REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 5, Pages 93–110 (May 17, 2001)
S 1088-4165(01)00127-3

STRICTLY SMALL REPRESENTATIONS AND A REDUCTION THEOREM FOR THE UNITARY DUAL

SUSANA A. SALAMANCA-RIBA AND DAVID A. VOGAN, JR.

Abstract. To any irreducible unitary representation X of a real reductive Lie group we associate in a canonical way, a Levi subgroup G_{su} and a representation of this subgroup. Assuming a conjecture of the authors on the infinitesimal character of X, we show that X is cohomologically induced from a unitary representation of the subgroup G_{su}. This subgroup is in some cases smaller than the subgroup G_u that the authors attached to X in earlier work. In those cases this provides a further reduction to the classification problem.

1. Introduction

For a Lie group G in Harish-Chandra's class, the authors outlined in [3] a program for classifying its unitary dual. This is the set of equivalence classes of unitary irreducible representations of G. In that paper we partition the unitary dual into disjoint subsets parametrized by a discrete set denoted by Λ_u. Roughly, to each element $\lambda_u \in \Lambda_u$ and its centralizer G_u in G we attach a set $\Pi_u^{\lambda_u}(G)$ of unitary irreducible representations of G related to some representations of the subgroup G_u. This is done so that, under an assumption on the infinitesimal characters of unitary representations (Conjecture 18), the problem of classifying the unitary dual is reduced to classifying the unitarily small representations (Definition 10). These are representations attached to a parameter λ_u whose centralizer in G is G itself.

However, this is still a very large set and it contains representations which we already know how to construct from smaller groups. In the case when G is $SL(2,\mathbb{R})$, the set of unitarily small representations consists of the trivial representation, the two discrete series with lowest K types 2 and -2, the two limits of discrete series, the complementary series, and the unitary principal series.

In the present paper we give a refinement of this partition. Still assuming Conjecture 18, we show that for each irreducible unitary representation X of G associated to a given subgroup G_u we can find a subgroup $G_{su} \subseteq G_u$ so that X is cohomologically induced from some unitary representation of G_{su}. This further reduces the classification problem to those representations for which G_{su} is still all of G. Our main result is Theorem 19.

In this new framework, the subgroup G_{su} attached to the first two discrete series of $SL(2,\mathbb{R})$ is the torus. For the rest of the unitarily small representations of $SL(2,\mathbb{R})$, G_{su} is still G.

Received by the editors December 1, 2000 and, in revised form, March 30, 2001.
2000 *Mathematics Subject Classification.* Primary 22E46.
Supported in part by NSF grant DMS-9721441.

This suggests that we can separate the discrete series of the group G—for which we should expect a reduction—from the unitarily small representations. What we gain is that we have fewer unitary representations left to classify.

More precisely, we associate a group G_{su} to a representation of the maximal compact subgroup K parametrized by a weight μ (see (1)). The same group is associated to any unitary representation containing μ as a lowest K type (see Definition 1 below).

In the same way, the group G_a (resp. G_u) in [3] is the same for any admissible (resp. unitary) representation containing a lowest K type μ.

The precise definition of the map $\mu \to G_{su}$ is given in Definition 2; but the idea behind it is fairly simple. The semisimple part of the subgroups G_u and G_a are each determined by a set of simple roots which are zero on the weight λ_u or, respectively on λ_a, as described in [3] (for convenience, we recall the definitions below). The relationship between these two weights is that all the roots which vanish on λ_a also vanish on λ_u and consequently $G_a \subset G_u$.

The groups G_a and G_u are part of an increasing family of subgroups G_t parametrized by a scalar $t \in [1, 2]$, so that $G_1 = G_a$, $G_2 = G_u$ and G_t is constant except for a finite number of points t in that interval (see Definition 2). We define G_{su} to be equal to $G_{2-\varepsilon}$ for any small positive number ε. This is made more precise in Definition 2.

The point is that for this subgroup G_{su} we can implement a bijection between a set of unitary representations of G_{su} and unitary representations of G with lowest K type μ.

This approach fits with the general shape of the reduction theorems of this type. Typically one fixes a representation of K with highest weight μ and associate to μ a pair (H, μ_H) with $H \subseteq G$, a subgroup of G and μ_H, a highest weight of a representation of $H \cap K$ and such that there is a bijection between the sets

$$\left\{ \begin{array}{c} \text{irreducible unitary} \\ \text{representations of } G \\ \text{with lowest } K \text{ type } \mu \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{irreducible unitary} \\ \text{representations of } H \\ \text{with lowest } H \cap K \text{ type } \mu_H \end{array} \right\}.$$

Such a theorem gives no information at those K types μ for which $H = G$ (and $\mu_H = \mu$). We call such K types *non-reducing.*

For example, in [3] the non-reducing K types are the unitarily small ones. Their extremal weights are those lying in a certain closed convex polygon around zero. What we accomplish here is to prove a reduction theorem for K types on the boundary of this polygon.

The general goal is to look for ways to shrink the set of non-reducing K types; possibly by making the group H associated to μ even smaller so that the case $H = G$ happens less often than before. In this paper the smaller subgroup is G_{su} and non-reducing K types are called strictly unitarily small or strictly small. These are precisely the K types whose highest weights lie in the interior of the closed convex polygon corresponding to the unitarily small K types (cf. Lemma 29 and the remark following). Because the highest weights lie in a lattice, extending the reduction theorem to the boundary of the polygon can be a significant improvement, especially for groups of low rank. In the case of $Sp\,(4, \mathbb{R})$, for example, the theorem of [3] provides a reduction theorem except for 25 lowest K-types (Example 14 below). The result in this paper provides a reduction theorem for 11 of those 25 remaining cases (Example 25).

2. Unitarily small K types

Denote by K the maximal compact subgroup of G and let T be a maximal torus in K. As is the case for the group G_u in [3], the construction of G_{su} for X will be in terms of a lowest K type of X. We first describe the construction of G_u and include some results from [3].

We will denote Lie groups by roman uppercase letters, their Lie algebras by the corresponding lower case gothic letter with the subscript 0 and their complexified Lie algebras by the same gothic letter without the subscript. Recall that we have an inclusion

$$\widehat{T} \subset i\mathfrak{t}_0^*.$$

For any T-invariant subspace $\mathfrak{v}$ of $\mathfrak{g}$, denote by $\Delta(\mathfrak{v},\mathfrak{t})$ the set of weights of T in $\mathfrak{v}$. We fix a positive root system $\Delta^+(\mathfrak{k},\mathfrak{t})$ and denote by ρ_c half the sum of the positive roots for $\mathfrak{k}$.

Denote by the same θ the Cartan involution on G, $\mathfrak{g}_0$ and $\mathfrak{g}$, defined by K so that $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ is the corresponding Cartan decomposition of $\mathfrak{g}_0$. Here $\mathfrak{s}_0$ is the -1 eigenspace of θ on $\mathfrak{g}_0$.

We also fix a θ-invariant symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}_0$, negative definite on $\mathfrak{k}_0$ and positive definite on $\mathfrak{s}_0$. We use the same notation for its complexification (defined on $\mathfrak{g}$) as well as its restrictions and dualizations.

Choose a weight $\mu \in i\mathfrak{t}_0^*$. If μ is the highest weight of an irreducible representation of K, then μ is $\mathfrak{k}$-dominant and integral. That is,

$$\begin{aligned} \langle \mu, \alpha\rangle &\geq 0, \quad \alpha \in \Delta^+(\mathfrak{k},\mathfrak{t}),\\ \langle \mu, \alpha^\vee\rangle &\in \mathbb{Z}, \quad \alpha \in \Delta(\mathfrak{k},\mathfrak{t}). \end{aligned} \tag{1}$$

Definition 1. Suppose that X is an irreducible, admissible $(\mathfrak{g},K)$-module.

a) A K type of X is an irreducible representation $\delta \in \widehat{K}$ that appears as a subrepresentation of X when X is viewed as a representation of K.
b) A K type $\delta \in \widehat{K}$ of X is a lowest K type if for any highest weight $\mu \in i\mathfrak{t}_0^*$ of δ, the length

$$\langle \mu + 2\rho_c, \mu + 2\rho_c\rangle$$

is minimal for all K types of X.

