in \text{conv}(\phi_\sigma(S(\sigma')))$$

for all $v' \in \sigma'$. From this last equation, (ii') follows directly.

2. Let us fix now a face $\sigma \in \mathcal{C} \setminus \{\emptyset\}$, and a face $\tau' \in S^{-1}(\mathcal{C}|_\sigma)$. Let ψ_σ be the geometric representation of $S^{-1}(\mathcal{C}|_\sigma)$ described in Definition 8. Let Q be a fixed relative interior point of $\text{conv}(\psi_\sigma(\tau'))$. We claim that there is a $\sigma' \in S^{-1}(\mathcal{C}|_\sigma)$ such that $\text{conv}(\psi_\sigma(\sigma'))$ contains both Q and a $P \in \text{relint}(\text{conv}(\phi_\sigma(\sigma)))$. In fact, if for a $\sigma'' \in S^{-1}(\mathcal{C}|_{S(\sigma)})$, the set $\text{conv}(\psi_\sigma(\sigma''))$ avoids Q, then there is an open neighborhood $U_{\sigma''}$ of Q in $\text{conv}(\phi_\sigma(\sigma))$ which is disjoint from $\text{conv}(\psi_\sigma(\sigma''))$. Let P be a relative interior point of $\text{conv}(\phi_\sigma(\sigma))$ in the intersection of all such $U_{\sigma''}$-s. By condition (iii), $(S^{-1}(\mathcal{C}|_\sigma), \psi_\sigma)$ is a geometric subdivision of $(\mathcal{C}|_\sigma, \phi_\sigma)$, and so condition (2) of Definition 6 guarantees the existence of a $\sigma' \in S^{-1}(\mathcal{C}|_{S(\sigma)})$ such that $\text{conv}(\psi_\sigma(\sigma'))$ contains P. By the choice of P we must have $Q \in \psi_\sigma(\sigma')$. But then from the geometric representation ψ_σ we observe $\tau' \subseteq \sigma'$.

We claim that this σ' satisfies $S(\sigma') = \sigma$. In fact, by condition (ii') we have $\text{conv}(\psi_\sigma(\sigma')) \subseteq \text{conv}(\phi_\sigma(S(\sigma')))$, and so the face $\text{conv}(\phi_\sigma(S(\sigma')))$ contains $P \in \text{relint}(\text{conv}(\phi_\sigma(\sigma)))$. Therefore $S(\sigma')$ cannot be a proper face of σ.

3. Finally we show that $\sigma' \cap V \subseteq S(\sigma')$ holds for every $\sigma' \in \mathcal{C}'$. Assume the contrary. Then there is a vertex $v \in S(\sigma') \setminus \sigma'$ which belongs to V. We have already shown that S is monotone, and so $\{v\} \subseteq \sigma'$ implies $S(\{v\}) \subseteq S(\sigma')$. But then we have $v \notin S(\{v\})$, contradicting condition (B) of Definition 8.

QED

Corollary 1 *When $(\mathcal{C}', \Phi')$ and $S : \mathcal{C}' \longrightarrow \mathcal{C}$ is a subdivision of $(\mathcal{C}, \Phi)$ then the geometric realizations ψ_σ in Definition 8 are uniquely determined by $(\mathcal{C}', \Phi')$, S and $(\mathcal{C}, \Phi)$.*

Proof: Let us fix a face $\sigma \in \mathcal{C} \setminus \{\emptyset\}$, and a vertex v' of $S^{-1}(\mathcal{C}|_\sigma)$. By Lemma 1, there is a $\sigma' \in \mathcal{C}'$ such that $v' \in \sigma'$ and $\sigma = S(\sigma')$. Thus, by condition (i) in Definition 8 we must have $\psi_\sigma(v') = \psi_\sigma|_{\sigma'}(v') = \phi'_{\sigma'}(v')$. **QED**

Lemma 1 also allows us to compute the carrier map from its values on the singleton. In order to give a formula we need the notion of *polyhedral span.*

Definition 9 *For a set of vertices $X \subset V$ in a polyhedral complex $\mathcal{C}$ we define the* polyhedral span $\operatorname{Pspan}(X)$ *of X to be the smallest face containing the set X. (If there is no such face then we leave* $\operatorname{Pspan}(X)$ *undefined.) In the special case when $\mathcal{C}$ is a cubical complex we also call the polyhedral span of X the* cubical span *of X and denote it by* $\operatorname{Cspan}(X)$. *For a pair of vertices $\{u, v\}$ and a face $\tau \in \square$ satisfying* $\operatorname{Cspan}(\{u, v\}) = \tau$ *we say that $\{u, v\}$ is a* diagonal *of τ.*

Corollary 2 (Fundamental equation of the carrier map) *When $(\mathcal{C}', \Phi')$ and $S : \mathcal{C}' \to \mathcal{C}$ is a subdivision of $(\mathcal{C}, \Phi)$, then the carrier map $S : \mathcal{C}' \to \mathcal{C}$ is uniquely defined by its values on the singletons: for every $\sigma' \in \mathcal{C}'$ we have*

$$S(\sigma') = \operatorname{Pspan}\left(\bigcup_{v' \in \sigma'} S(\{v'\})\right). \qquad (1.1)$$

Proof: Let us set $\tau := \mathrm{Pspan}\left(\bigcup_{v' \in \sigma'} S(v')\right)$. By Lemma 1, the carrier map is monotone and so we have $S(\{v'\}) \subseteq S(\sigma')$ for every $v' \in \sigma'$. This implies $S(\sigma') \supseteq \tau$. On the other hand, by definition $S^{-1}(\mathcal{C}|_{\tau})$ is a full subcomplex of $\mathcal{C}'$ and so σ' must belong to $S^{-1}(\mathcal{C}|_{\tau})$, because all of its vertices belong there. Thus we also have $S(\sigma') \subseteq \tau$. **QED**

Clearly, in the case when both Φ and Φ' are principal, induced by the geometric realizations ϕ and ϕ', $(\mathcal{C}', \phi')$ is a geometric subdivision of $(\mathcal{C}, \phi)$, and S assigns to each $\sigma' \in \mathcal{C}' \setminus \{\emptyset\}$ the smallest face $\sigma \in \mathcal{C}$ satisfying $\mathrm{conv}(\phi'(\sigma')) \subseteq \mathrm{conv}(\phi'(\sigma))$, then $(\mathcal{C}', \Phi')$ and S is a subdivision of $(\mathcal{C}, \Phi)$. In the next proposition we show a weak converse of this statement.

Proposition 2 *Let $\mathcal{C}, \mathcal{C}', \Phi$ and Φ' be as before, and assume that Φ' is principal, induced by the geometric realization ϕ'. Assume furthermore that for all $\sigma, \tau \in \mathcal{C}$ we have*

$$\mathrm{conv}(\phi'(\sigma)) \cap \mathrm{conv}(\phi'(\tau)) = \mathrm{conv}(\phi'(\sigma \cap \tau)), \tag{1.2}$$

and that $(\mathcal{C}', \Phi')$ together with a map $S : \mathcal{C}' \longrightarrow \mathcal{C}$ is a subdivision of $(\mathcal{C}, \Phi)$.

Then $(\mathcal{C}, \Phi)$ is principal, induced by a geometric representation ϕ, $(\mathcal{C}', \phi')$ is a geometric subdivision of $(\mathcal{C}, \phi)$, and for every $\sigma' \in \mathcal{C}' \setminus \{\emptyset\}$ the face $S(\sigma') \in \mathcal{C}$ is the smallest face in the set $\{\sigma \in \mathcal{C} \,:\, \mathrm{conv}(\phi(\sigma)) \supseteq \mathrm{conv}(\phi'(\sigma'))\}$.

Proof: Let us fix any $\sigma \in \mathcal{C} \setminus \{\emptyset\}$ and a $v \in \sigma$. By Lemma 1, there is a $\sigma' \in S^{-1}(\sigma)$ such that $v \in \sigma'$. For such a σ' condition (i) of Definition 8 gives us

$$\psi_\sigma(v) = \phi'_{\sigma'}(v) = \phi'(v).$$

The second equality follows from the fact that Φ' is induced by ϕ'. On the other hand, from condition (iii) we obtain

$$\phi_\sigma(v) = \psi_\sigma(v).$$

The previous two equations imply

$$\phi_\sigma(v) = \phi'(v) \text{ for all } v \in V.$$

This equation together with (1.2) shows that Φ is principal, induced by $\phi'|_V$. In particular, we obtained that (1) in Definition 6 is satisfied by $(\mathcal{C}', \phi')$ and $(\mathcal{C}, \phi)$.

Observe next that condition (ii) of Definition 8 applied to $S(\sigma')$ (where $\sigma' \in \mathcal{C} \setminus \{\emptyset\}$ is an arbitrary nonempty face) gives us

$$\text{conv}(\phi'(\sigma')) \subseteq \text{conv}(\phi(S(\sigma'))).$$

Hence condition (3) of Definition 6 is fulfilled.

In order to prove that $(\mathcal{C}', \phi')$ is a geometric subdivision of $(\mathcal{C}, \phi)$, it only remains to show that condition (2') in the remark following Definition 6 is fulfilled. By (iii) in Definition 8 we obtain

$$\bigcup_{\tau \subseteq \sigma} \text{conv}(\phi(\tau)) \subseteq \bigcup_{\tau' \in S^{-1}(\mathcal{C}|_\sigma)} \text{conv}(\phi'(\tau')) \subseteq \bigcup_{\tau' \in \mathcal{C}'} \text{conv}(\phi'(\tau'))$$

for all $\sigma \in \mathcal{C} \setminus \{\emptyset\}$. Taking the union of both sides over all nonempty faces $\sigma \in \mathcal{C} \setminus \{\emptyset\}$, we obtain condition (2').

Finally, let us fix a $\sigma' \in \mathcal{C}' \setminus \{\emptyset\}$. As noted above, we have

$$\text{conv}(\phi'(\sigma')) \subseteq \text{conv}(\phi(S(\sigma'))).$$

We only need to show that no proper face $\sigma \subset S(\sigma')$ satisfies $\text{conv}(\phi'(\sigma')) \subseteq \text{conv}(\phi(\sigma))$. Assume the contrary, and consider any $P \in \text{relint}(\text{conv}(\phi'(\sigma')))$. This point belongs to $\text{conv}(\phi(\sigma))$ and so condition (iii) of Definition 8 for σ implies the existence of a $\tau' \in S^{-1}(\mathcal{C}|_\sigma)$ such that $P \in \text{conv}(\phi'(\tau'))$. But then from the geometric realization of $\mathcal{C}'$ we can observe $\tau' \supseteq \sigma'$, and so we have $\sigma' \in S^{-1}(\mathcal{C}|_\sigma)$ because $S^{-1}(\mathcal{C}|_\sigma)$ is a subcomplex of $\mathcal{C}'$. Thus we have $S(\sigma') \subseteq \sigma \subset S(\sigma')$, a contradiction. **QED**

In the following we restrict ourselves to the special case when $V = V'$.

Proposition 3 *Let $\mathcal{C}$ and $\mathcal{C}'$ be polyhedral complexes on the same vertex set V with weak realizations $\Phi = \{\phi_\sigma : \sigma \in \mathcal{C} \setminus \{\emptyset\}\}$ and $\Phi' = \{\phi'_{\sigma'} : \sigma' \in \mathcal{C}' \setminus \{\emptyset\}\}$. If $(\mathcal{C}', \Phi')$ together with the carrier map $S : \mathcal{C} \longrightarrow \mathcal{C}'$ is a subdivision of $(\mathcal{C}, \Phi)$, then*

$$\phi'_{\sigma'} = \phi_{S(\sigma')}\Big|_{\sigma'} \tag{1.3}$$

holds for every nonempty face $\sigma' \in \mathcal{C}'$. Moreover $S(\sigma')$ is the smallest face of $\mathcal{C}$ containing σ'.

Proof: Take any nonempty face $\sigma' \in \mathcal{C}'$ and let $\psi_{S(\sigma')}$ be a geometric representation of $S^{-1}\left(\mathcal{C}|_{S(\sigma')}\right)$, satisfying the conditions of Definition 8. Lemma 1 and $V = V'$ implies $\sigma' \subseteq S(\sigma')$. By condition (i) we have

$$\phi'_{\sigma'} = \psi_{S(\sigma')}\Big|_{\sigma'}.$$

Condition (iii), together with condition (1) of Definition 6 gives us

$$\phi_{S(\sigma')} = \psi_{S(\sigma')}\Big|_{S(\sigma')}$$

Thus both $\phi'_{\sigma'}$ and $\phi_{S(\sigma')}$ agree with $\psi_{S(\sigma')}$ on σ'. This gives us (1.3).

Finally equation (1.1) and condition (B) of Definition 8 give us

$$S(\sigma') = \text{Pspan}\left(\bigcup_{v' \in \sigma'} S(\{v'\})\right) = \text{Pspan}\left(\bigcup_{v' \in \sigma'} \{v'\}\right).$$

QED

In particular, for geometric subdivisions of polyhedral complexes with geometric realizations we obtain the following.

Proposition 4 *Let C and C' be polyhedral complexes on the same vertex set V. Assume that ϕ and ϕ' are geometric realizations of C and C' and $S : C \longrightarrow C'$ is a carrier map such that (C', ϕ') and S are a subdivision of (C, ϕ). Then $\phi = \phi'$, and for every $\sigma' \in C'$, $S(\sigma')$ is the smallest face of C containing σ'.*

Proof: Condition (1) of Definition 6 implies $\phi' = \phi$. The second claim is implied by the third statement in Proposition 2. **QED**

By Proposition 3, in the case when $V = V'$ we do not need to give the carrier map (because there is only one way to define it) and the geometric realizations of the faces of the subdivided complex determine the geometric realizations of the faces of the subdividing complex. Hence the question arises: is there a notion of subdivision which is independent of the geometric realizations? The answer is yes, we can define *natural subdivisions.*

Definition 10 *Let C and C' be polyhedral complexes on the same vertex set V. For every $\sigma' \in C'$ let $S(\sigma')$ be the smallest face of C containing the set σ'. We say that C' is a* natural subdivision *of C, if the following hold.*

1. *Every $\sigma' \in C'$ is contained in some $\sigma \in C$.*
2. *For every weak geometric representation $\Phi = \{\phi_\sigma \ : \ \sigma \in C \setminus \{\emptyset\}\}$ the set $\Psi := \{\, \phi_{C|_{S(\sigma')}}\Big|_{\sigma'} \ : \ \sigma' \in C' \setminus \{\emptyset\}\}$ is a weak geometric realization of C', and (C', Φ'), together with S, a subdivision of (C, Φ).*

When C' is a simplicial complex, then we say that C' is a natural triangulation *of C.*

Of course this definition is only interesting, if we have an example of such subdivisions. Fortunately, there exist an important class of natural triangulations: the *triangulations via pulling the vertices.*

Definition 11 *Let $\mathcal{C}$ be a polyhedral complex on the vertex set V and $<$ a linear order on V. Let us denote the smallest vertex of a face $\sigma \in \mathcal{C} \setminus \{\emptyset\}$ by $\delta_<(\sigma)$. We define the* natural triangulation of $\mathcal{C}$ via pulling the vertices in order $<$ *to be the family of all sets $\{v_1, \ldots, v_k\}$ such that $k \in \mathbb{N}$, $v_1 > \cdots > v_k$, and for $i = 1, \ldots, k$ we have*

$$v_i = \delta_<(\mathrm{Pspan}(\{v_1, \ldots, v_i\}))$$

Remark In [27], Stanley gives an apparently different definition for triangulations of convex polytopes via pulling the vertices. He considers all *full flags* $\sigma_0 \subset \cdots \subset \sigma_d$ of faces of a d-dimensional polytope satisfying $\delta_<(\sigma_i) \notin \sigma_{i-1}$ for $i = 1, 2, \ldots, d$, and then claims that the sets $\{\delta_<(\sigma_0), \ldots, \delta_<(\sigma_d)\}$ are the maximal faces of a triangulation of the polytope.

Definition 11 can be shown to be equivalent to Stanley's for the face complex of a convex polytope. For the record, we sketch the proof.

First, it is easy to check that every set $\{v_1, \ldots, v_k\}$ satisfying the conditions of Definition 11 may be extended to a $(d+1)$-element set of vertices satisfying the same conditions. This can be done by induction on dimension: when $k < d + 1$ then there is a "jump in dimension" in the chain $\mathrm{Pspan}(\{v_1\}) \subset \cdots \subset \mathrm{Pspan}(\{v_1, \ldots, v_k\})$, i.e., there is an i such that

$$\dim \mathrm{Pspan}(\{v_1, \ldots, v_i\}) - \dim \mathrm{Pspan}(\{v_1, \ldots, v_{i-1}\}) \geq 2$$

holds. Then there is a face σ strictly between $\mathrm{Pspan}(\{v_1, \ldots, v_{i-1}\})$ and $\mathrm{Pspan}(\{v_1, \ldots, v_i\})$. Applying the induction hypothesis to σ we get an u such that the set $\{v_1, \ldots, v_{i-1}, u\}$ satisfies the conditions of Definition 11. By our construction, we have $u > v_i$ and so it is easy to verify that $\{v_1, \ldots, v_k, u\}$ also satisfies the conditions of Definition 11. Iterating this argument we may obtain a $(d+1)$-element set of vertices containing $\{v_1, \ldots, v_k\}$ and belonging to $\triangle_<(\mathcal{C})$.

Observe next that the way we defined $\triangle_<(\mathcal{C})$, we really obtain a simplicial complex, i.e., $\triangle_<(\mathcal{C})$ is closed under taking subsets and it contains the singletons contained in V. Thus we only need to show that the sets $\{\delta_<(\sigma_0), \ldots, \delta_<(\sigma_d)\}$, where $\sigma_0 \subset \cdots \subset \sigma_d$ is a full flag satisfying $\delta_<(\sigma_i) \notin \sigma_{i-1}$ for $i = 1, 2, \ldots, d$, are exactly the $(d+1)$-element faces of $\triangle_<(\mathcal{C})$. It is easy to check that these sets in fact belong to $\triangle_<(\mathcal{C})$, and that conversely, every $\{v_0, \ldots, v_d\} \in \triangle_<(\mathcal{C})$ comes from a flag $\sigma_0 \subset \cdots \subset \sigma_d$ satisfying $\delta_<(\sigma_i) \notin \sigma_{i-1}$ for $i = 1, 2, \ldots, d$. In fact, assuming $v_0 > \cdots > v_d$, the flag defined by $\sigma_i := \mathrm{Pspan}(\{v_0, \ldots, v_i\})$ for $i = 1, 2, \ldots, d$ will have the required properties.

We have chosen to use the formulation of Definition 11 because it makes transparent the fact that the restriction of $\triangle_<(\mathcal{C})$ to a face σ is just the triangulation of $\mathcal{C}|_\sigma$ via pulling the vertices with respect to the order induced by $<$ on σ. Hence the fact that $\triangle_<(\mathcal{C})$ is a natural triangulation of $\mathcal{C}$ becomes a straight consequence of [27, Lemma 1.1].

1.3 Shellable polyhedral complexes

In the definition of shellable polyhedral complexes we will need the notions of *ball* and *sphere*.

Definition 12 *Let $(\mathcal{C}, \phi)$ be a geometrically represented polyhedral complex. Assume that the family of sets $\{\mathrm{conv}(\phi(\sigma)) : \sigma \in \mathcal{C}\}$ consists of the boundary faces of a convex polytope $\mathcal{P}$. A collection $\{F_1, F_2, \ldots, F_k\}$ of facets of $\mathcal{C}$ is called an* $(n-1)$-dimensional ball *or* $(n-1)$-dimensional sphere *respectively, if the set $\bigcup_{i=1}^{k} \mathrm{conv}(\phi(F_i))$ is homeomorphic to an $(n-1)$-dimensional ball or sphere respectively.*

Apparently, the notion of ball or sphere depends on the geometric representation ϕ. In reality, this is not the case, because of the following lemma.

Lemma 2 *Let $\mathcal{C}$ be identifiable with the collection of vertex sets of faces of the boundary of an n-dimensional convex polytope. Let ϕ and ϕ' be two geometric representations of $\mathcal{C}$. Then there is a homeomorphism η between the polytopes* $\mathrm{conv}(\phi(V))$ *and* $\mathrm{conv}(\phi'(V))$ *such that for every $\sigma \in \mathcal{C}$ we have*

$$\eta(\mathrm{conv}\phi(\sigma)) = \mathrm{conv}\phi'(\sigma).$$

(V denotes the vertex set of $\mathcal{C}$.)

Proof: Let us fix a linear order $<$ on V. It is easy to see that the geometric realizations ϕ and ϕ' are also geometric realizations of the natural triangulation $\triangle_<(\mathcal{C})$. (Note that triangulations via pulling the vertices were first defined on geometrically represented polytopes.) It is sufficient to define a homeomorphism $\eta : \mathrm{conv}(\phi(V)) \to \mathrm{conv}(\phi'(V))$ such that

$$\eta(\mathrm{conv}\phi(\sigma')) = \mathrm{conv}\phi'(\sigma')$$

be satisfied for every $\sigma' \in \triangle_<(\mathcal{C})$. We can define such an η as follows. For any face $\sigma' \in \triangle_<(\mathcal{C})$ and any point $Q \in \mathrm{conv}(\phi(\sigma'))$, we can write Q uniquely as the convex combination of the vertices of $\mathrm{conv}(\phi(\sigma'))$, i.e. we have

$$Q = \sum_{v' \in \sigma'} \alpha_{v'} \cdot \phi(v')$$

where the coefficients $\alpha_{v'}$ are nonnegative and their sum is 1. Let us set now

$$\eta(Q) := \sum_{v' \in \sigma'} \alpha_{v'} \cdot \phi'(v').$$

It is easy to verify that the η above is well-defined and has the required properties. **QED**

Now we are in the position to define shellable polyhedral complexes.

Definition 13 *A polyhedral complex $\mathcal{C}$ is* pure *if all facets of $\mathcal{C}$ have the same dimension. We define* shellable polyhedral complexes *as follows.*

1. *The empty set is a ((-1)-dimensional) shellable polyhedral complex.*

2. *A point is a (0-dimensional) shellable complex.*

3. *A d-dimensional pure complex $\mathcal{C}$ is shellable if its facets can be listed in a linear order*

   $F_0, F_1, \ldots, F_n$ such that for each $k \in \{1, 2, \ldots, n\}$ the subcomplex $\mathcal{C}|_{F_k} \cap (\mathcal{C}|_{F_1} \cup \cdots \cup \mathcal{C}|_{F_{k-1}})$ is a pure complex of dimension $(d-1)$ such that its maximal dimensional faces form a $d-1$ dimensional ball or sphere.

Chapter 2

Generalizations of Baxter's theorem and cubical homology

2.1 Baxter's theorem and its dual

Consider a 3-regular planar graph. Arbitrarily assign one of 4 colors to each of its regions. Let A_1 be the number of vertices surrounded by regions of colors 234 in clockwise order, minus the number of vertices surrounded by regions of colors 234 in counterclockwise order. Define A_2, A_3, A_4 similarly. In [2] Baxter proves the following result:

Theorem 1

$$A_1 = -A_2 = A_3 = -A_4.$$

In this chapter we generalize this theorem to triangulated and cubically subdivided manifolds of arbitrary dimension in several ways. From our cubical results we obtain a way to construct a cubical analogue of simplicial homology.

For this purpose, consider the dual statement: given a triangulation of the sphere, and a coloring of the vertices with 4 colors, we denote by A_1 the number of triangles whose vertices are colored with 234 in clockwise order minus the number of triangles whose vertices are colored with 234 in counterclockwise order. We define A_2, A_3, A_4 similarly. The dual of Baxter's theorem gives the same identity for the A_i-s as before. This statement may be generalized in an obvious way to colorings of triangulations of n-dimensional orientable (pseudo-)manifolds with $n+2$ colors, which we call the generalized Baxter's Theorem.

2.2 Simplicial generalizations

2.2.1 Simplicial polytopes and compact orientable manifolds

We begin by proving our generalization of Baxter's theorem in a special case, to give the flavor of how the topological construction works. Observe that the surface of a $(n+1)$-dimensional convex simplicial polytope may be viewed as triangulation of $\mathbb{S}^n$.

Theorem 2 (Generalized Baxter's theorem for simplicial polytopes) *Color arbitrarily the vertices of a given $(n+1)$-dimensional convex simplicial polytope P with $n+2$ colors. Let A_i be the number of facets with vertices colored with $\{1, 2, \ldots, n+2\} \setminus \{i\}$, where we count a facet with multiplicity 1 or -1 when the sign of the multiplicity is the sign of the orientation of the vertices of the facet listed in increasing order of colors. Then we have*

$$A_1 = -A_2 = A_3 = \cdots = (-1)^{n+1} \cdot A_{n+2}.$$

Proof: The proof will use some facts from algebraic topology. (See e.g. [8, Chapter VIII, Section 4].) Let M and N be two n-dimensional manifolds, which are compact and orientable. Furthermore let N be a connected manifold. For any continuous function $f : M \longrightarrow N$, one can define the degree $\deg(f)$. Moreover, for any $q \in N$, one can define the local degree $\deg_q(f)$. In the event when f is locally a homeomorphism around a small enough neighbourhood of each point of the fiber $f^{-1}(q)$, the local degree $\deg_q(f)$ may be viewed as "number of points in the fiber $f^{-1}(q)$, counted with multiplicities", where the multiplicity is 1 if the function f at that point in the fiber preserves orientation, and -1 otherwise. Finally, the local degree $\deg_q(f)$ is equal to the degree $\deg(f)$.

A coloring of the vertices of a polytope P is a function

$$\phi : \mathrm{vert}(P) \longrightarrow \mathrm{vert}(\triangle^{n+1}),$$

where $\mathrm{vert}(P)$ denotes the vertex set of P and $\triangle^{n+1}$ is the standard $(n+1)$-simplex with vertices $e_1, \ldots, e_{n+2}$. We can extend ϕ to a continuous map

$$f : \partial P \longrightarrow \partial\triangle^{n+1},$$

from the surface of P to the surface of $\triangle^{n+1}$ as follows. Every $p \in \partial P$ can be written uniquely as the barycentric combination of the vertices of the smallest face (say $\mathrm{conv}(\{v_1, \ldots, v_k\})$) containing p:

$$p = \sum_{i=1}^{k} \alpha_i \cdot v_i \text{ where all } \alpha_i \geq 0 \text{ and } \sum_{i=1}^{k} \alpha_i = 1.$$

Now define $f(p)$ by

$$f(p) := \sum_{i=1}^{k} \alpha_i \cdot \phi(v_i).$$

Clearly f is a continuous function between two homeomorphic images of $\mathbb{S}^n$. Observe that the sphere $\mathbb{S}^n$ is an oriented n-dimensional manifold, which is both compact and connected.

Consider a point q_i in the interior of the facet $\mathrm{conv}(\{e_1, \ldots, e_{n+2}\} \setminus \{e_i\})$, and let us compute the local degree $\deg_{q_i}(f)$. The fiber $f^{-1}(q_i)$ is a finite set, and there is exactly one element of $f^{-1}(q_i)$ in each facet of P that has vertices of colors $\{1, 2, \ldots, n+2\} \setminus \{i\}$. Hence the local degree $\deg_{q_i}(f)$ is equal to the number of facets in P with colors

$\{1, \ldots, n+2\} \setminus \{i\}$, counting multiplicities. The multiplicity is 1 if the orientation of the facet containing the element in the fiber $f^{-1}(q_i)$ has its vertices colored in the same orientation as $\text{conv}(\{e_1, \ldots, e_{n+2}\} \setminus \{e_i\})$, and the multiplicity is -1 otherwise. Hence we have

$$\deg_{q_i}(f) = A_i \cdot (-1)^{n-i},$$

because the orientation of $\text{conv}(\{e_1, \ldots, e_{n+2}\} \setminus \{e_i\})$ in $\triangle^{n+1}$ is $(-1)^{n-i}$. Therefore we obtain

$$(-1)^{n-i} \cdot A_i = \deg_{q_i}(f) = \deg(f) \quad \text{for } i = 1, 2, \ldots, n+2,$$

and this implies the theorem. **QED**

Theorem 3 (Generalized Baxter's theorem for compact orientable manifolds) *Let M be a compact oriented n-dimensional manifold. Let τ be a finite triangulation (i.e. ordered simplicial atlas) of M. Assign arbitrarily the colors $1, \ldots, n+2$ to the vertices of τ. Let A_i be the number of facets of τ with vertices colored with $\{1, \ldots, n+2\} \setminus \{i\}$, where we count a facet with multiplicity 1 or -1 when the sign of the multiplicity is the sign of the orientation of the vertices of the facet listed in increasing order of colors. Then we have*

$$A_1 = -A_2 = A_3 = \cdots = (-1)^{n+1} \cdot A_{n+2}.$$

In the previous proof the fact that our triangulated manifold is the surface of a convex simplicial polytope was only used when we extended the coloring of the vertices to a continuous function between the manifold and $\partial\triangle^{n+1}$. Hence, in order to obtain the desired generalization, the only thing we have to do is to extend the coloring $\phi : \text{vert}(\tau) \longrightarrow \text{vert}(\triangle^{n+1})$ to a continuous function $f : M \longrightarrow \partial\triangle^{n+1}$.

It is a well-known fact that any map between the vertices of two simplicial spaces that takes faces into faces can be extended uniquely to a simplicial (and hence continuous) map between the same two spaces. (See, e.g., [8, Chapter V, Proposition 7.11].) In our case the triangulation of M defines a simplicial space, and the simplex $\triangle^{n+1}$ is trivially a simplicial space. The rest of the proof goes through exactly as in the case of convex polytopes.

Note that the results of this subsection hold evidently for colored triangulations of disjoint unions of compact orientable n-dimensional manifolds.

2.2.2 Orientable simplicial pseudomanifolds

Let us recall the most important definitions.

Definition 14 *An* n-dimensional simplicial pseudomanifold *is a simplicial complex* $\triangle$ *satisfying the following conditions:*

(i) every facet of $\triangle$ *is* n*-dimensional,*

(ii) every subfacet is contained in at most two facets,

(iii) if F *and* F' *are facets of* $\triangle$ *then there is a sequence of facets* $F = F^1, F^2, \ldots, F^m = F'$ *such that* F^i *and* F^{i+1} *have a subfacet in common.*

We call the subcomplex generated by the subfacets contained in exactly one facet the boundary *of* $\triangle$*, and we denote it by* $\partial\triangle$*. If* $\partial\triangle = \emptyset$*, then we call* $\triangle$ *a* simplicial pseudomanifold without boundary*. For a boundary subfacet* σ *we denote the unique facet containing it by* $\Omega(\sigma)$*.*

We will define orientation on ordered faces.

Definition 15 *An* ordered face *is a list* $(u_1, \ldots, u_{k+1})$*, where* $\{u_1, \ldots, u_{k+1}\}$ *is a* k*-face. We denote the set of ordered* k*-faces by* $\mathrm{Ord}_k(\triangle)$*. If* $\triangle$ *is a simplicial pseudomanifold and* $\sigma = \{u_1, \ldots, u_n\}$ *is a boundary subfacet, then we denote by* $\Omega(u_1, \ldots, u_{n+1})$ *the ordered facet* $(u_1, \ldots, u_n, u_{n+1})$ *where* u_{n+1} *is the only element of* $\Omega(\sigma) \setminus \sigma$*.*

Now we are able to define orientable simplicial pseudomanifolds.

Definition 16 *Let $\triangle$ an n-dimensional simplicial pseudomanifold. We call $\triangle$* orientable *when there exists a map $\varepsilon : \mathrm{Ord}_n(\triangle) \longrightarrow \mathbb{Z}$ such that the following hold:*

(i) For every facet $F = \{v_1, \ldots, v_{n+1}\}$ we have

$$\varepsilon(v_1, v_2, \ldots, v_{n+1}) = \pm 1.$$

(ii) For every permutation $\pi \in S_{n+1}$ and every facet $\{v_1, \ldots, v_{n+1}\}$ we have

$$\varepsilon(v_1, v_2, \ldots, v_{n+1}) = \mathrm{sign}(\pi) \cdot \varepsilon(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n+1)}),$$

where $\mathrm{sign}(\pi)$ is -1 for odd and 1 for even permutations.

(iii) For every non-boundary subfacet $\{v_1, \ldots, v_n\} \notin \partial\triangle$ and the two facets $\{v_1, v_2, \ldots, v_n, v'_{n+1}\}$ and $\{v_1, \ldots, v_n, v''_{n+1}\}$ containing it, we have

$$\varepsilon(v_1, \ldots, v_n, v'_{n+1}) = -\varepsilon(v_1, \ldots, v_n, v''_{n+1}).$$

We call ε an orientation *of $\triangle$.*

It is an easy consequence of Definition 16 and of condition (iii) of Definition 14 that an orientable simplicial pseudomanifold has exactly two orientations which differ only by a constant factor of -1.

In the following we will investigate the number of those facets and subfacets of a simplicial pseudomanifold, which are colored in a prescribed way.

Definition 17 *Let $I = (i_1, \ldots, i_{n+1})$ be an ordered $(n+1)$-tuple of $n+1$ distinct colors, and let $\triangle$ be an n-dimensional oriented simplicial pseudomanifold. We will say that an* *ordered facet* $(v_1, \ldots, v_{n+1})$ *of* $\triangle$ *is* I-colored *when the color of* v_t *is* i_t *for* $t = 1, 2, \ldots, n+1$. *We denote the fact that* $(v_1, \ldots, v_{n+1})$ *is* I*-colored by* $(v_1, \ldots, v_{n+1}) \mapsto I$ *We define*

$$A_I := \sum_{(v_1, \ldots, v_{n+1}) \mapsto I} \varepsilon(v_1, \ldots, v_{n+1}).$$

Similarly, for $J = (j_1, \ldots, j_n)$ *we call an ordered boundary subfacet* $(u_1, \ldots, u_n)$ J*-colored when the color of* u_t *is* j_t *for* $t = 1, 2, \ldots, n$. *Again, we denote the fact that* $(u_1, \ldots, u_n)$ *is* J*-colored by* $(u_1, \ldots, u_n) \mapsto J$. *We define*

$$B_J := \sum_{(u_1, \ldots, u_n) \mapsto J} \varepsilon(\Omega(u_1, \ldots, u_n)).$$

In the following we will investigate the relations between the numbers A_I and B_J for an arbitrarily given coloring. Note that an analogous situation to that of Theorem 3 may be included in this setting, as the special case when $\triangle$ is an n-dimensional oriented pseudomanifold without boundary, and $\triangle$ is colored with the color set $\{1, 2, \ldots, n+2\}$. Then the numbers B_J will all be zero, and the color set will have $n+2$ different $(n+1)$-subsets, each of which can be written in the form $\{1, 2, \ldots, n+2\} \setminus \{i\}$. The number $A_{(1,2,\ldots,i-1,i+1,\ldots,n+2)}$ will be the analogue of the number A_i in the generalized Baxter's theorem. We will pay special attention to colorings of oriented simplicial pseudomanifolds without boundary. For these pseudomanifolds all the B_J-s are zero, thus we restrict our attention to the relations among the A_I-s.

