The Generalized Harish-Chandra Homomorphism for Hecke Algebras of Real Reductive Lie Groups

by

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Abstract

For complex reductive Lie algebras $\mathfrak{g}$, the classical Harish-Chandra homomorphism allows to link irreducible finite dimensional representations of $\mathfrak{g}$ to those of certain subalgebras $\mathfrak{l}$. The Casselman-Osborne theorem establishes an extension of this link to infinite dimensional irreducible representations. In this paper we present a generalized Harish-Chandra homomorphism construction for Hecke algebras, and establish the corresponding generalized Casselman-Osborne theorem. This homomorphism can be used to link representations of $(\mathfrak{g}, L \cap K)$-pairs to those of $(\mathfrak{l}, L \cap K)$-pairs, where $\mathfrak{l}$ is a certain subalgebra of $\mathfrak{g}$ as in the classical case. Since representations of such pairs are closely related to those of the underlying Lie group G, this construction is a good first approximation to lifting the Harish-Chandra homomorphism from the Lie algebra to the Lie group level.

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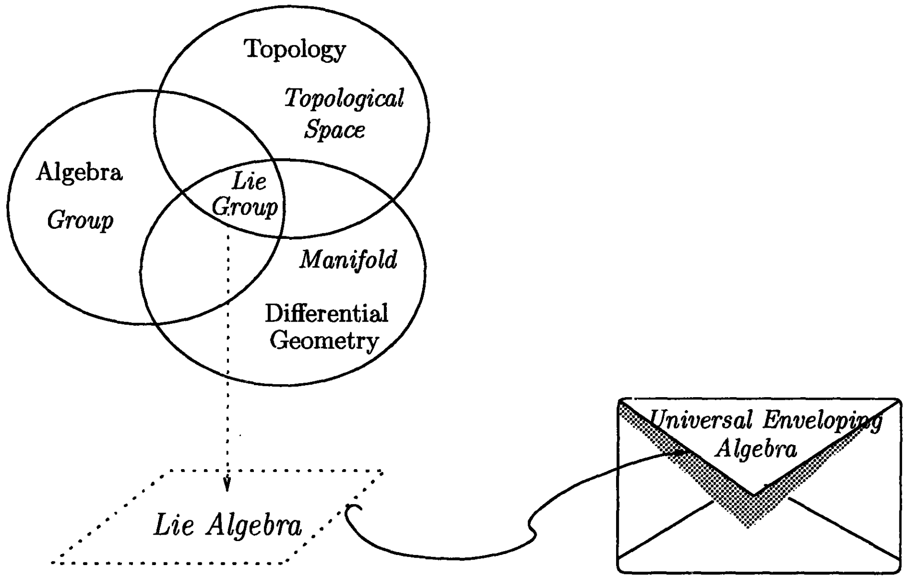
1 Introduction

Lie groups sit in the center of mathematics. Lie algebras are a linearization of Lie groups. Universal enveloping algebras associated to Lie algebras often provide an even more convenient algebraic setting.



Lie algebras associated to a Lie group capture much of the group structure and its local properties as a manifold. Because their underlying structure is that of a vector space, they are easier to study than their group counterparts. Lie algebras are an important tool in understanding Lie groups, and universal enveloping algebras are an important tool in understanding Lie algebras.

However, some properties of a Lie group are not reflected in its Lie algebra. In these cases, one often attempts to extend the construction used for an analogous question on the Lie algebra level to the Lie group. This paper presents work on one such extension, one regarding representations.

A representation of a Lie group or a Lie algebra is a vector space together with a homomorphism from the group or algebra to the endomorphism space of the vector space. Representations are easier to study than the original group because of the convenient linear structure of the underlying vector space. As a trade-off, a representation may not reflect all of the properties of a group we might be interested in. Instead of the group itself we only observe the action of its "shadow", that is the image of the group in the endomorphism space, on the vector space. However, just as one can learn much about an object by considering its shadows from many different angles, one can learn much about a Lie group or Lie algebra by studying their representations. This is why we care about representations, and their classification.

The classical Harish-Chandra homomorphism is a useful tool in such a classification for it allows us to relate finite dimensional irreducible representations of a Lie algebra to those of certain subalgebras. It turns out that the representation of the Lie subalgebra is the zero-degree Lie algebra cohomology. The Casselman-Osborne theorem generalizes the relation to infinite dimensional representations by extending the statement to higher cohomology degrees.

In this paper, we are interested in establishing similar tools and results for general (meaning possibly disconnected) real reductive Lie groups. As we will discuss in more detail in section (5), the problem in doing so is that the group does not posses the linear structure that enables the construction of the Harish-Chandra homomorphism in the Lie algebra case. A compromise is needed which reflects all the relevant properties of the group and simultaneously has enough algebraic structure to allow an analogous construction. The compromise is two-fold. For one, we will study pairs $(\mathfrak{g}, L \cap K)$ of the group's Lie algebra and compact subgroup $L \cap K$ instead of the group G itself. Their representations while not in one-to-one correspondence are intimately related. The framework for the second part of the compromise will be the algebra of distributions on the group. The specific object of study will be certain subalgebras of the distribution algebra - so called Hecke algebras, which will take the place of the universal enveloping algebra in the classical construction. For these pairs $(\mathfrak{g}, L \cap K)$ and Hecke algebras we will present a Harish-Chandra homomorphism construction.

The presentation is structured as follows.

	zero-degree cohomology	higher-degree cohomology
classical Lie algebra construction	section (2)	section (4)
$(\mathfrak{g}, K)$-pair construction	section (7)	section (8)

First we present the classical Harish-Chandra-construction, and some of its implications for representations. In this, we mostly follow [7]. We then review some definitions and properties of Lie algebra cohomology. In particular, we will see how Lie algebra cohomology relates to representations, and how the classical Harish-Chandra-construction is a statement about zero-degree Lie algebra cohomology. Next, using higher degree cohomologies, we exhibit the extension to infinite dimensional representations via the Casselman-Osborne theorem, more or less as in [9]. In section (6) we set the stage for the generalization by providing a short introduction to pairs, distributions, Hecke algebras, and their properties. The following sections (7) and (8) are used to present the generalizations of the Harish-Chandra homomorphism. As in the classical case, we first discuss the zero-degree cohomology case, and then argue similarly as in the Casselman-Osborne theorem to extend it to higher degree cohomologies. We conclude with some remarks on some open questions, and possible directions of further research.

Wherever possible, we provide a reference for the statements and claims we make. These do not so much indicate the original source of that statement or claim, as a starting point for the interested reader who would like to follow up or pursue this statement and its context further. However, for some statements we could not find an appropriate reference. The lack of such reference does not imply that the statement is trivial. In many cases it means on the contrary, that the statement is part of mathematical folklore, but it is so subtle that no-one has yet written it up clearly.

Also, we use footnotes to a larger extent than is common practice. We found their use convenient in separating the main story line from side comments, generalizations, or painful details. We hope the reader finds them as convenient.

2 The Classical Harish-Chandra Homomorphism

In this section we review the Harish-Chandra homomorphism associated to a complex reductive Lie algebra. For our purposes, one can consider a reductive Lie algebra to be the direct sum of a semisimple Lie algebra and an abelian Lie algebra. The semisimple summand is isomorphic to the derived subalgebra, and the abelian summand is isomorphic to the center of the reductive Lie algebra. In other words, given a complex reductive Lie algebra $\mathfrak{g}$

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z(\mathfrak{g})$$

as Lie algebras, $[\mathfrak{g}, \mathfrak{g}]$ is semisimple, and the center $Z(\mathfrak{g})$ is abelian.[1]

In the construction of the Harish-Chandra homomorphism we will have to use some results from the structure theory of semisimple Lie algebras. We state these results here without proof.

Proposition 2.1 If $\mathfrak{g}'$ is a complex semisimple Lie algebra, then $\mathfrak{g}'$ has a decomposition

$$\mathfrak{g}' = \mathfrak{u}^- \oplus \mathfrak{l}' \oplus \mathfrak{u} \qquad \text{as Lie algebras}$$

[1]statement 1.56 in [7]

such that

a) $\mathfrak{q}^- = \mathfrak{u}^- \oplus \mathfrak{l}'$ and $\mathfrak{q} = \mathfrak{l}' \oplus \mathfrak{u}$ are parabolic subalgebras of $\mathfrak{g}$

b) $\mathfrak{u}$ is an ideal in $\mathfrak{q}$, and $\mathfrak{u}^-$ is an ideal in $\mathfrak{q}^-$

c) $[\mathfrak{l}', \mathfrak{u}] \neq \{0\}$, and $[\mathfrak{l}', \mathfrak{u}^-] \neq \{0\}$. [2]

Moreover, there exists an element $H \in \mathfrak{l}'$ such that all eigenvalues of the adjoint action of H on $\mathfrak{g}'$ are real valued, and

d) $\mathfrak{u}$ is the sum of eigenspaces of $\mathrm{ad}(H)$ for positive eigenvalues
$\mathfrak{l}'$ is the eigenspace of $\mathrm{ad}(H)$ for the eigenvalue 0
$\mathfrak{u}$ is the sum of eigenspaces of $\mathrm{ad}(H)$ for negative eigenvalues. [3]

For a complex reductive Lie algebra $\mathfrak{g}$ we can write

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z(\mathfrak{g}) = \mathfrak{u}^- \oplus \mathfrak{l}' \oplus \mathfrak{u} \oplus Z(\mathfrak{g}) = u^- \oplus (\mathfrak{l}' \oplus Z(\mathfrak{g})) \oplus \mathfrak{u}.$$

By setting $\mathfrak{l} = \mathfrak{l}' \oplus Z(\mathfrak{g})$ we obtain

$$\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}.$$

This is a decomposition similar to the one described above for semisimple Lie algebras. The above stated properties, in particular the existence of the element H, carry over to this decomposition of reductive Lie algebras.

This decomposition is often referred to as *triangular decomposition.* The underlying reason is that for example for $\mathfrak{g} = \mathfrak{sl}_n = \{X \in \mathfrak{gl}_n \mid tr(X) = 0\}$ (and other classical Lie algebras) in sloppy notation

$$\begin{aligned}
\mathfrak{sl}_n &= \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n} \\
&= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \star & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \star & \cdots & \star & 0 \end{pmatrix} \oplus \begin{pmatrix} \star & 0 & \cdots & 0 \\ 0 & \star & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 0 & \star \end{pmatrix} \oplus \begin{pmatrix} 0 & \star & \cdots & \star \\ \vdots & \ddots & & \star \\ 0 & \cdots & 0 & \star \\ 0 & \cdots & 0 & 0 \end{pmatrix}
\end{aligned}$$

[2]statement 5.94 in [7]
[3]statement 5.101 in [7]

is a decomposition with the above listed properties. For example, for $\mathfrak{sl}_2$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is an element as described by above proposition (2.1). It will be useful to keep these matrices in mind as hands-on example.

The triangular decomposition lifts to universal enveloping algebras.

Proposition 2.2 Let $\mathfrak{g}$ be a complex reductive Lie algebra, and $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}$ a triangular decomposition. Then the universal enveloping algebras of the summands satisfy

$$\mathcal{U}(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{u}^-) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{l}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{u}) \quad \text{as vector spaces.} \tag{1}$$

Proof: We recall that the Poincarè-Birkhoff-Witt theorem states: If $\{X_i\}_{i\in A}$ is a basis of $\mathfrak{g}$ for some index set A with a total ordering, then the set of monomials

$$X_{i_1}^{i'_1} X_{i_2}^{i'_2} \cdots X_{i_n}^{i'_n}$$

with $i_1 < i_2 < \ldots < i_n$ for all k form a basis of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$[4].

Now let $\{Y_i\}$, $\{Z_j\}$, and $\{X_k\}$ be a basis of $\mathfrak{u}^-$, $\mathfrak{l}$, and $\mathfrak{u}$, respectively. The union of these sets forms a basis for $\mathfrak{g}$. Applying the Poincarè-Birkhoff-Witt theorem to this basis of $\mathfrak{g}$, establishes the homomorphism generated by

$$Y_{i_1}^{i'_1} Y_{i_2}^{i'_2} \cdots Y_{i_n}^{i'_n} Z_{j_1}^{j'_1} Z_{j_2}^{j'_2} \cdots Z_{j_m}^{j'_m} X_{k_1}^{k'_1} X_{k_2}^{k'_2} \cdots X_{k_n}^{k'_n} \mapsto Y_{i_1}^{i'_1} Y_{i_2}^{i'_2} \cdots Y_{i_n}^{i'_n} \otimes Z_{j_1}^{j'_1} Z_{j_2}^{j'_2} \cdots Z_{j_m}^{j'_m} \otimes X_{k_1}^{k'_1} X_{k_2}^{k'_2} \cdots X_{k_n}^{k'_n}$$

as vector space isomorphism. □

For our subsequent discussion, the emphasis of this proposition is on establishing a basis of the algebra $\mathcal{U}(\mathfrak{g})$ in terms of the bases of $\mathcal{U}(\mathfrak{u}^-)$, $\mathcal{U}(\mathfrak{l})$ and $\mathcal{U}(\mathfrak{u})$, respectively. Having established this basis, we can rewrite $\mathcal{U}(\mathfrak{g})$ as in the following statement:

Corollary 2.3 Under the assumptions of the previous proposition, $\mathcal{U}(\mathfrak{g})$ can be expressed as the direct sum of spaces spanned by the different types of monomials

$$\begin{aligned} \mathcal{U}(\mathfrak{g}) &\cong \mathcal{U}(\mathfrak{l}) \oplus (\mathfrak{u}^- \cdot \mathcal{U}(\mathfrak{u}^- \oplus \mathfrak{l})) \oplus (\mathfrak{u}^- \cdot \mathcal{U}(\mathfrak{g}) \cdot \mathfrak{u}) \oplus (\mathcal{U}(\mathfrak{l} \oplus \mathfrak{u}) \cdot \mathfrak{u}) \\ &\cong \mathcal{U}(\mathfrak{l}) \oplus (\mathfrak{u}^- \cdot \mathcal{U}(\mathfrak{q}^-)) \oplus (\mathfrak{u}^- \cdot \mathcal{U}(\mathfrak{g}) \cdot \mathfrak{u}) \oplus (\mathcal{U}(\mathfrak{q}) \cdot \mathfrak{u}) \end{aligned} \tag{2}$$

for $\mathfrak{q}^- = \mathfrak{u}^- \oplus \mathfrak{l}$, and $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. Equivalently,

$$\mathcal{U}(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{l}) \oplus (\mathcal{U}(\mathfrak{g}) \cdot \mathfrak{u} + \mathfrak{u}^- \cdot \mathcal{U}(\mathfrak{g})) \tag{3}$$

where the multiplication is the multiplication in $\mathcal{U}(\mathfrak{g})$, of which $\mathfrak{u}$ and $\mathfrak{u}^-$ can be considered subalgebras.

[4]statement 3.8 in [7]

As we will see in a moment, this decomposition is preserved under the adjoint action of $\mathfrak{l}$ on $\mathcal{U}(\mathfrak{g})$. It is the starting point for the construction of the Harish-Chandra homomorphism, which is derived from the projection of $\mathcal{U}(\mathfrak{g})^{ad(\mathfrak{l})}$ to $\mathcal{U}(\mathfrak{l})^{ad(\mathfrak{l})}$. Before diving into the details of this construction, we want to briefly and rather informally lay out the scope and relevance of it.

Our interest in the classical Harish-Chandra homomorphism lies in the way it relates irreducible representations of $\mathfrak{g}$ to those of $\mathfrak{l}$. Recall that

Proposition 2.4 Irreducible representations of a Lie algebra $\mathfrak{g}$ stand in one-to-one correspondence with irreducible unital left $\mathcal{U}(\mathfrak{g})$ modules. [5]

Proposition 2.5 (Schur's Lemma) If (V, π) is an irreducible finite dimensional representation of a Lie algebra $\mathfrak{g}$, and $\varphi : V \to V$ is a linear map which commutes with the action of $\mathfrak{g}$, that is $\pi(X) \circ \varphi = \varphi \circ \pi(X)$ for all $X \in \mathfrak{g}$, then φ is a scalar. [6]

Corollary 2.6 If (V, π) is an irreducible finite dimensional representation of a Lie algebra $\mathfrak{g}$, then all elements in the center $\mathcal{Z}_{\mathfrak{g}}$ of $\mathcal{U}(\mathfrak{g})$ act by scalars on V.

This action by scalars induces a map $\chi_{\mathfrak{g}} : \mathcal{Z}_{\mathfrak{g}} \to \mathbb{C}$.

Proposition 2.7 An irreducible finite dimensional representation (V, π) of a complex reductive Lie algebra $\mathfrak{g}$ is uniquely determined up to isomorphism by the scalar action of the elements of $\mathcal{Z}_{\mathfrak{g}}$, that is by the map $\chi_{\mathfrak{g}}$.

Given a triangular decomposition $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}$ of a Lie algebra $\mathfrak{g}$, the Harish-Chandra homomorphism is a map $\xi : \mathcal{Z}_{\mathfrak{g}} \to \mathcal{Z}_{\mathfrak{l}}$ for which

$$\chi_{\mathfrak{g}} = \chi_{\mathfrak{l}} \circ \xi. \quad [7]$$

It thus relates irreducible representations of $\mathfrak{g}$ and of $\mathfrak{l}$.

2.1 Construction

We now return to the decomposition of the universal enveloping algebra. We previously established in corollary 2.3 that for $\mathfrak{q}^- = \mathfrak{u}^- \oplus \mathfrak{l}$, and $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$

$$\mathcal{U}(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{l}) \oplus \left(\mathfrak{u}^- \cdot \mathcal{U}(\mathfrak{q}^-)\right) \oplus \left(\mathfrak{u}^- \cdot \mathcal{U}(\mathfrak{g}) \cdot \mathfrak{u}\right) \oplus \left(\mathcal{U}(\mathfrak{q}) \cdot \mathfrak{u}\right).$$

[5] statement 3.6 in [7]

[6] statement 5.2 in [7]

[7] Note that this is not a precise statement, as we have not specified yet which representation of $\mathfrak{l}$ the map $\chi_{\mathfrak{l}}$ refers to. It will be made precise in section 2.2. below.

Elements of $\mathfrak{l} \subseteq \mathfrak{g}$, considered as a subalgebra of $\mathcal{U}(\mathfrak{g})$, act on elements in $\mathcal{U}(\mathfrak{g})$ by the adjoint action. We claim that this action respects the above direct sum of spans of monomials.

Proposition 2.8 Let $\mathfrak{g}$ be a complex reductive Lie algebra, $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}$ a triangular decomposition, and $\mathfrak{q}^- = \mathfrak{u}^- \oplus \mathfrak{l}$, and $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. Then the adjoint action of $\mathfrak{l}$ preserves the above decomposition of $\mathcal{U}(\mathfrak{g})$. In other words, for all $z \in \mathfrak{l}$

i) $\mathrm{ad}(z)\mathcal{U}(\mathfrak{l}) \subseteq \mathcal{U}(\mathfrak{l})$

ii) $\mathrm{ad}(z)\mathfrak{u}^- \cdot \mathcal{U}(\mathfrak{q}^-) \subseteq \mathfrak{u}^- \cdot \mathcal{U}(\mathfrak{q}^-)$

iii) $\mathrm{ad}(z)\mathfrak{u}^- \cdot \mathcal{U}(\mathfrak{g}) \cdot \mathfrak{u} \subseteq \mathfrak{u}^- \cdot \mathcal{U}(\mathfrak{g}) \cdot \mathfrak{u}$

iv) $\mathrm{ad}(z)\mathcal{U}(\mathfrak{q}) \cdot \mathfrak{u} \subseteq \mathcal{U}(\mathfrak{q}) \cdot \mathfrak{u}$.

Proof: These claims are direct implications of properties of the triangular decomposition. Recall from proposition 2.1, that $[\mathfrak{l}, \mathfrak{u}^-] \subseteq \mathfrak{u}^-$, $[\mathfrak{l}, \mathfrak{l}] \subseteq \mathfrak{l}$, and $[\mathfrak{l}, \mathfrak{u}] \subseteq \mathfrak{u}$. This implies by linearity of the commutator, that $[\mathfrak{l}, \mathfrak{q}^-] \subseteq \mathfrak{q}^-$, $[\mathfrak{l}, \mathfrak{q}] \subseteq \mathfrak{q}$. These inclusions carry over to the adjoint action of $\mathfrak{l}$ on the universal enveloping algebras of $\mathfrak{q}^-$, $\mathfrak{l}$, and $\mathfrak{q}$.[8] In other words, $[\mathfrak{l}, \mathcal{U}(\mathfrak{q}^-)] \subseteq \mathcal{U}(\mathfrak{q}^-)$, $[\mathfrak{l}, \mathcal{U}(\mathfrak{l})] \subseteq \mathcal{U}(\mathfrak{l})$, and $[\mathfrak{l}, \mathcal{U}(\mathfrak{q})] \subseteq \mathcal{U}(\mathfrak{q})$. The claims follow. □

Corollary 2.9 The above direct sum is preserved under passing to invariants[9] of the adjoint action of $\mathfrak{l}$ on $\mathcal{U}(\mathfrak{g})$, that is

$$\mathcal{U}(\mathfrak{g})^{ad(\mathfrak{l})} = \mathcal{U}(\mathfrak{l})^{ad(\mathfrak{l})} \oplus (\mathfrak{u}^- \cdot \mathcal{U}(\mathfrak{q}^-))^{ad(\mathfrak{l})} \oplus (\mathfrak{u}^- \cdot \mathcal{U}(\mathfrak{g}) \cdot \mathfrak{u})^{ad(\mathfrak{l})} \oplus (\mathcal{U}(\mathfrak{q}) \cdot \mathfrak{u})^{ad(\mathfrak{l})}.$$

[8] This follows from the following fact: If A is an associative algebra, and $[\ ,\ ]$ is the commutator on A defined by $[a_1, a_2] = a_1 a_2 - a_2 a_1$ for $a_1, a_2 \in A$, then for any $a_i, x \in A$

$$[a_1 a_2 ... a_n, x] = \sum_{i=1}^{n} a_1 ... a_{i-1} [a_i, x] a_i ... a_n.$$

[9] "Invariants" is one of those words in mathematics that means different things in different contexts. In this context, it refers to elements that are sent to 0 by the adjoint action of $\mathfrak{l}$. The motivation for calling such elements invariants comes from the corresponding group action for matrix groups. A linear Lie groups (or matrix group) acts on the elements of its Lie algebra by conjugation

$$\mathrm{Ad}(g)(X) = gXg^{-1} \qquad \text{for all } X \in \mathfrak{g}, g \in G.$$

The differentiated version of this action is the adjoint action of the Lie algebra on itself. For an element to be invariant under the adjoint action of the group means the group acts on it by identity. The differential of acting by identity is acting by zero.

It turns out that this sum can be simplified as two of the summands are trivial.[10]

Proposition 2.10 In the above setting,

$$(\mathfrak{u}^- \cdot \mathcal{U}(\mathfrak{q}^-))^{ad(\mathfrak{l})} = (\mathcal{U}(\mathfrak{q}) \cdot \mathfrak{u})^{ad(\mathfrak{l})} = \{0\}.$$

Proof: These inclusions are direct implications of the existence of an element $H \in \mathfrak{l}$, as indicated in proposition 2.1.

By proposition 2.1 there exists an $H \in \mathfrak{l}$ such that $\mathfrak{l}$ is the zero eigenspace, and $\mathfrak{u}$ is the space spanned by all eigenspaces corresponding to positive eigenvalues under the action of ad(H). Therefore, we can write any element in

$$\mathcal{U}(\mathfrak{q})\mathfrak{u} = \mathcal{U}(\mathfrak{l} \oplus \mathfrak{u})\mathfrak{u} = (\mathcal{U}(\mathfrak{l}) \otimes \mathcal{U}(\mathfrak{u}))\,\mathfrak{u}$$

as sum of elements of the form

$$Z_I X_{\lambda_{i_1}}^{i'_1} \dots X_{\lambda_{i_n}}^{i'_n},$$

with at least one $i'_j > 0$, and where $X_{\lambda_{i_j}}$ is in the eigenspace corresponding to the eigenvalue λ_{i_j}, and $Z_I \in \mathcal{U}(\mathfrak{l})$. Then for any element $ZX \in \mathcal{U}(\mathfrak{q})\mathfrak{u}$[11]

$$\begin{aligned} \text{ad}(z)ZX &= \sum_{I=\{i_1 \dots i_n\}} \sum_{j=1}^{n} [z, Z_I] X_{\lambda_{i_1}}^{i'_1} \dots X_{\lambda_{i_n}}^{i'_n} + X_{\lambda_{i_1}}^{i'_1} \dots [z, X_{\lambda_{i_j}}^{i'_j}] \dots X_{\lambda_{i_n}}^{i'_n} Z_I \\ &= \sum_{I=\{i_1 \dots i_n\}} \sum_{j=1}^{n} \lambda_{i_j}^{i'_j} Z_I X_{\lambda_{i_1}}^{i'_1} \dots X_{\lambda_{i_n}}^{i'_n}. \end{aligned}$$

Since all λ_{i_j} are positive by assumption, this last expression cannot be zero unless $ZX = 0$.[12]

Similarly, $\mathfrak{u}^-$ is spanned by eigenspaces corresponding to positive eigenvalues for the element $-H$. Hence an analogous argument holds. □

[10] This is not only a side comment. In fact, we will need the simplified equality in the proof of theorem (2.16).

[11] We use once more the fact: If A is an associative algebra, and $[\ ,\ ]$ is the commutator on A defined by $[a_1, a_2] = a_1 a_2 - a_2 a_1$ for $a_1, a_2 \in A$, then for any $a_i, x \in A$

$$[a_1 a_2 \dots a_n, x] = \sum_{i=1}^{n} a_1 \dots a_{i-1} [a_i, x] a_i \dots a_n.$$

[12] Note in particular that different summands can not cancel out, for their component in $\mathfrak{u}$ lies in different subspaces of $\mathfrak{u}$.

Corollary 2.11 In the above setting,

$$\mathcal{U}(\mathfrak{g})^{ad(\mathfrak{l})} = \mathcal{U}(\mathfrak{l})^{ad(\mathfrak{l})} \oplus \left(\mathfrak{u}^- \cdot \mathcal{U}(\mathfrak{g}) \cdot \mathfrak{u}\right)^{ad(\mathfrak{l})}.$$

This equation implies that the projection onto the first summand is a vector space homomorphism. Moreover, it is an algebra homomorphism.

Proposition 2.12 In the above setting, the projection map

$$P : \mathcal{U}(\mathfrak{g})^{ad(\mathfrak{l})} \to \mathcal{U}(\mathfrak{l})^{ad(\mathfrak{l})}$$

is an algebra homomorphism.

Proof: This follows directly from the decomposition. We need to show that for any two elements A and B in $\mathcal{U}(\mathfrak{g})^{ad(\mathfrak{l})}$, $P(A)P(B) = P(AB)$.

