

**The Value of Flexibility**

**by**

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## Abstract

This paper develops a framework to evaluate the economic value derived from a firm's ability to switch between different modes of production in the face of uncertain prices. The model, cast as a set of simultaneous stochastic dynamic programs, is solved for the ex-ante value of flexibility, the optimal technology choice, and critical prices at which switching is optimal.

This general model of flexibility is used to synthesize several recent studies of real options encountered in capital budgeting. For example, the model yields as special cases (a) the value of waiting to invest, (b) the option to abandon, (c) the value of having an option to shut down, (d) the replacement timing and technology choice, and (e) the "time to build" option for irreversible projects that require sequential outlays.

We use an illustrative example with two modes to show that the value of flexibility is monotonically increasing with price variability and switching frequency. The value of flexibility can contribute about a 15 percent improvement over the better fixed technology. Early in the life of the project it is optimal to switch modes when the difference between values under one mode (for the current period and optimal switching thereafter) and the other mode exceeds the switching cost. Towards the end of the economic life, the above difference must be significantly larger for switching to occur.



## INTRODUCTION

In this paper we model investment behavior when firms face stochastic relative prices and are allowed to switch between production modes. We consider firms which can operate with one of several technology modes, where each technology will be preferred over the others under some states of the world. If the firm is already producing with one mode then a change in conditions may make it optimal to switch to a different mode and incur switching costs. We derive the value from this flexibility and the critical values of the state variables at which it is optimal to switch between modes.

A very similar problem also arises when considering investments in new projects where some have irreversibly fixed technologies and other, more expensive ones, are flexible systems. The flexible systems allow for changes in production modes without large switching costs. This model derives the incremental value due to flexibility which when compared with the incremental cost of the flexible system will determine the choice of technology.

We cast the general problem as that of solving a set of simultaneous markov decision problems and derive the ex-ante value of flexibility, the optimal technology choice, and the critical values of the states of the world at which switching modes is optimal.

This approach unifies several real options arising in capital budgeting as special cases of flexibility.<sup>1</sup> When one mode of the flexible system is the no production mode our problem simplifies to that of valuing projects in the presence of an option to shut down (McDonald and Siegel [1985]). When flexibility is limited to a single switch our problem yields the optimal investment timing problem (McDonald and Siegel [1983]) and the optimal abandonment problem (Myers and Majd [1984]) as special cases. When switching to a previously employed mode is excluded our problem yields the "time to build" option for irreversible projects that require sequential investment outlays

(Majd and Pindyck [1985]). Although all of these real options must be included in most investment decisions, previous papers only looked at one of them at a time, thus, precluding interactions between the various options. Our general model of flexibility allows for the simultaneous treatment of all real options in the capital budgeting process.

The model is best elucidated through an example. Consider an electric power generation plant which can be fired with coal or oil.<sup>2</sup> Suppose relative prices at the time of the initial investment are such that the expected profit is greatest when operated with oil. This decision can, however, be reversed if conditions change. For instance, if the relative price of oil increases sufficiently then it may be better to switch to coal and incur retooling costs. Our model solves for the relative price at which it is optimal to switch.

When switching between technologies is costly, making the current choice requires a value maximizing firm to look ahead at all future price contingencies and simultaneously solve for the entire path of decisions. This also implies that optimal choice depends on the technology which was in place during the previous period. In other words, a switching decision will affect not only the cash flows from the immediately following period but also affect the switching decisions and cash flows during all future periods. However, the entire path is summarized by the mode of use in the previous period, thus, yielding a markov decision process.

Now consider an electric utility which is planning to build a new power plant. One of the choices is to use a fixed technology which is specially designed to operate under one type of fuel. Suppose that based on current prices and forecasts the best fixed technology is a coal fired one. In the menu of available choices to the utility is also a flexible technology where the fuel type can be switched easily and at relatively little cost. Such a plant no doubt would incur a higher initial investment than a comparable plant with a



fixed technology. However, given the uncertainty in energy prices the value derived from flexibility to switch fuels may offset the extra investment. The difference between values of the flexible and the best of the fixed technologies gives the value of flexibility. A simple comparison of the incremental investment with the value of flexibility yields the technology choice.

One can imagine situations in which switching is based on input prices, output prices or both. Other examples where relative input prices determine the appropriate production process are numerous: Tire manufacturers will shift the production technology based on the relative price of natural and synthetic rubber; Automobile makers will use different metal alloys or plastics in certain components based on relative prices; In many modern manufacturing applications it is possible to switch rapidly between production processes based on relative price of energy inputs. In areas where electricity is priced on a spot market, switching within a production shift may be feasible.

In many other production situations output price and quantity demand conditions determine the technology choice. The prototypical example is the choice between a job shop (with human machinists) having low setup costs and a line process such as a "mold and stamp" machine which has high setup costs. The conventional wisdom among production planners is that for applications with high unit costs, low volumes, and with frequent design changes the extremely flexible job shop may be more desirable. The value of flexibility will be high for products such as automobiles where model changes occur frequently and, thus, justify investments in flexible manufacturing systems. However, for products with low unit costs, high values, and infrequent design changes a line process may be more desirable. Our approach formalizes this intuition and provides a quantifiable framework that is useful in capital budgeting.

Another example of where relative output prices determine the optimal production process can be found in the petroleum refining industry. The value

of a broad range refinery will be higher when the relative prices of the refined products is more uncertain. Hence during periods of high output price volatility a broad range refinery will be profitable, while in periods of relatively stable prices a refinery with a narrower range of products will be preferred.

We model relative prices with a known stochastic process. In the numerical simulations we use a mean reverting stochastic process. Since flexibility is derived from an ability to switch between substitutes it is reasonable to expect that market forces will drive the relative price to fluctuate randomly but revert towards some mean value.<sup>3</sup>

We assume that firms are price takers in both input and output markets characterize technologies by profit functions.<sup>4</sup> In the power generation example, a change in the type of fuel will be reflected by a change in the profit function. The technology choice decisions are made between small discrete intervals. At these decision points the firm contracts prices for the duration of a single period but all future prices follow the known stochastic process. One interpretation of this stylization is that contracting periods are given exogenously to the firm and, therefore, mode choice decisions need only be made at the beginning of such contracting periods. Alternatively, we can think of the this as a discretization of the continuous endogenous contract-decision points.

We study the sensitivity of the value of flexibility and the critical relative prices at which switching must take place on the various model parameters. Our results show that the value of flexibility increases with increasing price volatility, increasing price elasticity, and decreasing switching costs. The value also increases when the switching interval is shortened. We approximate the continuous switching case by numerically studying the convergence limit of the value of flexibility for very small switching

intervals.

Our results bear a close relationship to the option pricing literature. For example in the case with no switching costs, the expected value of maximum profit for a future period is analogous to the payoff from a call option with a stochastic exercise price. Hence the value of the firm can be obtained as the sum of a series of such options. Although there are known closed form solutions for some stochastic processes such solutions are not available in general. In another special case when switching is costly and when new process must be installed at every switching decision point our problem resembles a compound option.

The rest of the paper is organized as follows; In the next section we outline the basic structure of the model by describing the production processes and price dynamics. In section 2, we derive the value of projects under fixed technologies and use it as a frame of reference to compare the value under a flexible technology. In section 3, we derive the value of a flexible project. In section 4, we compute the value of flexibility as the difference between the flexible and fixed projects, discuss implications for the capital budgeting decision, and derive previously studied real options as special cases of flexibility. Section 5 reports results from a numerical example and investigates comparative static relationships. Finally, in section 6 we make some concluding remarks.

### 1. The Profit Function and Price Dynamics

Consider a price taking firm which faces one stochastic price while all other prices are deterministic.<sup>5</sup> We model the stochastic dynamics of the relative price  $P_t$  by the mean reverting continuous time stochastic process

$$(1) \quad dP_t = \lambda(\bar{P} - P_t) dt + \sigma_P dZ_P, \quad \lambda > 0$$

where  $dZ_p$  is a standard Gauss Weiner process.<sup>6</sup>

In this analysis, we assume that  $P_t$  does not contain any systematic risk, and thus the equilibrium rate of return for an asset with similar risk characteristics to  $P_t$  will be the risk free interest rate  $r$ .<sup>7</sup> An application of Ito's lemma reveals that any differentiable function of  $P_t$  will also contain no systematic risk.

We characterize a project by its instantaneous dollar (flow) profit function  $G^f(P_t)$ .<sup>8</sup> At time  $t$  the firm observes the realization of  $P_t$  and fixes it contractually for a short period  $(t, t+\tau)$ .<sup>9</sup> We find the optimal switching strategies for a given value of  $\tau$  and then vary  $\tau$  to study the comparative statics. When contracting arrangements are given exogenously we can study the effects changing contract duration on the value of flexibility. When contract length is within the control of the firm, we can approximate the limiting continuous time case by very small values of  $\tau$ .

The flow of profits during this period is constant and its present value (at time  $t$ ) gives the profit function  $G(P_t, \tau)$ :

$$(2) \quad G(P_t, \tau) = \int_t^{t+\tau} G^f(P_t) e^{-r(s-t)} ds$$

$$= G^f(P_t) \Theta(\tau)$$

where  $\Theta(\tau) = (1 - e^{-r\tau})/r$ , and  $r$  is the risk free interest rate. Varying functional forms and parameters can represent the  $M$  alternative production processes  $G^j$ :  $j \in (1, \dots, M)$ . In the power plant example, the generation process corresponding to each energy type can be represented by its accompanying profit function.

## 2. Value without Flexibility: $V_0^R$

In order to form a frame of reference to compare flexible technologies and thereby obtain the value of flexibility, we first derive the value of the project under a fixed (rigid) technology. Let the economic life be  $T$  thus having  $N (= T/\tau)$  decision periods. The present value of the project,  $V_0^R$ , is the discounted sum of present value profit functions:

$$(3) \quad V^R(0) = E_0 \sum_{i=1}^N G^f(P_{i\tau}) \theta(\tau) e^{-rit}$$

Since closed form solutions to the expected value are not available in general, we can use a backward recursion formula to assure a computationally feasible method to evaluate  $V_0^R$ .

Since prices are contracted at beginning period values, the value function at the beginning of the last period (at time  $T = (N-1)\tau$ ) is simply the profit during that period:

$$(4) \quad V^R(T) = G(P_T).$$

The value function at the start of the previous period will be the sum of profits from that period and the discounted value  $V^R(T)$ :

$$(5) \quad V^R(T-\tau) = G(P_{T-\tau}) + \rho E_T[V^R(T)],$$

where  $\rho (= e^{-r\tau})$  is the discount factor. Continuing this backward recursion gives the value function at time 0 as

$$(6) \quad V^R(0) = G(P_0) + \rho E_0[V^R(\tau)].$$

Expectations are computed numerically by discretizing the transition probability matrix of the price process and forming a probability weighted sum. Details are given in an appendix.

### 3. Value with Flexibility: $V^F$

Under a flexible production system the firm must evaluate the stream of future profits to determine the optimal technology mode for the coming period. The choice is reconsidered upon the arrival of new price information at the beginning of the next period. If production decisions call for a change in process type then switching can be accomplished instantly incurring a cost  $\delta$ .<sup>10</sup>

Consider the last period of operation beginning at time  $T-\tau$ . The value of the project for the remaining life ( $\tau$ ) will depend on the price  $P_{T-\tau}$  and the mode that was used during the previous period. The latter dependence stems from the presence of switching costs. The mode in use during the immediately preceding period sufficiently summarizes the entire path of modes for purposes of the current decision rule. Therefore, the value function at time  $T-\tau$  will consist of the arguments time, the relative price realization,  $P_{T-\tau}$ , and the state,  $S_{T-\tau}$ , which represents the mode employed in the previous period.