The following equalities are straightforward consequences of condition (ii) of Definition 16.

Lemma 3 *We have*

$$A_{(i_{\pi(1)}, \ldots, i_{\pi(n+1)})} = \operatorname{sign}(\pi) \cdot A_{(i_1, \ldots, i_{n+1})} \tag{2.1}$$

for all permutations $\pi \in \mathcal{S}_{n+1}$, *and*

$$B_{(j_{\tau(1)},\dots,j_{\tau(n)})} = \text{sign}(\tau) \cdot B_{(j_1,\dots,j_n)} \tag{2.2}$$

for all permutations $\tau \in \mathcal{S}_n$.

The simplest nontrivial situation where nonzero A_I-s and B_J-s might arise is when we color with $n+1$ colors. The following lemma deals with this case.

Lemma 4 (Fundamental coloring lemma for orientable simplicial pseudomanifolds) *Let* $\triangle$ *be an oriented* n*-dimensional simplicial pseudomanifold. Color the vertices of* $\triangle$ *arbitrarily with the color set* $\{1, 2, \dots, n+1\}$. *Let us fix two permutations* $\alpha = (\alpha_1, \dots, \alpha_{n+1})$ *and* $\beta = (\beta_1, \dots, \beta_{n+1})$ *of* $\{1, 2, \dots, n+1\}$. *Then we have*

$$\text{sign}(\alpha) \cdot A_\alpha = \text{sign}(\beta) \cdot B_{(\beta_1,\dots,\beta_n)}. \tag{2.3}$$

In particular, when $\triangle$ *is a simplicial pseudomanifold without boundary, then*

$$A_\alpha = 0 \tag{2.4}$$

holds.

Proof: By Lemma 3, the equation does not change if we replace α or β by other permutations. Thus, without loss of generality we may assume $\alpha = \beta = \text{id}$, that is, $\alpha_i = \beta_i = i$ for all $i = 1, \dots, n+1$.

We construct a graph $G = (V, E)$ associated to $\triangle$ and its coloring. The vertex set is defined by

$$V := \{(u_1, \dots, u_{n+1}) \in \text{Ord}_n(\triangle) \, : \, \text{color}(u_i) = i \text{ for } i = 1, 2, \dots, n\}.$$

Note that the color of u_{n+1} is arbitrary, and so V is the disjoint union of

$$V_1 = \{(u_1, \ldots, u_{n+1}) \in V \ : \ \text{color}(u_{n+1}) = n+1\},$$

and

$$V_2 = \{(u_1, \ldots, u_{n+1}) \in V \ : \ \text{color}(u_{n+1}) \neq n+1\}.$$

Furthermore, we can split both V_1 and V_2 into the disjoint union of smaller sets. We have

$$V_i = V_i' \uplus V_i'' \ \ (i = 1, 2)$$

where

$$V_i' = \{(u_1, \ldots, u_{n+1}) \in V_i \ : \ \{u_1, \ldots, u_n\} \notin \partial\triangle\},$$

and

$$V_i'' = \{(u_1, \ldots, u_{n+1}) \in V_i \ : \ \{u_1, \ldots, u_n\} \in \partial\triangle\}$$

We define the edge set of G as a disjoint union $E := E_1 \cup E_2$ where

$$E_1 := \{((u_1, \ldots, u_n, s), (u_1, \ldots, u_n, t)) \ : \ s \neq t\},$$

and

$$E_2 := \{((u_1, \ldots, u_{i-1}, s, u_{i+1} \ldots, u_n, t), (u_1, \ldots, u_{i-1}, t, u_{i+1} \ldots, u_n, s)) : s \neq t, 1 \leq i \leq n\}.$$

Observe that the ε-values on adjacent vertices of G have opposite signs. In fact, for edges in E_1 resp. E_2, this follows from condition (iii) resp. (ii) of Definition 16. Observe furthermore that every vertex is adjacent to at most one other vertex in E_1. In fact, for every $\underline{u} = (u_1, \ldots, u_{n+1}) \in V$, there is at most one facet different from $\{u_1, \ldots, u_{n+1}\}$ that contains the subfacet $\{u_1, \ldots, u_n\}$. The vertex $\underline{u}$ has no neighbor in E_1 if and only if $\{u_1, \ldots, u_n\}$ belongs to the boundary, i.e. iff $\underline{u} \in V_1'' \cup V_2''$. Note at last that E_2 is a matching on the set V_2. In fact, if $\underline{u} = (u_1, \ldots, u_{n+1}) \in V_2$ then there is exactly one way to exchange u_1 with the unique u_i having the same color.

Therefore the degree of a vertex in E_1 or E_2, depending on the fact, to which subset of V it belongs, can be found in the following table.

Degree in	V_1'	V_1''	V_2'	V_2''
E_1	1	0	1	0
E_2	0	0	1	1

Since every vertex in the graph G has degree at most 2, the graph G is a disjoint union of singletons, paths and cycles. The singletons are the elements of the set V_1''. The cycles lie inside the set V_2'. The two sets V_1' and V_2'' contain the the endpoints of all the paths, and the internal nodes of all the paths lie in V_1''.

Observe that a path that connects two vertices $x, y \in V_1'$ will have odd length. Hence we know that $\varepsilon(x) = -\varepsilon(y)$. Similarly, a path that connects two vertices in V_2'' will also have odd length and thus the two endpoints of the path have opposite signs. But a path that connects a vertex in V_1' with a vertex in V_2'' will have even length. The two end points of such a path will have the same sign. We conclude that

$$\sum_{x \in V_1'} \varepsilon(x) = \sum_{x \in V_2''} \varepsilon(x).$$

The $(1, \ldots, n+1)$-colored ordered facets are represented in the graph G by the vertices in V_1. Hence

$$A_{(1,\ldots,n+1)} = \sum_{x \in V_1} \varepsilon(x).$$

Similarly the $(1,\ldots,n)$-colored boundary faces are represented by the vertices in $V_1'' \uplus V_2''$, and thus

$$B_{(1,\ldots,n)} = \sum_{x \in V_1'' \uplus V_2''} \varepsilon(x).$$

By combining the three above equations we get

$$\begin{aligned} A_{(1,\ldots,n+1)} &= \sum_{x \in V_1'} \varepsilon(x) + \sum_{x \in V_1''} \varepsilon(x) \\ &= \sum_{x \in V_2''} \varepsilon(x) + \sum_{x \in V_1''} \varepsilon(x) \\ &= B_{(1,\ldots,n)}. \end{aligned}$$

Observe that the paths and the singletons in the graph G describe a bijection between the signed set of $(1,\ldots,n+1)$-colored facets and the signed set of $(1,\ldots,n)$-colored subfacets. Hence the proof is bijective. **QED**

From this lemma we obtain the strongest possible master theorem in a surprisingly straightforward way.

Theorem 4 (Master theorem for colored triangulations)

Let $\triangle$ be an oriented n-dimensional simplicial pseudomanifold and $m \geq n+1$. Color the vertices of $\triangle$ arbitrarily with the color set $\{1, 2, \ldots, m\}$. Let $\{C_1, C_2, \ldots, C_{n+1}\}$ be a partition of the colors $\{1, \ldots, m\}$ into $n+1$ blocks. Then we have

$$\sum_{i_1 \in C_1, \ldots, i_{n+1} \in C_{n+1}} A_{(i_1,\ldots,i_{n+1})} = \sum_{j_1 \in C_1, \ldots, j_n \in C_n} B_{(j_1,\ldots,j_n)}. \tag{2.5}$$

In particular, for n-dimensional oriented simplicial pseudomanifolds without boundary we have

$$\sum_{i_1 \in C_1, \ldots, i_{n+1} \in C_{n+1}} A_{(i_1, \ldots, i_{n+1})} = 0. \tag{2.6}$$

Proof: Construct a function $\lambda : \{1, \ldots, m\} \longrightarrow \{1, \ldots, n+1\}$ by setting $\lambda(i) = j$ if $i \in C_j$. A coloring of the vertices of $\triangle$ can be viewed as a function $\phi : \text{vert}(\triangle) \longrightarrow \{1, \ldots, m\}$. Hence, the composition $\lambda \circ \phi$ is also a coloring of $\triangle$, but with $n+1$ colors. Apply Lemma 4 with $\alpha = \beta = (1, 2, \ldots, n)$ to this coloring. We obtain $A^{\lambda \circ \phi}_{(1,2,\ldots,n+1)} = B^{\lambda \circ \phi}_{(1,2,\ldots,n)}$. Obviously, $A^{\lambda \circ \phi}_{(1,2,\ldots,n+1)}$ is equal to the left hand side, and $B^{\lambda \circ \phi}_{(1,2,\ldots,n)}$ is equal to the right hand side of (2.5). **QED**

Examples

1. When $m = n+2$, then a partition of the color set $\{1, 2, \ldots, n+2\}$ into $n+1$ blocks consists of n singletons and a block of size 2. If we assume that the only 2-element block is $\{i, i+1\}$, then in the case of a simplicial pseudomanifold without boundary, equation (2.6) gives us

   $$A_{(i,1,2,\ldots,i-1,i+1,\ldots,n+1)} + A_{(i+1,1,2,\ldots,i,i+2,\ldots,n+1)} = 0$$

   which is equivalent to $A_i = -A_{i+1}$ in the formulation of Theorem 3.

2. Consider the coloring of an $(n+1)$-dimensional convex simplicial polytope P with $n+2$ colors. Let us divide P with hyperplanes spanned by vertices of P into $(n+1)$-dimensional simplices. This way we obtain an $(n+1)$-dimensional oriented simplicial pseudomanifold $\triangle$, for which $\partial\triangle$ consists of the facets of ∂P. Lemma 4 gives us yet another proof of the generalized Baxter's theorem for convex polytopes. Note that this time the numbers $B_{j_1,\ldots,j_{n+1}}$ will agree up to sign with the numbers

A_i of Theorem 2.

Observe that in the proofs of this subsection we never used the connectedness property of simplicial pseudomanifolds (condition (iii) in Definition 14). Hence the results hold also for disjoint unions of n-dimensional orientable pseudomanifolds, or even for those unions of n-dimensional orientable pseudomanifolds in which any two pseudomanifolds intersect in less than $(n-1)$-dimensional faces.

2.2.3 The linear space of conditions on A_I-s and B_J-s

In this subsection we show that Theorem 4 exhausts all linear relations among the numbers $A_{(i_1,\ldots,i_{n+1})}$ and $B_{(j_1,\ldots,j_n)}$.

Let us introduce for notational convenience symbols $A_{(i_1,\ldots,i_{n+1})}$ resp. $B_{(j_1,\ldots,j_n)}$ for all vectors $(i_1,\ldots,i_{n+1})$ resp. $(j_1,\ldots,j_n)$ with entries in $\{1,2,\ldots,m\}$, agreeing immediately that $A_{(i_1,\ldots,i_{n+1})} = 0$ resp. $B_{(j_1,\ldots,j_n)} = 0$ whenever $(i_1,\ldots,i_{n+1})$ resp. $(j_1,\ldots,j_n)$ has repeated entries.

Then we can define

$$A_{C_1,\ldots,C_{n+1}} := \sum_{i_1\in C_1,\ldots,i_{n+1}\in C_{n+1}} A_{(i_1,\ldots,i_{n+1})},$$

and

$$B_{C_1,\ldots,C_n} := \sum_{j_1\in C_1,\ldots,j_n\in C_n} B_{(j_1,\ldots,j_n)}$$

for arbitrary $n+1$-tuples of sets $C_1,\ldots,C_{n+1}$.

Thus (2.5) may be written as

$$A_{C_1,\ldots,C_{n+1}} = B_{C_1,\ldots,C_n} \tag{2.7}$$

for all partitions of the set $\{1,2,\ldots,m\}$ into $n+1$ blocks $C_1,\ldots,C_{n+1}$.

Lemma 5 *The system of equations (2.5) is equivalent to the following:*

$$\sum_{i=1}^{m} A_{(i_1,i_2,\dots,i_n,i)} = B_{(i_1,\dots,i_n)} \tag{2.8}$$

for all n-element subsets $\{i_1,\dots,i_n\} \subset \{1,2,\dots,m\}$.

In the boundariless case, the equations (2.6) are equivalent to

$$\sum_{i=1}^{m} A_{(i_1,i_2,\dots,i_n,i)} = 0 \tag{2.9}$$

for all n-element subsets $\{i_1,\dots,i_n\} \subset \{1,2,\dots,m\}$.

Proof: We show the statement for oriented simplicial pseudomanifolds in general; the reasoning is essentially the same in the boundariless case. Clearly, the equations of (2.8) are special case of (2.7), with $C_1 = \{i_1\}, C_2 = \{i_2\}, \dots, C_n = \{i_n\}$ and $C_{n+1} = \{1,2,\dots,m\} \setminus \{i_1,\dots,i_n\}$. Thus we only have to show that (2.8) implies (2.7).

It is an easy consequence of (2.2) that $A_{C_1,\dots,C_{n+1}} = 0$ whenever two $C_i - s$ are equal. Hence we may write

$$A_{C_1,\dots,C_{n+1}} = \sum_{j=1}^{n+1} \underbrace{A_{C_1,\dots,C_n,C_j}}_{0 \quad \text{if } j\neq n+1} = \sum_{i=1}^{m} A_{C_1,\dots,C_n,\{i\}} = \sum_{(i_1,\dots,i_n)\in C_1\times\cdots\times C_n} \left(\sum_{i=1}^{m} A_{(i_1,\dots,i_n,i)}\right).$$

On the other hand by the definition of $B_{C_1,\dots,C_n}$ we have

$$B_{C_1,\dots,C_n} = \sum_{j_1\in C_1,\dots,j_n\in C_n} B_{(j_1,\dots,j_n)} = \sum_{(j_1,\dots,j_n)\in C_1\times\cdots\times C_n} B_{(j_1,\dots,j_n)}.$$

Therefore (2.7) is the sum of equations of the form (2.8). **QED**

Definition 18 *Let $\mathcal{V}$ be an m-dimensional vector space with basis $e_1,\dots,e_m$. Given a coloring $\phi : \text{vert}(\triangle) \longrightarrow \{1,2,\dots,m\}$ of an orientable n-dimensional simplicial pseudo-*

manifold $\triangle$*, we define the* weight vector *of the coloring as the following skew-symmetric tensor*

$$\mathbf{w}_\phi := \sum_{i_1<\cdots<i_{n+1}} A_{(i_1,\ldots,i_{n+1})} \cdot e_{i_1} \wedge \cdots \wedge e_{i_{n+1}} + \sum_{j_1<\cdots<j_n} B_{(j_1,\ldots,j_n)} \cdot e_{j_1} \wedge \cdots \wedge e_{j_n},$$

which lies in $\mathrm{Ext}_{n+1}(\mathcal{V}) \oplus Ext_n(\mathcal{V})$*. We denote the* $\mathbb{Z}$*-module resp. vector space generated by all weight vectors by* $\mathcal{M}$ *resp.* $\mathcal{W}$*. Moreover, the submodule resp. subspace generated by the weight vectors of colorings of oriented* n*-dimensional simplicial pseudomanifolds without boundary will be denoted by* $\mathcal{M}_0$ *resp.* $\mathcal{W}_0$*. (Observe that* $\mathcal{M}_0$ *and* $\mathcal{W}_0$ *are contained in* $\mathrm{Ext}_{n+1}(\mathcal{V})$*.)*

Note that conditions (2.2), (2.1), and our convention about $A_{(i_1,\ldots,i_{n+1})}$ resp. $B_{(j_1,\ldots,j_n)}$ being zero in case of repeated indices allows us to think of $A_{(i_1,\ldots,i_{n+1})}$ as the coefficient of $e_{i_1}\wedge\cdots\wedge e_{i_{n+1}}$, and of $B_{(j_1,\ldots,j_n)}$ as the coefficient of $e_{j_1}\wedge\cdots\wedge e_{j_n}$, for arbitrary $(i_1,\ldots,i_{n+1})$ and $(j_1,\ldots,j_n)$.

A linear condition on the A_I-s and B_J-s may be transcribed as

$$\langle \mathbf{v}^* \mid \mathbf{w}_\phi \rangle = 0$$

for all vectors $\mathbf{w}_\phi$ and a fixed $\mathbf{v}^* \in (\mathrm{Ext}_{n+1}(\mathcal{V}) \oplus \mathrm{Ext}_n(\mathcal{V}))^* = (\mathrm{Ext}_{n+1}(\mathcal{V}))^* \oplus (\mathrm{Ext}_n(\mathcal{V}))^*$. When we restrict ourselves to pseudomanifolds without boundary, we may take v^* to be an element of $(\mathrm{Ext}_{n+1}(\mathcal{V}))^*$.

It is a well-known fact that the dual space $(\mathrm{Ext}_k(\mathcal{V}))^*$ is isomorphic to $\mathrm{Ext}_k(\mathcal{V}^*)$, where we define the value of $u_1^* \wedge \cdots \wedge u_k^* \in \mathrm{Ext}_k(\mathcal{V}^*)$ on $v_1 \wedge \cdots \wedge v_k \in \mathrm{Ext}_k(\mathcal{V})$ to be the determinant of the matrix $(\langle u_i^* \mid v_j \rangle)_{i,j=1}^k$. Using this isomorphism for $k = n, n+1$, and introducing the dual basis $e_i^* \in \mathcal{V}^*$ defined as usual by $\langle e_i^* \mid e_j \rangle := \delta_{i,j}$, we can write the equations (2.8) in the following equivalent form:

$$\left\langle e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \wedge (e_1^* + e_2^* + \cdots + e_m^* + 1) \,\middle|\, \mathbf{w}_\phi \right\rangle = 0 \tag{2.10}$$

for all $\{i_1, \ldots, i_n\} \subset \{1, 2, \ldots, m\}$ and all weight vectors $\mathbf{w}_\phi$. Similarly, the equations (2.9) are equivalent to all conditions of the form

$$\left\langle e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \wedge (e_1^* + e_2^* + \cdots + e_m^*) \,\middle|\, \mathbf{w}_\phi \right\rangle = 0. \tag{2.11}$$

Lemma 6 *The vectors* $e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \wedge (e_1^* + e_2^* + \cdots + e_m^* + 1)$ *span an* $\binom{m}{n}$*-dimensional subspace of* $\mathrm{Ext}_{n+1}(\mathcal{V}^*) \oplus \mathrm{Ext}_n(\mathcal{V}^*)$*. Furthermore, the dimension of the subspace of* $\mathrm{Ext}_{n+1}(\mathcal{V}^*)$ *generated by the vectors* $e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \wedge (e_1^* + e_2^* + \cdots + e_m^*)$ *is* $\binom{m-1}{n}$.

Proof: Note that the linear span of the vectors $e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \wedge (e_1^* + e_2^* + \cdots + e_m^* + 1)$ is exactly the image of the linear map

$$\begin{array}{ccc} \mathrm{Ext}_n(\mathcal{V}^*) & \longrightarrow & \mathrm{Ext}_{n+1}(\mathcal{V}^*) \oplus \mathrm{Ext}_n(\mathcal{V}^*) \\ u_1^* \wedge \cdots \wedge u_n^* & \longmapsto & u_1^* \wedge \cdots \wedge u_n^* \wedge (e_1^* + \cdots + e_m^* + 1). \end{array}$$

This map is injective, for its projection onto $\mathrm{Ext}_n(\mathcal{V}^*)$ is the identity. This shows the first statement of the lemma. On the other hand, the linear span of the vectors $e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \wedge (e_1^* + e_2^* + \cdots + e_m^*)$ is equal to the image of the linear map

$$\begin{array}{ccc} \mathrm{Ext}_n(\mathcal{V}^*) & \longrightarrow & \mathrm{Ext}_{n+1}(\mathcal{V}^*) \\ u_1^* \wedge \cdots \wedge u_n^* & \longmapsto & u_1^* \wedge \cdots \wedge u_n^* \wedge (e_1^* + \cdots + e_m^*). \end{array}$$

If we take now another basis $f_1^*, \ldots, f_m^*$ of $\mathcal{V}^*$ with $f_m^* = e_1^* + \cdots + e_m^*$, then the image of the above linear map will be the linear span of

$$\{f_{i_1}^* \wedge \cdots \wedge f_{i_n}^* \wedge f_m^* \,:\, \{i_1, \ldots, i_n\} \subseteq \{1, 2, \ldots, m-1\}\}$$

which is clearly $\binom{m-1}{n}$-dimensional. **QED**

Corollary 3 *We have* $\dim \mathcal{W} \leq \binom{m}{n+1}$, *and* $\dim \mathcal{W}_0 \leq \binom{m-1}{n+1}$.

Proof: Let

$$\mathcal{X} = \text{span}\{e^*_{i_1} \wedge \cdots \wedge e^*_{i_n} \wedge (e^*_1 + e^*_2 + \cdots + e^*_m + 1) \ : \ \{i_1, \ldots, i_n\} \subset \{1, 2, \ldots, m\}\}.$$

Thus $\mathcal{X}$ is a subspace of $\text{Ext}_{n+1}(\mathcal{V}^*) \oplus \text{Ext}_n(\mathcal{V}^*)$. Define the orthogonal space $\mathcal{X}^\perp$ by

$$\mathcal{X}^\perp = \{\mathbf{v} \in \text{Ext}_{n+1}(\mathcal{V}) \oplus \text{Ext}_n(\mathcal{V}) \ : \ \forall \mathbf{v}^* \in \mathcal{X}, \ \langle \mathbf{v}^* \mid \mathbf{v} \rangle = 0\}.$$

By (2.10) we have that $\mathcal{W} \subseteq \mathcal{X}^\perp$. Hence

$$\begin{aligned} \dim \mathcal{W} &\leq \dim \mathcal{X}^\perp \\ &= \dim (\text{Ext}_{n+1}(\mathcal{V}) \oplus \text{Ext}_n(\mathcal{V})) - \dim \mathcal{X} \\ &= \binom{m}{n+1} + \binom{m}{n} - \binom{m}{n} \\ &= \binom{m}{n+1}. \end{aligned}$$

Similarly, let

$$\mathcal{X}_0 = \text{span}\{e^*_{i_1} \wedge \cdots \wedge e^*_{i_n} \wedge (e^*_1 + e^*_2 + \cdots + e^*_m) \ : \ \{i_1, \ldots, i_n\} \subset \{1, 2, \ldots, m\}\},$$

and define the orthogonal space $\mathcal{X}_0^\perp \subseteq \text{Ext}_{n+1}(\mathcal{V})$. By (2.11) we know that $\mathcal{W}_0 \subseteq \mathcal{X}_0^\perp$. Thus

$$\begin{aligned} \dim \mathcal{W}_0 &\leq \dim \mathcal{X}_0^\perp \\ &= \dim \text{Ext}_{n+1}(\mathcal{V}) - \dim \mathcal{X}_0 \end{aligned}$$

$$= \binom{m}{n+1} - \binom{m-1}{n}$$
$$= \binom{m-1}{n+1}.$$

QED

Definition 19 *Let $\{i_1, \ldots, i_{n+2}\}$ be an arbitrary $n+2$ element subset of $\{1, \ldots, m\}$. Color the vertices of the standard $n+1$ simplex $\triangle^{n+1}$ with the colors $\{i_1, \ldots, i_{n+2}\}$. Consider this coloring as a coloring of the n-dimensional simplicial pseudomanifold without boundary $\partial\triangle^{n+1}$, and denote the weight vector of this coloring by $\mathbf{p}_{(i_1,\ldots,i_{n+2})}$. Let us call these weight vectors $\mathbf{p}_{(i_1,\ldots,i_{n+2})}$* elementary weight vectors of the first kind.

Let $\{j_1, \ldots, j_{n+1}\}$ be an arbitrary $n+1$ element subset of $\{1, \ldots, m\}$. Consider the coloring of a standard n-simplex with the colors $\{j_1, \ldots, j_{n+1}\}$. Let us denote the weight vector of this coloring by $\mathbf{q}_{(j_1,\ldots,j_{n+1})}$. We call the weight vectors $\mathbf{q}_{(j_1,\ldots,j_{n+1})}$ elementary weight vectors of the second kind.

Obviously $\mathbf{p}_{(i_1,\ldots,i_{n+2})}$ is a linear combination of skew tensors of the form $e_{i_1} \wedge \cdots \wedge e_{i_{s-1}} \wedge e_{i_{s+1}} \wedge \cdots \wedge e_{i_{n+2}}$, where $s \in \{1, 2, \ldots, n+2\}$. The coefficient of each of the above-mentioned skew tensors is ± 1. Similarly, $\mathbf{q}_{(j_1,\ldots,j_{n+1})}$ is the linear combination of $e_{j_1} \wedge \cdots \wedge e_{j_{n+1}}$ and skew tensors of the form $e_{j_1} \wedge \cdots \wedge e_{j_{t-1}} \wedge e_{j_{t+1}} \wedge \cdots \wedge e_{j_{n+1}}$, where $t \in \{1, 2, \ldots, n+1\}$. The coefficients of these tensors are also ± 1. Observe that elementary weight vectors of the first kind belong to oriented simplicial pseudomanifolds without boundary.

Lemma 7 *The set $M_0 := \{\mathbf{p}_{(i_1,\ldots,i_{n+1},m)} : \{i_1, \ldots, i_{n+1}\} \subseteq \{1, 2, \ldots, m-1\}\}$ is a $\mathbb{Z}$-basis of $\mathcal{M}_0$, and the set $M := M_0 \cup \{\mathbf{q}_{(j_1,j_2,\ldots,j_n,m)} : \{j_1, \ldots, j_n\} \subseteq \{1, 2, \ldots, m-1\}\}$ is a $\mathbb{Z}$-basis of $\mathcal{M}$.*

Proof: First we show that M_0 is a $\mathbb{Z}$-basis of $\mathcal{M}_0$. Consider the antilexicographic order on the basis $\{e_{i_1} \wedge \cdots \wedge e_{i_{n+1}} : i_1 < i_2 < \cdots < i_{n+1}\}$ of $\mathrm{Ext}_{n+1}(\mathcal{V})$. That is, we set

$$e_{i_1} \wedge \cdots \wedge e_{i_{n+1}} \overset{\text{def}}{<} e_{j_1} \wedge \cdots \wedge e_{j_{n+1}}$$

when

$$i_{n+1} = j_{n+1}, i_n = j_n, \ldots, i_{s+1} = j_{s+1} \text{ and } i_s < j_s \text{ for some } s \in \{1, \ldots, n+1\}.$$

With respect to this order, the smallest term of $\mathbf{p}_{(i_1,\ldots,i_{n+2})}$ for $i_1 < i_2 < \cdots < i_{n+2}$ is $e_{i_1} \wedge \cdots \wedge e_{i_{n+1}}$. Hence the antilexicographically first terms of the elements of M_0 are pairwise different, and so M_0 is an independent set. Thus we have only to show that M_0 also generates the $\mathbb{Z}$-module $\mathcal{M}$. Assume by way of contradiction that there is a $\mathbf{r} \in \mathcal{M}$ which is not a $\mathbb{Z}$-linear combination of elements from M. We can take $\mathbf{r}$ to be a counterexample that has the largest possible antilexicographically first term. Then the first term of $\mathbf{r}$ cannot be a multiple of $e_{i_1} \wedge \cdots \wedge e_{i_n} \wedge e_{i_{n+1}}$ with $i_1 < \cdots < i_{n+1} \leq m-1$, because then we can subtract a multiple of $\mathbf{p}_{(i_1,\ldots,i_{n+1},m)}$ and get a counterexample with larger antilexicographically first term. Hence we can suppose that

$$\mathbf{r} = \sum_{\{i_1,\ldots,i_n\} \subseteq \{1,2,\ldots,m-1\}} A_{(i_1,\ldots,i_n,m)} \cdot e_{i_1} \wedge \cdots \wedge e_{i_n} \wedge e_m.$$

The vector $\mathbf{r}$ is the linear combination of weight vectors, so the equations (2.11) hold for $\mathbf{r}$, giving

$$A_{(i_1,\ldots,i_n,m)} = 0$$

for all $\{i_1, \ldots, i_n\} \subseteq \{1, 2, \ldots, m-1\}$ and so $\mathbf{r} = 0$, a contradiction.

The proof of the fact that M is a generating system of the $\mathbb{Z}$-module $\mathcal{M}$ is analogous to the above reasoning. We introduce the same antilexicographic order on the basis $\{e_{i_1} \wedge \cdots \wedge e_{i_{n+1}} : i_1 < i_2 < \cdots < i_{n+1}\}$ of $\mathrm{Ext}_{n+1}(\mathcal{V})$ as before. Again we look for a counterexample $\mathbf{r} \in \mathcal{M}$ that has a largest possible antilexicographically first term in $\mathrm{Ext}_{n+1}(\mathcal{V})$. (Note that $\mathbf{r}$ does not have to be an element of $\mathrm{Ext}_{n+1}(\mathcal{V})$, but it has a uniquely defined $\mathrm{Ext}_{n+1}(\mathcal{V})$-component). As before, we can show that the $\mathrm{Ext}_{n+1}(\mathcal{V})$-component of $\mathbf{r}$ is of the form

$$\sum_{\{i_1,\dots,i_n\}\subseteq\{1,2,\dots,m-1\}} A_{(i_1,\dots,i_n,m)}\cdot e_{i_1}\wedge\cdots\wedge e_{i_n}\wedge e_m.$$

But then, subtracting

$$\sum_{\{i_1,\dots,i_n\}\subseteq\{1,2,\dots,m-1\}} A_{(i_1,\dots,i_n,m)}\cdot \mathbf{q}_{(i_1,\dots,i_n,m)}$$

from $\mathbf{r}$, we obtain a $\tilde{\mathbf{r}}\in\mathcal{M}$ which is not generated by M, and has zero $\mathrm{Ext}_{n+1}(\mathcal{V})$-component. From the equations (2.10), which hold by linearity for all elements of $\mathcal{M}$, we obtain $\tilde{\mathbf{r}}=0$, a contradiction.

Finally, in order to show the independence of the elements of M, observe that the antilexicographically first term of the $\mathrm{Ext}_{n+1}(\mathcal{V})$-component of $\mathbf{q}_{(i_1,\dots,i_n,m)}$ is $e_{i_1}\wedge\cdots\wedge e_{i_n}\wedge e_m$, and so the antilexicographically first terms of $\mathrm{Ext}_{n+1}(\mathcal{V})$-components of M are pairwise different. **QED**

Corollary 4 *M_0 is a vector space basis of $\mathcal{W}_0$, M is a vector space basis of $\mathcal{W}$. Therefore we have*

$$\dim\mathcal{W}_0=\binom{m-1}{n+1},$$

and

$$\dim\mathcal{W}=\binom{m-1}{n+1}+\binom{m-1}{n}=\binom{m}{n+1}.$$

The previous lemmas add up to the proof of the fact that Theorem 4 and the sign-relations expressed in Lemma 3 imply all linear relations between the A_I-s and B_J-s in the general case as well as in the boundariless case.

Theorem 5 *The vector space of linear equations of the A_I-s and B_J-s is spanned by the equations (2.5), (2.1) and (2.2). Similarly, for oriented simplicial pseudomanifolds without boundary, all linear relations among the A_I-s are implied by the equations (2.6) and (2.1).*

Proof: By Corollary 3 the linear conditions of Theorem 4 allow $\mathcal{W}_0$ to be at most $\binom{m-1}{n+1}$-dimensional, and $\mathcal{W}$ to be at most $\binom{m}{n+1}$-dimensional. On the other hand, Corollary 4 guarantees that $\mathcal{W}_0$ has dimension $\binom{m-1}{n+1}$ and $\mathcal{W}$ has dimension $\binom{m}{n+1}$. Therefore there cannot be any additional linear conditions on the A_I-s and B_J-s, resp. A_I-s in either case. **QED**

As noted at the end of the last subsection, the weight vectors of colorings of disjoint unions of n-dimensional orientable simplicial pseudomanifolds also belong to $\mathcal{M}$. For such a disjoint union, the weight vector of the coloring is equal to the sum of the weight vectors of the restrictions of the coloring to the connected components. On the other hand, reversing the orientation on a component multiplies the weight vector belonging to that component by -1. From these elementary facts we can deduce that $\mathcal{M}$ is not simply the $\mathbb{Z}$-linear span of the weight vectors, but it is *equal as a set* to the set of weight vectors of colorings, when we allow disconnectedness. It is also true that the set of weight vectors of colorings of (connected) n-dimensional orientable pseudomanifolds is equal to $\mathcal{M}$. We leave the proof to the reader.

2.2.4 Simplicial homology

The results of Subsection 2.2.2 and 2.2.3 can be more easily proved using simplicial homology. In this subsection we outline how to do this, and how to get coloring theorems when we color with the vertices of a simplicial complex. A good reference about simplicial homology is [21]. Here is a reminder about the definitions.

Definition 20 *Let $\triangle$ be an arbitrary simplicial complex. We define $S_k(\triangle)$ to be the free $\mathbb{Z}$-module generated by the basis $\{\sigma : \sigma \in \triangle_k\}$. We represent $S_k(\triangle)$ as the factor of the free $\mathbb{Z}$-module generated by the ordered k-faces, and the following relations:*

$$\left[(v_{\pi(1)}, \ldots, v_{\pi(k+1)})\right] = \operatorname{sign}(\pi) \cdot [(v_1, \ldots, v_{k+1})]$$

for every $\pi \in \mathcal{S}_n$. (The symbol $\left[(v_{\pi(1)}, \ldots, v_{\pi(k+1)})\right]$ stands for the class represented by the ordered face $(v_1, \ldots, v_{k+1})$.) We turn $S_\bullet(\triangle)$ into a chain complex by defining the boundary map

$$\partial([(v_1, \ldots, v_{k+1})]) := \sum_{i=1}^{k+1} (-1)^{i-1} \cdot [(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{k+1})].$$

We call the homology groups $H_n(S_\bullet(\triangle))$ the (absolute) simplicial homology groups of $\triangle$.

We define the cochain complex $S^\bullet(\triangle)$ to be the dual complex $S_\bullet(\triangle)$, consisting of the dual $\mathbb{Z}$-modules $S^k(\triangle) = \operatorname{Hom}(S_k(\triangle), \mathbb{Z})$, and connected by the coboundary map δ^k, which is the adjoint of ∂_{k+1}. The cohomology groups of $S^\bullet(\triangle)$ are the (absolute) simplicial cohomology groups *of* $\triangle$.