In more detail, let

$$\begin{aligned} A &= A_{\mathfrak{l}} + \sum_s Y_s A_s X_s \quad \text{and} \\ B &= B_{\mathfrak{l}} + \sum_t Y_t' B_t X_t' \end{aligned}$$

with $A_{\mathfrak{l}}, B_{\mathfrak{l}} \in \mathcal{U}(\mathfrak{l})$, $Y_s, Y_t' \in \mathfrak{u}^-$, $A_s, B_t \in \mathcal{U}(\mathfrak{g})$, and $X_s, X_t' \in \mathfrak{u}$ be the decompositions of two elements in $\mathcal{U}(\mathfrak{g})^{ad(\mathfrak{l})}$. Then $P(A)P(B) = A_{\mathfrak{l}}B_{\mathfrak{l}}$. On the other hand, we compute

$$AB = A_{\mathfrak{l}}B_{\mathfrak{l}} + \sum_s Y_s A_s X_s B_{\mathfrak{l}} + \sum_t A_{\mathfrak{l}} Y_t' B_t X_t' + \sum_{s,t} Y_s A_s X_s Y_t' B_t X_t'.$$

We briefly mentioned before, that if A is an associative algebra, and $[\ ,\ ]$ is the commutator on A defined by $[a_1, a_2] = a_1a_2 - a_2a_1$ for $a_1, a_2 \in A$, then for any $a_i, x \in A$

$$[a_1a_2...a_n, x] = \sum_{i=1}^{n} a_1...a_{i-1}[a_i, x]a_i...a_n.$$

In the context of this proposition, this fact together with $[\mathfrak{l}, \mathfrak{u}^-] \subseteq \mathfrak{u}^-$ and $[\mathfrak{l}, \mathfrak{u}] \subseteq \mathfrak{u}$ implies that $A_{\mathfrak{l}}Y_t'$ can be rewritten as $\sum_{p_t} Y_{p_t}' A_{p_t}^{\mathfrak{l}}$, and similarly $X_s B_{\mathfrak{l}}$ can be rewritten as $\sum_{q_s} B_{q_s}^{\mathfrak{l}} X_{q_s}'$ for some $Y_{p_t}' \in \mathfrak{u}^-$, $X_{q_s}' \in \mathfrak{u}$, and $A_{p_t}^{\mathfrak{l}}, B_{q_s}^{\mathfrak{l}} \in \mathcal{U}(\mathfrak{l})$.[13] In other words,

$$AB = A_{\mathfrak{l}}B_{\mathfrak{l}} + \sum_{s,q_s} Y_s A_s B_{q_s}^{\mathfrak{l}} X_{q_s}' + \sum_{t,p_t} Y_{p_t}' A_{p_t}^{\mathfrak{l}} B_t X_t + \sum_{s,t} Y_s A_s X_s Y_t B_t X_t.$$

[13]Both, the above quoted fact, and this implication can be proven by induction on n or the degree of the monomial, respectively.

As none of the last three sums can be non-zero and in $\mathcal{U}(\mathfrak{l})$, $P(AB) = A_{\mathfrak{l}}B_{\mathfrak{l}}$. This completes the proof. □

The final step in constructing the Harish-Chandra homomorphism is to note that $\mathcal{Z}_{\mathfrak{g}} = \mathcal{U}(\mathfrak{g})^{ad(\mathfrak{g})}$ is a subalgebra of $\mathcal{U}(\mathfrak{g})^{ad(\mathfrak{l})}$. So the above projection map descends to an algebra homomorphism

$$\boxed{\xi : \mathcal{Z}_{\mathfrak{g}} \to \mathcal{Z}_{\mathfrak{l}} = \mathcal{U}(\mathfrak{l})^{ad(\mathfrak{l})}} \tag{4}$$

This map is the *Harish-Chandra homomorphism associated to* $\mathfrak{g}$.

As a concluding remark regarding this construction, note that it was necessary to pursue this analysis on the level of universal enveloping algebras. While the analysis could similarly have been done using the Lie algebra $\mathfrak{g}$ and its triangular decomposition directly, the result would not have been as strong: if the reductive Lie algebra $\mathfrak{g}$ is semi-simple, its center is zero. In this case, the above map would just be the zero-map. The center of its universal enveloping algebra, however, cannot be zero. It always contains at least the elements of degree zero, and for reductive Lie algebras it also always contains the Casimir element.[14]. The latter is of degree two, and thus the center of the universal enveloping algebra is not only non-zero, but also non-trivial.

2.2 Relating Representations of Reductive Lie Algebras

We still need to formally establish the Harish-Chandra homomorphism's relation to representations, that is to make the earlier claim that

$$\chi_{\mathfrak{g}} = \chi_{\mathfrak{l}} \circ \xi$$

precise and prove it.

Let $\mathfrak{g}$ be a complex reductive Lie algebra, and $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}$ a triangular decomposition as before. Recall again that if (V, π) is a finite dimensional irreducible representation of $\mathfrak{g}$, then the center $\mathcal{Z}_{\mathfrak{g}}$ of the universal enveloping algebra of $\mathfrak{g}$ acts by scalars on V. This induces an algebra homomorphism, which we denote by

$$\begin{aligned} \chi_{\mathfrak{g}}^V : \mathcal{Z}_{\mathfrak{g}} &\to \mathbb{C} \\ z &\mapsto \lambda \qquad \text{such that } \pi(z).v = \lambda \cdot v \text{ for all } v \in V, z \in \mathcal{Z}_{\mathfrak{g}}. \end{aligned}$$

[14]statement 5.24 in [7]

We will prove below, that for finite dimensional irreducible representations this map $\chi_{\mathfrak{g}}^V$ uniquely determines the representation V.

To relate irreducible representations of $\mathfrak{g}$ and $\mathfrak{l}$, we would also like $\chi_{\mathfrak{l}}$ to correspond to an irreducible representation of $\mathfrak{l}$. The space V itself may not be irreducible as a representation of $\mathfrak{l}$, but V contains an irreducible $\mathfrak{l}$-sub-representation, namely

$$V^{\mathfrak{u}} = \{v \in V | \pi(X).v = 0 \quad \forall X \in \mathfrak{u}\}.$$

This is a representation of $\mathfrak{l}$ since $[\mathfrak{l}, \mathfrak{u}] \subseteq \mathfrak{u}$, and hence a (unital left) $\mathcal{U}(\mathfrak{l})$-module.

The precise version of our main equation is

$$\pi(z).v = \pi|_{V^{\mathfrak{u}}}(\xi(z)).v \qquad \forall z \in \mathcal{Z}_{\mathfrak{g}}, v \in V^{\mathfrak{u}} \tag{5}$$

or equivalently

$$\chi_{\mathfrak{g}}^V = \chi_{\mathfrak{l}}^{V^{\mathfrak{u}}} \circ \xi \qquad \forall z \in \mathcal{Z}_{\mathfrak{g}} \tag{6}$$

We will conclude this section by proving these three claims, and some remarks.

Proposition 2.13 A finite dimensional irreducible representation V of $\mathfrak{g}$ is uniquely determined by $\chi_{\mathfrak{g}}^V$.[15]

Proof: There are various ways to prove this statement. Here is an almost purely algebraic version. Its crucial ingredient is the Jacobson-Chevalley Density theorem[16].

Elements in $\mathfrak{g}$ act on the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ by the adjoint action. This action preserves the degree of elements in $\mathcal{U}(\mathfrak{g})$.[17] Therefore, monomials (with respect to a fixed basis of $\mathfrak{g}$) of a fixed degree span invariant subspaces. These subspaces are finite dimensional, and therefore completely reducible. It follows that under this adjoint action the universal enveloping algebra can be written as the direct sum of irreducible subspaces:

$$\mathcal{U}(\mathfrak{g}) =_{\text{ad}} \bigoplus_{\substack{\text{a finite dim'l} \\ \text{irred. repr } \tau}} m(\tau)\tau \;=\; m(\mathbb{C})\mathbb{C} \oplus \bigoplus_{\tau \neq \mathbb{C}} m(\tau)\tau$$

[15]Note that this is only true for scalar actions of the center $\mathcal{Z}_{\mathfrak{g}}$ of the universal enveloping algebra that arrive from a finite dimensional irreducible representation of $\mathfrak{g}$. The algebra $\mathcal{Z}_{\mathfrak{g}}$ can also act by scalars on vector spaces in ways that do not correspond to a representation of $\mathfrak{g}$.

[16]For more details see, for example, statement 11.16 in [12].

[17]This is true, because as mentioned earlier in any associative algebra A, where the commutator $[a_1, a_2] = a_1a_2 - a_2a_1$ for $a_1, a_2 \in A$, for any $a_i, x \in A$

$$[a_1a_2...a_n, x] = \sum_{i=1}^{n} a_1...a_{i-1}[a_i, x]a_i...a_n.$$

where $m(\tau)$ is a certain multiplicity factor. Under this identification, the center $\mathcal{Z}_{\mathfrak{g}}$ of $\mathcal{U}(\mathfrak{g})$ corresponds to the trivial representation $m(\mathbb{C})\mathbb{C}$. Now let (V,π) and (V',π') be two inequivalent irreducible $\mathcal{U}(\mathfrak{g})$-modules, such that elements in $\mathcal{Z}_{\mathfrak{g}}$ act by the same scalar on both spaces. Then the Jacobson-Chevalley Density theorem implies that there exists a $X \in \mathcal{U}(\mathfrak{g})$ such that $\pi(X) = \mathrm{id}$, and $\pi'(X) = 0$.

According to the above decomposition of $\mathcal{U}(\mathfrak{g})$, X can be uniquely written as a sum of an element $X_Z \in m(\mathbb{C})\mathbb{C} \cong \mathcal{Z}_{\mathfrak{g}}$ and $X_o \in \bigoplus_{\tau \neq \mathbb{C}} m(\tau)\tau$. Then $Tr(\pi(X_o)) = Tr(\pi'(X_o)) = 0$ because all such elements are commutators, and the trace on commutators is in general zero. Thus on one hand

$$Tr(\pi(X)) = Tr(\pi(X_Z)) + Tr(\pi(X_o)) = Tr(\chi_{\mathfrak{g}}^{V}(X_Z)1) = \dim(V) \cdot \chi_{\mathfrak{g}}^{V}(X_Z)$$

and similarly

$$Tr(\pi'(X)) = Tr(\pi'(X_Z)) + Tr(\pi'(X_o)) = Tr(\chi_{\mathfrak{g}}^{V'}(X_Z)1) = \dim(V') \cdot \chi_{\mathfrak{g}}^{V'}(X_Z).$$

On the other hand

$$Tr(\pi(X)) = Tr(\mathbf{1}) = \dim(V) \neq 0$$

but

$$Tr(\pi'(X)) = Tr(\mathbf{0}) = 0.$$

This contradicts the assumption $\chi_{\mathfrak{g}}^{V}(X_Z) = \chi_{\mathfrak{g}}^{V'}(X_Z)$. Therefore, $\chi_{\mathfrak{g}}^{V}$ uniquely determines a finite dimensional irreducible representation. □

The following is an auxiliary lemma we will use in the subsequent proposition.

Lemma 2.14 Let $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}$ be a triangular decomposition of the complex reductive Lie algebra $\mathfrak{g}$, and let H be an element in $\mathfrak{l}$ for which $\mathfrak{u}^-$,$\mathfrak{u}$, and $\mathfrak{l}$ are the spaces spanned by eigenspaces corresponding to negative, positive, and zero eigenvalues, respectively, under the action of $\mathrm{ad}(H)$ on $\mathfrak{g}$. Such an element always exists according to proposition 2.1.

Now let (V,π) be a representation of $\mathfrak{g}$. Then H acts on V. Let

$$V_\beta = \{v \in V \mid \pi(H)v = \beta v\}$$

denote the subspaces of V corresponding to the eigenvalue $\beta \in \mathbb{C}$ under the action of H. If $X \in \mathfrak{g}$ is such that $\mathrm{ad}(H)X = [H,X] = \alpha X$ for some $\alpha \in \mathbb{R}$, then $\pi(X)v \in V_{\alpha+\beta}$.

Proof: We compute

$$\pi(H)(\pi(X)v) = \pi(HX)v \;=\; \pi([H,X])v + \pi(X)\pi(H)v$$

$$\begin{aligned} &= \alpha\pi(X)v + \beta\pi(X)v \\ &= (\alpha+\beta)\pi(X)v \end{aligned}$$

This completes the proof. □

Proposition 2.15 If (V, π) is a finite dimensional irreducible representation of $\mathfrak{g}$, then the space of $\mathfrak{u}$-invariants $V^{\mathfrak{u}} = \{v \in V \mid \pi(X).v = 0 \,\forall X \in \mathfrak{u}\}$ is non-trivial and is irreducible as a representation of $\mathfrak{l}$.

Proof: For this proof, we once more make use of the properties of the triangular decomposition, in particular of the element $H \in \mathfrak{l}$ for which $\mathfrak{u}$ is the space spanned by eigenspaces associated to positive eigenvalues, which exists according to proposition 2.1. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of the action of H on V, and let $V_{\lambda_1}, ..., V_{\lambda_n}$ be the corresponding eigenspaces. Note in particular that since V is finite dimensional there are finitely many such eigenvalues. Let $\bar{\lambda}$ be the one with the largest real part.

i) We first check that $V^{\mathfrak{u}}$ is a representation of $\mathfrak{l}$. Let $Z \in \mathfrak{l}$, $X \in \mathfrak{u}$, and $v \in V^{\mathfrak{u}}$. Then

$$\pi(X)\left(\pi(Z)v\right) = \pi([X, Z])v + \pi(Z)\pi(X)v = 0$$

because $[\mathfrak{l}, \mathfrak{u}] \subseteq \mathfrak{u}$, and by definition of $V^{\mathfrak{u}}$, $\pi(\mathfrak{u})V^{\mathfrak{u}} = 0$.

ii) We next prove that $V^{\mathfrak{u}}$ is non-trivial. By construction of H, it acts on $\mathfrak{u}$ with real positive eigenvalues. Let $X \in \mathfrak{u}$ be such that $[H, X] = \alpha X$ for some real $\alpha > 0$. According to above lemma 2.14, $\pi(X)V_{\lambda_i} \subseteq V_{\lambda_i+\alpha}$. This implies that $\pi(\mathfrak{u})V_{\bar{\lambda}} = 0$ by maximality of $\bar{\lambda}$. In other words, $V_{\bar{\lambda}} \subseteq V^{\mathfrak{u}}$, hence $V^{\mathfrak{u}}$ is non-trivial.

iii) Finally, one can use H in a similar vein to show irreducibility. This part is more technical, so we only give a brief outline. Assume $W \subsetneqq V^{\mathfrak{u}}$ is a non-zero $\mathfrak{l}$-invariant proper subspace of $V^{\mathfrak{u}}$, and denote the intersection with V_λ by W_λ. Then each W_λ is also an $\mathfrak{l}$-invariant subspace by above lemma. If W_λ is non-zero, one shows that $\pi(\mathcal{U}(\mathfrak{u}^-))W_\lambda$ is a $\mathfrak{g}$-invariant subspace of V. So by irreducibility of V,

$$\pi(\mathcal{U}(\mathfrak{u}^-))W_\lambda = V.$$

Moreover, the previous lemma 2.14 implies that the action of $\mathfrak{u}^-$, and hence the action of $\mathcal{U}(\mathfrak{u}^-)$, "decreases" the real part of the eigenvalue by which $\mathrm{ad}(H)$ acts. Therefore,

$$\pi(\mathcal{U}(\mathfrak{u}^-))W_\lambda = W_\lambda + \{\text{elements in } V_\mu \text{ with } \Re(\mu) < \Re(\lambda)\}.$$

In particular, $W_{\bar{\lambda}} \subseteq W_\lambda$, so $\lambda = \bar{\lambda}$. This implies for one, that $V^{\mathfrak{u}}$ only contains vectors from $V_{\bar{\lambda}}$, and two, if W_λ is non-zero, than it is all of $V^{\mathfrak{u}}$. This completes the proof. □

Finally we prove the main theorem.

Theorem 2.16 $$\boxed{\chi_{\mathfrak{g}}^{V} = \chi_{\mathfrak{l}}^{V^{\mathfrak{u}}} \circ \xi}$$

Proof: To prove this equation, it suffices to show that for all $v \in V^{\mathfrak{u}}$, and $z \in \mathcal{Z}_{\mathfrak{g}}$

$$0 = (\pi(z) - \pi(\xi(z)))\, v = \pi\,(z - \xi(z))\, v.$$

We defined ξ to be the projection onto the first summand of the decomposition of $\mathcal{U}(\mathfrak{g})$ according to corollary 2.11

$$\mathcal{U}(\mathfrak{g})^{ad(\mathfrak{l})} = \mathcal{U}(\mathfrak{l})^{ad(\mathfrak{l})} \oplus \left(\mathfrak{u}^{-} \cdot \mathcal{U}(\mathfrak{g}) \cdot \mathfrak{u}\right)^{ad(\mathfrak{l})}$$

This implies that $z - \xi(z) \in \mathfrak{u}^{-} \cdot \mathcal{U}(\mathfrak{g}) \cdot \mathfrak{u}$. By definition of $V^{\mathfrak{u}}$, this in turn means that $\pi(z - \xi(z)).v = 0$ for all $v \in V^{\mathfrak{u}}$. This finishes the proof. □

One way to interpret this theorem is that it says that no two distinct finite dimensional representations V of $\mathfrak{g}$ can induce the same $\mathfrak{l}$-module $V^{\mathfrak{u}}$. Another interpretation is that by means of this theorem ξ provides an injective map from the space of finite dimensional irreducible representations of $\mathfrak{g}$ to the analogous space of $\mathfrak{l}$.

Finally, note that at no point in the argument did we need the fact that the element z was actually in the center $\mathcal{Z}_{\mathfrak{g}}$ of the universal enveloping algebra. Indeed, the proof would hold for any element $z \in \mathcal{U}(\mathfrak{g})^{ad(\mathfrak{l})}$. However, this does not yield any additional information since $\chi_{\mathfrak{g}}^{V}$ already completely characterizes the representation.

3 Excursion: Lie Algebra Cohomology

In the next section, we will discuss what of the previous analysis carries over to infinite dimensional representations. For this, we will need the machinery of Lie algebra cohomology. To have everything set up in consistent notation we will review the main definitions and all properties we will need in the following sections.

Lie algebra cohomology can be approached as an analogue to de Rham cohomology, that is using cochain complexes, or via resolutions of representations. For the joy of it, we will present both definitions.

3.1 Lie Algebra Cohomology with Cochain Complexes

To warm up to Lie algebra cohomology, recall that in general, whenever one has a sequence of spaces

$$\ldots \quad C^{k-1} \xrightarrow{d_k} C^k \xrightarrow{d_{k+1}} C^{k+1} \quad \ldots$$

with

$$d_{k+1} \circ d_k = 0 \qquad \forall k,$$

one can form the cohomology of this sequence

$$H^k = \frac{\ker\{d_{k+1} : C^k \to C^{k+1}\}}{\operatorname{im}\{d_k : C^{k-1} \to C^k\}}.$$

For example, on a smooth manifold M one can consider the space C^k of differential k-forms. The maps d_{k+1} sending k-forms to $k+1$-forms are exterior differentiation, usually denoted just by d. For a k-form $\omega = \sum_{I=\{i_1,..,i_k\}} f_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ exterior differentiation is given by

$$d\omega = \sum_{I=\{i_1,..,i_k\}} df_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

This map d has the property $d^2 = 0$. The *de Rham cohomology* of a manifold is the cohomology defined according to above recipe

$$H^k_{deRham}(M) = \frac{\ker\{d : C^k \to C^{k+1}\}}{\operatorname{im}\{d : C^{k-1} \to C^k\}} = \frac{\text{closed } k\text{-forms}}{\text{exact } k\text{-forms}}.$$

Analogously, let $\mathfrak{g}$ be a Lie algebra, and consider the space C^k of alternating multilinear maps from $\mathfrak{g}$ to the real numbers

$$C^k = \Lambda^k \mathfrak{g}^* = \{f : \underbrace{\mathfrak{g} \times \ldots \times \mathfrak{g}}_{k} \to \mathbb{R} \,|\, f \text{ is an alternating } \mathbb{R}\text{-linear map}\}$$

We can define $d : C^k \to C^{k+1}$ by

$$df(X_0, ..., X_k) = \sum_{i<j} (-1)^{i+j} f([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_k) \qquad \forall f \in C^k, X_i \in \mathfrak{g}$$

where a hat over an argument means that the argument is omitted. One can check by computation that $d^2 = 0$. The *Lie algebra cohomology of* $\mathfrak{g}$ *with coefficients in* $\mathbb{R}$ is then given by

$$H^k(\mathfrak{g}, \mathbb{R}) = \frac{\ker\{d : C^k \to C^{k+1}\}}{\operatorname{im}\{d : C^{k-1} \to C^k\}}.$$

The Lie algebra cohomology and the de Rham cohomology are related as follows

Theorem 3.1 If $\mathfrak{g}$ is the Lie algebra of a compact connected Lie group G, then [18,19]

$$H^k(\mathfrak{g}, \mathbb{R}) = H^k_{deRham}(G)$$

Since no particular property of $\mathbb{R}$ was needed in the definition of the Lie algebra cohomology above, we can more generally define the Lie algebra cohomology of $\mathfrak{g}$ with coefficients in any vector V space that carries a representation of $\mathfrak{g}$.[20] In the following, we will exclusively be interested in complex representations V of complex Lie algebras $\mathfrak{g}$. [21]

Let $\mathfrak{g}$ be a complex Lie algebra, and let V be a complex vector space which is a representation of $\mathfrak{g}$. Analogous to the specific case above, let C^k be the space of alternating $\mathbb{C}$-multilinear maps from $\mathfrak{g}$ to V

$$C^k = Hom_{\mathbb{C}}(\Lambda^k \mathfrak{g}, V)$$

and define $d : Hom_{\mathbb{C}}(\Lambda^k \mathfrak{g}, V) \to Hom_{\mathbb{C}}(\Lambda^{k+1}\mathfrak{g}, V)$ for all $f \in C^k, X_i \in \mathfrak{g}$ by[22]

$$\begin{aligned} df(X_0, ..., X_k) &= \sum_{i=0}^{k} (-1)^i X_i . \Big(f(X_0, ..., \hat{X}_i, ..., X_k) \Big) \\ &\quad + \sum_{i<j} (-1)^{i+j} f([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_k). \end{aligned} \tag{7}$$

Note that if we consider the trivial action of $\mathfrak{g}$ on $\mathbb{R}$ (ie. every element acts by zero), this map reduces to the map previously defined for Lie algebra cohomology with coefficients in $\mathbb{R}$.

Proposition 3.2 The above defined map d has the property $d^2 = 0$.

Corollary 3.3 The *Lie algebra cohomology of* $\mathfrak{g}$ *with coefficients in* V is well-defined by

$$H^k(\mathfrak{g}, V) = \frac{\ker\{d : Hom_{\mathbb{C}}(\Lambda^k \mathfrak{g}, V) \to Hom_{\mathbb{C}}(\Lambda^{k+1}\mathfrak{g}, V)\}}{\mathrm{im}\{d : Hom_{\mathbb{C}}(\Lambda^{k-1}\mathfrak{g}, V) \to Hom_{\mathbb{C}}(\Lambda^k \mathfrak{g}, V)\}}. \tag{8}$$

[18] statement 26.1 in [16]

[19] This theorem is included here to exemplify how Lie algebra cohomology relates to other cohomologies. We will not need it anywhere later on.

[20] This representation may, of course, be the trivial representation, as in the case with coefficients in $\mathbb{R}$ above.

[21] Lie algebra cohomology carries a representation of $\mathfrak{g}$ via its action on V together with the adjoint action on $\Lambda^k\mathfrak{g}$, but this action is always automatically zero. Nonetheless, Lie algebra cohomology is relevant in the context of representation theory: One typically considers the Lie algebra cohomology of a subalgebra of $\mathfrak{g}$ with coefficients in a representation V of $\mathfrak{g}$ which carries an action by certain other subalgebras of $\mathfrak{g}$. For example, if $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}$ is a triangular decomposition of $\mathfrak{g}$ as discussed in section 2, then $\mathfrak{l}$ acts on $H^k(\mathfrak{u}, V)$ non-trivially. We will see this action again in later sections.

[22] Recall that the action of a Lie algebra on a vector space V is often written as $X.v$ for $X \in \mathfrak{g}$, $v \in V$.

Example 3.4 Let $\mathfrak{g}$ be a complex Lie algebra, and V a representation of $\mathfrak{g}$. We want to compute $H^0(\mathfrak{g}, V)$. First, recall that $\Lambda^0\mathfrak{g} = \mathbb{C}$, $\Lambda^1\mathfrak{g} = \mathfrak{g}$, and by convention $\Lambda^{-1}\mathfrak{g} = 0$. Therefore

$$\begin{aligned} Hom_{\mathbb{C}}(\Lambda^{-1}\mathfrak{g}, V) &= Hom_{\mathbb{C}}(0, V) \cong \{0\} \\ Hom_{\mathbb{C}}(\Lambda^{0}\mathfrak{g}, V) &= Hom_{\mathbb{C}}(\mathbb{C}, V) \cong V \\ Hom_{\mathbb{C}}(\Lambda^{1}\mathfrak{g}, V) &= Hom_{\mathbb{C}}(\mathfrak{g}, V). \end{aligned}$$

Thus the image of d in $Hom_{\mathbb{C}}(\Lambda^0\mathfrak{g}, V)$ is just the 0-map. Applying d to an element in $f_v : 1 \mapsto v \in Hom_{\mathbb{C}}(\Lambda^0\mathfrak{g}, V)$ yields according to (7)

$$df_v(X) = (-1)^0 X.f_v(1) + 0 = X.v.$$

Therefore, the map $f_v : 1 \mapsto v$ is in the kernel of d if and only if $X.v = 0$ for all $X \in \mathfrak{g}$. We found

$$H^0(\mathfrak{g}, V) = ker\{d : Hom_{\mathbb{C}}(\Lambda^0\mathfrak{g}, V) \to Hom_{\mathbb{C}}(\Lambda^1\mathfrak{g}, V)\} \cong \{v \in V | X.v = 0 \, \forall X \in \mathfrak{g}\} = V^{\mathfrak{g}}.$$

As it is the case in other cohomology theories, short exact sequences of spaces give rise to long exact sequences in Lie algebra cohomology.[23] More specifically, let

$$0 \longrightarrow U \xrightarrow{\varphi} V \xrightarrow{\psi} W \longrightarrow 0$$

be a short exact sequence of representations of a Lie algebra $\mathfrak{g}$. It induces a short exact sequence of corresponding Hom-spaces

$$0 \longrightarrow Hom_{\mathbb{C}}(\Lambda^k\mathfrak{g}, U) \xrightarrow{\varphi_*} Hom_{\mathbb{C}}(\Lambda^k\mathfrak{g}, V) \xrightarrow{\psi_*} Hom_{\mathbb{C}}(\Lambda^k\mathfrak{g}, W) \longrightarrow 0$$

where the maps are given by

$$\begin{aligned} \varphi_*(f)(X_1, ..., X_k) &= \varphi(f(X_1, ..., X_k)) \qquad \text{and} \\ \psi_*(f)(X_1, ..., X_k) &= \psi(f(X_1, ..., X_k)). \end{aligned}$$

This sequence of Hom-spaces in turn gives rise to the following commutative diagram of spaces, where all vertical maps are exact

[23] Indeed, this is one of the reasons that make the cohomology theories attractive to work with.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & & Hom_{\mathbb{C}}(\Lambda^{k-1}\mathfrak{g}, U) & \xrightarrow{d} & Hom_{\mathbb{C}}(\Lambda^{k}\mathfrak{g}, U) & \xrightarrow{d} & Hom_{\mathbb{C}}(\Lambda^{k+1}\mathfrak{g}, U) & \longrightarrow & \cdots \\
 & & \downarrow{\varphi_*} & & \downarrow{\varphi_*} & & \downarrow{\varphi_*} & & \\
\cdots \longrightarrow & & Hom_{\mathbb{C}}(\Lambda^{k-1}\mathfrak{g}, V) & \xrightarrow{d} & Hom_{\mathbb{C}}(\Lambda^{k}\mathfrak{g}, V) & \xrightarrow{d} & Hom_{\mathbb{C}}(\Lambda^{k+1}\mathfrak{g}, V) & \longrightarrow & \cdots \\
 & & \downarrow{\psi_*} & & \downarrow{\psi_*} & & \downarrow{\psi_*} & & \\
\cdots \longrightarrow & & Hom_{\mathbb{C}}(\Lambda^{k-1}\mathfrak{g}, W) & \xrightarrow{d} & Hom_{\mathbb{C}}(\Lambda^{k}\mathfrak{g}, W) & \xrightarrow{d} & Hom_{\mathbb{C}}(\Lambda^{k+1}\mathfrak{g}, W) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We can define a correspondence[24]

$$\ker \mathrm{d}{:}\{Hom_{\mathbb{C}}(\Lambda^{k-1}\mathfrak{g}, W) \to Hom_{\mathbb{C}}(\Lambda^{k}\mathfrak{g}, W)\} \xrightarrow{\rho} \ker \mathrm{d}{:}\{Hom_{\mathbb{C}}(\Lambda^{k}\mathfrak{g}, U) \to Hom_{\mathbb{C}}(\Lambda^{k+1}\mathfrak{g}, U)\}$$

which descends to a well-defined map on cohomology as follows. Let $f \in Hom_{\mathbb{C}}(\Lambda^{k-1}\mathfrak{g}, W)$ be such that $df = 0$. Since ψ_* is surjective by exactness, there exists a $g \in Hom_{\mathbb{C}}(\Lambda^{k-1}\mathfrak{g}, V)$ such that $\psi_*(g) = f$. For this element g,

$$\psi_*(dg) = d(\psi_*(g)) = df = 0$$

so $dg \in Hom_{\mathbb{C}}(\Lambda^{k}\mathfrak{g}, V)$ is in the kernel of ψ_*, and therefore by exactness in the image of φ_*. Let $h \in Hom_{\mathbb{C}}(\Lambda^{k}\mathfrak{g}, U)$ be such that $\varphi_*(h) = dg$. We set

$$\rho(f) = \{h \mid \exists g \text{ s.t. } \psi_*(g) = f, \varphi_*(h) = dg\}.$$

This correspondence descends to a well-defined map $\hat{\rho}$ on cohomology.[25] It is called the *connecting homomorphism*, because it glues subsequent degrees of cohomology together to form the long exact sequence

$$\ldots\ H^{k-1}(\mathfrak{g}, U) \xrightarrow{\hat{\varphi}} H^{k-1}(\mathfrak{g}, V) \xrightarrow{\hat{\psi}} H^{k-1}(\mathfrak{g}, W) \xrightarrow{\hat{\rho}} H^{k}(\mathfrak{g}, U) \xrightarrow{\hat{\varphi}} H^{k}(\mathfrak{g}, V) \xrightarrow{\hat{\psi}} H^{k}(\mathfrak{g}, W)\ \ldots$$

where the maps $\hat{\varphi}$, and $\hat{\psi}$ are induced on the cohomology spaces by the maps φ_*, and ψ_* on the Hom-spaces.