Let $\Delta(\mathfrak{g},\mathfrak{t})$ be the set of nonzero weights of T in $\mathfrak{g}$, a possibly non-reduced root system. Fix a system of positive roots making $\mu + 2\rho_c$ dominant:

$$\Delta^+(\mathfrak{g},\mathfrak{t}) \subseteq \{\alpha \in \Delta(\mathfrak{g},\mathfrak{t}) \mid \langle \mu + 2\rho_c, \alpha\rangle \geq 0\}, \tag{2}$$

Let ρ be half the sum of these positive roots and C, the positive Weyl chamber defined by the system

$$C = \left\{\chi \in i\mathfrak{t}_0^* \mid \langle \chi, \beta\rangle \geq 0,\ \forall \beta \in \Delta^+(\mathfrak{g},\mathfrak{t})\right\}, \tag{3}$$

a closed convex cone in $i\mathfrak{t}_0^*$. Define

$$P\colon i\mathfrak{t}_0^* \to C \tag{4}$$

to be the orthogonal projection onto C with respect to $\langle\ ,\ \rangle$.

We will use a family of different projections onto this convex cone, taken from [1] and [3, Section 1]).

Definition 2. Let $\mu \in i\mathfrak{t}_0^*$ be as in Definition 1 and P as in (4).

1. For $t \in [1,2] \subset \mathbb{R}$, define
$$\begin{aligned} \lambda_t &= \lambda_t(\mu) = P(\mu + 2\rho_c - t\rho) \in C, \text{ and} \\ G_t &= G_t(\mu) = G(\lambda_t) \end{aligned}$$
to be the centralizer in G of λ_t.
2. Define

(5) $$G_{su}(\mu) = \max\ \{G_t(\mu) \mid t < 2\}$$

As was seen in [3, Proposition 1.4], λ_t is well defined, that is, independent of the choice of roots $\Delta^+(\mathfrak{g}, \mathfrak{t})$ positive on $\mu + 2\rho_c$.

Remark 3. At the endpoints we have

(6) $$\lambda_1(\mu) = \lambda_a(\mu) = \lambda_a, \text{ and } G_1(\mu) = G_a,$$

(7) $$\lambda_2(\mu) = \lambda_u(\mu) = \lambda_u, \text{ and } G_2(\mu) = G_u$$

as defined in [3, (0.5d) and Proposition 2.3]. We also denote

(8) $$\Lambda_a = \left\{\lambda_a(\mu) \middle| \mu \in \widehat{T} \text{ dominant}\right\},$$

(9) $$\Lambda_u = \left\{\lambda_u(\mu) \middle| \mu \in \widehat{T} \text{ dominant}\right\}.$$

Remark 4. Note that both $\lambda_a(\mu)$ and $\lambda_u(\mu)$ determine not only their centralizers in G but also a subset of roots of $\mathfrak{t}$ in $\mathfrak{g}$ that are positive on them. There is a non-empty interval $(s, 2)$, so that for all $t \in (s, 2)$, the weight $\lambda_t(\mu)$ determines the same group $G_{su}(\mu)$ and the same subset of roots of $\mathfrak{t}$ in $\mathfrak{g}$ that are positive on $\lambda_t(\mu)$. We will pick one of these weights and call it $\lambda_{su}(\mu)$.

In the examples below we define $\operatorname{sgn}(0) = +1$.

Example 5. In the case when G is $SL(2, \mathbb{R})$, $K = SO(2) = T$ and $\widehat{K} \longleftrightarrow \mathbb{Z} \subset \mathbb{R} \cong i\mathfrak{t}_0^*$. Then $\Delta(\mathfrak{k}, \mathfrak{t}) = \emptyset$, $\Delta(\mathfrak{g}, \mathfrak{t}) = \{\pm 2\}$. Let $\mu \longleftrightarrow n$.
1. If $|n| \leq 2$, then $\lambda_u = 0$ and $G_u = G$.
2. If $|n| > 2$, then $\lambda_u = n - 2\operatorname{sgn}(n)$ and $G_u = T$.

However, both G_{su} and G_a are T if $|n| > 1$. If $|n| \leq 1$, then for all $t \in [1, 2]$, $\lambda_t = 0$ and $G_t = G$.

Example 6. If $G = U(1,1)$, then $K = U(1) \times U(1) = T$ and $\widehat{K} \longleftrightarrow \mathbb{Z} \times \mathbb{Z}$.

$\Delta(\mathfrak{k}, \mathfrak{t}) = \phi$, $\Delta(\mathfrak{g}, \mathfrak{t}) = \{\pm(1, -1)\}$. Let $\mu \longleftrightarrow (a, b) \in \widehat{K}$. Define $p = a - b$. Then $\Delta^+(\mathfrak{g}, \mathfrak{t}) = \{\operatorname{sgn}(p)(1, -1)\}$. Then if α is the positive root, $\rho = \frac{\alpha}{2} = \frac{\operatorname{sgn}(p)}{2}(1, -1)$,
$$\mu + 2\rho_c - t\rho = \mu - t\rho = (a, b) - \operatorname{sgn}(p)\, t\left(\frac{1}{2}, -\frac{1}{2}\right)$$
and
$$\langle \mu - t\rho, \alpha\rangle = \operatorname{sgn}(p)(a - b) - t = |a - b| - t.$$
Then
$$\lambda_t(\mu) = \begin{cases} \mu - t\rho & \text{if } |a - b| \geq t, \\ \left(\frac{a+b}{2}, \frac{a+b}{2}\right) & \text{if } |a - b| \leq t. \end{cases}$$
Therefore, for $1 \leq t < 2$,
1. if $|a - b| \geq 2$, $\lambda_a(\mu) = \mu - \rho$, $\lambda_t(\mu) = \mu - t\rho$, and $G_a(\mu) = G_{su}(\mu) = T$;
2. if $|a - b| \leq 1$, $\lambda_a(\mu) = \lambda_t(\mu) = \left(\frac{a+b}{2}, \frac{a+b}{2}\right)$.

For $t = 2$,

1. if $|a-b| > 2$ $\lambda_u(\mu) = \mu - 2\rho$ and $G_u(\mu) = T$;
2. if $|a-b| \le 2$ $\lambda_u(\mu) = \left(\frac{a+b}{2}, \frac{a+b}{2}\right)$ and $G_u(\mu) = G$.

The following results are useful when calculating these parameters.

Proposition 7 ([1, Prop. 1.2] and [3, Prop. 1.1])**.** *Let V be a finite-dimensional real vector space with a positive definite inner product $\langle\,,\rangle$. Let C be a closed convex cone in V. Denote by C^0 the dual cone of C:*

$$C^0 = \{v \in V \,|\langle v, c\rangle \ge 0, \textit{ for all } c \in C\}.$$

Then, for any $v \in V$ there is a unique element c_0 of C closest to v. The following conditions characterize c_0:

1. *For any $c \in C$, $|v - c_0| \le |v - c|$.*
2. *For any $c \in C$, $\langle v - c_0, c - c_0\rangle \le 0$.*
3. *The vector $c_0 - v \in C^0$ and $\langle c_0 - v, c_0\rangle = 0$.*

In the context of (3), c_0 is precisely Pv (4), the projection of v onto C. Then any $v \in V$ is uniquely written as an orthogonal decomposition

$$v = v_{dom} + v_{neg} \tag{10}$$

where $v_{dom} = Pv = c_0 \in C$ and $v_{neg} = v - Pv \in -C^0$.

Proposition 8. *Suppose $\mu \in i\mathfrak{t}_0^*$ and a decomposition of μ as in* (10)*. Denote by Φ the set of simple roots of $\Delta^+(\mathfrak{g},\mathfrak{t})$ and set $S(\mu) = \{\alpha \in \Phi \,|\langle \alpha, \mu_{dom}\rangle = 0\}$. Suppose $T \subset S(\mu)$, $T = \{\alpha_{i_1}, \alpha_{i_2} \ldots \alpha_{i_t}\}$, define*

$$\begin{aligned} P^T\mu &= \textit{orthogonal projection of } \mu \textit{ onto } (\textit{Span}\, T)^{\perp}, \\ \mu^T &= P^T\mu. \end{aligned}$$

Then $\left(\mu^T\right)_{dom} = \mu_{dom}$

Proof. Write

$$\mu_{neg} = -\left(\sum_{\substack{\alpha\in S(\mu)\\ m_\alpha \ge 0}} m_\alpha \alpha\right).$$

Clearly,

$$\begin{aligned} \mu^T &= \mu_{dom} - \left(\sum_{\substack{\alpha\in S(\mu)\\ m_\alpha \ge 0}} m_\alpha P^T\alpha\right) \\ &= \mu_{dom} - \left(\sum_{\substack{\alpha\in S(\mu)-T\\ m_\alpha \ge 0}} m_\alpha P^T\alpha\right) \end{aligned} \tag{11}$$

since P^T is 0 on all roots in T.

