Local Complex Singularity Exponents for Isolated Singularities

by

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Abstract

In this thesis, I studied the stability of local complex singularity exponents (*lcse*) for holomorphic functions whose zero sets have only isolated singularities. For a given holomorphic function f defined on a neighborhood of the origin in $\mathbb{C}^n$, the lcse $c_0(f)$ is defined as the supremum of all positive real number λ for which $1/|f|^{2\lambda}$ is integrable on some neighborhood of the origin. It has been conjectured that $c_0(f)$ should not decrease if f is deformed small enough. Using J. Mather and S.S.T. Yau's result on the classification of isolated hypersurface singularities, together with a well known result on the stability of $c_0(f)$ when f is deformed in a finite dimension base space, I proved that if the zero set of f has only isolated singularity at the origin, then $c_0(g) \geq c_0(f)$ for g close enough to f with respect to the C^0 norm over a neighborhood of the origin, thus gave a partial solution to the conjecture. Using the stability results, I also computed the holomorphic invariant $\alpha(M)$ for some special Fano manifold M.

Thesis Supervisor: Gang Tian
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Dedicated to my parents

Contents

Chapter 1

Introduction

We work over the complex number field $\mathbb{C}$. Given a complex manifold X and a compact subset $K \subset X$, let $f_1, \dots, f_n$ be holomorphic functions defined over an open neighborhood of K. We define the *local complex singularity exponent (lcse)* of f_i's over K as:

$$c_K(f_1, \dots, f_n) := \sup\{\lambda \in \mathbb{R}_+ \mid \frac{1}{(\sum_i |f_i|)^{2\lambda}} \in L_{\text{loc}}(K)\}$$

When the complex space $Z(f)$ defined by f_i's is not smooth around K, $c_K(f_1, \dots, f_n)$ can be a measurement on how singular $Z(f)$ is around K.

1.1 Complex singular indexes and Bernstein-Sato polynomials

Early research on local complex singularity exponent was focused on its connections with the asymptotic expansion of oscillatory integrations over Milnor fibers and Bernstein-Sato polynomials for convergent power series.

Suppose $f : (X^{n+1}, x) \to (\mathbb{C}, 0)$ is a germ of holomorphic function on a complex manifold X of dimension $n+1$. For $t \in \mathbb{C}$, denote $X_t = f^{-1}(t)$. Arnold considered the oscillatory integration of the form

$$\int_{\sigma(t)} \omega$$

where $\sigma(t) \in H_n(X_t, \mathbb{Q})$ is a continuous section, and ω is a section of $K_{X/\mathbb{C}}$. He proved that when $t \to 0$, there is an asymptotic expansion

$$\int_{\sigma(t)} \omega = \sum_{\alpha \in \mathbb{Q}, k \in \mathbb{N}} b(\sigma, \omega, \alpha, k) t^{\alpha} (\log t)^{k}.$$

Then the *complex singular index* $\beta(f)$ is defined as:

$$\beta(f) := 1 + \inf\{\, \alpha \mid b(\sigma, \omega, \alpha, k) \neq 0 \,\}$$

Connection between lcse $c_x(f)$ and complex singular index $\beta(f)$ is established by an observation due to Varchenko [24] which states that

$$c_x(f) = \min\,(1, \beta(f)\,)$$

For more information on oscillatory integration and complex singular index, see [2].

Bernstein-Sato polynomial was introduced by Bernstein [3] for polynomials and later generalized by Björk [4] for convergent power series and formal power series.

Let f be a formal power series, then there is a nonzero ideal $\mathcal{I}_f \subset \mathbb{C}[t]$ with the property that $b(t) \in \mathcal{I}_f$, iff there exists a formal linear differential operator

$$P_{b,f} = \sum_{I,j} f_{I,j} \cdot t^{j} \cdot \frac{\partial^{|I|}}{\partial z^{I}} \tag{1.1}$$

where $f_{I,j}$ are also formal power series, such that

$$b(t) f^{t} = P_{b,f} f^{t+1} \tag{1.2}$$

We call the unique monic generator of $\mathcal{I}_f$ as the *Bernstein-Sato polynomial* of f, and denote it as $b_f(t)$.

Remark. The Equation 1.2 is just a formal equality, f^t and f^{t+1} are not real functions. When f is polynomial or convergent power series, the $f_{I,j}$'s in Equation 1.1 are also polynomials or convergent power series respectively.

Kashiwara [8] proved one of the most important results on Bernstein-Sato polynomials.

Theorem 1.1. *Let $\pi : X \to U \subset \mathbb{C}^n$ be a proper map, and $f' = f \circ \pi$. Let $b_f(t)$ and $b_{f'}(t)$ be the Bernstein-Sato polynomials of f and f' respectively. Then $b_f(t)$ is a divisor of $b_{f'}(t)b_{f'}(t+1)\cdots b_{f'}(t+N)$ for N large enough.*

If we choose a special π, such as the log resolution of the log pair $(U, Z(f))$ where $Z(f)$ is the divisor defined by f, then we can reduce the computation of $b_f(t)$ for general f to the case when f is just monomial. Especially, we can show that all the roots of $b_f(t)$ are negative rational numbers. Along this way, Lichtin [10] showed that the largest root of $b_f(t)$ is exactly the negative of the lcse of f.

Based on these two interpretations of lcse, Varchenko [25] proved a stability result for $c(f)$ when f is deformed in a finite dimensional base space.

Theorem 1.2. *Let X be a complex manifold and S be a reduced complex space. Let $f(x, s)$ be a holomorphic function on $X \times S$. Then $\forall\, (x_0, s_0) \in X \times S$, there exists a Zariski neighborhood U of s_0 in S, such that $\forall\, s \in U$,*

$$c_{x_0}(f|_{X\times s}) \geq c_{x_0}(f|_{X\times s_0})$$

Later, this important results was reproved independently by Demailly and Kollár in [6] and Phong and Sturm in [15] using L^2 extension theorem. Mustaţă [12] also reproved this theorem using the method of jet scheme and motivic integration. We will sketch their proof in Section 4.1

1.2 Kähler-Einstein metrics and multiplier ideal sheaves

One of the most important applications of lcse is Tian's work on the existence of Kähler-Einstein metric on complex manifolds with positive first Chern classes.

Let (M, g) be a compact Kähler manifold with $c_1(M) > 0$, and G be a compact subgroup of $\mathrm{Aut}(M,g)$ we define

$$P_G(M,g) = \{\, \phi \in C^\infty(M) \mid \phi \text{ is } G\text{-invariant}, \ \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi > 0, \ \sup_M \phi = 0 \,\}$$

In [20] Tian introduced following global holomorphic invariant

$$\alpha_G(M) = \sup\{\, \alpha \mid \exists\, C_\alpha > 0, \text{s.t.} \int_M e^{-\alpha\phi}\omega_g^n \le C_\alpha\,, \ \forall\, \phi \in P_G(M,g) \,\}$$

and proved a sufficient condition for the existence of Kähler-Einstein metric.

Theorem 1.3. *If $\alpha_G(M) > n/n+1$, then M admits a G-invariant Kähler-Einstein metric.*

Tian [20] also computed $\alpha_G(M)$ when M is the Fermat hypersurface in $\mathbb{CP}^{n+1}$ of degree n or $n+1$. In these two cases, $\alpha_G(M) > n/n+1$, therefore by Theorem 1.3 there exists Kähler-Einstein metric on M.

Remark. If we compare the definition of $\alpha(M)$ and the definition of lcse in Section 2.1, we can regard $\alpha(M)$ as the *global* complex singularity exponent of M.

In some applications, $P_G(M)$ is just too large. So Tian [21] considered following approximations for $P(M)$ and $\alpha(M)$.

$$P_{m,k,G}(M) = \{\frac{1}{2}\log(\sum_1^k ||s_i||_g^2) \in P_G(M) \mid s_i \in H^0(M, K_M^{-m}) \text{ and } < s_i, s_j >_g = \delta_{ij}\,\}$$

$$\alpha_{m,k,G}(M) := \sup\{\alpha \mid \exists\, C_\alpha \text{ s.t. } \int_M e^{-2\alpha\varphi}\omega_g^n \le C_\alpha \ \forall \varphi \in P_{m,k,G}\}$$

where $||\cdot||_g$ is the metric for $-mK_X = K_X^{-m}$ induced by g and $<,>_g$ is the inner product on $H^0(M, -mK_M)$ also induced by g. He also proved a variation of Theorem 1.3.

Proposition 1.4. *Let M be a smooth Fano surface. If for some m*

$$\frac{1}{\alpha_{m,1}(M)} + \frac{1}{\alpha_{m,2}(M)} < 3$$

then there exists a Kähler-Einstein metric on M.

In [22, 21], Tian computed the α invariants for smooth Fano surface which is the blownup of $\mathbb{CP}^2$ at k points in general position, with $3 \leq k \leq 8$. They either satisfies the requirement in Theorem 1.3 or Proposition 1.4, hence there exists a Kähler-Einstein metric on such surface.

Nadel [13, 14] generalized the constructions above and introduced the concept of multiplier ideal sheaf.

For a pluri-subharmonic (*psh*) function φ defined on a complex manifold X, the *multiplier ideal sheaf* $\mathcal{I}(\varphi)$ is the ideal sheaf of $\mathcal{O}_X$ whose stalk at point x is generated by germs of functions f for which $|f|^2 \exp(-2\varphi)$ is locally integrable around x. Nadel proved that $\mathcal{I}(\varphi)$ is a coherent ideal sheaf, and there is a Kodair type vanishing theorem for multiplier ideal sheaves.

Theorem 1.5 (Nadel Vanishing Theorem). *Let (X, ω) be a compact Kähler manifold, and let F be a holomorphic line bundle with a singular hermitian metric $e^{2\varphi}$. Assume that $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi \geq \varepsilon\omega$ for some positive continuous function ε on X. Then*

$$H^q(X, \mathcal{O}(K_X + F) \otimes \mathcal{I}(\varphi)) = 0 \qquad \text{for all} \quad q > 0$$

Nadel Vanishing Theorem has lots of important applications in higher dimension complex geometry. For an introduction to multiplier ideal sheaf and its applications based on an analytic approach, see [5].

1.3 Log canonical thresholds and Shokurov's conjectures

Local complex singularity exponent also appears as log canonical threshold in log Minimal Model Program.