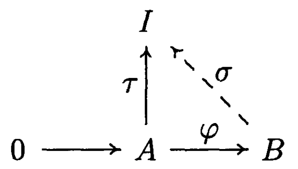
[24]Note that this is not a well-defined map.

[25]This statement has to be checked, of course. We skip this computations, as this construction is standard in Algebraic Topology. See for example, p. 116 in [4] for details of the argument.

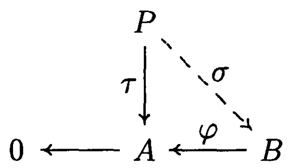
3.2 Lie Algebra Cohomology with Resolutions

There are alternative but equivalent ways to define Lie algebra cohomology using either injective or projective resolutions of $\mathcal{U}(\mathfrak{g})$-modules.[26]

Recall that a left module I over a ring S is an *injective* left S-module if for any injective S-homomorphism $\varphi : A \to B$ and S-homomorphism $\tau : A \to I$



there exists an S-homomorphism $\sigma : B \to I$ such that $\varphi \circ \sigma = \tau$.
A *projective* left S-module P is a left module over a ring S such that for any surjective S-homomorphism $\varphi : B \to A$ and S-homomorphism $\tau : P \to A$



there exists an S-homomorphism $\sigma : P \to B$ such that $\sigma \circ \varphi = \tau$.[27]

Here are the three main examples of injective modules which will be interesting to us in the context of Lie algebra cohomology.

Example 3.5 Any complex vector space V is an injective $\mathbb{C}$-module. This follows directly from the definition: For all $b \in \operatorname{im}(\varphi)$, choose $\sigma(b) = (\tau \circ \varphi^{-1})(b)$ as map from B to V. Since φ is by assumption injective, its inverse is well-defined on its image. As all spaces involved are vector spaces, σ can be extended to all of B.

Example 3.6 If S and R are two rings, and $\phi : S \to R$ a ring homomorphism, then any R-module can be considered an S-module. Moreover, if I is an injective S-module, then $Hom_S(R, I)$ is an injective R-module.
The idea is the following: For any R-module M, $Hom_R(M, Hom_S(R, I)) \cong Hom_S(M, I)$.

[26]In fact, it will turn out that the previously considered cochain complex is a special case of a projective resolution.

[27]For a brief review of injective and projective modules in the context of Lie theory, see chapter V.1 and VI in [9]. For a more general development of the algebraic theory of injective and projective modules, see [13], in particular sections 1.2 and 1.3.

So in particular, this is true for $M = A$ and $M = B$. By injectivity of I, there is a map from $Hom_S(A, I)$ to $Hom_S(B, I)$, which by the previous equivalence induces a map from $Hom_R(A, Hom_S(R, I))$ to $Hom_R(B, Hom_S(R, I))$.

Example 3.7 As a special case of the previous example, if $\mathfrak{g}$ is a Lie algebra, $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra, and V a complex vector space, then $Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{g}), V)$ is an injective $\mathcal{U}(\mathfrak{g})$-module.

The three main example for projective modules relevant to our context are very similar.

Example 3.8 Any complex vector space V is a projective $\mathbb{C}$-module. Similarly to the argument made in example 3.5 above, this follows directly from the definition.

Example 3.9 Again, if S and R are two rings, and $\phi : S \to R$ a ring homomorphism, then any R-module can be considered an S-module. Moreover, if P is a projective S-module, then $R \otimes_S P$ is a projective R-module. Again, the argument is analogous to the one made for the injective case.

Example 3.10 As a special case of the previous example, if $\mathfrak{g}$ is a Lie algebra, $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra, and $\Lambda^k\mathfrak{g}$ its exterior algebra, then $\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} \Lambda^k\mathfrak{g}$ is a projective $\mathcal{U}(\mathfrak{g})$-module.

We now proceed to defining Lie algebra cohomology using injective and projective resolutions. As usually, let $\mathfrak{g}$ be a Lie algebra, $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra, and V a representation of $\mathfrak{g}$, or equivalently a (unital left) $\mathcal{U}(\mathfrak{g})$ module.

An *injective resolution of* V is an exact sequence of injective $\mathcal{U}(\mathfrak{g})$-modules I_k starting in V

$$0 \longrightarrow V \xrightarrow{\varphi_0} I_0 \xrightarrow{\varphi_1} I_1 \xrightarrow{\varphi_2} I_2 \xrightarrow{\varphi_3} \cdots$$

such that the action of $\mathcal{U}(\mathfrak{g})$ on these modules commutes with the maps φ_k.[28] Exactness means that $\operatorname{im}(\varphi_k) = \ker(\varphi_{k+1})$, and in particular, $\varphi_k \circ \varphi_{k+1} = 0$ for all k.

Since $\mathcal{U}(\mathfrak{g})$ acts on all of these modules, we can pass to a sequence of $\mathfrak{g}$-invariant subspaces with $I_k^{\mathfrak{g}} = \{v \in I_k \,|\, X.v = 0 \;\; \forall X \in \mathfrak{g}\}$

$$0 \longrightarrow V^{\mathfrak{g}} \xrightarrow{\varphi_0^*} I_0^{\mathfrak{g}} \xrightarrow{\varphi_1^*} I_1^{\mathfrak{g}} \xrightarrow{\varphi_2^*} I_2^{\mathfrak{g}} \xrightarrow{\varphi_3^*} \cdots .$$

[28] Alternatively, one can define an injective resolution of V to be a sequence of injective modules

$$0 \longrightarrow I_0 \xrightarrow{\varphi_1} I_1 \xrightarrow{\varphi_2} I_2 \xrightarrow{\varphi_3} \cdots$$

which is exact everywhere except at I_0 and for which $\ker(\varphi_1) \cong V$.

For this sequence it is still true that $\varphi_k^* \circ \varphi_{k+1}^* = 0$ for all k, but the sequence is no longer necessarily exact for all k. However, the sequence still is exact at $V^{\mathfrak{g}}$ [29] and $I_0^{\mathfrak{g}}$ [30].

At the points where the sequence is not exact, we can consider the well-defined quotient $\ker(\varphi_{k+1}^*)/\mathrm{im}(\varphi_k^*)$ to measures to what extent the sequence is not exact. In fact, we can instead consider the sequence

$$0 \xrightarrow{\hat{\varphi^*}_0} I_0 \xrightarrow{\varphi_1^*} I_1^{\mathfrak{g}} \xrightarrow{\varphi_2^*} I_2^{\mathfrak{g}} \xrightarrow{\varphi_3^*} \cdots$$

which is not necessarily exact at any point, and "measure" non-exactness by the quotients of kernel over image at all points. It turns out that all the quotients ker/im in this sequence only depend on V and $\mathfrak{g}$ and not on the particular resolution chosen.[31] Thus the quotients

$$H_{inj}^0(\mathfrak{g}, V) = \frac{\ker\left(\varphi_1^* : I_0^{\mathfrak{g}} \to I_1^{\mathfrak{g}}\right)}{\mathrm{im}\left(\hat{\varphi}^*{}_0 : 0 \to I_1^{\mathfrak{g}}\right)} \tag{9}$$

$$H_{inj}^k(\mathfrak{g}, V) = \frac{\ker\left(\varphi_{k+1}^* : I_k^{\mathfrak{g}} \to I_{k+1}^{\mathfrak{g}}\right)}{\mathrm{im}\left(\varphi_k^* : I_{k-1}^{\mathfrak{g}} \to I_k^{\mathfrak{g}}\right)} \tag{10}$$

are well defined.

Example 3.11 For any Lie algebra $\mathfrak{g}$, and any representation V of $\mathfrak{g}$, we find

$$H_{inj}^0(\mathfrak{g}, V) = \ker\left(\varphi_1^* : I_0^{\mathfrak{g}} \to I_1^{\mathfrak{g}}\right) = \mathrm{im}\left(\varphi_0^* : V^{\mathfrak{g}} \to I_1^{\mathfrak{g}}\right) = V^{\mathfrak{g}}$$

because by exactness $\mathrm{im}(\varphi_0^*) = \ker(\varphi_1^*)$, and $\ker(\varphi_0^*) = 0$. This coincides with our computation of $H^0(\mathfrak{g}, V)$ in example 3.4.

Example 3.12 If V is an injective $\mathcal{U}(\mathfrak{g})$-module, then $H_{inj}^k(\mathfrak{g}, V) = 0$ for all $k > 0$. This is true because if V is injective, then $0 \to V \to V \to 0$ is an exact sequence of injective

[29] In detail: Exactness of the original sequence implies $\ker(\varphi_0) = 0$. The map φ_0^* results from the original map φ_0 via restriction to the subspace $V^{\mathfrak{g}} \subseteq V$. Therefore, $\ker(\varphi_0^*) \subseteq \ker(\varphi_0) = 0$. In other words, the new sequence is exact at $V^{\mathfrak{g}}$.

[30] To prove this, one needs to show that $\mathrm{im}(\varphi_0^*) = \ker(\varphi_1^*)$. The inclusion $\mathrm{im}(\varphi_0^*) \subseteq \ker(\varphi_1^*)$ is immediate from the exactness of the original sequence, and $\varphi_1^* = \varphi_1|_{I_0^{\mathfrak{g}}}$. To show the converse inclusion, let $v \in \ker(\varphi_1^*)$. Then $v \in \mathrm{im}(\varphi_0^*)$ by exactness of the original sequence. So there exists $w \in V$ such that $\varphi_0(w) = v$. Because $v \in I_0^{\mathfrak{g}}$ and the action of $\mathfrak{g}$ commutes with all φ^k, we have

$$0 = X.v = X.\varphi_0(w) = \varphi_0(X.w) \quad \forall X \in \mathfrak{g}.$$

Since the original sequence is exact in V, $\ker(\varphi_0) = 0$. This implies, $X.w = 0 \ \forall X \in \mathfrak{g}$. Thus, $w \in V^{\mathfrak{g}}$, and $\mathrm{im}(\varphi_0^*) = \ker(\varphi_1^*)$

[31] This argument, given in the language of categories and functors, can be found in almost any book on homological algebra, for example in [17].

modules, that is, an injective resolution of V. Since the cohomology does not depend on the choice of the resolution, we can consider this resolution, and compute $H^k_{inj}(\mathfrak{g}, V) = 0$ for all $k > 0$.

Example 3.13 As a special case of the previous example, we find that

$$H^k_{inj}(\mathfrak{g}, Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{g}), V)) = 0 \quad \forall k > 0.$$

Recall from example 3.7 above that $Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{g}), V))$ is an injective $\mathcal{U}(\mathfrak{g})$-module.

Note we have not shown that every $\mathcal{U}(\mathfrak{g})$-module V has such an injective resolution. While this is not obvious, it is true, but the proof of it is beyond this work.[32]

In a somewhat similar manner, we can define cohomology using a projective resolution of $\mathbb{C}$. A *projective resolution of* $\mathbb{C}$ is an exact sequence of projective $\mathcal{U}(\mathfrak{g})$-modules P_k starting in $\mathbb{C}$

$$\cdots \longrightarrow P_3 \xrightarrow{\phi_3} P_2 \xrightarrow{\phi_2} P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\phi_0} \mathbb{C} \longrightarrow 0$$

such that the action of $\mathcal{U}(\mathfrak{g})$ on these modules commutes with the maps ϕ_k.[33] As before, exactness means that $\text{im}(\phi_k) = \ker(\varphi_{k-1})$, and in particular, $\phi_k \circ \varphi_{k-1} = 0$ for all k. Considering $\mathcal{U}(\mathfrak{g})$-homomorphisms of each of these modules into a representation V of $\mathfrak{g}$ yields the sequence

$$\begin{aligned} \cdots \leftarrow Hom_{\mathcal{U}(\mathfrak{g})}(P_3, V) \xleftarrow{\phi_3^*} Hom_{\mathcal{U}(\mathfrak{g})}(P_2, V) \xleftarrow{\phi_2^*} Hom_{\mathcal{U}(\mathfrak{g})}(P_1, V) \xleftarrow{\phi_1^*} \\ \xleftarrow{\phi_1^*} Hom_{\mathcal{U}(\mathfrak{g})}(P_0, V) \xleftarrow{\phi_0^*} Hom_{\mathcal{U}(\mathfrak{g})}(\mathbb{C}, V) \leftarrow 0 \end{aligned}$$

Here the ϕ_k^* send $\mathcal{U}(\mathfrak{g})$-homomorphisms to their composition with ϕ_k. Also, $\phi_k \circ \varphi_{k-1} = 0$ implies that $\phi_k^* \circ \phi_{k+1}^* = 0$.

Similarly, to the injective resolution case, the sequence is exact in $Hom_{\mathcal{U}(\mathfrak{g})}(P_0, V)$ and in $Hom_{\mathcal{U}(\mathfrak{g})}(\mathbb{C}, V)$. So we consider

$$\cdots \leftarrow Hom_{\mathcal{U}(\mathfrak{g})}(P_3, V) \xleftarrow{\phi_3^*} Hom_{\mathcal{U}(\mathfrak{g})}(P_2, V) \xleftarrow{\phi_2^*} Hom_{\mathcal{U}(\mathfrak{g})}(P_1, V) \xleftarrow{\phi_1^*} Hom_{\mathcal{U}(\mathfrak{g})}(P_0, V) \xleftarrow{\widehat{\phi_0^*}} 0$$

and define

$$H^0_{proj}(\mathfrak{g}, V) = \frac{\ker(\phi_1^*)}{\text{im}\left(\widehat{\phi_0^*}\right)} \tag{11}$$

[32]For more detail on this question see chapter V in [9], and the notion of a "good category" therein.

[33]The action of $\mathcal{U}(\mathfrak{g})$ on $\mathbb{C}$ here is given by multiple of the unit in $\mathcal{U}(\mathfrak{g})$ by themselves, and everything else in $\mathcal{U}(\mathfrak{g})$ acting by zero.

$$H^k_{proj}(\mathfrak{g}, V) = \frac{\ker\left(\phi^*_{k+1} : Hom_{\mathcal{U}(\mathfrak{g})}(P_k, V) \to Hom_{\mathcal{U}(\mathfrak{g})}(P_{k+1}, V)\right)}{\mathrm{im}\left(\phi^*_k : Hom_{\mathcal{U}(\mathfrak{g})}(P_{k-1}, V) \to Hom_{\mathcal{U}(\mathfrak{g})}(P_k, V)\right)} \tag{12}$$

As with injective resolutions, these quotients are independent of the projective resolution chosen.

Example 3.14 We have seen in example 3.10 that for any Lie algeba $\mathfrak{g}$, $\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} \Lambda^k\mathfrak{g}$ is a projective $\mathcal{U}(\mathfrak{g})$-module for all k. In fact,

$$\cdots \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} \Lambda^3\mathfrak{g} \xrightarrow{\phi_3} \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} \Lambda^2\mathfrak{g} \xrightarrow{\phi_2} \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} \Lambda^1\mathfrak{g} \xrightarrow{\phi_1} \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} \Lambda^0\mathfrak{g} \xrightarrow{\phi_0} \mathbb{C} \to 0$$

is a projective resolution of $\mathbb{C}$ with

$$\begin{aligned}\phi_k(u \otimes X_1 \wedge \ldots \wedge X_k) &= \sum_{i=1}^{k}(-1)^{i+1}(uX_i \otimes X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge X_k) \\ &\quad + \sum_{i<j}(-1)^{i+j}(u \otimes [X_i, X_j] \wedge X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge \hat{X}_j \wedge \ldots \wedge X_k)\end{aligned}$$

where as before a hat over an argument means that it is omitted. Observing that[34]

$$Hom_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} \Lambda^k\mathfrak{g}, V) = Hom_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}), Hom_{\mathbb{C}}(\Lambda^k\mathfrak{g}, V)) = Hom_{\mathbb{C}}(\Lambda^k\mathfrak{g}, V)$$

we find that we recovered the cochain complex definition of Lie algebra cohomology for this particular choice of a projective resolution.

Proposition 3.15 For any Lie algebra $\mathfrak{g}$, all of the three cohomologies defined above are equivalent.

$$H^k(\mathfrak{g}, V) = H^k_{proj}(\mathfrak{g}, V) = H^k_{inj}(\mathfrak{g}, V) \quad \forall k \geq 0$$

We already showed that the cochain-complex definition of Lie algebra cohomology is a special case of the projective resolution definition. To completely establish the equivalence of all three of these definitions, it remains to show that $H^k_{proj} = H^k_{inj}$ for all degrees and spaces. This is true in general for cohomology (not just Lie algebra cohomology), and can be shown using the machinery of spectral sequences. However, this proof is beyond the scope of this work. □

[34]The first isomorphism follows from $Hom_S(A \otimes_R B, C) \cong Hom_S(A, Hom_R(B, C))$, which is true in general.

3.3 Actions on Lie Algebra Cohomology

As we briefly indicated in the beginning of this section, Lie algebra cohomology is relevant in the context of representation theory: One typically considers the Lie algebra cohomology of a subalgebra of $\mathfrak{g}$ with coefficients in a representation V of $\mathfrak{g}$ which carries an action by certain other subalgebras of $\mathfrak{g}$. In the following few paragraphs we will discuss the action of $\mathfrak{l} \subseteq \mathfrak{g}$ and $\mathcal{Z}_{\mathfrak{g}} \subseteq \mathcal{U}(\mathfrak{g})$ on cohomology spaces of $\mathfrak{u}$ in preparation for the Casselman-Osborne Theorem in the next section. For this, we return to the setting where $\mathfrak{g}$ is a reductive complex Lie algebra, $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}$ a triangular decomposition as discussed in section 2, $\mathcal{U}(\mathfrak{g})$, $\mathcal{U}(\mathfrak{u}^-)$, $\mathcal{U}(\mathfrak{l})$, and $\mathcal{U}(\mathfrak{u})$ the respective universal enveloping algebras, $\mathcal{Z}_{\mathfrak{g}}$ the center of $\mathcal{U}(\mathfrak{g})$, and V a representation of $\mathfrak{g}$, in other words a left unital $\mathcal{U}(\mathfrak{g})$-module.

First (to make the obvious explicit), since $\mathfrak{u}$, $\mathfrak{l}$, and $\mathfrak{u}^-$ are subalgebras of $\mathfrak{g}$, they act on V, and we can consider their respective cohomologies with coefficients in V.

Next, not only elements in the Lie algebra of which the cohomology is considered may act on the cohomology. For example,

Proposition 3.16 In the setting described above, $H^k(\mathfrak{u}, V)$ is a $\mathcal{Z}_{\mathfrak{g}}$-module.

Proof: Since V is a representation of $\mathfrak{g}$, elements of $\mathcal{U}(\mathfrak{g})$ act on V, and hence on $Hom_{\mathbb{C}}(\Lambda^k \mathfrak{u}, V)$. The action of elements in the center of $\mathcal{U}(\mathfrak{g})$ commutes with the chain complex map $d : Hom_{\mathbb{C}}(\Lambda^k \mathfrak{u}, V) \to Hom_{\mathbb{C}}(\lambda^{k+1}\mathfrak{u}, V)$. In particular, the action preserves images and kernels of d, and consequently descends to the cohomology. □

Note in particular that if V is irreducible, the action of $\mathcal{Z}_{\mathfrak{g}}$ on the cohomology $H^k(\mathfrak{u}, V)$ is by the same scalars by which $\mathcal{Z}_{\mathfrak{g}}$ acts on V (by Schur's lemma).

Also, acting on the coefficient space is not the only way a Lie algebra can act on a cohomology space. For example,

Proposition 3.17 In the setting described above, $\mathfrak{l}$ acts on $Hom_{\mathbb{C}}(\Lambda^k \mathfrak{u}, V)$ by

$$(L.f)(X_1 \wedge ... \wedge X_k) = L.(f(X_1 \wedge ... \wedge X_k)) + \sum_{i=1}^{k} (-1)^i f([L, X_i] \wedge X_1 \wedge ... \wedge \hat{X}_i \wedge ... \wedge X_k$$

for $L \in \mathfrak{l}$, $f \in Hom_{\mathbb{C}}(\Lambda^k \mathfrak{u}, V)$ and $X_i \in \mathfrak{u}$, and this action descends to cohomology.[35] This induces an action of $\mathcal{U}(\mathfrak{l})$ on the cohomology.

[35] Note that for this to be a well-defined map, $[\mathfrak{l}, \mathfrak{u}] \subseteq$ is required. This condition is in particular satisfied for $\mathfrak{l}$ and $\mathfrak{u}$ coming from a triangular decomposition.

Proof: To show that this is a well-defined action, we need to check that for $[L_1, L_2].f = L_1.(L_2.f) - L_2.(L_1.f)$ for all $L_1, L_2 \in \mathfrak{l}$, $f \in Hom_{\mathbb{C}}(\Lambda^k \mathfrak{u}, V)$. This is a straightforward computation, using the Jacobi-identity $[[L_1, L_2], X_i] = +[L_1, [L_2, X_i]] + [[L_1, X_i], L_2]$ and the fact that f is $\mathbb{C}$-linear. To establish that it descends to cohomology, we need to prove that this action commutes with the cochain complex map d as defined in (7):

$$\begin{aligned} df(X_0, ..., X_k) &= \sum_{i=0}^{k} (-1)^i X_i. \left(f(X_0, ..., \hat{X}_i, ..., X_k) \right) \\ &\quad + \sum_{i<j} (-1)^{i+j} f([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_k). \end{aligned}$$

Paying attention to the changing indices, this is also a plug-and-play computation. □

Finally, one of the advantages of cohomology theory is that short exact sequences give rise to long exact sequences. As we saw at the end of section 3.1, this is also true for Lie algebra cohomology. This will be crucial to the proof of the Casselman-Osborne Theorem in the next section. Since we consider the action of Lie algebras and subalgebras of the universal enveloping algebra on cohomologies, we may ask which of these actions commute with the maps in the long exact sequence.

Proposition 3.18 In the setting as before, let

$$0 \to U \to V \to W \to 0$$

be a short exact sequence of $\mathcal{U}(\mathfrak{g})$ modules. Then as $\mathcal{U}(\mathfrak{u})$-modules this gives rise to a long exact sequence

$$\ldots\ H^{k-1}(\mathfrak{u}, U) \to H^{k-1}(\mathfrak{u}, V) \to H^{k-1}(\mathfrak{u}, W) \to H^k(\mathfrak{u}, U) \to H^k(\mathfrak{u}, V) \to H^k(\mathfrak{u}, W)\ \ldots.$$

The action of $\mathcal{Z}_{\mathfrak{g}}$ on these cohomologies as discussed in proposition 3.16 above commutes with the maps in the long exact sequence.[36,37]

Proposition 3.19 In the setting as before, let

$$0 \to U \to V \to W \to 0$$

be a short exact sequence of $\mathcal{U}(\mathfrak{g})$ modules. As in the previous proposition, as $\mathcal{U}(\mathfrak{u})$-modules this gives rise to a long exact sequence

$$\ldots\ H^{k-1}(\mathfrak{u}, U) \to H^{k-1}(\mathfrak{u}, V) \to H^{k-1}(\mathfrak{u}, W) \to H^k(\mathfrak{u}, U) \to H^k(\mathfrak{u}, V) \to H^k(\mathfrak{u}, W)\ \ldots,$$

[36] statement 6.20 in [9]

[37] Note that we need to start with $\mathcal{U}(\mathfrak{g})$ modules in order to have a well-defined action of $\mathcal{Z}_{\mathfrak{g}}$ on the cohomologies.

and the action of $\mathcal{U}(\mathfrak{l})$ on $H^k(\mathfrak{u}, V)$ commutes with the maps in the long exact sequence. [38,39]

4 The Casselman-Osborne Theorem

We discussed in section 2 how the Harish-Chandra homomorphism relates finite dimensional irreducible representations of a reductive complex Lie algebra $\mathfrak{g}$ and those of a certain subalgebra $\mathfrak{l}$. Naturally, the question arises what can be said in similar vein about infinite dimensional irreducible representations. Most of the above statements carry over to this case. In particular, there is a version of Schur's lemma, due to Dixmier, for infinite dimensional representations. It implies that central elements of the universal enveloping algebra act by scalars on any irreducible $\mathcal{U}(\mathfrak{g})$-module.[40]

However, for infinite dimensional representations the analysis of section 2 breaks down in two places: One, the subspace $V^{\mathfrak{u}}$ of $\mathfrak{u}$-invariants is no longer necessarily non-trivial.[41] Two, the map $\chi_{\mathfrak{g}} : \mathcal{Z}_{\mathfrak{g}} \to \mathbb{C}$ which is induced by the scalar action of elements in $\mathcal{Z}_{\mathfrak{g}}$, no longer uniquely determines the representation.