A simplicial map $\phi : \triangle \longrightarrow \triangle'$ *is a map*

$$\phi : \operatorname{vert}(\triangle) \longrightarrow \operatorname{vert}(\triangle')$$

which takes faces into faces. We define the chain map induced by ϕ *to be the linear map satisfying*

$$S_k(\phi)([(v_1, \ldots, v_{k+1})]) = \begin{cases} [(\phi(v_1), \ldots, \phi(v_{k+1}))] & \text{if } \phi(v_1), \ldots, \phi(v_{k+1}) \text{ are distinct,} \\ 0 & \text{otherwise.} \end{cases}$$

Next let us recall the definition of relative simplicial homology.

Definition 21 *Given a subcomplex* $\tilde{\triangle}$ *of* $\triangle$, *we define the* relative *complex of* $\mathbb{Z}$*-modules* $S_\bullet(\triangle, \tilde{\triangle})$ *by*

$$S_\bullet(\triangle, \tilde{\triangle}) := S_\bullet(\triangle)/S_\bullet(\tilde{\triangle}),$$

where we embed $S_\bullet(\tilde{\triangle})$ *into* $S_\bullet(\triangle)$ *using the chain map induced by the inclusion* $\tilde{\triangle} \longrightarrow \triangle$.

We call the homology groups $H_n(S_\bullet(\triangle, \tilde{\triangle}))$ *the* relative simplicial homology groups of the pair $(\triangle, \tilde{\triangle})$. *We introduce the cochain complex* $S^\bullet(\triangle, \tilde{\triangle})$ *in perfect analogy to the absolute case. We call the cohomology groups of* $S^\bullet(\triangle, \tilde{\triangle})$ *the* relative simplicial cohomology groups of the pair $(\triangle, \tilde{\triangle})$.

When a simplicial map $\phi : \triangle \longrightarrow \triangle'$ *takes the subcomplex* $\tilde{\triangle}$ *of* $\triangle$ *into the subcomplex* $\tilde{\triangle}'$ *of* $\triangle'$, *then we say that* ϕ *is a simplicial map from the pair* $(\triangle, \tilde{\triangle})$ *to the pair* $(\triangle', \tilde{\triangle}')$. *In this case we can consider* $S_\bullet(\phi)$ *as a chain map* $S_\bullet(\phi) : S_\bullet(\triangle, \tilde{\triangle}) \longrightarrow S_\bullet(\triangle', \tilde{\triangle}')$.

Introducing the convention that $S_\bullet(\emptyset)$ be the zero complex, we obtain the absolute simplicial homology and cohomology as the special case of the relative analogues with $\tilde{\triangle} = \emptyset$.

When we reformulate our results in the language of simplicial homology, the notion of *mapping cone* will be indispensable.

Definition 22 *If* $f : K_\bullet \longrightarrow L_\bullet$ *is a chain map of* $\mathbb{Z}$*-modules, we define the* mapping cone of f *to be the complex* $C(f)_\bullet$ *defined by*

$$C(f)_n := L_n \oplus K_{n-1},$$

and

$$\partial^{C(f)}(y, x) := \left(\partial^L y + f(x), -\partial^K(x)\right).$$

The mapping cone of id : $K_\bullet \longrightarrow K_\bullet$ *is called* the cone of $K_\bullet$.

We will be interested only in the special case when f is an inclusion map. In this case we have the following lemma. (See eg. [16, page 46, Proposition 1.7.5].)

Lemma 8 *If $f : K_\bullet \longrightarrow L_\bullet$ is injective, then we have*

$$H_k(C(f)) \cong H_k(L_\bullet / K_\bullet)$$

for all k.

Definition 23 *Let $\triangle'$ be a subcomplex of the simplicial complex $\triangle$. We denote by $C_\bullet(\triangle, \triangle')$ the mapping cone of $S_\bullet(\rho)$, where ρ is the inclusion map $\rho : \triangle' \longrightarrow \triangle$.*

As a special case of Lemma 8 we obtain the following equality for all pairs of simplicial complexes $(\triangle, \triangle')$.

$$H_k(C_\bullet(\triangle, \triangle')) \cong H_k(S_\bullet(\triangle, \triangle')) \tag{2.12}$$

for all k.

The following lemma shows that a simplicial map between pairs of complexes induces a chain map between the mapping cones.

Lemma 9 *Let $f : K_\bullet \longrightarrow L_\bullet$ and $f' : K'_\bullet \longrightarrow L'_\bullet$ be chain maps. Assume furthermore that we are given chain maps $g : K_\bullet \longrightarrow K'_\bullet$ and $h : L_\bullet \longrightarrow L'_\bullet$, such that the diagram of chain complexes*

$$\begin{array}{ccc} K_\bullet & \xrightarrow{f} & L_\bullet \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ K'_\bullet & \xrightarrow{f'} & L'_\bullet \end{array}$$

is commutative. Then the pair (g, h) induces a chain map

$$\begin{array}{ccc} C(f)_\bullet & \longrightarrow & C(f')_\bullet \\ (y, x) & \longmapsto & (h(y), g(x)) \end{array}.$$

The proof is straightforward substitution into the definitions.

Corollary 5 *Let $\tilde{\triangle}$ be a subcomplex of $\triangle$ and $\tilde{\triangle}'$ a subcomplex of $\triangle'$. Assume that the simplicial map $\phi : \triangle \longrightarrow \triangle'$ takes the pair $(\triangle, \tilde{\triangle})$ into the pair $(\triangle', \tilde{\triangle}')$. Then ϕ (more precisely the pair $(S_\bullet(\phi), S_\bullet(\phi|_{\tilde{\triangle}}))$) induces a chain map*

$$C_\bullet(\phi) : C_\bullet(\triangle, \tilde{\triangle}) \longrightarrow C_\bullet(\triangle', \tilde{\triangle}').$$

In analogy with the cochain complex $S^\bullet(\triangle, \tilde{\triangle})$ we also introduce the dual complex of $C_\bullet(\triangle, \tilde{\triangle})$.

Definition 24 *We define $C^\bullet(\triangle, \tilde{\triangle})$ to be the dual complex of $C_\bullet(\triangle, \tilde{\triangle})$, consisting of the $\mathbb{Z}$-modules $C^k(\triangle, \tilde{\triangle}) = \mathrm{Hom}(C_k(\triangle, \tilde{\triangle}), \mathbb{Z})$, and connected by the coboundary map $\delta^k = \mathrm{Hom}(\partial_{k+1}, \mathbb{Z})$.*

We can easily compute the coboundary map of $C^\bullet(\triangle, \tilde{\triangle})$ from the coboundary map of $C^\bullet(\triangle)$ and $C^\bullet(\tilde{\triangle})$, as the following lemma shows.

Lemma 10 *Given a chain map $f : K_\bullet \longrightarrow L_\bullet$, the dual of the mapping cone $C(f)_\bullet$ is isomorphic to the mapping cone $C(f^*)^\bullet$ of the dual map $f^* : \mathrm{Hom}(K_\bullet, \mathbb{Z}) \longrightarrow \mathrm{Hom}(L_\bullet, \mathbb{Z})$. Using the isomorphism $\mathrm{Hom}(L_k \oplus K_{k-1}, \mathbb{Z}) \cong \mathrm{Hom}(L_k, \mathbb{Z}) \oplus \mathrm{Hom}(K_{k-1}, \mathbb{Z})$ we can describe the action of the coboundary map δ by*

$$\delta(y^*, x^*) = (\delta_L(y^*), f^*(y^*) - \delta_K(x^*)) \tag{2.13}$$

for all $y^* \in \mathrm{Hom}(L_k, \mathbb{Z})$ and $x^* \in \mathrm{Hom}(K_{k-1}, \mathbb{Z})$. Moreover, when f is injective and $K_\bullet, L_\bullet$ are complexes of free modules, then $\mathrm{Im}\left(\delta_{C(f^\bullet)}\right)$ is generated by the elements of the form $\delta((y^*, 0))$.

Proof: It is straightforward substitution into the definitions. The isomorphism,

$$\mathrm{Hom}(L_k \oplus K_{k-1}, \mathbb{Z}) \cong \mathrm{Hom}(L_k, \mathbb{Z}) \oplus \mathrm{Hom}(K_{k-1}, \mathbb{Z}),$$

gives us the result that the dual of a mapping cone is isomorphic to the mapping cone of the dual map. To see the formula for the coboundary map, observe that we have

$$\begin{aligned} \langle \delta(y^*, x^*) \mid (y, x) \rangle = \langle (y^*, x^*) \mid \partial(y, x) \rangle = \Big\langle (y^*, x^*) \;\Big|\; (\partial^L y + f(x), -\partial^K(x)) \Big\rangle \\ = \Big\langle y^* \;\Big|\; \partial^L y + f(x) \Big\rangle + \Big\langle x^* \;\Big|\; -\partial^K(x) \Big\rangle = \langle \delta_L(y^*) \mid y \rangle + \langle f^*(y^*) \mid x \rangle + \langle -\delta_K(x^*) \mid x \rangle \\ = \langle (\delta_L(y^*), f^*(y^*) - \delta_K(x^*)) \mid (y, x) \rangle. \end{aligned}$$

Assume now that f is injective. It is sufficient to show that for all $x^* \in \mathrm{Hom}(K_{k-1}, \mathbb{Z})$ there is a $y^* \in \mathrm{Hom}(L_k, \mathbb{Z})$ such that we have

$$\delta(y^*, 0) = \delta(0, x^*).$$

By (2.13) this equation is equivalent to

$$(\delta_L(y^*), f^*(y^*)) = (0, -\delta_K(x^*)).$$

This holds iff we have $y^* \circ \partial^L_{k+1} = 0$ and $y^* \circ f_k = -x^* \circ \partial^K_k$. When we consider K_k as a submodule of L_k, these two relations prescribe the value of y^* on the submodules K_k and $\mathrm{Im}\left(\partial^L_{k+1}\right)$. The prescription is consistent in the sense that both formulas require the same value for y^* on $K_k \cap \mathrm{Im}\left(\partial^L_{k+1}\right)$. Hence a map $\tilde{y} : K_k + \mathrm{Im}\left(\partial^L_{k+1}\right) \longrightarrow \mathbb{Z}$ may be defined, in accordance with the requirements for y^*. The submodule $K_k + \mathrm{Im}\left(\partial^L_{k+1}\right)$ of L_k is free, and so a direct summand of L_k. Therefore $\tilde{y}$ may be extended to a map $y^* : L_k \longrightarrow \mathbb{Z}$ which satisfies our requirements. **QED**

We establish an important equivalence for the orientability of simplicial pseudomanifolds.

Lemma 11 *For an n-dimensional simplicial pseudomanifold $\triangle$, the following are equivalent.*

(i) $\triangle$ *is orientable,*

(ii) $H_n(S_\bullet(\triangle, \partial\triangle)) \cong \mathbb{Z}$.

(iii) $H_n(C_\bullet(\triangle, \partial\triangle)) \cong \mathbb{Z}$.

Proof: By (2.12) it is sufficient to verify the equivalence of (i) and (ii). Observe that since $S_n(\partial\triangle) = 0$, we have $S_n(\triangle, \partial\triangle) = S_n(\triangle)$. An element of $S_n(\triangle)$ is a linear combination $\sum_\sigma \alpha(\sigma) \cdot [\sigma]$ of facets. Here each facet σ is represented as an equivalence class of ordered facets. Using the sign rules of this factorization, we can extend the definition of $\alpha(\sigma)$-s to ordered facets $(v_1, \ldots v_{n+1})$. The rules for change of sign will be identical to condition (ii) of Definition 16. Now it is easy to verify that

$$\partial \left(\sum_\sigma \alpha(\sigma) \cdot [\sigma] \right) = 0$$

is equivalent to condition (iii) of Definition 16. Using condition (iii) of Definition 14 we can convince ourselves that the value of α on an arbitrarily fixed ordered facet $(v_1, \ldots, v_{n+1})$ determines α completely. Therefore we have $H_n(S_\bullet(\triangle, \partial\triangle)) = \mathbb{Z}$ or $H_n(S_\bullet(\triangle, \partial\triangle)) = 0$ for all n-dimensional simplicial pseudomanifolds. The n-th relative simplicial homology group of $(\triangle, \partial\triangle)$ is non zero, if and only if there is a nonzero function α satisfying conditions (ii) and (iii) of Definition 16. This is equivalent to having an orientation ε, and all functions α of the above kind will be linear multiples of ε. **QED**

Remark The usual definition of orientability of simplicial pseudomanifolds is condition (ii) in the above lemma. See for example [26].

From now on we shall assume that $\triangle$ is an oriented n-dimensional simplicial pseudomanifold.

Lemma 12 *The generator of $H_n(C_\bullet(\triangle, \partial\triangle))$ may be represented by the vector*

$$\mathbf{w}_\triangle := \sum_{\sigma\in\triangle_n} \varepsilon(\sigma)\cdot[\sigma] - \sum_{\tau\in(\partial\triangle)_{n-1}} \varepsilon(\Omega(\tau))\cdot[\tau].$$

Proof: Note first that by condition (iii) of Definition 16, $\varepsilon(\sigma)\cdot[\sigma]$ is the always the same vector, regardless of the order in which we list the vertices of σ. The same holds for $\varepsilon(\Omega(\tau))\cdot[\tau]$. Thus $\mathbf{w}_\triangle$ is well defined.

Again we have $S_n(\partial\triangle) = 0$, and so $H_n(C_\bullet(\triangle, \partial\triangle))$ is equal to $\mathrm{Ker}(\partial_n^{C_\bullet(\triangle,\partial\triangle)})$. It is obviously true in general for mapping cones of embeddings $f : K_\bullet \longrightarrow L_\bullet$ that we have

$$\mathrm{Ker}(\partial_n^{C(f)_\bullet}) = \{(y, -\partial^L y) \in L_n \oplus K_{n-1} \ : \ \partial^L y \in K_{n-1}\}.$$

Thus an element of $\mathrm{Ker}(\partial_n^{C_\bullet(\triangle,\partial\triangle)})$ is of the form $y - \partial^{S_\bullet(\triangle)} y$, where y is a linear combination of facets such that $\partial^{S_\bullet(\triangle)} y$ is a linear combination of boundary subfacets. We have seen in the proof of Lemma 11 that the set of such y-s is equal to the set of linear multiples of $\sum_{\sigma\in\triangle_n} \varepsilon(\sigma)\cdot[\sigma]$. On the other hand, straightforward substitution into the definitions shows

$$\partial^{S_\bullet(\triangle)}\left(\sum_{\sigma\in\triangle_n} \varepsilon(\sigma)\cdot[\sigma]\right) = \sum_{\tau\in(\partial\triangle)_{n-1}} \varepsilon(\Omega(\tau))\cdot[\tau].$$

QED

Now we can consider a coloring of an orientable pseudomanifold with $n+1$ colors as a simplicial map $\phi : \triangle \longrightarrow \triangle^n$ from the pseudomanifold to the standard n-simplex. This map takes the pair $(\triangle, \partial\triangle)$ into the pair $(\triangle^n, \triangle^n)$. Then $H_n(C_\bullet(\triangle^n, \triangle^n)) = H_n(S_\bullet(\triangle^n, \triangle^n)) = 0$ implies

$$H_n(C_\bullet(\phi)) \left(\sum_{\sigma \in \triangle_n} \varepsilon(\sigma) \cdot [\sigma] - \sum_{\tau \in (\partial\triangle)_{n-1}} \varepsilon(\Omega(\tau)) \cdot [\tau] \right) = 0,$$

which is Lemma 4.

When we use m colors, we can think of the coloring as a simplicial map $\phi : \triangle \longrightarrow \triangle^{m-1}$. This map sends the pair $(\triangle, \partial\triangle)$ into the pair $(\triangle^{m-1}, \triangle^{m-1})$. It is well-known that the chain complex $S_\bullet(\triangle^{m-1})$ can be embedded into the exterior algebra $\mathrm{Ext}(\mathcal{V})$, where $\mathcal{V}$ is again an m-dimensional vector space with basis $\{e_v : v \in \mathrm{vert}(\triangle^{m-1})\}$. Thus $C_n(\triangle^{m-1}, \triangle^{m-1}) = S_n(\triangle^{m-1}) \oplus S_{n-1}(\triangle^{m-1})$ can be embedded into $\mathrm{Ext}_n(\mathcal{V}) \oplus Ext_{n-1}(\mathcal{V})$. The weight vector of the coloring ϕ is the image of

$$C_n(\phi)\,(\mathbf{w}_\triangle) = C_n(\phi) \left(\sum_{\sigma \in \triangle_n} \varepsilon(\sigma) \cdot [\sigma] - \sum_{\tau \in (\partial\triangle)_{n-1}} \varepsilon(\Omega(\tau)) \cdot [\tau] \right)$$

under this embedding. Hence we can think of the weight vectors as elements of $C_n(\triangle^{m-1}, \triangle^{m-1})$, or, in the boundariless case, as elements of $C_n(\triangle^{m-1}, \emptyset) = S_n(\triangle^{m-1})$. By abuse of notation we will write $\mathbf{w}_\phi = C_n(\phi)\,(\mathbf{w}_\triangle)$, resp. $\mathbf{w}_\phi \in S_n(\triangle^{m-1})$ in the boundariless case. The fact that $\mathbf{w}_\triangle$ belongs to $\mathrm{Ker}(\partial_n^{C_\bullet(\triangle, \partial\triangle)})$, implies

$$\mathbf{w}_\phi = C_n(\phi)\,(\mathbf{w}_\triangle) \in \mathrm{Ker}\left(\partial_n^{C_\bullet(\triangle^{m-1}, \triangle^{m-1})}\right),$$

or, in the boundariless case

$$\mathbf{w}_\phi = S_n(\phi)\,(\mathbf{w}_\triangle) \in \mathrm{Ker}\left(\partial_n^{S_\bullet(\triangle^{m-1})}\right).$$

The elementary weight vectors $\mathbf{p}_{(i_1,\ldots,i_{n+2})}$ correspond to the vectors $\partial(([i_1,\ldots,i_{n+2}]),0)$. The elementary weight vectors $\mathbf{q}_{(j_1,\ldots,j_{n+1})}$ correspond to the vectors $\partial((0,[j_1,\ldots,j_{n+1}]))$. Thus the elementary weight vectors generate $\mathrm{Im}(\partial_{n+1}^{C_\bullet(\triangle^{m-1},\triangle^{m-1})})$ in the general case, and the elementary weight vectors of the first kind generate $\mathrm{Im}\left(\partial_{n+1}^{S_\bullet(\triangle^{m-1})}\right)$ in the boundariless case. Thus in both general and boundariless cases, $\mathcal{M}$ resp. $\mathcal{M}_0$ is contained in the kernel of the respective ∂_n, and the elementary weight vectors generate the image of the respective ∂_{n+1}. Therefore the fact that the $\mathbb{Z}$-module $\mathcal{M}$ is generated by the elementary weight vectors follows from the trivial equality $H_n(C_\bullet(\triangle^{m-1},\triangle^{m-1}))=0$. Similarly the fact that the $\mathbb{Z}$-module $\mathcal{M}_0$ is generated by the elementary weight vectors of the first kind is an immediate consequence of $H_n(S_\bullet(\triangle^{m-1}))=0$. Note that the vanishing of $H_n(S_\bullet(\triangle^{m-1}))$ is a geometric property of the standard $(m-1)$-simplex, but we could replace $\triangle^{m-1}$ by any other simplicial complex in $H_n(C_\bullet(\triangle^{m-1},\triangle^{m-1}))=0$.

Partitioning the colors $\{1,2,\ldots,m\}$ into $n+1$ blocks is equivalent to defining a simplicial map $\lambda : \triangle^{m-1} \longrightarrow \triangle^n$, which maps the pair $(\triangle^{m-1},\triangle^{m-1})$ into the pair $(\triangle^n,\triangle^n)$. This induces a cochain map $C^\bullet(\lambda) : C^\bullet(\triangle^n,\triangle^n) \longrightarrow C^\bullet(\triangle^{m-1},\triangle^{m-1})$ in the general case, and a cochain map $S^\bullet(\lambda) : S^\bullet(\triangle^n) \longrightarrow S^\bullet(\triangle^{m-1})$ in the boundariless case. In particular, by $H^n(C^\bullet(\triangle^n,\triangle^n))=0$ and $H^n(S^\bullet(\triangle^n))=0$, the maps $\delta^{n-1} : C^{n-1}(\triangle^n,\triangle^n) \longrightarrow C^n(\triangle^n,T^n)$ and $\delta^{n-1} : S^{n-1}(\triangle^n) \longrightarrow S^n(\triangle^n)$ are surjective, and so we get

$$\mathrm{Im}(C^n(\lambda)) \subseteq \mathrm{Im}(\delta^{n-1})$$

in $C^n(\triangle^{m-1},\triangle^{m-1})$ in the general case, and

$$\mathrm{Im}(S^n(\lambda)) \subseteq \mathrm{Im}(\delta^{n-1})$$

in $S^n(\triangle^{m-1})$ in the boundariless case.

Theorem 4 follows from the orthogonality relation

$$\operatorname{Im}(\delta^{n-1})^{\perp} = \operatorname{Ker}(\partial_n). \tag{2.14}$$

which holds in both $C^{\bullet}(\triangle^{m-1}, \triangle^{m-1})$ and $S^{\bullet}(\triangle^{m-1})$. Using (2.13), the equations (2.8) may be translated into

$$\left\langle \delta^{n-1}\left(([j_1, i_2, \ldots, j_n]^*, 0)\right) \;\middle|\; C_n(\phi)\left(\mathbf{w}_{\triangle}\right)\right\rangle = 0.$$

By Lemma 10 the vectors $\delta^{n-1}\left(([j_1, i_2, \ldots, j_n]^*, 0)\right)$ generate $\operatorname{Im}(\delta^{n-1})$ in $C^{\bullet}(\triangle^{m-1}, \triangle^{m-1})$. Therefore the orthogonality relation (2.14) implies the first statement of Lemma 5.

In the boundariless case the equations (2.9) correspond to

$$\left\langle \delta^{n-1}([i_1, i_2, \ldots, i_n]^*) \;\middle|\; S_n(\phi)\left(\sum_{\sigma \in \triangle_n} \varepsilon(\sigma) \cdot [\sigma]\right)\right\rangle = 0.$$

Clearly, the vectors $\delta^{n-1}([i_1, i_2, \ldots, i_n]^*)$ generate $\operatorname{Im}(\delta^{n-1})$ in $S^{\bullet}(\triangle^{m-1})$ and so we have the the statement of Lemma 5 for pseudomanifolds without boundary.

Given the fact that $\mathcal{M}$ resp. $\mathcal{M}_0$ may be identified with $\operatorname{Ker}(\partial_n)$ in $C_{\bullet}(\triangle^{m-1}, \triangle^{m-1})$ resp. $S_{\bullet}(\triangle^{m-1})$, and that under this identification $\mathcal{X}$ resp. $\mathcal{X}_0$ becomes $\operatorname{Im}(\delta^{n-1})$ in $C^{\bullet}(\triangle^{m-1}, \triangle^{m-1})$ resp. $S^{\bullet}(\triangle^{m-1})$, the orthogonality relation (2.14) also implies Theorem 5.

Observe that in the general case we did not use any specific property of $\triangle^{m-1}$, and that in the boundariless case the only fact we needed to know about $\triangle^{m-1}$ is that $H_n(S_{\bullet}(\triangle^{m-1})) = 0$. If we now replace the standard simplex with any other simplicial complex $\tilde{\triangle}$, we can restate the analogues of Theorem 4 and Theorem 5 for colorings of orientable n-dimensional pseudomanifolds (with boundary) with colors from $\tilde{\triangle}$. If we want the analogues of the boundariless versions to hold, we have only to require $H_n(S_\bullet(\tilde{\triangle})) = 0$. (We define colorings to be simplicial maps into $\tilde{\triangle}$.) Thus we have a complete theory in these cases. In particular, we obtain the analogues of Theorem 4 and Theorem 5 for octahedral colorings. (Note that an octahedron has zero n-th simplicial homology group.) The octahedral analogue of Theorem 4 contains [12, Theorem 1] as a special case.

2.2.5 A proof of Sperner's lemma

As an application of our results we present a proof of Sperner's lemma.

Lemma 13 (Sperner's lemma) *Assume we are given a triangulation τ of the simplex $\triangle^{m-1} = \mathrm{conv}(\{e_1, e_2, \ldots, e_m\})$ and a coloring of $\mathrm{vert}(\tau)$ with colors $1, 2, \ldots, m$ such that the color of a vertex lying on the face $\mathrm{conv}(\{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\})$ is chosen from the set $\{i_1, \ldots, i_k\}$.*

Let T^+ be the number of facets of color $\{1, 2, \ldots, m\}$ in positive order and T^- be the number of the facets of color $\{1, 2, \ldots, m\}$ in negative order. (Here we consider the order $e_1, \ldots, e_m$ to be positive.) Then we have

$$T^+ - T^- = 1.$$

This implies that the number of facets of τ where the vertices are colored with all m colors is odd. In particular, there is at least one facet of τ that has all m colors.

Proof: The standard simplex $\triangle^{m-1}$ is a facet of the octahedron

$$\mathbb{O} := \mathrm{conv}\left(\{\pm e_i \,:\, i = 1, 2, \ldots, m\}\right).$$

For a subset I of $\{1, 2, \ldots, m\}$, define the following notation.

$$\begin{aligned} e^+(I) &= \{e_i : i \in I\}, \\ e^-(I) &= \{-e_i : i \in I\}. \end{aligned}$$

Also denote the complement of I, that is, $\{1, \ldots, m\} \setminus I$ by $\overline{I}$.

It is easy to check that the triangulation τ of the facet $\text{conv}(\{e_1, e_2, \ldots, e_m\})$ extends uniquely to a triangulation σ of the boundary of the octahedron. Let the vertex set of the triangulation σ be the set $\text{vert}(\tau) \cup \{-e_1, \ldots, -e_m\}$. Let the facets of σ be of the form

$$e^-(I) \cup F,$$

where $I \subseteq \{1, \ldots, m\}$, and F is a maximal face of τ in $\text{conv}(e^+(\overline{I}))$. Note that in the event when $I = \emptyset$, the facets described above are the facets of τ. At the other extreme, when $I = \{1, \ldots, m\}$ there is only one facet, namely $\text{conv}(\{-e_1, \ldots, -e_m\})$.

Extend the coloring of the vertices of τ to a coloring of the vertices of the triangulation σ, by coloring the vertex $-e_i$ with color $i+1$ for $i = 1, \ldots, m-1$, and coloring $-e_m$ with color 1.

Assume that a facet G of σ has its vertices colored with the colors $\{1, \ldots, m\}$. We know that $\text{vert}(G) = e^-(I) \cup \text{vert}(F)$, for some $I \subseteq \{1, \ldots, m\}$ and some maximal face F of τ in $\text{conv}(e^+(\overline{I}))$. Thus the colors of the vertex set $e^-(I)$ are distinct, and the same holds for the set $\text{vert}(F)$. Moreover, the colors of $e^-(I)$ and $\text{vert}(F)$ are disjoint. By the condition of the lemma, the set of colors of the vertices of F is $\overline{I}$. In our coloring of $\{-e_1, \ldots, -e_m\}$, the set of colors of $e^-(I)$ is $J = \{i + 1 \pmod m : i \in I\}$. Unless $I = \emptyset$ or $I = \{1, \ldots, m\}$ we have that

$$\overline{I} \cap J \neq \emptyset.$$

But this contradicts the fact that the colors of the vertices of F and the colors of $e^{-}(I)$ are disjoint. Thus we conclude that if a facet G of the triangulation σ has all colors, then either G is a facet of τ or G is equal to conv($\{-e_1, \ldots, -e_m\}$).

Recall that $\partial\mathbb{O}$ is homeomorphic to $\mathbb{S}^{m-1}$, and that we have a triangulation σ of $\partial\mathbb{O}$, where the vertices are colored with m different colors. At this point we have two options to finish the proof. The first one is to apply Theorem 3 with $n = m - 1$. We observe that in the coloring of σ we do not use the color $m + 1$, and thus we have that $A_1 = \cdots = A_m = 0$ for this coloring. By the theorem, we then have $A_{m+1} = 0$, i.e., in this triangulation σ there is an equal number of positively and negatively oriented facets that are colored with $\{1, 2, \ldots, m\}$. Since the coloring of the facet conv($\{-e_1, -e_2, \ldots, -e_m\}$) has negative orientation, we conclude that

$$0 = A_{m+1} = T^{+} - T^{-} - 1.$$

We now have a proof of Sperner's lemma that relies on the equivalence of degree and local degree of a continuous function.

The other way to finish the proof is to apply the boundariless version of Lemma 4, with $n = m - 1$. Again we find that in the triangulation σ there is an equal number of positively and negatively oriented facets colored with $\{1, 2, \ldots, m\}$. From here the reasoning goes exactly the same way as in the previous ending. Together with the proof of Lemma 4 we have a purely combinatorial proof of Sperner's lemma that does not use induction on dimension. **QED**

Remark As it is shown in [7], we can also prove Sperner's lemma by induction on dimension, applying Lemma 4 directly to the triangulation τ.

2.3 Cubical generalizations

2.3.1 Preliminaries about the standard n-cube

In this subsection we recall some notations and facts about the standard n-cube, which was defined in Definition 5. We also recall some technical lemmas that will be necessary in the proof of the cubical analogues of our results about coloring simplicial pseudomanifolds. Readers familiar with the topic are encouraged to go ahead to the next subsection and come back to read the lemmas only when they are cited.

In the geometric representation we will denote the vector $(0,\ldots,0,1,0,\ldots,0)$, where 1 is at the i-th place, by e_i, and the vector $(0,\ldots,0)$ by $\mathbf{0}$.

Definition 25 *We define the* Hamming distance $\mathrm{d}(.,.)$ *on* $\{0,1\}^n$ *by*

$$\mathrm{d}(\mathbf{x},\mathbf{y}) := |\{i \ : \ x_i \neq y_i\}| \, .$$

It is straightforward that $\mathrm{vert}(\square^n)$ is a finite metric space with d, and that edge-preserving maps preserve the Hamming distance.

We can encode all nonempty faces of $\square^n$ with vectors $(u_1,u_2,\ldots,u_n) \in \{0,1,*\}^n$ in the following way. We set $u_i = 0$ or 1 respectively if the i-th coordinate of every element of the face is 0 or 1 respectively. Otherwise we set $u_i = *$. This bijection is described for example in [20].

Lemma 14 *Let* $\phi : \mathrm{vert}(\square^n) \longrightarrow \mathrm{vert}(\square^m)$ *be an injective map. If* ϕ *takes edges into edges then the set* $\phi(\square^n)$ *is a* n*-face of* $\square^m$*.*

Proof: Without loss of generality we shall assume that $\phi(\mathbf{0}) = \mathbf{0} = (0,\ldots,0)$. Since ϕ is edge preserving we have that $\phi(e_i) = f_i$, where f_i is a unit vector. (Not necessarily the ith unit vector.) Moreover since ϕ is injective, $f_1,\ldots,f_n$ are distinct. We show now by induction on $\mathrm{d}(\mathbf{0},x)$ that if $\mathbf{x} = e_{i_1} + \cdots + e_{i_k}$ then $\phi(x) = f_{i_1} + \cdots + f_{i_k}$. Since x is adjacent to $x - e_{i_j}$, we know that $\phi(x)$ is adjacent to $\phi(x - e_{i_j})$. Thus $\phi(x)$ is adjacent to $\phi(x) - f_{i_j}$ for all index j. Since ϕ is injective, we conclude that $\phi(x) = f_{i_1} + \cdots + f_{i_k}$, and the induction step is proven.

It follows that the image of $\square^n$ is a n-face of $\square^m$. **QED**

It is known that the group $\mathcal{B}_n$ of symmetries of the standard n-cube form a *Coxeter group*. We can define $\mathcal{B}_n$ as the set of those bijections of $\square^n$ that take faces into faces. In fact, by Lemma 14, it is sufficient to require that the bijections take edges into edges. $\mathcal{B}_n$ has $2^n \cdot n!$ elements, and it is generated by an n-element set $\{s_1, s_2, \ldots, s_n\}$ of reflections, called *simple reflections*. We define the sign of π to be

$$\mathrm{sign}(\pi) := (-1)^{l(w)},$$

where $l(w)$ is the length of the shortest word $w = s_{i_1} s_{i_2} \ldots s_{i_l}$ that represents π. This sign function behaves similarly to the sign function of the symmetric group: it is a group-homomorphism from $\mathcal{B}_n$ to $\{-1, 1\}$, and the sign of any reflection is -1. For more detailed information, see [15].

We will use the following elementary observations about the symmetries of the cube.

Lemma 15 *Every $\pi \in \mathcal{B}_n$ is uniquely determined, provided we know either one of the following:*

- *its restriction to a facet, or*
- *its value on $n + 1$ vertices that span an n-simplex in the geometric realization.*

Proof: Assume first we know the value of π on a facet. Without loss of generality we may suppose that this facet is $(*, *, \ldots, *, 0)$, and that the restriction of π to this facet is the identity. Let us now take any $\mathbf{x} \in (*, *, \ldots, *, 1)$. The map π takes edges into edges, and so

$$\begin{aligned}&\{\pi((x_1, x_2, \ldots, x_{n-1}, 1)), \pi((x_1, x_2, \ldots, x_{n-1}, 0))\}\\&= \{\pi((x_1, x_2, \ldots, x_{n-1}, 1)), (x_1, x_2, \ldots, x_{n-1}, 0)\}\end{aligned}$$

is an edge. But the only vector that is adjacent to $(x_1, x_2, \ldots, x_{n-1}, 0)$, and is not in the image of $(*, *, \ldots, *, 0)$ under π, is $\mathbf{x}$ itself. Hence we must have $\pi(\mathbf{x}) = \mathbf{x}$, and π is uniquely determined.

In the second case the lemma follows from the fact that π corresponds to an affine transformation of $\mathbb{R}^n$ in the geometric representation. **QED**

Lemma 16 *Let* $\Phi : \{0,1\}^n \longrightarrow \{0,1\}^n$ *be a map of the standard n-cube such that*

(i) $\mathrm{d}(\Phi(\mathbf{x}), \Phi(\mathbf{y})) \leq \mathrm{d}(\mathbf{x}, \mathbf{y})$ *for all* $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$*, and*

(ii) $\mathrm{d}(\mathbf{x}, \Phi(\mathbf{x})) \leq 1$ *for all* $\mathbf{x} \in \{0,1\}^n$.

Then either $\Phi \in \mathcal{B}_n$*, or there is a facet* F *and a vector* $\mathbf{e} \in \{+e_i, -e_i\}_{i=1,\ldots,n}$ *such that* $\mathbf{e} \perp F$ *and* $\Phi(\mathbf{x}) = \mathbf{x} + \mathbf{e}$ *for all* $\mathbf{x} \in F$.