We claim that for $\alpha \notin T$, $P^T\alpha = \alpha + \sum_{j=1}^{t} c_j \alpha_{i_j}$, with all $c_j \geq 0$. In fact, the identity is true with all the $c_j \in \mathbf{R}$. Now, for any $\beta \in T$,

$$\begin{aligned}\left\langle \sum_{j=1}^{t} c_j \alpha_{i_j}, \beta \right\rangle &= \left\langle P^T \alpha, \beta \right\rangle - \langle \alpha, \beta \rangle \\ &= -\langle \alpha, \beta \rangle \geq 0.\end{aligned}$$

The last inequality is true since $\alpha \neq \beta$. This shows that $\sum_{j=1}^{t} c_j \alpha_{i_j}$ is a dominant weight in $SpanT$ and hence in the positive cone. So, all the constants c_j are positive. Then

$$\mu^T = \mu_{dom} - \left(\sum_{\alpha \in s(\mu) - T} m_\alpha \left(\alpha + \sum_{j=1}^{t} c_j \alpha_{i_j} \right) \right).$$

The last term is a positive linear combination of simple roots perpendicular to μ_{dom}. By the uniqueness of the decomposition given by (10), Proposition 8 follows. □

Proposition 8 gives us an algorithm to find $\lambda_t(\mu)$. That is, we just need to find the set $S(\mu + 2\rho_c - t\rho)$. We proceed as follows. Write $\mu_t = \mu + 2\rho_c - t\rho$.

1. If μ_t is dominant, then $S(\mu_t) = \phi$ and $\lambda_t(\mu) = \mu_t$.
2. If μ_t is not dominant, choose $T_1 = \{\alpha \in \Phi \,|\, \langle \alpha, \mu \rangle \leq 0\} \subseteq S(\mu)$. Then the previous proposition says that $S\left(\mu^{T_1}\right) = S(\mu)$.
3. If μ^{T_1} is dominant, we are done and $\lambda_t(\mu) = \mu^{T_1}$.
4. If μ^{T_1} is not dominant, set $T_2 = \left\{\alpha \in \Phi \,\middle|\, \left\langle \alpha, \mu^{T_1} \right\rangle \leq 0 \right\}$. Then $T_1 \subseteq T_2 \subseteq S(\mu)$.
5. Proceed in this way until μ^{T_j} is dominant.

Example 9. For $G = SO_e(4,1)$, $K = SO(4)$, $T = SO(2) \times SO(2)$. We can fix

$$\Delta^+(\mathfrak{k}, \mathfrak{t}) = \{e_1 \pm e_2\},$$

so that $\widehat{K} \longleftrightarrow \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a \geq |b|\} \subset \mathbb{R}^2 \cong i\mathfrak{t}_0^*$ and $2\rho_c = (2,0)$. Let $\mu \longleftrightarrow (a,b) \in \widehat{K}$. Then $\mu + 2\rho_c = (a+2, b)$, $a + 2 > |b|$ and we can choose

$$\Delta^+(\mathfrak{g}, \mathfrak{t}) = \{e_1 \pm e_2; e_1; \operatorname{sgn}(b)\, e_2\}.$$

Then the set of simple roots of $\mathfrak{t}$ in $\mathfrak{g}$ is $\Phi = \{e_1 - \operatorname{sgn}(b)\, e_2; \operatorname{sgn}(b)\, e_2\}$ and

$$\begin{aligned} t\rho &= t\left(\frac{3}{2}, \operatorname{sgn}(b)\frac{1}{2}\right), \\ \mu_t &= \mu + 2\rho_c - t\rho = \left(a + 2 - \frac{3}{2}t, b - \frac{\operatorname{sgn}(b)}{2}t\right). \end{aligned}$$

Denote by $\alpha_1 = e_1 - \operatorname{sgn}(b)\, e_2$, and by $\alpha_2 = \operatorname{sgn}(b)\, e_2$. Then

$$\langle \mu_t, \alpha_1 \rangle = a - |b| + 2 - t \geq a - |b|, \text{ for all } t \in [1,2], \tag{12}$$

$$\langle \mu_t, \alpha_2 \rangle = |b| - \frac{1}{2}t \geq |b| - 1, \text{ for all } t \in [1,2]. \tag{13}$$

1. First suppose $t = 2$. Then
   (a) If $a > |b| > 1$, then $\lambda_2(\mu) = \mu_2 = \lambda_u$ is dominant and $G_u = T$.
   (b) If $a = |b| > 1$, following Proposition 8, set $T_1 = \{\alpha_1\}$. Then

$$\lambda_u = (|b| - 1, b - \operatorname{sgn}(b))$$

   So $G_u \cong SL(2)_{\alpha_1} \subseteq G$.

(c) When $a = |b| \leq 1$, then by (c) in Proposition 7, $\lambda_u = 0$, $G_u = G$ and $\mu \in \{(0,0), (1, \pm 1)\}$.

(d) If $a > |b|$ and $|b| \leq 1$, set $T_1 = \{\alpha_2\}$. Then

$$\mu^{T_1} = (a-1, 0).$$

Clearly, $\langle \mu^{T_1}, \alpha_2 \rangle = 0$, but $\langle \mu^{T_1}, \alpha_1 \rangle$ is negative if and only if $a \leq 1$. Then $G_u = G$ only if $\mu = (1,0)$.

2. Now assume that $t = 2 - \varepsilon$.

(a) For $a \geq |b| > 1$, $\lambda_t(\mu) = \mu_t$ is strictly dominant and $G_{su} = T$.

(b) When $a = |b| \leq 1$, then $a - |b| + 2 - t = a - |b| + \varepsilon > 0$ but $|b| - \frac{1}{2}t = |b| - 1 + \frac{\varepsilon}{2} \leq 0$ only if $|b| = 0$. In that case set $T_1 = \{\alpha_2\}$. Then

$$\mu^{T_1} = \left(-1 + \frac{3}{2}\varepsilon, 0\right).$$

Clearly, $\langle \mu^{T_1}, \alpha_1 \rangle$ is negative for $\varepsilon < \frac{2}{3}$. Hence, if $\mu = (0,0)$, $\lambda_{su} = 0$ and $G_{su} = G$.

On the other hand, if $a = |b| = 1$, μ_t is dominant for $t = 2 - \varepsilon$ and $G_{su} = T$.

(c) When $a > |b|$ but $|b| \leq 1$, we proceed as in 2(b). If $b = 0$, $\mu^{T_1} = \left(a - 1 + \frac{3}{2}\varepsilon, 0\right) > 0$ on α_1 since $a > |b|$. Hence $T \subsetneq G_{su} \subsetneq G$ in this case.

Definition 10 (see [3, Definition 6.1])**.** Let $\mu \in i\mathfrak{t}_0^*$ as in Definition 2 and let $\lambda_u(\mu)$ and $G_u(\mu)$ as in (7).

1. We say that $\lambda_u(\mu)$ is unitarily small if $G_u(\mu) = G$. If that is the case, we say that μ is a unitarily small K type associated to $\lambda_u(\mu)$ and any representation X with a unitarily small K type is unitarily small as well.
2. Denote by $B_u^{\lambda_u}(G)$, the set of unitarily small K types associated to λ_u by Definition 2.

It is clear from the definition that if G is semisimple, then the only unitarily small weight in Λ_u (see (9)) is $\lambda_u = 0$. In general, if λ_u is unitarily small, it lies in the center of $\mathfrak{g}$ (under the identification of $\mathfrak{t}^*$ with $\mathfrak{t}$ via the invariant bilinear form $\langle\ ,\ \rangle$).