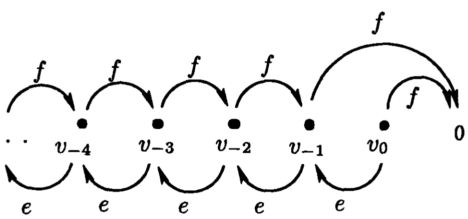
Example 4.1 The subspace $V^{\mathfrak{u}}$ of $\mathfrak{u}$-invariants is no longer necessarily non-trivial. Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. It is spanned by $\langle e, f, h\rangle$ with the relations

$$[h, e] = 2e, [h, f] = -2f, [e, f] = h.$$

A triangular decomposition of $\mathfrak{sl}_2(\mathbb{C})$ is given by

$$\mathfrak{u}^- = \langle f\rangle, \quad \mathfrak{l} = \langle h\rangle, \quad \mathfrak{u} = \langle e\rangle.$$

Let W be the vector space spanned by $\langle v_0, v_{-1}, v_{-2}\ldots\rangle$, and define an action π of $\mathfrak{sl}_2(\mathbb{C})$ on W by



$$\begin{aligned} \pi(h)v_{-i} &= (2i)v_{-i} \\ \pi(e)v_{-i} &= v_{-i-1} \\ \pi(f)v_{-i} &= -i(i+1)v_{-i+1} \end{aligned}$$

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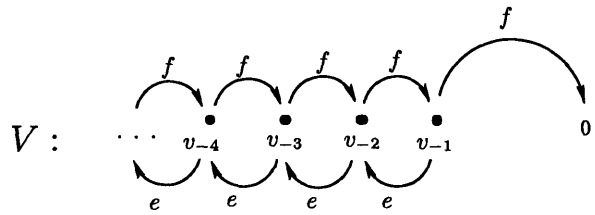
[38] statement 6.20 in [9]

[39] Here, too, we need to start with $\mathcal{U}(\mathfrak{g})$ modules in order to have a well-defined action of $\mathcal{U}(\mathfrak{l})$ on the cohomologies.

[40] statement 5.19 in [7]

[41] A crucial point in the argument for finite dimensional spaces was the existence of an eigenvalue with a maximal real part. If V is infinite dimensional, such an eigenvalue does not necessarily have to exist.

with $v_{+1} = 0$.[42] The subspace $V = \langle v_{-1}, v_{-2} \ldots \rangle$ is an irreducible $\mathfrak{sl}_2(\mathbb{C})$-subspace.



The space of $\mathfrak{u}$-invariants for this representation $(V, \pi|_V)$ of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ is

$$V^{\mathfrak{u}} = \{v \in V \mid \pi(e).v = 0\}.$$

This shows that for infinite dimensional representations $V^{\mathfrak{u}}$ can be trivial.

Example 4.2 Scalar action of elements in $\mathcal{Z}_{\mathfrak{g}}$ no longer uniquely determines the representation.

We give an example by computing the scalars for two non-isomorphic representations. As in the previous example, consider the action of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ on $W = \langle v_0, v_{-1}, v_{-2} \ldots \rangle$ given by

$$\begin{aligned} \pi(h)v_{-i} &= (2i)v_{-i} \\ \pi(e)v_{-i} &= v_{-i-1} \\ \pi(f)v_{-i} &= -i(i+1)v_{-i+1} \end{aligned}$$

with $v_1 = 0$. We mentioned before that $V = \langle v_{-1}, v_{-2} \ldots \rangle$ is an infinite dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. The universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$ is generated as an algebra by its Casimir element $\Omega = \frac{1}{2}h^2 + ef + fe$. We compute

$$\begin{aligned} \pi(\Omega)v_{-i} &= \frac{1}{2}\pi(h)\pi(h)v_{-i} + \pi(e)\pi(f)v_{-i} + \pi(f)\pi(e)v_{-i} \\ &= \frac{1}{2}4i^2 v_{-i} - i(i+1)v_{-i} - (i-1)(i)v_{-i} \\ &= 0. \end{aligned}$$

So the zero-degree elements in the center $\mathcal{Z}_{\mathfrak{g}}$ of the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$ act by themselves, and all other elements in $\mathcal{U}(\mathfrak{sl}_2)$ act by zero on the irreducible representation V. In other words, $\mathcal{Z}_{\mathfrak{sl}_2}$ acts on this space in the same manner as it does on the one-dimensional trivial representation. So not only do the scalars by which $\mathcal{Z}_{\mathfrak{g}}$ acts no longer determine the module, they do not even distinguish between finite and infinite dimensional representations.[43]

[42] This is more or less the familiar $\mathfrak{sl}_2(\mathbb{C})$-action, defined on an infinite dimensional space. For more detail see for example section 1.9 in [7].

[43] This construction is a particular example of the discussion in section 23.2 in [6].

The latter point implies that the Harish-Chandra homomorphism can no longer be easily used to match irreducible representations of the Lie algebra $\mathfrak{g}$ and its subalgebra $\mathfrak{l}$. One could try to solve this problem by looking for additional properties to aid in the classification or distinction of irreducible representations. For now, however, we shall be content with what can be learnt about the representation from the action of $\mathcal{Z}_{\mathfrak{g}}$.

The former caveat means that the scalars by which $\mathcal{Z}_{\mathfrak{l}}$ act on $V^{\mathfrak{u}}$, and thus the map $\chi_{\mathfrak{l}}^{V^{\mathfrak{u}}}$ are no longer well-defined. This problem we can not ignore. We can hope to replace $V^{\mathfrak{u}}$ with another - hopefully non-trivial - $\mathcal{U}(\mathfrak{l})$-module.

Since we have seen in (3.4) that

$$V^{\mathfrak{u}} = H^0(\mathfrak{u}, V)$$

there is reason for this hope: We might consider Lie algebra cohomologies of higher degrees; especially since according to [1] for any reductive Lie algebra there exists a triangular decomposition such that the cohomology of some degree is non-zero.

In fact, this hope is not in vain. This is the content of the Casselman-Osborne Theorem.

Theorem 4.3 Casselman-Osborne Theorem
Let $\mathfrak{g}$ be a reductive Lie algebra, $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}$ a triangular decomposition, $\mathcal{U}(\mathfrak{g})$ its enveloping algebra, $\mathcal{Z}_{\mathfrak{g}}$ the center of $\mathcal{U}(\mathfrak{g})$, and V any representation of $\mathfrak{g}$, that is an $\mathcal{U}(\mathfrak{g})$ module. Then we have seen in 3.16 that $\mathcal{Z}_{\mathfrak{g}}$ acts on $H^k(\mathfrak{u}, V)$ by acting on V, and in 3.17 that $\mathfrak{l}$ (and therefore $\mathcal{U}(\mathfrak{l})$) acts on $H^k(\mathfrak{u}, V)$ by simultaneously acting on $\mathfrak{u}$ and on V. Denote the action of $\mathcal{Z}_{\mathfrak{g}}$ on $H^k(\mathfrak{u}, V)$ by π and the one of $\mathcal{U}(\mathfrak{l})$ on $H^k(\mathfrak{u}, V)$ by σ.
Further, let $\xi : \mathcal{Z}_{\mathfrak{g}} \to \mathcal{Z}_{\mathfrak{l}}$ be the Harish-Chandra homomorphism as constructed in section 2.1. Then

$$\pi(z)\omega = (\sigma \circ \xi)(z)\omega$$

for all $z \in \mathcal{Z}_{\mathfrak{g}}$, and $\omega \in H^k(\mathfrak{u}, V)$.

We will follow the simplified version of the proof A. Knapp gave in section VI.6 of [9].

Proof: We proof this statement by induction on the cohomology degree k, using a technique known as "dimension shifting".

For $k = 0$ we established in example 3.4 and 3.11 that $H^0(\mathfrak{u}, V) = V^{\mathfrak{u}}$. In this case, the claim reduces to

$$\pi(z)\omega = (\sigma \circ \xi)(z)\omega \quad \forall z \in \mathcal{Z}_{\mathfrak{g}}, \forall \omega \in V^{\mathfrak{u}}.$$

This is like the statement in 2.16, and since neither irreducibility of V nor $\dim V < \infty$ was needed in the proof of 2.16, the proof carries over to this situation here.[44]

[44] The fact that the proof carries over is an indication that the theorem 2.16 maybe should have been

Now assume that the statement is true for degree $k-1$ for all (unital left) $\mathcal{U}(\mathfrak{g})$-modules. We want to consider a short exact sequence of $\mathcal{U}(\mathfrak{g})$-modules. Such a sequence gives rise to a long exact sequence with a connecting homomorphism linking cohomologies of different degrees. This will facilitate the induction step.

A convenient short exact sequence for these purposes is

$$0 \to V \xrightarrow{\varphi} Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{g}), V) \xrightarrow{\psi} Q \to 0$$

with $\varphi : v \mapsto f_v$ such that $f_v(X) = X.v$[45] for all $X \in \mathcal{U}(\mathfrak{g})$, and ψ being the projection map on the quotient $Q = Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{g}), V)/\mathrm{im}(\varphi)$. Thus in some sense, this sequence is very "hands-on". Notice however, that in the following argument all we need is that $Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{g}), V)$ is an injective $\mathcal{U}(\mathfrak{g})$-module, and that the $\mathcal{U}(\mathfrak{g})$ acts on the quotient Q. Indeed, the argument would work for any sequence of this form

$$0 \to V \to I \to Q_I \to 0$$

where I is an injective $\mathcal{U}(\mathfrak{g})$-module.[46] Trivially, since all of these spaces are $\mathcal{U}(\mathfrak{g})$-modules, they are also $\mathcal{U}(\mathfrak{u})$-modules, and the sequence is exact as sequence of $\mathcal{U}(\mathfrak{u})$-modules.

We argued at the end of section 3.1 that such a short exact sequence gives rise to a long exact sequence

$$\cdots \to H^{k-1}(\mathfrak{u}, Q) \xrightarrow{\hat{\rho}} H^k(\mathfrak{u}, V) \xrightarrow{\hat{\varphi}} H^k(\mathfrak{u}, Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{g}), V)) \xrightarrow{\hat{\psi}} H^k(\mathfrak{u}, Q) \to \cdots.$$

Next we want to argue that $H^k(\mathfrak{u}, Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{g}), V)) = 0$ for all $k > 0$. Since we know by example 3.12 that this is true for all injective modules, it suffices to show that $Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{g}), V)$ is injective - as an $\mathcal{U}(\mathfrak{u})$-module. We saw in example 3.4 that for any vector space W, $Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{u}), W)$ is an injective $\mathcal{U}(\mathfrak{u})$-module. Using the triangular decomposition of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{u}^-) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{l}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{u})$ discussed in section 2 we thus find for $W = Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{u}^-) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{l}), V)$

$$\begin{aligned} Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{g}), V)) &= Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{u}^-) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{l}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{u}), V)) \\ &= Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{u}), Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{u}^-) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{l}), V)) = Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{u}), W). \end{aligned}$$

Since the right hand side of this equation is an injective $\mathcal{U}(\mathfrak{u})$-module, so must the left hand side be. In particular,

$$H^k(\mathfrak{u}, Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{g}), V)) = 0 \quad \forall k > 0$$

formulated in terms of cohomology in the first place.

[45]Or in more detail $f_v(1) = v$ and $f_v(X) = f_{X.v}(1)$, which demonstrates that f respects the $\mathcal{U}(\mathfrak{g})$ action.

[46]Note that it would not suffice for I to be an injective $\mathcal{U}(\mathfrak{u})$-module, as we will need the actions of $\mathcal{U}(\mathfrak{l})$ and $\mathcal{Z}_{\mathfrak{g}}$ to be well-defined on it.

Therefore the long exact sequence simplifies for $k > 0$ to

$$\cdots \to H^{k-1}(\mathfrak{u}, Q) \xrightarrow{\hat{\rho}} H^k(\mathfrak{u}, V) \xrightarrow{\hat{\varphi}} 0 \xrightarrow{\hat{\psi}} H^k(\mathfrak{u}, Q) \to \cdots$$

Exactness of this sequence implies that the connecting homomorphism $\hat{\rho}$ is surjective.

We are now ready to argue the inductive step. Given $\omega \in H^k(\mathfrak{u}, V)$ there must exist an $\widetilde{\omega} \in H^{k-1}(\mathfrak{u}, Q)$ such that $\hat{\rho}(\widetilde{\omega}) = \omega$. By induction hypothesis, the claim is true for all $\omega \in H^{k-1}(\mathfrak{u}, Q)$, that is

$$\pi(z)\widetilde{\omega} = (\sigma \circ \xi)(z)\widetilde{\omega}$$

By propositions 3.18 and 3.19 the actions of $\mathcal{Z}_{\mathfrak{g}}$ and $\mathcal{U}(\mathfrak{l})$ commute with the maps of the long exact sequence. Therefore applying $\hat{\rho}$ to both sides of the previous equation yields

$$\pi(z)\omega = \pi(z)(\hat{\rho}(\widetilde{\omega})) = \hat{\rho}(\pi(z)\widetilde{\omega}) = \hat{\rho}((\sigma \circ \xi)(z)\widetilde{\omega}) = (\sigma \circ \xi)(z)(\hat{\rho}(\widetilde{\omega})) = (\sigma \circ \xi)(z)\omega$$

This completes the induction step, and thus the proof. □

Having established this theorem, we extended the application of the Harish-Chandra homomorphism to representations on infinite dimensional vector spaces. We conclude this discussion with some comments.

For one, notice that unlike $V^{\mathfrak{u}}$ in the case of finite dimensional representations V, $H^k(\mathfrak{u}, V)$ is not necessarily irreducible as a representation of $\mathfrak{l}$. This means that for infinite dimensional representations, unlike for finite dimensional ones, the Harish-Chandra homomorphism relates irreducible representations of $\mathfrak{g}$ not necessarily to irreducible ones of $\mathfrak{l}$. To get to irreducible representations of $\mathfrak{l}$, one has to consider irreducible subspaces of $H^k(\mathfrak{u}, V)$

In the finite dimensional case, the Harish-Chandra homomorphism led to an injective map from irreducible representations of $\mathfrak{g}$ to those of $\mathfrak{l}$. Now, in the infinite dimensional case, we not only can no longer speak of an injective map since the action of $\mathcal{Z}_{\mathfrak{g}}$ no longer uniquely determines the representation, we do not even necessarily have a well-defined map.

Further, one may wonder about the different actions on both sides of the equation: $\mathcal{Z}_{\mathfrak{g}}$ acting on V alone, and $\mathcal{U}(\mathfrak{l})$ acting on simultaneously on $\mathfrak{u}$ and V. The basic explanation is that those are the actions for which the argument works. Note that in particular, $\mathcal{Z}_{\mathfrak{l}}$ acting on V only is not a well-defined action on $H^k(\mathfrak{u}, V)$: Since $\mathcal{Z}_{\mathfrak{l}} \not\subseteq \mathcal{Z}_{\mathfrak{g}}$, this action of $\mathcal{Z}_{\mathfrak{l}}$ usually does not commute with the map d on Hom-spaces, and hence does not descend to cohomology.

Finally, the structure of the proof seems to suggest that all information about V as a representation is encoded in Q. Since for any space W, we have the interpretation of

$H^0(\mathfrak{u}, W)$ as $\mathfrak{u}$-invariants of W, one might consider working with $H^0(\mathfrak{u}, Q_k)$ instead of $H^k(\mathfrak{u}, V)$ in the application of this theorem. However, as a quotient Q is rather complicated, and becomes more so if one wants to repeat the step and lower the cohomology degree further. Thus the gain is not worth the price.

5 A Harish-Chandra Homomorphism for Groups? - Some Remarks

Having established the Harish-Chandra homomorphism for reductive Lie algebras, the question arises whether a similar construction can be found for reductive groups. For example, one may consider the Iwasawa or some other decomposition of G, and attempt to mirror the Lie algebra construction.

However, these attempts fail, mostly because the center of the Lie group is to small to yield enough information for an analogous construction to the Harish-Chandra homomorphism above to be useful for representation theory.

In general, the group decomposition will in one way or another not have "enough algebraic structure" to allow a similar construction. The Lie algebra and its universal enveloping algebra, on the other hand, have "enough" algebraic structure, but they only capture local information of the group. For the purpose of classifying representations of a possibly disconnected group, local information is insufficient. The crucial point in generalizing the Harish-Chandra homomorphism to groups is to find an object that has enough algebraic structure to allow a similar construction, and at the same time captures enough information about the group to be useful for representation theory. It turns out that $(\mathfrak{g}, K)$-pairs[47] in the group's stead in place of the Lie algebra $\mathfrak{g}$ and Hecke algebras $\mathcal{R}(\mathfrak{g}, K)$ as a subalgebra of the distribution algebra on G in place of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ satisfy this criterion.

Two typical algebras one may consider on a Lie group G are its Lie algebra and the space of functions on G as a manifold. The Lie algebra reflects local and algebraic information of the Lie algebra, like the multiplicative structure of the Lie group. The space of functions captures analytic and global properties, like the number of connected components.[48] The

[47] Here, $\mathfrak{g}$ is the complexified Lie algebra of G, and K is a compact subgroup of G. Part of our discussion in section 7 will be about choosing an appropriate subgroup.

[48] For those eager for detail: One can consider the ring of analytic functions on the Lie group, and in it those which are idempotent, that is functions for which $f = f^2$. These functions can only take the values 0 and 1, and have to be constant on any connected component. Thus, if there are 2^n distinct such functions, then the group has n components.

algebra of distributions contains both algebras as subalgebras[49], and seems therefore a good candidate for a framework in which we can discuss representations of G employing both information. In passing to the distribution algebra we lose algebraic structure - multiplication is no longer well-defined. However, convolution of distributions is well-defined, and provides enough structure to turn the space of distributions into an algebra.

In principle, most of our analysis to follow could probably be done using the distribution algebra. However, computations simplify and we see the "meat" of what is going on by restricting our discussion to the Hecke subalgebras of the distribution algebra. The Hecke algebras have some convenient properties: Their dimension is at most countable infinite[50], on compact groups we know an explicit relation to representations on that group[51], and very similarly to the correspondence between representations of a Lie algebra $\mathfrak{g}$ and unital left $\mathcal{U}(\mathfrak{g})$-modules, there is a one-to-one correspondence between representations of $(\mathfrak{g}, K) - pairs$ and approximately unital left $R(\mathfrak{g}, K)$-modules.

To make it work, we will consider $(\mathfrak{g}, K)$-pairs related to the Lie group G instead of the group itself. Representations of G and those of $(\mathfrak{g}, K)$ are not in one-to-one correspondence, but they are closely related. This relation was extensively studied by Harish-Chandra, and we will not concern ourselves with the details of this relation for now.

In other words, this set-up is not the ideal set-up one might wish for. However, until a better construction is found, this is the best compromise we have.

In the next section, we will introduce the machinery of $(\mathfrak{g}, K)$-pairs, distributions and Hecke algebras that will be needed in the generalization of the Harish-Chandra homomorphism discussed in the following sections. Before we do so, a word of warning is appropriate. The analysis of Harish-Chandra homomorphism on the level of Lie algebras and universal enveloping algebras compares to the generalization we are about to venture into like a room in a hut to a room in a palace: The rooms themselves have many properties in common, they may be of similar size and shape. However, finding the room or describing its location with the hut or palace is very different. It is comparatively easy to orient oneself within the hut with its two or three rooms - just as it is relatively easy

[49]The universal enveloping algebra of the Lie algebra is isomorphic to the subalgebra of distributions supported at the identity. Also, for a fixed Haar measure $d\mu$ on G, compactly supported smooth functions t on G corresponds to a distribution T_t defined by

$$\langle T, f \rangle = \int_G f(g)t(g)d\mu.$$

[50]This allows an analogous statement of Schur's lemma to hold.

[51]Namely, $R(K) = \oplus_{\gamma \in \hat{K}} End(E_\gamma)$. See further down for explanation of notation.

to work with the universal enveloping algebras once one has mastered linear algebra and the Poincaré-Birkhoff-Witt theorem. In the palace, however, you may live for years and still discover rooms you have never seen before. There are cathedral-like halls, twisted staircases, quirky small rooms, and secret passages. Similarly, the distribution algebra on a group is very large and rich in structures, interesting subalgebras, and relations to all kinds of other mathematical objects. For the purpose of a generalized Harish-Chandra homomorphism we only need to analyze some special properties of a special subalgebra. Speaking within the picture, we only need to get to one particular room. It is beyond the scope of this work to discuss the complete outlay of the palace. Instead we will climb one staircase step by step, and find our way to the one room we are interested in. For those familiar with the palace, this may be a slow process, as for them the direction "Let's go up this staircase" would suffice. For those new to this part of the palace, it might at times be mysterious how we take each next step, how we know that the step is there and nowhere else, and they may wonder what is inside all of the other rooms we will not enter. Both readers will need some patience in the following discussion.

6 Excursion: $(\mathfrak{g}, K)$-Pairs, Distribution and Hecke Algebras

6.1 $(\mathfrak{g}, K)$-Pairs

As indicated in the previous section 5, the generalized Harish-Chandra homomorphism we will construct in section 7 will be for $(\mathfrak{g}, K)$-pairs rather than Lie groups G. In the following paragraphs, we give a definition for $(\mathfrak{g}, K)$-pairs, and outline some of their properties.

A *$(\mathfrak{g}, K)$-pair* is a pair consisting of a finite dimensional complex Lie algebra $\mathfrak{g}$ and a compact, possibly disconnected Lie group K such that[52]

i) the complexified Lie algebra $\mathfrak{k} = \mathrm{Lie}(K)^{\mathbb{C}} := \mathrm{Lie}(K) \oplus i\,\mathrm{Lie}(K)$ of K is a subalgebra of $\mathfrak{g}$[53],

ii) K acts on $\mathfrak{g}$ extending the adjoint action of K on $\mathfrak{k}$, we denote this action by $\mathrm{Ad}(k)$ for $k \in K$,

[52]see p. 67 in [11]

[53]This last part of the condition means, of course, "is isomorphic to a subalgebra of $\mathfrak{g}$". However, this abuse of notation is harmless, so we will continue to use it. Moreover, in all the cases we will consider in the following, $\mathfrak{k}$ will indeed be a proper subalgebra of $\mathfrak{g}$.

iii) the differential of this action $\mathrm{Ad}(K)$ is $\mathrm{ad}(\mathfrak{k}) \subseteq \mathrm{ad}(\mathfrak{g})$.

These conditions are often referred to as *compatibility conditions* of $\mathfrak{g}$ and K.

Example 6.1 If G is a Lie group, $\mathfrak{g}$ its complexified Lie algebra, and K a compact subgroup, then with the usual adjoint action of the G on $\mathfrak{g}$ restricted to K, $(\mathfrak{g}, K)$ is a pair satisfying the above compatibility conditions. We will call such pairs *directly derived from G.*

However, not every $(\mathfrak{g}, K)$-pair arrives this way from a Lie group G.

Example 6.2 Let $\mathfrak{g}$ be the complex Lie algebra spanned over the complex numbers $\mathbb{C}$ by $\langle x, y, z\rangle$ with the relations

$$[x, y] = 0, \qquad [x, z] = x, \qquad [y, z] = 2iy$$

and let $K = \{1\}$ be the trivial compact Lie group. Then $(\mathfrak{g}, K)$ satisfies the above conditions, almost trivially. However, since $\mathfrak{g}$ is not of the form $\mathfrak{g}_0 \oplus i\mathfrak{g}_0$ for some real Lie algebra $\mathfrak{g}_0$[54] no Lie Group G exists such that $\mathfrak{g}$ is its complexified Lie algebra.

The previous example notwithstanding, within the following discussion we will always treat $(\mathfrak{g}, K)$-pairs as being directly derived from a Lie group G in the same way as described in example 6.1. For the purpose of this work, this is not a restriction because

[54]To see that there is no such real Lie algebra, suppose there were a real Lie algebra $\mathfrak{g}_0$ such that $\mathfrak{g} \cong \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. So in particular, $z = a + ib$ for some $a, b \in \mathfrak{g}_0$. Writing a and b in terms of the basis $\langle x, y, z\rangle$,

$$a = a_1 x + a_2 y + a_3 z, \quad \text{and} \quad b = b_1 x + b_2 y + b_3 z$$

implies

$$a_1 + ib_1 = 0, \qquad a_2 + ib_2 = 0, \qquad a_3 + ib_3 = 1.$$

So at least one of a_3 or b_3 has to be non-zero. Choosing a or b, depending on which of a_3 or b_3 is non-zero, we have shown that the real Lie algebra $\mathfrak{g}_0$ contains an element $A = cz + dx + ey$, with c not equal to zero.

Now $\mathrm{ad}(A)|_{\mathfrak{g}_0}$ is a real-linear transformation of the three-dimensional vector space $\mathfrak{g}_0$. It therefore has three real eigenvalues, or else one real eigenvalue and a pair of conjugate complex eigenvalues. These eigenvalues are exactly the same as the eigenvalues of the complex-linear transformation $\mathrm{ad}(A)$ acting on $\mathfrak{g}$. We find those with respect to the basis $\langle x, y, A\rangle$

$$[A, x] = -cx, \qquad [A, y] = -2icy, \qquad [A, A] = 0.$$

In other words, the eigenvalues of $\mathrm{ad}(A)$ are 0, $-c$, and $-2ic$. Since by construction, c is not zero, the two non-zero eigenvalues $-c$ and $-2ic$ are neither both real nor complex conjugates of each other. This contradiction shows that no such real Lie algebra $\mathfrak{g}_0$ can exist.

we are interested in $(\mathfrak{g}, K)$-pairs and their representations only in so far as they help us understand representations of the Lie group G the pair is derived from.

A *locally K-finite representation of* $(\mathfrak{g}, K)$, or a *locally K-finite* $(\mathfrak{g}, K)$*-module* is a triplet (V, π, σ) of a complex vector space V carrying representations $\pi : \mathfrak{g} \to End(V)$ and $\sigma : K \to Aut(V)$ of $\mathfrak{g}$ and K, respectively, such that[55]

i) as a representation of K, (V, σ) is locally K-finite,

ii) the differential of the action of K is the restriction to $\mathfrak{k}$ of the action of $\mathfrak{g}$, that is $d\sigma = \pi|_{\mathfrak{k}}$,

iii) $\pi\left((\mathrm{Ad}(k).X)\right) v = \sigma(k)\left(\pi(X)(\sigma(k^{-1})v)\right)$ for all $k \in K$, $X \in \mathcal{U}(\mathfrak{g})$, and $v \in V$.[56]

Example 6.3 If (V, σ) is a finite dimensional representation of a Lie group G, and $(\mathfrak{g}, K)$ is a pair directly derived from G, then $(V, d\sigma, \sigma|_K)$ is a locally K-finite representation of $(\mathfrak{g}, K)$.

However, even if $(\mathfrak{g}, K)$ is a pair directly derived from a Lie group G, not every $(\mathfrak{g}, K)$-module V extends to a representation of G on V.