Let X be a normal algebraic variety with log canonical singularity. $K \subset X$ is a compact subset and D is an effective $\mathbb{Q}$-Cartier divisor on X. The *log canonical threshold* of D over K is defined as the supremum of all positive real number c for which (X, cD) is log canonical in an open neighborhood of K, and it is denoted as $c_K(X, D)$. In Chapter 3 we will show that when D is the divisor defined by a holomorphic function f, $c_K(X, D)$ is exactly $c_K(f)$.

In log Minimal Model Program, we are often required to show that certain operation must stop after finitely many steps, such as flop or flip. A common strategy is to assign those objects certain numerical invariants which will increase after we perform the operation. If we can show that the set of all possible value of the numerical invariant satisfies the ACC condition, i.e. any non decreasing chain terminates, then the operation must stop after finitely many steps. This is the case how we proved the termination of flips for three folds using difficulty.

Based on this general *philosophical* principle, Shokurov[17] proposed many conjectures concerning the ACC property of certain invariants. The simplest one is following:

Conjecture 1.6. *Let S_n be the set of all possible $c_0(\mathbb{C}^n, D)$, where D is an effective divisor passing through the origin. Then S_n satisfies the ACC condition.*

Remark. This conjecture has only been proved for $n = 2$ by Alexeev [1] using an algebraic method. Phong and Sturm [16] also gave an analytic proof later.

1.4 Stability of local complex singularity exponent

By stability, we *expect* that for any given holomorphic function f, if g is close enough to f in some suitable sense, then

$$c_x(g) \geq c_x(f) \tag{1.3}$$

Though the simplicity of Equation 1.3, we do not know much about the stability of lcse. Basically, there are four different methods to approach this problem.

The first approach is oscillatory integration and Bernstein-Sato polynomial. We've already cited Theorem 1.2 by Varchenko on the stability of $c_x(f)$ if f is deformed in a finite dimension base space. Noticed we can also define the local *real* singularity exponent for real analytic function by the same way we define local *complex* singularity exponent. And similar relation hold between local real singularity exponent and oscillatory integration. However, there is no stability for local real singularity exponents. Varchenko [23], gave an example of real analytic functions $f(x,t)$ defined on $\mathbb{R}^3 \times \mathbb{R}$ such that $c_0(f|_{\mathbb{R}^3\times\{0\}}) = \frac{5}{8}$ while $c_0(f|_{\mathbb{R}^3\times\{t\}}) \to \frac{1}{2}$ when $t < 0$ and $t \to 0$.

The second method is using Newton Polyhedron associated with f. Using this method, Tian [21] proved that for surface $\int 1/|f|^{2\delta}$ is continuous at f if $\delta < c_x(f)$.

Theorem 1.7. *Let f be a holomorphic function defined on the unit ball $B_1 \subset \mathbb{C}^2$ and $\delta > 0$ such that $\int_{B_1} 1/|f|^{2\delta}$ is finite. Then for any $r < 1$ and $f_i \in \mathcal{O}_{\mathbb{C}^2}(B_1)$, such that $\lim_{i\to\infty} f_i = f$ uniformly on B_1, we have*

$$\lim_{i\to\infty} \int_{B_r} \frac{1}{|f_i|^{2\delta}} = \int_{B_r} \frac{1}{|f|^{2\delta}}$$

The third approach is the L^2 extension theorem due to Ohsawa-Takegoshi. Many deep results was obtained as applications of L^2 extension theorem. For example, Demailly and Kollár [6] proved following sub-additive formula.

Theorem 1.8. *Let f and g be holomorphic functions defined on a neighborhood of the origin in $\mathbb{C}^n$, then*

$$c_0(f+g) \le c_0(f) + c_0(g)$$

Theorem 1.2 was also reproved by L^2 extension method in [6]. In fact, what Demailly and Kollár proved is more than just the stability, they also got an effective bound for the integrals.

Theorem 1.9. *Let $f(x,s)$ be a holomorphic function defined on $X \times S$ where X is a complex manifold and S is a reduced complex space. Let K be a compact subset of X. Then for any $s_0 \in S$ and $c < c_K(f|_{X\times\{s_0\}})$, there is an open neighborhood $U(c)$ of K and a constant $M(c)$ depends on c, such that*

$$\int_{U(c)} \frac{1}{|f(x,s)|^{2c}}\, dV(x) \le M(c)$$

for s in a neighborhood of s_0 in S.

Combining Theorem 1.8 and Theorem 1.9, Demailly and Kollár proved a weaker stability result.

Theorem 1.10. *Let f be a holomorphic function defined on a complex manifold X. K is a compact subset of X. Then for every open set L containing K in its interior and every $\varepsilon > 0$, there is $\delta > 0$, such that for holomorphic function g defined on L,*

$$\sup_L |g - f| < \delta \quad \Rightarrow \quad c_K(g) \geq c_K(f) - \varepsilon.$$

Following stability conjecture was also proposed by Demailly and Kollár.

Conjecture 1.11 (Stability Conjecture). *Notation as in Theorem 1.10. For every nonzero holomorphic function f, there exists $\delta = \delta(f, K, L) > 0$, such that*

$$\sup_L |g - f| < \delta \quad \Rightarrow \quad c_K(g) \geq c_K(f)$$

It is easy to see that Stability Conjecture is implied by Conjecture 1.6 and Theorem 1.10. By the remark after Conjecture 1.6, since Conjecture 1.6 is true for surfaces, so the Stability Conjecture is also true for surfaces.

Phong and Sturm [15] proved Theorem 1.2 by L^2 extension method too. In the same paper, using local cluster-scale, they also developed a sharp algebraic estimate for the integral of the form $\int_{B_r} |g(z)|^\varepsilon / |f(z)|^\delta$, where both $f(z)$ and $g(z)$ are polynomials of one variable with fixed degree. Let A be the affine space of all polynomials g and f with fixed degree, they showed that their is a stratification of A by algebraic subvarieties depends only on ε and δ, i.e.

$$A = U_0 \supset U_1 \supset \ldots \supset U_N = \phi$$

where all U_λ's are algebraic subvarieties in A, and if $(f, g) \in U_\lambda \setminus U_{\lambda+1}$, then there is an estimate

$$C_\lambda \frac{|A_\lambda(b, b')|^{\varepsilon'}}{|B_\lambda(b, b')|^{\delta'}} \quad \leq \quad \int_{B_r} \frac{|g(z)|^\varepsilon}{|f(z)|^\delta} \quad \leq \quad D_\lambda \frac{|A_\lambda(b, b')|^{\varepsilon'}}{|B_\lambda(b, b')|^{\delta'}}$$

where (b, b') is the coefficients of f and g, and polynomials A_λ, B_λ and the real positive numbers C_λ, D_λ, ε' and δ' depend only on ε and δ. Using this algebraic estimate, Phong and Sturm reproved and generalized some known results. In [15] they reproved Theorem 1.7, and generalized it to the case $n = 3$ with some extra conditions by induction on dimension; in [16], they reproved Conjecture 1.6 and Conjecture 1.11 for surfaces simultaneously.

The fourth approach is jet scheme and motivic integration.

For every scheme Y of finite type over $\mathbb{C}$ and every $m \in \mathbb{N}$, the m-th jet scheme for Y is a scheme Y_m characterized by

$$\mathrm{Hom}\big(\mathrm{Spec}(A), Y_m\big) = \mathrm{Hom}\big(\mathrm{Spec}(A[t]/(t^{m+1})), Y\big)$$

for every $\mathbb{C}$-algebra A. Let $\rho_{m,Y}$ be the canonical projection from Y_m to Y.

Mustaţă [12] proved a formula which compute log canonical threshold by dimensions of certain jet schemes.

Theorem 1.12. *Let (X, Y) be a (log) pair, with $Z \subset X$ an nonempty closed subset. Let $\dim_Z Y_m$ be the dimension of Y_m along $Y_m \cap \rho_m^{-1}(Z)$. If X is smooth, then*

$$c_Z(X, Y) = \dim X - \sup_{m \geq 0} \frac{\dim_Z Y_m}{m+1}$$

Using Theorem 1.12, Mustaţă reproved Theorem 1.2. (a theorem which has been reproved over and over again by so many different ways.)

1.5 Main Results

In this thesis, We will prove following two stability results.

Main Theorem 1. *Given holomorphic functions $f_1, \ldots, f_N$ defined on a neighborhood U of the origin in $\mathbb{C}^n$ such that $f_i(0) = 0$. If $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0}/(f_1, \ldots, f_N)$ is finite, then there exists $\varepsilon > 0$, such that for any holomorphic functions $g_1, \ldots, g_N$ defined on U,*

$$\sup_U \sum_i |g_i - f_i| < \varepsilon \quad \Rightarrow \quad c_0(g_1, \ldots, g_N) \geq c_0(f_1, \ldots, f_N)$$

where ε depends only on f_i's and U.

Main Theorem 2. *Let f be a holomorphic function defined on a neighborhood U of the origin in $\mathbb{C}^n$. If the complex space $Z(f)$ defined by f has only isolated singularity at the origin, then there exists $\delta > 0$, such that for any holomorphic function g defined on U, we have*

$$\sup_U |g - f| < \delta \quad \Rightarrow \quad c_K(g) \geq c_K(f)$$

where δ depends only on U and f.

We will also use the stability result to recompute $\alpha(M)$ when M is the blownup of $\mathbb{CP}^n$. Especially, we will show that when M is the blownup of $\mathbb{CP}^n$ at one points, then $\alpha_{m,k}(M) \geq \min(1/2, 2/n+1)$, and when M is the blownup of $\mathbb{CP}^n$ at n points, then $\alpha_{m,k}(M) \geq 1/(n+1)$, and when M is the blownup of $\mathbb{CP}^n$ at $n+1$ points, then $\alpha_{m,k}(M) \geq 3/(n+1)$. Though Tian already computed $\alpha(M)$ for Fano surface M in more general cases, our computation here is in the hope to try to simplify some of his works.

The rest of this thesis is organized as follows. In Chapter 2, we will introduce the analytic side of the theory of local complex singularity exponents and multiplier ideal sheaves. In Chapter 3, log canonical singularities and log canonical thresholds will be introduced. The equivalence of local complex singularity exponent and log canonical threshold will also be established in Chapter 3. In Chapter 4, we will sketch the proof of stability over finite dimension base space and then prove the two main theorems. In Chapter 5, we will compute $\alpha_{m,k}(M)$ for some special smooth Fano manifolds.