If mode  $j$  was used in the previous period (i.e.  $S_{T-\tau} = j$ ) and price  $P_{T-\tau}$  was observed then the value function at  $T-\tau$  can be written as

$$(6) \quad V^F(T-\tau, P_{T-\tau}, j) = \sup \left[ G^1(P_{T-\tau}) - \delta, \dots, G^j(P_{T-\tau}), \dots, G^M(P_{T-\tau}) - \delta \right],$$

where  $G^j$  is the profit function for the  $j^{\text{th}}$  mode.<sup>11</sup>

Now consider the decisions at time  $T-2\tau$ . If  $S_{T-2\tau} = j$ , then the firm should choose to switch processes only if the present value (at time  $T-2\tau$ ) of all future cash flows when using another mode during the coming period (and optimal future decisions) is greater than the corresponding cash flows when continuing to use mode  $j$ .

$$(7) \quad V^F(T-2\tau, P_{T-2\tau}, j) = \sup \left[ \begin{array}{l} G^1(P_{T-2\tau})^{-\delta+\rho} E_{T-2\tau} [V^F(T-\tau, P_{T-\tau}, 1)] \dots\dots\dots \\ G^j(P_{T-2\tau})^{+\rho} E_{T-2\tau} [V^F(T-\tau, P_{T-\tau}, j)] \dots\dots\dots \\ G^M(P_{T-2\tau})^{-\delta+\rho} E_{T-2\tau} [V^F(T-\tau, P_{T-\tau}, M)] \end{array} \right].$$

There will be  $M$  such equations, for  $j = 1, \dots, M$ , at point in time. The backward recursion is continued until time 0, where the firm observes and contracts prices  $P_0$ . The simultaneous system of stochastic dynamic programs can be solved for the value of the flexible system and the optimal switching strategies.

We drop the time arguments and list the general form of the dynamic programming equations for future reference:

$$\begin{aligned} V^F(1) &= \sup [G^1 + \rho E[V^F(1)], \dots, G^j - \delta + \rho E[V^F(j)], \dots, G^M - \delta + \rho E[V^F(M)]] \\ &\cdot \\ (8) \quad V^F(j) &= \sup [G^1 - \delta + \rho E[V^F(1)], \dots, G^j + \rho E[V^F(j)], \dots, G^M - \delta + \rho E[V^F(M)]] \\ &\cdot \\ V^F(M) &= \sup [G^1 - \delta + \rho E[V^F(1)], \dots, G^j - \delta + \rho E[V^F(j)], \dots, G^M + \rho E[V^F(M)]] \end{aligned}$$

#### 4. Applications

In this section we apply the general model of flexibility to several previously studied applications of real options in capital budgeting. The

values of the option to wait to invest, to abandon, and to shut down are derived as special cases when there are only two alternative modes of operation of which one mode becomes a null technology. When investments are incurred in stages, the resulting time to build option also is shown to be a special case.

#### 4.1 The Value of Flexibility

When making the initial investment at  $t=0$ ,  $V^F$  does not depend on the state  $S_0$ . The present value of the flexible project can be expressed as

$$(9) \quad V^F(0, P_0) = \sup \left[ \begin{array}{l} G^1(P_0) + \rho E_0[V^F(\tau, P, 1)], \dots\dots\dots \\ \dots\dots, G^j(P_0) + \rho E_0[V^F(\tau, P, j)], \dots\dots\dots \\ \dots\dots, G^M(P_0) + \rho E_0[V^F(\tau, P, M)] \end{array} \right].$$

The value of flexibility is the difference between  $V^F(0, P_0)$  and the maximum of the  $M$  fixed projects:

$$(10) \quad V(0) = V^F(0, P_0) - \text{Max} [V_1^R, \dots, V_j^R, \dots, V_M^R].$$

If  $\Delta_i$ ,  $i=1, \dots, M$ , are the initial investment for fixed processes and  $\Delta_F$  is the investment for the flexible process the capital budgeting decision rule is to compare  $V(0)$  with the incremental investment requirement for the flexible system. In other words, invest in the flexible system only if

$$V^F - \Delta_F > \text{Max} [ [V_1^R - \Delta_1], \dots, [V_M^R - \Delta_M] ].$$

In cases where there is no truly flexible technology, the above solutions are also useful in forming mode choice when faced with multiple technologies: For example, when faced with two technologies A and B the decision rule is



Invest in mode A if  $V^F(0, P_0) - \Delta_A > V^F(0, P_0) - \Delta_B$  and in mode B if the inequality is reversed.

One further caveat regarding the switching cost should be mentioned. For convenience, we have assumed switching costs to be a constant. In the above case, the switching cost associated with the first time a processor is brought on line will include its purchase and installation costs. Thereafter the switching cost will only involve retooling and reorganization costs. We have also assumed switching costs to be a constant amount, whether the switch is from A to B or from B to A. In practice this need not be so. For instance in the power generation example, switching from coal to oil may be more costly than switching from oil to coal. This can be easily accommodated in the dynamic programming algorithm by replacing  $\delta$  by  $\delta_{AB}$  and  $\delta_{BA}$ , the costs that take into account the direction of the switch.

#### 4.2 Replacement Decisions

If the project involves replacing one machine with another, then the initial mode  $S_0$  will be the one corresponding to the existing processor. If  $S_0 = A$ , then the value of the project is given by

$$(11a) \quad V^F(0, P_0, A) = \sup \left[ \begin{array}{l} G^A(P_0) + \rho E_0[V^F(\tau, P, A)] , \\ G^B(P_0) - \delta_B + \rho E_0[V^F(\tau, P, B)] \end{array} \right],$$

where  $\delta_B$  is the cost of mode B.

Furthermore, a simple comparison of the arguments of the sup [...] determines the choice of the technology. If  $G^A(P_0) + \rho E_0[V^F(\tau, P, A)] > G^B(P_0)$

$-\delta + \rho E_0[V^F(\tau, P, B)]$ , then continue with mode A; otherwise replace it with mode B.

Similarly, if  $S_0 = B$ , then the value of the project is given by

$$(11b) \quad V^F(0, P_0, B) = \sup \left[ \begin{array}{l} G^A(P_0) - \delta_A + \rho E_0[V^F(\tau, P, A)] , \\ G^B(P_0) + \rho E_0[V^F(\tau, P, B)] \end{array} \right].$$

and mode B should be replaced only if  $G^A(P_0) - \delta_A + \rho E_0[V^F(\tau, P, A)] > G^A(P_0) + \rho E_0[V^F(\tau, P, B)]$ .

#### 4.3 Valuation when there is an option to shut down

A type of flexibility that is always available to manufacturers is the option to shut down. When the value of a project is stochastic, it might be optimal to operate a currently unprofitable facility in order to save shut down and startup costs. In a recent paper, McDonald and Siegel [1983] studied this problem and derived the value of having such an option to shut down. Their analysis, based on an infinitely lived project whose value followed geometric brownian motion, derived a closed form solution to the above value. It is easy to see that the presence of the option to shut down is merely one form of flexibility, and the value thus derived can be obtained as a special case of our formulation.

If the second mode of our flexible manufacturing system is the no production mode then replacing  $G^B$  with 0 will yield the value of the project under a shut down option. Shut down and startup costs are represented by the switching costs  $\delta_{AB}$  and  $\delta_{BA}$ . The algorithm presented in this paper can be used to evaluate the value of the shut down option under a variety of price processes and switching cost scenarios. It is not limited to infinite lived projects.

#### 4.4 The Value of Waiting to Invest

Our model can be reinterpreted to study the value of waiting to invest, which was investigated by McDonald and Siegel [1983]. When the price of investments and the present value from the project are stochastic, the optimal investment decision may not be to invest as soon as the net present value is positive. By waiting until the investment price and the project value reach some critical levels, the firm can derive a higher NPV. McDonald and Siegel solved for this value in a continuous time model where the project value follows a log normal process with a drift and where projects are infinitely lived.

With minor reinterpretations we can address above problem within our framework. Let the initial mode of operation be the null mode (i.e.  $S_0 = 0$ ).<sup>12</sup> The switching cost from the null to the production mode (mode 1) will equal the initial investment cost (i.e.  $\delta_{01} = I$ ). If shut down is not allowed the switching cost from the production to the null mode should be set at a very large value (i.e.  $\delta_{10} = \infty$ ).<sup>13</sup> The value of the project under the above parameter values will include the value due to the optimal timing of the investment. Hence, the difference between the value of a fixed technology and that of above flexible technology will be the value of waiting to invest.

#### 4.5 Value of the Option to Abandon

A very similar option to that of waiting to invest is available at the end of a project. When the salvage value and the value of the project over its remaining life are stochastic, the optimal time to abandon a project can be solved in a manner similar to that above. This problem was solved by Myers and Majd [1984] when the values were assumed to follow geometric brownian motion and

when project life was finite. Since closed form solutions exist only for the infinite time case, they used numerical methods to solve for the value of the abandonment option.

The abandonment option can be incorporated in the value of the project by considering the following parameter values in our general model of flexibility; let the initial state be the production mode (mode 1), the alternative mode be the null mode (mode 0),  $\delta_{10}$  = - salvage value, and  $\delta_{01} = \infty$ .

### The "Time To Build" Option

In a recent paper Majd and Pindyck [1985] model the value of projects when (a) spending decisions and cash outlays occur sequentially over time, (b) there is a maximum rate at which outlays and construction can proceed and (c) the project yields no cash flows until it is completed. The pattern of investment outlays can be flexible but will be subject to a maximum rate constraint.

In order to study this problem we can specialize our general model of flexibility in the following ways; Let the maximum rate at which investment outlays can be made be  $\delta$  per  $\tau$  units of time and  $G^i(\cdot)$  be the profit function of the project after  $i$  such investments of capital has been incurred. The total investment required is  $M\delta$  (i.e.  $M$  investments of  $\delta$ ). In the case where no cash flows are derived until all investments are made  $G^i = 0$  for all  $i < M$  and  $G^M$  will be the profit function of the project. Once an investment has been committed it is irreversible. In our model this implies that the firm can not revert to a mode of operation which was used during a previous period. These conditions specialize the general system of dynamic programs in equation (8) as follows;

$$V(1) = \sup \left[ G^1 + E[V(1)], G^2 - \delta + E[V(2)], \dots, G^M - \delta + E[V(M)] \right]$$

$$\begin{aligned}
 & \vdots & & \vdots \\
 (12) \quad V(j) &= \sup \left[ G^j + E[V(j)], G^{j+1} - \delta + E[V(j+1)], \dots, G^M - \delta + E[V(M)] \right] \\
 & \vdots & & \vdots \\
 V(M) &= \sup \left[ G^M - \delta + E[V(M)] \right].
 \end{aligned}$$

Since the profit functions are ranked according to the level of outlays, eliminating the possibility of switching to a previously used state is achieved by including the value functions corresponding only to higher levels of investment in the sup[.]. In the first period when no investment has been made  $j = 1$ , and thus we have the equation for the full flexibility case. After all investment has been incurred  $j = M$ , and the project becomes a fixed technology.