Example 6.4 Let G be the Lie group consisting of two circles in the quaternions $\mathbb{H}$[57], given by

$$G = \{\cos(\alpha) + i\sin(\alpha)\} \sqcup \{j\cos(\beta) + k\sin(\beta)\}$$

with group multiplication being multiplication in $\mathbb{H}$. Let K be the identity component of G, and $\mathfrak{g}$ the complexified Lie algebra, i.e.

$$K = G_0 = \{\cos(\alpha) + i\sin(\alpha)\}, \quad \mathfrak{g} = \mathrm{Lie}(G) = \mathrm{Lie}(K) = (\mathbb{R})^{\mathbb{C}} = \mathbb{C}.$$

In particular, $(\mathfrak{g}, K)$ is directly derived from G. Also, K acts on $c \in \mathbb{C}$, by rotation, that is

$$\sigma(\cos(\alpha) + i\sin(\alpha)).c = e^{i\alpha}c,$$

[55]p. 75 in [11]. Note that [11] use "representation" instead of "locally K-finite representation". We choose the more specific name, because more general representations of $(\mathfrak{g}, K)$-pairs exist, although there are technicalities involved in defining them, for example a topology has to be fixed to ensure that the differential in point ii) exists. Moreover, in our later discussion, the property of being locally K-finite is crucial, and hence should be emphasized. We will define "locally K-finite" in section 6.4.

[56]Recall that as a representation of $\mathfrak{g}$, V is a $\mathcal{U}(\mathfrak{g})$-module.

[57]Recall that the quaternions are spanned as vector space over $\mathbb{R}$ by $\langle 1, i, j, k\rangle$. Multiplication is defined by 1 acting as identity, and

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

and $\mathfrak{g}$ acts by the differential of this action: $d\sigma(X)c = e^{iX}c$ for $X \in \mathfrak{g} = \mathbb{C}$. The action of K and $\mathfrak{g}$ is compatible by construction. However, there is no action of G on $\mathbb{C}$ that restricts to this action of K and $\mathfrak{g}$.[58]

6.2 Distributions: Definition and Examples

Distributions on a Lie group G form an algebra with respect to convolution. This algebra is rich in both analytical and algebraic structures. Because of these structures one can approach the theory of distributions from a range of different angles. In working practice, of course, one often has to readily switch between these different perspectives.

From an analytical perspective, the distribution algebra is a generalization of the space of functions. The space of functions is a well-studied space used to analyze the manifold structure underlying a Lie group, but in many ways it is often insufficient. One of these insufficiencies is that differentiation is not always well-defined, even when one thinks it should be. For example, let f be an integrable function on the real line. Its Fourier transform

$$F(f)(\zeta) = \int_{\infty}^{\infty} e^{-ix\zeta} f(x)dx \quad \text{for } \zeta \in \mathbb{R}$$

has the property that

$$F\left(-i\frac{d}{dx}f\right) = x \cdot f.$$

The right hand side of this equation is always defined. So the same should be true for the left hand side. In particular, $-i\frac{d}{dx}f$ should be well-defined for all functions f on the real line.

As a second example, consider the following two partial differential equations in two

[58]In detail: The automorphism group $Aut(\mathbb{C})$ is commutative, but G is non-abelian. Thus, if there was such an extension $\sigma : G \to Aut(\mathbb{C})$, then

$$\begin{aligned}
\sigma\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) &= \sigma\left(j \cdot (\frac{1}{\sqrt{2}}j + \frac{1}{\sqrt{2}k})\right) \\
&= \sigma(j)\sigma\left(\frac{1}{\sqrt{2}}j + \frac{1}{\sqrt{2}k}\right) \\
&= \sigma\left(\frac{1}{\sqrt{2}}j + \frac{1}{\sqrt{2}}k\right)\sigma(j) \\
&= \sigma\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right).
\end{aligned}$$

But rotation by $\theta = 3/4\pi$, for which $\cos(\alpha) + i\sin(\alpha) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ is not the same action as rotation by $\alpha' = -3/4\pi$, for which $\cos(\alpha') + i\sin(\alpha') = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$. This is a contradiction, and therefore no extending representation of G on $\mathbb{C}$ can exist.

variables

$$(a)\ \frac{\partial^2}{\partial x^2}f(x,y)+\frac{\partial^2}{\partial y^2}f(x,y)=0 \qquad (b)\ \frac{\partial^2}{\partial x^2}g(x,y)-\frac{\partial^2}{\partial y^2}g(x,y)=0.$$

Solutions to either of these equations are twice continuously differentiable functions satisfying the equations everywhere. It seems reasonable to expect uniform limits of solutions to be solutions. While this is true for solutions to (a), it is not for solutions to (b). [59]

The distribution algebra is in some sense the smallest extension of the space of continuous function such that differentiation is always well-defined. Convolution of distributions in this context is the extension of the convolution of functions.

On the other hand, from an algebraic point of view, the distribution algebra on a Lie group G generalizes other algebras associated with the Lie group. In particular, it contains subalgebras isomorphic to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, and to the group algebra $\mathbb{C}[G]$[60], respectively. Thus it contains all the information about G that these algebras contain. Convolution in the algebraic perspective arrives as multiplication in the algebra; to coincide with the multiplication of these relevant subalgebras.

After these introductory remarks we now move to the definition, or one possible one, of distributions. On a Lie group G, the *algebra of distributions with compact support* $\mathfrak{D}_0(G)$[61] is defined to be the continuous[62] dual of the space of smooth functions, that is[63]

$$\mathfrak{D}_0(G)=\{T : C^\infty(G)\to\mathbb{C} \mid T \text{ is continuous}\}.$$

In other words, we think of distributions on G as maps sending smooth functions on G to the complex numbers. Typically, we denote a distribution T being applied to a function f by

$$\langle T, f\rangle \quad \text{or} \quad \int_G f(g)dT(g).$$

[59] These two examples were taken from the introductory chapter of [5].

[60] Here, $\mathbb{C}[G]$ is the algebra formally generated by δ_g with $g\in G$, and $\delta_{g_1}*\delta_{g_2}=\delta g_1g_2$.

[61] There is a more general definition of distributions on manifolds. For details about this, see for example section 6.3 in [5]. Starting with this more general definition, one can define the notion of "compact support", and prove that the distributions of compact support are isomorphic to the continuous dual of $C^\infty(G)$. However, we are only interested in distributions with compact support, so we skip the more general definition. In the following, distribution will always mean distribution with compact support.

[62] Here, continuous means continuous with respect to the topology given by a family of seminorms. However, the topology will play no role in our discussion.

[63] see p. 39 in [11] for further reference.

Example 6.5 Let $d\mu$ be a fixed (Haar-) measure on the Lie group G, and let $\varphi : G \to \mathbb{C}$ be a smooth function on G with compact support. Then

$$\langle \Phi, f \rangle = \int_G f(g)\varphi(g)d\mu(g)$$

is a distribution on G.

Example 6.6 Let g be a fixed element in G. Then $f \mapsto f(g)$ is a distribution on G. This is the *Dirac distribution* at g. It is denoted by δ_g.

Example 6.7 Let D be a directional derivative in the neighborhood of the identity $1 \in G$. Then $f \mapsto Df(g)|_{g=1}$ is a distribution on G.

Example 6.8 The Lie algebra $\mathfrak{g}$ of a Lie group G can be identified with the tangent space of G at the identity, and thus with the space directional derivatives in a neighborhood of the identity. In other words, by the previous example we can think of elements in the Lie algebra as elements in the distribution algebra on G. In particular, for $X \in \mathfrak{g}$

$$\langle \partial_X, f \rangle = \frac{d}{dt}(f(\exp tX))|_{t=0}. \tag{13}$$

Example 6.9 Generalizing the previous example, we would like to associate distributions to general elements in the universal enveloping algebra of the Lie algebra $\mathfrak{g}$. One way to do this is via a detour to right actions. The above equation (13) associating distributions to elements in the Lie algebra is a special case of the right action of the Lie algebra on the space of smooth functions. Indeed, elements X in the Lie algebra $\mathfrak{g}$ act on functions $f \in C^\infty(G)$ by

$$(r(X)f)(g) = \frac{d}{dt}(f(g \cdot \exp tX))|_{t=0}.$$

Restricting this equation to evaluation at $g = 1$ recovers the above equation (13). Now, by universality, this right action of $\mathfrak{g}$ extends to a right action of elements of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ on the space of smooth functions. Restricting this action to evaluation at $g = 1$, we have come full circle, and find that

$$\langle \partial_X, f \rangle = (r(X)f)(1)$$

associates distributions to elements $X \in \mathcal{U}(\mathfrak{g})$.

The space of distributions on a Lie group G is an algebra with respect to convolution. The definition of convolution of distributions can be derived via duality from the convolution of functions, or as a push-forward of the group multiplication $m : G \times G \to G$[64]. Concretely, the convolution of two distributions S and T is defined by

$$\langle S * T, f \rangle = \int_{G\times G} f(gh) dS(g) dT(h).$$

Example 6.10 Let $d\mu$ be a fixed (Haar-) measure, and let Φ and Ψ be two distributions associated to two smooth, compactly supported, complex-valued functions φ and ψ, as in example 6.5. Then

$$\begin{aligned}
\langle \Phi * \Psi, f \rangle &= \int_{G\times G} f(gh)\varphi(g)\psi(h) d\mu(g) d\mu(h) \\
&= \int_{G\times G} f(g')\varphi(g)\psi(g^{-1}g') d\mu(g) d\mu(g') \\
&= \int_G f(g') \left[\int_G \varphi(g)\psi(g^{-1}g') d\mu(g) \right] d\mu(g') = \int_G f(g')(\varphi * \psi)(g') d\mu(g').
\end{aligned}$$

This establishes an algebra homomorphism from the algebra of compactly supported smooth functions with convolution to the subalgebra generated by all distributions of the form $\varphi(g)d\mu(g)$. In fact, this subalgebra is isomorphic to the space $C^{\infty}_{com}(G)$ of such functions.

Example 6.11 For the Dirac distributions δ_g considered in example 6.6, it turns out that

$$\delta_{g_1} * \delta_{g_2} = \delta_{g_1 g_2}.$$

Moreover, the subalgebra generated by all these Dirac distributions is isomorphic to $\mathbb{C}[G]$.

Example 6.12 For distributions associated to elements X in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ as in example 6.9, convolution yields[65]

$$\partial_{X_1} * \partial_{X_2} = \delta_{X_1 \cdot X_2}$$

where $X_1 \cdot X_2$ is the multiplication in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. This establishes an algebra homomorphism from $\mathcal{U}(\mathfrak{g})$ to the subalgebra generated by all distributions associated to elements in the $\mathcal{U}(\mathfrak{g})$. Not surprisingly, this subalgebra of $\mathfrak{D}_0(G)$ is isomorphic to $\mathcal{U}(\mathfrak{g})$.

As these examples show, the distribution algebra indeed contains subalgebras isomorphic to the space of $C^{\infty}_{\mathrm{com}}(G)$ with convolution, the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, and the group algebra $\mathbb{C}[G]$ as indicated in the beginning of this section.

[64]see p. 41 and theorem B.20 in [11] for more details of this derivation.

[65]see p.42 of [11] for details

6.3 Some Properties of Distributions

In this subsection we assemble some of the properties of distributions that will be relevant to the subsequent discussion.

First, we introduce the notion of the support of a distribution. For any distribution $T \in \mathfrak{D}_0(G)$ the *support of* T is the set[66]

$$\mathrm{supp}(T) = \{g \in G \mid \nexists \text{ open } U \subseteq G \text{ with } g \in U \text{ and } T|_U \equiv 0\}.$$

As indicated in the definition of $\mathfrak{D}_0(G)$, in this work we only discuss distributions with compact support.

Example 6.13 The Dirac distribution $\delta_g : f \mapsto f(g)$ for $g \in G$, as considered in example 6.6, has support at $\{g\}$.

Example 6.14 The distributions ∂_X associated to an element X in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, as introduced in example 6.9, have support at the identity. In fact, every distribution with support at the identity is of the form ∂_X for some $X \in \mathcal{U}(\mathfrak{g})$[67].

From the definition of the support of a distribution it follows that for any two distributions $S, T \in \mathfrak{D}_0(G)$

$$\mathrm{supp}(S * T) \subseteq \overline{(\mathrm{supp}(S)) \cdot (\mathrm{supp}(T))}$$

where multiplication on the right hand side is group multiplication, and the bar denotes closure.[68] This inclusion implies that the algebra of distributions of compact support is well-defined.

Next, we find distributions on a Lie group G can not only be applied to complex-valued functions on G, but to functions with values in any finite dimensional complex vector space. In detail, let $T \in \mathfrak{D}_0(G)$ a distribution as defined above, and let V be such a finite dimensional complex vector space. Then T induces a map

$$\widetilde{T} : \{\varphi : G \to V \mid \varphi \text{ is smooth}\} \to V$$

[66]Equivalently, the support can be defined as the complement of the set

$$G \backslash \mathrm{supp}(T) = \{g \in G \mid \exists \text{ open } U \subseteq G \text{ with } g \in U \text{ and } T|_U \equiv 0\}.$$

This definition is indirect, but often more convenient to work with.

[67]For more details see statement B.33 in [11] (with $X = \{1\}$, $Y = G$, and $p = 1$)

[68]To prove this, show that the reverse inclusion holds for the complements.

by

$$\langle \widetilde{T}, \varphi \rangle = \int_G \varphi(g) dT(g) .$$

In other words, we can think of a distribution as an assignment of weights to the elements of the group. Any smooth function on G with values in the complex vector space V can be integrated against these weights. [69]

As an application of the previous paragraph we observe that distributions supported in a compact Lie group K act on any finite dimensional smooth representation of K: Let (V, σ) be such a representation of K, and T a distribution supported in K. Then T acts on $v \in V$ by sending the V-valued function $g \mapsto \sigma(g)v$ to a value in V. That is,

$$\widetilde{\sigma}(T)v = \langle \widetilde{T},\, g \mapsto \sigma(g)v \rangle = \int_G \sigma(g) v dT(g). \tag{14}$$

To conclude this section, we exhibit the left and right action by the group G and by the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$[70] on the distribution algebra $\mathfrak{D}_0(G)$. Let $g \in G$, $X \in \mathcal{U}(\mathfrak{g})$, and $T \in \mathfrak{D}_0(G)$, then

$$l(g)T = \delta_g * T \quad \text{and} \quad r(g)T = T * \delta_{g^{-1}}, \tag{15}$$

and

$$l(X)T = \partial_X * T \quad \text{and} \quad r(X)T = T * \delta_{X^T}{}^{71} \tag{16}$$

are well-defined actions on distributions in $\mathfrak{D}_0(G)$.

It is worth remarking, that these left and right actions arise from the regular left and right actions on the space of smooth functions via duality. In detail, the right and left action of group elements $g \in G$ on functions $f \in C^\infty(G)$ are given by

$$(l(g)f)(h) = f(g^{-1}h) \quad \text{and} \quad (r(g)f)(h) = f(hg).$$

[69] On the side: In the special case where $\langle P, 1 \rangle = \int_G dP(g) = 1$, the distribution P is sometimes called *probability distribution*. The support of such probability distributions is often defined as the smallest closed set such that the probability outside this set is zero. This is consistent with our more general definition. Also, if P_1 and P_2 are two probability distributions, then

$$\langle P_1 * P_2, 1 \rangle = \int_{G \times G} dP_1(g) dP_2(g') = 1.$$

So probability distributions are closed under convolution - however, they do not form a subalgebra, as they are not closed under addition or scalar multiplication.

[70] Where $\mathfrak{g}$ is the complexified Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)^{\mathbb{C}}$ of G.

[71] Here, ^{T} refers to the unique antiautomorphism of $\mathcal{U}(\mathfrak{g})$ characterized by $Z^T = -Z$ and $(XY)^T = Y^T X^T$ for all $Z \in \mathfrak{g}$, and all $X, Y \in \mathcal{U}(\mathfrak{g})$. This automorphism is often called transpose. See statement 3.7 in [7] for a proof of existence and uniqueness of this map.

These induce representations on the distribution algebra $\mathfrak{D}_0(G)$, which after some rewriting and manipulation can be seen to be the above defined left and right actions. In other words,

$$\langle l(g)T, f\rangle = \langle T, l(g^{-1})f\rangle \quad \text{and} \quad \langle r(g)T, f\rangle = \langle T, r(g^{-1})f\rangle. \tag{17}$$

Similarly, left and right actions by elements X in the real Lie algebra $\mathfrak{g}_0 = \mathrm{Lie}(G)$ on functions $f \in C^\infty(G)$ are given by

$$(l(X)f)(g) = \frac{d}{dt}(f([\exp tX]^{-1} \cdot g))|_{t=0} = \frac{d}{dt}(f(\exp -tX \cdot g))|_{t=0}$$

$$\text{and} \quad (r(X)f)(g) = \frac{d}{dt}(f(g \cdot \exp tX))|_{t=0}$$

These extend to left and right actions by elements in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, and the thus induced actions on the distribution algebra are precisely those defined above.

6.4 Hecke Algebras

When one is looking for an object with certain properties, it occasionally works out to build some of these properties into the definition. The same is true here. Our object of study - Hecke algebras - are subalgebras of the distribution algebra on elements of which a compact subgroup acts in a certain way. This particular way of the group acting on the Hecke algebra implies certain nice properties which will simplify our computations.

This "certain" way in which the compact subgroup K acts is called "K-finite". The following two paragraphs introduce this notation.

Let K be a compact, possibly disconnected Lie group, and (V, σ) a representation of K. The vector space V might be infinite dimensional, and we do not require σ to be continuous.[72] For every vector $v \in V$, we can consider the subspace $\langle\sigma(K)v\rangle$ generated by the action of K on v. If this space is finite dimensional, then it has a natural topology, namely the one of $\mathbb{C}^{\dim(V)}$, and we may ask whether σ is smooth on this subspace. If $\langle\sigma(K)v\rangle$ is finite dimensional and $\sigma|_{\langle\sigma(K)v\rangle}$ is smooth, we call v a *K-finite* vector in V. The space of all K-finite vectors is denoted by V_K. If all vectors in V are K-finite, then V is called *locally K-finite.*[73]

Example 6.15 Every continuous finite dimensional representation is locally K-finite.

[72] This is called to consider the representation in the algebraic sense.

[73] see p.45 in [11]

Example 6.16 Let $K = \mathbf{O}_2$ be the group of real orthogonal matrices. Let V be the vector space formally spanned by elements of the form $\{e_{ia}, e_{ib}\}_{i\in\mathbb{N}}$. Further let σ be the following homomorphism

$$\sigma(k)(e_{ia}) = \begin{cases} e_{ia} & \text{if } \det(k) = 1 \\ e_{ib} & \text{if } \det(k) = -1 \end{cases} \quad \text{and} \quad \sigma(k)(e_{ib}) = \begin{cases} e_{ib} & \text{if } \det(k) = 1 \\ e_{ia} & \text{if } \det(k) = -1 \end{cases}$$

Then $\dim(V) = \infty$, but $\dim(\langle\sigma(K)v\rangle) < \infty$[74] This is a locally K-finite representation.[75]

Example 6.17 As seen in the previous section, an element k of a compact Lie subgroup $K \subseteq G$ acts from the left and the right on distributions $T \in \mathfrak{D}_0(G)$ by

$$l(k)T = \delta_k * T \quad \text{and} \quad r(k)T = T * \delta_{k^{-1}}$$

We denote the corresponding subspaces of K-finite distributions by $\mathfrak{D}_0(G)_{K,l}$ and $\mathfrak{D}_0(G)_{K,r}$, respectively.

As a special case of the previous example, let $\mathfrak{D}_K(G)$ denote distributions on a Lie group G with support in a compact subgroup $K \subseteq G$[76]. Then K acts on it from the left and the right as above. However, for this special case, the distinction between left- and right-K-finiteness does not matter:

Proposition 6.18 For the right and left action of a compact Lie group $K \subseteq G$ on distributions on G with support in K [77],

$$\mathfrak{D}_K(G)_{K,l} = \mathfrak{D}_K(G)_{K,r}$$

We are now ready to define Hecke algebras. Let G be a Lie algebra, $\mathfrak{g}$ its complexified Lie algebra, K a compact subgroup of G. Then the Hecke algebra $\mathcal{R}(\mathfrak{g}, K)$ is the algebra of K-finite distributions on G with support in K.[78,79]

[74] Namely, $\dim(\langle\sigma(K)v\rangle) \in \{0,1,2\}$ for all $v \in V$.

[75] This V consists just of infinitely many copies of the determinant representation of $\mathbf{O}_2$ together with infinitely many copies of the trivial representation of $\mathbf{O}_2$. More complicated examples are, of course, possible.

[76] Note from the definition of distributions and support that a distribution on G with support in K is not the same thing as a distribution on K.

[77] statement 1.83 in [11]

[78] Note that by the previous proposition 6.18, elements in $\mathcal{R}(\mathfrak{g}, K)$ are K-finite with respect to left and right action of K.

[79] Note that this defines the Hecke algebra for any directly derived pair. Hecke algebras can also be defined for pairs not directly derived from any Lie group G. While we are not concerned with such pairs in this work, we will in the periphery of some of our arguments encounter Hecke algebras on such pairs. We will comment on those where they appear.

Proposition 6.19 The space $\mathcal{R}(\mathfrak{g}, K)$ of K-finite distributions on G is a subalgebra of $\mathfrak{D}_0(G)$.

Proof: Since K-finiteness and having the support in K are preserved under addition of and scalar multiplication with distributions, we only need to show that $\mathcal{R}(\mathfrak{g}, K)$ is closed under convolution.

First, if S and T are two distributions on G, and S is K-finite, then so is $S * T$, because

$$\begin{aligned} \dim\langle\sigma(K)S * T\rangle &= \dim\langle\delta_k * (S * T) \mid k \in K\rangle \\ &= \dim\langle(\delta_k * S) * T \mid k \in K\rangle \\ &\leq \dim\langle\delta_k * S \mid k \in K\rangle \\ &< \infty. \end{aligned}$$

The last (strict) inequality holds by assumption.

Also, we noted in the previous section that for any two distributions $S, T \in \mathfrak{D}_0(G)$

$$\operatorname{supp}(S * T) \subseteq \overline{(\operatorname{supp}(S)) \cdot (\operatorname{supp}(T))}$$

where multiplication on the right hand side is group multiplication. Since K is a compact, so in particular closed subgroup of G, this shows that having support in K is preserved under convolution.

This completes the proof. □

Example 6.20 For a complex Lie algebra $\mathfrak{g}$, and $K = \{1\}$, $\mathcal{R}(\mathfrak{g}, \{1\}) \cong \mathcal{U}(\mathfrak{g})$. In fact, $\mathcal{R}(\mathfrak{g}, \{1\})$ is the algebra of K-finite distributions supported at the identity. We noted in example 6.14 that these can be identified with the universal enveloping algebra of $\mathfrak{g}$.[80]

Example 6.21 For $G = K$, $\mathcal{R}(\mathfrak{k}, K)$ is the algebra of K-finite distributions on K. This special case is denoted by $\mathcal{R}(K)$. This Hecke algebra has a nice structure in terms of representations of K, as we shall see in proposition 6.24.

As algebras, the usual notions of left and right modules apply to Hecke algebras $\mathcal{R}(\mathfrak{g}, K)$. It is a known fact that representations of a Lie algebra $\mathfrak{g}$ on complex vector spaces stand in one-to-one correspondence with unital left $\mathcal{U}(\mathfrak{g})$-modules[81]. Our interest in these Hecke algebras arrives mainly from the fact, that they carry an analogous property in a more general setting.

[80]This example is one of the reasons why the Hecke algebra can be considered a generalization of the universal enveloping algebra. Theorem 6.22 is another one.

[81]see, for example, statement 3.6 in [7].

Theorem 6.22 For any complex Lie algebra $\mathfrak{g}$ and compact Lie group K forming a $(\mathfrak{g}, K)$-pair, locally K-finite representations of $(\mathfrak{g}, K)$ stand in one-to-one correspondence with approximately unital left $\mathcal{R}(\mathfrak{g}, K)$-modules.[82]

We will discuss the notion of "approximately unital" in the next section.

To conclude this section, we mention the concept of a "K-type". Having established the notion of locally K-finite representations, we can consider the set of equivalence classes of such representations. This set is denoted by $\hat{K}$. An element in $\hat{K}$ is called a *K-type* γ. If a representation (V, σ) is in the equivalence class γ, then we say (V, σ), or V, is *of type* γ. We write E_γ to indicate a representation of K-type γ.

Observe that the cardinality of the set $\hat{K}$ is at most countable infinite. This is true because for any dimension n there are at most countable infinitely many non-isomorphic irreducible representations on n-dimensional vector spaces, and there are a countable infinite number of dimensions under consideration.

Example 6.23 We know that every finite dimensional representation (V, σ) of a (compact) Lie group K is completely reducible. We can now express this fact in the notation of K-types. Indeed,

$$V \cong \bigoplus_{\gamma \in \hat{K}} Hom(E_\gamma, V) \otimes E_\gamma.$$

The isomorphism from the right to the left is $\varphi \otimes v \mapsto \varphi(v)$. By Schur's lemma, the space $Hom(E_\gamma, V)$ is non-zero if and only if V contains K-invariant subspaces of K-type γ. If it is non-zero, its dimension measures the multiplicity with which E_γ occurs in V. On the other hand, every irreducible K-invariant subspace of V is finite dimensional, hence K-finite, and therefore of type γ for some $\gamma \in \hat{K}$. Thus it is not unreasonable that the above described map is indeed the claimed isomorphism.

6.5 Some Properties of Hecke Algebras

As Hecke algebras $\mathcal{R}(\mathfrak{g}, K)$ will be our main object of study for sections to come, we here discuss some of their relevant structural properties, the notion of "approximate identity", as well as the action of K and of $\mathcal{U}(\mathfrak{g})$ on $\mathcal{R}(\mathfrak{g}, K)$.

First, for the special case of $\mathcal{R}(K) = \mathcal{R}(\mathfrak{k}, K)$ we have a very explicit description of this Hecke algebra's structure.

[82]see p.75 in [11] for the statement for pairs directly derived from a Lie group G, and Theorem 1.117 on p. 90 for details of the proof in the more general setting.

Proposition 6.24 Let K be a compact Lie group, $\hat{K}$ the set of all equivalence classes of irreducible finite dimensional locally K-finite representations, and (E_γ, γ) a representative of type γ. Then[83]

$$\mathcal{R}(K) \cong \bigoplus_{\gamma \in \hat{K}} \mathrm{End}(E_\gamma).$$

This proposition has the following two implications.

Corollary 6.25 The dimension of $\mathcal{R}(K)$ is at most countable infinite.[84]

Proof: We observed in the previous section that the cardinality of $\hat{K}$ is at most countable infinite. Since every E_γ, and hence every summand $\mathrm{End}(E_\gamma)$ is finite dimensional, the claim follows. □

Corollary 6.26 If the compact Lie group K is infinite, the Hecke algebra $\mathcal{R}(K)$ does not have a multiplicative identity.

Proof: If an identity existed, it would have to act by 1 on all E_γ. It therefore would have to be an infinite sum of elements in the $\mathrm{End}(E_\gamma)$, in contradiction to the definition of a direct sum. □

Similar statements are true for the more general Hecke algebras $\mathcal{R}(\mathfrak{g}, K)$, as we will see below.

Before moving on, notice that the latter corollary is a problem in treating the Hecke algebra as a generalization of the *unital* universal enveloping algebra. The next few paragraphs show that this can be repaired by using the notion of an "approximate identity" instead of "identity".

Let R be a ring, and S a directed set[85]. A collection $\{\varsigma_s \in R \,|\, s \in S\}$ of elements in R is called an *approximate identity* if it satisfies the following conditions

[83] The map Υ from left to right is given by

$$(\Upsilon(T))(v_\gamma) = \gamma(T)(v_\gamma) = \int_K \gamma(k) v_\gamma dT(k) \quad \forall\, v_\gamma \in E_\gamma.$$

For details in establishing Υ as isomorphism, see section 1.2 in [11].