Chapter 2

Local Complex Singularity Exponents

This chapter presents the basic theory of local complex singularity exponents and multiplier ideal sheaves by analytic approach.

2.1 Basic definitions and properties

Let X be a complex manifold with a fixed volume form dV_X so that we can integrate functions on X; let K be a compact subset of X and φ be a pluri-subharmonic (*psh*) function on a neighborhood of K.

Definition 2.1. Notations as above, we define the *local complex singularity exponent (lcse)* of φ over K as:

$$c_K(\varphi) := \sup \{ \lambda \geq 0 \mid \exp(-2\lambda\varphi) \in \mathrm{L}^1_{\mathrm{loc}}(K, dV_X) \}$$

Remark. Even though we fixes a voluming form dV_X on X in the definition above, the value of $c_K(\varphi)$ is actually independent of the volume form we choose. Since for any two different continuous volume forms dV_1 and dV_2, there exist two positive real numbers M and N, such that

$$M dV_1 \leq dV_2 \leq N dV_1$$

holds on a neighborhood U of K, hence

$$\int_U \exp(-2\lambda\varphi) dV_1 < \infty \quad \Longleftrightarrow \quad \int_U \exp(-2\lambda\varphi) dV_1 < \infty$$

So $c_K(\varphi)$ is independent of the choice of volume form.

Since K is compact, we have following simple observation:

$$c_K(\varphi) = \inf\{\, c_x(\varphi) \,|\, x \in K\} \tag{2.1}$$

In fact, we will show that the infimum in Equation 2.1 is actually a minimal. This observation reduces our computation for general K to the case when K is just a single point.

There are some variations of Definition 2.1.

1. Let $f_1, f_2, \ldots, f_N$ be holomorphic functions defined on a neighborhood of K. For real number $q > 0$, we define

$$c_K\big((f_1,\ldots,f_N)^q\big) := c_K\big(\, q \cdot \log(|f_1| + \cdots + |f_N|)\,\big)$$

2. Let $\mathcal{I}$ be a coherent ideal sheaf. If $f_1, f_2, \ldots, f_N$ generate $\mathcal{I}$ around $x \in X$, then we define

$$c_x\big(\mathcal{I}^q\big) := c_x\big(\, q \cdot \log(|f_1| + \cdots + |f_N|)\,\big) \tag{2.2}$$

For general compact set K, we use equation 2.1 to define

$$c_K\big(\mathcal{I}^q\big) = \inf\ \big\{\, c_x(\mathcal{I}^q) \,|\, x \in K\big\}$$

Suppose that $\{f_i\}$ and $\{g_j\}$ are two different sets of generators around x. Since $\{f_i\}$ generate $\mathcal{I}_x$, so

$$g_j = \sum_i h_{j,i}\, f_i$$

for some $h_{j,i} \in \mathcal{O}_x$, hence $\exists\, C > 0$, such that

$$\sum |g_j| \leq C \sum |f_i|$$

in a neighborhood of x. Therefore

$$\int \frac{1}{(\sum |g_j|)^{2q\lambda}} \geq \frac{1}{C^{2q\lambda}} \int \frac{1}{(\sum |f_i|)^{2q\lambda}}$$

So

$$c_x\big(\,q\log(\sum_j |g_j|)\,\big) \le c_x\big(\,q\log(\sum_i |f_i|)\,\big)$$

Repeat the argument for $\{g_j\}$, we have

$$c_x\big(\,q\cdot\log(\sum_i |f_i|)\,\big) = c_x\big(\,q\cdot\log(\sum_j |g_j|)\,\big)$$

So $c_x(\mathcal{I}^q)$ is well defined by Equation 2.2.

3. Let (L,h) be a Hermitian line bundle with positive (in the current sense) curvature form. Choosing a local trivialization, we can write $h = \exp(-2\varphi)$ with φ being a psh function. Then we define

$$c_K(h) := c_K(\varphi)$$

Noticed that if we change the trivialization, φ will be changed to $\varphi + \phi$ with ϕ being a locally bounded function, hence $c_K(\varphi) = c_K(\varphi + \phi)$. So $c_K(h)$ is well defined.

4. Let L be a line bundle, and $s_1,\ldots,s_N$ be sections of L. After choosing a local trivialization, s_i's correspond to holomorphic functions $f_1,\ldots,f_N$, so we define

$$c_K\big(\,(s_1,\ldots,s_N)^q\,\big) := c_K\big(\,(f_1,\ldots,f_N)^q\,\big)$$

Easy to check that this is well defined. Another way to define $c_K\big(\,(s_1,\ldots,s_N)^q\,\big)$ is to introduce a Hermitian metric h for the line bundle L first, and then define

$$c_K\big(\,(s_1,\ldots,s_N)^q\,\big) := c_K\big(\,q\log(\sum_1^N |s_i|_h)\,\big)$$

Easy to check that these two definitions are equivalent.

Remark. In all these definitions, we assume that X is a smooth manifold. In fact, X can be a reduced complex space with singularity. Because the singularity set of X is is a measure zero set which will not affect the integration behavior, we can go through all the definitions above without any problem.

We can reduce the computation of $c_x(f_1,\ldots,f_N)$ to $c_x(f)$ for just one function f.

Proposition 2.2. *Let $f_1, \dots, f_N$ be holomorphic functions defined around x such that $f_i(x) = 0$. Then for any nonzero $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{C}^N$, we have*

$$c_x(\sum_{i=1}^{N} \alpha_i f_i) \leq \min \left\{c_x(f_1, \dots, f_N), 1\right\}$$

and equality hold for almost all α except a measure zero set.

Proof. Following the argument of [6]. The inequality is trivial, because

$$|\sum_{i=1}^{N} \alpha_i f_i| \leq \big(\sum |\alpha_i|\big)\big(\sum |f_i|\big)$$

so

$$\int \frac{1}{\big(\sum |f_i|\big)^{2\delta}} \leq \int \frac{\big(\sum |\alpha_i|\big)^{2\delta}}{|\sum_{i=1}^{N} \alpha_i f_i|^{2\delta}}$$

So $c_x\big(\sum \alpha_i f_i\big) \leq c_x(f_1, \dots, f_N)$. Also since $\sum \alpha_i f_i(x) = 0$, so $c_x\big(\sum \alpha_i f_i\big) \leq 1$. On the other hand, for any $c < 1$, there is $A_c > 0$ depends only on c such that

$$\int_{|\alpha|=1} \frac{dV(\alpha)}{|\sum_i \alpha_i f_i|^{2c}} = \frac{A_c}{\big(\sum |f_i|\big)^{2c}}$$

Hence if $c < \min \left\{c_x(f_1, \dots, f_N), 1\right\}$,

$$\int_{|\alpha|=1} dV(\alpha) \int_U \frac{1}{|\sum \alpha_i f_i|^{2c}} dV(z) = A_c \int_U \frac{1}{(\sum |f_i|)^{2c}} dV(z) < \infty$$

So for α outside of measure zero set M_c, $\int_U 1/|\sum \alpha_i f_i|^{2c}$ is finite. Let c_n be a sequence approach to $c_x(f_1, \dots, f_N)$, then $M = \cup_n M_{c_n}$ is the union of countable many measure zero sets, which is also measure zero. And for $\alpha \in \mathbb{C}^N \setminus M$, we have

$$c_x\big(\sum \alpha_i f_i\big) = \min \left\{c_x(f_1, \dots, f_N), 1\right\}$$

□

Proposition 2.3. *Let φ, ψ be psh functions on X, and $\mathcal{I}, \mathcal{J}$ be coherent ideals on X, $K \subset X$ be a compact set.*

1. *The function $x \mapsto c_x(\varphi)$ is lower semi-continuous with respect to the holomorphic Zariski topology;*

2. *If $\varphi \leq \psi$, then $c_K(\varphi) \leq c_K(\psi)$;*

3. $c_K(q \cdot \varphi) = \frac{1}{q} c_K(\varphi)$ *and*
$c_K(\mathcal{I}^q) = \frac{1}{q} c_K(\mathcal{I})$*;*

4. Let $Z(\mathcal{I}_x)$ *be the germ of the subscheme defined by* $\mathcal{I}_x$*. If* $Z(\mathcal{I}_x)$ *contains a p-codimensional irreducible component, then* $c_x(\mathcal{I}) \leq p$*;*

5. If $\mathcal{I}$ *is the ideal sheaf of a p-codimensional subvariety* $Y \subset X$*, then* $c_x(\mathcal{I}) = p$ *for every smooth point* x *of* Y*;*

6. Define the multiplicity $mult_x(\mathcal{I})$ *of* $\mathcal{I}$ *at* x *to be the maximal of all integers* k *such that* $\mathcal{I} \subset \mathfrak{m}_x^k$*, where* $\mathfrak{m}_x$ *is the maximal ideal corresponding to* x*. Then*

$$\frac{1}{mult_x(\mathcal{I})} \leq c_x(\mathcal{I}) \leq \frac{n}{mult_x(\mathcal{I})}$$

where n *is the dimension of* X*.*

Proof. See Demailly and Kollár [6]

(1) Fix a relatively compact subset $B \subset\subset X$. For $c > 0$, considering the Hilbert Space $\mathcal{H}_{c\varphi}$ of the holomorphic functions with finite norm:

$$||f||_{c\varphi} := \int_B |f|^2 e^{-2c\varphi} dV_X$$

By Hörmander's L^2 estimates, whenever $e^{-2c\varphi}$ is locally integrable around x there exists $f \in \mathcal{H}_{c\varphi}$, such that $f(x) = 1$. So

$$\{\, x \in B \mid c_x(\varphi) \leq c_0 \} \cap B = \bigcap_{f \in \cup_{c > c_0} \mathcal{H}_{c\varphi}} f^{-1}(0)$$

and this completes the proof of (1).

(2) and (3) are trivial.

(4) and (5) are the consequences of the fact that $\int 1/(\sum_{i \leq p} |z_i|)^{2c} < \infty$ iff $c < p$.

(6) is essentially a result due to Skoda [18]. We will discuss $1/\text{mult}_x(\mathcal{I}) \leq c_x(\mathcal{I})$ in more detail in Proposition 4.4. □

Remark. By property (1), it is easy to see that in equation (2.1), the infimum can be attained, so it is in fact a minimal.