Proof: Clearly, if Φ is a bijection then we have $\Phi \in \mathcal{B}_n$.

Hence we may assume that there is a vertex $\mathbf{x}$ that does not belong to $\mathrm{Im}\Phi$. Without loss of generality we may assume $\mathbf{x} = \mathbf{0}$. By (ii), $\Phi(\mathbf{0})$ is adjacent but not equal to $\mathbf{0}$, w.l.o.g. we may assume that $\Phi(\mathbf{0}) = e_n$. We claim that in this case $\Phi((*, *, \ldots, *, 0)) = (*, *, \ldots, *, 1)$.

We show by induction on $\mathrm{d}(\mathbf{0}, \mathbf{x})$ that for every $\mathbf{x} \in (*, *, \ldots, *, 0)$ we have

$$\Phi(\mathbf{x}) = \mathbf{x} + e_n \tag{2.15}$$

This is true for $\mathbf{x} = 0$. Assume (2.15) holds for all $\mathbf{x} \in (*, *, \dots, *, 0)$ at Hamming distance at most $k-1$ from $\mathbf{0}$. Let us take an $\mathbf{x} \in (*, *, \dots, *, 0)$ with $\mathrm{d}(\mathbf{0}, \mathbf{x}) = k$. Then $\mathbf{x}$ may be written in the form

$$\mathbf{x} = e_{i_1} + e_{i_2} + \cdots + e_{i_k}$$

where $1 \le i_1 < i_2 < \cdots < i_k \le n-1$. By (i), $\Phi(\mathbf{x})$ must be adjacent to the Φ-image of the neighbouring vertices, in particular, to the vertices $\Phi(\mathbf{x} - e_{i_j})$ for $j = 1, 2, \dots, k$. By our induction hypothesis, we have

$$\Phi(\mathbf{x} - e_{i_j}) = \mathbf{x} - e_{i_j} + e_n$$

for $j = 1, 2, \dots, k$. Moreover, according to (ii), $\Phi(\mathbf{x})$ must also be adjacent to $\mathbf{x}$. The only vector fulfilling these conditions is $\mathbf{x} + e_n$, therefore we have

$$\Phi(\mathbf{x}) = \mathbf{x} + e_n$$

as stated. **QED**

In the following lemma we describe exactly the bijective Φ-s. Recall that the symmetries $\pi \in \mathcal{B}_n$ preserve the Hamming distance.

Lemma 17 *Assume $\pi \in \mathcal{B}_n$ satisfies $\mathrm{d}(\mathbf{x}, \pi(\mathbf{x})) \le 1$ for all $\mathbf{x} \in \{0,1\}^n$. Then either $\pi = \mathrm{id}$, or a reflection in the hyperplane $x_i = \frac{1}{2}$, defined by the formula*

$$\pi(\mathbf{x})_k = \begin{cases} 1 - x_i & \textit{for } k = i \\ x_k & \textit{otherwise} \end{cases},$$

for some fixed i, or a 90° rotation around the affine space $x_i = x_j = \frac{1}{2}$, defined by the *formula*

$$\pi(\mathbf{x})_k = \begin{cases} 1 - x_j & \textit{for } k = i \\ x_i & \textit{for } k = j \\ x_k & \textit{otherwise} \end{cases} .$$

for some fixed i, j.

Proof: Assume first that for some $\mathbf{x} \in \{0,1\}^n$ we have $\pi(\mathbf{x}) = \mathbf{x}$. Without loss of generality we may suppose $\mathbf{x} = \mathbf{0}$. By the distance-preserving property of π we have

$$\mathrm{d}(\pi(e_1), \mathbf{0}) = \mathrm{d}(\pi(e_1), \pi(\mathbf{0})) = \mathrm{d}(e_1, \mathbf{0}),$$

and so $\pi(e_1) = e_i$ for some $i \in \{1, 2, \ldots, n\}$. By the assumption of the lemma about π, we must also have

$$1 \geq \mathrm{d}(e_1, \pi(e_1)) = \mathrm{d}(e_1, e_i) = 2 \cdot \delta_{1,i},$$

and so $i = 1$. We can show $\pi(e_j) = e_j$ similarly for $j = 2, \ldots, n$. Hence π agrees with the identity on $\mathbf{0}, e_1, e_2, \ldots, e_n$ and so $\pi = \mathrm{id}$ by Lemma 15. Therefore from now on we may assume that π does not fix any vertex, or in other words

$$\mathrm{d}(\mathbf{x}, \pi(\mathbf{x})) = 1 \tag{2.16}$$

for all $\mathbf{x} \in \{0,1\}^n$.

Assume next that $\pi(\pi(\mathbf{x})) = \mathbf{x}$ for some vertex $\mathbf{x}$. By (2.16), $\mathbf{x}$ and $\pi(\mathbf{x})$ are different and adjacent. Without loss of generality we may suppose that we have $\pi(\mathbf{0}) = e_1$ and $\pi(e_1) = \mathbf{0}$. Let us take any e_i with $i \geq 2$. By the edge-preserving property of π and by our assumption about $\pi(\mathbf{0})$ we have

$$1 = \mathrm{d}(\mathbf{0}, e_i) = \mathrm{d}(\pi(\mathbf{0}), \pi(e_i)) = \mathrm{d}(e_1, \pi(e_i)).$$

Considering also the fact that $\pi(e_i) \neq \pi(e_1) = \mathbf{0}$ it follows that

$$\pi(e_i) = e_1 + e_k$$

for some $k \geq 2$. On the other hand, from $\mathrm{d}(e_i, \pi(e_i)) = 1$ it follows that $e_k = e_i$. Therefore in this case we have that

$$\pi(e_i) = e_i + e_1$$

for all $i \geq 2$. We refer again to Lemma 15 to conclude that π is of the first type.

From now on we may assume that $\pi(\pi(\mathbf{x})) \neq \mathbf{x}$ for every vertex $\mathbf{x}$. Without loss of generality we may assume that $\pi(\mathbf{0}) = e_1$ and $\pi^{-1}(\mathbf{0}) = e_2$. Let us take an e_i with $i \geq 3$. From

$$1 = \mathrm{d}(\mathbf{0}, e_i) = \mathrm{d}(\pi(\mathbf{0}), \pi(e_i) = \mathrm{d}(e_1, \pi(e_i))$$

and from $\pi(e_i) \neq \pi(e_2) = \mathbf{0}$ we infer that

$$\pi(e_i) = e_1 + e_k$$

for some $k \neq 1$. Again by $\mathrm{d}(e_i, \pi(e_i)) = 1$ we have $e_k = e_i$, and so

$$\pi(e_i) = e_i + e_1$$

holds for all $i \geq 3$. Let us calculate $\pi(e_1)$. From $\mathrm{d}(e_1, \pi(e_1)) = 1$ and from $\pi(e_1) \neq \pi(e_2) = \mathbf{0}$ it follows that $\pi(e_1)$ is also of the form $\pi(e_1) = e_1 + e_j$ for some $j \neq 1$. Given the fact that $\pi(e_1)$ must be different from all $\pi(e_i) = e_i + e_1$ for $i \geq 3$, we have

$$\pi(e_1) = e_1 + e_2.$$

By Lemma 15, π is of the second type. **QED**

2.3.2 Orientable cubical pseudomanifolds

In this subsection we show results analogous to the statements in Subsection 2.2.2.

The definition of cubical pseudomanifolds is analogous to the definition of their simplicial counterparts.

Definition 26 *An n-dimensional* cubical pseudomanifold *is a cubical complex $\square$ satisfying the following conditions:*

(i) every facet is an n-cube of $\square$,

(ii) every subfacet is contained in at most two facets,

(iii) if F and F' are facets of $\square$ then there is a sequence of facets $F = F^1, F^2, \dots, F^m = F'$ such that F^i and F^{i+1} have a subfacet in common.

We call the subcomplex generated by the subfacets contained in exactly one facet the boundary *of $\square$, and we denote it by $\partial\square$. If $\partial\square = \emptyset$, then we call $\square$ a* cubical pseudomanifold without boundary. *For a boundary subfacet σ we denote the unique facet containing it by $\Omega(\sigma)$.*

Next we define ordered faces. Note that in the simplicial case we could think of the ordering of an n-dimensional face σ as a bijection between the vertices of the standard simplex $\triangle^n$ and the vertices of σ. The cubical analogue will run as follows.

Definition 27 *Let $\sigma \in \square$ be an n-dimensional face of a cubical complex $\square$. We define a* cubical order *on the face σ to be a bijection $f : \mathrm{vert}(\square^n) \longrightarrow \sigma$ between the faces of the standard cube $\square^n = [0,1]^n$ and the vertices of σ that takes faces into faces. We will call the pair (f, σ) a* (cubically) ordered face. *We denote the set of ordered n-dimensional faces of $\square$ by $\mathrm{Ord}_n(\square)$.*

Usually we will refer to the ordered face (f, σ) as f. There cannot be any confusion because σ is the image-set of f.

The symmetries $\pi \in \mathcal{B}_n$ of the standard n-cube act faithfully and transitively on the cubical orders of an n-dimensional face σ by assigning $f \circ \pi$ to f. Hence the number of cubical orders on a face is $2^n \cdot n!$. Note that we have $\mathrm{Ord}_n(\Box^n) = \mathcal{B}_n$.

Analogously to the simplicial case, we define the Ω-operation on ordered boundary subfacets.

Definition 28 *Let g be an ordered boundary subfacet of the n-dimensional cubical pseudomanifold $\Box$, where $\mathrm{Im}(g) = \tau$. For every vertex $v \in \tau$ there is a unique vertex $\Omega(\tau, v) \in \Omega(\tau) \setminus \tau$ such that $\{v, \Omega(\tau, v)\}$ is an edge of $\Box$. We define $\Omega(g)$ to be the following cubical order on the facet $\Omega(\tau)$.*

$$\Omega(g)(x_1, \ldots, x_{n-1}, x_n) := \begin{cases} g(x_1, \ldots, x_{n-1}) & \textit{when } x_n = 0 \\ \Omega(\tau, g(x_1, \ldots, x_{n-1})) & \textit{when } x_n = 1 \end{cases}$$

Let us denote by ι_k the the embedding $\iota_k : \Box^k \longrightarrow \Box^{k+1}$ that takes $(x_1, \ldots, x_k)$ into $(x_1, \ldots, x_k, 0)$. We will drop the index of ι whenever there is no risk of confusion. Observe that we have

$$\Omega(g) \circ \iota_{n-1} = g$$

for all $g \in \mathrm{Ord}_{n-1}(\partial \Box)$.

Definition 29 *Let $\Box$ be an n-dimensional cubical pseudomanifold. We call $\Box$* orientable *when there exists a map $\varepsilon : \mathrm{Ord}_n(\Box) \longrightarrow \mathbb{Z}$ such that the following hold:*

(i) For every ordered facet $f \in \mathrm{Ord}_n(\Box)$ we have

$$\varepsilon(f) = \pm 1.$$

(ii) For every ordered facet $f \in \mathrm{Ord}_n(\Box)$, and every $\pi \in \mathcal{B}_n$,

$$\varepsilon(f \circ \pi) = \mathrm{sign}(\pi) \cdot \varepsilon(f)$$

holds, where sign *is the sign function defined on* $\mathcal{B}_n$*.*

(iii) Given a non-boundary subfacet τ *and the two facets* σ, σ' *containing it, and given* f, f' *cubical orders on* σ, σ' *respectively such that* $f|_\tau = f'|_\tau$*, we have*

$$\varepsilon(f) = -\varepsilon(f').$$

We call ε *an* orientation *of* $\square$*.*

We may think of the colorings of a simplicial pseudomanifold as a simplicial map from the pseudomanifold to a standard simplex. Simplicial maps are those maps between the vertex sets of two simplicial complexes which take faces into faces of same or less dimension. It turns out that in the cubical case we can allow a broader class of functions.

Definition 30 *Let* $\square$ *and* $\square'$ *be cubical complexes. A* cubical map $\phi : \square \to \square'$ *is a map*

$$\phi : \text{vert}(\square) \longrightarrow \text{vert}(\square')$$

subject to the following conditions.

(i) for every $\sigma \in \square$*,* $\phi(\sigma)$ *is contained in some* $\tau \in \square'$*.*

(ii) ϕ *takes adjacent vertices into adjacent vertices or the same vertex.*

Note that every cubical order $f : \square^k \longrightarrow \square$ is an injective cubical map. It is useful to notice that the converse is true as well.

Lemma 18 *Let* $f : \square^k \longrightarrow \square$ *be an injective cubical map. Then* f *is an ordered face.*

Proof: By condition (i) of Definition 30, $\text{Im}(f)$ is contained in some face σ. By Lemma 14 $\text{Im}(f)$ is a face of σ. **QED**

Now we can define "cubical colorings" in an analogous way to the simplicial case.

Definition 31 *Let $\square$ be an n-dimensional cubical pseudomanifold. We define a* cubical coloring of $\square$ with an m-cube *to be a cubical map ϕ from the cubical pseudomanifold $\square$ to the standard m-cube $\square^m$.*

The following two lemmas are necessary to prove the cubical analogue of the Fundamental Lemma for colored triangulations. The first-time reader might want to jump to the definition of the cubical analogue of the A_I and B_J parameters and return to these lemmas when quoted.

Lemma 19 *Let $\phi : \partial\square^n \longrightarrow \square^{n-1}$ be an arbitrary cubical coloring of the surface of the standard n-cube. Consider the set of ordered facets $S = \{f \in \mathrm{Ord}_{n-1}(\partial\square^n) \ : \ \phi \circ f = \mathrm{id}\}$. These ordered facets correspond to facets that are colored with all colors. Then*

$$\sum_{f \in S} \varepsilon(f) = 0,$$

and the set S contains at most 4 elements.

Proof: Let us first note two properties about the set S. If $f, g \in S$ such that $\mathrm{Im}(f) = \mathrm{Im}(g)$, then $f = g$. If $f, g \in S$ such that the two facets $\mathrm{Im}(f)$ and $\mathrm{Im}(g)$ are neighbouring facets, then $f = \tau \circ g$, where $\tau \in \mathcal{B}_n$ is a reflection that leaves $\mathrm{Im}(f) \cap \mathrm{Im}(g)$ fixed. Observe that in this case $\varepsilon(f) = -\varepsilon(g)$.

When S is empty there is nothing to prove. Hence, without loss of generality we may assume that the facet $(*, *, \ldots, *, 0)$ is one of those colored with all colors, and that this facet keeps the orientation. That is, $\phi \circ \iota = \mathrm{id}$.

Observe next that for an arbitrary cubical coloring of the surface of an n-cube, there cannot be three ordered facets $f, g, h \in S$ meeting in one vertex. In fact, when we assume the contrary, without loss of generality we may suppose that $\mathrm{Im}(f) = (0, *, \ldots, *)$, $\mathrm{Im}(g) = (*, 0, *, \ldots, *)$, and $\mathrm{Im}(h) = (*, *, 0, *, \ldots, *)$. Let $x = \phi(e_1)$. Observe that x is adjacent to 0 in $\square^{n-1}$. We have that $\phi(e_1) = x = \phi(g(x))$. Since g is a bijection between

$\square^{n-1}$ and the facet $(*, 0, *, \ldots, *)$, we get $e_1 = g(x)$. Similarly we have $e_1 = h(x)$. Since x is adjacent to 0, $f(x)$ is also adjacent to 0. Hence $f(x) = e_i$ for some $i \geq 2$. Now $\phi(e_1) = x = \phi(f(x)) = \phi(e_i)$. If $i \geq 3$, then we reach a contradiction, since ϕ is a bijection between $\square^{n-1}$ and $(*, 0, *, \ldots, *)$. Similarly, if $i = 2$, we reach a contradiction by the fact that ϕ is a bijection between $\square^{n-1}$ and $(*, *, 0, *, \ldots, *)$.

Therefore, besides the ordered facet ι, there can be at most three other ordered facets in S. If there is only one other such ordered facet f, and if $\mathrm{Im}(f)$ is a neighbouring facet of $(*, *, \ldots, 0)$, then our statement is an easy consequence of condition (iii) of Definition 29.

Assume next that besides $\iota \in S$, there are $f, g \in S$ such that $\mathrm{Im}(f)$ and $\mathrm{Im}(g)$ are neighbouring facets to $(*, \ldots, *, 0)$. We may suppose that $\mathrm{Im}(f) = (*, \ldots, *, 0, *)$ and $\mathrm{Im}(g) = (, *, \ldots, *, 1, *)$, since $\mathrm{Im}(f)$ and $\mathrm{Im}(g)$ are not adjacent. In this case we can easily deduce from $\phi \circ \iota = \mathrm{id}$ that

$$\begin{aligned} f((x_1, \ldots, x_{n-1})) &= (x_1, \ldots, x_{n-2}, 0, x_{n-1}), \\ g((x_1, \ldots, x_{n-1})) &= (x_1, \ldots, x_{n-2}, 1, 1 - x_{n-1}). \end{aligned}$$

Consider $h \in \mathrm{Ord}_{n-1}(\partial \square^n)$ defined by

$$h((x_1, \ldots, x_{n-1})) = (x_1, \ldots, x_{n-2}, 1 - x_{n-1}, 1).$$

We claim that $h \in S$.

$$\begin{aligned} \phi(h((x_1, \ldots, x_{n-2}, 0))) &= \phi((x_1, \ldots, x_{n-2}, 1, 1)) \\ &= \phi(g((x_1, \ldots, x_{n-2}, 0))) \\ &= (x_1, \ldots, x_{n-2}, 0). \end{aligned}$$

Similarly, we find that $\phi(h((x_1, \ldots, x_{n-2}, 1))) = (x_1, \ldots, x_{n-2}, 1)$, and thus the claim is proved. Observe now that $\varepsilon(\iota) = -\varepsilon(f) = -\varepsilon(g) = \varepsilon(h)$, and we obtain the statement of our lemma.

Assume now that there is no $f \in S$ such that $\mathrm{Im}(f)$ is a neighbouring facet of $(*,\dots,*,0)$. Construct a map $\Phi : \{0,1\}^{n-1} \longrightarrow \{0,1\}^{n-1}$ by

$$\Phi((x_1,\dots,x_{n-1})) = \phi((x_1,\dots,x_{n-1},1)).$$

The map Φ satisfies the conditions of Lemma 16. Thus either we have $\Phi \in \mathcal{B}_{n-1}$, or there is a facet F of $\square^{n-1}$ and a vector $\mathbf{e} \in \{+e_i, -e_i\}_{i=1,\dots,n-1}$ such that $\mathbf{e} \perp F$ and $\Phi(\mathbf{x}) = \mathbf{x}+\mathbf{e}$ for all $\mathbf{x} \in F$. Let us consider the second case first. Define $f \in \mathrm{Ord}_{n-1}(\partial\square^n)$ by

$$f((x_1,\dots,x_{n-1})) = \begin{cases} (x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_{n-1},x_i) & \text{if } \mathbf{e} = e_i \\ (x_1,\dots,x_{i-1},1,x_{i+1},\dots,x_{n-1},1-x_i) & \text{if } \mathbf{e} = -e_i \end{cases}$$

It is easy to check that $f \in S$, and that $\mathrm{Im}(f)$ is a neighbouring facet of $(*,\dots,*,0)$. We have already excluded this case, so we may assume $\Phi \in \mathcal{B}_{n-1}$. Lemma 17 gives a full description of all possible Φ. Let us check the three different possibilities.

The map Φ cannot be a reflection. In fact, if Φ switches the i-th coordinate, then $f \in S$ and $\mathrm{Im}(f)$ is a neighbouring facet of $(*,\dots,*,0)$, where $f((x_1,\dots,x_{n-1})) = (x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_{n-1},x_i)$.

If $\Phi = \mathrm{id}$ or Φ is a rotation, then it is easy to see that there exist $f \in S$ such that $\mathrm{Im}(f) = (*,\dots,*,1)$. Since $\mathrm{sign}(\Phi) = 1$, it follows that $\varepsilon(f) = -\varepsilon(\iota)$. This finishes the last possible case, and thus the proof is complete. **QED**

Lemma 20 *Given a cubical map $\psi : \square^n \longrightarrow \square^n$ such that ψ is not a bijection, let*

$T = \{f \in \mathrm{Ord}_n(\square^n) \ : \ \psi \circ f \circ \iota_{n-1} = \iota_{n-1}\}$. *Then*

$$\sum_{f \in T} \varepsilon(f) = 0,$$

and T contains at most 4 elements.

Proof: If T is empty, there is nothing to prove. We can assume that $\psi \circ \iota = \iota$, since there exists a $f' \in T$, and we can consider the lemma with the map $\eta = \psi \circ f'$. Since f' is a bijection, there is a one to one correspondence between T_ψ and T_η. Moreover $\eta \circ \iota = \iota$.

Consider the projection $\rho : \square^n \longrightarrow \square^{n-1}$ defined by

$$\rho((x_1, \ldots, x_n)) = (x_1, \ldots, x_{n-1}).$$

Obviously ρ is a cubical map and $\rho \circ \iota = \mathrm{id}$.

Observe that there is a bijection between $\mathrm{Ord}_n(\square^n)$ and $\mathrm{Ord}_{n-1}(\partial(\square^n))$ that preserves signs. Given $f \in \mathrm{Ord}_n(\square^n)$ we have that $f \circ \iota \in \mathrm{Ord}_{n-1}(\partial(\square^n))$. Also given $g \in \mathrm{Ord}_{n-1}(\partial(\square^n))$, Lemma 15 gives us a unique $f \in \mathrm{Ord}_n(\square^n)$ such that $f \circ \iota = g$. Moreover it is easy to check that $\varepsilon(f) = \varepsilon(g)$.

Apply Lemma 19 to the cubical map $\rho \circ \psi : \square^n \longrightarrow \square^{n-1}$. (Observe that we can view $\rho \circ \psi$ as a cubical map from $\partial(\square^n)$.) Let $S = \{g \in \mathrm{Ord}_{n-1}(\partial \square^n) \ : \ \rho \circ \psi \circ g = \mathrm{id}\}$.

Consider the map $P : T \longrightarrow S$, defined by $P(f) = f \circ \iota$. The map P is well defined since $f \circ \iota \in \mathrm{Ord}_{n-1}(\square^n)$ and $\rho \circ \psi \circ f \circ \iota = \rho \circ \iota = \mathrm{id}$. If we can prove that P is bijective, then the conclusion of the lemma follows.

For every $g \in S$ there is a unique $f \in \mathrm{Ord}_n(\square^n)$ such that $f \circ \iota = g$. Hence P is at least injective. In order to prove the surjectivity of P, we only have to show that this f is always an element of T when $g \in S$.

From $f \circ \iota = g$ we have $\psi \circ f \circ \iota = \psi \circ g$. Since $\rho \circ \psi \circ g = \mathrm{id}$, we know that $(\psi \circ g)((x_1, \ldots, x_{n-1})) = (x_1, \ldots, x_{n-1}, z)$ where z may depend upon $x_1, \ldots, x_{n-1}$. We claim that z is constant. If not, then we have $(\psi \circ g)(\mathbf{x}) = (x_1, \ldots, x_{n-1}, 0)$ and $(\psi \circ g)(\mathbf{y}) = (y_1, \ldots, y_{n-1}, 1)$ for some $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$. Since $\psi \circ g$ is a cubical map, it does not increase Hamming distance. Thus

$$\begin{aligned} \mathrm{d}(\mathbf{x}, \mathbf{y}) + 1 &= \mathrm{d}((x_1, \ldots, x_{n-1}, 0), (y_1, \ldots, y_{n-1}, 1)) \\ &= \mathrm{d}((\psi \circ g)(\mathbf{x}), (\psi \circ g)(\mathbf{y})) \\ &\leq \mathrm{d}(\mathbf{x}, \mathbf{y}), \end{aligned}$$

and this contradiction proves the claim.

Next we show that z is always equal to 0. Assume the opposite, that is, $z = 1$ for some $g \in S$. Consider first the case when $\mathrm{Im}(g)$ is equal to the facet $(*, \ldots, *, 0)$ or a neighbouring facet of $(*, \ldots, *, 0)$. Then there are some values $\mathbf{x}, \mathbf{y}$ such that $g((x_1, \ldots, x_{n-1})) = (y_1, \ldots, y_{n-1}, 0)$. Hence,

$$\begin{aligned} (x_1, \ldots, x_{n-1}, 1) &= (\psi \circ g)((x_1, \ldots, x_{n-1})) \\ &= \psi((y_1, \ldots, y_{n-1}, 0)) \\ &= (y_1, \ldots, y_{n-1}, 0), \end{aligned}$$

where the last equality follows from $\psi \circ \iota = \iota$. But this is a contradiction, since the last coordinate of the first vector is different from the last coordinate of the last vector in these equalities.

Thus we are left with the case when $\mathrm{Im}(g)$ is the opposite face of $(*, \ldots, *, 0)$, that is, $\mathrm{Im}(g) = (*, \ldots, *, 1)$. Define $\psi_0 : \Box^n \longrightarrow \Box^n$ by

$$\begin{aligned} \psi_0((x_1, \ldots, x_{n-1}, 0)) &= (x_1, \ldots, x_{n-1}, 0), \\ \psi_0((x_1, \ldots, x_{n-1}, 1)) &= g((x_1, \ldots, x_{n-1})). \end{aligned}$$

Now $\psi \circ \psi_0 = \psi_0 \circ \psi = \text{id}$, and this contradicts the assumption that ψ is not bijective.

Thus we have proven that z is always equal to 0. So $\psi \circ g = \iota$ and we conclude that $\psi \circ f \circ \iota = \psi \circ g = \iota$, and so $f \in T$. Hence $P : T \longrightarrow S$ is bijective. Thus $|T| = |S| \leq 4$ and

$$\sum_{f \in T} \varepsilon(f) = \sum_{f \in T} \varepsilon(P(f)) = \sum_{g \in S} \varepsilon(g) = 0.$$

QED

In the following we will investigate the linear relations between coloring parameters, which are defined in analogy to the numbers A_I and B_J of the simplicial case.

Definition 32 *Let $f \in \text{Ord}_n(\square^m)$ be an ordered n-face of a standard m cube, $\square$ an n-dimensional oriented cubical pseudomanifold, and $\phi : \square \longrightarrow \square^m$ a cubical coloring. We say that an ordered facet $\tilde{f}$ of $\square$ is* f-colored *when $\phi \circ \tilde{f} = f$ holds. We define*

$$A_f := \sum_{\phi \circ \tilde{f} = f} \varepsilon(\tilde{f}).$$

Similarly, for an ordered boundary $(n-1)$-face $g \in \text{Ord}_{n-1}(\partial \square^m)$ of the standard m-cube, we say that an ordered boundary subfacet $\tilde{g}$ of $\square$ is g-colored *when $\phi \circ \tilde{g} = g$ holds. We define*

$$B_g := \sum_{\phi \circ \tilde{g} = g} \varepsilon(\Omega(\tilde{g})).$$

Observe that $\tilde{f} \in \text{Ord}_n(\square)$ satisfies $\phi \circ \tilde{f} = f$ if and only if the facet $F := \text{Im}(\tilde{f})$ satisfies $\text{Im}\,\phi|_F = \text{Im}(f)$ and $\tilde{f} = \phi|_F{}^{-1}$. This allows us to write

$$A_f = \sum_{\phi(F) = \text{Im}(f)} \varepsilon(\phi|_F{}^{-1}),$$

where F ranges over the facets of $\Box$. Similarly we can write

$$B_g = \sum_{\phi(G)=\mathrm{Im}\, g} \varepsilon(\Omega(\phi|_G{}^{-1})),$$

where G ranges over the boundary subfacets of $\Box$.

As in the simplicial case, condition (ii) of Definition 29 implies the following sign relations.

Lemma 21 *We have*

$$A_{f\circ\pi} = \mathrm{sign}(\pi) \cdot A_f \tag{2.17}$$

for all $\pi \in \mathcal{B}_n$, *and*

$$B_{g\circ\tau} = \mathrm{sign}(\tau) \cdot B_g \tag{2.18}$$

for all $\tau \in \mathcal{B}_{n-1}$.

Lemma 22 (Fundamental coloring lemma for cubical pseudomanifolds) *Let* $\Box$ *be an orientable* n*-dimensional cubical pseudomanifold. Color the vertices of* $\Box$ *arbitrarily with the vertices of the standard* n*-cube* $\Box^n$*, i.e., define a cubical map* $\phi : \Box \to \Box^n$*. Then we have*

$$A_h = B_{h\circ\iota}$$

for every ordered n*-face* $h \in \mathrm{Ord}_n(\Box^n)$*. In particular, when* $\Box$ *is an ordered cubical pseudomanifold without boundary, then we have*

$$A_h = 0.$$

Proof: Our reasoning will be analogous to the proof of Lemma 4. Construct a graph

$G = (V, E)$ associated to $\square$ and its coloring ϕ. The vertex set will be

$$V := \{f \in \mathrm{Ord}_n(\square) \ : \ \phi \circ f \circ \iota = h \circ \iota\}.$$

We can write V as the disjoint union of

$$V_1 := \{f \in V \ : \ \phi \circ f = h\},$$

and

$$V_2 := \{f \in V \ : \ \phi \circ f \neq h\}.$$

Split V_1 and V_2 into the disjoint union of smaller sets. We have

$$V_i = V_i' \uplus V_i'' \ \ (i = 1, 2)$$

where

$$V_i' = \{f \in V_i \ : \ f \circ \iota \notin \mathrm{Ord}_{n-1}(\partial \square)\},$$

and

$$V_i'' = \{f \in V_i \ : \ f \circ \iota \in \mathrm{Ord}_{n-1}(\partial \square)\}.$$

We define the edge set of G as a disjoint union $E := E_1 \cup E_2$ where

$$E_1 := \{(f_1, f_2) \ : \ f_1 \circ \iota = f_2 \circ \iota, \ f_1 \neq f_2\},$$

and

$$E_2 := \{(f_1, f_2) \ : \mathrm{Im}(f_1) = \mathrm{Im}(f_2), \ f_1 \neq f_2\}.$$

Consider the subgraph of G consisting of the edges in E_1. We claim that this subgraph is a matching on the vertices $V_1' \uplus V_2'$, and has the vertices $V_1'' \uplus V_2''$ as singletons. Since the vertices V_i'' are on the boundary, they cannot be adjacent to any other vertex through an edge in E_1. Thus for the moment we can restrict our attention to the vertices $V_1' \uplus V_2'$. Since $f_1 \circ \iota = f_2 \circ \iota$ and $\phi \circ f_1 \circ \iota = h \circ \iota$, the intersection of $\mathrm{Im}(f_1)$ and $\mathrm{Im}(f_2)$ is the subfacet $\mathrm{Im}(\phi \circ f_1 \circ \iota)$. Hence given f_1 there is exactly one way to choose f_2, because of condition (ii) of Definition 26. Thus the claim holds. Moreover, by condition (iii) of Definition 29, the signs of adjacent vertices in this subgraph are opposite.

It is clear that the edges in E_2 only connects vertices in V_2. Consider the subgraph that consists of vertices in V_2 and the edges in E_2. Since two different vertices $f_1, f_2 \in V_2$ are adjacent if $\mathrm{Im}(f_1) = \mathrm{Im}(f_2)$, this graph consists of vertex-disjoint cliques. Each clique is a complete graph and corresponds to a facet of $\Box$ that is not completely colored. Apply Lemma 20 with $\psi = h^{-1} \circ \phi$. Since the set T in the lemma has cardinality less than or equal to four, and since the cardinality is even, we have that each clique is isomorphic to either K_2 or K_4. (Recall that K_n is the complete graph on n vertices.) Thus the possible degrees in this subgraph are 1 and 3.

We can now present the results about the different degrees in the following table.

Degree in	V_1'	V_1''	V_2'	V_2''
E_1	1	0	1	0
E_2	0	0	1 or 3	1 or 3

In order to make a bijection argument we will consider a subgraph $\widehat{G} = (V, \widehat{E})$ of the graph G. For each clique in E_2 that is isomorphic to K_4, select two edges that are vertex-disjoint, such that vertices connected with these two edges have opposite sign. By Lemma 20 it is possible to make such a choice. The set T in the lemma contains as many positively oriented ordered facets as negatively oriented ones. Let $\widehat{E}_2$ be a subset of E_2 consisting of these selected edges and the cliques isomorphic to K_2. Each vertex in V_2 has degree exactly 1 in $\widehat{E}_2$. Thus the table looks like

Degree in	V_1'	V_1''	V_2'	V_2''
E_1	1	0	1	0
$\widehat{E}_2$	0	0	1	1

Let $\widehat{G}$ be the subgraph of G consisting of the edges $\widehat{E} = E_1 \uplus \widehat{E}_2$. Observe that two adjacent vertices, f_1 and f_2, which are adjacent in the graph $\widehat{G}$ have different signs. That is, $\varepsilon(f_1) = -\varepsilon(f_2)$.

The rest of the proof is now the same as the end of the proof of Lemma 4. The graph $\widehat{G}$ consists of singletons, paths and cycles. A path in $\widehat{G}$ that connects two vertices $f_1, f_2 \in V_1'$ will have odd length. Hence we know that $\varepsilon(f_1) = -\varepsilon(f_2)$. The same is true of a path that connects two vertices in V_2''. A path that connects a vertex in V_1' with a vertex in V_2'' will have even length, and hence the two end points of such a path will have the same sign. We conclude that

$$\sum_{f \in V_1'} \varepsilon(f) = \sum_{f \in V_2''} \varepsilon(f).$$

The h-colored ordered facets are represented in the graph $\widehat{G}$ by the vertices in V_1. Hence

$$A_h = \sum_{\phi \circ f = h} \varepsilon(f) = \sum_{f \in V_1} \varepsilon(f).$$

Similarly the $h \circ \iota$-colored boundary faces are represented by the vertices in $V_1'' \uplus V_2''$, and thus

$$\begin{aligned} B_{h \circ \iota} &= \sum_{\phi \circ g = h \circ \iota} \varepsilon(\Omega(g)) \\ &= \sum_{\phi \circ f \circ \iota = h \circ \iota} \varepsilon(f) \\ &= \sum_{f \in V_1'' \uplus V_2''} \varepsilon(f), \end{aligned}$$

where in the first sum we are summing over an ordered boundary subfacet g, and in the second sum an ordered facet f, such that $f \circ \iota$ is an ordered boundary subfacet.

By combining the three above equations we get

$$\begin{aligned} A_h &= \sum_{f \in V_1'} \varepsilon(f) + \sum_{f \in V_1''} \varepsilon(f) \\ &= \sum_{f \in V_2''} \varepsilon(f) + \sum_{f \in V_1''} \varepsilon(f) \\ &= B_{h \circ \iota}. \end{aligned}$$

Observe that the paths and the singletons in the graph $\widehat{G}$ describe a bijection between the signed set of h-colored facets of $\Box$ and the signed set of $h \circ \iota$-colored subfacets of $\Box$. Hence the proof is bijective. **QED**

Observe that Lemma 19 follows from the above lemma, by considering $\partial \Box^n$ as an orientable $(n-1)$-dimensional cubical pseudomanifold. In fact, we could have proven Lemma 19, Lemma 20, and Lemma 22 at the same time by using induction on dimension, without referring to Lemma 16.