Example 11. $B_u^0(SL(2,\mathbb{R})) = \{0, \pm 1, \pm 2\}$.

Example 12. $B_u^{\left(\frac{m}{2},\frac{m}{2}\right)}(U(1,1)) = \left(\frac{m}{2}, \frac{m}{2}\right) + \begin{cases} \{(0,0), \pm(1,-1)\} & \text{for } m \text{ even,} \\ \left\{\pm\left(\frac{1}{2}, -\frac{1}{2}\right)\right\} & \text{for } m \text{ odd.} \end{cases}$

Example 13. $B_u^0(SO_e(4,1)) = \{(0,0), (1,0), (1,\pm 1)\}$.

Example 14. $B_u^0(Sp(4,\mathbb{R})) = \{(a,b) \in i\mathfrak{t}_0^* \mid 3 \geq a \geq b \geq -3, a - b \leq 4\}$.

The first three examples above are clear from the previous calculations for these groups. For $G = Sp(4,\mathbb{R})$, the calculation was partially done in [3, example 6.3].

Remark 15. The example of $U(1,1)$ illustrates the rôle of the center. Since $U(1,1)$ is not semisimple, the set of unitarily small K types is not finite. It is the set of integral weights inside the product of the lattice of half integral weights in the center of the group and a finite set of K types in $SU(1,1)$. This finite set is the set of unitarily small K types of $SU(1,1)$ associated to $(0,0)$. More precisely, the set of all the unitarily small K types of $U(1,1)$ is the set of integral points on the lines $y = x$, $y = x \pm 1$ and $y = x \pm 2$.

A similar phenomenon happens for a general reductive group. We will make this precise in Proposition 27. However, we need to use the rôle of the center of G to state Theorem 19. Therefore we will fix some notation we need right now and postpone the explanation of the relationship of the center of G with the unitarily small K types until we need it. We will use the same notation as in [3], (6.1) and Lemma 6.5:

Denote by Z the identity component of the center of G. We have

$$\begin{aligned} Z &= (Z \cap K)\exp(\mathfrak{z}_0 \cap \mathfrak{s}_0) = Z_c Z_h, \\ G_s &= \text{derived group of } G_0, \\ T_s &= G_s \cap T, \text{ and } \mathfrak{t}_0 = \mathfrak{t}_{0,s} + \mathfrak{z}_{0,c}. \end{aligned} \tag{14}$$

Here $\mathfrak{z}_{0,c} = Lie\,(Z_c)$ and $\mathfrak{t}_{0,s} =$ span of the roots of $\mathfrak{t}$ in $\mathfrak{g}$. Then we can write any K-dominant weight $\mu \in \widehat{T}$ as a decomposition of its restrictions to Z_c and T_s. If we call the differentials of these restrictions $\mu_z \in i\mathfrak{z}^*_{0,c}$ and $\mu_s \in i\mathfrak{t}^*_{0,s}$ respectively, then

$$\mu = \mu_z + \mu_s. \tag{15}$$

3. Main Theorem.

Let $\mathfrak{h} = \mathfrak{t} + \mathfrak{a} \subset \mathfrak{g}$ be the maximally compact Cartan subalgebra of $\mathfrak{g}$.

Notation 16. For the purpose of stating and proving Theorem 19, we fix once and for all a weight $\mu \in i\mathfrak{t}_0^*$ dominant for $\Delta^+(\mathfrak{k},\mathfrak{t})$. Whenever it is clear from the context, we will drop the variable μ from all the parameters attached to this fixed weight. For example, we will write G_{su} for $G_{su}\,(\mu)$ and λ_{su} for $\lambda_{su}\,(\mu)$, etc. as in Definition 2 and Remark 4.

Denote by $\mathfrak{g}_{su} = \mathfrak{g}_{su}\,(\mu)$ the complexified Lie algebra of G_{su}. It is clear from the definitions that

$$\mathfrak{g}_{su} = \mathfrak{g}\,(\lambda_{su}) = \mathfrak{h} + \sum_{\langle \lambda_{su}, \alpha\rangle = 0} \mathbb{C}X_\alpha. \tag{16}$$

Set

$$\mathfrak{u}_{su} = \sum_{\langle \lambda_{su}, \alpha\rangle > 0} \mathbb{C}X_a \tag{17}$$

and

$$\mathfrak{q}_{su} = \mathfrak{q}_{su}\,(\mu) = \mathfrak{q}\,(\lambda_{su}) = \mathfrak{g}_{su} + \mathfrak{u}_{su} \tag{18}$$

the θ-stable parabolic subalgebra of $\mathfrak{g}$ defined by λ_{su} (see [2], Section 4.6). Denote by

$$\Delta\,(\mathfrak{u}_{su} \cap \mathfrak{s}) = \{\alpha \in \Delta\,(\mathfrak{s},\mathfrak{t}) \mid (\lambda_{su}, \alpha) > 0\}$$

and

$$2\rho\,(\mathfrak{u}_{su} \cap \mathfrak{s}) = \sum_{\alpha \in \Delta(\mathfrak{u}_{su} \cap \mathfrak{s})} \alpha. \tag{19}$$

Definition 17. Given any weight $\lambda \in i\mathfrak{t}_0^*$, let $G(\lambda)$ be the centralizer in G of λ and let $\mathfrak{q}(\lambda) = \mathfrak{g}(\lambda) + \mathfrak{u}(\lambda)$ be the θ-stable parabolic subalgebra defined by λ as in [2], Section 4.6, and $s(\lambda) = \dim(\mathfrak{u}(\lambda) \cap \mathfrak{k})$. Define the functors

$$\begin{aligned} \mathcal{L}_{s(\lambda)}(\lambda) :\ (\mathfrak{g}(\lambda), G(\lambda) \cap K) \text{ modules} &\rightarrow (\mathfrak{g}, K) \text{ modules}, \\ \mathcal{L}^K_{s(\lambda)}(\lambda) :\ (G(\lambda) \cap K) \text{ modules} &\rightarrow K \text{ modules} \end{aligned}$$

as in [2], Section 5.1.

Denote by W the Weyl group of $\Delta(\mathfrak{g}, \mathfrak{t})$ and for any weight $\gamma \in \mathfrak{t}^*$, let

$$\langle W \cdot \gamma \rangle \tag{20}$$

be the convex hull of the orbit of γ under W. Our Main Theorem will still assume the following.

Conjecture 18. *Let X be an irreducible Hermitian Harish-Chandra module of G; $\mathfrak{h}$, a θ-stable Cartan subalgebra of $\mathfrak{g}$ and $\phi \in \mathfrak{h}^*$, a weight representing the infinitesimal character of X. Assume X is unitarily small (Definition 10) and let $\lambda_u(X)$ be a weight associated to X as in Definition 2 and $B_u^{\lambda_u(X)}(G)$, the set of unitarily small K types associated to $\lambda_u(X)$, (Definition 10). Suppose further that the canonical real part $RE\phi$ of ϕ does not belong to*

$$\lambda_u(X) + \langle W \cdot \rho \rangle .$$

Then, the Hermitian form on X is indefinite on $B_u^{\lambda_u(X)}(G)$.