Example. Some examples on the computation of $c_x(f)$

1. If X is a smooth curve, then $\mathcal{O}_{X,x}$ is a DVR, hence all the ideal of $\mathcal{O}_{X,x}$ must be of the form $\mathfrak{m}_x^N$ for some $N \in \mathbb{N}$. By Property (6) in Proposition (2.3) above, we know that

$$c_x(\mathfrak{m}_x^N) = \frac{1}{N}$$

2. Let X be $\mathbb{C}^n$, with $f = z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n}$. Direct computation gives

$$c_0(f) = \min\left\{\frac{1}{i_\alpha} | \alpha = 1, \ldots, n\right\}$$

Let's concentrate on the computation of $c_0(f)$ for $f \in \mathcal{O}_{\mathbb{C}^n,0}$. The computation becomes really involved as n increases. When $n = 1$, we know that $c_0(f) = 1/\mathrm{mult}_0 f$. When $n = 2$, the computation is closely related to the Newton Polyhedron of f.

Definition 2.4. Given a local coordinate $z = (z_1, \ldots, z_n)$ of $\mathbb{C}^n$. Let f be a holomorphic function defined on a neighborhood of the origin, we define $N(f, z)$ the *Newton Polyhedron* of f with respect to z, to be the convex hull of the set $\{\alpha \in \mathbb{Z}^n_{\geq 0} \mid f_\alpha \neq 0\}$ in $\mathbb{R}^n_{\geq 0}$, where f_α is the coefficient of z^α term in the Taylor expansion of f with respect to z. Let the line $\{z_1 = \cdots = z_n\}$ intersects $N(f, z)$ at point $(r, \ldots, r)$, we call r the remotedness of $N(f, z)$, and denoted it as $r(N(f, z))$.

Easy to prove that $c_0(f) \leq 1/r(N(f, z))$ and strict inequality does happen sometimes. However Tian [21] proved that if $n = 2$ equality holds after taking a limit.

Proposition 2.5. *Let f be a holomorphic function defined on a neighborhood of the origin in $\mathbb{C}^2$, then there exist a sequence of coordinate systems $z^\beta = (z_1^\beta, z_2^\beta)$, such that*

$$r(N(f, z^\beta)) \leq r(N(f, z^\gamma)) \quad \text{if} \quad \beta \leq \gamma$$

and

$$c_0(f) = \lim_{\beta \to \infty} r(N(f, z^\beta)^{-1}.$$

Proof. see Tian [21]. □

In higher dimensions, the most efficient way to compute $c_x(f)$ is using resolution of singularity whose existence is due to Hironaka when the base field is $\mathbb{C}$.

Theorem 2.6 (Hironaka [7]). *Let (X,x) be a germ of reduced complex space such that K_X is $\mathbb{Q}$-Cartier. Let $\mathcal{I} \subset \mathcal{O}_{X,x}$ be an ideal sheaf. Then there exists a proper birational map $\pi : Y \to X$, such that*

1. *$\pi : Y \setminus \pi^{-1}\big(\mathrm{supp}(\mathcal{I}) \cup \mathrm{Sing}(X)\big) \to X \setminus \big(\mathrm{supp}(\mathcal{I}) \cup \mathrm{Sing}(X)\big)$ is isomorphism, where $\mathrm{Sing}(X)$ is the singular set of X;*

2. *Y is smooth and $K_{Y/X} = K_Y - \pi^* K_X = \sum_i b_i F_i$ where F_i's are distinct irreducible divisors on Y;*

3. *$\pi^*(\mathcal{I}) = \mathcal{I}\mathcal{O}_Y = \mathcal{O}_Y(-E)$ with $E = \sum a_i F_i$, where $a_i \geq 0$;*

4. *$\sum_i F_i$ is a simple normal crossing (snc) divisor whose support is contained in $Ex(\pi) \cup \pi^{-1}(supp(\mathcal{I}))$.*

Any proper birational map satisfying the condition above is called as log resolution *of $(X, \mathcal{I})$ or $(X, Z(\mathcal{I}))$.*

Proposition 2.7. *Notation as in Theorem 2.6, then we have*

$$c_x(\mathcal{I}^q) = \min\left\{ \frac{b_i + 1}{q a_i} \mid x \in \pi(F_i) \right\} \tag{2.3}$$

Especially, we have $c_x(\mathcal{I}) \in \mathbb{Q}$.

Proof. Let $\{f_1, \ldots, f_N\}$ be a set of generators of $\mathcal{I}$, and U be an open neighborhood of x on which all f_j's are defined. Since π is proper, $\pi^{-1}(x)$ is a compact subset. Cover $\pi^{-1}(x)$ with finitely many open set U_α such that over each U_α, $\mathcal{O}_Y(-F_i)$ is generated by $h_{\alpha,i} \in \mathcal{O}_Y(U_\alpha)$. Shrink both U and U_α if necessary, we may assume that $U = \cup_\alpha \pi(U_\alpha)$. By the formula of change of variables, we have

$$\int_U |f|^{-2\lambda} dV_X = \int_{\pi^{-1}(U)} |f \circ \pi|^{-2\lambda} |J_\pi|^2 dV_Y$$

where J_π is the Jacobian of π. So

$$\int_U |f|^{-2\lambda} dV_X < \infty \quad \Longleftrightarrow \quad \int_{U_\alpha} |f \circ \pi|^{-2\lambda} |J_\pi|^2 dV_Y < \infty \quad \forall \alpha$$

For two functions F and G, we denote $F \sim G$, if both F/G and G/F are bounded by positive constants. Since $K_{Y/X} = \sum b_i F_i$, we have

$$|J_\pi| \sim \prod |h_{\alpha,i}|^{b_i} \qquad \text{on } U_\alpha$$

Also by $\mathcal{I}\mathcal{O}_Y = \mathcal{O}_Y(-\sum a_i F_i)$, we have

$$|f \circ \pi| \sim \prod |h_{\alpha,i}|^{a_i} \qquad \text{on } U_\alpha$$

Therefore

$$\int_{U_\alpha} |f \circ \pi|^{-2\lambda} |J_\pi|^2 dV_Y \sim \int_{U_\alpha} \prod |h_{\alpha,i}|^{-2(a_i\lambda - b_i)} dV_Y$$

Now that $\sum F_i$ is snc divisor, i.e. $\exists\, I_\alpha$, such that $\{\, h_{\alpha,i} \mid i \in I_\alpha \,\}$ can be extended to be a coordinated system on U_α, and $|h_{\alpha,j}| > 0$ for $j \notin I_\alpha$. Hence

$$\int_{U_\alpha} \prod |h_{\alpha,i}|^{-2(a_i\lambda - b_i)} dV_Y < \infty \quad \Longleftrightarrow \quad \lambda < \min\left\{\, \frac{b_i + 1}{a_i} \mid i \in I_\alpha \,\right\}$$

Noticed that $\cup_\alpha I_\alpha = \{\, i \mid x \in \pi(F_i) \,\}$ hence

$$\int_U |f|^{-2\lambda} dV_X < \infty \quad \Longleftrightarrow \quad \lambda < \min\left\{\, \frac{b_i + 1}{a_i} \mid x \in \pi(F_i) \,\right\}$$

Therefore

$$c_x(\mathcal{I}^q) = c_x(|f|^q) = \min\left\{\, \frac{b_i + 1}{q a_i} \mid x \in \pi(F_i) \,\right\}$$

□

Remark. By the proof above, we can see that the set

$$\{\, \lambda \,|\, (|f_1| + \cdots + |f_N|)^{-2\lambda} \in L_{\text{loc}} \}$$

is in fact an open set. So whenever

$$\int_U \frac{1}{|f|^{2\lambda}} dV < \infty$$

we can always find $\varepsilon > 0$, such that after shrink U if necessary, we have

$$\int_U \frac{1}{|f|^{2(\lambda+\varepsilon)}} dV < \infty$$

Example. Let $(X, x) = (\mathbb{C}^2, 0)$ and $f(x, y) = x^2 + y^3 \in \mathcal{O}_{\mathbb{C}^2,0}$, successive blowup over smooth point 3 times, we can get a proper birational map $\pi : Y \to X$ which resolve the singularity of $\mathcal{I} = (f)$, and

$$K_{Y/X} = E_1 + 2E_2 + 4E_3 \quad \text{and} \quad \pi^{-1}(f) = \mathcal{O}_Y(-(E + 2E_1 + 3E_2 + 6E_3))$$

where E_i's are the divisors correspond to the blowup, and E is the strict transform of $Z(f)$. Hence

$$c_0(x^2 + y^3) = 5/6.$$

Similarly, one can get following result which is known for a long time.

$$c_0(x^m + y^n) = \min\left\{1\,, \frac{1}{m} + \frac{1}{n}\right\}$$

Another way to compute $c_x(f)$ is using weighted blowup.

Proposition 2.8. *Let $f \in \mathcal{O}_{\mathbb{C}^n,0}$. Assign integral weights $\omega(z_i)$ to variable z_i, and let $\omega(f)$ be the weighted multiplicity of f, that is, the lowest weight of the monomials occurring in f's Taylor expansion. Then s*

$$c_0(f) \leq \frac{\sum \omega(z_i)}{\omega(f)}$$

Let f_ω be the weighted homogeneous leading term of f. Assume that

$$Z(f(z_1^{\omega z_1}, \ldots, z_n^{\omega z_n})) \subset \mathbb{C}P^{n-1} \quad \textit{is smooth}$$

Then

$$c_0(f) = \min\left\{1, \frac{\sum \omega(z_i)}{\omega(f)}\right\}.$$

Proof. For proof, see [9] □

Remark. In fact, the remotedness of the Newton Polyhedron is given by:

$$r(N(f, z)) = \min_{\omega} \frac{\omega(f)}{\sum \omega(z_i)}$$

Corollary 2.9. *For* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, *we have*

$$c_0(x_1^{\alpha_1} + \cdots + x_n^{\alpha_n}) = \min\left\{1, \sum_i \frac{1}{\alpha_i}\right\}$$

2.2 Multiplier ideal sheaves

Let φ be a psh function defined on a complex manifold X.

Definition 2.10. The *multiplier ideal sheaf* associated with φ is defined as the ideal sheaf $\mathcal{I}(\varphi) \subset \mathcal{O}_X$, such that for any $x \in X$, the germ $\mathcal{I}(\varphi)_x$ consists of germs of holomorphic functions $f \in \mathcal{O}_{X,x}$ for which $|f|^2 \exp(-2\varphi)$ is locally integrable around x.