### 5. An Example and Comparative Statics

Since our model does not yield closed form solutions for the value of flexibility, we use a simple example to illustrate the solution technique and numerically study comparative statics.

We use the following mean reverting stochastic process to model the relative price path;<sup>14</sup>

$$dP_t = \lambda(\bar{P} - P_t) dt + \sigma_p dZ_p.$$

The instantaneous drift term  $(\bar{P} - P_t)$  acts as an elastic force which produces mean reversion. For example, if  $\bar{P} = 1$ , when  $P_t > 1$  project A yields higher profits, but when  $P_t < 1$  the dominance is reversed. In the power generation example we can treat the relative price of oil with respect to coal as  $P_t$ . Prior to OPEC price increases  $P_t < 1$ , thus making oil burning more cost

efficient. Since OPEC, the relative price of coal became cheaper, thus making coal the preferred fuel. The recent dramatic reduction in oil prices have reversed this relationship. In general, although less dramatic, it is likely that market forces will move to make the relative price of substitutes revert toward the mean value of  $\bar{P}$ .

The stochastic term  $dZ_p$ , with variance  $\sigma_p$ , causes continuous fluctuations about  $P_t$ .  $\sigma_p$  measures the volatility of relative prices. We think that such processes will more accurately depict price paths of production inputs and outputs.

As the base case we chose parameter values  $\lambda = 0.1$ ,  $\sigma_p = 0.20$ ,  $\bar{P} = 1$ . For purposes of numerical computations values of  $P_t$  are restricted to the range  $[0.5, 1.5]$ . This range is divided into 100 discrete price levels.<sup>15</sup> Details of constructing the discrete probability transition matrix are given in Appendix 1.

The firm can operate with one of two technology modes:

$$G^A = a_1 + a_2 P_t^\alpha$$

$$G^B = b_1 + b_2 P_t^\beta$$

The parameter values were chosen such that  $G^A = G^B$  when  $P_t = \bar{P}$  (=1) and monotonicity and convexity conditions are satisfied. The base case values were

$$a_1 = 0 \quad a_2 = 5 \quad \alpha = 1.5$$

$$b_1 = 4.5 \quad b_2 = 0.5 \quad \beta = 1.1$$

$a_1$  and  $b_1$  capture the effects due to fixed costs while  $a_2$ ,  $b_2$ ,  $\alpha$  and  $\beta$  reflect the price elasticities. The fixed mode B is relatively insensitive to price while that of A is very sensitive. Mode A can be thought of as having higher fixed costs than B:  $a_1 < b_1$ .

We set the switching cost  $\delta = 0.05$  which is about 1 percent of the mean annual profit. The project has an economic life of 30 years which we divide into 300 steps, thus, under our contractual arrangement firms contract for a period ( $\tau=1$ ) that is little over a month.<sup>16</sup> Switching decisions are made at

the beginning of the contract period. In the comparative static analysis we numerically study the impact of changes in contract length ( $\tau$ ), switching cost ( $\delta$ ), volatility of the relative price ( $\sigma_p$ ), mean reversion parameter ( $\lambda$ ), and relative price elasticities ( $\alpha$ ) on the value of flexibility and the critical prices for switching modes.

Since we assume zero systematic risk in prices, we discount all cash flows at the risk free interest rate.<sup>17</sup> An instantaneous risk free rate of 2 percent is used in the base case.

In figure 1 we plot values of the fixed technologies and the flexible technologies under each of the two starting states against initial price realizations. For low values of  $P_0$ , fixed mode B is preferred over fixed mode A. For higher values of  $P_t$  the preference ordering is reversed. Although  $G_A(1) = G_B(1)$  the price at which the firm is indifferent between A and B (i.e.  $V_A = V_B$ ) is not at  $P_0 = 1$ . An application of the Jensen's inequality shows that the expected values of  $V_A$  and  $V_B$  do not have to equal since the profit functions have different levels of convexity. The indifference point,  $P_I$ , is approximately 0.93. The  $V_A^F$  and  $V_B^F$  plots are very close together indicating that values of the flexible technology are not affected much by the starting mode.

The value of flexibility, however, is significantly affected by the price realization,  $P_0$ . This is shown in Figure 2, where the Percentage Increase in Value due to Flexibility (PIVF),  $[(V^F/\text{Max}(V_A, V_B)) - 1] \times 100$ , is plotted against the initial price. PIVF is maximized when the two modes of production are similar (at  $P_0 = P_I$ ). Flexibility produces about a 15 percent improvement over the better of the two fixed modes.

The reason for this peaking of  $V^F$  can be explained as follows; For a moment ignore switching costs. Suppose  $P_0 < P_I$  and the firm uses the preferred mode B. For a change in mode to be warranted the discrete price change (over a

discrete time interval) must be greater than  $(P_I - P_0)$ . But the probability of a price change is inversely related to its magnitude and, hence, the probability of a switch in mode. This situation is symmetrically opposite for  $P_0 > P_I$  in that the firm will operate with mode A and require a price change greater than  $P_0 - P_I$  in order to switch to B. Therefore, the further away  $P_0$  is from  $P_I$  the smaller is the probability of switching modes. With switching costs the threshold price change for switching modes becomes even larger.

An obvious analogy can be drawn to the options pricing literature. When  $P_0 < P_I$  the firm will employ mode B and hold a call option to switch to mode A. This option is out-of-the-money until  $P_0$  becomes larger than  $P_I$ . However, for  $P_0 > P_I$  the firm will employ mode A and hold an out-of-the-money option to switch to B. Hence the only value of  $P_0$  at which the option is in-the-money is when  $P_0 = P_I$ . The value of this option (the value of flexibility) is maximized at  $P_0 = P_I$ . As  $P_0$  moves away from  $P_I$  the option goes deeper out-of-the-money.

In Figure 3, we study the optimal switching points over the life of the project. The critical price at which it is optimal to switch from mode A to mode B is plotted against time in the top line (SW1). The critical switching point from B to A is the lower line (SW2). In a single period profit maximizing world, the critical switching point will be when the difference between  $V_A$  and  $V_B$  is equal to the switching cost. As the plot indicates, towards maturity of the project it is optimal to switch only if the price change yields a larger difference in value than the switching cost. This is because of the possibility of running out of time to switch back in case of a price change which requires a mode reversal. As a consequence when there is little time remaining the switching costs exceed the value of flexibility (derived at the steady state critical prices) and it becomes simply not worth switching. In earlier periods SW1\* and SW2\* are at their steady state values where switching takes place as soon as the difference between  $V_A^F$  and  $V_B^F$  exceeds the switching cost.



We now turn to the comparative static analysis and vary the base case parameter values and investigate the impact on the value of flexibility and the critical switch points. In Figure 4 we plot the values of the fixed and flexible systems against  $\sigma_p$  for  $P_0 = \bar{P}$ . A ceteris paribus increase in  $\sigma_p$  increases the expected value of the convex profit functions. Since we measure profits at  $P_0 = E(P) = \bar{P}$ , an application of the Jensen's inequality shows that  $E[G^i]$  is an increasing function of  $\sigma$  for convex  $G^i$ . For very small values of  $\sigma$ ,  $G^A$  is slightly greater than  $G^B$ . This is probably due to the finite contracting lengths and the approximations introduced by discretizing the price process. In this model flexibility is derived from the firm's increased ability to cope with price uncertainty. Hence, the value under flexible technologies increase with increasing volatility.

Figure 5 shows the responsiveness of PIVF to changes in  $\sigma$ . As expected the value of flexibility increases with increasing volatility. The intuition behind this result is similar to that which explains the increase of call option prices with increasing volatility of the stock price. Like the exercise price in a call option, the presence of flexibility provides down side protection in the ability to switch to the alternate mode. Since profit under the alternative mode itself is affected by relative prices this is similar to an option with a stochastic exercise price. However, due to the state dependency introduced by switching costs our problem is more complex.<sup>18</sup>

We next turn to the impact of changes in switching cost  $\delta$ . Figure 6 shows PIVF plotted against  $\delta$ . As we expect, ceteris paribus increases in  $\delta$  lowers the value of flexibility. The impact of  $\delta$  on critical price is depicted in Figure 7. As the switching cost is reduced the critical prices move towards  $P_I$ .<sup>19</sup>

Values of rigid and flexible systems and PIVF against  $\lambda$  are plotted in figures 8 and 9, respectively. As  $\lambda$  increases the value of fixed technologies increase but those of flexible technologies decrease at a faster rate. Hence,

the value of flexibility decreases. This further illustrates that the value of flexibility is derived from the presence of uncertainty. When  $\lambda = 0$  the prices are purely random. As  $\lambda$  is increased it imposes an increasing tendency to move prices towards the mean,  $\bar{P}$ , and introduces a deterministic component.

The responsiveness of the value of flexibility to increases in the risk free rate is demonstrated in Figures 10 and 11. An increasing discount rate reduces the value of both rigid and flexible technologies. However, the value of flexible technologies decrease at a faster rate than the rigid ones as the seen by the converging curves in Figure 10. Therefore, PIVF is reduced as the risk free rate increases.

Figures 12 and 13 show the values of the rigid and flexible systems and the value of flexibility plotted against  $\alpha$ . Since we hold all other production function characteristics constant, this experiment captures the effect of a wider disparity in the price elasticities between the two modes.<sup>20</sup>  $G^A$  is a monotonically increasing function of  $\alpha$  and the value of the flexible systems also are increasing monotonically with  $\alpha$ . Therefore, within this range of  $\alpha$ , PIVF is monotonically increasing and concave.

The intuition behind this result is that, the degree to which the two modes differ at the indifference price ( $P_I$ ) determines the value added due to flexibility. Since the probability of smaller price changes (during discrete intervals) is greater than those of larger ones, as the disparity between profits under the two modes grows the value derived from each switch will be greater.

Finally, we study the comparative statics with respect to the decision interval  $\tau$ . In an attempt to minimize computational costs we considered a project with a one period life and partitioned this interval into smaller contract periods. There are two effects taking place with changing  $\tau$ : (i) Due to the contract length and (ii) due to the decision interval. If contract

periods are exogenous then these must be equal since it only makes sense to switch when a new price is revealed. Since in our model more frequent decisions do not incur higher switching costs, we expect the value of flexibility to be maximized when continuous decisions are permitted. This is illustrated in Figure 14 where the value of flexibility increases with increasing frequency of decisions (decreasing values of  $\tau$ ). Since the prices are reverting to a constant mean and since the  $P_0$  is chosen to be the mean contracting length will not affect the values. If, however, the prices had an increasing trend then increasing contract lengths will lock into a lower price which will result in reduced values of flexibility. When  $P_0$  is away from the mean this effect can be significant even under mean reverting processes.

As the step size was reduced the values increased but levelled off as  $\tau$  became very small. For very small step sizes the values can be thought of as approximations to the continuous time problem with endogenous contracting. However, for  $\tau$  smaller than  $10^{-3}$  the values started falling. This is due to rounding off errors in the numerical computations.<sup>21</sup>

Figure 16 shows the behavior of critical prices to changing  $\tau$ . As  $\tau$  is made smaller (approaching continuous time) the critical switch points diverge.<sup>22</sup>

Similarly, we can also investigate the effect of ceteris paribus changes in other technology characteristics. This analysis is restricted to price paths containing no systematic risk. With minor modifications our model can incorporate the risk characteristics of the cash flows by adjusting the discount rates used in the present value calculations.<sup>23</sup>

## 6. Concluding Remarks

In this paper we model a firm which faces stochastic prices (for inputs and/or outputs) and has the ability to switch between modes of production in

response to price realizations. We derive the ex ante value of flexibility and the critical prices at which modes must be switched. The reaction of the value of flexibility and the optimal switching between modes to changes in price volatility, switching costs, and frequency of decisions is studied via a numerical example.