Theorem 6 (Master theorem for cubical colorings)

Let $\Box$ be an orientable n-dimensional cubical pseudomanifold, and $\phi : \Box \longrightarrow \Box^m$ be a cubical coloring of it with colors of the standard m-cube. Let $\lambda : \Box^m \longrightarrow \Box^n$ be any cubical map. Then we have

$$\sum_{\substack{f \in \mathrm{Ord}_n(\Box^m) \\ \lambda \circ f = \mathrm{id}}} A_f = \sum_{\substack{g \in \mathrm{Ord}_{n-1}(\Box^m) \\ \lambda \circ g = \iota}} B_g. \tag{2.19}$$

In particular, for n-dimensional oriented cubical pseudomanifolds without boundary we *have*

$$\sum_{\substack{f \in \mathrm{Ord}_n(\Box^m) \\ \lambda \circ f = \mathrm{id}}} A_f = 0. \tag{2.20}$$

Proof: We apply Lemma 22 to the coloring $\lambda \circ \phi : \Box \longrightarrow \Box^n$, and the ordered n-face $\mathrm{id} \in \mathrm{Ord}_n(\Box^n)$. We obtain $A_{\mathrm{id}}^{\lambda\circ\phi} = B_{\iota}^{\lambda\circ\phi}$. As in the simplicial case, $A_{\mathrm{id}}^{\lambda\circ\phi}$ is equal to the left hand side, and $B_{\iota}^{\lambda\circ\phi}$ is equal to the right hand side of (2.19). **QED**

2.3.3 The vector space of A_f-s and B_g-s

In this subsection we show that in general, in analogy to the simplicial results of Subsection 2.2.3, there cannot be more linear relations among the numbers A_f and B_g than those implied by Theorem 6.

For this purpose we need to define a cubical analogue of the exterior power of a vector space.

Definition 33 *Let* Cube(m, k) *stand for a vector space with basis*

$$\{e_\sigma \,:\, \sigma \in \Box_k^m\},$$

i.e., Cube(m, k) *has a basis indexed with the k-faces of the standard m-cube. We represent the faces of $\Box^m$ with vectors $(u_1, \ldots, u_m) \in \{0, 1, *\}^m$ in the usual way. Given a k-face $\sigma = (u_1, \ldots, u_m)$ of $\Box^m$ we define the* standard embedding ι_σ *of σ by*

$$\iota_\sigma : (x_1, x_2, \ldots, x_k) \longmapsto (u_1, u_2, \ldots, x_1, \ldots, x_2, \ldots, x_k, \ldots, u_m)$$

where the coordinates $x_1, x_2, \ldots, x_n$ are substituted into the $*$ signs from left to the right in this order. Given a cubical order f on a k-face σ, there is a unique $\pi \in \mathcal{B}_k$ such that

$f = \iota_\sigma \circ \pi$. *We introduce the notational convention*

$$e_f := \text{sign}(\pi) \cdot e_\sigma.$$

Now we can define the weight vector of a coloring as in the simplicial case.

Definition 34 *Given a coloring* $\phi : \square \longrightarrow \square^m$ *of an* n*-dimensional cubical pseudomanifold, we define the* weight vector *of the coloring as the following element of* $\text{Cube}(m, n) \oplus \text{Cube}(m, n-1)$.

$$\mathbf{w}_\phi := \sum_{\sigma \in \square_n^m} A_{\iota_\sigma} \cdot e_\sigma + \sum_{\tau \in \square_{n-1}^m} B_{\iota_\tau} \cdot e_\tau.$$

We denote the $\mathbb{Z}$*-module resp. vector space generated by all weight vectors by* $\mathcal{M}$ *resp.* $\mathcal{W}$*. Moreover, as in the simplicial case, we denote the submodule resp. subspace generated by the weight vectors of colorings of oriented cubical pseudomanifolds without boundary by* $\mathcal{M}_0$ *resp.* $\mathcal{W}_0$*. (Note that both* $\mathcal{M}_0$ *and* $\mathcal{W}_0$ *are subsets of* $\text{Cube}(m, n)$*.)*

Again, as in the simplicial case, the equations (2.17) and (2.18) allow us to think of A_f resp. B_g as the "coefficient of e_f resp. e_g in $\mathbf{w}_\phi$".

Introducing $A_\sigma := A_{\iota_\sigma}$ and $B_\tau := B_{\iota_\tau}$ we may write the equations (2.19) in the following form.

$$\sum_{\substack{\sigma \in \square_n^m \\ \lambda(\sigma) = \{0,1\}^n}} \text{sign}(\lambda \circ \iota_\sigma) \cdot A_\sigma = \sum_{\substack{\tau \in \square_{n-1}^m \\ \lambda(\tau) = (*, *, \ldots, *, 0)}} \text{sign}(\lambda \circ \iota_\tau) \cdot B_\tau. \qquad (2.21)$$

Similarly, the equations (2.20) are equivalent to

$$\sum_{\substack{\sigma \in \square_n^m \\ \lambda(\sigma) = \{0,1\}^n}} \operatorname{sign}(\lambda \circ \iota_\sigma) \cdot A_\sigma = 0. \tag{2.22}$$

Among the equations (2.21) and (2.22) there is an important type of special case.

Definition 35 *Let $\tau = (u_1, \ldots, u_m)$ be an $(n-1)$-face of $\square^m$. Assume that the $*$ signs in $(u_1, \ldots, u_m)$ are $u_{i_1}, u_{i_2}, \ldots, u_{i_{n-1}}$. Define the cubical map $\lambda_\tau : \square^m \longrightarrow \square^n$ as follows.*

$$\lambda_\tau((x_1, x_2, \ldots, x_m)) := \begin{cases} (x_{i_1}, x_{i_2}, \ldots, x_{i_{n-1}}, 0) & \textit{when } \mathbf{x} \in \tau \\ (x_{i_1}, x_{i_2}, \ldots, x_{i_{n-1}}, 1) & \textit{when } \mathbf{x} \notin \tau \end{cases}$$

We verify that λ_τ is a cubical map. If both $\mathbf{x}$ and $\mathbf{y}$ are adjacent vertices, and if they belong or don't belong to τ at the same time, then the last coordinate of their λ_τ image will agree, so $\lambda_\tau(\mathbf{x})$ and $\lambda_\tau(\mathbf{y})$ will be adjacent or equal. If $\mathbf{x}$ and $\mathbf{y}$ are adjacent, and say $\mathbf{x} \in \tau, \mathbf{y} \notin \tau$, then there is an $i \notin \{i_1, i_2, \ldots, i_{n-1}\}$ such that $x_i \neq y_i$. Thus we must have $(x_{i_1}, x_{i_2}, \ldots, x_{i_{n-1}}) = (y_{i_1}, y_{i_2}, \ldots, y_{i_{n-1}})$, and so $\lambda_\tau(\mathbf{x})$ and $\lambda_\tau(\mathbf{y})$ are adjacent.

Let us now evaluate (2.21) and (2.22) for $\lambda = \lambda_\tau$. First we have to find those n-faces σ of $\square^m$ for which $\lambda_\tau(\sigma) = \{0,1\}^n$. It is easy to check that this holds iff $\lambda_\tau(\sigma)$ contains $(*, \ldots, *, 0)$, which holds iff $\sigma \supset \tau$. Thus the right hand side of both equations will be

$$\sum_{\sigma \supset \tau} \operatorname{sign}(\lambda_\tau \circ \iota_\sigma) \cdot A_\sigma.$$

On the other hand, for an $(n-1)$-face τ' of $\square^m$, we have $\lambda_\tau(\tau') = (*, *, \ldots, *, 0)$ iff $\tau = \tau'$. Therefore we obtain

$$\sum_{\sigma \supset \tau} \operatorname{sign}(\lambda_\tau \circ \iota_\sigma) \cdot A_\sigma = B_\tau \tag{2.23}$$

in the general case, and

$$\sum_{\sigma \supset \tau} \mathrm{sign}(\lambda_\tau \circ \iota_\sigma) \cdot A_\sigma = 0 \tag{2.24}$$

for oriented cubical pseudomanifolds without boundary.

Note next that the maps ι_μ, where μ is an $(n+1)$-dimensional face of $\square^m$, may be considered as a coloring of the surface of the standard $(n+1)$-cube. The weight vector $\mathbf{p}_\mu$ of this coloring is a cubical analogue of the notion of elementary weight vector of the first kind defined in the simplicial case. Similarly, the maps ι_ν, where ν is an n-dimensional face of $\square^m$, may be considered as a coloring of a standard n-cube. The weight vector $\mathbf{q}_\nu$ of this coloring is the cubical analogue of an elementary weight vector of the second kind.

Theorem 7 *Let M_0 be the set of those elementary weight vectors $\mathbf{p}_\mu$ of the first kind for which the code $(u_1, \ldots, u_m)$ of the $(n+1)$-face μ has no 0-s between the last two $*$ sign. Let M be the union of M_0 with the set of those elementary weight vectors $\mathbf{q}_\nu$ of the second kind, for which the code $(v_1, \ldots, v_m)$ of the n-face ν has only 1-s after the last $*$ sign. Then M_0 is a $\mathbb{Z}$-basis of $\mathcal{M}_0$, and M is a $\mathbb{Z}$-basis of $\mathcal{M}$. Moreover, the equations (2.23) together with (2.17) and (2.18) span the vector space of all linear relations among the A_f-s and B_g-s, and the analogous statements hold for the equations (2.24) and (2.17) when we restrict ourselves to oriented n-dimensional cubical pseudomanifolds without boundary.*

Proof: In analogy to the simplicial case, we prove the theorem first for the boundariless case. We introduce an antilexicographic order on the vectors $e_\sigma \in \mathrm{Cube}(m,n)$ as follows. Let us write σ in the form $(u_1, \ldots, u_m) \in \{0, 1, *\}^m$. We set $0 < * < 1$, and define $(u_1, \ldots, u_m) < (u'_1, \ldots, u'_m)$ iff for the largest i satisfying $u_i \neq u'_i$ we have $u_i < u'_i$.

With this order, the antilexicographically first term of a $\mathbf{p}_\mu$ can be obtained by replacing the last $*$ of $\mu = (u_1, \ldots, u_m)$ by a 0. If we restrict ourselves to the $\mathbf{p}_\mu$-s which have no 0 between the last two $*$-s, then we get every $(u_1, \ldots, u_m)$ that contains a 0 after the last $*$ exactly once as the antilexicographically first term of some $\mathbf{p}_\mu$. (To get this μ we have to replace the first 0 after the last $*$ by $*$.) Hence our chosen $\mathbf{p}_\mu$-s are linearly independent.

Assume now that there is a $\mathbf{r} \in \mathcal{M}_0$ that is not a $\mathbb{Z}$-linear combination of our $\mathbf{p}_\mu$-s. Let us take such a counterexample $\mathbf{r}$ with the largest possible antilexicographically first term. The antilexicographically first term of $\mathbf{r}$ must be of the form $(u_1, u_2, \ldots, u_k, *, 1, 1, \ldots, 1)$ for some $k \geq n-1$, because if there is a 0 after the last $*$, then we can subtract a multiple of one of our $\mathbf{p}_\mu$-s such that all terms of the difference will be antilexicographically larger. Also, by the definition of our antilexicographic order, all antilexicographically larger terms must be of the form $(u'_1, \ldots, u'_k, u'_{k+1}, 1, 1, \ldots, 1)$, where $u'_{k+1} \in \{*, 1\}$, because $u'_{k+1} \geq *$. We claim that only the face represented by the antilexicographically first term of $\mathbf{r}$ contains the $(n-1)$-face $\tau := (u_1, \ldots, u_k, 0, 1, 1, \ldots, 1)$. In fact, if $u'_{k+1} = 1$, then $(u'_1, \ldots, u'_k, u'_{k+1}, 1, 1, \ldots, 1) \not\supset \tau$, and if $u'_{k+1} = *$ then the only way $(u'_1, \ldots, u'_k, u'_{k+1}, 1, 1, \ldots, 1)$ can contain τ is to have $u_1 = u'_1, \ldots, u_k = u'_k$.

The vector $\mathbf{r}$ is in the linear span of weight vectors, and the coefficients of its terms must satisfy the equations (2.23). Applying (2.24) with τ it follows that the antilexicographically smallest term of $\mathbf{r}$ is 0, a contradiction.

Therefore our $\mathbf{p}_\mu$-s are a $\mathbb{Z}$-basis for $\mathcal{M}_0$ and a vector space basis of $\mathcal{W}_0$. Note that, in proving this, we used only the equations (2.24), and implicitly (2.18). Hence the kernel defined by these linear relations is not larger than $\mathcal{W}_0$, and so these linear relations span the vector space of all linear relations.

The proof of the fact that M generates $\mathcal{M}$ runs in analogy to the boundariless case. We introduce the same antilexicographic order on the basis $e_\sigma \in \mathrm{Cube}(m,n)$ as before. Again we assume the existence of a counterexample $\mathbf{r} \in \mathcal{M}$ that has a largest possible antilexicographically first term in $\mathrm{Cube}(m,n)$. (In analogy with the simplicial case, it may happen that we have $\mathbf{r} \notin \mathrm{Cube}(m,n)$, but $\mathbf{r}$ has a uniquely defined $\mathrm{Cube}(m,n)$-component.) As before, we can show that the antilexicographically first term of the Cube(m,n)-component of this $\mathbf{r}$ is the multiple of e_ν, where ν is of the form $(u_1, u_2, \ldots, u_k, *, 1, 1, \ldots, 1)$ for some $k \geq n-1$. Subtracting an appropriate multiple of $\mathbf{q}_\nu$ we obtain a counterexample with larger antilexicographically smallest term, a contradiction. The only way out is to assume that the Cube(m,n)-component of $\mathbf{r}$ is 0, but then equations (2.23) guarantee that the Cube$(m,n-1)$-component of $\mathbf{r}$ is zero as well. This shows that M generates $\mathcal{M}$. On the other hand, the antilexicographically first terms of the elements of M are pairwise different, and so M is a $\mathbb{Z}$-basis. Therefore M is also a vector space basis of $\mathbf{w}_\phi$. In our reasoning we have used only equations (2.23), (2.18) and (2.17). Hence these equations generate the space of all linear relations. **QED**

Remark The fact that the equations (2.17), (2.18), and (2.23) generate all linear relations among the A_f-s and B_g-s also may be shown directly, in analogy to Lemma 5.

Remark In both cases (simplicial and cubical) our theorems and lemmas remain valid if we define the orientation ε to map into $\mathbb{Z}_p$. Thus the $A_{(i_1,i_2,\ldots,i_{n+1})}$-s and A_f-s, resp. the $B_{(j_1,j_2,\ldots,j_n)}$ and B_g, lie in $\mathbb{Z}_p$, and we obtain mod p congruences from our coloring theorems. We get an interesting degeneration in the case of $p=2$: here we may consider every manifold "orientable" by defining a function ε that assigns 1 to every ordered facet. Hence the mod 2 analogues of our coloring theorems hold even for non-orientable pseudomanifolds. In the cubical case we can reproduce this way the cubical results published in [11].

2.4 Cubical homology

2.4.1 Cubical homology groups and chain maps

If there were an appropriate cubical analogue of simplicial homology, we could instantly generalize the results of Subsection 2.2.4 to the cubical case. In this subsection we show that one can define such a homology. It is sufficient to build the theory of absolute cubical homology, because then we may obtain relative homology and mapping cones using exclusively the methods of homological algebra.

In the definition of the cubical homology groups we will embed the groups $\mathcal{B}_k$ into each other in the following way. Recall that ι_k denotes the embedding $\iota_k : \square^k \longrightarrow \square^{k+1}$ that takes $(x_1, \ldots, x_k)$ into $(x_1, \ldots, x_k, 0)$. This induces an embedding $\jmath_k : \mathcal{B}_k \longrightarrow \mathcal{B}_{k+1}$ defined by

$$\jmath_k(\pi)((x_1, \ldots, x_k, x_{k+1})) := (\pi((x_1, \ldots, x_k)), x_{k+1}).$$

Obviously, $\jmath_k(\mathcal{B}_k)$ is the stabilizer of the facet $(*, *, \ldots, *, 0)$ in $\mathcal{B}_{k+1}$. We will often use the following straightforward identities that hold for all $\pi \in \mathcal{B}_k$. We have

$$\text{sign}(\jmath_k(\pi)) = \text{sign}(\pi) \tag{2.25}$$

and

$$\jmath_k(\pi) \circ \iota_k = \iota_k \circ \pi. \tag{2.26}$$

Definition 36 *Given a cubical complex* $\square$, *let* $S_k(\square)$ *stand for the free* $\mathbb{Z}$*-module generated by the basis* $\{\sigma : \sigma \in \square_k\}$. *We represent* $S_k(\square)$ *as the module generated by* $\{f : f \in \text{Ord}_k(\square)\}$ *modulo the relations*

$$[f \circ \pi] = \text{sign}(\pi) \cdot [f]$$

for all $\pi \in \mathcal{B}_k$. *The symbol* $[f]$ *stands for the equivalence class represented by the ordered face* f.

We define the boundary map $\partial_k : S_k(\square) \longrightarrow S_{k-1}(\square)$ *as follows. We choose a system* *of representatives* $\pi_1, \ldots, \pi_{2k}$ *for the set of left cosets* $[\mathcal{B}_k : \jmath_{k-1}(\mathcal{B}_{k-1})]$. *We set*

$$\partial_k([f]) := \sum_{i=1}^{2k} \operatorname{sign}(\pi_i) \cdot [f \circ \pi_i \circ \iota_{k-1}]. \tag{2.27}$$

We call the homology groups $H_k(C_\bullet(\Box))$ *the* cubical homology groups of $\Box$.

When we show that ∂ is well defined and that it is a boundary map, we will use some elementary group-theoretical facts, stated in the following lemma.

Lemma 23 *Let* G *be a group,* H *a subgroup of finite index in* G*, and* $g_1, \ldots, g_k$ *a system of representatives for the set of left cosets* $[G : H]$. *Then the following statements hold.*

1. *For all* $g \in G$ *the set* $\{g \cdot g_1, \ldots, g \cdot g_k\}$ *is a system of representatives.*

2. *If* g *belongs to the normalizer* $\mathrm{N}_G(H)$ *of* H *in* G*, which is defined as*

$$\mathrm{N}_G(H) := \{g \in G \ : \ g \cdot H \cdot g^{-1} = H\},$$

then the set $\{g_1 \cdot g, \ldots, g_k \cdot g\}$ *is a system of representatives. Moreover, if* $g \notin H$ *then the action* $g_i \cdot H \mapsto g_i \cdot g \cdot H$ *of* g *on the left cosets is fixed-point free.*

Proof: Because of the finiteness of $[G : H]$, it is sufficient to show in both cases that the listed elements belong to pairwise different left cosets. In the first case, we have $g \cdot g_i \cdot H = g \cdot g_j \cdot H$ iff $g_i \cdot H = g_j \cdot H$, which holds iff $i = j$. In the second case $g_i \cdot g \cdot H = g_j \cdot g \cdot H$ is equivalent to $g_i \cdot g \cdot H \cdot g^{-1} = g_j \cdot g \cdot H \cdot g^{-1}$, and so we reach the same conclusion by $g \cdot H \cdot g^{-1} = H$. Finally if $g_i \cdot g \cdot H = g_i \cdot H$ then we have $H = g \cdot H$ and $g \in H$. **QED**

Lemma 24 *The map* ∂ *is a well-defined boundary map.*

Proof: Note first, that the definition of ∂ does not depend on the choice of the coset representatives $\pi_1, \ldots, \pi_{2k}$. In fact, by (2.25) and (2.26) we have

$$\operatorname{sign}(\pi \circ \jmath(\gamma)) \cdot [f \circ \pi \circ \jmath(\gamma) \circ \iota)] = \operatorname{sign}(\pi) \cdot \operatorname{sign}(\jmath(\gamma)) \cdot [f \circ \pi \circ \iota \circ \gamma] = \operatorname{sign}(\pi) \cdot [f \circ \pi \circ \iota]$$

for every $\pi \in \mathcal{B}_k$ and $\gamma \in \mathcal{B}_{k-1}$. Hence, if π and π' belong to the same left coset, i.e., $\pi' = \pi \circ \jmath(\mathcal{B}_{k-1})$ for some $\gamma \in \jmath(\mathcal{B}_{k-1})$, then they contribute the same term to $\partial([f])$.

Next we show that ∂ does not depend on the choice of the representative f for $[f]$ either. By Lemma 23, if $\pi_1, \ldots, \pi_{2k}$ are left coset representatives then $(\pi \circ \pi_1), \ldots, (\pi \circ \pi_{2k})$ are also left coset representatives. Thus we have

$$\begin{aligned} \sum_{i=1}^{2k} \operatorname{sign}(\pi_i) \cdot [(f \circ \pi) \circ \pi_i \circ \iota] &= \operatorname{sign}(\pi) \cdot \sum_{i=1}^{2k} \operatorname{sign}(\pi \circ \pi_i) \cdot [f \circ (\pi \circ \pi_i) \circ \iota] \\ &= \operatorname{sign}(\pi) \cdot \sum_{i=1}^{2k} \operatorname{sign}(\pi_i) \cdot [f \circ \pi_i \circ \iota]. \end{aligned}$$

Therefore the definition of $\partial([f])$ gives the same result for f and $f \circ \pi$, for all $\pi \in \mathcal{B}_k$. Hence ∂ is well-defined.

Finally we show that ∂ is a boundary map. Let $\pi_1, \ldots, \pi_{2k}$ be a system of representatives for $[\mathcal{B}_k : \jmath(\mathcal{B}_{k-1})]$ and $\pi'_1, \ldots, \pi'_{2k-2}$ be a system of representatives for $[\mathcal{B}_{k-1} : \jmath(\mathcal{B}_{k-2})]$. Then for all $f \in \operatorname{Ord}_k(\square)$ we have

$$\begin{aligned} \partial^2([f]) &= \partial\left(\sum_{i=1}^{2k} \operatorname{sign}(\pi_i) \cdot [f \circ \pi_i \circ \iota]\right) \\ &= \sum_{i=1}^{2k} \operatorname{sign}(\pi_i) \cdot \sum_{j=1}^{2k-2} \operatorname{sign}(\pi'_j) \cdot \left[f \circ \pi_i \circ \iota \circ \pi'_j \circ \iota\right] \\ &= \sum_{i=1}^{2k} \sum_{j=1}^{2k-2} \operatorname{sign}(\pi_i \circ \jmath(\pi'_j)) \cdot \left[f \circ \pi_i \circ \jmath(\pi'_j) \circ \iota^2\right]. \end{aligned}$$

Here the elements $\pi_i \circ \jmath(\pi'_j)$ form a system of representatives for $[\mathcal{B}_k : \jmath^2(\mathcal{B}_{k-2})]$. Similarly to the case of $\partial(f)$, we can show that if we replace $\pi_i \circ \jmath(\pi'_j)$ with any other system of representatives for $[\mathcal{B}_k : \jmath^2(\mathcal{B}_{k-2})]$, the result remains the same. Thus we have

$$\partial^2([f]) = \sum_{i=1}^{4k(k-1)} \operatorname{sign}(\pi''_i) \cdot \left[f \circ \pi''_i \circ \iota^2\right],$$

where $\pi''_1, \pi''_2, \ldots, \pi''_{4k(k-1)}$ is any system of representatives for $[\mathcal{B}_k : \jmath^2(\mathcal{B}_{k-2})]$.

Consider $\tau \in \mathcal{B}_k$ defined by

$$\tau(x_1, \ldots, x_k) := (x_1, \ldots, x_{k-2}, x_k, x_{k-1}).$$

Geometrically, τ can be represented as the reflection in the hyperplane $x_k = x_{k-1}$, and so we have $\tau^2 = 1$ and $\operatorname{sign}(\tau) = -1$. Observe that $\tau \circ \iota_{k-1} \circ \iota_{k-2} = \iota_{k-1} \circ \iota_{k-2}$. Moreover τ commutes with all elements in $\jmath^2(\mathcal{B}_{k-2})$, thus we have

$$\tau \circ \jmath^2(\mathcal{B}_{k-2}) \circ \tau^{-1} = \jmath^2(\mathcal{B}_{k-2}).$$

Hence τ is in the normalizer, and thus by Lemma 23, $(\pi''_1 \circ \tau), \ldots, (\pi''_{4k(k-1)} \circ \tau)$ is also a system of left coset representatives. Therefore

$$\partial^2([f]) = \sum_{i=1}^{4k(k-1)} \operatorname{sign}(\pi''_i \circ \tau) \cdot \left[f \circ \pi''_i \circ \tau \circ \iota^2\right] = - \sum_{i=1}^{4k(k-1)} \operatorname{sign}(\pi''_i) \cdot \left[f \circ \pi''_i \circ \iota^2\right] = -\partial^2([f])$$

holds, implying $\partial^2([f]) = 0$. **QED**

Remarks

1. We only have to modify the end of the last proof, if we also want it to work in the case of chain-complexes with coefficient-field of characteristic 2. By Lemma 23, $\tau \notin \jmath^2(\mathcal{B}_{k-2})$ induces a fixed point free permutation of the left cosets $[\mathcal{B}_k : \jmath^2(\mathcal{B}_{k-2})]$.

Thus we can choose a system of representatives such that $\pi_i'' \circ \tau$ will be equal to a π_j'' with $j \neq i$ for each i. For this j we also have $\pi_j'' \circ \tau = \pi_i''$, because τ is an involution. Hence we can arrange these coset representatives into pairs $\{\pi_i'', \pi_j''\}$, with $\pi_i'' \circ \iota^2 = \pi_j'' \circ \iota^2$ and $\mathrm{sign}(\pi_i'') = -\mathrm{sign}(\pi_j'')$. The terms $\mathrm{sign}(\pi_i'') \cdot [f \circ \pi_i'' \circ \iota^2]$ and $\mathrm{sign}(\pi_j'') \cdot [f \circ \pi_i'' \circ \iota^2]$ cancel for each pair $\{\pi_i'', \pi_j''\}$, and so we have $\partial^2([f]) = 0$.

2. Consider the system of representatives $\pi_1^0, \pi_2^0, \ldots, \pi_k^0, \pi_1^1, \pi_2^1, \ldots, \pi_k^1$ for $[\mathcal{B}_k : \jmath(\mathcal{B}_{k-1})]$, defined as follows,

$$\begin{aligned} \pi_i^0(x_1, \ldots, x_k) &:= (x_1, \ldots, x_{i-1}, x_k, x_i, x_{i+1}, \ldots, x_{k-1}), \qquad \text{and} \\ \pi_i^1(x_1, \ldots, x_k) &:= (x_1, \ldots, x_{i-1}, 1 - x_k, x_i, x_{i+1}, \ldots, x_{k-1}) \end{aligned}$$

for $i = 1, 2, \ldots, k$.

We can obtain the representatives π_i^0 by reflecting one after the other in the following hyperplanes: $x_k = x_{k-1}, x_{k-1} = x_{k-2}, \ldots, x_{i+1} = x_i$. Hence π_i^0 is the product of $k - i$ reflections, and we have $\mathrm{sign}(\pi_i^0) = (-1)^{k-i}$. Similarly, we can obtain π_i^1 from π_i^0 by reflecting in the hyperplane $x_i = \frac{1}{2}$. Thus we have $\mathrm{sign}(\pi_i^1) = (-1)^{k-i+1}$. Introducing

$$\begin{aligned} a_i &:= \pi_i^0 \circ \iota_{k-1}, \\ b_i &:= \pi_i^1 \circ \iota_{k-1} \end{aligned}$$

for $i = 1, 2, \ldots, k$, we obtain the following formula,

$$\partial([f]) = \sum_{i=1}^{k} (-1)^{k-i} \cdot ([f \circ a_i] - [f \circ b_i]). \tag{2.28}$$

Here a_i is the embedding of $\square^{k-1}$ into the facet $(*, *, \ldots, *, 0, *, \ldots, *)$ of $\square^k$, where 0 stands at the ith place, and b_i is the embedding of $\square^{k-1}$ into the facet

$(*, *, \ldots, *, 1, *, \ldots, *)$, where 1 stands at the ith place. The continuous analogue of this definition can be found in [24] and [18].

It is worthwhile noticing that each left coset from $[\mathcal{B}_k : \iota(\mathcal{B}_{k-1})]$ is equal to the set of cubical orders on a facet of $\square^k$.

3. One can define simplicial homology similarly to the way we defined cubical homology, using a system of representatives for $[\mathcal{S}_n : \mathcal{S}_{n-1}]$, where we embed $\mathcal{S}_{n-1}$ into $\mathcal{S}_n$ as the stabilizer of a point. It is easy to show that this definition agrees with the usual one.

In the exact same way as in the simplicial case, we can define the cochain complex $S^\bullet(\square)$ and the *cubical cohomology groups* $H^k(S^\bullet(\square))$.

Our next step is to define chain maps induced by cubical maps.

Definition 37 *Let* $\phi : \square \longrightarrow \square'$ *be a cubical map. We define* the chain map $S_\bullet(\phi)$ *induced by* ϕ *as follows. For every* $f \in \mathrm{Ord}_k(\square)$ *we set*

$$S_k(\phi)([f]) = \begin{cases} [\phi \circ f] & \text{if } \phi \circ f \text{ is injective} \\ 0 & \text{otherwise.} \end{cases}$$

We extend the definition by linearity to all elements of $S_k(\square)$.

The map $\pi \circ f$ is injective iff the restriction of ϕ to $\mathrm{Im}(f)$ is injective, and in this event $\phi \circ f$ is an ordered face of $\square'$. Thus $S_\bullet(\phi)$ is well defined. Similarly a simple injectivity-check shows that for cubical maps $\phi : \square \longrightarrow \square'$ and $\psi : \square' \longrightarrow \square''$ we have

$$S_\bullet(\phi \circ \psi) = S_\bullet(\phi) \circ S_\bullet(\phi).$$

We only have to check that $S_\bullet(\phi)$ is really a chain map.

The definition of chain map suggests extending the usage of the symbol $[f]$ to the case when f is not an ordered face, but only a (non-injective) cubical map $\square^k \longrightarrow \square$.

Definition 38 *For a cubical map $f : \square^k \longrightarrow \square$, we define*

$$[f] := 0$$

whenever f is not injective.

If we can show that the definition of ∂ is consistent with the extended definition of $[f]$, then we can reformulate the definition of $S_\bullet(\phi)$ by setting

$$S_k(\phi)([f]) := [\phi \circ f]$$

for all cubical maps $f : \square^k \longrightarrow \square$. From this definition it is straightforward that $S_\bullet(\phi)$ is a chain map.

Thus we only have to show the following lemma.

Lemma 25 *Given a system of representatives $\pi_1, \ldots, \pi_{2k}$ for $[\mathcal{B}_k : \jmath(\mathcal{B}_{k-1})]$, the defining equation (2.27) specializes to an identity of the form $0 = 0$ when we substitute a cubical map $f : \square^k \longrightarrow \square$ which is not an ordered face.*

Proof: We have to show that

$$\sum_{i=1}^{2k} \operatorname{sign}(\pi_i) \cdot [f \circ \pi_i \circ \iota_{k-1}] = 0 \tag{2.29}$$

whenever $f : \square^k \longrightarrow \square$ is not an ordered face. Note that the right hand side gives the same result irrespective of the choice of left coset representatives.

Condition (ii) of Definition 30 guarantees that $\mathrm{Im}(f)$ is contained in some face of $\square$. W.l.o.g. we can assume that $\square$ is equal to this face, i.e., we can restrict ourselves to the case of cubical maps $f : \square^k \longrightarrow \square^m$ for some fixed $m \in \mathbb{N}$. The equation (2.29) is trivially true when none of the maps $f \circ \pi_i \circ \iota_{k-1} : \square^{k-1} \longrightarrow \square^m$ is an ordered $(k-1)$ face.

Assume therefore, that for some $\tau \in \square^m_{k-1}$, at least one of the maps $f \circ \pi_i \circ \iota$ is a cubical order on τ. It is sufficient to show that the sum of those terms in (2.29) corresponding to a cubical order on τ is 0. The same proof will work for any other $(k-1)$-face of $\square^m$.

W.l.o.g. we may assume $\tau = (*, \dots, *, 0, \dots, 0)$, where $k-1$ stars are followed by $m-k+1$ zeros. Also we may assume that $f \circ \pi_1 \circ \iota$ is a cubical order on τ, that $\pi_1 = \mathrm{id}$, and that f maps $(x_1, \dots, x_{k-1}, 0)$ into $(x_1, \dots, x_{k-1}, 0, 0, \dots, 0)$. We show that in this situation we may assume $m \leq k$. This is obviously true when $\mathrm{Im}(f) = \tau$, hence we may suppose that some $\mathbf{x} = (x_1, \dots, x_{k-1}, 1)$ satisfies $f(\mathbf{x}) \notin \tau$. From the Hamming distance preserving property of f, and from $f((x_1, \dots, x_{k-1}, 0)) = (x_1, \dots, x_{k-1}, 0, \dots, 0)$, we infer that $f(\mathbf{x}) = (x_1, \dots, x_{k-1}, 0, \dots, 0) + e_j$ for some $j \geq k$. Now for any other $\mathbf{y} \in (*, \dots, *, 1)$ with $f(\mathbf{y}) \notin \tau$, we must have $f(\mathbf{y}) = (y_1, \dots, y_{k-1}, 0, \dots, 0) + e_j$ with the same j as before, since $\mathrm{d}(\mathbf{x}, \mathbf{y}) \geq \mathrm{d}(f(\mathbf{x}), f(\mathbf{y}))$. Thus $\mathrm{Im}(f)$ is contained in the k-cube $\tau \cup (\tau + e_j)$.

Therefore we may assume that f is a map from $\square^k$ to $\square^k$, and $\tau = (*, \dots, *, 0)$. Our statement then becomes –after an eventual change of coset representatives– a reformulation of Lemma 20. **QED**

Hence we have a cubical analogue of simplicial homology, and every result of Subsection 2.2.4 can be repeated in the cubical setting. The only thing that remains to be shown, in order to have the boundariless version of our theorem, is that $H_n(S_\bullet(\square^m)) = 0$. This will follow from the fact that in the cubical case we can define homotopy equivalence of cubical maps such that homotopic cubical maps induce chain-homotopic chain maps. This result will be the subject of the next subsection.