[84] Because of this, one can establish an analogous to Schur's lemma, which implies that elements in $\mathcal{R}(K)^K$ act by scalars on irreducible $\mathcal{R}(K)$-modules. We will need this in the generalized construction later on.

[85] A *directed set* S is a set with a partial ordering $\leq$ for which $\forall x, y \in S\ \exists z \in S$ such that $x \leq z \wedge y \leq z$.

i) If $s \leq s'$ for two elements $s, s' \in S$, then $\varsigma_s \varsigma_{s'} = \varsigma_{s'} \varsigma_s = \varsigma_s$. Here multiplication is the ring multiplication in R.

ii) If $r \in R$, then there is an $s \in S$ such that $\varsigma_s r = r \varsigma_s = r$.

We say that a ring R *has an approximate identity* if there exist a non-empty directed set S and a collection of elements in R satisfying the above conditions. A left R-module M is called *approximately unital* if for any $m \in M$ there exists an $s \in S$ such that $\varsigma_s m = m$.[86]

Example 6.27 Every ring with an identity has an approximate identity. To see this, choose S to be the one-element set with the trivial partial ordering, and let $\varsigma_s = 1$.

Example 6.28 The Hecke algebra $\mathcal{R}(K)$ has an approximate identity: Let S be the set of finite subsets of $\hat{K}$ with inclusion as partial ordering, and 1_γ the identity element in $\mathrm{End}(E_\gamma)$. Then

$$\left\{ \varsigma_s = \sum_{\gamma \in s} 1_\gamma \;\middle|\; s \subseteq \hat{K}, |s| < \infty \right\}$$

is an approximate identity.

Somewhat sloppy formulated, this means that since the elements T in $\mathcal{R}(K)$ are only finite sums of elements in $\mathrm{End}(E_\gamma)$, it suffices for the "identity" to be a collection of finite sums.

Now moving on, we can specify the structure of the more general Hecke algebras $\mathcal{R}(\mathfrak{g}, K)$.

Proposition 6.29 Let G be a Lie group, $\mathfrak{g}$ its complexified Lie algebra, $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, K a compact subgroup of G, and $\mathcal{R}(\mathfrak{g}, K)$ the corresponding Hecke algebra of K-finite distributions on G with support in K. Then[87]

$$\mathcal{R}(\mathfrak{g}, K) \cong \mathcal{R}(K) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{U}(\mathfrak{g}).$$

[86] see section 1.3 in [11] for further discussion on approximate identities

[87] The proof consists of four and a half steps, proving

1 that $(T, X) \mapsto T * \partial_X$ is a bilinear map from $\mathcal{R}(K) \times \mathcal{U}(\mathfrak{g})$ to $\mathcal{R}(\mathfrak{g}, K)$,

$1\frac{1}{2}$ this map induces a $\mathbb{C}$-linear map $\mathcal{R}(K) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g}) \to \mathcal{R}(\mathfrak{g}, K)$,

2 it descends to a map $\mathcal{R}(K) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{U}(\mathfrak{g}) \to \mathcal{R}(\mathfrak{g}, K)$,

3 the map $\mathcal{R}(K) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{U}(\mathfrak{g}) \to \mathcal{R}(\mathfrak{g}, K)$ is injective, and

4 it is onto $\mathcal{R}(\mathfrak{g}, K)$.

For more technical details see section 1.4 in [11].

As a side remark, this proposition provides means for defining Hecke algebras for pairs $(\mathfrak{g}, K)$ which are not directly derived from a Lie group G - one can read the proposition's equation as definition.[88]

Back to our main discussion, the proposition has the same two implications for $\mathcal{R}(\mathfrak{g}, K)$ as the previous proposition 6.24 had for $\mathcal{R}(K)$.

Corollary 6.30 The dimension of $\mathcal{R}(\mathfrak{g}, K)$ is at most countable infinite.[89]

Proof: We argued previously that the dimension of $\mathcal{R}(K)$ is at most countable infinite. We also know that the same is true for the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ because by the Poincaré-Birkhoff-Witt theorem we know an explicit countable basis of $\mathcal{U}(\mathfrak{g})$. The claim follows. □

Corollary 6.31 If the compact Lie group K is infinite, then the Hecke algebra $\mathcal{R}(\mathfrak{g}, K)$ does not have a multiplicative identity.

Proof: This follows directly from the previously proven fact, that $\mathcal{R}(K)$ does not have a multiplicative identity. □

However, the Hecke algebra $\mathcal{R}(\mathfrak{g}, K)$ has an approximate identity. Analogous to the approximate identity of $\mathcal{R}(K)$, let S be the set of finite subsets of $\hat{K}$ with inclusion as partial ordering, 1_γ be the identity in $\text{End}(E_\gamma)$, and $1_{\mathcal{U}(\mathfrak{g})}$ be the identity of the universal enveloping algebra. Then

$$\left\{ \varsigma_s = \sum_{\gamma \in s} 1_\gamma \otimes 1_{\mathcal{U}(\mathfrak{g})} \,\middle|\, s \subseteq \hat{K}, |s| < \infty \right\}$$

is an approximate identity.

Analogous to the above decomposition we can of course also consider $\mathcal{R}(\mathfrak{g}, K) \cong \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{R}(K)$. In fact, we will at various times have to work with both decompositions. In particular, we will need explicit expressions for the left and right action of the compact subgroup $K \subseteq G$ on both decompositions of $\mathcal{R}(\mathfrak{g}, K)$. We round off this section by establishing those actions in the next few paragraphs.

[88] Note that in the following we will use that $\mathcal{R}(K) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{U}(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{R}(KK)$. In the case, where the Hecke algebra is defined with respect to a directly derived pair, this isomorphism is clear. However, if one uses the above proposition to define more general Hecke algebras, then this isomorphism requires proof.

[89] Again, this is relevant, because it allows one to establish an analogous to Schur's lemma, which implies that elements in $\mathcal{R}(\mathfrak{g}, K)^{\mathfrak{g}, K}$ act by scalars. We will need this in the generalized construction later on.

For each of the decompositions, we can easily define one of the left and right action by K. The trouble is defining the respective other action. So the question is how to switch back and forth between these two decompositions. In other words, we would like to have an explicit isomomorphism[90]

$$\underbrace{\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{R}(K) \cong \mathcal{R}(\mathfrak{g}, K) \cong \mathcal{R}(K) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{U}(\mathfrak{g})}_{\Psi}$$

that identifies the same element in $\mathcal{R}(\mathfrak{g}, K)$ with its different representations in the two decomposition. In particular, this means that $X \otimes T$ and $\Psi(X \otimes T)$ act in the same way on any $\mathcal{R}(\mathfrak{g}, K)$-module.

By theorem 6.22, we know that approximately unital left $\mathcal{R}(\mathfrak{g}, K)$ modules correspond to locally K-finite representations of $(\mathfrak{g}, K)$-pairs: Let (V, π, σ) be a locally K-finite representation of $(\mathfrak{g}, K)$. Then $X \otimes T$ acts on $v \in V$ by[91]

$$\begin{aligned}(X \otimes T).v = \pi(X)\widetilde{\sigma}(T)v &= \int_K \pi(X)\sigma(k)v dT(k) \\ &= \int_K \sigma(k)\pi(\mathrm{Ad}(k^{-1})X)v dT(k).\end{aligned}$$

This latter equation describes the action of an element of $\mathcal{R}(K) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{U}(\mathfrak{g})$ on $v \in V$. This means we know how an element $k_0 \in K$ acts on it by left action on the $\mathcal{R}(K)$-component. Rewriting this left action using equation (17), which said $\langle l(k_0)T, f\rangle = \langle T, l(k_0^{-1})f\rangle$[92] , we obtain

$$(l(k_0)(X \otimes T)).v = \int_K \sigma(k_0 k)\pi(\mathrm{Ad}(k)X)v dT(k).$$

We can use condition (iii) in the definition of locally K-finite representations of $(\mathfrak{g}, K)$-pairs once more to rewrite this expression in terms of the decomposition $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{R}(K)$.

$$(l(k_0)(X \otimes T)).v = \int_K \sigma(k_0 k)\pi(\mathrm{Ad}(k^{-1})X)v dT(k)$$

[90] We will not actually make this homomorphism explicit. However, it will help us conceptually to compute the desired actions.

[91] The first equality holds by definition of $\widetilde{\sigma}$ in (14). The second equality follows from a variation of condition (iii) in the definition of locally K-finite representations of $(\mathfrak{g}, K)$-pairs.

[92] So in particular, for the function $\sigma_v(k) = \sigma(k)v$, we have

$$\langle l(k_0)T, \sigma_v\rangle = \langle T, l(k_0^{-1})\sigma_v\rangle = \int_K \sigma_v(k_0 k) dT(k).$$

$$
\begin{aligned}
&= \int_K \sigma(k')\pi(\mathrm{Ad}(k'^{-1}k_0)X)v\,dT(k_0^{-1}k') \\
&= \int_K \sigma(k')\pi(\mathrm{Ad}(k'^{-1})\,\mathrm{Ad}(k_0)X)v\,dT(k_0^{-1}k') \\
&= \int_K \pi(\mathrm{Ad}(k_0)X)\sigma(k')v\,\underbrace{dT(k_0^{-1}k')}_{\delta_{k_0}*T} \\
&= (\mathrm{Ad}(k_0)X \otimes \delta_{k_0} * T)\,.v.
\end{aligned}
$$

This argument has to hold for all $v \in V$, and in fact for all locally K-finite representations (V, π, σ) of $(\mathfrak{g}, K)$-pairs. Therefore,

$$
l(k_0)(X \otimes T) = \mathrm{Ad}(k_0)X \otimes \delta_{k_0} * T.
$$

An analogous computation can be done for the other cases. We summarize for $X \in \mathcal{U}(\mathfrak{g})$, $k \in K$, and $T \in \mathcal{R}(K)$

$$
\begin{array}{rclrcl}
r(k)(X \otimes T) & = & X \otimes T * \delta_{k^{-1}} & r(k)(T \otimes X) & = & T * \delta_{k^{-1}} \otimes \mathrm{Ad}(k^{-1})X \\
l(k)(X \otimes T) & = & \mathrm{Ad}(k)X \otimes \delta_k * T & l(k)(T \otimes X) & = & \delta_k * T \otimes X.
\end{array} \tag{18}
$$

For completeness we also list the actions by elements in the universal enveloping algebra which we will need in our subsequent arguments. Let $X, Y \in \mathcal{U}(\mathfrak{g})$, and $T \in \mathcal{R}(K)$, then[93]

$$
l(Y)(X \otimes T) \;=\; YX \otimes T \qquad\qquad r(Y)(T \otimes X) \;=\; T \otimes XY^T. \tag{19}
$$

7 The Generalized Harish-Chandra Homomorphism

Having assembled all the pieces, we can finally proceed to constructing a generalized Harish-Chandra homomorphism. In first approximation, the construction itself and many of the proofs are very similar to those in section 2. This is no coincidence as the algebras we consider are chosen precisely so that arguments similar to the classical case are possible. In "second order", however, the arguments in this section are different: they are more subtle and require more care.

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[93]Recall that ^{T} is the unique anti-homomorphism, i.e. $(XY)^T = Y^T X^T$ for all $X, Y \in \mathcal{U}(\mathfrak{g})$, on the universal enveloping algebra for which $Z^T = -Z$ for all $Z \in \mathfrak{g} \subseteq \mathcal{U}(\mathfrak{g})$. For more details, see for example, statement 3.7 in [7].

We will use the following notations:

G	a real reductive Lie group
K	a maximal compact subgroup of G[94]
$\mathfrak{g}_0$, $\mathfrak{k}_0$	the Lie algebras of G, and K, respectively
$\mathfrak{g}$, $\mathfrak{k}$	the complexified Lie algebras of G, and K, respectively
θ	Cartan involution such that $\mathfrak{k}_0$ is the $+1$-eigenspace in $\mathfrak{g}_0$ under θ[95]
$\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}$	a θ-stable triangular decomposition of $\mathfrak{g}$[96]
L	a subgroup of G which preserves the triangular decomposition under the adjoint action Ad, and such that $\mathfrak{l}$ is the complexified Lie algebra of L
$\mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{u}^-), \mathcal{U}(\mathfrak{l}), \mathcal{U}(\mathfrak{u})$	the universal enveloping algebras of these Lie algebras
$\mathcal{R}(K), \mathcal{R}(L \cap K), \mathcal{R}(\mathfrak{g}, K), \ldots$	various Hecke algebras

[94]Notice the change in notation. Up to this point, K denoted a general compact subgroup of G. From now on, it stands for a maximal compact subgroup.

[95]Technically speaking, a Cartan involution comes "with the package" of a reductive group. In other words, it is part of the definition of a reductive group. For details, see for example chapter VII.2 in [7]. Practically, we need the triangular decomposition of the Lie algebra $\mathfrak{g}$ to be stable under the Cartan involution θ. This ensures that the triangular decomposition of $\mathfrak{g}$ induces a triangular decomposition of $\mathfrak{k}$. We will not need θ beyond this.

[96]For a θ-stable triangular decomposition of $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}$, we find

$$\mathfrak{k} = (\mathfrak{u}^- \cap \mathfrak{k}) \oplus (\mathfrak{l} \cap \mathfrak{k}) \oplus (\mathfrak{u} \cap \mathfrak{k})$$

is a triangular decomposition of $\mathfrak{k}$, because by definition of θ, $\mathfrak{k} = \mathfrak{g}^\theta$. Note that this is a necessary requirement. For example, for $\mathfrak{g} = \mathbf{sl}_2(\mathbb{C})$ and $\mathfrak{k} = \mathbf{su}_2$,

$$\mathfrak{g}_0 = \underbrace{\begin{pmatrix} 0 & 0 \\ \star & 0 \end{pmatrix}}_{\mathfrak{u}^-} \oplus \underbrace{\begin{pmatrix} \star & 0 \\ 0 & \star \end{pmatrix}}_{\mathfrak{l}} \oplus \underbrace{\begin{pmatrix} 0 & \star \\ 0 & 0 \end{pmatrix}}_{\mathfrak{u}}$$

is a triangular decomposition. However,

$$\mathfrak{u}^- \cap \mathfrak{k} = \mathfrak{l} \cap \mathfrak{k} = \mathfrak{u} \cap \mathfrak{k} = 0$$

So this triangular decomposition of $\mathfrak{g}$ does not restrict to a triangular decomposition of $\mathfrak{k}$. In this example, θ is conjugation transpose, and it does not preserve the triangular decomposition of $\mathfrak{g}$.
It remains to argue that such a θ-stable decomposition always exist. According to proposition 2.1, we can find an $H \in \mathfrak{k}$ such that the spaces spanned by the eigenspaces associated to negative, positive and zero eigenvalues under the adjoint action of H form a triangular decomposition. Forming the corresponding spaces under the adjoint action of H on $\mathfrak{g}$, we obtain a θ-stable triangular decomposition of $\mathfrak{g}$.

We use the property that G is reductive only to ensure that its complexified Lie algebra $\mathfrak{g}$ has a triangular decomposition. We do not use the fact that G is a real group. All of our arguments holds for complex groups as well. However, real groups are generally more complicated to analyze. It is therefore a feature of this discussion that it also holds for real groups.

7.1 Some Remarks on the Generalized Construction

In the next few paragraphs we will discuss some of the aspects that led to the generalized construction as we will present it in the following sections. Those eager to see the construction, can safely skip ahead.

The overall goal of the generalized construction is to relate irreducible locally finite representations of a pair to those of a certain "sub-pair"[97]. Because of the close relation between locally finite representations of pairs and modules of the Hecke-algebra associated to the pair, given by theorem 6.22, the idea is to find an algebra homomorphism between subalgebras of the Hecke algebras associated to the pair and a "sub-pair". This generalizes the classical construction, where the Harish-Chandra homomorphism between subalgebras of the universal enveloping algebras associated to the Lie algebras under consideration, allowed us to link irreducible representations of these Lie algebras.

Ideally, we would like to establish such a relationship between the representations of the pairs $(\mathfrak{g}, K)$ and $(\mathfrak{l}, L \cap K)$, where the notation is as set up above. In other words, we would like to find an algebra homomorphism

$$\mathcal{R}(\mathfrak{g}, K)^K \to \mathcal{R}(\mathfrak{l}, L \cap K)^{L \cap K}$$

which is compatible with the module action of $\mathcal{R}(\mathfrak{g}, K)$ and $\mathcal{R}(\mathfrak{l}, L \cap K)$. However, at least so far, we have not been able to establish such a map. One crucial step in the classical construction, as well as in the generalized construction below is the triangular decomposition which gives a more or less explicit basis for the algebra under consideration. So far, we could not find such a triangular decomposition for $\mathcal{R}(\mathfrak{g}, K)$.[98]

[97]The notation here is admittedly a bit vague. It will become more precise in the next paragraph. The term "sub-pair" refers to a pair which consists of a subalgebra of of the Lie algebra of the original pair, and an appropriate subgroup of the original group.

[98]The closest we could find is

$$\mathcal{R}(\mathfrak{g}, K) \cong \mathcal{R}(K) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{U}(\mathfrak{g}) \cong \mathcal{R}(K) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{U}(\mathfrak{l}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{u}^-) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{u}).$$

It is unfortunately not clear from this decomposition how to project from $\mathcal{R}(\mathfrak{g}, K)^K$ to $\mathcal{R}(\mathfrak{l}, L \cap K)^{L \cap K}$, or alternatively, how to consider $\mathcal{R}(\mathfrak{l}, L \cap K)$ as a subalgebra of $\mathcal{R}(\mathfrak{g}, K)$.

The way in which the triangular decomposition for $\mathcal{R}(\mathfrak{g}, K)$ fails, suggests to consider the pairs $(\mathfrak{g}, L\cap K)$ and $(\mathfrak{l}, L\cap K)$ instead. If $L\cap K$ meets every connected component of K[99], then there is an injective map from the $\hat{K}$[100] to $\widehat{L\cap K}$.[101] So using this adaptation would still allow us to study representations of $(\mathfrak{g}, K)$-pairs. With this approach we would try to construct a map

$$\mathcal{R}(\mathfrak{g}, L\cap K)^{L\cap K} \to \mathcal{R}(\mathfrak{l}, L\cap K)^{L\cap K}$$

which is compatible with the actions of both modules. In this case, one can find such a map.[102] Unfortunately, this map does not yield much information helpful in relating representations - in some sense, $\mathcal{R}(\mathfrak{g}, L\cap K)^{L\cap K}$ is "too large", or $\mathcal{R}(\mathfrak{l}, L\cap K)^{L\cap K}$ is "too small".

One way one might try to salvage this, is to pass some subalgebra of $\mathcal{R}(\mathfrak{g}, L\cap K)^{L\cap K}$ which is "smaller" than $\mathcal{R}(\mathfrak{g}, L\cap K)^{L\cap K}$. A natural choice seems to be to consider invariants under the Lie algebra action. Unfortunately, this does not work either, because in most cases, $\mathcal{R}(\mathfrak{g}, L\cap)^{\mathfrak{l}, L\cap K} = \mathcal{R}(\mathfrak{l}, L\cap K)^{\mathfrak{l}, L\cap K} = 0$.[103]

The case for which we do find a working construction, is another subalgebra of $\mathcal{R}(\mathfrak{g}, L\cap K)$, called the step algebra. This is generally non-zero, and it is small and large enough so that the generalized Harish-Chandra homomorphism is meaningful for studying representations.[104]

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[99]This phrase "meets every connected component" is frequent in the representation theory of possibly disconnected groups. The reason is the following. The action of a disconnected compact group has two aspects to it - the local one, given by the restriction of the action to its identity component, or its Lie algebra, and a global one, reflected by the action of the component group (and how the component group "twists" with the identity component. If one wants to study such representation both aspects need to be considered. By requiring a subgroup to meet every connected component, one ensures that the information regarding the component group is included.

[100]Recall that $\hat{K}$ denotes the set of equivalence classes of irreducible finite dimensional representations.

[101]For finite dimensional representations, this map sends (E_γ, γ) to $(E_\gamma^{\mathfrak{u}\cap\mathfrak{k}}, \gamma|_{L\cap K})$.

[102]See next section 7.2 for details of this construction.

[103]This holds because the support of a distribution on G which commutes with the action of the Lie algebra $\mathfrak{l}$ must be invariant under the conjugation by elements in the identity component L_0 of L. The support for a distribution in $\mathcal{R}(\mathfrak{g}, L\cap K)^{\mathfrak{l}, L\cap K}$ must therefore be a union of conjugacy classes of L_0. On the other hand, it must also be contained in $L\cap K$; and by $L\cap K$-finiteness, it must be a union of connected components of $L\cap K$. In most cases, there are no components of $L\cap K$ which are unions of L_0 conjugacy classes.

[104]This is not to say that this is the only algebra for which this construction works. There may be other algebras for which the construction may be interesting, and in fact it is quite likely that there are.

7.2 Construction

The story of this construction consists of two threads that at the end intertwine and merge to a happy end. One of these threads is establishing an algebra homomorphism

$$\mathcal{R}(\mathfrak{g}, L \cap K)^{L\cap K} \to \mathcal{R}(\mathfrak{l}, L \cap K)^{L\cap K}$$

starting with a decomposition for Hecke algebras analogous to the triangular decomposition of universal enveloping algebras. The other thread is defining step algebras as subalgebras of the Hecke algebras. We find the generalized Harish-Chandra homomorphism by applying the algebra homomorphism to a subalgebra of invariants in the step algebra.

We begin by finding the algebra homomorphism $\mathcal{R}(\mathfrak{g}, L \cap K)^{L\cap K} \to \mathcal{R}(\mathfrak{l}, L \cap K)^{L\cap K}$. We do so by following the classical construction step by step. To begin with, there is "triangular" decomposition for the Hecke algebra $\mathcal{R}(\mathfrak{g}, L \cap K)$.

Proposition 7.1 With the notation set up as above (on page 54),

$$\mathcal{R}(\mathfrak{g}, L \cap K) \cong \mathcal{U}(\mathfrak{u}^-) \otimes_{\mathbb{C}} \mathcal{R}(\mathfrak{l}, L \cap K) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{u}). \tag{20}$$

Proof: We saw in proposition 6.29, that in general $\mathcal{R}(\mathfrak{g}, K) \cong \mathcal{R}(K) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{U}(\mathfrak{g})$. We know from section 2, that the triangular decomposition of $\mathfrak{g}$ lifts to the level of the universal enveloping algebra. Thus

$$\begin{aligned}
\mathcal{R}(\mathfrak{g}, L \cap K) &\cong \mathcal{R}(L \cap K) \otimes_{\mathcal{U}(\mathfrak{l}\cap\mathfrak{k})} \mathcal{U}(\mathfrak{g}) \\
&\cong \mathcal{R}(L \cap K) \otimes_{\mathcal{U}(\mathfrak{l}\cap\mathfrak{k})} \mathcal{U}(\mathfrak{l}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{u}^-) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{u}) \\
&\cong \mathcal{U}(\mathfrak{u}^-) \otimes_{\mathbb{C}} (\mathcal{R}(L \cap K) \otimes_{\mathcal{U}(\mathfrak{l}\cap\mathfrak{k})} \mathcal{U}(\mathfrak{l})) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{u}).
\end{aligned}$$

The last equivalence requires proof. In [11] the authors prove the equivalence by developing the theory of pairs and associated Hecke algebras for general pairs, that is for those that are not directly derived from a Lie group G. They can then apply this theory to the pair $(\mathfrak{q}^-, L\cap K)$, where $\mathfrak{q}^- = \mathfrak{u}^- \oplus \mathfrak{l}$. They find $\mathcal{R}(L\cap K) \otimes_{\mathcal{U}(\mathfrak{l}\cap\mathfrak{k})} \mathcal{U}(\mathfrak{q}^-) \cong \mathcal{U}(\mathfrak{q}^-) \otimes_{\mathcal{U}(\mathfrak{l}\cap\mathfrak{k})} \mathcal{R}(L\cap K)$, which implies the required equivalence.[105] Alternatively, one can show that the earlier established isomorphism of $\mathcal{R}(L \cap K) \otimes_{\mathcal{U}(\mathfrak{l}\cap\mathfrak{k})} \mathcal{U}(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{l}\cap\mathfrak{k})} \mathcal{R}(L \cap K)$ as algebras restricts to an isomorphism of $\mathcal{R}(L \cap K) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{u}^-) \cong \mathcal{U}(\mathfrak{u}^-) \otimes_{\mathbb{C}} \mathcal{R}(L \cap K)$.

In any case, we once more apply proposition 6.29 to the expression in the last line, and conclude

$$\mathcal{R}(\mathfrak{g}, L \cap K) \cong \mathcal{U}(\mathfrak{u}^-) \otimes_{\mathbb{C}} \mathcal{R}(\mathfrak{l}, L \cap K) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{u}).$$

[105] see section 1.5 and 1.6 in [11] for details.

This completes the proof.[106] □

Next, we find an additive decomposition of the Hecke algebra.

Corollary 7.2 In the above setting, we formally write[107]

$$\begin{aligned}\mathcal{R}(\mathfrak{q}^-, L\cap K) &= \mathcal{U}(\mathfrak{u}^-)\otimes_{\mathbb{C}}\mathcal{R}(\mathfrak{l}, L\cap K)\\ \mathcal{R}(\mathfrak{q}, L\cap K) &= \mathcal{R}(\mathfrak{l}, L\cap K)\otimes_{\mathbb{C}}\mathcal{U}(\mathfrak{u}).\end{aligned}$$

Then the Hecke algebra $\mathcal{R}(\mathfrak{g}, L\cap K)$ has the following additive decomposition analogous to the classical decomposition of $\mathcal{U}(\mathfrak{g})$ in corollary 2.3

$$\begin{aligned}&\mathcal{R}(\mathfrak{g}, L\cap K)\\ \cong\ &\mathcal{R}(\mathfrak{l}, L\cap K)\oplus\mathfrak{u}^-\left(\mathcal{U}(\mathfrak{u}^-)\otimes_{\mathbb{C}}\mathcal{R}(\mathfrak{l}, L\cap K)\right)\oplus\mathfrak{u}^-\mathcal{R}(\mathfrak{g}, L\cap K)\mathfrak{u}\oplus\left(\mathcal{R}(\mathfrak{l}, L\cap K)\otimes_{\mathbb{C}}\mathcal{U}(\mathfrak{u})\right)\mathfrak{u}\\ \cong\ &\mathcal{R}(\mathfrak{l}, L\cap K)\oplus\mathfrak{u}^-\mathcal{R}(\mathfrak{q}^-, L\cap K)\oplus\mathfrak{u}^-\mathcal{R}(\mathfrak{l}, L\cap K)\mathfrak{u}\oplus\mathcal{R}(\mathfrak{q}, L\cap K)\mathfrak{u}\end{aligned}$$

where multiplication is convolution inside the distribution algebra.

The adjoint action of $\mathfrak{l}$, we considered in the classical case, is replaced by a conjugation action by the group. Since $L\cap K$ is a subgroup of K, $L\cap K$, it acts on distributions supported on K from the left and the right[108] This allows us to define a *conjugation* action of elements in $L\cap K$ on distributions T by

$$\mathrm{conj}(k)(T) = l(k)r(k)(T) \qquad \forall k\in L\cap K.$$

As the next two examples show, this conjugation action of the group on distributions is consistent with the conjugation and adjoint action of the group on itself and its Lie algebra, respectively.