This approach has wide applications in capital budgeting problems. When deciding between investments into fixed and flexible manufacturing systems we can compute the value derived from flexibility and compare this with the incremental investment requirement for the flexible system. Once a technology is in place and it is costly to switch, our model gives the critical price changes that would warrant such a switch.

An important contribution of this paper is to synthesize and interpret several recent applications of contingent claim analysis to capital budgeting within a more general framework of flexibility. In particular we show that the options to shut down, to wait to invest, to abandon projects, and that due to "time to invest" become special cases of our model.

Although we use a mean reverting stochastic process in the illustrative application the numerical techniques are quite general and can be applied to a wide range of processes. This analysis was also restricted to price paths containing no systematic risk. This can easily be remedied to include the risk characteristics of the cash flows by adjusting the discount rates used in the present value calculations.

Another seeming limitation of this analysis is that we consider the project in a series of discrete steps. The switching decisions are restricted to those steps and prices are held constant over the intervals. However, by shrinking the decision interval we studied the limiting case which approximates continuous time. Nevertheless, the model with contracting warrants consideration on its own merit. We plan to study this in a future paper.

Although the model seems best suited to study a broad class of industrial projects it can also be applied to several purely financial problems. For example, the flexibility to switch between assets (incurring transaction costs) in managing a portfolio represents a direct application of this model. Exposure management problems, particularly in a multicurrency setting, can also be analyzed within this context.

Furthermore, the extreme generality and flexibility of the model makes it a prime candidate for an expert system aimed at making complex capital budgeting decisions.

Footnotes

- <sup>1</sup> Recent papers by Mason and Merton [1984] and Baldwin, Mason, and Ruback [1983] discuss the use of contingent claims analysis in capital budgeting applications. They note the existence of a value from flexibility that is similar to that discussed here.
- <sup>2</sup> This description applies to truly flexible plants which are designed to switch between fuel types and also to cases where the plant is originally designed to operate on one type of fuel but can later be modified to operate under other fuel types. Both scenarios can be handled within this framework with proper adjustment of the switching cost structure.
- <sup>3</sup> This mean may be a constant or change over time. For example, if technical or market characteristics change, then the mean relative price if the substitutes will change accordingly.
- <sup>4</sup> We can allow for some monopoly power and use cost functions to characterize technologies.
- <sup>5</sup> The stochastic price could be for a factor input or an output. Although conceptually similar, the case of multiple stochastic prices makes the computations substantially more complex.
- <sup>6</sup> When we model the price by geometric Brownian motion we can obtain closed form solutions to some of the cases studied in this paper. Those results are reported in Kulatilaka [1985b].
- <sup>7</sup> i.e.  $dZ_p$  is uncorrelated  $dZ_m$ , the stochastic term of the returns on a well diversified portfolio.
- <sup>8</sup> If  $P_t$ , the stochastic price, is an output price then  $G^f$  is monotonically increasing, concave and homogenous of degree one in  $P_t$ . If  $P_t$  is an input price  $G^f$  is monotonically increasing, concave, and homogenous of degree one in  $P_t$ .
- <sup>9</sup> In many applications firms tend to go into contractual arrangements for purchasing inputs and outputs. This stylization will accurately depict such firms. Quasi-contractual arrangements, sluggish price adjustments and transactions in forward and futures purchases are also common in output goods.
- <sup>10</sup> In a recent paper Majd and Pindyck [1985] have considered the staggered nature of investments. We can relax the instantaneous switching assumption by modeling switching as taking place over several steps where the decision can be reconsidered at each of these sub-decision point.
- <sup>11</sup> When the value was not state dependent and when considering only two alternative modes then our problem simplifies to that in Stulz [1982] where he derives the value of a security whose payoff is the maximum (or minimum) value of two assets. Such is the case when there are no switching costs.
- <sup>12</sup> Although our model allows for other starting modes, we chose a null initial mode to illustrate the McDonald - Siegel case.
- <sup>13</sup> If we allow for shut down then  $\delta_{10}$  = shut down cost.

- 14 We also used the process:  $dP_t = \lambda (\bar{P} - P_t) dt + \sigma_P P_t dZ_P$ , where the instantaneous variance is proportional to current price. Although the numerical values were changed the qualitative results remained invariant.
- 15 We also experimented with finer grid sizes but the value added in obtaining smoother transitions did not, in my opinion, justify the increases in computational costs.
- 16 Smaller time increments are used in later comparative static experiments. Although this value of  $\tau$  does not give a good approximation to the continuous time case the results are qualitatively similar. We can still interpret this case a realistic exogenously determined contract length.
- 17 Alternatively we can treat this as a risk neutral world.
- 18 In a separate paper (Kulatilaka [1986]) we follow the conventional contingent claim pricing literature and model flexibility, in the absence of switching cost and when prices follow geometric Brownian motion, as a call option with stochastic exercise price.
- 19 At  $P_0 = P_I$  the value functions are identical.
- 20 The range of allowable values of  $\alpha$  is constrained so that the regularity conditions are satisfied and that neither technology dominates over the entire range of possible prices.
- 21 As  $\tau$  becomes extremely small the value contribution in one time step also becomes extremely small. Even with double precision calculations on a 32 bit computer the rounding off errors start to become significant at these values.
- 22 The flat regions of the plots are due to the discrete approximations used in the price generation process.
- 23 See Kulatilaka [1986] for such a treatment when prices follow geometric brownian motion but earn below equilibrium rates of return.

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Appendix 1The Discrete Transition Probability Matrix for a Mean Reverting Process

Consider the mean reverting stochastic process

$$(A1) \quad dX_t = \lambda (\bar{X} - X_t) dt + \sigma dZ_X$$

where  $dZ_X$  is a standard wiener process. The first step is to chose a range  $\{X_{\min}, X_{\max}\}$  within which the discretization is to be performed. Depending on the required precision, then divide this range into  $N$  discrete states (i.e.  $N-1$  intervals).



where  $xs = (X_{\min} - X_{\max})/2(N-1)$ .

Without loss of generality we can let the discrete time interval  $\Delta t = 1$ .

In order to bring about a transition from state  $i$  to  $i+1$  ( $X_i$  to  $X_{i+1}$ ) the following conditions must be satisfied;

$$(A2) \quad \Delta X > xs \quad \implies \quad Z > [xs - \lambda(\bar{X} - X_i)]/\sigma$$

and

$$(A3) \quad \Delta X < 3 xs \quad \implies \quad Z < [3xs - \lambda(\bar{X} - X_i)]/\sigma$$

Hence the transition probability  $P_{i, i+1}$  is

$$(A4) \quad P_{i, i+1} = \text{Prob} \left[ Z \in \left\{ \frac{[xs - \lambda(\bar{X} - X_i)]}{\sigma}, \frac{[3xs - \lambda(\bar{X} - X_i)]}{\sigma} \right\} \right]$$

Define  $Z_0 = -\lambda(\bar{X} - X_1)/\sigma$  and  $Z_d = xs/\sigma$ . Then A4 can be rewritten as

$$(A5) \quad P_{i,i+1} = N[Z_0 + \{2(i+1-1)-1\}Z_d] - N[Z_0 + \{2(i+1-i)+1\}Z_d]$$

where  $N[\cdot]$  is the cumulative normal distribution. In general the transition from state  $i$  to  $j$  is given by

$$(A6) \quad P_{i,j} = N[Z_0 + \{2(j-i)-1\}Z_d] - N[Z_0 + \{2(j-i)+1\}Z_d].$$

Special care must be taken with the end points  $P_0$  and  $P_N$ . Lumping all exterior values to the boundary we obtain the transition probabilities

$$(A7) \quad P_{i,n} = 1 - N[Z_0 + \{2(n-i)-1\}Z_d]$$

and

$$(A8) \quad P_{i,1} = N[Z_0 + \{2(1-i)+1\}Z_d].$$

Note that for  $\Delta t \neq 1$  we must set  $\lambda = \lambda \Delta t$  and  $\sigma = \sigma (\Delta t)^{1/2}$ .

Once the above discrete probabilities are available the expected values (such as those encountered in the dynamic programming problems discussed in sections 2 and 3) are trivially obtained as a probability weighted sum. For example, if  $X_{t-1} = X_j$  then  $E_{t-1}[V(X_t)]$  is

$$(A8) \quad E_{t-1}[V(X_t)] = \sum_{i=1}^N V(X_i) P_{j,i}.$$

Figure 1: Values of Fixed and Flexible Technologies plotted against  $P_0$

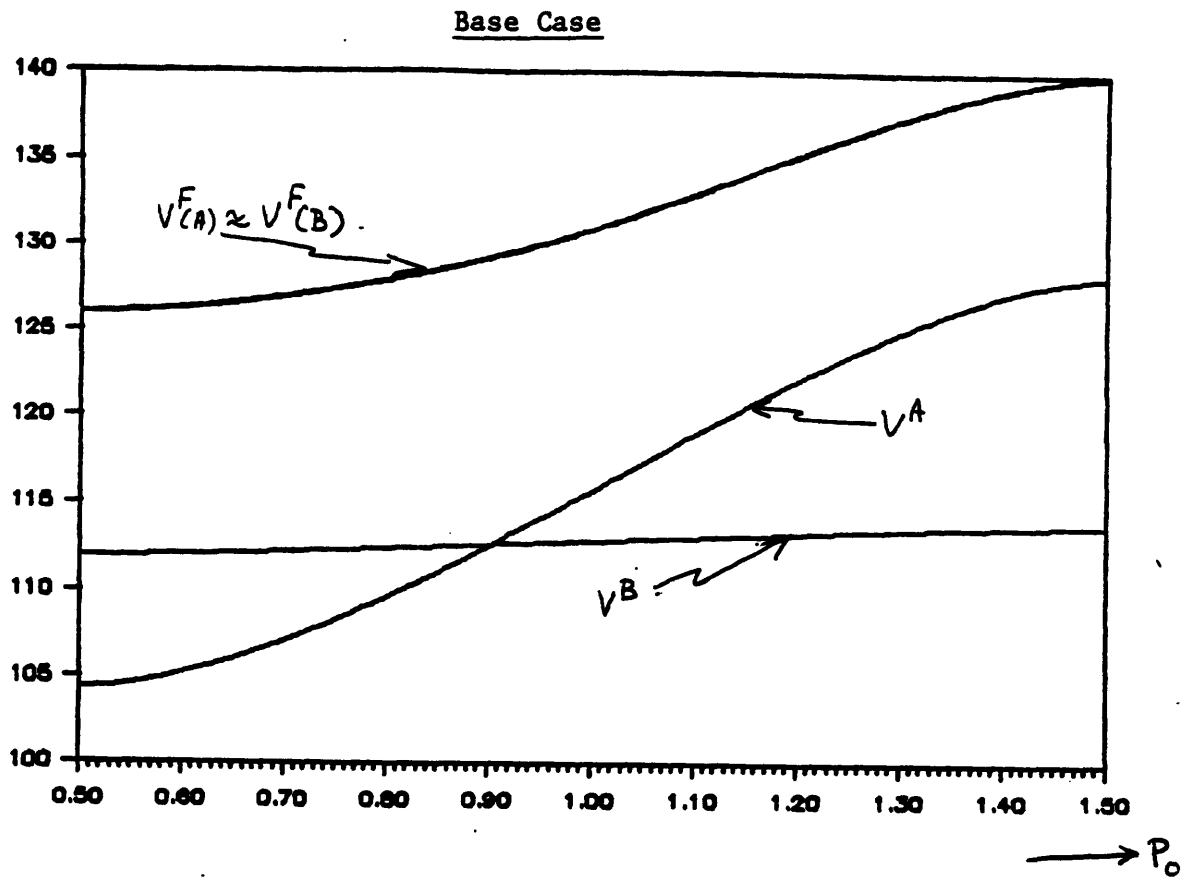


Figure 2: Percentage Increase in Value due to Flexibility (PIVF)