2.4.2 Homotopy equivalence of cubical maps

Definition 39 *Given two cubical complexes* $\square$ *and* $\square'$ *we define their* direct product $\square \times \square'$ *on the vertex set* $\text{vert}(\square) \times \text{vert}(\square')$ *to be the family of faces* $\{\sigma \times \tau : \sigma \in \square, \tau \in \square'\}$. *Given the cubical map* $\phi : \square \longrightarrow \square'$ *and another cubical map* $\psi : \tilde{\square} \longrightarrow \tilde{\square}'$*, we define the* direct product of the maps ϕ and ψ *by the formula*

$$\phi \times \psi((u, v)) := (\phi(u), \psi(v)),$$

for all $u \in \text{vert}(\square)$ *and* $v \in \text{vert}(\tilde{\square})$.

It is easy to check that the direct product of cubical maps is a cubical map. In particular, the direct product of an ordered k-face of $\square$ and an ordered l-face of $\square'$ is an ordered $k + l$-face of $\square \times \square'$.

Before we give our definition of homotopy equivalence, note that a graph consisting of a path is a one-dimensional cubical complex, and a natural analogue of a continuous path in a topological space.

Definition 40 *Two cubical maps* $\phi, \psi : \square \longrightarrow \square'$ *are* homotopic *when there is a path* $I = v_0 \ldots v_n$ *and cubical map* $\Phi : \square \times I \longrightarrow \square'$ *such that for every* $v \in \text{vert}(\square)$ *we have* $\phi(v) = \Phi(v, v_0)$ *and* $\psi(v) = \Phi(v, v_n)$. *If we can take* I *to be a path of length one, i.e., a standard* 1*-cube, then we call* ϕ *and* ψ elementarily homotopic *maps.*

Obviously the above notion of homotopy is an equivalence relation, and it is the transitive closure of the elementary homotopy relation.

In this finite setting it is relatively easy to prove that homotopic maps induce chain homotopic chain maps.

Lemma 26 *If the cubical maps* $\phi, \psi : \square \longrightarrow \square'$ *are homotopic, then the induced chain maps* $S_\bullet(\phi), S_\bullet(\psi) : S_\bullet(\square) \longrightarrow S_\bullet(\square')$ *are chain homotopic.*

Proof: By transitivity, we may restrict ourselves to the case when ϕ and ψ are elementarily homotopic. On the other hand, by the compatibility of the composition of cubical maps with the operation $S_\bullet(\dot{})$, we may assume that $\Box' = \Box \times \{0,1\}$, $\Phi = \mathrm{id}$, $\phi(v) = (v,0)$ and $\psi(v) = (v,1)$.

The direct product $f \times \mathrm{id}$ of $f \in \mathrm{Ord}_k(\Box)$ and the identity map id of $\{0,1\}$ is an ordered $(k+1)$-face of $\Box \times \{0,1\}$. For any $\pi \in \mathcal{B}_k$, we have

$$(f \circ \pi) \times \mathrm{id} = (f \times \mathrm{id}) \circ (\pi \times \mathrm{id}),$$

where π and $\pi \times \mathrm{id}$ have the same sign. Thus we can define

$$\begin{array}{rccc} s_k : & S_k(\Box) & \longrightarrow & S_{k+1}(\Box \times \{0,1\}) \\ & [f] & \longmapsto & [f \times \mathrm{id}]. \end{array}$$

Now, using (2.28) we obtain

$$\begin{aligned}(s \circ \partial + \partial \circ s)([f]) &= s\left(\sum_{i=1}^{k}(-1)^{k-i} \cdot ([f \circ a_i] - [f \circ b_i]\right) + \partial([f \times \mathrm{id}]) \\ &= \sum_{i=1}^{k}(-1)^{k-i} \cdot ([(f \circ a_i) \times \mathrm{id}] - [(f \circ b_i) \times \mathrm{id}]) \\ &\quad + \sum_{i=1}^{k+1}(-1)^{k+1-i} \cdot ([(f \times \mathrm{id}) \circ a_i] - [(f \times \mathrm{id}) \circ b_i]) .\end{aligned}$$

As a straightforward consequence of the definition of a_i and b_i we have $(f \circ a_i) \times \mathrm{id} = (f \times \mathrm{id}) \circ a_i$ and $(f \circ b_i) \times \mathrm{id} = (f \times \mathrm{id}) \circ b_i$ for $i = 1, 2, \ldots, k$. (We add the coordinate for id as the last coordinate). After the cancellations we obtain

$$(s \circ \partial + \partial \circ s)([f]) = [(f \times \mathrm{id}) \circ a_{k+1}] - [(f \times \mathrm{id}) \circ b_{k+1}] = S_\bullet(\phi)([f]) - S_\bullet(\psi)([f]).$$

Therefore we have $s \circ \partial + \partial \circ s = S_\bullet(\phi) - S_\bullet(\psi)$. **QED**

As a consequence we find that homotopy-equivalent cubical complexes have isomorphic homology and homotopy groups. In particular, $\square^m$ is homotopy-equivalent to $\square^0$ consisting of only one point. Therefore the positive degree homology groups of a standard cube vanish.

Let us mention finally that we have the following cubical analogue of Mayer-Vietoris sequences for cubical complexes.

Theorem 8 *Let $\square'$ and $\square''$ be subcomplexes of a cubical complex $\square$. Then there exists an exact sequence of chain complexes*

$$0 \longrightarrow S_\bullet(\square' \cap \square'') \longrightarrow S_\bullet(\square') \oplus S_\bullet(\square'') \longrightarrow S_\bullet(\square' \cup \square'') \longrightarrow 0$$

inducing a long exact sequence of homology groups

$$\cdots \to H_k(S_\bullet(\square' \cap \square'')) \to H_k(S_\bullet(\square')) \oplus H_k(S_\bullet(\square'')) \to H_k(S_\bullet(\square' \cup \square'')) \to$$
$$\to H_{k-1}(S_\bullet(\square' \cap \square'')) \to \cdots$$

The proof is straightforward.

Chapter 3

On the Stanley ring of cubical complexes

3.1 Elementary properties of the Stanley ring

R. Stanley suggested investigating the following ring associated to cubical complexes.

Definition 41 *Let* $\square$ *be a cubical complex,* K *a field. Associate a variable* x_v *to each vertex* $v \in V$. *The* Stanley ring $K[\square]$ *of the complex* $\square$ *over the field* K *is the factor ring* $K[x_v : v \in V] \big/ I(\square)$, *where the ideal* $I(\square)$ *is generated by the following elements.*

(i) $x_{v_1} \cdot x_{v_2} \cdots x_{v_k}$ *for all* $v_1, \ldots, v_k \in V$ *such that* $\{v_1, \ldots, v_k\}$ *is not contained in any face of* $\square$.

(ii) $x_u \cdot x_v - x_{u'} \cdot x_{v'}$ *for all* $u, u', v, v' \in V$ *such that* $\{u, v\}$ *and* $\{u', v'\}$ *are diagonals of the same face* $\mathrm{Cspan}(\{u, v\}) = \mathrm{Cspan}(\{u', v'\}) \in \square$.

We denote the ideal generated by the elements of type (i), (ii) by $I_1(\square)$, $I_2(\square)$ *respectively. We call* $I(\square)$ *the* face ideal *of the cubical complex* $\square$.

In this section we will show that condition (i) can be weakened to requiring the product of at most three variables to be in $I(\square)$, whenever the set of their indices is not contained in any face. In Section 3.4 we will prove that for some important classes of cubical complexes (like boundary complexes of convex cubical polytopes), it is even sufficient to set the product of pairs to be zero in $K(\square)$ when they are not diagonals of a face. In doing so, the following equivalence relation defined on multisets of vertices will be instrumental. (Entries between brackets "$\lfloor$" and "$\rfloor$" are to be read as a list of elements of a multiset.)

Definition 42 *We call the multisets of vertices* $\lfloor u_1, u_2, \ldots, u_k \rfloor$ *and* $\lfloor v_1, v_2, \ldots, v_l \rfloor$ equivalent, *if* $k = l$ *and* $\lfloor v_1, v_2, \ldots, v_k \rfloor$ *can be obtained from* $\lfloor u_1, u_2, \ldots, u_k \rfloor$ *by repeated application of the following operation. If* $\mathrm{Cspan}(\{u_1, u_2\})$ *exists, replace* $\lfloor u_1, u_2, u_3, \ldots, u_k \rfloor$ *with* $\lfloor u_1', u_2', u_3, \ldots, u_k \rfloor$*, where* $\lfloor u_1', u_2' \rfloor$ *is any diagonal of* $\mathrm{Cspan}(\{u_1, u_2\})$.

The operation of replacing a diagonal with another one is reversible, and so the relation defined above is in fact an equivalence relation. Clearly, if a face $\tau \in \square$ contains $\{u_1, \ldots, u_k\}$ then the same holds for all equivalent multisets $\lfloor v_1, \ldots, v_k \rfloor$. *Hence we can say that a face* τ *contains or does not contain a given equivalence class of multisets.* In particular, $\mathrm{Cspan}(\lfloor u_1, u_2, \ldots, u_k \rfloor)$ is simultaneously defined or not defined for all multisets of an equivalence class, and its value is constant on an equivalence class, on which it is defined. The definitions yield immediately the following connections between the equivalence classes of multisets and monomials.

Lemma 27 *The monomials of* $K[x_v : v \in V]$ *have the following properties.*

1. *We have* $x_{u_1} \cdots x_{u_k} \in I_1(\square)$ *if and only if* $\mathrm{Cspan}(\lfloor u_1, \ldots, u_k \rfloor)$ *does not exist.*

2. *The differences* $x_{\underline{u}} - x_{\underline{v}}$*, where* $\underline{u}$ *and* $\underline{v}$ *are equivalent multisets of vertices, form a generating system of the* K*-vectorspace* $I_2(\square)$*. Consequently, monomials of degree*

k indexed by equivalent multisets of vertices represent the same element modulo $I_2(\square)$.

The following theorem is the key to understanding the role of the equivalence of multisets of vertices.

Theorem 9 *Monomials not belonging to $I_1(\square)$ and associated to multisets from different equivalence classes are linearly independent modulo $I(\square)$.*

Proof: Assume that we have a linear combination of monomials $\sum_{\underline{v}} \lambda_{\underline{v}} \cdot x_{\underline{v}} \in I(\square)$ with coefficients $\lambda_{\underline{v}} \in K$ such that all the multisets $\underline{v} = \lfloor v_1, \ldots, v_l \rfloor$ occurring in this sum belong to different equivalence classes, and for all occurring $\underline{v} = \lfloor v_1, \ldots, v_l \rfloor$ the face $\text{Cspan}(\lfloor v_1, \ldots, v_l \rfloor)$ exists. Let us fix one $x_{\underline{u}} = x_{u_1} \cdots x_{u_k}$ and show that we must have $\lambda_{\underline{u}} = 0$. By $x_{\underline{u}} \notin I_1(\square)$ the face $\text{Cspan}(\lfloor u_1, \ldots, u_k \rfloor)$ must exist. Observe that the factor of $K[\square]$ by the ideal $(x_v \ : \ v \notin \text{Cspan}(\lfloor u_1, \ldots, u_k \rfloor))$ is the Stanley ring of the complex $\square\restriction_{\text{Cspan}(\lfloor u_1,\ldots,u_k \rfloor)}$, and we have $x_{\underline{u}} \notin I_1(\square\restriction_{\text{Cspan}(\lfloor u_1,\ldots,u_k \rfloor)})$. Observe furthermore that if two multisubsets of $V(\square\restriction_{\text{Cspan}(\lfloor u_1,\ldots,u_k \rfloor)})$ are not equivalent in $\square$ then they are not equivalent in $\square\restriction_{\text{Cspan}(\lfloor u_1,\ldots,u_k \rfloor)}$ either. Thus without loss of generality we may assume that $\square = \square\restriction_{\text{Cspan}(\lfloor u_1,\ldots,u_k \rfloor)}$, i.e., $\square$ is a standard n-cube $\square^n$ for some $n \in \mathbb{N}$.

For a standard n-cube $\square^n$ we have $I_1(\square^n) = 0$, and so $I(\square^n) = I_2(\square^n)$. Let us fix a standard geometric representation ϕ of $\square^n$. Then the vectors $\{\phi(v) \ : \ v \in V(\square^n)\}$ are the characteristic vectors of the subsets of $\{1, 2, \ldots, n\}$. Let us denote by $\text{Set}(v)$ the subset of $\{1, 2, \ldots, n\}$ with characteristic vector $\phi(v)$. A subset X of $V(\square^n)$ is a face iff. $\{\text{Set}(v) \ : \ v \in X\}$ is an interval of the boolean algebra $P(\{1, 2, \ldots, n\})$. Hence we have $\text{Cspan}(\{u, v\}) = \text{Cspan}(\{u', v'\})$ if and only if for the corresponding subsets

$$\text{Set}(u) \cap \text{Set}(v) = \text{Set}(u') \cap \text{Set}(v') \text{ and } \text{Set}(u) \cup \text{Set}(v) = \text{Set}(u') \cup \text{Set}(v') \qquad (3.1)$$

holds.

We define a K-linear map from the K-vectorspace $K[x_v : v \in V(\square^n)]$ to the K-vectorspace with basis $\mathbb{N}^{n+1}$ as follows. We associate to each monomial $x_{\underline{v}} = x_{v_1} \cdots x_{v_l}$ the vector $\underline{\alpha}(x_{\underline{v}}) = (\alpha_0, \alpha_1, \ldots, \alpha_n)$, where α_0 is l and for $i \geq 1$, α_i is the number of j-s such that $i \in \mathrm{Set}(v_j)$. (We count repeated vertices with their multiplicity.)

If two multisets $\underline{v} = \lfloor v_1, \ldots, v_l \rfloor$ and $\underline{v'} = \lfloor v'_1, \ldots, v'_l \rfloor$ are equivalent then $\underline{\alpha}(x_{\underline{v}})$ is equal to $\underline{\alpha}(x_{\underline{v'}})$. In fact, when we replace two sets $\mathrm{Set}(u), \mathrm{Set}(v)$ in a multiset of subsets of $\{1, 2, \ldots, n\}$ with the sets $\mathrm{Set}(u'), \mathrm{Set}(v')$ such that (3.1) is satisfied then neither the cardinality of the multiset of sets nor the number of sets in the multiset containing a given element $i \in \{1, 2, \ldots, n\}$ does change. Hence the kernel of $\underline{\alpha}$ contains $I(\square^n) = I_2(\square^n)$ by the second statement of Lemma 27.

Therefore in order to prove $\lambda_{\underline{u}} = 0$ we only need to show that for a multiset $\underline{v} = \lfloor v_1, \ldots, v_l \rfloor$ not equivalent to $\underline{u}$ we have $\underline{\alpha}(x_{\underline{v}}) \neq \underline{\alpha}(x_{\underline{u}})$.

Let $\underline{v} = \lfloor v_1, \ldots, v_l \rfloor$ be an arbitrary multiset of vertices. Replacing any pair of vertices (v_i, v_j) with the pair

$$(\mathrm{Set}^{-1}(\mathrm{Set}(v_i) \cap \mathrm{Set}(v_j)), \mathrm{Set}^{-1}(\mathrm{Set}(v_i) \cup \mathrm{Set}(v_j))),$$

we obtain an equivalent multiset of vertices. Using this operation repeatedly, we can reach an equivalent multiset $\underline{v'} = \lfloor v'_1, \ldots, v'_l \rfloor$ such that $\mathrm{Set}(v'_1) \subseteq \cdots \subseteq \mathrm{Set}(v'_l)$ holds. (We can prove this by induction on l.) Now the statement follows from the obvious fact that in the event when $\mathrm{Set}(v'_1) \subseteq \cdots \subseteq \mathrm{Set}(v'_l)$ holds, we must have

$$\mathrm{Set}(v_j) = \{i \in \{1, 2, \ldots, n\} : \alpha_i \leq l - i\}.$$

Therefore $\underline{\alpha}$ assigns different vectors to different equivalence classes of multisets of vertices. **QED**

Corollary 6 *We have* $x_{u_1} \cdots x_{u_k} \in I(\Box)$ *if and only if* $\text{Cspan}(\lfloor u_1, \ldots, u_k \rfloor)$ *does not exist.*

Corollary 7 *Two monomials* $x_{u_1} \cdots x_{u_k} \notin I(\Box)$ *and* $x_{v_1} \cdots x_{v_l} \notin I(\Box)$ *represent the same class modulo* $I(\Box)$ *if and only if* $k = l$ *and the multisets* $\lfloor u_1, \ldots, u_k \rfloor$ *and* $\lfloor v_1, \ldots, v_k \rfloor$ *are equivalent.*

Part of the proof of Theorem 9 may be used to show the following lemma.

Lemma 28 *Let* $\Box$ *be an arbitrary cubical complex and* $k \geq 2$. *Then any monomial* $x_{u_1} \cdot x_{u_2} \cdots x_{u_k}$ *such that* $\text{Cspan}(\{u_1, \ldots, u_k\})$ *exists, is equivalent modulo* $I_2(\Box)$ *to a monomial* $x_{v_1} \cdot x_{v_2} \cdots x_{v_k}$ *such that*

$$\text{Cspan}(\{v_1, v_2\}) = \text{Cspan}(\{u_1, \ldots, u_k\}) = \text{Cspan}(\{v_1, \ldots, v_k\})$$

holds.

Proof: Without loss of generality we may assume $\Box = \text{Cspan}(\{u_1, \ldots, u_k\})$, i.e., that $\Box$ is a standard n-cube $\Box^n$. Let us fix again a geometric realization ϕ and denote by $\text{Set}(v)$ the subset of $\{1, 2, \ldots, n\}$ with characteristic vector $\phi(v)$. We have shown in the proof of Theorem 9 that $\lfloor u_1, \ldots, u_k \rfloor$ is equivalent to a multiset $\lfloor v_1, \ldots, v_k \rfloor$ such that $\text{Set}(v_1) \subseteq \cdots \subseteq \text{Set}(v_k)$ holds. This $\lfloor v_1, \ldots, v_k \rfloor$ will have the required properties.
QED

Using Lemma 28 we can show the following

Theorem 10 *Let* $\Box$ *be an arbitrary cubical complex. Let* $I_1'(\Box)$ *be the ideal of* $K[x_v : v \in V]$ *generated by all monomials* $x_{v_1} \cdots x_{v_k}$ *such that* $k \leq 3$ *and* $\{v_1, \ldots, v_k\}$ *is not contained in any face of* $\Box$. *Then we have*

$$I(\Box) = I_1'(\Box) + I_2(\Box).$$

Proof: By definition, $I_1'(\square)$ is contained in $I_1(\square)$. Hence it is sufficient to show that if $\{v_1, \ldots, v_k\}$ is not contained in any face of $\square$ then $x_{v_1} \cdots x_{v_k}$ is congruent modulo $I_2(\square)$ to a monomial from $I_1'(\square)$. We prove this statement by induction on k. For $k = 2, 3$ we have $x_{v_1} \cdots x_{v_k} \in I_1'(\square)$. Assume we know the statement for k and we are given $v_1, v_2, \ldots, v_{k+1}$ such that $\{v_1, \ldots, v_{k+1}\}$ is not contained in any face of $\square$. If $\{v_1, \ldots, v_k\}$ is not contained in any face, then we have $x_{v_1} \cdots x_{v_k} \in I_1(\square)$, by induction hypothesis we get $x_{v_1} \cdots x_{v_k} \in I_1'(\square)$, and so $x_{v_1} \cdots x_{v_k} \cdot x_{v_{k+1}} \in I_1'(\square)$. Hence we may assume that $\mathrm{Cspan}(\{v_1, \ldots, v_k\})$ exists. By Lemma 28, the monomial $x_{v_1} \cdots x_{v_k}$ is congruent modulo $I_2(\square)$ to a monomial $x_{v_1'} \cdots x_{v_k'}$ such that we have

$$\mathrm{Cspan}(\{v_1', v_2'\}) = \mathrm{Cspan}(\{v_1, \ldots, v_k\}).$$

But then $\mathrm{Cspan}(\{v_1', v_2', v_{k+1}\})$ does not exist and we get

$$x_{v_1'} \cdot x_{v_2'} \cdot x_{v_{k+1}} \in I_1'(\square).$$

This implies

$$x_{v_1'} \cdots x_{v_k'} \cdot x_{v_{k+1}} \in I_1'(\square),$$

and so $x_{v_1} \cdots x_{v_k} \cdot x_{v_{k+1}}$ is congruent modulo $I_2(\square)$ to an element of $I_1'(\square)$. **QED**

Theorem 9 and its corollaries also allow us to compute the *Hilbert-series* of the Stanley-ring of a cubical complex. Recall, that the Hilbert-series of a finitely generated $\mathbb{N}$-graded K-algebra A is usually defined as

$$\mathcal{H}(A, t) = \sum_{n=0}^{\infty} \dim_K(A_n) \cdot t^n,$$

where A_n is the vectorspace generated by the homogeneous elements of degree n, and the operator $\dim_K$ stands for taking the vectorspace dimension. (For details, see e.g. [26, p. 33].)

Theorem 11 *Let $\square$ be a d-dimensional cubical complex and let f_i be the number of i-dimensional faces of $\square$. Then the Hilbert-series $\mathcal{H}(K[\square],t)$ of the graded algebra $K[\square]$ is given by*

$$\mathcal{H}(K[\square],t) = 1 + \sum_{i=0}^{d} f_i \cdot \sum_{k=1}^{\infty}(k-1)^i \cdot t^k. \tag{3.2}$$

Proof: $K[\square]$ may be written as a direct sum of K-vectorspaces as follows.

$$K[\square] = \bigoplus_{\sigma\in\square}\bigoplus_{k=0}^{\infty}\langle x_{u_1}\cdots x_{u_k} \ : \ \mathrm{Cspan}(\lfloor u_1,\ldots,u_k\rfloor) = \sigma\rangle. \tag{3.3}$$

(Note that this sum includes the vectorspace generated by the empty product 1 for $\sigma = \emptyset$ and $k = 0$.) It is a consequence of Theorem 9 and its corollaries that for an i-dimensional face $\sigma \in \square$ and a positive integer k the dimension of $\langle x_{u_1}\cdots x_{u_k} \ : \ \mathrm{Cspan}(\lfloor u_1,\ldots,u_k\rfloor) = \sigma\rangle$ is equal to the number of multisets $\lfloor X_1,\ldots,X_k\rfloor$ of subsets of $\{1,2,\ldots,i\}$ such that we have

$$\emptyset = X_1 \subseteq X_2, \subseteq \cdots \subseteq X_k = \{1,2,\ldots,i\}.$$

(For $i = 0$ we write $\emptyset$ instead of $\{1,2,\ldots,i\}$.) The number of such multisets is 1 for $i = 0$, otherwise it is equal to the number of functions $f : \{1,2,\ldots,i\} \longrightarrow \{1,2,\ldots,k-1\}$. (A bijection is defined by setting $f^{-1}(j) := X_j \setminus X_{j-1}$ for $j = 1,2,\ldots,i$ when $(X_1,\ldots,X_k)$ is given and setting $X_j := \bigcup_{l=1}^{j} f^{-1}(l)$ when f is given.) Thus we have

$$\dim\left(\langle x_{u_1}\cdots x_{u_k} \ : \ \mathrm{Cspan}(\lfloor u_1,\ldots,u_k\rfloor) = \sigma\rangle\right) = (k-1)^i,$$

and so the theorem follows. **QED**

Introducing

$$\Phi_0(t) := \sum_{k\geq 0} t^k = \frac{1}{1-t} \text{ and } \Phi_r(t) := \sum_{k\geq 0} k^r \cdot t^k \text{ for } r \geq 1$$

we may rewrite (3.2) into the following form.

$$\mathcal{H}(K[\square], t) = 1 + \sum_{i=1}^{d} f_i \cdot t \cdot \Phi_i(t). \tag{3.4}$$

Let D denote the derivation operator of the polynomial ring $\mathbb{Z}[t]$ defined by $D : t \mapsto 1$. Then we have $t \cdot D(\Phi_r(t)) = \Phi_{r+1}(t)$. It is well-known that D satisfies the following operator identity. We have

$$(t \cdot D)^n = \sum_{k=0}^{n} S(n,k) \cdot t^k \cdot D^k,$$

where the letters $S(n,k)$ denote the Stirling numbers of the second kind. (See e.g. [22, p. 218, Section 6.6, formula (34)].) Using this formula for D allows for us to obtain

$$\Phi_i(t) = \sum_{j=0}^{i} S(i,j) \cdot t^j \cdot D^j \left(\frac{1}{1-t}\right) = \sum_{j=0}^{i} S(i,j) \cdot t^j \cdot \frac{j!}{(1-t)^{j+1}}.$$

This formula holds even for $i = 0$ if we assume $S(0,0) = 1$. Thus (3.4) is equivalent to

$$\mathcal{H}(K[\square], t) = 1 + \sum_{i=0}^{d} f_i \cdot t \cdot \sum_{j=0}^{i} S(i,j) \cdot t^j \cdot \frac{j!}{(1-t)^{j+1}}. \tag{3.5}$$

We may transform (3.5) into an even more familiar form, using the following notation.

Definition 43 *Let $\square$ be a d-dimensional cubical complex and let us denote by f_i the number of i-dimensional faces of $\square$. We define the* triangulation f-vector $(f_{-1}^{\triangle}, f_0^{\triangle}, \ldots, f_d^{\triangle})$ of $\square$ *as follows. We set*

$$f_j^{\triangle} := \begin{cases} \sum\limits_{i=j}^{d} f_i \cdot S(i,j) \cdot j! & \textit{when } j \geq 0 \\ 1 & \textit{when } j = -1 \end{cases},$$

where the numbers $S(i,j)$ are the Stirling numbers of the second kind and we assume $S(0,0) = 1$.

Using the triangulation f-vector, (3.5) may be written into the following equivalent form.

$$\mathcal{H}(K[\square], t) = \frac{\sum\limits_{i=-1}^{d} f_i^{\triangle} \cdot t^{i+1} \cdot (1-t)^{d-i-1}}{(1-t)^d}. \tag{3.6}$$

The expression "triangulation f-vector" is justified by the fact that it is equal to the f-vector of any triangulation via pulling the vertices.

Lemma 29 *Let $\square$ be a d-dimensional cubical complex, and $(f_{-1}^{\triangle}, f_0^{\triangle}, \ldots, f_d^{\triangle})$ its triangulation f-vector. Let $<$ be an arbitrary linear order on the vertex set V, and $\triangle_{<}(\square)$ the triangulation via pulling the vertices in order $<$. Then $f_j^{\triangle}$ is the number of j-dimensional faces of $\triangle_{<}(\square)$.*

Proof: The restriction of the triangulation $\triangle_{<}(\square)$ to a face $\sigma \in \square$ is the triangulation via pulling the vertices of $\square\downarrow_{\sigma}$ in the order $<|_{\sigma}$. Thus we only need to show the following identity for every triangulation $\triangle_{<}(\square^n)$ via pulling the vertices of every standard cube $\square^n$ and every $n \in \mathbb{N}$.

$$|\{\sigma' \in \triangle_{<}(\square^n) \, : \, \dim(\sigma') = j, \, \mathrm{Cspan}(\sigma') = V(\square^n)\}| = S(n,j) \cdot n! \tag{3.7}$$

In order to show (3.7) it is sufficient to prove the original statement of the lemma in that special case when $\square$ is a standard n-cube for some n. In fact, every nonempty face of a standard n-cube is a standard k-cube for some $k \leq n$ and we may apply the Möbius inversion formula to the partially ordered set of the faces of the standard n-cube.

Thus we are left with the task of proving our lemma for standard cubes.

Remember that by [27, Lemma 1.1], the combinatorial type of $\triangle_{<}(\square^n)$ remains the same, no matter which geometric representation of $\square^n$ we choose. Let us fix a standard geometric representation ϕ, and for every vertex v let $\mathrm{Set}(v)$ denote the subset of $\{1,\ldots,n\}$ with characteristic vector $\phi(v)$. Let us define $u \leq_\phi v$ whenever $\mathrm{Set}(u) \subseteq \mathrm{Set}(v)$ holds. Clearly $\leq_\phi$ is a partial order on $\square^n$. Using [27, Theorem 2.3] we may show that the standard n-cube is compressed. Thus by [27, Corollary 2.7] the f-vector of $\triangle_{<}(\square)$ does not depend on the choice of the order $<$. Hence we may assume that $<$ is a linear extension of the partial order $<_\phi$. Then for any set of vertices $\{u_1,\ldots,u_r\}$ we have

$$\delta_{<}(\mathrm{Cspan}(\{u_1,\ldots,u_r\})) = \mathrm{Set}^{-1}(\mathrm{Set}(u_1) \cap \cdots \cap \mathrm{Set}(u_r)).$$

Thus a set $\{v_1,\ldots,v_k\}$ satisfying $v_1 > \cdots > v_k$ is a face of $\triangle_{<}(\square^n)$ if and only if for every j we have

$$\mathrm{Set}(v_j) = \mathrm{Set}(v_1) \cap \cdots \cap \mathrm{Set}(v_j),$$

which is equivalent to

$$\mathrm{Set}(v_1) \supset \cdots \supset \mathrm{Set}(v_k).$$

Therefore the number of j-faces of $\triangle_{<}(\square^n)$ is equal to the number of $(j+1)$ element increasing chains in the Boolean algebra of subsets of $\{1,2,\ldots,n\}$. There is a bijection between the set of chains $\{S_0 \subset \cdots \subset S_j : S_j \subseteq \{1,2,\ldots,n\}\}$, and the triples of the form (X,Y,η) where X and Y are subsets of $\{1,2,\ldots,n\}$ satisfying $X \subset Y$ and $|Y \setminus X| \geq j$, and $\eta : Y \setminus X \longrightarrow \{1,\ldots,j\}$ is a surjective function. This bijection is defined as follows. We set $X := S_0, Y := S_j$, and $f^{-1}(i) := S_i \setminus S_{i-1}$ for $i = 1,2,\ldots,j$. For a given $i \geq j$ the number of pairs (X,Y) satisfying $X \subset Y \subset \{1,2,\ldots,n\}$ and $|Y \setminus X| = i$ is $f_i(\square^n)$.

For a fixed pair (X, Y) satisfying $X \subset Y \subset \{1, 2, \ldots, n\}$ and $|Y \setminus X| = i$ the number of surjective maps from $Y \setminus X$ to $\{1, \ldots, j\}$ is $S(i,j) \cdot j!$. Therefore we have

$$f_j(\triangle_<(\square^n)) = \sum_{i=j}^{n} f_i(\square^n) \cdot S(i,j) \cdot j!$$

and this is exactly what we wanted to prove. **QED**

Remark The defining equations of Definition 43 may summarized in the following polynomial equation for a $(d-1)$ dimensional cubical complex $\square$.

$$\sum_{i=0}^{d-1} f_i \cdot x^i = \sum_{j=0}^{d-1} f_j^{\triangle} \cdot \binom{x}{j}. \tag{3.8}$$

Let us recall now the definition of the *Stanley-Reisner ring of a simplicial complex* $\triangle$. (See e.g. [26].)

Definition 44 *Given a simplicial complex $\triangle$ with vertex set V, we define the* Stanley-Reisner ring $K[\triangle]$ *of $\triangle$ to be the factor ring* $K[x_v : v \in V] \big/ I(\triangle)$, *where the ideal $I(\triangle)$ is generated by the set $\{\mathbf{x}_{v_1} \cdots x_{v_k} : k \in \mathbb{N}, \{v_1, \ldots, v_k\} \notin \triangle\}$. We call $I(\triangle)$ the* face ideal *of $\triangle$.*

Comparing (3.6) and Lemma 29 with [26, 1.4 Theorem] we may notice that every cubical complex $\square$ and any triangulation via pulling the vertices $\triangle_<(\square)$ of $\square$ satisfy the following identity.

$$\mathcal{H}(K[\square], t) = \mathcal{H}(K[\triangle_<(\square)], t). \tag{3.9}$$

In words, the Hilbert function of the Stanley ring of a cubical complex is equal to the Hilbert function of the Stanley-Reisner ring of any of its triangulations via pulling the vertices.

3.2 Initial ideals and triangulations

In this section we investigate the connection between the Stanley-Reisner ring of a triangulation of a cubical complex $\Box$ via pulling the vertices, and the Stanley ring of this cubical complex. Note that both rings are the factors of the same polynomial ring $K[x_v : v \in V]$. We will apply the equality of Hilbert-series (3.9) to express the connection between the face ideal of a triangulation $\triangle_<(\Box)$ and the face ideal of $\Box$, using the following notions of Gröbner basis theory.

Definition 45 *Consider an arbitrary polynomial ring $K[X]$ over a field K. A* monomial order *on the set of monomials of $K[X]$ is a linear order $<$ on the semigroup of monomials such that if m_1, m_2 and n are monomials, and $n \neq 1$ holds then*

$$m_1 > m_2 \text{ implies } n \cdot m_1 > n \cdot m_2.$$

Given a monomial order $<$, for every polynomial $p \in K[X]$ we define the initial term $\text{init}_<(p)$ *of p to be the largest term with respect to the term order $<$. Given an ideal I of $K[X]$ we denote by* $\text{init}_<(I)$ *the ideal generated by the initial terms of elements of I. A generating system $\{p_1, \ldots, p_k\}$ of I is called a* Gröbner basis *with respect to the term order $<$, if* $\text{init}_<(I)$ *is generated by the set* $\{\text{init}_<(p_1), \ldots, \text{init}_<(p_k)\}$.

In particular, we will use *reverse lexicographic term orders*, which are defined as follows.

Definition 46 *Let $K[X]$ be a polynomial ring and $<$ a linear order on the set of variables X. We define the* reverse lexicographic order $<_{\text{rlex}}$ induced by $<$ *as follows. Given two monomials m and n, we write both of them in the form $m = x_1^{a_1} \cdots x_k^{a_k}, n = x_1^{b_1} \cdots x_k^{b_k}$ where $x_1 > \cdots > x_k$. We set $m <_{\text{rlex}} n$ iff* $\deg(m) < \deg(n)$ *holds or we have* $\deg(m) = \deg(n)$ *and $a_i > b_i$ for the last index i with $a_i \neq b_i$.*

Using the above definitions, the main theorem of this section may be stated in the following way.

Theorem 12 *Let $\square$ be a cubical complex on the vertex set V and $<$ any linear order on the vertices. Then we have the following identity.*

$$\mathrm{init}_{<_{\mathrm{rlex}}}(I(\square)) = I(\triangle_<(\square)).$$

In words, the initial ideal of the face ideal of $\square$ with respect to the reverse lexicographic order induced by $<$ is the face ideal of the triangulation of $\square$ via pulling the vertices with respect to the order $<$.