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[106]Note that there is an underlying action of $\mathcal{U}(\mathfrak{l}\cap\mathfrak{k})$ on $\mathcal{U}(\mathfrak{u}^-)$ and $\mathcal{U}(\mathfrak{u})$, which is not reflected in the notation.

[107]One can consider these equations as definitions for algebras we denote by $\mathcal{R}(\mathfrak{q}^-, L\cap K)$ and $\mathcal{R}(\mathfrak{q}, L\cap K)$. Since we previously denoted $\mathfrak{q}^- = \mathfrak{u}^-\oplus\mathfrak{l}$, and $\mathfrak{q} = \mathfrak{l}\oplus\mathfrak{u}$, these definitions are consistent with the main structural equality of Hecke algebras from proposition 6.29. Alternatively, one can work through the definition of general Hecke algebras, for example in section 1.5 and 1.6 of [11], and find these equations to be true.

[108]Recall from (15) and (17), that

$$\begin{aligned}\langle l(k)T, f\rangle &= \langle T, l(k^{-1})f\rangle &&\text{or equivalently} && l(k)T = \delta_k * T\\ \langle r(k)T, f\rangle &= \langle T, r(k^{-1})f\rangle &&\text{or equivalently} && r(k)T = T*\delta_{k^{-1}}.\end{aligned}$$

Example 7.3 Let k be an element in $L \cap K$, and let T be a Dirac distribution, that is $T = \delta_{k'}$ for some $k' \in L \cap K$. Then by (15), and by example 6.11

$$\mathrm{conj}(k)(\delta_{k'}) = l(k)r(k)(\delta_{k'}) = \delta_k * \delta_{k'} * \delta_{k^{-1}} = \delta_{kk'k^{-1}}.$$

Example 7.4 Let k be an element in $L \cap K$, and let T be a distribution of the form ∂_X for some $X \in \mathcal{U}(\mathfrak{g})$. Then according to (15)

$$\mathrm{conj}(k)(\partial_X) = l(k)r(k)(\partial_X) = \delta_k * \partial_X * \delta_{k^{-1}} = \partial_{\mathrm{Ad}(k)X}. \tag{21}$$

The second equality can be shown to hold, by first reasoning for elements in the real Lie algebra $\mathfrak{g}_0$, and then extending the result to general elements $X \in \mathcal{U}(\mathfrak{g})$.

The differential of the conjugation action is given by

$$\mathrm{ad}(X)T = (d\mathrm{conj})(X)T = l(X)T + r(X)T.$$

This is the *adjoint* action of a Lie algebra on distributions. It is compatible with the adjoint action of the universal enveloping algebra on itself:

Example 7.5 First, let Y be an element in the Lie algebra $\mathfrak{l} \cap \mathfrak{k}$, and let T be a distribution of the form ∂_X for some X in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Then by differentiating both sides of above equation (21) we obtain

$$\mathrm{ad}(Y)(\partial_X) = \partial_{\mathrm{ad}(Y)X}.$$

By universality, we can extend this action to the universal enveloping algebra.

As the adjoint action of $\mathfrak{l}$ in the classical case, the conjugation action of $L \cap K$ leaves each of the summands in corollary 7.2 invariant.

Proposition 7.6 With the notation outlined on page 54, and

$$\begin{aligned} \mathcal{R}(\mathfrak{q}^-, L \cap K) &= \mathcal{U}(\mathfrak{u}^-) \otimes_{\mathbb{C}} \mathcal{R}(\mathfrak{l}, L \cap K) \\ \mathcal{R}(\mathfrak{q}, L \cap K) &= \mathcal{R}(\mathfrak{l}, L \cap K) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{u}), \end{aligned}$$

the conjugation action of $L \cap K$ on distributions leaves each of the direct summands in the decomposition in above corollary invariant. In other words, for all $k \in L \cap K$

i) $\mathrm{conj}(k).\mathcal{R}(\mathfrak{l}, L \cap K) \subseteq \mathcal{R}(\mathfrak{l}, L \cap K)$

ii) $\mathrm{conj}(k).(\mathfrak{u}^-)\mathcal{R}(\mathfrak{q}^-, L \cap K) \subseteq (\mathfrak{u}^-)\mathcal{R}(\mathfrak{q}^-, L \cap K)$

iii) $\mathrm{conj}(k).(\mathfrak{u}^-)\mathcal{R}(\mathfrak{g}, L\cap K)(\mathfrak{u}) \subseteq (\mathfrak{u}^-)\mathcal{R}(\mathfrak{g}, L\cap K)(\mathfrak{u})$

iv) $\mathrm{conj}(k).\mathcal{R}(\mathfrak{q}, L\cap K)(\mathfrak{u}) \subseteq \mathcal{R}(\mathfrak{q}, L\cap K)(\mathfrak{u})$.

Proof: These claims can be proven by explicit computation, using the formulas (15) for right and left action of $L\cap K$ on distributions, and the expressions (18) for action on products $X\otimes T \in \mathcal{R}(\mathfrak{g}, L\cap K)$.

Exemplarily, we prove iii). It suffices to prove the claim for expressions of the form $\partial_Y \otimes T \otimes \partial_X = \partial_Y * T * \partial_X$, for $Y \in \mathfrak{u}^-$, $X \in \mathfrak{u}$, and $T \in \mathcal{R}(\mathfrak{g}, L\cap K)$. We compute[109]

$$\begin{aligned} \mathrm{conj}(k)\,(\partial_Y \otimes T \otimes \partial_X) &= l(k)r(k)\,(\partial_Y * T \otimes \partial_X) \\ &= l(k)\,\left[r(k)(\partial_Y * T) \otimes \partial_{\mathrm{Ad}(k^{-1})X}\right] \\ &= l(k)(\partial_Y * (T * \delta_{k^{-1}})) \otimes \partial_{\mathrm{Ad}(k^{-1})X} \\ &= \partial_{\mathrm{Ad}(k)Y} \otimes \delta_k * T * \delta_{k^{-1}} \otimes \partial_{\mathrm{Ad}(k^{-1})X}. \end{aligned} \tag{22}$$

Since elements in $\mathcal{R}(\mathfrak{g}, L\cap K)$ are $(L\cap K)$-finite by definition, $\delta_k * T * \delta_{k^{-1}} \in \mathcal{R}(\mathfrak{g}, L\cap K)$. Also, by assumption on L, see page 54, the adjoint action Ad of L respects the triangular decomposition of $\mathfrak{g}$. So in particular, $\mathfrak{u}^-$ and $\mathfrak{u}$ are preserved under the adjoint action of any subgroup of L. Therefore,

$$\mathrm{conj}(k)\,(\partial_Y \otimes T \otimes \partial_X) \in (\mathfrak{u}^-)\mathcal{R}(\mathfrak{g}, L\cap K)(\mathfrak{u}).$$

This establishes the claim. □

Corollary 7.7 The above direct sum is preserved under passing to invariants under the conjugation action of K, that is

$$\begin{aligned} \mathcal{R}(\mathfrak{g}, L\cap K)^{\mathrm{conj}(L\cap K)} &\cong \mathcal{R}(\mathfrak{l}, L\cap K)^{\mathrm{conj}(L\cap K)} \oplus \left(\mathfrak{u}^-\mathcal{R}(\mathfrak{q}^-, L\cap K)\right)^{\mathrm{conj}(L\cap K)} \\ &\oplus \left(\mathfrak{u}^-\mathcal{R}(\mathfrak{g}, L\cap K)\mathfrak{u}\right)^{\mathrm{conj}(L\cap K)} \oplus \left(\mathcal{R}(\mathfrak{q}, L\cap K)\mathfrak{u}\right)^{\mathrm{conj}(L\cap K)}. \end{aligned}$$

Again, parallel to the classical case, two of the summands turn out to be zero.

Proposition 7.8 In the notation of this section,

$$\left(\mathfrak{u}^-\mathcal{R}(\mathfrak{q}^-, L\cap K)\right)^{\mathrm{conj}(L\cap K)} = 0 \qquad \text{and} \qquad \left(\mathcal{R}(\mathfrak{q}, L\cap K)\mathfrak{u}\right)^{\mathrm{conj}(L\cap K)} = 0.$$

Proof: The proof of this proposition is a bit trickier than it was in the classical case in proposition 2.10, but the idea is the same.

[109]Note that the order in which we apply the actions to which part of this expression does not matter.

Recall that according to proposition 2.1, there exists an element $H \in \mathfrak{l}$ such that $\mathfrak{l}$ is the zero-eigenspace, and $\mathfrak{u}$ is the space spanned by eigenspaces associated to positive eigenvalues under the adjoint action of H on $\mathfrak{l}$. One might try to lift this element to the group level, and use the image of H by this lift to argue similarly to the classical case in proposition 2.10. However, this does not work for a variety of reasons.[110] So instead, we need to "translate" the invariance under conjugation by $L \cap K$ into a statement of Lie algebra action. In particular, we will consider the action of one element in the Lie algebra.

This particular element in the Lie algebra $\mathfrak{l} \cap \mathfrak{k}$, that will "do the job", is $\widetilde{H} = H + \theta(H)$. It is in $\mathfrak{l}$[111], as well as in $\mathfrak{k}$[112], so hence in $\mathfrak{l} \cap \mathfrak{k}$.

Now let T be any distribution in $\mathcal{R}(\mathfrak{g}, L \cap K)^{L \cap K}$. By differentiation, T is also invariant under the action of the real Lie algebra $\mathfrak{l}_0 \cap \mathfrak{k}_0$. Because the Hecke algebra $\mathcal{R}(\mathfrak{g}, L \cap K)$ is complex, the distribution T commutes with the complexified Lie algebra $\mathfrak{l} \cap \mathfrak{k}$.[113] In particular, T commutes with $\widetilde{H}$, so

$$\mathcal{R}(\mathfrak{g}, L\cap K)^{L\cap K} \subseteq \mathcal{R}(\mathfrak{g}, L\cap K)^{\mathrm{ad}(\mathfrak{l}\cap\mathfrak{k})} \subseteq \mathcal{R}(\mathfrak{g}, L\cap K)^{\mathrm{ad}(\widetilde{H})} = \{T \in \mathcal{R}(\mathfrak{g}, L\cap K) \mid \mathrm{ad}(\widetilde{H})T = 0\}.$$

Finally, we prove the second equation. Let $X_{\lambda_{i_j}}$ be in the eigenspace corresponding to the eigenvalue λ_{i_j} under the action of $\mathrm{ad}(\widetilde{H})$ on $\mathfrak{l}$. Then every element in

$$\mathcal{R}(\mathfrak{q}, L \cap K)\mathfrak{u} = (\mathcal{R}(\mathfrak{l}, L \cap K) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{u}))\,\mathfrak{u}$$

can be written as sum of elements of the form

$$T_I \otimes \partial^{i'_1}_{X_{\lambda_{i_1}}} \cdots \partial^{i'_n}_{X_{\lambda_{i_n}}} = T_I \otimes \partial_{X^{i'_1}_{\lambda_{i_1}} \cdots X^{i'_n}_{\lambda_{i_n}}}$$

[110]For example, the exponential map as a natural choice for the lift has the real algebra as domain. But H is an element in the complex Lie algebra $\mathfrak{g}$, and there is no analogous statement like proposition 2.1 for real Lie algebras. In fact, elements of the real Lie algebras of any compact group act as skew-Hermitian operators in any representation, so in particular by ad. They therefore have purely imaginary eigenvalues, and element H with properties as implied by proposition 2.1 for complex Lie algebras can not exist for real Lie algebras of compact groups.

[111]Since the triangular decomposition of $\mathfrak{g}$ is by assumption θ-stable, $\theta(H)$ will act like H on $\mathfrak{l}$ by zero, with positive eigenvalues on $\mathfrak{u}$, and with negative eigenvalues on $\mathfrak{u}^-$. Consequently, $\widetilde{H}$ acts in the same manner as H and θH. In other words, $\mathfrak{l}$ consists of all elements in $\mathfrak{g}$ on which $\mathrm{ad}(\widetilde{H})$ acts by zero. Since $[\widetilde{H}, \widetilde{H}] = 0$, the element $\widetilde{H}$ is in $\mathfrak{l}$.

[112]This is implied by $\widetilde{H}$ being θ-stable, and $\mathfrak{k}$ being the $+1$-eigenspace in $\mathfrak{g}$ under θ.

[113]More specifically, given an action of the real Lie algebra $\mathfrak{g}_0$ on $\mathcal{R}(\mathfrak{g}, L \cap K)$, we can define an action of $i\mathfrak{g}_0$ on this Hecke algebra. Combining the two action, we get an action by $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ on $\mathcal{R}(\mathfrak{g}, L\cap K)$.

with at least one $i'_j > 0$, and $T_I \in \mathcal{R}(L \cap K)$. We compute[114]

$$\begin{aligned} & \mathrm{ad}(\widetilde{H}) \left(\sum_{I=\{i_1 \dots i_n\}} T_I \otimes \partial_{X_{\lambda_{i_1}}^{i'_1}} \dots \partial_{X_{\lambda_{i_n}}^{i'_n}} \right) \\ = & \sum_{I=\{i_1 \dots i_n\}} \mathrm{ad}(\widetilde{H}) \left(T_I \otimes \partial_{X_{\lambda_{i_1}}^{i'_1}} \dots \partial_{X_{\lambda_{i_n}}^{i'_n}} \right) \\ = & \sum_{I=\{i_1 \dots i_n\}} \mathrm{ad}(\widetilde{H})(T_I) + \sum_{j=1}^{n} \cdot T_I \otimes \partial_{X_{\lambda_{i_1}}^{i'_1}} \dots \mathrm{ad}(\widetilde{H})(\partial_{X_{\lambda_{i_j}}^{i'_j}}) \dots \partial_{X_{\lambda_{i_n}}^{i'_n}} \\ = & \sum_{I=\{i_1 \dots i_n\}} \sum_{j=1}^{n} \lambda_{i_j}^{i'_j} \cdot T_I \otimes \partial_{X_{\lambda_{i_1}}^{i'_1} \dots X_{\lambda_{i_n}}^{i'_n}} . \end{aligned}$$

Since all the λ_{i_j} are positive, this last sum can only be zero, if all summands $T_I \otimes \partial^{i'_1}_{X_{\lambda_{i_1}}} \dots \partial^{i'_n}_{X_{\lambda_{i_n}}}$ are zero. This proves the second equality. The first equation is proven analogous. □

Corollary 7.9 The direct sum of invariants under the conjugation action of K simplifies to

$$\mathcal{R}(\mathfrak{g}, L \cap K)^{\mathrm{conj}(L \cap K)} \cong \mathcal{R}(\mathfrak{l}, L \cap K)^{\mathrm{conj}(L \cap K)} \oplus \left(\mathfrak{u}^- \mathcal{R}(\mathfrak{l}, L \cap K)\mathfrak{u}\right)^{\mathrm{conj}(L \cap K)} .$$

As in the classical case, this equation implies that the projection onto the first summand is a vector space homomorphism. We claim it is an algebra homomorphism.

Proposition 7.10 In the notation of this section, the projection map

$$P : \mathcal{R}(\mathfrak{g}, L \cap K)^{\mathrm{conj}(L \cap K)} \to \mathcal{R}(\mathfrak{l}, L \cap K)^{\mathrm{conj}(L \cap K)}$$

is an algebra homomorphism.

[114]Recall that for any associative algebra A, on which an commutator $[\ ,\ ]$ is defined by $[a_1, a_2] = a_1 a_2 - a_2 a_1$ for $a_1, a_2 \in A$, and for any $a_i, x \in A$

$$[a_1 a_2 \dots a_n, x] = \sum_{i=1}^{n} a_1 \dots a_{i-1} [a_i, x] a_i \dots a_n .$$

This is a purely algebraic statement, proven by induction on n, and holds for the operator $[a_1, a_2] = a_1 a_2 + a_2 a_1$ as well. Thus it applies to the adjoint action of elements in the Lie algebra on distributions, too.

Proof: The proof is similar to the one in proposition 2.12 in the classical case, since the argument is purely algebraic. □

Having established this algebra homomorphism, we pick up the second "thread" of this story, and define the subalgebras for which the generalized Harish-Chandra construction will carry through.

For the pair $(\mathfrak{g}, L\cap K)$ directly derived from G, and a θ-stable triangular decomposition $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}$, the so-called *step algebra*[115] is defined by

$$S(\mathfrak{g}, L\cap K) = \{T \in \mathcal{R}(\mathfrak{g}, L\cap K) \,|\, \partial_X * T \in \mathcal{R}(\mathfrak{g}, L\cap K)(\mathfrak{u}\cap\mathfrak{k}) \text{ for all } X \in (\mathfrak{u}\cap\mathfrak{k})\} \quad (23)$$

Example 7.11 For $\mathfrak{l}$ as in the triangular decomposition of $\mathfrak{g}$, the step algebra associated to the pair $(\mathfrak{l}, L\cap K)$, is

$$\begin{aligned} & S(\mathfrak{l}, L\cap K) \\ = \; & \{T \in \mathcal{R}(\mathfrak{l}, L\cap K) \,|\, \partial_X * T \in \mathcal{R}(\mathfrak{l}, L\cap K)(\mathfrak{u}\cap(\mathfrak{l}\cap\mathfrak{k})) \text{ for all } X \in \mathfrak{u}\cap(\mathfrak{l}\cap\mathfrak{k}) = \{0\}\} \\ = \; & \mathcal{R}(\mathfrak{l}, L\cap K). \end{aligned}$$

Proposition 7.12 For the pair $(\mathfrak{g}, L\cap K)$, and the associated step algebra $S(\mathfrak{g}, L\cap K)$

a) $S(\mathfrak{g}, L\cap K)$ is an algebra with respect to convolution.

b) $\mathcal{R}(L\cap K) \subseteq S(\mathfrak{g}, L\cap K)$.

Proof:

a) We only need to check that $S(\mathfrak{g}, L\cap K)$ is closed under convolution. So let $T_1, T_2 \in S(\mathfrak{g}, L\cap K)$, and $X \in \mathfrak{u}\cap\mathfrak{k}$. Then

$$\begin{aligned} \partial_X * (T_1 * T_2) &= (\partial_X * T_1) * T_2 \\ &= \left(\sum_i T_i' * \partial_{X_i'}\right) * T_2 && \text{for some } T_i' \in \mathcal{R}(\mathfrak{g}, L\cap K), X_i' \in (\mathfrak{u}\cap\mathfrak{k}) \\ &= \sum_i T_i' * \left(\sum_j T_{ij}'' * \partial_{X_j''}\right) && \text{for some } T_{ij}'' \in \mathcal{R}(\mathfrak{g}, L\cap K), X_j'' \in (\mathfrak{u}\cap\mathfrak{k}) \\ &= \sum_j \left(\sum_i T_i' * T_{ij}''\right) * \partial_{X_j''} \end{aligned}$$

[115]Step-algebras were introduced for Lie algebras by J. Mickelsson in [15]. Another - but hard to find - reference is [18]. The idea in this context is to consider a construction that is similar to requiring invariants under the adjoint action of $\mathfrak{l}$ in the classical case without being too restrictive and making the resulting algebra too small.

So in particular, $T_1 * T_2 \in S(\mathfrak{g}, L \cap K)$. In other words, $S(\mathfrak{g}, L \cap K)$ is a subalgebra of $\mathcal{R}(\mathfrak{g}, L \cap K)$.

b) This point is true, because $\mathcal{R}(L \cap K)$ normalizes $(\mathfrak{u} \cap \mathfrak{k})$.[116]

Thus the statement is completely proven. □

Merging the two threads, we argue that the projection map from proposition 7.10 descends to an algebra homomorphism on the step algebras.

Proposition 7.13 The projection map

$$P : \mathcal{R}(\mathfrak{g}, L \cap K)^{L\cap K} \to \mathcal{R}(\mathfrak{l}, L \cap K)^{L\cap K}$$

as discussed in proposition 7.10, descends to an algebra homomorphism

$$\boxed{\xi : S(\mathfrak{g}, L \cap K)^{L\cap K} \to S(\mathfrak{l}, L \cap K)^{L\cap K}.} \tag{24}$$

Proof: We saw in example 7.11 that $S(\mathfrak{l}, L\cap K) = \mathcal{R}(\mathfrak{l}, L\cap K)$, and hence $S(\mathfrak{l}, L\cap K)^{L\cap K} = \mathcal{R}(\mathfrak{l}, L \cap K)^{L\cap K}$. Therefore, the image of $S(\mathfrak{g}, L \cap K)^{L\cap K}$ under the projection map P is contained in $S(\mathfrak{l}, L \cap K)^{L\cap K}$. In other words, ξ is a well-defined map. □

This algebra homomorphism ξ is a *generalized Harish-Chandra homomorphism.*

Note that in particular, this not a trivial map. We saw that $\mathcal{R}(L\cap K) = \bigoplus_{\gamma\in\widehat{L\cap K}} End(E_\gamma)$. Therefore

$$\left(\bigoplus_{\gamma\in\widehat{L\cap K}} End(E_\gamma)\right)^{L\cap K} = \mathcal{R}(L \cap K)^{L\cap K} \subseteq S(\mathfrak{l}, L \cap K)^{L\cap K} \neq \{0\}.$$

So the domain of above map is almost always non-zero.

Note that it was necessary to pass to $L \cap K$-invariants in order to establish ξ as an algebra homomorphism.

[116]More generally, one can show by a somewhat explicit computatuion, that if W is an $\mathrm{Ad}(L \cap K)$-invariant subspace of $\mathcal{U}(\mathfrak{g})$, then

$$W\mathcal{R}(L \cap K) \cong \mathcal{R}(L \cap K)W.$$

So this is in particular true for $W = \mathfrak{u} \cap \mathfrak{k}$.

7.3 Relating Hecke Algebra Modules

We now move on to establishing the generalized Harish-Chandra homomorphism as a link between (irreducible) Hecke algebra modules. For this, we first prove the link provided by the generalized Harish-Chandra homomorphism, and then discuss what information this link yields in the context of representation theory. We discuss here the basic, that is the zero-degree Lie algebra cohomology case. In the next section, we present the extension to general finite cohomology degrees.

Let $(V, \widetilde{\sigma})$ be a locally $L \cap K$-finite $\mathcal{R}(\mathfrak{g}, L \cap K)$-module. Then

$$V^{\mathfrak{u} \cap \mathfrak{k}} = \{v \in V \mid \pi(X)v = 0 \quad \text{for all } X \in \mathfrak{u} \cap \mathfrak{k}\}$$

is a locally $L \cap K$-finite $S(\mathfrak{g}, L \cap K)$-module, so in particular it is a $S(\mathfrak{g}, L \cap K)^{L \cap K}$-module.[117] Also,

$$V^{\mathfrak{u}} = \{v \in V \mid \pi(X)v = 0 \quad \text{for all } X \in \mathfrak{u}\} \subseteq V^{\mathfrak{u} \cap \mathfrak{k}}$$

is a locally $L \cap K$-finite $\mathcal{R}(\mathfrak{l}, L \cap K)$-, and hence a $S(\mathfrak{l}, L \cap K)^{L \cap K}$-module.

The following theorem links the actions by $S(\mathfrak{g}, L \cap K)^{L \cap K}$ and $S(\mathfrak{l}, L \cap K)^{L \cap K}$ via the generalized Harish-Chandra homomorphism.

Theorem 7.14 Using the notation set up on page 54, let $(V, \widetilde{\sigma})$ be any $L \cap K$-finite $\mathcal{R}(\mathfrak{g}, L \cap K)$-module, and let $V^{\mathfrak{u}}$ and $V^{\mathfrak{u} \cap \mathfrak{k}}$ be as defined in the previous paragraph. Then for all $T \in S(\mathfrak{g}, L \cap K)^{L \cap K}$, and for all $v \in V^{\mathfrak{u}}$

$$\boxed{\widetilde{\sigma}|_{V^{\mathfrak{u} \cap \mathfrak{k}}}(T)v = \left(\widetilde{\sigma}|_{V^{\mathfrak{u}}} \circ \xi\right)(T)v.}$$

Equivalently, we can denote the inclusion map $V^{\mathfrak{u}} \hookrightarrow V^{\mathfrak{u} \cap \mathfrak{k}}$ by res, and write

$$\boxed{\widetilde{\sigma}(T)\,\mathrm{res}(v) = \mathrm{res}\left((\widetilde{\sigma} \circ \xi)(T)v\right).}$$

Proof: We prove the first version of the theorem's statement. Since $V^{\mathfrak{u}} \subseteq V^{\mathfrak{u} \cap \mathfrak{k}}$, we have somewhat trivially $\widetilde{\sigma}|_{V^{\mathfrak{u}}} = \widetilde{\sigma}|_{V^{\mathfrak{u} \cap \mathfrak{k}}}|_{V^{\mathfrak{u}}}$. So to prove above equation, it suffices to show that for all $v \in V^{\mathfrak{u}}$, and $T \in S(\mathfrak{g}, L \cap K)^{L \cap K}$

$$0 = \left(\widetilde{\sigma}(T) - \widetilde{\sigma}(\xi(T))\right)v = \widetilde{\sigma}\left(T - \xi(T)\right)v.$$

[117] Considering the classical case, and the definition of the step algebra $S(\mathfrak{g}, L \cap K)^{L \cap K}$, the subspace $V^{\mathfrak{u} \cap \mathfrak{k}}$ seems the appropriate subrepresentation to consider here.

We defined ξ to be the projection of $\mathcal{R}(\mathfrak{g}, L\cap K)^{L\cap K}$ to $\mathcal{R}(\mathfrak{l}, L\cap K)^{L\cap K}$, according to the decomposition

$$\mathcal{R}(\mathfrak{g}, L\cap K)^{\mathrm{conj}(L\cap K)} \cong \mathcal{R}(\mathfrak{l}, L\cap K)^{\mathrm{conj}(L\cap K)} \oplus \left(\mathfrak{u}^{-}\mathcal{R}(\mathfrak{l}, L\cap K)\mathfrak{u}\right)^{\mathrm{conj}(L\cap K)}.$$

Therefore, $T-\xi(T)\in \mathfrak{u}^{-}\mathcal{R}(\mathfrak{g}, L\cap K)\mathfrak{u}$. By definition of $V^{\mathfrak{u}}$, this means that for all $v\in V^{\mathfrak{u}}$

$$\widetilde{\sigma}(T-\xi(T)).v = 0.$$

This finishes the proof. □

The theorem establishes a relation between $S(\mathfrak{g}, L\cap K)^{L\cap K}$- and $S(\mathfrak{l}, L\cap K)^{L\cap K}$-modules. What we are interested in, however, is to relate $\mathcal{R}(\mathfrak{g}, L\cap K)$- and $\mathcal{R}(\mathfrak{l}, L\cap K)$-modules, or even better between $\mathcal{R}(\mathfrak{g}, K)$- and $\mathcal{R}(\mathfrak{l}, L\cap K)$-modules. So naturally, the question arises to what extent the link between modules of the step algebras lifts to a link of modules of the Hecke algebras. In other words, to what extent is V as a $\mathcal{R}(\mathfrak{g}, L\cap K)$-module - or as an $\mathcal{R}(\mathfrak{g}, K)$-module - determined by the $S(\mathfrak{g}, L\cap K)^{L\cap K}$-module $V^{\mathfrak{u}\cap\mathfrak{p}}$, and likewise, to what extent is $V^{\mathfrak{u}}$ as a $\mathcal{R}(\mathfrak{l}, L\cap K)$-module determined by $V^{\mathfrak{u}}$ as a $S(\mathfrak{l}, L\cap K)^{L\cap K}$-module.[118]

For the algebras on the right hand side of the Harish-Chandra homomorphism, the question can be answered completely.