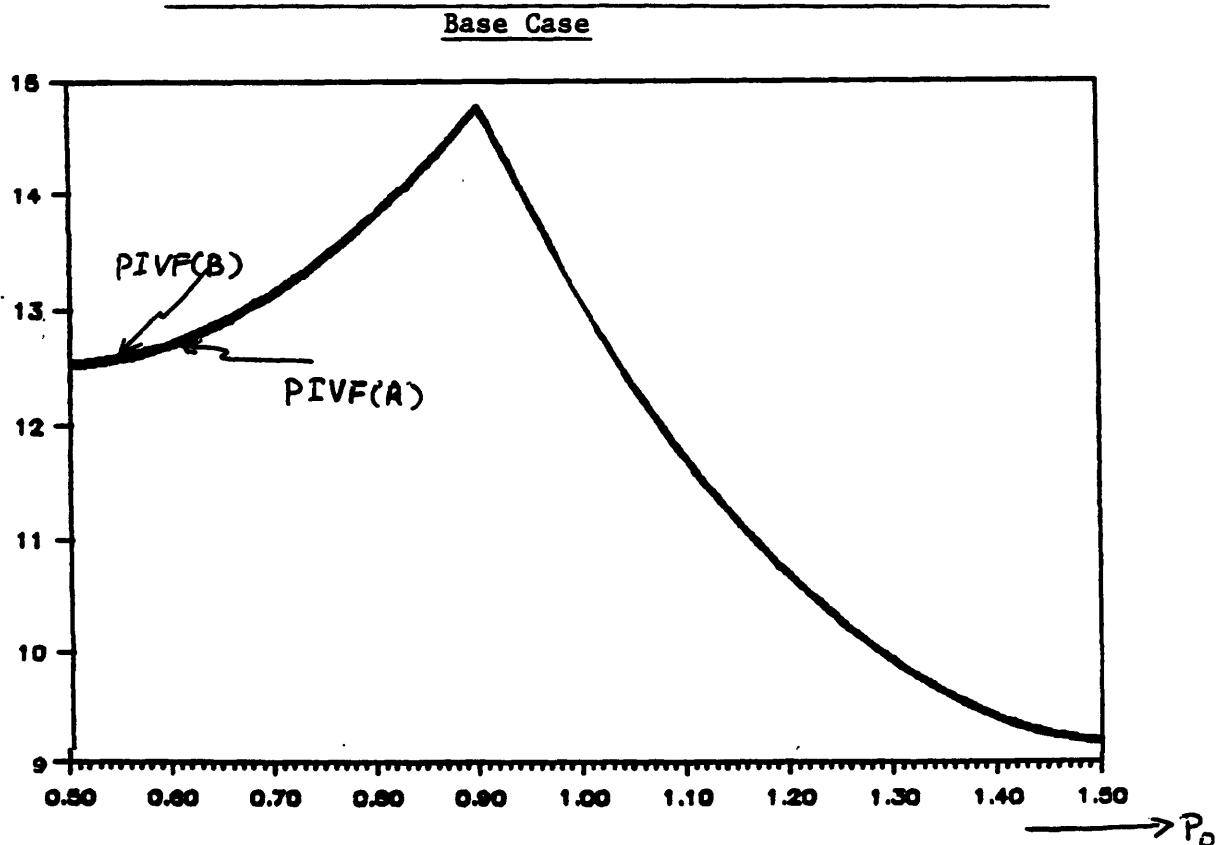


Figure 3: Critical Prices to Switch Modes Plotted against Time

Base Case

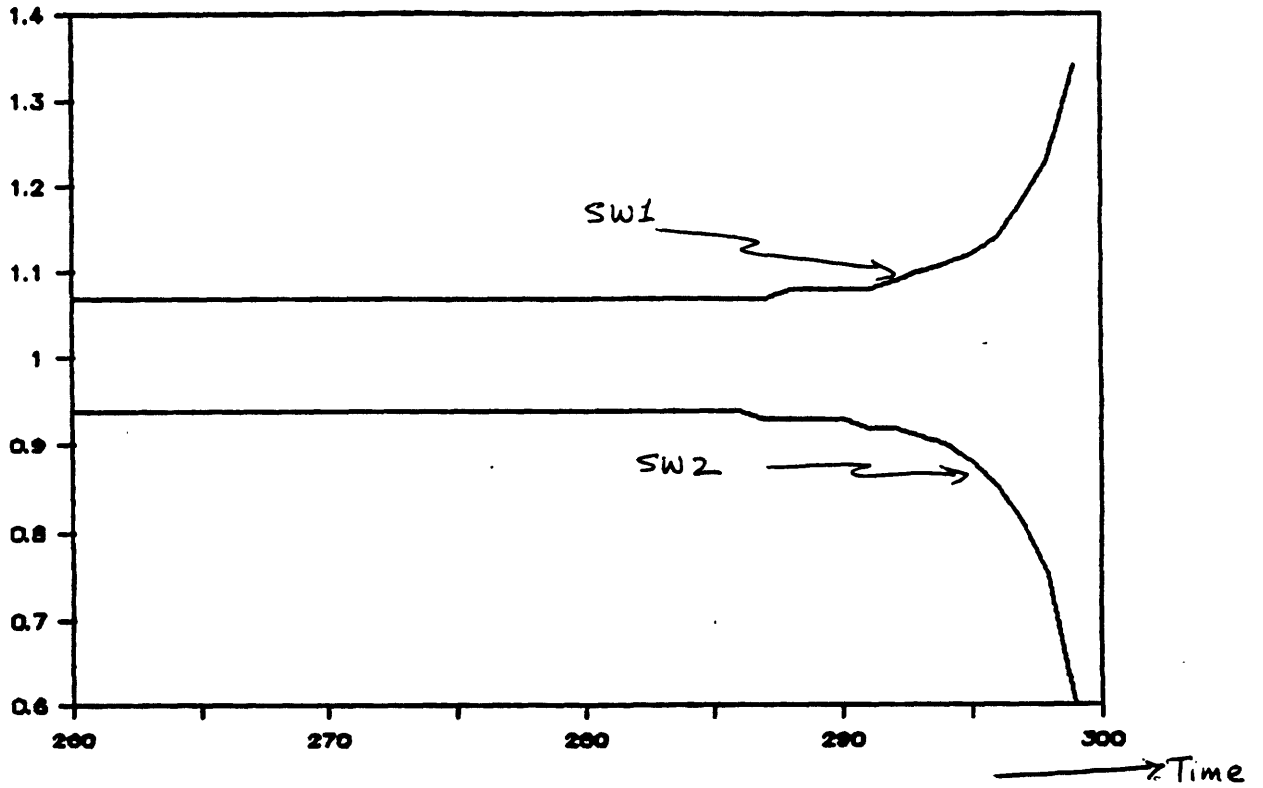


Figure 4: Values of Fixed and Flexible Technologies plotted against

Instantaneous Price Variance,  $\sigma_p$

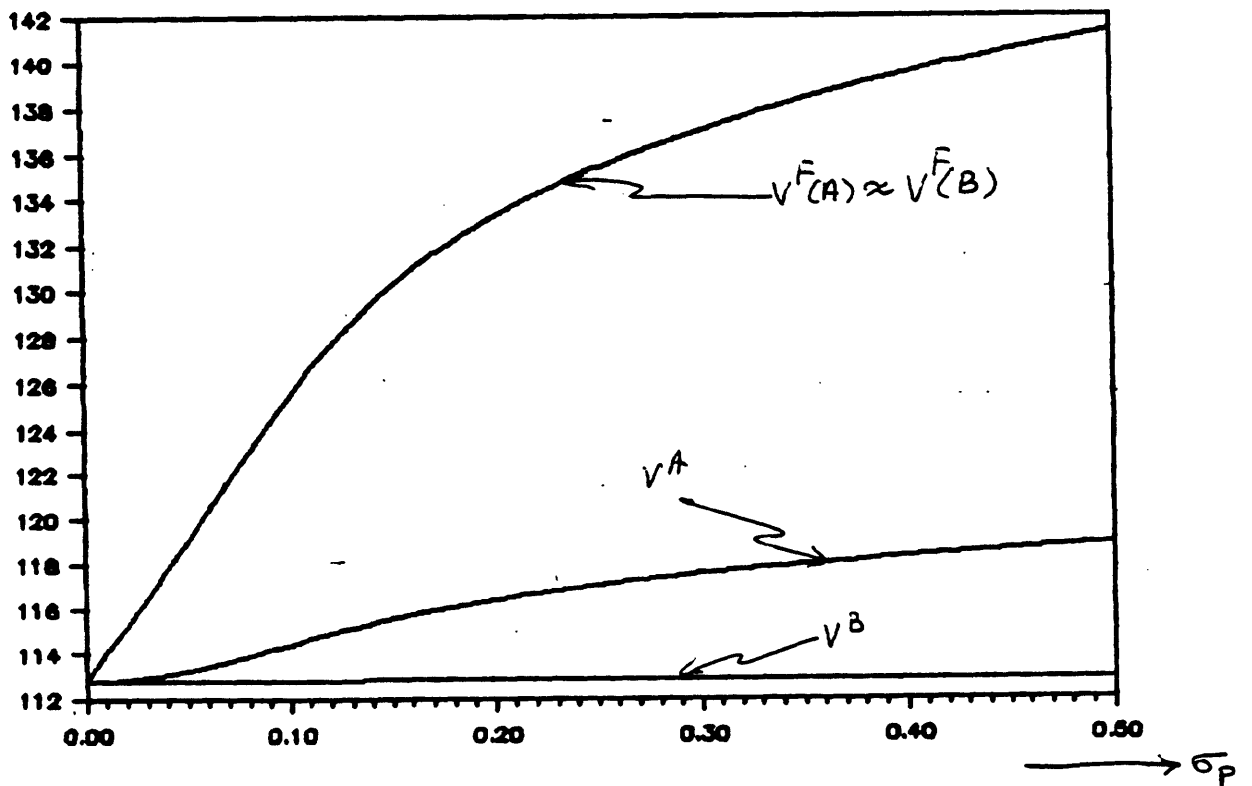


Figure 5: Percentage Increase in Value due to Flexibility (PIVF) plotted against