Let us consider first the special case when the cubical complex $\square$ is a standard n-cube $\square^n$. Let us fix a standard geometric realization $\phi : V(\square^n) \longrightarrow \mathbb{R}^n$, and let $\mathrm{Set}(v)$ be again the subset of $\{1, \ldots, n\}$ with characteristic vector $\phi(v)$. Let us denote again by $u \leq_\phi v$ when the relation $\mathrm{Set}(u) \subseteq \mathrm{Set}(v)$ holds. We will show that the Stanley ring $K[\square^n]$ with respect to the partial ordered set $(\{x_v : v \in V\}, \leq_\phi)$ is *an algebra with straightening law* over the field K. Let us recall the definition of algebras with straightening law from [5, p. 38].

Definition 47 *Let A be a B-algebra and $\Pi \subset A$ a finite subset with partial order $\leq$. A is a* graded algebra with straightening law *(on Π, over B) if the following conditions hold.*

(H_0) $A = \bigoplus_{i\geq 0} A_i$ is a graded B-algebra, such that $A_0 = B$, Π consists of elements of positive degree and generates A as a B-algebra.

(H_1) The products $\xi_1 \cdots \xi_m$ $m \in \mathbb{N}, \xi_i \in \Pi$, such that $\xi_1 \leq \cdots \leq \xi_m$ are linearly independent. They are called standard monomials.

(H_2) (Straightening law) For all incomparable $\xi, \nu \in \Pi$ the product $\xi \cdot \nu$ has a representation

$$\xi \cdot \nu = \sum a_\mu \cdot \mu, a_\mu \in B, a_\mu \neq 0, \mu \text{ standard monomial}$$

satisfying the following condition: every μ contains a factor $\zeta \in \Pi$ such that $\zeta \leq \xi, \zeta \leq \nu$. (It is of course allowed that $\xi \cdot \nu = 0$ the sum $\sum a_\mu \cdot \mu$ being empty.)

Lemma 30 *$K[\square^n]$ is an algebra with straightening law on $(\{x_v : v \in V\}, \leq_\phi)$, over K.*

Proof: Axiom (H_0) is obviously satisfied. Consider next the product $x_u \cdot x_v$ where $u, v \in V(\square^n)$. Let us denote $\text{Set}^{-1}(\text{Set}(u) \cap \text{Set}(v))$ by u' and $\text{Set}^{-1}(\text{Set}(u) \cup \text{Set}(v))$ by v'. Then we have $x_u \cdot x_v = x_{u'} \cdot x_{v'}$ and $x_{u'} \leq_\phi x_u, x_v$. Hence the straightening law (H_2) holds. Finally, standard monomials are of the form $x_{u_1} \cdots x_{u_k}$ with $\text{Set}(u_1) \subseteq \cdots \subseteq \text{Set}(u_k)$, and so they are linearly independent by the proof of Theorem 9. **QED**

Recall, that an algebra A with straightening law over K on Π may be represented as the factor of a polynomial ring $K[x_\xi : \xi \in \Pi]$ modulo the ideal I_Π generated by the elements $x_\xi \cdot x_\nu - \sum_{\underline{\mu}} a_{\underline{\mu}} \cdot x_{\underline{\mu}}$ representing the straightening relations. (Cf. [5, Proposition (4.2)].)

Brian Taylor has provided me with the proof of the following lemma.([30].)

Lemma 31 *Let K be a field and A be a graded algebra with straightening law over K on Π, represented as $K[x_\xi : \xi \in \Pi] \big/ I_\Pi$, where I_Π is the ideal generated by the representatives of straightening relations. Let $<$ be any extension of the partial order on Π to a linear order. Then the straightening relations form a Gröbner basis of I_Π with respect to the reverse lexicographic order induced by the order $<$.*

Proof: Assume by way of contradiction that there is a polynomial $p \in I_\Pi$ such thet $\text{init}_{<_{\text{rlex}}}(p)$ is is not in the ideal generated by the initial terms of the representatives of the straightening relations. If p is a linear combination of standard monomials then $p \in I_\Pi$ and the fact that the standard monomials are linearly independent modulo I_Π imply $p = 0$, a contradiction. Let m be the largest monomial with respect to the $<_{\text{rlex}}$ order which is not standard and appears with a nonzero coefficient in p. Then m is of the form $m = a \cdot n \cdot x_\xi \cdot x_\nu$ where $a \in K$, n is a monomial, and μ and ν are incomparable in the partial order of Π. Thus we have a straightening relation represented the form $x_\xi \cdot x_\nu - \sum_{\underline{\mu}} a_{\underline{\mu}} \cdot x_{\underline{\mu}} \in I_\Pi$. Here every monomial $x_{\underline{\mu}}$ contains a factor x_ζ which is less than ξ or μ in the partial order of Π, and so also in the order $<$. Thus we have

$$\text{init}_{<_{\text{rlex}}}(x_\xi \cdot x_\nu - \sum_{\underline{\mu}} a_{\underline{\mu}} \cdot x_{\underline{\mu}}) = x_\xi \cdot x_\nu.$$

Hence $m \neq \text{init}_{<_{\text{rlex}}}(p)$, otherwise $\text{init}_{<_{\text{rlex}}}(p)$ would be the multiple of of $\text{init}_{<_{\text{rlex}}}(x_\xi \cdot x_\nu - \sum_{\underline{\mu}} a_{\underline{\mu}} \cdot x_{\underline{\mu}})$. Thus for

$$p' := p - a \cdot n \cdot \left(x_\xi \cdot x_\nu - \sum_{\underline{\mu}} a_{\underline{\mu}} \cdot x_{\underline{\mu}} \right)$$

we have $p' \in I_\Pi$, $\text{init}_{<_{\text{rlex}}}(p') = \text{init}_{<_{\text{rlex}}}(p)$, and with respect to $<_{\text{rlex}}$, the largest nonstandard monomial of p' is smaller than than m. Every term order satisfies the descending chain condition, and every polynomial has finitely many terms, so iterating the above reasoning finitely many times yields a polynomial $q \in I_\Pi$ with $\text{init}_{<_{\text{rlex}}}(q) = \text{init}_{<_{\text{rlex}}}(p)$ which is the linear combination of standard monomials. Therefore we have reached a contradiction. **QED**

Corollary 8 *Let ϕ be a standard geometric representation of the standard n-cube $\square^n$ and $<_\phi$ the partial order on $V(\square^n)$ induced by ϕ. Let $<$ be any linear extension of $<_\phi$, and let* $\text{init}_{<_{\text{rlex}}}$ *be the reverse lexicographic order on $K[x_v : v \in V(\square^n)]$ induced by $<$.*

Then the set

$$\{x_u \cdot x_v - x_{u'} \cdot x_{v'} \ : \ \mathrm{Cspan}(\{u,v\}) = \mathrm{Cspan}(\{u',v'\})\}$$

is a Gröbner basis with respect to the term order $\mathrm{init}_{<_{\mathrm{rlex}}}$.

Corollary 9 *The ideal* $\mathrm{init}_{<_{\mathrm{rlex}}}(I_\Pi)$ *is the linear span of all those monomials which are not standard.*

Lemma 32 *Consider* $K[\square^n]$ *as an algebra with straightening law on the poset* $(\{x_v \ : \ v \in \ V(\square^n)\}, <_\phi)$ *over* K. *(*ϕ *is a standard geometric realization of* $\square^n$*). Let* $<$ *be any linear extension of the partial order* $<_\phi$. *Then a monomial* $x_{v_1} \cdots \mathbf{x}_{v_k}$ *is standard if and only if the set* $\{v_1, \ldots, v_k\}$ *is a face of the triangulation* $\triangle_<(\square^n)$

Proof: Without loss of generality we may assume $v_1 \geq \cdots \geq v_k$. The proof of Lemma 29 implies that $\{v_1, \ldots, v_k\}$ is a face of $\triangle_<(\square^n)$ if and only if for every j we have

$$\mathrm{Set}(v_1) \supseteq \cdots \supseteq \mathrm{Set}(v_k),$$

But this hold if and only if $x_{v_1} \cdots \mathbf{x}_{v_k}$ is a standard monomial. **QED**

Lemma 32 and Corollary 9 imply the following special case of Theorem 12.

Proposition 5 *Let* $\square^n$ *be a standard n-cube with a fixed standard geometric representation* ϕ, *and* $<$ *be a linear extension of* $<_\phi$. *Then we have*

$$\mathrm{init}_{<_{\mathrm{rlex}}}(I(\square)) = I(\triangle_<(\square)).$$

Example Consider the standard 3-cube $\square^3$ with a standard geometric representation ϕ. By abuse of notation we will identify the vertices with their image under ϕ. Let $<$ be the following order of the vertices.

$$(0,0,0) < (1,1,0) < (1,0,1) < (0,1,1,) < (1,0,0) < (0,1,0) < (0,1,1) < (1,1,1).$$

Consider the reverse lexicographic order on $K[x_v : v \in V(\square^3)]$ induced by this order of $V(\square^3)$. Then

$$x_{(1,1,0)} \cdot x_{(1,0,1)} \cdot x_{(0,1,1)} - x_{(0,0,0)} \cdot x_{(1,1,1)} \cdot x_{(1,1,1)} \in I(\square^2),$$

and

$$\text{init}_{<_{\text{rlex}}} \left(x_{(1,1,0)} \cdot x_{(1,0,1)} \cdot x_{(0,1,1)} - x_{(0,0,0)} \cdot x_{(1,1,1)} \cdot x_{(1,1,1)} \right) = x_{(1,1,0)} \cdot x_{(1,0,1)} \cdot x_{(0,1,1)}$$

imply

$$x_{(1,1,0)} \cdot x_{(1,0,1)} \cdot x_{(0,1,1)} \in \text{init}_{<_{\text{rlex}}}(I(\square^3)).$$

It is easy to check that this term does not belong to the ideal generated by the initial terms of the set $\{x_u \cdot x_v - x_{u'} \cdot x_{v'} : \text{Cspan}(\{u,v\}) = \text{Cspan}(\{u',v'\})\}$.

Thus Corollary 8 cannot be generalized to an arbitrary order of vertices. On the other hand, Proposition 5 may be generalized to Theorem 12.

Proof of Theorem 12: Inspired by Lemma 32, let us call a monomial $x_{v_1} \cdots x_{v_k}$ and the underlying multiset $\lfloor v_1, \ldots, v_k \rfloor$ *standard* if the set $\{v_1, \ldots, v_k\}$ is a face of $\triangle_<(\square)$.

First we show by induction on k that every non-standard monomial $x_{v_1} \cdots x_{v_k}$ belongs to the ideal $\text{init}_{<_{\text{rlex}}}(I(\square))$. Without loss of generality we may assume that we have $v_1 \geq \cdots \geq v_k$. If $\text{Cspan}(\{v_1, \ldots, v_k\})$ does not exist then by definition $x_{v_1} \cdots x_{v_k}$ belongs to $I(\square)$, and so we have $x_{v_1} \cdots x_{v_k} \in \text{init}_{<_{\text{rlex}}}(I(\square))$. Hence we may assume that $\mathrm{Cspan}(\{v_1,\ldots,v_k\})$ exists. Assume next that

$$v_k \neq \delta_<(\mathrm{Cspan}(\{v_1,\ldots,v_k\}))$$

holds. By Lemma 28 the monomial $x_{v_1}\cdots x_{v_k}$ is congruent modulo $I(\square)$ to a monomial $x_{u_1}\cdots x_{u_k}$ satisfying

$$\mathrm{Cspan}(\{u_1,u_2\}) = \mathrm{Cspan}(\{v_1,\ldots,v_k\}) = \mathrm{Cspan}(\{u_1,\ldots,u_k\}).$$

Replacing, if necessary, $\lfloor u_1,u_2\rfloor$ with another diagonal of the face $\mathrm{Cspan}(\{u_1,\ldots,u_k\})$, we may assume that

$$u_k = \delta_<(\mathrm{Cspan}(\{u_1,\ldots,u_k\})) = \delta_<(\mathrm{Cspan}(\{v_1,\ldots,v_k\}))$$

and thus

$$\mathrm{init}_{<_{\mathrm{rlex}}}(x_{v_1}\cdots x_{v_k} - x_{u_1}\cdots x_{u_k}) = x_{v_1}\cdots x_{v_k}$$

holds. This last equality and $x_{v_1}\cdots x_{v_k} - x_{u_1}\cdots x_{u_k} \in I(\square)$ imply

$$x_{v_1}\cdots x_{v_k} \in \mathrm{init}_{<_{\mathrm{rlex}}} I(\square).$$

Therefore we may assume

$$v_k = \delta_<(\mathrm{Cspan}(\{v_1,\ldots,v_k\})).$$

But then $x_{v_1}\cdots x_{v_k}$ is non-standard iff $x_{v_1}\cdots x_{v_{k-1}}$ is non-standard, hence by induction hypothesis we have $x_{v_1}\cdots x_{v_{k-1}} \in \mathrm{init}_{<_{\mathrm{rlex}}}(I(\square))$ and so a fortiori $x_{v_1}\cdots x_{v_k} \in \mathrm{init}_{<_{\mathrm{rlex}}} I(\square)$.

Up to now we have shown the following inclusion.

$$I(\triangle_<(\square)) \subseteq \mathrm{init}_{<_{\mathrm{rlex}}}(I(\square)).$$

Hence in order to finish the proof it is sufficient to show that the Hilbert-series of the factor algebras modulo the above two ideals are the same, i.e.,

$$\mathcal{H}(K(\triangle_<(\square)),t) = \mathcal{H}\left(K[x_v \,:\, v \in V] \Big/ \mathrm{init}_{<_{\mathrm{rlex}}}(I(\square)), t\right) \tag{3.10}$$

holds. It is well-known in the theory of Gröbner bases that for every polynomial ring $K[X]$, every ideal I of this polynomial ring, and every term order $\prec$ the Hilbert-series of $K[X] \big/ I$ is equal to the Hilbert-series of $K[X] \big/ \mathrm{init}_{<_\prec}(I)$, thus we have

$$\mathcal{H}\left(K[x_v \,:\, v \in V] \Big/ \mathrm{init}_{<_{\mathrm{rlex}}}(I(\square)), t\right) = \mathcal{H}(K[\square],t).$$

Therefore (3.10) is equivalent to (3.9) and we are done. **QED**

Remark B. Sturmfels proves in [29] an analogous result to our Theorem 12 for initial ideals of toric ideals. His proof is however completely different.

3.3 Shellable cubical complexes

In this section we take a closer look at shellable cubical complexes. We have seen in section 1.3 that the question whether a collection of facets of a cube is a ball or sphere is purely combinatorial, not depending on the geometric realization. Thus our first goal is to give a description of such collections.

As in [20], we encode the nonempty faces of $\square^n$ with vectors $(u_1, u_2, \ldots, u_n) \in \{0,1,*\}^n$ in the following way. Consider a standard geometric realization $\phi : \square^n \longrightarrow \mathbb{R}^n$.

For a nonempty face $\sigma \in \square^n$ and $i \in \{1, 2, \ldots, n\}$ set $u_i = 0$ or 1 respectively if the i-th coordinate of every element of $\phi(\sigma)$ is 0 or 1 respectively. Otherwise we set $u_i = *$. Using this coding, the facets of $\square^n$ will correspond to the vectors $(u_1, \ldots, u_n)$ for which exactly one of the u_i-s is not a $*$-sign.

Definition 48 *Let A_i^0 resp. A_i^1 stand for the facet $(u_1, u_2, \ldots, u_n)$ with $u_i = 0$ resp. $u_i = 1$ and $u_j = *$ for $j \neq i$. Let $\{F_1, \ldots, F_k\}$ be a collection of facets of $\partial(\square^n)$. Let r be the number of i-s such that exactly one of A_i^0 and A_i^1 belong to $\{F_1, \ldots, F_k\}$, and let s be the number of i-s such that both A_i^0 and A_i^1 belong to $\{F_1, \ldots, F_k\}$. We call (r, s) the* type *of $\{F_1, \ldots, F_k\}$.*

Note that when the type of $\{F_1, \ldots, F_k\}$ is (r, s) then there are exactly $n - r - s$ coordinates i such that neither A_i^0 nor A_i^1 belong to $\{F_1, \ldots, F_k\}$.

The following lemma, originally due to Ron Adin [1], gives a full description of those collections of facets $\{F_1, \ldots, F_k\}$ which are an $(n-1)$-dimensional ball or sphere.

Lemma 33 *The collection of facets $\{F_1, \ldots, F_k\}$ of the boundary of an n-cube is an $(n-1)$-sphere if and only if it has type $(0, n)$ and it is an $(n-1)$-ball if and only if its type (r, s) satisfies $r > 0$.*

Proof: We show first by induction on $n + r + s$ that a collection of facets of type (r, s) with $r > 0$ is an $(n-1)$-dimensional ball. Suppose first that we have $r > 1$. Without loss of generality we may assume that $F_1 = A_1^0$, and A_1^1 does not belong to $\{F_1, \ldots, F_k\}$. Then $\{F_2, \ldots, F_k\}$ as type $(r-1, s)$ and so, by induction hypothesis, it is an $(n-1)$-dimensional ball. On the other hand, $\{F_2 \cap F_1, \ldots, F_k \cap F_1\}$ is a collection of $(n-1)$-dimensional faces of F_1 of type $(r-1, s)$. By induction hypothesis, $\{F_2 \cap F_1, \ldots, F_k \cap F_1\}$ is an $(n-2)$ dimensional ball. Finally, an $(n-1)$-cube is homeomorphic to an $(n-1)$-ball and so $\{F_1\}$ is an $(n-1)$-dimensional ball. Now the fact that $\{F_1, \ldots, F_k\}$ is an $(n-1)$-dimensional ball follows from the following observation:

Let Y_1 and Y_2 be topological subspaces of a topological space X such that the following hold.

(i) Y_1 and Y_2 are homeomorphic to an $(n-1)$-dimensional ball.

(ii) $Y_1 \cap Y_2$ is homeomorphic to an $(n-2)$-dimensional ball.

Then $Y_1 \cup Y_2$ is homeomorphic to an $(n-1)$-dimensional ball.

The proof of this observation is straightforward, and left to the reader.

Suppose next that we have $s > 0$. Then without loss of generality we may assume $F_1 = A_1^0$ and $F_2 = A_1^1$. The collection $\{F_3, \ldots, F_k\}$ has type $(r, s-1)$ and so it is an $(n-1)$-dimensional ball by induction hypothesis. The collection of $(n-2)$-faces $\{F_3 \cap F_1, \ldots, F_k \cap F_1\}$ has type $(r, s-1)$ in F_1 and so it is an $(n-2)$-ball by induction hypothesis. Similarly, $\{F_3 \cap F_2, \ldots, F_k \cap F_2\}$ is an $(n-2)$-ball in F_2. Finally, $\{F_1\}$ and $\{F_2\}$. Thus $\{F_1, \ldots, F_k\}$ is an $(n-1)$-ball because of the following straightforward observation:

Let Y_1, Y_2 and Y_3 be topological subspaces of a topological space X such that the following hold.

(i') Y_1, Y_2 and Y_3 are homeomorphic to an $(n-1)$-dimensional ball.

(ii') $Y_1 \cap Y_3$ and $Y_2 \cap Y_3$ are homeomorphic to an $(n-2)$-dimensional ball.

(iii') We have $Y_1 \cap Y_2 = \emptyset$.

Then $Y_1 \cup Y_2 \cup Y_3$ is homeomorphic to an $(n-1)$-dimensional ball.

Hence we are left with the only case when $(r, s) = (1, 0)$. Then $\{F_1\}$ is an $(n-1)$-ball.

Note next that $\{F_1, \ldots, F_k\}$ has type $(0, n)$ if and only if it is the collection of all facets of the n-cube. The surface of an n-cube is an $(n-1)$ dimensional sphere and so $\{F_1, \ldots, F_k\}$ must be a sphere.

It only remains to be shown that for all other types (r,s), the set $\bigcup_{i=1}^{k} \mathrm{conv}(\phi(F_i))$ is not homeomorphic to an $(n-1)$-ball or an $(n-1)$-sphere. (The map ϕ is a standard geometric realization of the n-cube.) The types not listed above are of the form $(0,s)$ with $0 \leq s < n$. Let us fix such a type. Consider all those coordinates i for which neither A_i^0 nor A_i^1 belong to $\{F_1, \ldots, F_k\}$. Without loss of generality we may assume that these coordinates are $i = 1, 2, \ldots, n-s$. Consider the continuous map $\psi : \mathbb{R}^n \times [0,1] \longrightarrow \mathbb{R}^n$, defined by

$$((x_1, x_2, \ldots, x_{n-s}, x_{n-s+1}, \ldots, x_n), t) \longmapsto (t \cdot x_1, t \cdot x_2, \ldots, t \cdot x_{n-s}, x_{n-s+1}, \ldots, x_n).$$

This map retracts $\bigcup_{i=1}^{k} \mathrm{conv}(\phi(F_i))$ to a collection of all facets of an s-cube, i.e. an $(s-1)$-dimensional sphere. Now our lemma follows from the fact that an $(s-1)$-sphere with $s < n$ is not homotopy equivalent to an $(n-1)$-ball or an $(n-1)$ sphere, because its homology groups are different. **QED**

By abuse of notation we will say that the attachment of $\Box|_{F_k}$ to $\Box|_{F_1} \cup \cdots \cup \Box|_{F_{k-1}}$ in a shelling $F_1, F_2, \ldots$ has type (r,s) if set of facets of $\Box|_{F_k} \cap (\Box|_{F_1} \cup \cdots \cup \Box|_{F_{k-1}})$ considered as a collection of facets of $\Box|_{F_k}$ has type (r,s).

Shellable complexes are of great importance because of the following theorem, which is a consequence of the proof of [14, Theorem 2°].

Theorem 13 *If $\Box$ is a shellable cubical complex, then the Stanley-ring $K[\Box]$ is a Cohen-Macaulay ring.*

Usually in commutative algebra, the Cohen-Macaulay property is first defined for local rings, and then an arbitrary commutative ring is called Cohen-Macaulay, if every localization of it by a prime ideal is Cohen-Macaulay. In the case of graded rings associated to combinatorial structures the following (equivalent) definitions are more often used in the literature. (See [26, §5].)

Definition 49 *Let A be and $\mathbb{N}$-graded K-algebra. The* Krull-dimension *of A is the length of the longest chain of prime ideals in A. A sequence $\theta_1, \ldots, \theta_r$ of homogeneous elements of positive degree is called a* regular sequence *if θ_{i+1} is a non-zero-divisor in $A \big/ (\theta_1, \ldots, \theta_i)$, where $0 \leq i < r$. The* depth of A *is the length of the longest regular sequence in A. The graded algebra A is* Cohen-Macaulay *if the depth of A is equal to the Krull-dimension of A.*

The Cohen-Macaulay property turned out to be extremely useful in the study of the Stanley-Reisner ring of simplicial complexes. We will use some properties of Cohen-Macaulay graded algebras over infinite fields in Section 3.6.

In the study of the Stanley-ring of shellable cubical complexes we will need the following elementary lemma.

Lemma 34 *The edge-graph of a shellable cubical complex of dimension at least 2 is bipartite.*

Proof: We use induction on the number of facets. Let $F_1, \ldots, F_k$ be a shelling of $\square$. By induction hypothesis, the complex $\square|_{F_1} \cup \cdots \cup \square|_{F_{k-1}}$ has a bipartite edge-graph. Clearly $\square|_{F_k}$ has a bipartite edge-graph: when we represent it's vertices, as vertices of the standard d-cube $[0,1]^d$, an appropriate coloring with 2 colors is to color the vertices according with the parity of the sum of their coordinates. It is easy to check that the edge-graph of $\square|_{F_k} \cap (\square|_{F_1} \cup \cdots \cup \square|_{F_{k-1}})$ is a connected graph. Thus the induction step follows from the following lemma. **QED**

Lemma 35 *Let G_1 resp. G_2 be bipartite graphs on vertex set V_1 resp. V_2 and edge set E_1, resp. E_2. Assume that we have*

$$E_1 \cap ((V_1 \cap V_2) \times (V_1 \cap V_2)) = E_2 \cap ((V_1 \cap V_2) \times (V_1 \cap V_2)),$$

i.e., both graphs have the same restriction to $V_1 \cap V_2$, and that the restriction of G_1 to $V_1 \cap V_2$ is connected. Then the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$ is bipartite.

Proof: The proof is straightforward, it depends on the fact that –up to permutation of colors–there is only one way to color a connected bipartite graph with 2 colors. **QED**

3.4 A homogeneous generating system of degree 2 for $I(\Box)$

Theorem 10 inspires the following question. Let $I_1''(\Box)$ be the ideal generated by the monomials $x_u \cdot x_v$ such that the pair $\{u, v\} \subseteq V$ is not contained in any face. When do we have $I(\Box) = I_1''(\Box) + I_2(\Box)$? For such complexes $I(\Box)$ is generated by homogeneous forms of degree 2.

Definition 50 *We call a cubical complex* well behaved *when it satisfies $I(\Box) = I_1''(\Box) + I_2(\Box)$.*

The following lemma is a straightforward consequence of Theorem 10 and the the trivial fact $I_1''(\Box) \subseteq I_1'(\Box)$.

Lemma 36 *A cubical complex is well behaved iff for every triple $\lfloor u_1, u_2, u_3 \rfloor$ either $\{u_1, u_2, u_3\}$ is contained in a face of $\Box$ or there is a $\lfloor v_1, v_2, v_3 \rfloor$ equivalent to $\lfloor u_1, u_2, u_3 \rfloor$ such that $\{v_1, v_2\}$ is not contained in any face of $\Box$.*

We will use the statement of the lemma as an equivalent definition of well behaved cubical complexes.

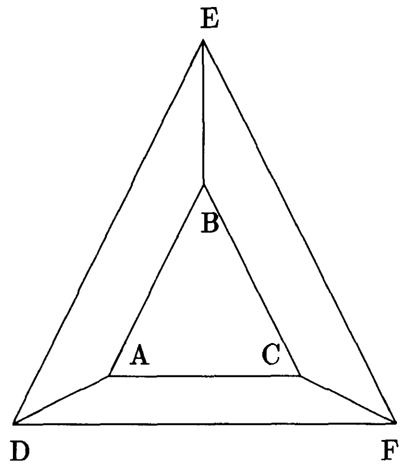

Figure 3-1: Not well behaved cubical complex

Example Figure 3-1 represents a not well behaved cubical complex. The facets of the complex are $ABED, BCEF$ and $ACDF$.

It is easy to verify that for the triple $\lfloor A, C, E \rfloor$ and any equivalent triple any two elements of the triple are contained in a face, but there is no face containing all three of them.

Conjecture 1 *Every shellable cubical complex is well behaved.*

The following lemmas are statements about the properties of an eventual minimal counterexample to Conjecture 1. (Minimality will always mean minimality of the number of facets.) At the end we will not get a proof of the conjecture, but the properties to be shown will allow us to exclude all shellable subcomplexes of a boundary complex of a convex cubical polytope from the class of shellable not well behaved cubical complexes.

Clearly, a *counterexample* to the conjecture is a shellable cubical complex $\square$ containing a triple $\lfloor u_1, u_2, u_3 \rfloor$ such that $\mathrm{Cspan}(\{u_1, u_2, u_3\})$ does not exist, but for every equivalent triple $\lfloor v_1, v_2, v_3 \rfloor$, the every pair $\{v_i, v_j\}$ is contained in some face of $\square$.

Definition 51 *Let $\Box$ be a not well behaved shellable cubical complex. We call the triple $\lfloor u_1, u_2, u_3 \rfloor$ a* counter-evidence *if any two of u_1, u_2 and u_3 is contained in some face of $\Box$, but no face contains the set $\{u_1, u_2, u_3\}$, and the same holds for all equivalent triples in $\Box$.*

Note that if a triple belongs to a shellable subcomplex $\Box'$, and it is a counter-evidence in $\Box$ then it is also a counter-evidence in $\Box'$. This observation is used and explained in the proof of the following lemma.

Lemma 37 *Let $\Box$ be a minimal not well behaved shellable cubical complex and $F_1, F_2, \ldots, F_k$ a shelling of $\Box$. Then every counter-evidence $\lfloor u_1, u_2, u_3 \rfloor$ has exactly one element outside F_k, and two elements in F_k.*

Proof: Clearly $\lfloor u_1, u_2, u_3 \rfloor$ cannot be a counter-evidence, if all three u_i-s lie in F_k. Thus at least one of them must lay outside F_k. If at least two of u_1, u_2, u_3 are not in F_k, then any face containing at least two of them is also contained in $\Box\!\restriction_{F_1} \cup \cdots \cup \Box\!\restriction_{F_{k-1}}$. In particular, we must have $\{u_1, u_2, u_3\} \subset F_1 \cup \cdots \cup F_{k-1}$. By minimality, $\lfloor u_1, u_2, u_3 \rfloor$ is not a counter-evidence in the shellable complex $\Box\!\restriction_{F_1} \cup \cdots \cup \Box\!\restriction_{F_{k-1}}$. Thus there is an equivalent $\lfloor v_1, v_2, v_3 \rfloor$ such that $\{v_1, v_2, v_3\}$ is contained in some face of $\Box\!\restriction_{F_1} \cup \cdots \cup \Box\!\restriction_{F_{k-1}}$. But then the same holds in $\Box$ and so $\lfloor u_1, u_2, u_3 \rfloor$ is not a counter-evidence. Hence the only way for $\lfloor u_1, u_2, u_3 \rfloor$ to be a counter-evidence is to contain exactly one element outside F_k.
QED

Corollary 10 *Let $\Box$ be a minimal not well behaved shellable cubical complex and $F_1, F_2, \ldots, F_k$ a shelling of $\Box$ and $\lfloor u_1, u_2, u_3 \rfloor$ a counter-evidence such that $u_1, u_2 \in F_k$ and $u_3 \notin F_k$ Then for every $u \in \mathrm{Cspan}(\{u_1, u_2\})$, the face $\mathrm{Cspan}(\{u_3, u\})$ has exactly half of its vertices in F_k.*

Proof: Clearly, we can replace the diagonal $\lfloor u_1, u_2 \rfloor$ by another diagonal of the face $\text{Cspan}(\{u_1, u_2\})$ such that $u = u_1$ holds, and so $\text{Cspan}(\{u_3, u\})$ exists, and we may assume $u_1 = u$. By $u_3 \notin F_k$, at most half of $\text{Cspan}(\{u_3, u\})$ belongs to F_k. If less then half is contained in F_k then the diagonal $\lfloor u_1, u_3 \rfloor$ may be replaced by a diagonal $\lfloor u_1', u_3' \rfloor$ such that both u_1' and u_3' are outside F_k. Thus the triple $\lfloor u_1', u_2, u_3' \rfloor$ (which is equivalent to $\lfloor u_1, u_2, u_3 \rfloor$ will be a counter-evidence not satisfying the criterion of Lemma 37. **QED**

Lemma 38 *Let $\square$ be a minimal not well behaved shellable cubical complex and $F_1, F_2, \dots, F_k$ a shelling of $\square$. Assume that there is a pair H, H' of subfacets which are opposite facets of $\partial(\square|_{F_k})$ and both belong to $\square|_{F_1} \cup \cdots \cup \square|_{F_{k-1}}$. Let $\lfloor u_1, u_2, u_3 \rfloor$ be a counter-evidence such that $u_1, u_2 \in F_k, u_3 \notin F_k$. Then there is a face of $\square$ containing $\text{Cspan}(\{u_1, u_2\}) \cap H$ and u_3.*

Proof: If $\text{Cspan}(\{u_1, u_2\}) \cap H = \emptyset$ then we have $\text{Cspan}(\{u_1, u_2\}) \subseteq H' \in \square|_{F_1} \cup \cdots \cup \square|_{F_{k-1}}$, meaning that already $\square|_{F_1} \cup \cdots \cup \square|_{F_{k-1}}$ was a counterexample. Similar contradiction with minimality arises when we assume $\text{Cspan}(\{u_1, u_2\}) \cap H' = \emptyset$. Thus we may suppose $u_1 \notin H$ and $u_2 \in H$, and we may renumber all equivalent triples $\lfloor v_1, v_2, v_3 \rfloor$ such that $v_3 \notin F_k$, $v_1 \in H'$ and $v_2 \in H$ holds. Let u_1' be the projection of u_1 onto H. Then we have $\text{Cspan}(\{u_1', u_2\}) = \text{Cspan}(\{u_1, u_2\}) \cap H$. If the set $\{u_1', u_2, u_3\}$ is contained in a face, then we are done. Otherwise, given the fact that the cubical span of any two of u_1', u_2 and u_3 is contained in $\square|_{F_1} \cup \cdots \cup \square|_{F_{k-1}}$, a well behaved shellable cubical complex, we obtain that the triple $\lfloor u_1', u_2, u_3 \rfloor$ is equivalent to a triple $\lfloor z_1, z_2, z_3 \rfloor$ such that there is no face of $\square|_{F_1} \cup \cdots \cup \square|_{F_{k-1}}$ containing $\{z_1, z_2\}$. Consider a sequence of replacing diagonals, which demonstrates the equivalence of $\lfloor u_1', u_2, u_3 \rfloor$ and $\lfloor z_1, z_2, z_3 \rfloor$. Assume that $\lfloor u_1, u_2, u_3 \rfloor$ was chosen from its equivalence class such that this derivation of equivalence is the shortest possible.

If the first step is replacing the triple $\lfloor u_1', u_2, u_3 \rfloor$ with $\lfloor w_1', w_2, u_3 \rfloor$ where $\{w_1', w_2\}$ is a diagonal Cspan($\{u_1', u_2\}$) then we get a contradiction with the minimality of the derivation. In fact, let w_1 be the projection of w_1' onto H'. It is easy to check that $\{w_1', w_2\}$ is a diagonal of Cspan($\{u_1, u_2\}$) and so $\lfloor w_1, w_2, u_3 \rfloor$ is equivalent to $\lfloor u_1, u_2, u_3 \rfloor$ and there is a shorter derivation of equivalence between $\lfloor w_1', w_2, u_3 \rfloor$ and $\lfloor z_1, z_2, z_3 \rfloor$. We also get a contradiction when we assume that our first step was to replace $\lfloor u_1', u_2, u_3 \rfloor$ with $\lfloor u_1', w_2, w_3 \rfloor$ where $\{w_2, w_3\}$ is a diagonal of Cspan($\{u_2, u_3\}$). In this case, the very same replacement can be performed on $\lfloor u_1, u_2, u_3 \rfloor$ and we obtain the equivalent triple $\lfloor u_1, w_2, w_3 \rfloor$, from which we have a shorter derivation.