Similarly we can define $\mathcal{I}(f^q)$ and $\mathcal{I}(\mathcal{J}^q)$ for holomorphic function f, coherent ideal sheaf $\mathcal{J}$ and real positive number q.

Multiplier ideal sheaf is a very convenient notation by which many classical theorems in complex analysis can be expressed in a quite nice way. For example, one of Skoda's important results in [18] can be expressed as:

Theorem 2.11. *Let* $\mathcal{J}$ *be a coherent ideal sheaf on* X *and* $\dim X = n$, *then for any* $s \geq n$, *we have*

$$\mathcal{I}(\mathcal{J}^s) \subset \mathcal{J} \cdot \mathcal{I}(\mathcal{J}^{s-1})$$

Multiplier ideal sheaf is indeed a coherent sheaf, an algebraic object in some sense, even though φ can be pure transcendental.

Theorem 2.12. *For any psh function* φ, $\mathcal{I}(\varphi)$ *is a coherent ideal sheaf.*

Proof. See Nadel [13, 14]. □

The power of multiplier ideal sheaf comes from two results about it: one is Nadel Vanishing Theorem which is cited in Chapter 1 as Theorem 1.5; the other is L^2 extension theorem due to Ohsawa and Takegoshi.

Theorem 2.13 (Ohsawa-Takegoshi). *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudo-convex domain, and H be a linear subspace defined by $z_1 = \cdots = z_p = 0$. Then there exists a constant $C_{n,\Omega}$ depending only on n and the diameter of Ω, such that for every psh function φ on Ω and $f \in \mathcal{O}(H \cap \Omega)$ with $\int_{H\cap\Omega} |f|^2 \exp(-2\varphi) dV_H < \infty$, there exists an extension $F \in \mathcal{O}(\Omega)$ of f with*

$$\int_{\Omega} |F|^2 \exp(-2\varphi) dV_{\mathbb{C}^n} \leq C_{n,\Omega} \int_{H\cap\Omega} |f|^2 \exp(-2\varphi) dV_H$$

where dV_L and $dV_{\mathbb{C}^n}$ are the Lebesgue volume in L and $\mathbb{C}^n$ respectively.

A simple but important application of this L^2 extension theorem is following corollary.

Corollary 2.14. *Let Y be a submanifold of X, then for any psh function φ, we have*

$$\mathcal{I}(\varphi|_Y) \subset \mathcal{I}(\varphi)|_Y$$

Especially, we have

$$c_x(\varphi|_Y) \leq c_x(\varphi)$$

Proof. Let $x \in Y$, and $f \in \mathcal{I}(\varphi|_Y)$, let Ω be a pseudo-convex domain in X which contains x, shrink Ω if necessary, we have

$$\int_{\Omega\cap Y} |f|^2 \exp(-2\varphi) dV_Y < \infty$$

By the L^2 extension theorem, there exists $F \in \mathcal{O}(\Omega)$, such that

$$\int_{\Omega} |F|^2 \exp(-2\varphi) dV_X \leq C \int_{\Omega\cap Y} |f|^2 \exp(-2\varphi) dV_Y < \infty$$

i.e. $F \in \mathcal{I}(\varphi)_x$ and $F|_Y = f$, so $f \in \mathcal{I}(\varphi)_x$. □

Using this fact, it is easy to prove the stability of lcse at some special functions. For example, let $f(z) = z_1^d$, then take L to be line $z_2 = \cdots = z_n = 0$, then $c_0(f) = c_0(f|_L) = 1/d$. Then fix an open neighborhood U of the origin, there is $\delta > 0$, such that for any holomorphic function g defined on U,

$$\sup_U |g - f| < \delta \quad \Rightarrow \quad \sup_{L\cap U} |g - f| < \delta \quad \Rightarrow \quad c_0(g) \geq c_0(g|_L) \geq c_0(f|_L) = c_0(f)$$

Using the notation of multiplier ideal sheaf, Theorem 1.10 can be interpreted as a stability result for multiplier ideal sheaves.

Proposition 2.15. *Let $f \in \mathcal{O}_{x,X}$ with X smooth and $x \in X$. If $1 \in \mathcal{I}(f^c)_x$, then there exists an open neighborhood U of x and $\varepsilon > 0$, such that for any holomorphic function defined over U,*

$$\sup_U |g - f| < \varepsilon \quad \Rightarrow \quad 1 \in \mathcal{I}(g^c)_x$$

Inspired by this results, we propose following conjecture.

Conjecture 2.16. *Let $f_1, \dots f_N$ be holomorphic functions defined around $x \in X$. If $h \in \mathcal{I}((f_1, \dots, f_N)^c)_x$, then there exists an open neighborhood U of x and $\varepsilon > 0$, such that for any holomorphic functions $g_1, \dots, g_N$ defined over U,*

$$\sup_U \sum_i |g_i - f_i| < \varepsilon \quad \Rightarrow \quad h \in \mathcal{I}((g_1, \dots, g_N)^c)_x$$

Chapter 3

Log canonical thresholds

When φ has only algebraic singularity, we can also define $c_x(\varphi)$ by algebraic method. By *algebraic singularity*, we mean that φ can be written locally as

$$\varphi = \frac{1}{q}\log\left(|f_1| + \cdots + |f_N|\right)$$

where q is a real number and f_i's are holomorphic functions. In this chapter, we will present the theory of log canonical thresholds, the algebraic counterpart of local complex singularity exponents.

3.1 Log canonical singularities

Definition 3.1. A *log pair* (X, D) is a pair consisting of a normal variety X and a $\mathbb{Q}$-divisor $D = \sum d_i D_i$, where D_i are distinct irreducible components. In this thesis, we also require that K_X is $\mathbb{Q}$-Cartier, so that it can be pulled back by morphism to X.

Let (X, D) be a log pair and ν be an algebraic valuation on the function field $K(X)$. By *algebraic*, we mean that the transcendental degree of $R_\nu/\mathfrak{m}_\nu$ over $\mathbb{C}$ is $\dim X - 1$, where $(R_\nu, \mathfrak{m}_\nu)$ is the DVR of $k(X)$ corresponding to ν. Zariski showed that every algebraic valuation can be realized geometrically as the valuation of certain divisor on a variety Y birational to X. After performing some blownup on Y if necessary, we may assume that Y is smooth and there is a proper birational map $\pi : Y \to X$. Let E be the divisor corresponds to ν, then

$$K_Y = \pi^*(K_X + D) + \sum a(E_i, X, D)E_i \tag{3.1}$$

where E_i's are distinct divisors on Y and their support is contained in the exceptional divisor and $\pi^{-1}(D)$.

Definition 3.2. Notation as above, if $E = E_i$ for some i, then we define the *discrepancy* of E or ν with respect to (X, D) as $a(E, X, D)$ in Equation 3.1, otherwise, we define the discrepancy of E as 0.

Definition 3.3. A log pair (X, D) has only *log canonical* singularities if for any algebraic valuation ν, the corresponding discrepancy $a(\nu, X, D) \geq -1$.

Remark. The importance of log canonical singularity is that it is the largest category in which log Minimal Model Program is possible.

Proposition 3.4. *(X, D) has only log canonical singularity iff for any log resolution* [1] *$\pi : Y \to (X, D)$, we have*

$$K_Y = \pi^*(K_X + D) + \sum_i b_i E_i$$

with $b_i \geq -1$, and E_i's are distinct irreducible reduced divisors whose support is contained in $Ex(\pi) \cup \pi^{-1}(D)$.

Proof. $\Rightarrow$ part is obviously. For $\Leftarrow$ part, let ν be an algebraic valuation of $k(X)$ which can be realized as $E \subset Z$. After blownup if necessary, we may also assume that there exists a map $\eta : Z \to Y$, such that $\pi \circ \eta$ is also a log resolution of (X, D). On Z, we have

$$\begin{aligned} K_Z &= \eta^* K_Y + \sum a_j F_j \\ &= \eta^* \pi^*(K_X + D) + \eta^*(\sum b_i E_i) + \sum a_j F_j \\ &= (\pi \circ \eta)^*(K_X + D) + \sum b_i \eta_*^{-1} E_i + \sum c_j F_j \end{aligned}$$

where $\sum F_j$ are the exceptional divisor of η. Since Y is smooth and $\sum E_i$ is snc, we know that $a_j + 1 \geq \#\{\, i \mid \eta(F_j) \cap E_i \neq \emptyset \}$. Therefore

$$\begin{aligned} c_j &= a_j + \sum_{\{i \mid \eta(F_j) \cap E_i \neq \emptyset\}} b_i \\ &\geq -1 \end{aligned}$$

[1] see Theorem 2.6 for the definition of log resolution

So we have that

$$\min\{b_i\} \geq -1 \quad \Longrightarrow \quad \min\{b_i, c_j\} \geq -1$$

Therefore, the discrepancy of E must be greater or equal to -1, hence (X, D) has only local canonical singularity. $\square$

3.2 Log canonical thresholds

Definition 3.5. Let (X, D) be a log pair, such that $(X, \emptyset)$ has only log canonical singularity. Let $Z \subset X$ be a closed subset. The *log canonical threshold* of (X, D) along Z is defined as

$$c_Z(X, D) := \sup\left\{c > 0 \mid (X, cD) \text{ is log canonical in a neighborhood of } Z\right\}$$

Proposition 3.6. *Let (X, D) be a log pair and Z is a closed subset of X. Let $\pi : Y \to X$ be a log resolution of (X, D) such that*

$$\pi^*(D) = \sum a_i F_i$$

and

$$K_{Y/X} = K_Y - \pi^* K_X = \sum b_i F_i$$

where F_i's are distinct irreducible reduced divisors whose support is contained in $Ex(\pi) \cup \pi^{-1}(D)$. We have

$$c_Z(X, qD) = \min\left\{\frac{b_i + 1}{qa_i} \mid \pi(F_i) \cap Z \neq \emptyset\right\}$$

Proof. From the condition in the proposition, we get

$$K_Y = \pi^*(K_X + cqD) + \sum_i (b_i - cqa_i) F_i$$

So by Proposition 3.4, (X, cqD) is log canonical iff

$$b_i - cqa_i \geq -1 \quad \forall\, i$$

that is

$$c \leq \frac{b_i + 1}{qa_i} \quad \forall\, i$$

So

$$c_Z(X, qD) = \min \left\{ \frac{b_i + 1}{qa_i} \mid \pi(F_i) \cap Z \neq \emptyset \right\}$$

□

Remark. In general, if T is a subscheme of X, using log resolution of (X, T), we can define the discrepancy of algebraic valuation with respect to (X, T) by the same way as what we did in Definition 3.2. And Proposition 3.6 go through word by word.