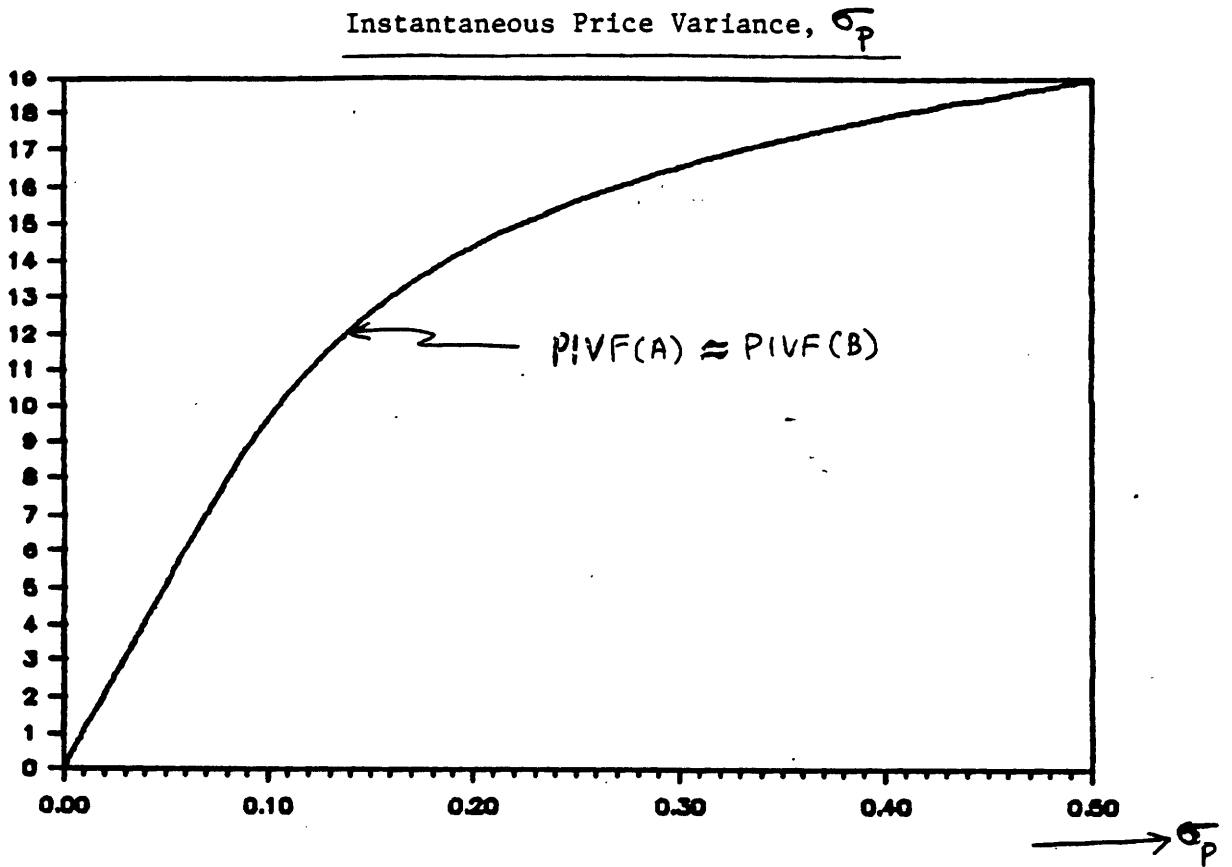


Figure 6: The Percentage Increase in Value due to Flexibility (PIVF) plotted

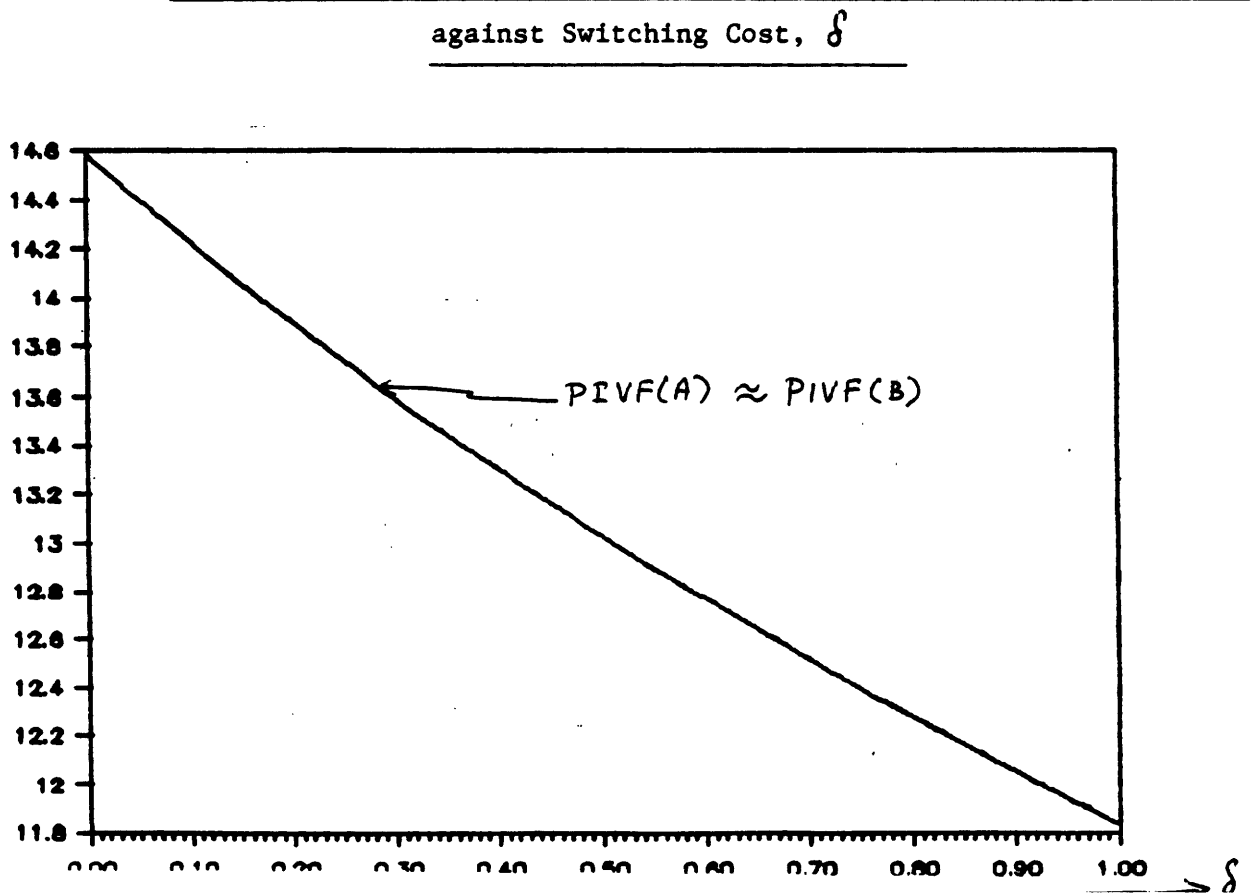


Figure 7: Critical Prices to Switch Modes plotted against Switching Cost,  $\delta$

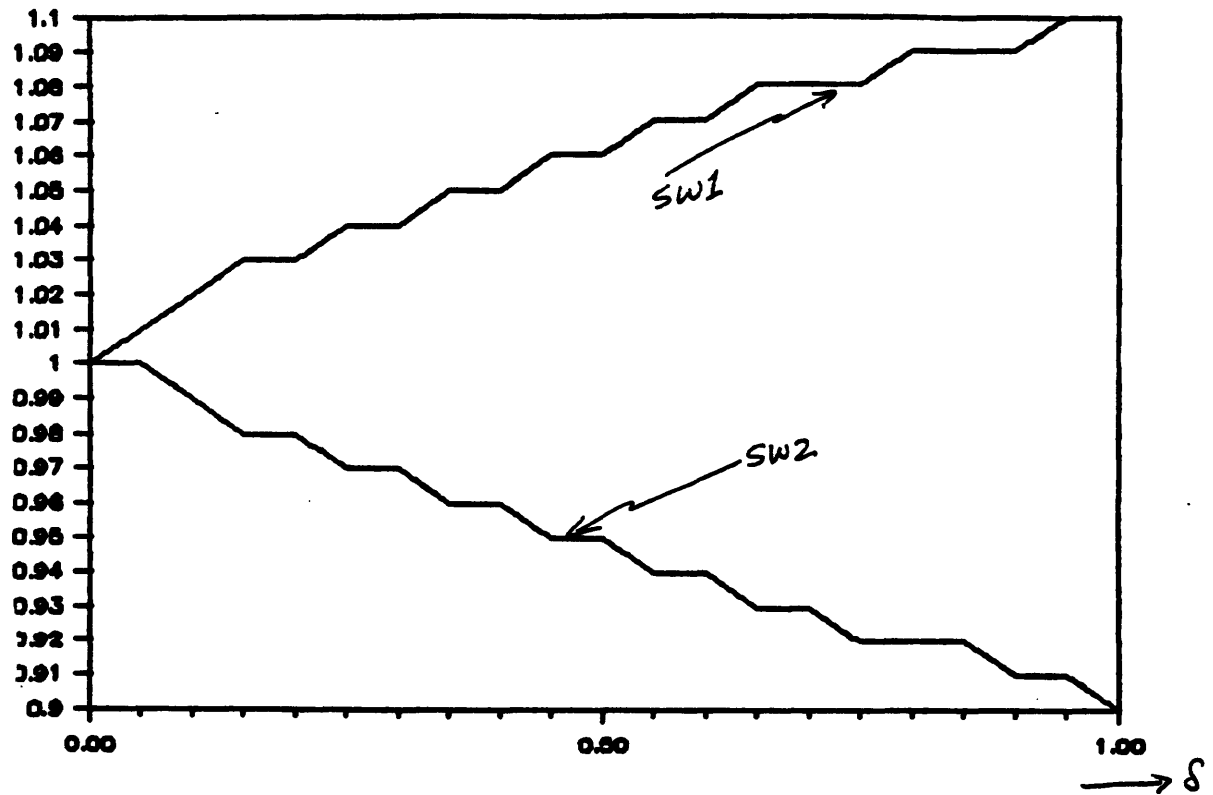


Figure 8: Values of Fixed and Flexible Technologies plotted against the Mean Reversion Parameter,  $\lambda$