Proposition 7.15 Let W be an irreducible $\mathcal{R}(\mathfrak{l}, L\cap K)$-module, and let containing the irreducible representation E_{γ_0} of $L\cap K$. Then $Hom_{L\cap K}(E_{\gamma_0}, W)$ is an irreducible $\mathcal{R}(\mathfrak{l}, L\cap K)^{L\cap K} = S(\mathfrak{l}, L\cap K)^{L\cap K}$-module, and as such it determines W as $\mathcal{R}(\mathfrak{l}, L\cap K)$-module up to equivalence.

For the left hand side, there are some cases for which we know how to answer the question.

[118] Recall that a module $(V,\widetilde{\sigma})$ of the Hecke algebra $\mathcal{R}(\mathfrak{g}, K)$ is the same as a representation (V,π,σ) of the pair $(\mathfrak{g}, K)$. In answering this question, it is sometimes convenient, to "decompose" the information describing such a (locally K-finite) representation in one of the following two ways.

For one, we can start by considering (V,σ) as a locally K-finite representation of K, and look how the Lie algebra $\mathfrak{g}$ acts via π on irreducible K-invariant subspaces of V. For example, if (V,π,σ) is an infinite dimensional irreducible locally K-finite representation of $(\mathfrak{g}, K)$, then as a representation of K, V has infinitely many irreducible K-invariant subspaces E_γ. By irreducibility, we conclude that for any non-zero E_γ, $\pi(\mathfrak{g})E_\gamma$ spans the whole space V.

Alternatively, we can consider (V,π,σ) as a representation (V,π) of $\mathfrak{g}$, or equivalently as $\mathcal{U}(\mathfrak{g})$-module. If V contains irreducible $\mathcal{U}(\mathfrak{g})$-subspaces, we ask how K acts on these subspaces via σ, how it "moves them around". For example, this perspective is convenient, when V is determined by given information as a $\mathcal{U}(\mathfrak{g})$-module, and we are interested in determining it as a $\mathcal{R}(\mathfrak{g}, K)$-module.

Example 7.16 If $L \cap K$ is a finite group, then the $S(\mathfrak{g}, L \cap K)^{L \cap K}$-module $V^{\mathfrak{u} \cap \mathfrak{k}}$, determines V as a $L \cap K$-finite $\mathcal{R}(\mathfrak{g}, L \cap K)$-module up to equivalence.[119]

Example 7.17 If K is finite, and $L \cap K$ meets every connected component of K, then $V^{\mathfrak{u} \cap \mathfrak{k}}$ as a $S(\mathfrak{g}, L \cap K)^{L \cap K}$-module determines V as a K-finite $\mathcal{R}(\mathfrak{g}, K)$-module.[120]

Example 7.18 If K is a connected group, (and $L \cap K$ meets every connected component of K,)[121] then $V^{\mathfrak{u} \cap \mathfrak{k}}$ as a $S(\mathfrak{g}, L \cap K)^{L \cap K}$-module determines V as K-finite $\mathcal{R}(\mathfrak{g}, K)$-module up to equivalence.[122]

In general, we think the following statement is true.

Conjecture 7.19 (D. Vogan) Assume that $L \cap K$ meets every connected component of K, and let V be an irreducible locally K-finite $\mathcal{R}(\mathfrak{g}, K)$-module [123].

a) Then $V^{\mathfrak{u} \cap \mathfrak{k}}$ is a non-trivial irreducible $S(\mathfrak{g}, L \cap K)$-module, and this module determines V as an $\mathcal{R}(\mathfrak{g}, K)$-module up to equivalence.

b) Moreover, let (E_{γ_0}, γ_0) be an irreducible representation of $L \cap K$ which appears in $V^{\mathfrak{u} \cap \mathfrak{k}}$.[124] Then
$$Hom_{L \cap K}(E_{\gamma_0}, V^{\mathfrak{u} \cap \mathfrak{k}})$$
is a non-trivial irreducible $S(\mathfrak{g}, L \cap K)^{L \cap K}$-module, and as such it determines V as an $\mathcal{R}(\mathfrak{g}, K)$-module up to equivalence.[125]

[119] By construction of L, and the triangular decomposition of $\mathfrak{g}$ being stable under the Cartan involution θ, the Lie group $L \cap K$ can only be finite if K is finite. The Lie algebra associated to a finite group is the zero-space. So $\mathfrak{k} = \{0\} = \mathfrak{u} \cap \mathfrak{k}$. Therefore, $S(\mathfrak{g}, L \cap K) = \mathcal{R}(\mathfrak{g}, L \cap K)$, and $V^{\mathfrak{u} \cap \mathfrak{k}} = V$. So it boils down to showing that the $L \cap K$-invariants of the Hecke algebra determines the action of the entire Hecke algebra. The argument to see this, is somewhat subtle.

[120] Of course, if K is finite, and $L \cap K$ meets every connected component of K, then $K = L \cap K$, and the argument is as in the previous example.

[121] This criterion is trivially satisfied in this setting.

[122] The proof of this consists of two steps. One, let $X \in \mathcal{U}(\mathfrak{g})^{\mathrm{Ad}(K)}$ and let ς be an approximate identity in $\mathcal{R}(L \cap K)^{L \cap K}$, then all elements of the form $\partial_X * \varsigma$ are in $S(\mathfrak{g}, L \cap K)^{L \cap K}$. Two, the action of $\mathcal{U}(\mathfrak{g})^{\mathrm{Ad}(K)}$ on $V^{\mathfrak{u} \cap \mathfrak{k}}$ determines V as a representation of $(\mathfrak{g}, K)$. The first step can be shown by explicit computation. The second step follows from an argument similar to the one given by Lepowsky and McCollum for Lemma 5.3 in [14].

[123] So in particular, V is an irreducible $\mathcal{R}(\mathfrak{g}, L \cap K)$-module.

[124] In other words, $V^{\mathfrak{u} \cap \mathfrak{k}}$ contains an irreducible $L \cap K$-invariant subspace isomorphic to E_{γ_0}. An equivalent way to formulate this condition is: the representation E_γ of K which has highest weight γ_0 appears in V.

[125] Using this notation, we can set
$$V_{\gamma_0}^{\mathfrak{u} \cap \mathfrak{k}} = Hom_{L \cap K}(E_{\gamma_0}, V^{\mathfrak{u} \cap \mathfrak{k}})$$

8 The Generalized Casselman-Osborne Theorem

We now extend the result from the previous section for zero-degree Lie algebra cohomology to higher cohomology-degrees. We use the notation as set up on page 54, and denote by $(V, \widetilde{\sigma})$ an $\mathcal{R}(\mathfrak{g}, L \cap K)$-module. In the classical discussion of the Harish-Chandra homomorphism for Lie algebras, the motivation to consider higher cohomology degrees was that the zero-degree cohomology could be zero for infinite dimensional representations. The same motivation is underlying the generalized discussion. While we claimed in above conjecture 7.19, that $V^{\mathfrak{u}\cap\mathfrak{k}}$ is non-trivial as a $S(\mathfrak{g}, L \cap K)^{L\cap K}$-module, if V is a locally K-finite module, this is not necessarily true for infinite dimensional spaces V, if V is only an $\mathcal{R}(\mathfrak{g}, L \cap K)$-module. Moreover, $V^{\mathfrak{u}}$ can be zero in either case.[126]
In the proof of the main theorem, we will need the following statements, which can be proven analogous to the corresponding classical cases.

Lemma 8.1 Recall from section 3.2 that one way to define Lie algebra cohomology was via an injective resolution. Let

$$0 \to V \to I_0 \to I_1 \to I_2 \to \cdots$$

be a resolution of injective $\mathcal{R}(\mathfrak{g}, L \cap K)$-modules. The step algebra $S(\mathfrak{g}, L \cap K)$ acts on the $\mathfrak{u} \cap \mathfrak{k}$-invariants of these injective modules. In other words, $S(\mathfrak{g}, L \cap K)$ acts on

$$0 \to V^{\mathfrak{u}\cap\mathfrak{k}} \to I_0^{\mathfrak{u}\cap\mathfrak{k}} \to I_1^{\mathfrak{u}\cap\mathfrak{k}} \to I_2^{\mathfrak{u}\cap\mathfrak{k}} \to \cdots .$$

This action descends to an action on cohomology $H^k(\mathfrak{u}\cap\mathfrak{k}, V)$. So in particular, $H^k(\mathfrak{u} \cap \mathfrak{k}, V)$ is a $S(\mathfrak{g}, L \cap K)^{L\cap K}$-module.

Lemma 8.2 The step algebra $S(\mathfrak{l}, L \cap K) = \mathcal{R}(\mathfrak{l}, L \cap K)$ acts on the Lie algebra cohomology $H^k(\mathfrak{u}, V)$ by simultaneously acting on $\mathfrak{u}$ with $\widetilde{\mathrm{Ad}}$[127], and on V as subalgebra of

$$V^{\mathfrak{u}}_{\gamma_0} = Hom_{L\cap K}(E_{\gamma_0}, V^{\mathfrak{u}}).$$

These spaces are $S(\mathfrak{g}, L \cap K)^{L\cap K}$- and $\mathcal{R}(\mathfrak{l}, L \cap K)^{L\cap K} = S(\mathfrak{l}, L \cap K)^{L\cap K}$-modules, respectively. In this context, theorem 7.14 states, that for all $T \in S(\mathfrak{g}, L \cap K)^{L\cap K}$, and for all $v \in V^{\mathfrak{u}}_{\gamma_0}$, we have

$$T.res(v) = res(\xi(T).v).$$

[126] One way to think about this, is the following. If V is a locally K-finite module, then for any $v \in V$, the action of $\mathfrak{u} \cap \mathfrak{k} \subseteq \mathfrak{k}$ preserves the finite-dimensional subspace $\langle K.v\rangle$. Thus, the argument we made in the finite dimensional classical case holds. This implies $(\langle K.v\rangle)^{\mathfrak{u}\cap\mathfrak{k}}$ is non-trivial, and hence neither is $V^{\mathfrak{u}\cap\mathfrak{k}}$. However, when we consider spaces which are only $L \cap K$-finite, then the action of the Lie algebra $\mathfrak{k}$ might not preserve the $L \cap K$-subspaces. Then one can run into problems as in example 4.1.

[127] That is, the group $L \cap K$ acts on $\mathfrak{u}$ by adjoint action Ad. This induces an action of the Hecke-algebra $\mathcal{R}(\mathfrak{l}, L \cap K)$ on $\mathfrak{u}$. As elsewhere in this work, we denote this action by $\widetilde{\mathrm{Ad}}$.

$\mathcal{R}(\mathfrak{g}, L\cap K)$. By restriction of the action to $L\cap K$-invariants, $H^k(\mathfrak{u}, V)$ is a $S(\mathfrak{l}, L\cap K)^{L\cap K}$-modules.

Proposition 8.3 Given a short exact sequence of $\mathcal{R}(\mathfrak{g}, L\cap K)$-modules, the above described actions of the step algebras $S(\mathfrak{g}, L\cap K)$ and $S(\mathfrak{l}, L\cap K)$ commute with the respective maps of the induced long exact sequence in cohomology. Hence, so do the actions of $S(\mathfrak{g}, L\cap K)^{L\cap K}$ and $S(\mathfrak{l}, L\cap K)^{L\cap K}$, respectively.

Now everything is ready to go:

Theorem 8.4 Generalized Casselman-Osborne Theorem
Let $(V, \widetilde{\sigma})$ be an $\mathcal{R}(\mathfrak{g}, L\cap K)$ - module. Then $S(\mathfrak{g}, L\cap K)^{L\cap K}$ acts on $H^k(\mathfrak{u}\cap\mathfrak{k}, V)$, and $S(\mathfrak{l}, L\cap K)^{L\cap K}$ act on $H^k(\mathfrak{u}, V)$. Denote these actions by σ' and σ'', respectively. Further, let $\xi : S(\mathfrak{g}, L\cap K)^{L\cap K} \to S(\mathfrak{l}, L\cap K)^{L\cap K}$ be the generalized Harish-Chandra homomorphism as constructed in section 7, and res : $H^k(\mathfrak{u}, V) \to H^k(\mathfrak{u}\cap\mathfrak{k}, V)$ be the map induced on cohomology by the inclusion of $\mathfrak{u}\cap\mathfrak{k}$ into $\mathfrak{u}$. Then

$$\sigma'(T)\,\mathrm{res}(\omega) = \mathrm{res}\,((\sigma''\circ\xi)(T)(\omega))$$

for all $T \in S(\mathfrak{g}, L\cap K)^{L\cap K}$, and $\omega \in H^k(\mathfrak{u}, V)$.

Proof: We prove this theorem analogous to the classical case, by induction on the cohomology degree k, using a technique known as "dimension shifting".

For $k = 0$ we established earlier in example 3.4 that $H^0(\mathfrak{u}\cap\mathfrak{k}, V) = V^{\mathfrak{u}\cap\mathfrak{k}}$, and likewise $H^0(\mathfrak{u}, V) = V^{\mathfrak{u}}$. In this case, the claim reduces to

$$\sigma'(T)\,\mathrm{res}(\omega) = \mathrm{res}\,((\sigma''\circ\xi)(T)(\omega))$$

for all $T \in S(\mathfrak{g}, L\cap K)^{L\cap K}$, and $\omega \in V^{\mathfrak{u}}$. This is the claim of the previously proven theorem 7.14.

Now assume that the statement is true for degree $k-1$ for all (approximately unital left) $\mathcal{R}(\mathfrak{g}, L\cap K)$-modules. We want to consider a short exact sequence of $\mathcal{R}(\mathfrak{g}, L\cap K)$-modules. Such a sequence gives rise to a long exact sequence with a connecting homomorphism linking cohomologies of different degrees. This will facilitate the induction step.

A convenient short exact sequence for these purposes is

$$0 \to V \xrightarrow{\varphi} Hom_{\mathbb{C}}(\mathcal{R}(\mathfrak{g}, L\cap K), V) \xrightarrow{\psi} Q \to 0$$

with $\varphi : v \mapsto f_v$ such that $f_v(T) = \widetilde{\sigma}(T)(v)$ for all $T \in \mathcal{R}(\mathfrak{g}, L\cap K)$, and ψ being the projection map on the quotient $Q = Hom_{\mathbb{C}}(\mathcal{R}(\mathfrak{g}, L\cap K), V)/\mathrm{im}(\varphi)$. Trivially, since all of these spaces are $\mathcal{R}(\mathfrak{g}, L\cap K) \cong \mathcal{R}(L\cap K) \otimes_{\mathcal{U}(\mathfrak{l}\cap\mathfrak{k})} \mathcal{U}(\mathfrak{g})$-modules, they are also $\mathcal{U}(\mathfrak{g})$. So in particular we can consider these spaces as $\mathcal{U}(\mathfrak{u})$- and $\mathcal{U}(\mathfrak{u}\cap\mathfrak{k})$-modules, respectively.

We argued before that such a short exact sequence gives rise to a long exact sequence. Here, we obtain two long exact sequences which are linked by the restriction map res.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^{k-1}(\mathfrak{u}, Hom_{\mathbb{C}}(\mathcal{R}(\mathfrak{g}, L\cap K), V)) & \xrightarrow{\hat{\psi}} & H^{k-1}(\mathfrak{u}, Q) & \xrightarrow{\hat{\rho}} & H^k(\mathfrak{u}, V) & \xrightarrow{\hat{\varphi}} & H^k(\mathfrak{u}, Hom_{\mathbb{C}}(\mathcal{R}(\mathfrak{g}, L\cap K), V)) & \longrightarrow \cdots \\
& \downarrow \text{res} & & \downarrow \text{res} & & \downarrow \text{res} & & \downarrow \text{res} & \\
\cdots \longrightarrow & H^{k-1}(\mathfrak{u}\cap\mathfrak{k}, Hom_{\mathbb{C}}(\mathcal{R}(\mathfrak{g}, L\cap K), V)) & \xrightarrow{\hat{\psi}} & H^{k-1}(\mathfrak{u}\cap\mathfrak{k}, Q) & \xrightarrow{\hat{\rho}} & H^k(\mathfrak{u}\cap\mathfrak{k}, V) & \xrightarrow{\hat{\varphi}} & H^k(\mathfrak{u}\cap\mathfrak{k}, Hom_{\mathbb{C}}(\mathcal{R}(\mathfrak{g}, L\cap K), V)) & \longrightarrow \cdots
\end{array}$$

The maps $\hat{\varphi}$ and $\hat{\psi}$ are induced by the maps in the short exact sequence, $\hat{\rho}$ is the connecting homomorphism discussed in section 3.1. Also, all the squares in this diagram are commutative.[128]

We claim that $H^k(\mathfrak{u}, Hom_{\mathbb{C}}(\mathcal{R}(\mathfrak{g}, L\cap K), V)) = 0$ for all $k > 0$. To prove this, we show that $Hom_{\mathbb{C}}(\mathcal{R}(\mathfrak{g}, L\cap K), V))$ is injective - as an $\mathcal{U}(\mathfrak{u})$-module.[129]

$$\begin{aligned}
Hom_{\mathbb{C}}(\mathcal{R}(\mathfrak{g}, L\cap K), V)) &\cong Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{u}^-) \otimes_{\mathbb{C}} \mathcal{R}(\mathfrak{l}, L\cap K) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{u}), V) \\
&\cong Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{u}), Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{u}^-) \otimes_{\mathbb{C}} \mathcal{R}(\mathfrak{l}, L\cap K), V)) \\
&\cong Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{u}), W)
\end{aligned}$$

for $W = Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{u}^-) \otimes_{\mathbb{C}} \mathcal{R}(\mathfrak{l}, L\cap K), V)$. By our earlier argument in example 3.4 we saw that for any vector space W, the space $Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{u}), W)$ is an injective $\mathcal{U}(\mathfrak{u})$-module. Since thus the right hand side of above equation is an injective $\mathcal{U}(\mathfrak{u})$-module, so must the left hand side be. In particular,

$$H^k(\mathfrak{u}, Hom_{\mathbb{C}}(\mathcal{R}(\mathfrak{g}, L\cap K), V)) = 0 \quad \forall k > 0.$$

Similarly, $H^k(\mathfrak{u}\cap\mathfrak{k}, Hom_{\mathbb{C}}(\mathcal{R}(\mathfrak{g}, L\cap K), V)) = 0$ for all $k > 0$.[130]

[128]This is a fact from algebraic topology. It follows from the functoriality of cohomology. Alternatively, one can prove this by direct computation.

[129]We use the fact, that in general $Hom_S(A \otimes_R B, C) \cong Hom_S(A, Hom_R(B, C))$.

[130]In more detail, we need to show that $Hom_{\mathbb{C}}(\mathcal{R}(\mathfrak{g}, L\cap K), V)$ is an injective $\mathcal{U}(\mathfrak{u}\cap\mathfrak{k})$-module. Recall that $\mathfrak{k}$ is the $+1$-eigenspace of the Cartan involution θ in $\mathfrak{g}$. The -1-eigenspace is commonly denoted by $\mathfrak{p}$. So $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as vector space. Since the triangular decomposition of $\mathfrak{g}$ is θ-stable, $\mathfrak{u} = \mathfrak{u}\cap\mathfrak{k} \oplus \mathfrak{u}\cap\mathfrak{p}$ as vector spaces. Analogous to the argument in proposition 2.2, this decomposition lifts to a decomposition of $\mathcal{U}(\mathfrak{u}) = \mathcal{U}(\mathfrak{k}) \otimes_{\mathbb{C}} \mathcal{P}$ as vector spaces, where $\mathcal{P}$ is the vector space spanned by monomials formed by basis elements in $\mathfrak{p}$. We can then continue above series of equation by rewriting

$$Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{u}), W) \cong Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{u}\cap\mathfrak{k}) \otimes_{\mathbb{C}} \mathcal{P}, W) \cong Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{u}\cap\mathfrak{k}), Hom_{\mathbb{C}}(\mathcal{P}, W)).$$

Therefore the double long exact sequence above breaks up for $k > 0$ into commutative squares

$$\begin{array}{ccc} H^{k-1}(\mathfrak{u}, Q) & \xrightarrow{\hat{\rho}} & H^k(\mathfrak{u}, V) \\ \downarrow \text{res} & & \downarrow \text{res} \\ H^{k-1}(\mathfrak{u} \cap \mathfrak{k}, Q) & \xrightarrow{\hat{\rho}} & H^k(\mathfrak{u} \cap \mathfrak{k}, V) \end{array}$$

where the horizontal maps, i.e. the connecting homomorphism $\hat{\rho}$, are surjective.

We are now ready to argue the inductive step. Given $\omega \in H^k(\mathfrak{u}, V)$ there must exist an $\widetilde{\omega} \in H^{k-1}(\mathfrak{u}, Q)$ such that $\hat{\rho}(\widetilde{\omega}) = \omega$. By commutativity of the diagram, $\hat{\rho}$ commutes with the restriction map, in particular

$$\hat{\rho}(\text{res}(\widetilde{\omega})) = \text{res}(\hat{\rho}(\widetilde{\omega})) = \text{res}(\omega).$$

By induction hypothesis, the claim is true for all $\widetilde{\omega} \in H^{k-1}(\mathfrak{u}, Q)$, that is for all $T \in S(\mathfrak{g}, L \cap K)^{L \cap K}$

$$\sigma'(T)\,\text{res}(\widetilde{\omega}) = \text{res}\left((\sigma'' \circ \xi)(T)(\widetilde{\omega})\right).$$

By propositions above, the actions of $S(\mathfrak{g}, L \cap K)^{L \cap K}$ and $S(\mathfrak{l}, L \cap K)^{L \cap K}$ commute with the maps of the respective long exact sequences. Therefore applying $\hat{\rho}$ to both sides of the previous equation, yields on the left side

$$\begin{aligned} \hat{\rho}(\sigma'(T)\,\text{res}(\widetilde{\omega})) &= \sigma'(T)(\hat{\rho}(\text{res}(\widetilde{\omega})) \\ &= \sigma'(T)(\text{res}(\omega)) \end{aligned}$$

and on the right side

$$\begin{aligned} \hat{\rho}\left(\text{res}\left((\sigma'' \circ \xi)(T)(\widetilde{\omega})\right)\right) &= \text{res}\left(\hat{\rho}\left((\sigma'' \circ \xi)(T)(\widetilde{\omega})\right)\right) \\ &= \text{res}\left((\sigma'' \circ \xi)(T)(\hat{\rho}(\widetilde{\omega}))\right) \\ &= \text{res}\left((\sigma'' \circ \xi)(T)(\omega)\right). \end{aligned}$$

By equality of left and right side we obtain

$$\sigma'(T)(\text{res}(\omega)) = \text{res}\left((\sigma'' \circ \xi)(T)(\omega)\right)$$

which is what we wanted to show.
This finishes the induction step, and thus the proof. □

By proving the generalized Casselman-Osborne theorem, we have completed the process of constructing a generalized Harish-Chandra homomorphism for Hecke algebras, and establishing it as a link between Hecke algebra modules.

Now by the same argument as before, $Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{u} \cap \mathfrak{k}), Hom_{\mathbb{C}}(\mathcal{P}, W))$ is an injective $\mathcal{U}(\mathfrak{u} \cap \mathfrak{k})$-module, and hence so is $Hom_{\mathbb{C}}(\mathcal{R}(\mathfrak{g}, L \cap K), V))$.

9 Final Remarks

In this work, we presented the classical Harish-Chandra homomorphism for Lie algebras, its relevance for representations theory, and a generalization of it for reductive Lie groups, by using Hecke algebras associated to the Lie group.

However, this area of representation and Lie theory is far from perfectly understood. Specifically, in the context of this work, several questions deserve further investigation. For one, it would be interesting to formally prove conjecture 7.19 that indeed $V^{\mathfrak{u}\cap\mathfrak{k}}$ as $S(\mathfrak{g}, L\cap K)^{L\cap K}$-module determines V as $\mathcal{R}(\mathfrak{g}, L\cap K)$-module, and to see whether the statement generalizes to higher cohomology degrees of $\mathfrak{u}\cap\mathfrak{k}$. Next, there is more structure to the step algebras than we could discuss here. It would be worthwhile to consider the implication for the module theory of Hecke algebras implied by these structures. For example, we would like to describe not only how components $Hom_{L\cap K}(E_\gamma, V)$ determine V, but also how the action of the Hecke algebra "moves" between these components for different γs. Finally, the construction we presented was only a "second-best", since we could not find an appropriate algebra homomorphism from $\mathcal{R}(\mathfrak{g}, L\cap K)^K$ to $\mathcal{R}(\mathfrak{l}, L\cap K)^{L\cap K}$, and the question whether such an algebra homomorphism exists remains open.

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References

[1] A. A. Beilinson, J. Bernstein: A Generalization of Casselmans Submodule Theorem, in Representation theory of reductive groups: Proceedings of the University of Utah Conference 1982, *Progress in Mathematics, v.40*, Birkhäuser Boston, Boston, MA, 1983.

[2] J. Dixmier: Enveloping Algebras, *Graduate Studies in Mathematics, v.11*, American Mathematical Society, 1996.

[3] W. Fulton, J. Harris: Representation Theory, Springer Verlag 1991.

[4] A. Hatcher: Algebraic Topology, Cambridge University Press 2002.

[5] L. Hörmander: The Analysis of Linear Partial Differential Operator I, *Grundlehren der mathematischen Wissenschaften, v.256*, Springer Verlag 1983.

[6] J. E. Humphreys: Introduction to Lie Algebras and Representation Theory, *Graduate Texts in Mathematics*, Springer Verlag 1997, 3rd Revision edition.

[7] A. W. Knapp: Lie Groups Beyond an Introduction *Progress in Mathematics, v.140*, Birkhäuser 2002, 2nd Edition.

[8] A. W. Knapp: Representation Theory of Semisimple Groups, *Princeton Landmarks of Mathematics*, Princeton University Press 1986.

[9] A. W. Knapp: Lie Groups, Lie Algebras, and Cohomology, *Mathematical Notes, v.34*, Princeton University Press 1988.

[10] A. W. Knapp, P. E. Trapa: Representations of Semisimple Lie Groups, In J. Adams, D. Vogan, editors, *Representation Theory of Lie Groups*, IAS/Park City Mathematics Series, Vol 8, 2000, 5–88.

[11] A. W. Knapp, D. A. Vogan: Cohomological Induction and Unitary Representations, *Princeton Mathematics Series, v.45*, Princeton Univeristy Press 1995.

[12] T. Y. Lam: A First Course in Noncommutative Rings, *Graduate Texts in Mathematics, v. 131*, Springer Verlag 2001, Second Edition.

[13] T. Y. Lam: Lectures on Modules and Rings, *Graduate Texts in Mathematics, v.189*, Springer Verlag 1998.

[14] J. Lepowsky, G. W. McCollum: On the Determination of Irreducible Modules by Restriction to a Subalgebra, *Transactions of the American Mathematical Society, v.176*, American Mathematical Society, 1973.

[15] J. Mickelsson: Discrete Series of Lie Superalgebras, *Reports on Mathematical Physics, 18*, 1980.

[16] A. C. da Silva: Lectures on Symplectic Geometry *Lecture Notes in Mathematics, v.1764*, Springer Verlag 2001.

[17] C. A. Weibel: Introduction to Homological Algebra, *Cambridge Studies in Advanced Mathematics, v.38*, Cambridge University Press 1994.

[18] D. P. Zhelobenko: An Introduction to the Theory of S-algebras over reductive Lie algebras, *Representation of Lie Groups and Related Topics, 155-221, Advanced Studies in Contemporary Mathematics, 7*, Gordon and Breach, New York 1990.