Proposition 3.7. *Let X be a smooth manifold and Y be the complex space defined by holomorphic functions $f_1, \ldots, f_N$. Then*

$$c_x\big((f_1, \ldots, f_N)^q \big) = c_x(X, qY)$$

Proof. Compare the formulas in Proposition 2.7 and Proposition 3.6. □

Chapter 4

Stabilities

In this chapter, we will first sketch a proof of Theorem 1.2 using both L^2 extension theorem and jet scheme, then we present the proof of the two Main Theorems stated in Chapter 1.

4.1 Stability over finite dimension base space

4.1.1 Analytic approach

The proof in this subsection is due to Demailly and Kollár [6].

Proposition 4.1. *Let $\Omega \subset \mathbb{C}^n$ and $S \subset \mathbb{C}^p$ be bounded open sets. Let $\varphi(z,s)$ be a psh function on $\Omega \times S$ which is Hölder continuous, i.e. there exists $\alpha > 0$ such that*

$$|\varphi(z_1,s_1) - \varphi(z_2,s_2)| < \big(|z_1 - z_2| + |s_1 - s_2|\big)^{\alpha}$$

Let $(z_0,s_0) \in \Omega \times S$ be a fixed point, then there exists an open neighborhood U of z_0 in Ω, such that for any $c < c_{z_0}(\varphi|_{\Omega\times\{s_0\}})$, there is constant $M(c)$ with

$$\int_U \exp(-2c\varphi(z,s))dV(z) \leq M(c)$$

for s in a neighborhood of s_0.

Proof. Shrink Ω if necessary, we may assume that

$$\int_\Omega \exp\big(-2c\varphi(z,s_0)\big)dV(z) < \infty$$

Let k be a positive integer, and set

$$\phi_{k,s}(z,t) = 2c\phi\big(z, s + (kt)^k(s_0 - s)\big) \quad \text{on } \Omega \times B$$

where $B \in \mathbb{C}$ is the unit disk. Clearly, when s is close enough to s_0, the function $\phi_{k,s}$ is well defined. Noticed that $\phi_{k,s}(z, 1/k) = \varphi(z, s_0)$, so by the L^2 extension theorem, there exists $F_{k,s}$ on $\Omega \times B$ such that $F_{k,s}(z, 1/k) = 1$ and

$$\int_{\Omega \times B} |F_{k,s}(z,t)|^2 \exp\big(-\phi_{k,s}(z,t)\big) dV(z) dV(t) \leq C_1 \tag{4.1}$$

Noticed by Equation 4.1, we can also bound the derivative of $F_{k,s}$, and hence by $F_{k,s}(x, 1/k) = 1$, it follows that when k is large enough, there exists a neighborhood U of z_0 in Ω and $B(0,\varepsilon)$ of the origin in $\mathbb{C}$, such that $F_{k,s} \geq 1/2$ on $U \times B(0,\varepsilon)$. Change of variable, we get

$$\int_{U \times B(0,(k\varepsilon)^k)} \frac{dV(z)dV(r)}{\exp\big(2c\varphi(z, s + r(s_0 - s))\big)|r|^{2(1-1/k)}} \leq C_2$$

By the Hölder continuity, we have

$$\exp\big(2c\varphi(z, s + r(s - s_0))\big) \leq C_3\big(\exp\big(2c\varphi(z,s)\big) + |r|^{2c\alpha}\big)$$

Therefore

$$\int_{U \times B(0,(k\varepsilon)^k)} \frac{dV(z)dV(r)}{\big(\exp(2c\varphi(z,s)) + |r|^{2c\alpha}\big)|r|^{2(1-1/k)}} dV(z)dV(r) \leq C_4$$

Now integral over the disk of $|r|^\alpha < C_5 \exp(\varphi(z,s))$, we get

$$\int_U \frac{dV(z)}{\exp\big(2(c - 1/k\alpha)\varphi(z,s)\big)} \leq C_6$$

Since $c - \frac{1}{k\alpha}$ can be arbitrarily close to c, so we might take a larger c at the beginning, and then take k large enough, and this completes the proof. □

By Proposition 4.1, the function $s \mapsto c_{z_0}(\varphi(\cdot, s))$ is lower semi-continuous for a Hölder continuous psh function φ. However, in order to get stability, we must assume that φ has only algebraic singularity. For simplicity, let's assume that $\varphi = \log(|f|)$. Using resolution of singularity over families, and Proposition 2.7, we can show that there is a stratification of S by finite many S_λ, where S_λ is an open subset of closed subspace of S, and $c_x(f|_{X\times s})$ depends on the stratum containing s. Then Proposition 4.1 implies the stability of lcse over finite dimension base space.

4.1.2 Jet scheme approach

The proof in the subsection is due to Mustaţă [12].

Proposition 4.2. *Let $\pi : W \to S$ be a family of schemes and $\sigma : S \to W$ be a section of π. For every $m \geq 1$, the function*

$$f(s) = \dim(\rho_m^{W_s})^{-1}(\sigma(s))$$

is upper semi-continuous on the set of closed points of S.

Proof. The basic ideal is to show that the family $s \mapsto (\rho_m^{W_s})^{-1}(\sigma(s))$ is the affine cone of some projective family over S, hence by the semi-continuity theorem for projective family, we get the results. □

Suppose (X, Y) is a pair with Y being a subscheme of X. Using Motivic integration, Mustaţă proved following theorem.

Theorem 4.3. *If X is smooth and Y is a closed subscheme, then we have*

$$c(X, Y) = \dim X - \sup_{m\geq 0} \frac{\dim Y_m}{m+1}$$

If $\pi : \tilde{X} \to (X, Y)$ is a log resolution with $\pi^{-1}(Y) = \sum_i a_i E_i$ where E_i's are distinct irreducible divisors on $\tilde{X}$, then

$$c(X, Y) = \dim X - \frac{\dim Y_M}{M+1}$$

where M is any positive integer such that $a_i|(M+1)$ for all i.

Now let X be a smooth variety, $Y \subset X \times S$ be a family of subschemes in X. By induction on the dimension of S, we can show that for any $s \in S$, there exists a log resolution of (X, Y_s) for which the corresponding a_i's are bounded. Hence there exists M divisible enough, such that $\forall\, s \in S$

$$c(X, Y_s) = \dim X - \frac{\dim Y_M}{M+1}$$

Therefore by the Proposition 4.2, $c(X, Y_s)$ is lower semi-continuous. This gives another proof of Theorem 1.2.

4.2 Limit Method

In this section, we will give an example showing how the stability can be utilized to get some useful information about lcse. This method will be used over and over again in Chapter 5.

Proposition 4.4. *Let f be a holomorphic function defined on an open neighborhood of the origin in $\mathbb{C}^n$. Suppose that $mult_0(f) = d$, then*

$$c_0(f) \geq \frac{1}{d}$$

Proof. Change the coordinate system from $(z_1, \ldots, z_n)$ to $(z_1, z_2 + z_1, \ldots, z_n + z_1)$ if necessary, we may assume that the Taylor expansion of f is of the form $z_1^d + \cdots$. Now consider the the family of germs of holomorphic function defined by

$$F(z,t) := \frac{1}{t^d} f(tz_1, t^2 z_2, \ldots, t^2 z_n)$$

When $t \neq 0$, the germ of $F(\cdot, t)$ at the origin is equivalent to the germ of f, so $c_0(F(\cdot, t)) = c_0(f)$. On the other hand, $F(z, 0) = z_1^d$, so $c_0(F(\cdot, 0)) = 1/d$. Therefore by stability over finite dimension base space, we have $c_0(f) \geq c_0(F(\cdot, 0)) = 1/d$. □

Remark. This proposition is just the first half of property (6) in Proposition 2.3

4.3 A Lemma

In this section, we will prove following lemma.

Lemma 4.5. *Fix a coordinate system $(z_1, \ldots, z_n)$ for $\mathbb{C}^n$. Let $f_1, \ldots, f_N \in \mathcal{O}_{\mathbb{C}^n,0}$ such that $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0}/(f_1, \ldots, f_N) < \infty$. Then there exists $M \in \mathbb{N}$, such that for any open neighborhood U of the origin on which f_i's are defined, there is $\varepsilon(U) > 0$, such that for any holomorphic functions $g = \{g_1, \ldots, g_N\}$ defined on U, we have*

$$\sup_U \sum_{i=1}^{N} |g_i - f_i| < \varepsilon(U) \quad \Rightarrow \quad z_j^M \in (g_1, \ldots, g_N) \quad j = 1, \ldots, n$$

where $(g_1, \ldots, g_N)$ is the ideal generated by $g_1, \ldots, g_N$ in $\mathcal{O}_{\mathbb{C}^n,0}$.