Theorem 19. *Fix notation as in Remark 4,* (16) *and* (18)*. Assume Conjecture 18 holds for all Levi factors of θ-stable parabolic subalgebras of G. Let λ_u and G_u as in* (7) *and let $\mathfrak{g}_u$ be the complexified Lie algebra of G_u. Let $\mathfrak{q}_u$ be the θ-stable parabolic subalgebra of $\mathfrak{g}$ determined by λ_u and with Levi factor $\mathfrak{g}_u$, and let $\mathfrak{q}_{su} \subseteq \mathfrak{q}_u$ be as in* (18)*.*

Then, the following are true:

1. *The weight $\mu_{su} = \mu - 2\rho(\mathfrak{u}_{su} \cap \mathfrak{s}) \in i\mathfrak{t}_0^*$ is a highest weight of a unitarily small finite-dimensional irreducible representation of $G_{su} \cap K$.*
2. *There is a bijection between the set of unitary irreducible Harish-Chandra modules of G_{su} containing μ_{su} as a lowest $G_{su} \cap K$ type and the set of irreducible unitary Harish-Chandra modules of G containing the K type μ as a lowest K type. The mapping between these sets is provided by the derived functor module map*

$$\mathcal{L}_s(\lambda_{su}) :\ (\mathfrak{g}_{su}, G_{su} \cap K) \text{ modules } \rightarrow (\mathfrak{g}, K) \text{ modules}$$

   of Definition 17. Here

$$s = s(\lambda_{su}) = \dim(\mathfrak{u}_{su} \cap \mathfrak{k}) . \tag{21}$$

3. *Moreover, if X is an irreducible Hermitian $(\mathfrak{g}, K)$ module with lowest K type μ corresponding to an irreducible Hermitian $(\mathfrak{g}_{su}, G_{su} \cap K)$ module X_{su} with lowest $G_{su} \cap K$ type μ_{su}, then:*
   (a) *For every representation η_{su} of $G_{su} \cap K$ in $B_u^{\lambda_u}(G_{su})$, the K type $\eta = \mathcal{L}_s^K(\lambda_{su})(\eta_{su})$ is nonzero.*
   (b) *The signature of the Hermitian form on X_{su} on a representation η_{su} in $B_u^{\lambda_u}(G_{su})$ is equal to the signature on η of the form on X.*

We will give the proof in Section 5.

There is one respect in which this theorem is not as clean as the one in [3]. There we had an identification of *all* representations of G attached to λ_u with all representations of G_u attached to λ_u. This theorem identifies just *some* of the representations of G attached to λ_u with those of G_{su} attached to λ_u. The case of $SL(2,\mathbb{R})$ illustrates what is going on.

Example 20. Suppose G is $SL(2,\mathbb{R})$ and $\lambda_u = 0$. The unitary representations of G corresponding to $\lambda_u = 0$ are the unitarily small representations of G: those with lowest K types $\mu = 0, \pm 1, \pm 2$. For these unitary representations the main theorem of [3] provides no information. However, if $\mu = 2$, then $G_{su} = T$ and $\mu_{su} = 0$. Theorem 19 says that we have a bijection between the set of irreducible, unitary, unitarily small representations of T with weight 0 (which is just the trivial representation of T) and the set of all unitary, unitarily small representations of G containing $\mu = 2$ as a lowest K type (which is just a single discrete series representation).

Notice, however, that we have only an inclusion of the unitary representations of T with $\lambda_u = 0$ (still just the trivial representation) into those of G with $\lambda_u = 0$ (which include also the principal series, the limits of discrete series, the complementary series, the trivial representation, and another discrete series representation).

4. Strictly unitarily small K types.

Definition 21. Let μ be the highest weight of a unitarily small K type (Definition 10).

1. We say that μ is strictly unitarily small, or strictly small if $G_{su}(\mu) = G$. We denote by $B^z_{su}(G)$, the set of strictly small K types of G.
2. If $\lambda \in \Lambda_u \cap \mathfrak{z}$ (see (9) and (14)), we denote by $B^{\lambda}_{su}(G)$ the set of strictly unitarily small K types with central character λ. If G is semisimple, then we denote $B^z_{su}(G) = B^0_{su}(G)$ simply by $B_{su}(G)$.

Example 22. $B^0_{su}(SL(2,\mathbb{R})) = \{0, \pm 1\}$.

Example 23. $B^z_{su}(U(1,1)) = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} | \ |a-b| \leq 1\}$.

Example 24. $B^0_{su}(SO_e(4,1)) = \{(0,0)\}$.

Example 25. $B^0_{su}(Sp(4,\mathbb{R})) = \{(a,b) | 2 \geq a \geq b \geq -2; a-b \leq 3\}$.

We collect here a series of results on projections onto C (3) and unitarily small K types that we will need. They are all proved in [3].

Proposition 26. *Let the notation be as in* (2), (3), (4) *and* (14). *Denote by* $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ *the set of simple roots for* $\Delta^+(\mathfrak{g},\mathfrak{t})$ *and let* $\{\xi_i\}$ *be the dual basis of fundamental weights. Then:*

1. *Denote by* C^0 *the dual cone of* C*:* $C^0 = \{\zeta \in C \mid \langle \zeta, \nu \rangle \geq 0,$ *for all* $\nu \in C\}$. *Then*

$$C = \mathfrak{z}_0 + \sum_{i=1}^{l} \mathbb{R}^{\geq 0} \xi_i,$$

$$C^0 = \sum_{i=1}^{l} \mathbb{R}^{\geq 0} \alpha_i.$$

2. *Suppose γ and δ are dominant weights in C and v is any weight in $i\mathfrak{t}_0^*$. Then*

$$P(\nu-\gamma-\delta) = P(P(\nu-\gamma)-\delta).$$

Proof. 1 is obvious from the definitions and 2 is Corollary 1.6 in [3]. □

Proposition 27. *With the notation as in (14) and Definition 10, suppose $\mu \in \widehat{T}$ is a highest weight of a representation of K. Let λ_a and λ_u be as in (6) and (7) and μ_s and μ_z as in (15).*

1. *$\lambda_u = \lambda_u^{G_s}(\mu_s) + \mu_z$. Here $\lambda_u^{G_s}(\mu_s)$ is the weight associated to μ_s, with respect to G_s, as a $G_s \cap K$ representation.*
2. *The following are equivalent:*
   (a) *μ is a highest weight of a unitarily small K type.*
   (b) *$\lambda_u = \mu_z$.*
   (c) *$\lambda_a \in \mu_z + \langle W \cdot \rho\rangle$.*
   (d) *Let $\Delta^+(\mathfrak{g},\mathfrak{t})$ be a choice of roots positive on $\mu + 2\rho_c$, Π, the set of simple roots of $\Delta^+(\mathfrak{g},\mathfrak{t})$ and $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{t})}\alpha$. Then*

$$\lambda_a = \mu_z + \rho - \sum_{\substack{\alpha\in\Pi\\ c_\alpha\geq 0}} c_\alpha \alpha.$$

Proof. 1 is Lemma 6.5 in [3]; 2 (a)–(c) is Theorem 6.7 (a)–(c) of the same paper and 2 (d) follows from Proposition 1.7 (a) and (b), again from the same reference. □

The next lemma will be a consequence of Proposition 27 and Definition 2. We need the following

Definition 28 (see [3, (2.3a)])**.** For weights $\lambda, \gamma \in i\mathfrak{t}_0^*$ we say that γ is a singularization of λ if for all $\alpha \in \Delta(\mathfrak{g},\mathfrak{t})$,

$$\langle\gamma,\alpha\rangle > 0 \Rightarrow \langle\lambda,\alpha\rangle > 0.$$

Lemma 29. *Let $\mu \in i\mathfrak{t}_0^*$ be the highest weight of a K-representation, λ_a and λ_u as in Definition 2, G_{su} and λ_{su} as in (16) and Remark 4. Denote by $W_{\mathfrak{g}_{su}}$ the Weyl group of the roots of $\mathfrak{t}$ in $\mathfrak{g}_{su}$. Write (approximately as in Proposition 27 (d))*

$$\lambda_a = \lambda_u + \rho - \sum_{\substack{\alpha\in\Pi\\ c_\alpha\geq 0}} c_\alpha \alpha.$$

(This will be explained in more detail in (22) and (23) below.) Then

1. *λ_{su} is a singularization of λ_a (Definition 28).*
2. *With the notation as above,*

$$\mathfrak{g}_{su} = \mathfrak{h} + \sum_{\beta\in Span\{\alpha | c_a > 0\}} \mathbb{C}X_\beta.$$

3. *λ_a belongs to the interior of the polygon $\lambda_u + \rho(\mathfrak{u}_{su}) + \langle W(\mathfrak{g}_{su},\mathfrak{t})\cdot\rho(\mathfrak{g}_{su})\rangle$. This is a boundary facet of the polygon $\lambda_u + \rho(\mathfrak{u}_u) + \langle W(\mathfrak{g}_u,\mathfrak{t})\cdot\rho(\mathfrak{g}_u)\rangle$.*
4. *The K type μ is strictly unitarily small—that is, $\mathfrak{g}_{su} = \mathfrak{g}$—if and only if λ_a belongs to the interior of the polygon $\mu_z + \langle W\cdot\rho\rangle$. In particular, μ_{su} is strictly unitarily small for G_{su}.*

Proof. Recall that (see (4) and Definitions 2 and 4)

$$\lambda_a = P(\mu + 2\rho_c - \rho),$$
$$\lambda_{su} = P(\mu + 2\rho_c - t\rho)$$

for some $1 < t < 2$, close to 2.