Hence we are left with the case when the first step of the derivation involves replacing $\lfloor u_1', u_2, u_3 \rfloor$ with $\lfloor w_1', u_2, w_3 \rfloor$ where $\{w_1', w_3\}$ is a diagonal of Cspan($\{u_1', u_3\}$) such that $w_1' \in H$ and $w_3 \notin F_k$. Let q_1 be the projection of u_2 onto H' and let us introduce the notations $q_1' := u_2$, $q_2 := u_1'$. Then the triple $\lfloor q_1, q_2, u_3 \rfloor$ is equivalent to $\lfloor u_1, u_2, u_3 \rfloor$, and $\lfloor q_1', w_1', w_3 \rfloor$ is equivalent to $\lfloor q_1', q_2, u_3 \rfloor$. From this second equivalence we obtain that also $\lfloor q_1, w_1', w_3 \rfloor$ is equivalent to $\lfloor q_1, q_2, u_3 \rfloor$, hence taking $\lfloor q_1, w_1', w_3 \rfloor$ instead of $\lfloor u_1, u_2, u_3 \rfloor$ makes the first step again unnecessary.

Therefore we may assume that at least two of u_1', u_2 and u_3 are not contained in a common face. This pair cannot be $\{u_1', u_2\} \subseteq F_k$ and it cannot be $\{u_2, u_3\}$ because then $\lfloor u_1, u_2, u_3 \rfloor$ is not a counterexample. Finally, if $\{u_1', u_2\}$ is not contained in any face, then we obtain a contradiction after we have replaced $\lfloor u_1, u_2, u_3 \rfloor$ by the triple $\lfloor q_1, q_2, u_3 \rfloor$ defined as above. **QED**

Proposition 6 *Assume that for the shelling $F_1, \ldots, F_k$ of $\square$, the attachment of $\square|_{F_k}$ to $\bigcup_{i=1}^{k-1} \square|_{F_i}$ has type $(r, 0)$. Then $\square$ can not be a minimal not well behaved shellable complex.*

Proof: Assume the contrary. When the type is $(r, 0)$ then $F_k \setminus (F_1 \cup \cdots \cup F_{k-1})$ is not empty: there is at least one vertex which was added when we added F_k. Let

$\lfloor u_1, u_2, u_3 \rfloor$ be a counterexample with $u_1, u_2 \in F_k$, $u_3 \notin F_k$. If $\mathrm{Cspan}(\{u_1, u_2\})$ contains a newly added vertex, then –after replacing eventually u_1 and u_2 with another diagonal of $\mathrm{Cspan}(\{u_1, u_2\})$– we may assume $u_2 \in F_k \setminus (F_1 \cup \cdots \cup F_{k-1})$. But then $\{u_2, u_3\}$ is not contained in any face of $\square$, and we get a contradiction. Thus we must have $\mathrm{Cspan}(\{u_1, u_2\}) \subset F_1 \cup \cdots \cup F_{k-1}$. In this case, however, any face containing at least two of u_1, u_2 and u_3 is contained in $\square|_{F_1} \cup \cdots \cup \square|_{F_{k-1}}$ and so $\lfloor u_1, u_2, u_3 \rfloor$ is a counter-evidence in this smaller complex already. **QED**

Lemma 39 *Assume $F_1, \ldots, F_k$ is a shelling of a minimal not well behaved complex $\square$, and $\lfloor u_1, u_2, u_3 \rfloor$ is a counter-evidence. Assume H and H' are opposite facets of $\partial(\square|_{F_k})$, such that they both belong to $\square|_{F_1} \cup \cdots \cup \square|_{F_{k-1}}$. Then there is no edge $\{v, w\} \in \mathrm{Cspan}(\{u_1, u_2\})$ that would satisfy $v \in H, w \in H'$ and $\mathrm{Cspan}(\{u_3, v, w\})$ exists.*

Proof: Assume the contrary. Since $\{v, w\}$ is an edge, either $\{u_3, v\}$ or $\{u_3, w\}$ is a diagonal of $\mathrm{Cspan}(\{u_3, v, w\})$. W.l.o.g. we may assume that $\{u_3, v\}$ is a diagonal. We also may assume that the triple $\lfloor u_1, u_2, u_3 \rfloor$ was chosen in such a way that $u_1 = v$ holds. (If not, we can replace the pair $\lfloor u_1, u_2 \rfloor$ with another pair containing $v \in \mathrm{Cspan}(\{u_1, u_2\})$.) Let u_3' be the vertex diagonally opposite to w in $\mathrm{Cspan}(\{u_3, v, w\})$. Then $\lfloor u_1, u_2, u_3 \rfloor$ is equivalent to $\lfloor w, u_2, u_3' \rfloor$ and here we have $\{w, u_2\} \subseteq H' \in \square|_{F_1} \cup \cdots \cup \square|_{F_{k-1}}$, contradicting the assumption about the minimality of $\square$. **QED**

Proposition 7 *If $\square$ has a shelling $F_1, \ldots, F_k$ such that the attachment of $\square|_{F_k}$ to $\bigcup_{i=1}^{k-1} \square|_{F_i}$ has type (r, s) with $s \geq 2$, then $\square$ is not a minimal not well behaved shellable complex.*

Proof: Assume the contrary. Let H_1, H_1' and H_2, H_2' be pairs of subfacets which are opposite in F_k and the all belong to $\square|_{F_k} \cap (\square|_{F_1} \cup \cdots \square|_{F_{k-1}})$. Assume furthermore that $\lfloor u_1, u_2, u_3 \rfloor$ is a counterexample satisfying $u_1, u_2 \in F_k$, $u_3 \notin F_k$. Then $\text{Cspan}(\{u_1, u_2\} \cap H_i$ and $\text{Cspan}(\{u_1, u_2\} \cap H_i'$ is non-empty by the minimality of $\square$. By Lemma 38, $\text{Cspan}(\{u_1, u_2\}) \cap H_1$ is contained in a face with u_3. But then we can find $v, w \in \text{Cspan}(\{u_1, u_2\}) \cap H_1$ such that $\{v, w\}$ is an edge and we have $v \in H_2, w \in H_2'$, contradicting Lemma 39. **QED**

Lemma 40 *Let $\square$ be a minimal not well behaved shellable complex. Let H and H' be opposite facets of $\partial(\square|_{F_k})$ such that H also belongs to $(\square|_{F_1} \cup \cdots \cup \square|_{F_{k-1}})$ but H' does not. Assume, there is a counter-evidence $\lfloor u_1, u_2, u_3 \rfloor$, such that $u_1, u_2 \in F_k, u_3 \notin F_k$, and $\text{Cspan}(\{u_1, u_2\}) \cap H \neq \emptyset$ hold. Then for any $u \in \text{Cspan}(\{u_1, u_2\}) \cap H'$ we have*

$$\text{Cspan}(\{u_3, u\}) \cap F_k \subseteq H'.$$

Proof: Note first that $\text{Cspan}(\{u_1, u_2\}) \cap H' \neq \emptyset$ otherwise $\lfloor u_1, u_2, u_3 \rfloor$ would also be a counter-evidence in $(\square|_{F_1} \cup \cdots \cup \square|_{F_{k-1}})$. W.l.o.g. we may assume $u = u_1$ and so $u_2 \in H$. By Corollary 10 the face $\text{Cspan}(\{u_3, u_1\})$ has exactly half of its vertices in F_k. In particular, u_3 is connected by an edge to a unique vertex $v \in F_k$ and we have $\text{Cspan}(\{u_3, u_1\}) \cap F_k = \text{Cspan}(\{u_1, v\})$. Thus we only need to show $v \in H'$. If not, then we can replace the diagonal $\lfloor u_3, u_1 \rfloor$ with a diagonal $\lfloor u_3', v \rfloor$ and obtain an equivalent triple $\lfloor v, u_2, u_3' \rfloor$ with $v, u_2 \in H$, contradicting the assumption of minimality of $\square$. **QED**

Proposition 8 *Let $\square$ be shellable d-dimensional minimal not well behaved cubical complex, with shelling $F_1, F_2, \ldots, F_k$. Then the type of the attachment of $\square|_{F_k}$ to $\square|_{F_1} \cup \cdots \cup \square|_{F_{k-1}}$ can not be $(r, d-r)$.*

Proof: Assume the contrary. By Proposition 6 and Proposition 7 we may assume that the type of the attachment of $\Box|_{F_k}$ to $\Box|_{F_1} \cup \cdots \cup \Box|_{F_{k-1}}$ is $(d-1,1)$. Thus, taking a standard geometric realization ϕ of $\Box|_{F_k}$, we may assume that exactly the following facets of $\Box|_{F_k}$ belong to $\Box|_{F_1} \cup \cdots \cup \Box|_{F_{k-1}}$: $A_1^0, A_2^0, \ldots, A_{d-1}^0, A_d^0$ and A_d^1.

Let $\lfloor u_1, u_2, u_3 \rfloor$ be a counter-evidence such that $u_1, u_2 \in F_k, u_3 \notin F_k$, and $\dim \mathrm{Cspan}(\{u_1, u_2\})$ is maximal under these conditions. Then, similarly to Corollary 10, we can show that for every $u \in \mathrm{Cspan}(\{u_1, u_2\})$ the face $\mathrm{Cspan}(\{u_3, u\})$ exists and has exactly half of its vertices in $\mathrm{Cspan}(\{u_1, u_2\})$. By minimality of $\Box$, for any $i \in \{1, 2, \ldots, d-1\}$ the vertices u_1 and u_2 cannot be both contained in A_i^0, otherwise the triple $\lfloor u_1, u_2, u_3 \rfloor$ is already a counter-evidence in $\Box|_{F_1} \cup \cdots \cup \Box|_{F_{k-1}}$. Similarly, the last coordinate of u_1 and u_2 can not agree. These considerations show that the vertices $u, v \in F_k$ defined by $\phi(u) := (1, 1, \ldots, 1, 0)$ and $\phi(v) := (1, 1, \ldots, 1, 1)$ both belong to $\mathrm{Cspan}(\{u_1, u_2\})$. We claim that $\mathrm{Cspan}(\{u_3, u\})$ and $\mathrm{Cspan}(\{u_3, v\})$ are edges. In fact, as noted above, half of $\mathrm{Cspan}(\{u_3, u\})$ is contained in $\mathrm{Cspan}(\{u_1, u_2\})$. It is sufficient to show therefore that $\mathrm{Cspan}(\{u_3, u\}) \cap \mathrm{Cspan}(\{u_1, u_2\})$ is zero dimensional. If not, then it contains a vertex u' for which $\phi(u')$ differs from $\phi(u)$ from exactly one coordinate, say the j-th one. When $j = d$ then we get a contradiction by Lemma 39, when $j \leq d-1$ we get a contradiction by Lemma 40. Hence $\{u_3, u\}$ is an edge and similarly $\{u_3, v\}$ is an edge. But then u_3, u and v form a triangle in the edge-graph of $\Box$, which cannot be bipartite therefore, contradicting Lemma 34. **QED**

Propositions 6, 7 and 8 imply the following theorem.

Theorem 14 *Every shellable cubical complex of dimension* 2 *is well behaved.*

Proof: Take a minimal counterexample $\Box$ with shelling $F_1, \ldots, F_k$. By Lemma 33, the possible types of attachments of $\Box|_{F_k}$ to $\Box|_{F_1} \cup \cdots \cup \Box|_{F_{k-1}}$ are the following: $(1,0), (2,0), (1,1)$ and $(0,2)$. The types $(1,0)$ and $(2,0)$ are excluded by Proposition

6, type $(0,2)$ is forbidden by Proposition 7, and finally type $(1,1)$ is disallowed by Proposition 8. **QED**

Finally, we prove the main theorem of this section.

Theorem 15 *Let $\Box$ be a $(d-1)$-dimensional shellable subcomplex of the boundary complex of a d-dimensional convex cubical polytope P. Then $\Box$ is well behaved.*

Proof: Let $\Box$ be a minimal counterexample and $F_1, F_2, \ldots, F_k$ a shelling of $\Box$. By Proposition 6 and Proposition 7 we may assume that the type of attachment of $\Box|_{F_k}$ to $\Box|_{F_1} \cup \cdots \cup \Box|_{F_{k-1}}$ is $(r,1)$. Here, by Theorem 14 we have $\dim(\Box) = d-1 > 2$ and so, 1 is not one less than the dimension of $\Box$. Hence, by Lemma 33, we must have $r > 0$. Let H_1 and H_2 be the only pair of opposite facets of $\partial(\Box|_{F_k})$ such that they both belong to $\Box|_{F_1} \cup \cdots \cup \Box|_{F_{k-1}}$.

Let us take a counter-evidence $\lfloor u_1, u_2, u_3 \rfloor$ such that $u_1, u_2 \in F_k$, $u_3 \notin F_k$ hold, and the dimension of $\mathrm{Cspan}(\{u_1, u_2\})$ be maximal under these conditions. Let us denote $\mathrm{Cspan}(\{u_1, u_2\})$ by τ_3. By Lemma 38, the faces $\tau_i := \mathrm{Cspan}(\{u_3\} \cup (\tau_3 \cap H_i))$ exist for $i = 1,2$. As in the proof of Theorem 14, the maximality of $\dim \mathrm{Cspan}(\{u_1, u_2\})$ implies that exactly half of the vertices of τ_1 or τ_2 belong to τ_3. Thus we have

$$\frac{|\tau_1|}{2} = |\tau_1 \cap \tau_3| = |\tau_3 \cap H_1| = \frac{|\tau_3|}{2} = |\tau_3 \cap H_2| = |\tau_2 \cap \tau_3| = \frac{|\tau_1|}{2},$$

and so τ_1, τ_2 and τ_3 have the same dimension. Let us denote this dimension by δ.

Let S be the affine hull of x_3 and τ_3. It is a $(\delta+1)$ dimensional plane, and it intersects the polytope P in a $(\delta+1)$-dimensional polytope P'. Clearly, S contains both τ_1 and τ_2, because half of these faces is a $(\delta-1)$-face of τ_3 (and so belongs to S) and the affine span of u_3 and $\tau_i \cap \tau_3$ contains τ_i. $(i = 1,2.)$

Consider the "pyramid" $Q := \text{conv}(x_3, \tau_3)$. We may assume that $\text{relint}(Q) \subset \text{int}\ (P)$ otherwise Q is contained in a face of $\partial(P)$, and so there is a face of $\partial(P)$ containing τ_1, τ_2 and τ_3. It is easy to convince ourselves, however, that no cube can contain three equidimensional faces with the intersection properties of τ_1, τ_2 and τ_3: if half of the vertices of τ_1 and τ_2 intersect τ_3 in opposite halves of τ_3 then $\tau_1 \cap \tau_2$ would be empty, and we need $x_3 \in \tau_1 \cap \tau_2$. The affine hull of Q is S.

W.l.o.g. we may assume that u_1 is diagonally opposite to u_3 in τ_1. (If not, we may replace $\lfloor u_1, u_2 \rfloor$ by another diagonal of τ_3.) Let u_3' be the vertex of $\tau_1 \setminus \tau_3$ which is connected to u_1 by an edge. (In other words let u_3' be the vertex diagonally opposite to u_3 in the face $\tau_1 \setminus \tau_3$.) Let u_1' be the vertex diagonally opposite to u_1 in $\tau_3 \cap \tau_1$. Then u_1' is diagonally opposite to u_3' in τ_1 and so $\lfloor u_1', u_2, u_3' \rfloor$ is equivalent to $\lfloor u_1, u_2, u_3 \rfloor$ and so there is a face containing u_3' and u_2. In particular, the line segment connecting u_3' and u_2 belongs to $\partial(P)$ and thus it cannot have any common point with $\text{relint}(Q)$. Consider now the supporting hyperplanes of the facets of Q in S. These are δ dimensional hyperplanes and they are either the affine hull of a $(\delta - 1)$-face of τ_3 and u_3, or the affine hull of τ_3. For each such hyperplane, let us call the half-space of S determined by the hyperplane which contains Q, the *positive half* of the hyperplane. We claim that for every supporting hyperplane K of a facet of Q which contains u_1', u_2 and u_3, the vertex u_3' is in the strict positive half of K. In fact, K when intersects τ_3 in a $(\delta - 1)$-face of τ_3 and so it intersects $\tau_1 \cap \tau_3$ in a $(\delta - 2)$-face. $K \cap \tau_1$ contains this $(\delta - 2)$-face and u_3 and so $K \cap \tau_1$ contains a $(\delta - 1)$-face of τ_1. Thus $K \cap \tau_1$ is this $(\delta - 1)$-face, because otherwise K contains the whole affine hull of τ_1 which does not contain u_2. The $(\delta - 1)$-face $K \cap \tau_1$ of τ_1 contains u_1' and so it cannot contain u_3' which is diagonally opposite to $u_1' \in K$. The only hyperplanes of facets of Q through u_2 which don't contain u_3 or u_1' are aff (τ_3) and aff (τ_2). The vertex u_3' cannot be in he strict positive half of both of them, because otherwise the line segment connecting u_3' with u_2 would contain a point of $\text{relint}(Q)$ close to u_2. Therefore either $u_3' \in \text{aff}\ (\tau_3)$ or $u_3' \in \text{aff}\ (\tau_2)$ must hold.

If u_3' belongs to τ_3 then, considering the fact that u_1' which is diagonally opposite to u_3' in τ_1 also belongs to τ_3, we get $\tau_1 \subseteq \tau_3$, a contradiction. Therefore u_3' must belong to aff (τ_2) and so $\tau_1 \cap \tau_2$ contains the $(\delta - 1)$-face Cspan$(\{u_2, u_3'\})$, and so $\tau_1 \cap \tau_2$ must also be $(\delta - 1)$-dimensional. We claim that in this case u_2 is connected to u_3 by an edge. In fact by what was said above, the line segment connecting u_3' and u_2 intersects relint(τ_2) and so u_3' and u_2 are diagonally opposite in τ_2, and u_3 is connected to u_2 by an edge. Therefore u_1', u_2 and u_3 form a triangle, contradicting Lemma 34. **QED**

Corollary 11 *The boundary complex of a convex cubical polytope is well behaved.*

Proof: As a special case of the results shown in [4], the boundary complex of a convex cubical polytope is shellable. We may apply therefore Theorem 15. **QED**

3.5 Edge-orientable cubical complexes

Definition 52 *We call two edges $\{u, v\}$ and $\{u', v'\}$ of a cubical complex* parallel *if there is a facet $F \in \Box$ and a subfacet $H \subset F$ and such that $|\{u, v\} \cap H| = |\{u', v'\} \cap H| = 1$.*

We can turn the edge-graph of $\Box$ into a directed graph by defining a function

$$\pi : V \times V \longrightarrow \{-1, 0, 1\},$$

satisfying the following properties

(i) $\pi(u, v) = -\pi(v, u)$ holds for all $u \neq v$,

(ii) $\pi(u, v) = 0$ if and only if $\{u, v\}$ is not an edge of $\Box$.

(We say when $\pi(u, v) = 1$ that "the edge points from u towards v".)

We call π *an* orientation of the edge-graph of $\square$ *or edge-orientation on* $\square$ *if it satisfies the following condition: given two parallel edges* $\{u,v\}$ *and* $\{u',v'\}$*, a facet* F *containing these edges and a subfacet* $H \subset F$ *such that* $\{u,v\} \cap H = u$*,* $\{u',v'\} \cap H = u'$ *we have*

$$\pi(u,v) = \pi(u',v')$$

We call $\square$ *edge-orientable, if its edge-graph has an orientation* π*.*

In plain English, edge-orientability means that we can direct the edges of $\square$ such that "parallel edges point in the same direction." As a consequence of Jordan's theorem, the boundary complex of a 3-dimensional cubical polytope is edge-orientable. In higher dimensions edge-orientability means that every $(d-2)$ dimensional manifold connecting midpoints of parallel edges is orientably embedded into the surface of the polytope.

The following lemma shows the existence of a labeling for a shellable and edge-orientable complex $\square$ which will have important applications.

Lemma 41 *Let* $\square$ *be a shellable and edge-orientable complex of dimension at least* 2 *and* π *an orientation of the edge-graph of* $\square$*. Then there is a labeling*

$$\theta : V \longrightarrow \mathbb{Z}$$

such that for every edge $\{u,v\}$ *we have*

$$\theta(v) - \theta(u) = \pi(u,v). \tag{3.11}$$

Proof: The proof is analogous to the proof of Lemma 34. Assume $\square$ is a counterexample with a minimal number of facets. Let $F_1, F_2, \ldots, F_k$ be a shelling of $\square$. The complex $\square|_{F_1} \cup \cdots \cup \square|_{F_{k-1}}$ is shellable, and the restriction of π provides an edge-orientation on it. Hence, by the minimality of $\square$, there is a labeling θ' on it which satisfies equation (3.11) for every pair of vertices of $\square\restriction_{F_1} \cup \cdots \cup \square\restriction_{F_{k-1}}$. On the other hand it is easy to see that there is a labeling θ'' on the cube $\square\restriction_{F_k}$: we can take a standard geometric realization $\phi : \square\restriction_{F_k} \to [0,1]^{\dim(\square)}$, such that the only vertex with no incoming edges in $\square\restriction_{F_k}$ goes into $(0,0,\ldots,0)$, and the only vertex with no outgoing edges in $\square\restriction_{F_k}$ goes into $(1,1,\ldots,1)$. Then we can set $\theta''(v)$ to be the sum of the coordinates of $\phi(v)$ for every $v \in F_k$. It is easy to check that this labeling will also satisfy (3.11) for every pair of vertices of F_k.

Clearly, if a labeling θ satisfies (3.11) in a complex then the same holds for $\theta + c$ where c is an arbitrary constant. Thus we may assume that we have a $v_0 \in F_k \cap (F_1 \cup \cdots \cup F_{k-1})$ such that $\theta'(v_0) = \theta''(v_0)$ holds. But then, as we have observed in the proof of Lemma 34, the edge-graph of the complex $\square\restriction_{F_k} \cap (\square\restriction_{F_1} \cup \cdots \cup \square\restriction_{F_{k-1}})$ is connected. It is easy to see that if θ' and θ'' are labelings satisfying (3.11) in a directed graph G, which has a connected graph as underlying undirected graph, then their difference is constant. Thus, by $\theta'(v_0) = \theta''(v_0)$, the restriction of θ' to $\square\restriction_{F_k} \cap (\square\restriction_{F_1} \cup \cdots \cup \square\restriction_{F_{k-1}})$ is equal to the restriction of θ'' to $\square\restriction_{F_k} \cap (\square\restriction_{F_1} \cup \cdots \cup \square\restriction_{F_{k-1}})$. Therefore we can define

$$\theta(v) := \begin{cases} \theta'(v) & \text{when } v \in F_1 \cup \cdots \cup F_{k-1} \\ \theta''(v) & \text{when } v \in F_k \end{cases}$$

and obtain a labeling for $\square$ that satisfies (3.11), contradicting our assumption. **QED**

Lemma 42 *Let $\square$ be a shellable, edge-orientable cubical complex, and π be an edge-orientation of $\square$. Then the transitive closure $<_\pi$ of the relation*

$$u \overset{\text{def}}{<_\pi} v \quad \textit{whenever } \pi(u,v) = 1$$

is a partial order on the vertex set V of $\square$.

Proof: We only need to show that there is no sequence of vertices $v_1, v_2, \ldots, v_k$ such that

$$\pi(v_1, v_2) = \pi(v_2, v_3) = \cdots = \pi(v_{k-1}, v_k) = \pi(v_k, v_1) = 1$$

would hold. If we had such a sequence then for a labeling θ satisfying (3.11) we would have $\theta(v_{i+1}) = \theta(v_i) + 1$ for $i = 1, 2, \ldots, k-1$, and $\theta(v_k) + 1 = \theta(v_1)$. But this would imply

$$\theta(v_1) = \theta(v_1) + k,$$

a contradiction. **QED**

Definition 53 *Let* $\square$ *a shellable, edge-orientable cubical complex and* π *an edge-orientation of* $\square$*. We call the partial order described in Lemma 42 the* partial order induced by π *and we denote it by* $<_\pi$*.*

We define the triangulation $\triangle_\pi(\square)$ of $\square$ induced by π *as follows.*

1. *We set* $V(\triangle_\pi(\square)) := V(\square)$*.*

2. *A set* $\{v_1, \ldots, v_k\} \subseteq V(\square)$ *is a face of* $\triangle_\pi(\square)$ *if and only if* $\text{Cspan}(\{v_1, v_2, \ldots, v_k\})$ *exists and* $\{v_1, v_2, \ldots, v_k\}$ *is a chain in the partially ordered set* $(V, <_\pi)$*.*

Lemma 43 *Given a shellable and edge-orientable cubical complex* $\square$ *and an edge-orientation* π *of* $\square$*, the simplicial complex* $\triangle_\pi(\square)$ *is a natural triangulation of* $\square$*. In fact we have* $\triangle_\pi(\square) = \triangle_<(\square)$ *for any linear extension* $<$ *of the partial order* $<_\pi$*.*

Proof: Take an arbitrary subset $\{v_1, \ldots, v_k\}$ of the vertex set V. W.l.o.g. we may assume $v_1 > \cdots > v_k$.

If $\text{Cspan}(\{v_1, \ldots, v_k\})$ does not exist then $\{v_1, \ldots, v_k\}$ does not belong to any of $\triangle_\pi(\square), \triangle_<(\square)$. Thus we may assume that $\text{Cspan}(\{v_1, \ldots, v_k\})$ exist, and w.l.o.g. we may even assume that there is no vertex outside $\text{Cspan}(\{v_1, \ldots, v_k\})$, i.e., $\square$ is a standard

n-cube $\Box^n$ for some n. Then $\Box$ has a unique vertex u such that u has no incoming edges in $\Box^n$. Take the standard geometric realization ϕ of $\Box^n$ which satisfies $\phi(u) = (0, \ldots, 0)$. Let $\mathrm{Set}(v)$ denote again the subset of $\{1, 2, \ldots, n\}$ with characteristic vector $\phi(v)$. Then $v <_\pi v'$ is equivalent to $\mathrm{Set}(v) \subset \mathrm{Set}(v')$, and $\{v_1, \ldots, v_k\}$ is a face of $\triangle_\pi(\Box)$ iff

$$\mathrm{Set}(v_1) \supset \cdots \supset \mathrm{Set}(v_k)$$

holds. As it is shown in the proof of Lemma 29, the same relation is equivalent to $\{v_1, \ldots, v_k\} \in \triangle_<(\Box)$. **QED**

Recall that a d-dimensional pure simplicial complex $\triangle$ is *completely balanced* if the vertex set of $\triangle$ may be colored with $d+1$ colors such that no two vertices of the same color belong to a common face.

Lemma 44 *Let $\Box$ be a d-dimensional shellable edge-orientable complex and π an edge-orientation of $\Box$. Then $\triangle_\pi(\Box)$ is a completely balanced simplicial complex.*

Proof: As shown in Lemma 41, there is a labeling of the vertices of $\Box$ θ satisfying (3.11). Color the vertex v with the modulo $(d+1)$ equivalence class of $\theta(v)$. We claim that $\triangle(\Box, \pi)$ becomes a completely balanced complex, with this coloring. In fact, let us take a a face $\{v_1, v_2, \ldots, v_k\} \in \triangle(\Box, \pi)$. By the definition of the triangulation, there is a face $\tau \in \Box$ containing $\{v_1, v_2, \ldots, v_k\}$, and w.l.o.g. we may assume that we have $v_1 <_\pi v_2 <_\pi \cdots <_\pi v_k$ It is an easy consequence of (3.11) that then we have

$$\theta(v_1) < \theta(v_2) < \cdots < \theta(v_k).$$

The values of θ on τ are $\dim \tau + 1 \leq d + 1$ consecutive integers, hence no two of the above $\theta(v_i)$-s can be congruent modulo $(d+1)$, and so $\triangle(\Box, \pi)$ is a balanced complex. **QED**

3.6 The Eisenbud-Green-Harris conjecture

Using the Stanley ring of the boundary complex $\square$ of an edge-orientable convex cubical polytope we may construct an interesting example to a conjecture of D. Eisenbud, M. Green and J. Harris. Before stating the conjecture, let us recall the definition of the *h-vector of a graded algebra.* It is a well known fact that the Hilbert-series of a Noetherian $\mathbb{N}$-graded algebra A may be written in the following form.

$$\mathcal{H}(A,t) = \frac{\sum_{i=0}^{l} h_i \cdot t^i}{\prod_{i=1}^{s}(1-t^{e_i})}, \tag{3.12}$$

where $d = \sum_{i=1}^{s} e_i$ is the Krull-dimension of A, i.e. the maximum length of an increasing chain of prime ideals. (See, e.g. [28].)

Definition 54 *We call the vector $(h_0, \ldots, h_l)$ in (3.12) the h-*vector *of the graded Noetherian algebra A.*

In particular, for a simplicial complex $\triangle$ or a cubical complex $\square$ we define the h-vector of the simplicial or cubical complex to be the h-vector of their Stanley rings.

Now we may formulate the Eisenbud-Green-Harris conjecture as follows. (See [9, Conjecture ($\mathrm{V_m}$)].)

Conjecture 2 *Let I be an ideal of a polynomial ring of the polynomial ring $K[x_1, \ldots, x_r]$ which contains a regular sequence of length r in degree 2. Then the h vector of the graded algebra $K[x_1, \ldots, x_r]/I$ is the f-vector of some simplicial complex.*

Example Let $\square$ be the boundary complex of a $(d+1)$-dimensional convex cubical polytope, and assume that $\square$ is edge-orientable with an edge-orientation π. Assume furthermore that K is an infinite field. Then the Stanley ring $K(\square)$ is a d-dimensional Cohen-Macaulay ring, and it contains a linear system of parameters $l_1, \ldots, l_d$. We claim that the polynomial ring $K[x_v : v \in V] \big/ (l_1, \ldots, l_d)$ and the natural image of the face ideal $I(\square)$ in this ring provides an example for Conjecture 2.

In fact, by Theorem 15 the face ideal $I(\square)$ is generated by homogeneous elements of degree 2, and so the same holds for the image $\overline{I(\square)}$ of $I(\square)$ in $K[x_v : v \in V] \big/ (l_1, \ldots, l_d)$. Thus $\overline{I(\square)}$ contains a maximal regular system of parameters in degree 2. The factor of $K[x_v : v \in V] \big/ (l_1, \ldots, l_d)$ by $\overline{I(\square)}$ is Artinian, isomorphic to $K[\square] \big/ (l_1, \ldots, l_d)$ and its h vector is the h-vector of the cubical complex $\square$. By Lemma 29 this h-vector is the also the h-vector of any triangulation via pulling the vertices $\triangle_<(\square)$ of $\square$. By Lemma 43 whenever we take a linear extension $<$ of the partial order $<_\pi$, the simplicial complex $\triangle_<(\square)$ is equal to the simplicial complex $\triangle_\pi(\square)$. Thus we are left to show that the h-vector of $\triangle_\pi(\square)$ is the f-vector of some other simplicial complex. But by Lemma 44 the simplicial complex $\triangle_\pi(\square)$ is completely balanced and being a triangulation of a sphere, it is a Cohen-Macaulay simplicial complex by [26, Corollary 4.4]. Therefore its h-vector is the f-vector of another simplicial complex by [25, 4.5 Corollary].

Bibliography

[1] R. ADIN, private communication.

[2] G. BAXTER, A Necessary Condition for Four-Coloring a Planar Graph, *J. Combin. Theory* **1** (1966), 375-384.

[3] P. ALEXANDROFF AND H. HOPF, "Topologie," Berlin, 1935.

[4] M. BRUGGESER AND P. MANI, Shellable decompositions of cells and spheres, *Math. Scand.* **29**, 197-205.

[5] W. BRUNS AND U. VETTER, "Determinantal Rings," *Springer Lecture Notes In Mathematics* **No. 1327**, Springer Verlag, 1988.

[6] C. CHAN, Plane trees and h-vectors of shellable cubical complexes, *SIAM J. Disc. Math.* **Vol. 4** (1991), No. 4, 568-574.

[7] DANIEL I. A. COHEN, On the Sperner lemma, *J. Combin. Theory* **2** (1967), 585-587.

[8] A. DOLD, "Lectures on Algebraic Topology," Springer Verlag, New York, 1980.

[9] D. EISENBUD, M. GREEN, J. HARRIS, Some Conjectures Extending Castelnuovo Theory, preprint.

[10] K. FAN, A Generalization of Tucker's Combinatorial Lemma with Topological Applications, *Annals of Math.* **Vol. 56** No. 3 (1952), 431-437.

[11] K. FAN, Combinatorial Properties of Certain Simplicial and Cubical Vertex Maps, *Arch. Math.* **Vol. XI** (1960), 368-377.

[12] K. FAN, Simplicial Maps from an Orientable n-Pseudomanifold into $\mathbb{S}^m$ with the Octahedral Triangulation, *J. Combin. Theory* **2** (1967), 588-602.

[13] B. GRÜNBAUM, "Convex Polytopes," Interscience Publishers, London, New York, Sydney, 1967.

[14] M. HOCHSTER, Rings of invariants of tori, Cohen-Macaulay rings generated by monomials and polytopes, *Annals of Math.* **96** (1972), 318-337.

[15] J. E. HUMPHREYS, "Reflection groups and Coxeter groups," Cambridge University Press, 1990.

[16] M. KASHIWARA AND P. SHAPIRA, "Sheaves on Manifolds," Springer-Verlag Berlin, New York 1990.

[17] S. LEFSCHETZ, "Topology," Am. Math. Soc. Colloq. Publ. No. 12 (1930). (Second edition, Chelsea, New York 1956.)

[18] W. S. MASSEY, "Singular Homology Theory," Springer Verlag, New York, 1980.

[19] H. MATSUMURA, "Commutative ring theory," Cambridge University Press, Cambridge, 1992.

[20] N. METROPOLIS AND G-C. ROTA, Combinatorial structure of the faces of the n-cube, *SIAM J. Appl. Math.* **Vol 35** No. 4, 689-694.

[21] J. R. MUNKRES, "Elements of Algebraic Topology," Addison-Wesley Publishing Co. Inc., 1984.

[22] J. RIORDAN, "Combinatorial identities," John Wiley & Sons, Inc., New York-London, Sidney, 1968.

[23] E. H. SPANIER, "Algebraic topology," McGraw Hill, New York, San Francisco, St. Louis, Toronto, London, Sydney, 1966.

[24] J-P. SERRE, "Homologie singulière des espaces fibrés", Thesis, Paris, 1951, and *Ann. of Math.* **54** (1951), 425-505.

[25] R. P. STANLEY, Balanced Cohen-Macaulay complexes, *Transactions of the AMS* **Vol. 249** (1979), No. 1, 139-157.

[26] R. P. STANLEY, "Combinatorics and Commutative Algebra," Birkhäuser, 1983.

[27] R. P. STANLEY, Decompositions of rational convex polytopes, *Annals of Discrete Mathematics* **6** (1980), 333-342.

[28] R. P. STANLEY, Hilbert Functions of Graded Algebras, *Advances in Mathematics* **28** (1978), 57-83.

[29] B. STURMFELS, Gröbner bases of toric varieties, *Tôhoku Math J.* **43** (1991), 249-261.

[30] B. TAYLOR, private communication.