Proof. Let $R = \mathcal{O}_{\mathbb{C}^n,0}$. Since $\dim_{\mathbb{C}} R/(f_1, \dots, f_N) < \infty$, for $j = 1, \dots, n$ there exists $M_j \in \mathbb{N}$, such that $z_j^{M_j} \in (f_1, \dots, f_N)$. Take

$$M = \prod_{j=1}^{n} M_j$$

Set $I = (z_1^{M+1}, z_2^{M+1}, \dots, z_n^{M+1})$ and $\overline{R} = R/I$. Denote $\overline{f} = \{\overline{f_1}, \dots, \overline{f_N}\}$ the image of $f = \{f_1, \dots, f_N\}$ under the map $R \to R/I$. Then

$$z_j^{M_j} \in (\overline{f_1}, \dots, \overline{f_N}) \subset \overline{R}$$

Therefore

$$\dim_{\mathbb{C}} \overline{R}/(\overline{f_1}, \dots, \overline{f_N}) \leq M$$

For any $\overline{g} = \{\overline{g_1}, \dots, \overline{g_N}\} \in \overline{R}^N$, there is a canonical $\overline{R}$-linear map

$$\varphi(\overline{g}) : \underbrace{\overline{R} \oplus \cdots \oplus \overline{R}}_{N \text{ copies}} \to \overline{R}$$

with

$$\varphi(\overline{g}) : (\overline{h_1}, \dots, \overline{h_N}) \mapsto \sum_{i=1}^{N} \overline{h_i} \overline{g_i}$$

For an algebraic family of $\mathbb{C}$-linear map between two fixed vector spaces, the dimension of the cokernel is an upper semi-continuous function. Hence, there exists a Zariski neighborhood V of $\overline{f}$ in $\overline{R}^N$, such that $\forall \overline{g} \in V$,

$$\dim_{\mathbb{C}} \overline{R}/(\overline{g_1}, \dots, \overline{g_N}) \leq \dim_{\mathbb{C}} \overline{R}/(\overline{f_1}, \dots, \overline{f_N}) \leq M$$

Claim. For any $\overline{g} \in V$, and any $j = 1, \dots, n$

$$z_j^M \in (\overline{g_1}, \dots, \overline{g_N}) \subset \overline{R}$$

Proof of the Claim For $\overline{g} \in V$, since

$$\dim_{\mathbb{C}} \overline{R}/(\overline{g_1}, \dots, \overline{g_N}) \leq M$$

Fix $j \in \{1, \dots, n\}$, consider $1, z_j, z_j^2, \dots, z_j^M$. They can not be $\mathbb{C}$-linear independent in $R/(\overline{g_1}, \dots, \overline{g_N})$, i.e. there are $b_{j,k} \in \mathbb{C}$, $k = 0, \dots, M$, such that not all of $b_{j,k}$ are zeros and

$$\sum_{k=0}^{M} b_{j,k}\, z_j^k \in (\overline{g_1}, \ldots, \overline{g_N})$$

Take k_0 to be the minimum of k for which $b_{j,k} \neq 0$, then

$$\sum_{k=0}^{M} b_{j,k}\, z_j^k = z_j^{k_0}(b_{j,k_0} + z_j(\cdots)) \in (\overline{g_1}, \ldots, \overline{g_N}) \tag{4.2}$$

Since $z_j^{M+1} = 0$ in $\overline{R}$, so the $b_{j,k_0} + z_j(\cdots)$ term in Equation 4.2 is an invertible element in $\overline{R}$. Hence $z_j^{k_0} \in (\overline{g_1}, \ldots, \overline{g_N})$ and therefore $z_j^M \in (\overline{g_1}, \ldots, \overline{g_N})$.

Take U to by any open neighborhood of the origin on which f_i's are defined, then by Cauchy Integral Formula, there exists $\varepsilon(U) > 0$, such that for any holomorphic functions $g = \{g_1, \ldots, g_N\}$ defined on U, we have

$$\sup_U \sum_i |g_i - f_i| < \varepsilon \quad \Rightarrow \quad \overline{g} = \{\overline{g_1}, \ldots, \overline{g_N}\} \in V$$

Hence, by the claim above, $z_j^M \in (\overline{g_1}, \ldots, \overline{g_N})$, i.e. there exists $\overline{h_{j,i}}$ in $\overline{R}$, such that

$$z_j^M = \sum_i \overline{h_{j,i}} \cdot \overline{g_i}$$

Lifting $\overline{h_{j,i}}$ to $h_{j,i} \in R$, we can find $b_{j,k} \in R$, such that

$$z_j^M = \sum_{i=1}^{N} h_{j,i} \cdot g_i + \sum_{k=1}^{n} b_{j,k} \cdot z_k^{M+1}$$

i.e.

$$\begin{pmatrix} 1 - z_1 b_{1,1} & -z_2 b_{1,2} & \ldots & -z_n b_{1,n} \\ -z_1 b_{2,1} & 1 - z_2 b_{2,2} & \ldots & -z_n b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ -z_1 b_{n,1} & -z_2 b_{n,2} & \ldots & 1 - z_n b_{n,n} \end{pmatrix} \begin{pmatrix} z_1^M \\ z_2^M \\ \vdots \\ z_n^M \end{pmatrix} = \begin{pmatrix} h_{1,1} & h_{1,2} & \ldots, & h_{1,N} \\ h_{2,1} & h_{2,2} & \ldots, & h_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ h_{j,1} & h_{j,2} & \ldots, & h_{j,N} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \end{pmatrix} \tag{4.3}$$

Noticed that the determinant of the most left matrix in Equation 4.3 is an invertible element in R, therefor we can multiple the inverse of it to both side of the equation above, hence get an expression which express z_j^M as R-linear combination of g_i's, and this completes the proof. $\square$

4.4 Proof of Main Theorems

Main Theorem 1 is just a direct application of Lemma 4.5 and Theorem 1.2.

Main Theorem 1. *Given holomorphic functions $f_1, \ldots, f_N$ defined on an open neighborhood U of the origin in $\mathbb{C}^n$ such that $f_i(0) = 0$. If $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0}/(f_1, \ldots, f_N)$ is finite, then there exists $\varepsilon > 0$, such that for any holomorphic functions $g_1, \ldots, g_N$ defined on U,*

$$\sup_U \sum_i |g_i - f_i| < \varepsilon \quad \Rightarrow \quad c_0(g_1, \ldots, g_N) \geq c_0(f_1, \ldots, f_N)$$

where ε depends only on f_i's and U.

Proof. By the proof of Lemma 4.5, we know that there exists an integer $M \in \mathbb{N}$, and a positive real number δ, such that for any holomorphic functions $g = (g_1, \ldots, g_N)$ defined on U,

$$\sup_U \sum_{i=1}^N |g_i - f_i| < \delta \quad \Rightarrow \quad z_j^M \in (g_1, \ldots, g_N)$$

Therefore

$$(g_1, \ldots, g_N) = (z_1^M, \ldots, z_n^M, (g_1)_{\leq nM}, \ldots, (g_N)_{\leq nM})$$

where for holomorphic function F, $F_{\leq d}$ is the degree $\leq d$ part in the Taylor expansion of F with respect to the coordinate system $\{z_1, \ldots, z_n\}$. So

$$c_0(g_1, \ldots, g_N) = c_0(z_1^M, \ldots, z_n^M, (g_1)_{\leq nM}, \ldots, (g_N)_{\leq nM})$$

Thus we reduce the general g to $g_{\leq nM}$ which is in a finite dimension space. So by Theorem 1.2, there exists a positive real number ε, such that

$$\sup_U \sum_{i=1}^n |g_i - f_i| < \varepsilon \quad \Rightarrow \quad c_0(g_1, \ldots, g_N) \geq c_0(f_1, \ldots, f_N)$$

□

Before we proceed to the proof of Main Theorem 2, let's review the classification theorem on hypersurface in $\mathbb{C}^n$ with only isolated singularities, a result due to J. Mather and S.S.T. Yau.

Definition 4.6. Two germs of holomorphic functions f and g on $(\mathbb{C}^n, 0)$ such that $f(0) = g(0) = 0$ are *equivalent* iff there exist germs of holomorphic automorphisms

H and h for $(\mathbb{C}^n, 0)$ and $(\mathbb{C}, 0)$ respectively, such that following diagram commute,

$$\begin{array}{ccc} (\mathbb{C}^n, 0) & \xrightarrow{f} & (\mathbb{C}, 0) \\ \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle h} \\ (\mathbb{C}^n, 0) & \xrightarrow{g} & (\mathbb{C}, 0) \end{array}$$

Proposition 4.7. *If f and g are equivalent, then $c_0(f) = c_0(g)$.*

Let $f \in R = \mathcal{O}_{\mathbb{C}^n,0}$ with $f(0) = 0$. Choose a coordinate system $z = \{z_1, \ldots, z_n\}$ for $\mathbb{C}^n$, define

$$A(f) = R/(\, f, \frac{\partial f}{\partial z_i} \mid i = 1, \ldots, n\,)$$
$$B(f) = R/(f, z_j \frac{\partial f}{\partial z_i} \mid i, j = 1, \ldots, n\,)$$

It is easy to see that for different choice of coordinate systems, the corresponding $A(f)$ and $B(f)$ are isomorphic as rings. If the hypersurface defined by f has only isolated singularity at the origin, then both $\dim_{\mathbb{C}} A(f)$ and $\dim_{\mathbb{C}} B(f)$ are finite.

J. Mather and S.S.T. Yau [11] classified of the germs of hypersurface $Z(f)$ with isolated singularities using $A(f)$ and $B(f)$.

Theorem 4.8. *Let $f, g \in \mathcal{O}_{\mathbb{C}^n,0}$ such that $f(0) = g(0) = 0$. If the hypersurfaces $Z(f)$ and $Z(g)$ defined by f and g respectively have only isolated singularities at the origin, then the following are equivalent.*

1. *f and g are equivalent as germs of holomorphic functions on $(\mathbb{C}^n, 0)$;*
2. *$A(f)$ is isomorphic to $A(g)$;*
3. *$B(f)$ is isomorphic to $B(g)$.*

Now, let's start the proof of Main Theorem 2.

Main Theorem 2. *Let f be a holomorphic function defined on a neighborhood U of the origin of $\mathbb{C}^n$. If the complex space $Z(f)$ defined by f has only isolated singularity* *at the origin, then there exists $\delta > 0$, such that for any holomorphic function g defined on U, we have*

$$\sup_U |g - f| < \delta \quad \Rightarrow \quad c_0(g) \geq c_0(f)$$

where δ depends only on U and f.

Proof. Since $Z(f)$ has only isolated singularity at the origin, so

$$\dim_{\mathbb{C}} R/(f, \frac{\partial f}{\partial z_j} \mid j = 1, \ldots, n) < \infty$$

By Lemma 4.5, there exists $M \in \mathbb{N}$, such that for any open neighborhood V of the origin contained in U, there is $\varepsilon(V) > 0$, such that for any $h, h_1, \ldots, h_n$ defined on V,

$$\sup_V \left(|h - f| + \sum_{j=1}^{n} |h_i - \frac{\partial f}{\partial z_j}|\right) < \varepsilon(V) \quad \Rightarrow \quad z_j^M \in (h, h_1, \ldots, h_n) \subset R$$

Let V be small enough so that its closure is contained in U, then by Cauchy Integral Formula, there exists $\delta > 0$, such that for any holomorphic function g defined on U,

$$\sup_U |f - g| < \delta \quad \Rightarrow \quad \sup_V \left(|g - f| + \sum_{j=1}^{n} |\frac{\partial g}{\partial z_j} - \frac{\partial f}{\partial z_j}|\right) < \varepsilon(V)$$

therefore $z_j^M \in (g, \partial g/\partial z_j \mid j = 1, \ldots, n)$, for $j = 1, \ldots, n$.