Then, by Proposition 1.5 in [3],

$$\begin{aligned} \lambda_{su} &= P(P(\mu + 2\rho_c - \rho) - (t-1)\rho) \\ &= P(\lambda_a - (t-1)\rho). \end{aligned} \tag{22}$$

Now assume that α is a positive root, but $\langle \lambda_a, \alpha \rangle \leq 0$. Then Lemma 1.3 in [3] implies that

$$\langle P(\lambda_a - (t-1)\rho), \alpha \rangle = \langle \lambda_{su}, \alpha \rangle = 0.$$

This proves 1.

Now, by Proposition 1.1 in [3], for every $t \in [1,2]$ there are constants $c_\alpha(t) \geq 0$ so that

$$\lambda_t = \lambda_a - (t-1)\rho + \sum_{\substack{\alpha \in \Pi \\ c_\alpha \geq 0}} c_\alpha(t)\,\alpha, \tag{23}$$

with the last sum orthogonal to λ_t. The constants c_α of the lemma are just $c_\alpha(2)$. Denote by $\mathfrak{g}_t = \mathfrak{g}(\lambda_t)$ the Lie algebra of $G(\lambda_t)$ from Definition 2. Then

$$\mathfrak{g}_t = \mathfrak{h} + \sum_{\langle \lambda_t, \alpha \rangle = 0} \mathbb{C} X_\alpha. \tag{24}$$

Set

$$\begin{aligned} \mathfrak{u}_t &= \sum_{\langle \lambda_t, \alpha \rangle > 0} \mathbb{C} X_\alpha, \\ \Delta(\mathfrak{g}_t, \mathfrak{t}) &= \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}) \,|\, \langle \lambda_t, \alpha \rangle = 0\}, \\ \Delta(\mathfrak{u}_t, \mathfrak{t}) &= \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}) \,|\, \langle \lambda_t, \alpha \rangle > 0\}. \end{aligned} \tag{25}$$

Then 2 of this lemma will follow from

Lemma 30. *With the notation as in* (23), (24) *and* (25), *then*

$$\Delta(\mathfrak{g}_t, \mathfrak{t}) = \Delta(\mathfrak{g}, \mathfrak{t}) \cap Span\,\{\alpha \in \Pi \,|\, c_\alpha(t') > 0,\ t' > t\}.$$

Moreover,

1. $c_\alpha(t)$ *is increasing for all* $\alpha \in \Pi$.
2. *If* $c_\alpha(t_0) > 0$ *for some* $t_0 \in [1,2]$, *then* $c_\alpha(t)$ *is strictly increasing on* $[t_0, 2]$.

We leave the proof of Lemma 30 for later. To finish 2 recall from Remark 4 that there is a nonempty interval $(s,2)$ so that for all $t \in (s,2)$, the weight λ_t determines $\mathfrak{g}_{su}$. This means that if α is a simple root of $\mathfrak{t}$ in $\mathfrak{g}_{su}$, then $\alpha \in \Delta(\mathfrak{g}_t, \mathfrak{t})$ for $t \in (s,2)$. From Lemma 30, $c_\alpha > 0$, since $c_\alpha(t) > 0$ for all $t > s$. Conversely, if $c_\alpha > 0$, Lemma 30 also says that for $t \in (s,2)$, $\langle \lambda_t, \alpha \rangle = 0$ so that $\alpha \in \Delta(\mathfrak{g}_{su}, \mathfrak{t})$. Now 2 follows.

We will now prove 3 of Lemma 29. By 2 of Lemma 29 we have

$$\lambda_a = \lambda_u + \rho - \sum_{\alpha \in \Pi(\mathfrak{g}_{su})} c_\alpha \alpha,$$

with $c_\alpha > 0$. In particular,

$$\lambda_a - \lambda_u \in \rho + Span\left(\Delta\left(\mathfrak{g}_{su}, \mathfrak{t}\right)\right).$$

Using the parabolic subalgebra $\mathfrak{q}_u = \mathfrak{g}_u + \mathfrak{u}_u$, we can write

$$\rho = \rho(\mathfrak{u}_u) + \rho(\mathfrak{g}_u).$$

On the other hand, by (3.2d) and the discussion following in [3],

$$\lambda_a - \lambda_u \in \rho(\mathfrak{u}_u) + \langle W(\mathfrak{g}_u) \cdot \rho(\mathfrak{g}_u)\rangle.$$

Now most of 3 follows by applying to $\mathfrak{g}_u$ to the following fact.

Claim 31. Suppose $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subseteq \mathfrak{g}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$. Let $W_\mathfrak{l} = W\left(\mathfrak{l}, \mathfrak{h}\right)$. Then

$$\langle W \cdot \rho\rangle \cap \left[\rho + Span\left(\Delta\left(\mathfrak{l}, \mathfrak{t}\right)\right)\right] = \langle W_\mathfrak{l} \cdot \rho\rangle = \rho\left(\mathfrak{u}\right) + \langle W_\mathfrak{l} \cdot \rho_\mathfrak{l}\rangle.$$

This is a boundary facet of the convex polygon $\langle W \cdot \rho\rangle$.

Proof of Claim. If $w \in W$,

$$\begin{aligned} w\rho &= \rho - \sum_{\substack{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{t}) \\ c_\alpha(w)=0,1}} c_\alpha(w)\alpha \\ &= \rho - \sum_{\substack{\beta\in\Delta^+(\mathfrak{l},\mathfrak{t}) \\ c_\beta(w)=0,1}} c_\beta(w)\beta - \sum_{\substack{\gamma\in\Delta(\mathfrak{u},\mathfrak{t}) \\ c_\gamma(w)=0,1}} c_\gamma(w)\gamma. \end{aligned}$$

The last sum is nonzero if and only if $w \notin W_\mathfrak{l}$. It follows first of all that the left-hand side in the claim contains the right-hand side. For the other inclusion, suppose that $\sum a_w w \cdot \rho$ is a convex combination in $\langle W \cdot \rho\rangle$. Then

$$\sum a_w w \cdot \rho = \rho - \sum_w \sum_{\beta\in\Delta^+(\mathfrak{l},\mathfrak{t})} a_w c_\beta(w)\beta - \sum_w \sum_{\gamma\in\Delta(\mathfrak{u},\mathfrak{t})} a_w c_\gamma(w)\gamma.$$

All the coefficients here are nonnegative, so this convex combination can belong to $\rho + Span\left(\Delta\left(\mathfrak{l}, \mathfrak{t}\right)\right)$ only if all the $a_w c_\gamma(w)$ are zero; that is, only if all the w appearing belong to $W_\mathfrak{l}$. This proves that the left side in the claim is contained in the right side, completing the proof of equality. The left side is the intersection of a convex polygon with an affine subspace containing a vertex (namely ρ). It is therefore a boundary facet, proving the claim.

To complete the proof of 3 in Lemma 29, it remains to show that λ_a is an interior point of the polygon. What we showed before Claim 31 was that

$$\lambda_a = \lambda_u + \rho\left(\mathfrak{u}_{su}\right) + \left[\rho(\mathfrak{g}_{su}) - \sum_{\alpha\in\Pi(\mathfrak{g}_{su})} c_\alpha \alpha\right]$$

with $c_\alpha > 0$. We are to show that the term in square brackets is in the interior of $\langle W(\mathfrak{g}_{su}) \cdot \rho(\mathfrak{g}_{su})\rangle$. Now λ_a is dominant, and the other terms on the right are orthogonal to the roots of $\mathfrak{g}_{su}$. It follows that the term in square brackets is dominant for $\mathfrak{g}_{su}$. So we can apply the following to $\mathfrak{g}_{su}$:

Claim 32. Suppose γ is a dominant weight of the form

$$\gamma = \rho - \sum_{\alpha\in\Pi} c_\alpha \alpha,$$

with all $c_\alpha > 0$. Then γ belongs to the interior of $\langle W \cdot \rho\rangle$.