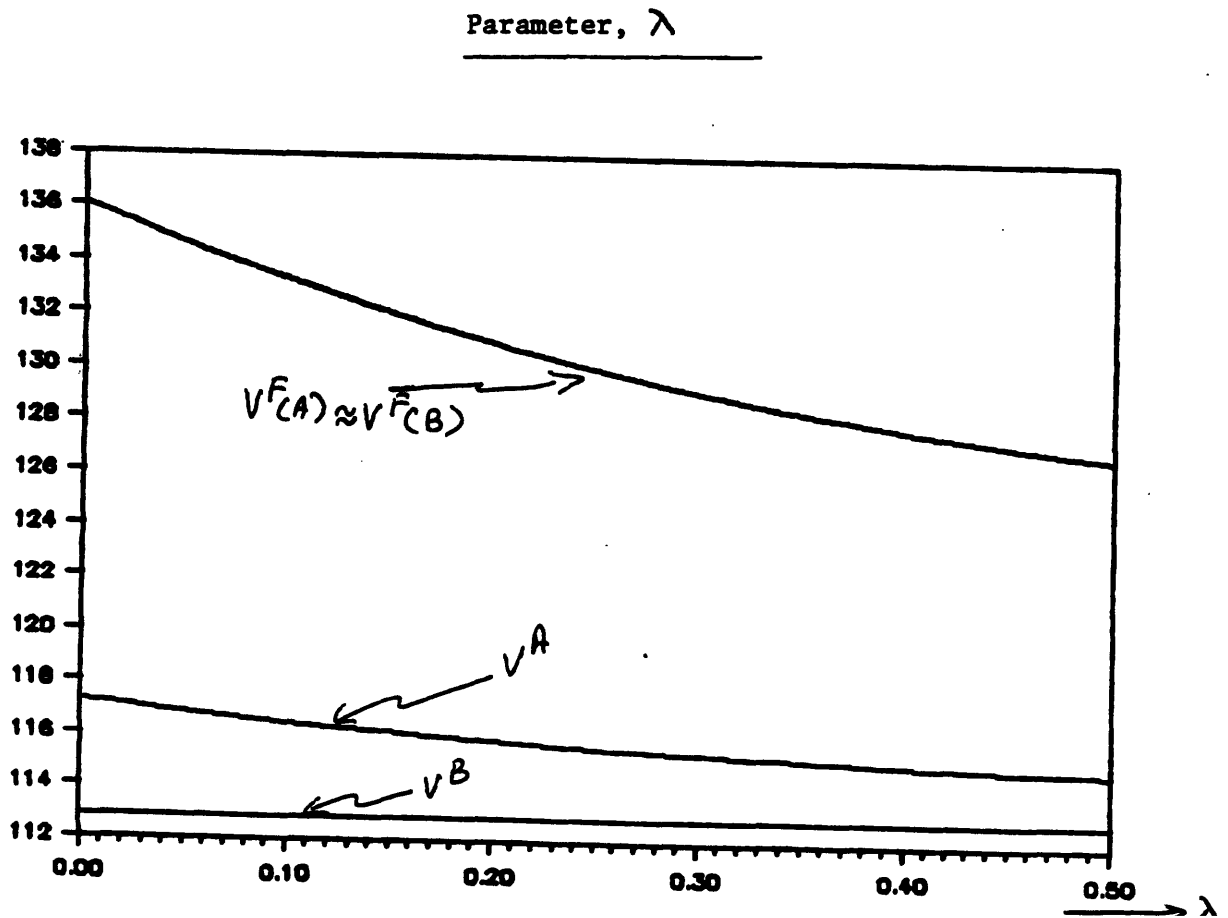


Figure 9: Percentage Increase in Value due to Flexibility (PIVF) plotted against the Mean Reversion Parameter,  $\lambda$

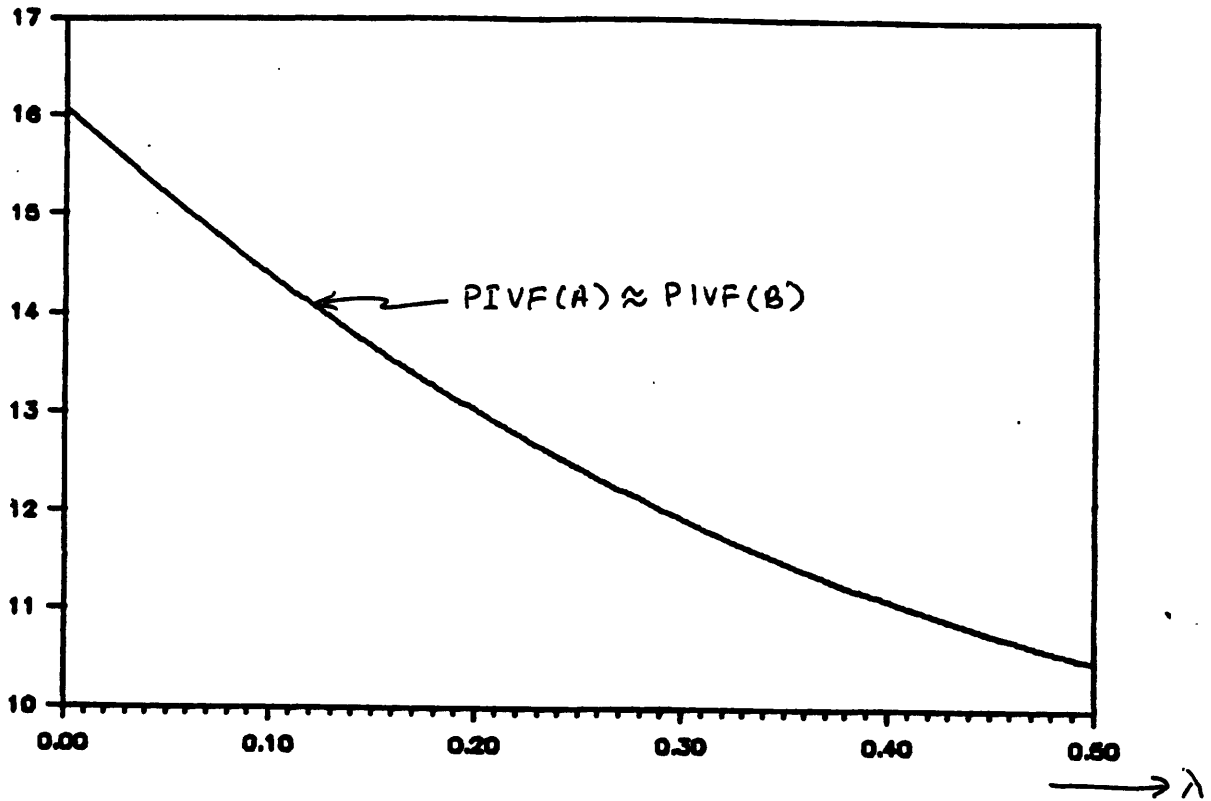


Figure 10: Values of Fixed and Flexible Technologies plotted against the Risk Free Interest Rate,  $r_f$

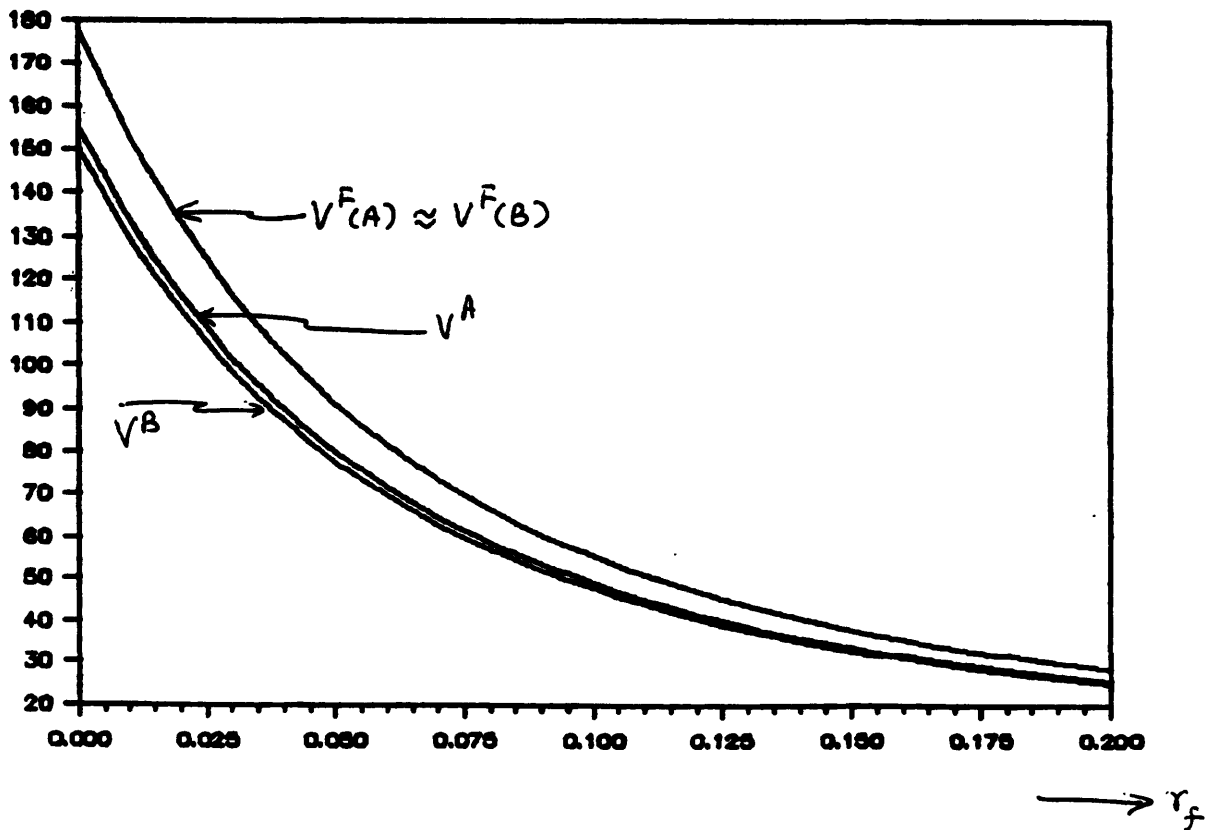


Figure 11: Percentage Increase in the Value due to Flexibility (PIVF) plotted against

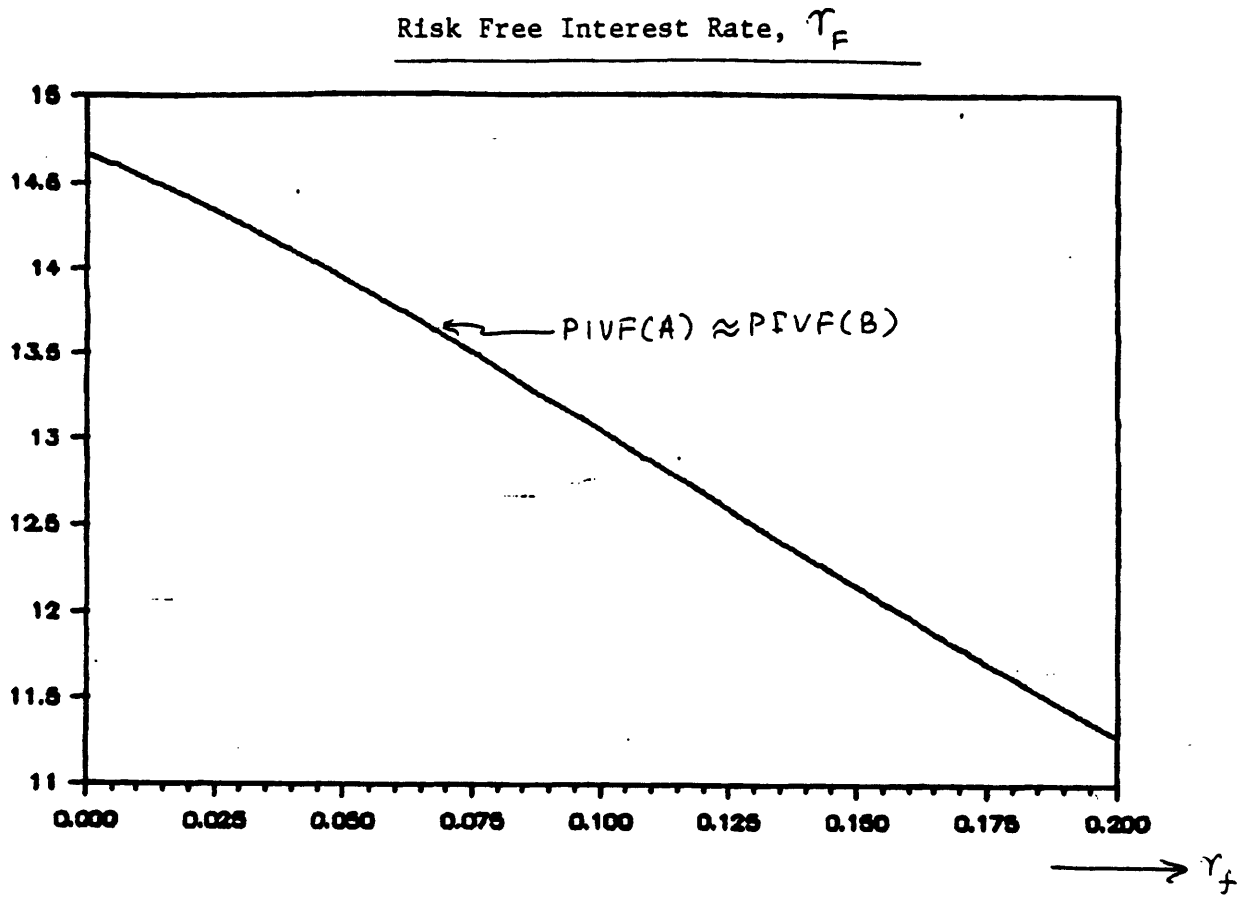


Figure 12: Values of Fixed and Flexible Technologies plotted against Price Elasticity