Proof of Claim. We must show that γ belongs to no codimension one boundary face of $\langle W\cdot\rho\rangle$. Such a face is given by the intersection with the polytope of a hyperplane

$$\{\gamma' \mid \langle\gamma',\lambda'\rangle = \langle\rho',\lambda'\rangle\}.$$

Here ρ' is half the sum of some set of positive roots, and λ' is a fundamental weight for that set. Given such a ρ' and λ', let λ be the corresponding fundamental weight for Δ^+; suppose it corresponds to the simple root α_0. Then we compute

$$\begin{aligned}\langle\gamma,\lambda'\rangle &\le \langle\gamma,\lambda\rangle \qquad \text{(since } \gamma \text{ is dominant)}\\ &= \langle\rho - \textstyle\sum c_\alpha\alpha,\lambda\rangle\\ &= \langle\rho,\lambda\rangle - c_{\alpha_0}\\ &< \langle\rho,\lambda\rangle = \langle\rho',\lambda'\rangle.\end{aligned}$$

This shows that γ does not belong to the boundary face defined by λ', and proves the claim.

We now turn to the proof of 4. Suppose first that μ is strictly unitarily small. That λ_a is in the interior of the indicated polygon is a special case of 3. Conversely, suppose that $\mathfrak{g}_{su}\neq\mathfrak{g}$. If $\mathfrak{g}_u\neq\mathfrak{g}$, then we know from Proposition 27 2(c) that λ_a is not even in the closed polygon $\mu_z+\langle W\cdot\rho\rangle$. If $\mathfrak{g}_u=\mathfrak{g}$, then $\lambda_u=\mu_z$; so 3 says that λ_a is on a boundary facet of $\mu_z+\langle W\cdot\rho\rangle$. The facet is proper since $\mathfrak{g}_{su}\neq\mathfrak{g}$ by hypothesis.

The last assertion of 4 follows from 3 and the first part, by an argument such as the proof of Lemma 3.2 in [3].

This completes the proof of Lemma 29. ∎

We now proceed to prove Lemma 30.

Proof. Since the last sum in (23) is orthogonal to λ_t, then

$$0=\sum c_\alpha(t)\langle\lambda_t,\alpha\rangle$$

but $c_\alpha(t)\ge 0$ and $\langle\lambda_t,\alpha\rangle\ge 0$. Hence,

$$c_\alpha(t)>0 \Longrightarrow \langle\lambda_t,\alpha\rangle = 0 \tag{26}$$

and α is a root in $\mathfrak{g}_t$. Moreover, α is also a root in $\mathfrak{g}_{t'}$, for all $t'\ge t$. Then (26) implies that $c_\beta(t)=0$ whenever $\langle\beta,\lambda_t\rangle=0$, so (23) becomes

$$\lambda_t=\lambda_a-(t-1)\rho+\sum_{\langle\alpha,\lambda_t\rangle=0} c_\alpha(t)\,\alpha.$$

We will be looking at what happens near a point $t_0\in[1,2]$. First write

$$\rho=\rho(\mathfrak{u}_{t_0})+\rho(\mathfrak{g}_{t_0})=\rho(\mathfrak{u}_{t_0})+\sum_{\langle\alpha,\lambda_t\rangle=0} m_\alpha\alpha,$$

with m_α positive. Then for any small $\varepsilon>0$,

$$\lambda_{t_0+\varepsilon}=\lambda_a-(t_0-1+\varepsilon)\,\rho+\sum c_\alpha(t_0+\varepsilon)\,\alpha.$$

On the other hand, the decomposition of ρ above gives

$$\begin{aligned}\lambda_{t_0} - \varepsilon\rho(\mathfrak{u}_{t_0}) &= \lambda_a - (t_0-1)\rho - \varepsilon\rho + \varepsilon\rho_{\mathfrak{g}_{t_0}} + \sum c_\alpha(t_0)\,\alpha \\ &= \lambda_a - (t_0 - 1 + \varepsilon)\,\rho + \sum_{\langle\alpha,\lambda_t\rangle=0} (c_\alpha(t_0) + m_\alpha\varepsilon)\,\alpha.\end{aligned}$$

Now, the last summand is orthogonal to both λ_{t_0} and $\rho(\mathfrak{u}_{t_0})$, and it belongs to the dual cone

$$C^0 = \{\gamma \in i\mathfrak{t}_0^* \,|\, \langle\gamma,\xi\rangle \geq 0, \text{ for all } \xi \in C\}$$

of C (see (3)). Whenever $\varepsilon \leq (\langle\lambda_{t_0},\beta\rangle / \langle\rho(\mathfrak{u}_{t_0}),\beta\rangle)$ (for all simple roots β that are positive on λ_{t_0}), the left-hand side is dominant. By Proposition 1.1 (c) in [3],

$$\lambda_{t_0+\varepsilon} = \lambda_{t_0} - \varepsilon\rho(\mathfrak{u}_{t_0}), \tag{27}$$

$$c_\alpha(t_0+\varepsilon) = c_\alpha(t_0) + m_\alpha\varepsilon \qquad (\langle\alpha,\lambda_t\rangle = 0), \tag{28}$$

$$c_\beta(t_0+\varepsilon) = 0 \qquad (\langle\beta,\lambda_t\rangle > 0). \tag{29}$$

These formulas establish Lemma 30. ☐

5. Proof of Theorem 19.

By the results in [3] (Theorems 5.4 (a)–(c) and 5.8 (a)), assuming Conjecture 18 holds for all Levi factors of θ-stable parabolic subalgebras of G, there is a bijection between unitary irreducible $(\mathfrak{g}, K)$ modules containing the K type with highest weight μ and irreducible unitary $(\mathfrak{g}_u, G_u \cap K)$ modules containing the (unitarily small) lowest $(G_u \cap K)$ type with highest weight μ_u (notation as in Theorem 19). Using this bijection, Theorem 19 can be reduced to the case when μ is unitarily small. Therefore, to prove the theorem we will only consider this case. To take care of one direction of the bijection we need the following

Proposition 33. *With the notation as in Theorem 19, suppose μ is unitarily small. For any $\eta_{su} \in B_u^{\lambda_u}(G_{su})$, write $\eta_{su} = (\eta_{su})_z + (\eta_{su})_s$. Then*

1. $\lambda_a^{G_{su}}(\eta_{su}) = (\eta_{su})_z + \lambda_a^{(G_{su})_s}(\eta_s)$.
2. $B_u^{\lambda_u}(G_{su}) = \left\{ \delta \in \widehat{(G_{su} \cap K)} \,\middle|\, \begin{array}{c}\delta \textit{ is unitarily small for } G_{su} \textit{ and the} \\ \textit{center of } G_{su}\cap K \textit{ acts by } (\mu_{su})_z \textit{ on } \delta\end{array} \right\}$.
3. *Set* $\lambda^G(\eta_{su}) = \lambda_a^{G_{su}}(\eta_{su}) + \rho(\mathfrak{u}_{su})$. *Then* $\langle\lambda^G(\eta_{su}),\alpha\rangle > 0$ *for all* α *in* $\Delta(\mathfrak{u}_{su},\mathfrak{t})$.
4. *The weight* $\eta = \eta_{su} + 2\rho(\mathfrak{u}_{su}\cap\mathfrak{s})$ *is dominant for* $\Delta^+(\mathfrak{k},\mathfrak{t})$ *and hence the highest weight of a representation of* K. *In particular, the map*

   $$\mathcal{L}_s^K(\lambda_{su}) : B_u^{\lambda_u}(G_{su}) \to B_u^{\lambda_u}(G)$$

   is never zero (see Theorem 19).
5. $\lambda^G(\eta_{su}) = \lambda_a(\eta)$.

Proof. 1 is immediate from the definitions. To prove 2 we use 2(a)–(c) of Lemma 27 applied to η_{su} and G_{su}. Then

$$(\eta_{su})_z = \lambda_u.$$

Now, in the proof of 4 in Lemma 29 we showed that

$$(\mu_{su})_z = \mu_z = \lambda_u.$$

This proves 2. Therefore, since $\eta_{su} = \lambda_u + (\eta_{su})_s$, then

$$\lambda^G(\eta_{su}) = \lambda_a^{G_{su}}((\eta_{su})_s) + \lambda_u + \rho(\mathfrak{u}_{su}),$$

λ_u is central in G, and hence it has no effect on the roots of $\mathfrak{u}_{su}$. So, to verify 3 we only need to check that

$$\left\langle \lambda_a^{G_{su}}((\eta_{su})_s) + \rho(\mathfrak{u}_{su}), \alpha \right\rangle > 0$$

for all α in $\Delta(\mathfrak{u}_{su}, \mathfrak{t})$.