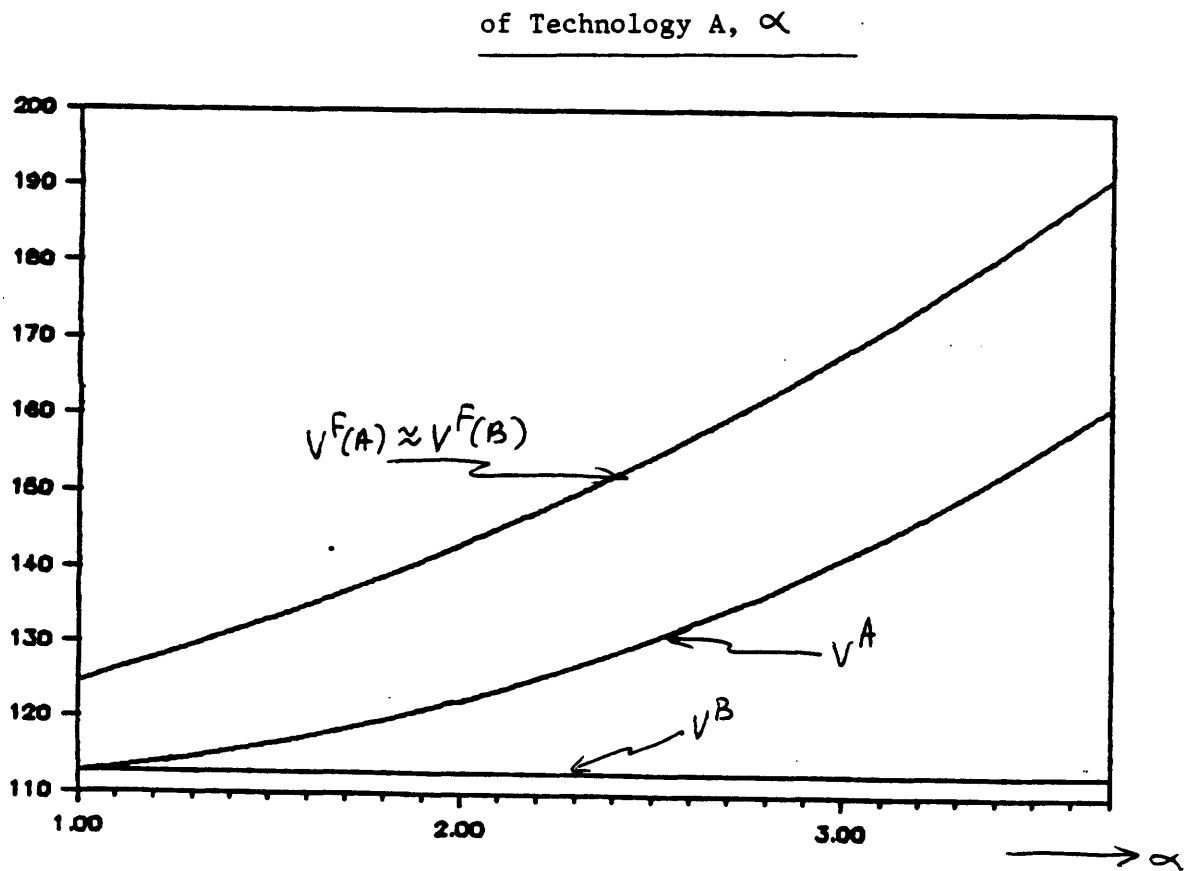




Figure 13: Percentage Increase in Value due to Flexibility (PIVF) plotted against

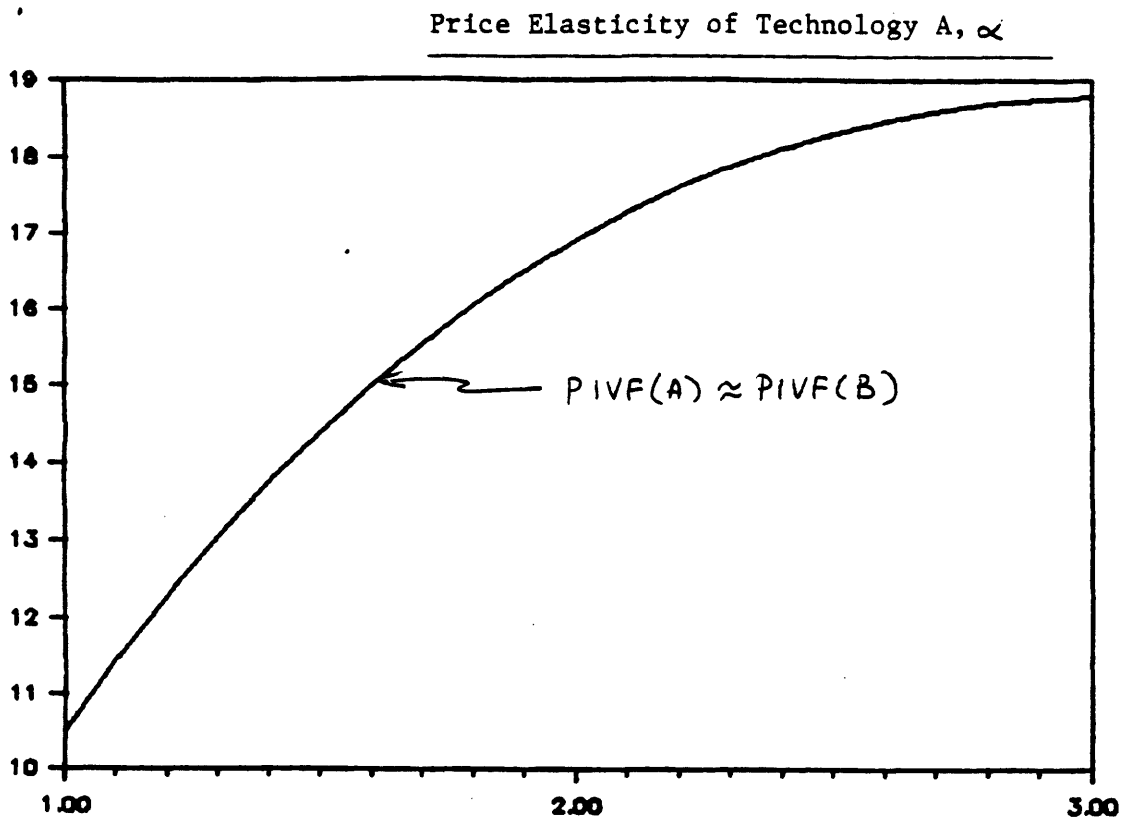


Figure 14: Percentage Increase in Value due to Flexibility (PIVF) plotted against

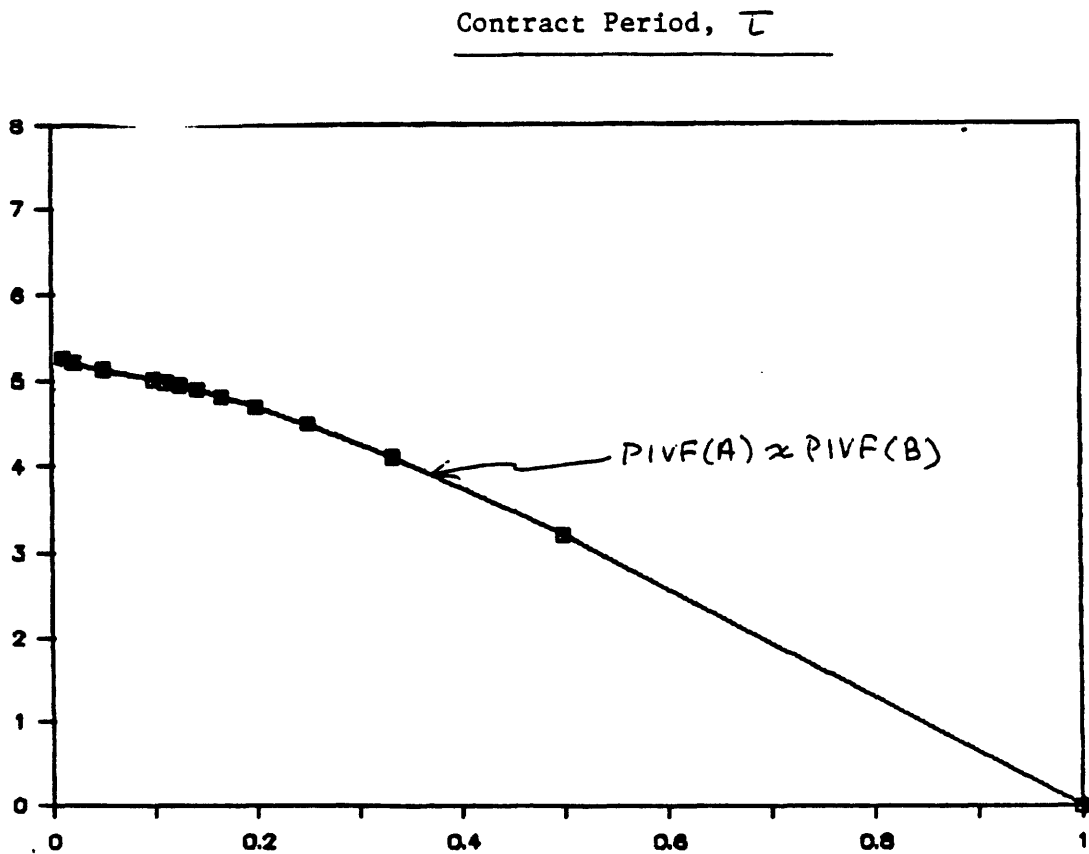


Figure 15: Critical Prices to Switch Modes plotted against Contract Period,  $\bar{t}$