For that we use the following.

Lemma 34. *Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subseteq \mathfrak{g}$ be any θ-stable parabolic subalgebra. Set $\rho_{\mathfrak{l}}$ as half the sum of some choice of positive roots $\Delta^+(\mathfrak{l}, \mathfrak{t})$ of $\mathfrak{t}$ in $\mathfrak{l}$. Denote by $W_{\mathfrak{l}}$ the Weyl group $W(\mathfrak{l}, \mathfrak{t})$ and let ν in $\langle W_{\mathfrak{l}} \cdot \rho_{\mathfrak{l}} \rangle$ (see Definition 20). Then*

$$\langle \nu + \rho(\mathfrak{u}), \alpha \rangle > 0$$

for all α in $\Delta(\mathfrak{u}, \mathfrak{t})$.

We leave the proof of this lemma for later. Now, since η_{su} is unitarily small for G_{su}, then $\lambda_a^{G_{su}}((\eta_{su})_s) \in \langle W_{\mathfrak{g}_{su}} \cdot \rho_{\mathfrak{g}_{su}} \rangle$ and using Lemma 34, 3 follows.

4 follows from 3 and from Lemma 6.3.23 of [4] and from Lemma 2.7 in [3]. Since by 4 η is dominant, then 5 follows from Lemma 6.5.4 in [4]. □

We turn now to the proof of Lemma 34.

Proof. Fix $\Delta^+(\mathfrak{g}, \mathfrak{t}) = \Delta^+(\mathfrak{l}, \mathfrak{t}) \cup \Delta(\mathfrak{u}, \mathfrak{t})$, so that $\rho = \rho_{\mathfrak{l}} + \rho(\mathfrak{u})$. Let $w \in W_{\mathfrak{l}} = W(\mathfrak{l}, \mathfrak{t})$. Then w preserves $\Delta(\mathfrak{u}, \mathfrak{t})$ and

$$w\rho = w\rho_{\mathfrak{l}} + \rho(\mathfrak{u}).$$

Hence $w\rho$ is strongly dominant for $\Delta(\mathfrak{u}, \mathfrak{t})$. Then, any sum

$$\sum_{\substack{a_w \geq 0 \\ w \in W_{\mathfrak{l}}}} a_w (w\rho_{\mathfrak{l}} + \rho(\mathfrak{u})) = \sum_{\substack{a_w \geq 0 \\ w \in W_{\mathfrak{l}}}} a_w (w\rho_{\mathfrak{l}}) + (\rho(\mathfrak{u})) \sum_{\substack{a_w \geq 0 \\ w \in W_{\mathfrak{l}}}} a_w$$

is strongly dominant for $\Delta(\mathfrak{u}, \mathfrak{t})$. Now if $\sum_{a_w \geq 0} a_w = 1$, the right-hand side becomes

$$\sum_{\substack{a_w \geq 0 \\ w \in W_{\mathfrak{l}}}} a_w (w\rho_{\mathfrak{l}}) + \rho(\mathfrak{u})$$

and hence strongly dominant for $\Delta(\mathfrak{u}, \mathfrak{t})$. This proves Lemma 34 and concludes the proof of Proposition 33. □

Proposition 35. *In the setting of Theorem 19, suppose Z is an irreducible module for $(\mathfrak{g}_{su}, G_{su} \cap K)$, endowed with a Hermitian form, with lowest $G_{su} \cap K$ type $\mu_{su} \in B_u^{\lambda_u}(G_{su})$. Assume Conjecture 18 holds for all Levi factors of θ-stable parabolic subalgebras of G, then:*

1. *$\mathcal{L}_s(\lambda_{su})(Z) = X$ contains the K type with highest weight μ as a lowest K type.*
2. *The Hermitian form on Z induces one on X by Proposition 2.6 in* [3].
3. *If, in addition, Z is unitary, then X is irreducible and unitary.*
4. *$\mathcal{L}_s(\lambda_{su})$ preserves signatures on $B_u^{\lambda_u}(G_{su})$. In other words, for these K types, the signature of the form on X equals the signature of the form on Z.*

Proof. μ_{su} is unitarily small for G_{su} and hence Z is a unitarily small Harish-Chandra module. By Conjecture 18 the canonical real part $RE\left(\gamma_{su}\right)$ of the weight γ_{su} associated to the infinitesimal character of Z is in $\lambda_u + \langle W_{\mathfrak{g}_{su}} \cdot \rho_{\mathfrak{g}_{su}} \rangle$. Then by Lemma 34,

$$RE\left\langle \gamma_{su} + \rho\left(\mathfrak{u}_{su}\right), \alpha \right\rangle > 0$$

for all $\alpha \in \Delta\left(\mathfrak{u}_{su}\right)$. By 2.13 (e) of [3], $\mathcal{L}_s\left(\lambda_{su}\right)(Z)$ is irreducible and hence contains the lowest K type μ. This proves 1. Now, 2 and 3 follow also from Theorem 2.13 (c) and (f) in [3]; and 4 follows from Proposition 3.1 in [3] and Theorem 2.13 (d) in [3] □

Now for the other direction in Theorem 19 we first need the following.

Proposition 36 ([3, Theorem 5.9])**.** *Let X be an irreducible Hermitian Harish-Chandra module of G, $\mathfrak{h}$ a θ-stable Cartan subalgebra of $\mathfrak{g}$, and $\phi \in \mathfrak{h}^*$ a weight representing the infinitesimal character of X. Assume X is unitarily small (Definition 10). Let λ_u be a weight associated to X as in Definition 2 and $B_u^{\lambda_u}(G)$ the set of unitarily small K types with the same central character λ_u (Definition 10). Suppose further that the canonical real part $RE\phi$ of ϕ belongs to*

$$\lambda_u + \langle W \cdot \rho \rangle .$$

Then, X is unitary if and only if the Hermitian form on X is positive definite on $B_u^{\lambda_u}(G)$.

Proposition 37. *With assumptions as in Theorem 19, suppose X is an irreducible unitary Harish-Chandra module of G with unitarily small lowest K type μ. Then:*

1. *There is a unique irreducible Hermitian $(\mathfrak{g}_{su}, G_{su} \cap K)$ module Z containing a lowest $G_{su} \cap K$ type μ_{su} so that X is the unique irreducible subquotient of $\mathcal{L}_s\left(\lambda_{su}\right)(Z)$.*
2. *The signature of the form on Z is strictly positive on all of $B_u^{\lambda_u}\left(G_{su}\right)$.*
3. *The real part of the infinitesimal character of Z lies in $\lambda_u + \langle W_{\mathfrak{g}_{su}} \cdot \rho_{\mathfrak{g}_{su}} \rangle$.*
4. *Z is unitary.*

Proof. 1 and 2 again follow from (b), (c) and (d) of Theorem 2.13 and Proposition 3.1 in [3]. 3 follows from 2 and Conjecture 18. Now, 4 follows from Proposition 36. □

Now Propositions 35 and 37 together prove Theorem 19.

References

[1] J. Carmona, Sur la Classification des Modules Admissibles Irréductibles, in *Non-commutative Harmonic Analysis and Lie Groups.* (J. Carmona and M. Vergne, eds.), 11–34, Lecture Notes in Mathematics **1020**, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983. MR **85i:**22022

[2] A. Knapp and D. A. Vogan Jr., *Cohomological Induction and Unitary Representations*, Princeton University Press, Princeton, New Jersey, 1995. MR **96c:**22033

[3] S. A. Salamanca-Riba and D. A. Vogan, Jr., On the Classification of Unitary Representations of Reductive Lie Groups, in *Ann. of Math*, **148** (1998), 1067–1133. MR **2000d:**22017

[4] D. A. Vogan, Jr., *Representations of Real Reductive Lie Groups*, Birkhäuser, 1981. MR **83c:**22022

Department of Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico 88003-0001

E-mail address: ssalaman@nmsu.edu

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

E-mail address: dav@math.mit.edu