Claim. For any g and h such that $z_k^M \in (g, \partial g/\partial z_1, \ldots, \partial g/\partial z_n)$ we have

$$A(g) = A(g - z_k^{M+1} h)$$

Proof of the claim Let $J(g)$ be the ideal in $\mathcal{O}_{\mathbb{C}^n,0}$ generated by $\partial g/\partial z_1, \ldots, \partial g/\partial z_n$ and g. Set $G = g - z_k^{M+1} h$. Since $g \in J(g)$ and $z_k^M \in J(g)$, so $G \in J(g)$. For $j \neq k$,

$$\frac{\partial G}{\partial z_j} = \frac{\partial g}{\partial z_j} + \frac{\partial (z_k^{M+1} h)}{\partial z_j} = \frac{\partial g}{\partial z_j} + z_k^{M+1} \frac{\partial h}{\partial z_j}$$

Since $\partial g/\partial z_j \in J(g)$ and $z_k^M \in J(g)$, so $\partial G/\partial z_j \in J(g)$. Also we have

$$\frac{\partial G}{\partial z_k} = \frac{\partial g}{\partial z_k} + z_k^M ((M+1)h + z_k \frac{\partial h}{\partial z_k})$$

So we also have $\partial G/\partial z_k \in J(g)$, Therefore $J(G) \subset J(g)$. Repeat the argument above for $g = G - z_k^{M+1}h$, we have $J(g) \subset J(G)$. So

$$J(g) = J(G)$$

Hence

$$A(g) = A(G)$$

And this completes the proof of the claim.

Since $z_j^M \in J(g)$ for $j = 1, \dots, n$, by the claim above we have

$$A(g) = A(g_{\leq nM})$$

Hence by Theorem 4.8 and Proposition 4.7

$$c_0(g) = c_0(g_{\leq nM})$$

Therefore we reduce the the problem to the case when the base space is the space of polynomials whose degree is less or equal to nM which is of finite dimension, so by Theorem 1.2, after shrink δ if necessary, we have

$$c_0(g) = c_0(g_{\leq nM}) \geq c_0(f_{\leq nM}) = c_0(f)$$

And this completes the proof of Main Theorem 2 $\square$

Chapter 5

α invariants for Fano manifolds

In this chapter, we will use the limit method in Section 4.2 to compute the α invariants for Fano manifolds obtained by blownup at point in $\mathbb{CP}^n$. In Section 5.1, we will show that $1/(n+1)$ is a lower bound of $\alpha_{m,k}(M)$ without considering the G action. In Section 5.2, with the consideration of the G action, we give a better lower bound for $\alpha_{m,k}(M)$ when M is the blownup of $\mathbb{CP}^n$ at one, n and $n+1$ points.

5.1 A lower bound

Let M be a n-dimension smooth Fano manifold which is obtained by blownup $\mathbb{CP}^n$ over some points $p_1, \ldots, p_k$. Let $\pi : M \to \mathbb{CP}^n$ be the blownup and $E_i = \pi^{-1}(p_i)$. Set $E = \sum E_i$, for any $m \in \mathbb{N}$, considering following exact sequence on M

$$0 \longrightarrow \pi^*(-mK_{\mathbb{CP}^n}) - m(n-1)E \longrightarrow \pi^*(-mK_{\mathbb{CP}^n}) \longrightarrow \pi^*(-mK_{\mathbb{CP}^n})|_{m(n-1)E} \longrightarrow 0$$

Noticed that

$$K_M = \pi^* K_{\mathbb{CP}^n} + (n-1)E$$

so we have

$$0 \longrightarrow -mK_M \longrightarrow \pi^*(-mK_{\mathbb{CP}^n}) \longrightarrow \pi^*(-mK_{\mathbb{CP}^n})|_{m(n-1)E} \longrightarrow 0$$

thus we get the long exact sequence

$$\begin{aligned} 0 \longrightarrow & H^0(M, -mK_M) \longrightarrow H^0(\mathbb{CP}^n, -mK_{\mathbb{CP}^n}) \longrightarrow \\ & H^0(\mathbb{CP}^n, -mK_{\mathbb{CP}^n}) \otimes \mathcal{O}_{p_1}/\mathfrak{m}_{p_1}^{m(n-1)} \otimes \cdots \otimes \mathcal{O}_{p_k}/\mathfrak{m}_{p_k}^{m(n-1)} \longrightarrow H^1(M, -mK_M) \end{aligned} \tag{5.1}$$

Since $-K_M$ is ample, so by Kodair Vanishing Theorem, the last term in Equation 5.1 should be zero. Therefore, $H^0(M, -mK_M)$ can be identified as the subspace of $H^0(\mathbb{CP}^n, -mK_{\mathbb{CP}^n})$ whose elements vanish at least to order $m(n-1)$ at each points p_i, for $i = 1, \ldots, k$. Noticed that π induced a well defined holomorphic map $\pi^* : \pi^* K_{\mathbb{CP}^n} \longrightarrow K_M$, hence we get a well defined meromorphic map from $-mK_{\mathbb{CP}^n}$ to $-mK_M$ which we also denote as π^*. Then π^* identify the element $s \in H^0(\mathbb{CP}^n, K_{\mathbb{CP}^n})$ which vanishes at least to order $m(n-1)$ at all p_i with $\pi^*(s) \in H^0(M, -mK_M)$.

Let's recall the definition of $\alpha_{m,k}(M)$ in Section 1.2. Suppose (M, g) is a Fano manifold with a Riemannian metric g. Then g induces metric $||\cdot||_g$ for $-mK_X$, and an inner product $<,>_g$ on $H^0(M, -mK_X)$. Let $G \subset \mathrm{Aut}(M)$ be a compact subgroup, a function φ is called *G-admissible* if

- $\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi \geq 0$
- φ is G-invariant.

Similarly, $s = \{s_1, \ldots, s_N\} \subset H^0(-mK_M)$ is called G-admissible if $\sum ||s_i||_g^2$ is G admissible. We define $\alpha_{m,k}(M)$ as the supremum of all $\alpha > 0$ for which there exists C_α such that for any G-admissible orthonormal set $s = \{s_1, \ldots, s_k\}$ in $H^0(M, -mK_M)$

$$\int_M \frac{\omega_g^n}{(||s_1||_g^2 + \cdots + ||s_k||_g^2)^{\alpha/m}} < C_\alpha$$

Observation. $\alpha_{m,k}(M) \geq \lambda_0$ iff for any $x \in M$ and any nonzero G-admissible section $s = \{s_1, \ldots, s_k\} \subset H^0(M, -mK_M)$,

$$c_x(s) = c_x(s_1, \ldots, s_k) \geq \frac{\lambda_0}{m} \tag{5.2}$$

Proof. Fix an arbitrary $0 < \lambda < \lambda_0$. For any G-admissible nonzero sections $s \subset H^0(M, -mK_M)$ and any $x \in M$, since $\lambda/m < \lambda_0/m \leq c_x(s)$ and $\dim H^0(M, -mK_M) < \infty$, by Theorem 4.1 there exists a neighborhood U_x of x in M, and a neighborhood $V_{x,s}$ of s in $H^0(M, -mK_M)^{\oplus k}$, such that

$$\int_{U_x} \frac{1}{||\tilde{s}||_g^{2c\lambda/m}} dV_g \leq C_{x,s,\lambda} \qquad \forall\, \tilde{s} \in V_{x,s}$$

Since M is compact, we can cover it by finitely many U_x, therefore we find a open neighborhood V_s of s, such that

$$\int_M \frac{1}{||\tilde{s}||_g^{2c\lambda/m}} dV_g \leq C_{s,\lambda} \qquad \forall\, \tilde{s} \in V_s$$

Since the set of all G-admissible orthonormal set in $H^0(M, -mK_M)^{\oplus k}$ is compact, so it can be covered by finitely many V_s, and take C_λ to be maximal of the corresponding $C_{s,\lambda}$, then C_λ satisfies the Equation 5.2. Hence $\alpha_{k,l}(M) \geq \lambda_0$. $\square$

On $\mathbb{CP}^n$, $K_{\mathbb{CP}^n} = -(n+1)H$ where H is the hyperplane section of $\mathbb{CP}^n$, so after choosing a homogeneous coordinate for $\mathbb{CP}^n$, any section of $H^0(\mathbb{CP}^n, -mK_{\mathbb{CP}^n})$ can be represented by a polynomial whose degree is less or equal to $m(n+1)$.

If we take $G = \{\mathrm{Id}_M\}$, then $\alpha_{m,k+1}(M) \geq \alpha_{m,k}(M)$. So in this case, it suffices to estimate $\alpha_{k,1}(M)$ only.

If $x \notin E$, then π is an isomorphism between an open neighborhood U of x and $\pi(U)$. So after choosing suitable local coordinates and local frames for $-mK_X$ and $-mK_{\mathbb{CP}^n}$, $s \in H^0(\mathbb{CP}^n, -mK_{\mathbb{CP}^n})$ and $\pi^*(s) \in H^0(M, -mK_M)$ can be represented by the same polynomial whose degree is less or equal to $m(n+1)$. Obviously

$$\mathrm{mult}_x(\pi^* s) \leq \deg_x(\pi^* s) \leq m(n+1)$$

So by Proposition 4.4, we have

$$c_x(\pi^* s) \geq \frac{1}{m(n+1)} \tag{5.3}$$

As for $c_{E_i}(\pi^* s)$, after choosing suitable local coordinate system around points on E_i and suitable local frame for $-mK_M$, locally $\pi^*(s)$ can be represented by polynomial

$$\frac{s(w_1 w_i, \ldots, w_{i-1} w_i, w_i, w_{i+1} w_i \ldots w_n w_i)}{w_i^{m(n-1)}}$$

Therefore

$$c_{E_i}(\pi^* s) > \delta \quad \Longleftrightarrow \quad \int_{U_{p_i}} \frac{(|z_1| + \cdots + |z_n|)^{2m(n-1)\delta}}{|s(z_1, \ldots, z_n)|^{2\delta}} \frac{dV(z)}{(|z_1| + \cdots + |z_n|)^{2(n-1)}} < \infty \tag{5.4}$$

where U_{p_i} is a neighborhood of p_i in $\mathbb{CP}^n$. When $\delta < 1/m$, we have

$$\begin{aligned}&\int_{U_{p_i}} \frac{(|z_1|+\cdots+|z_n|)^{2m(n-1)\delta}}{|s(z_1,\ldots,z_n)|^{2\delta}} \frac{dV(z)}{(|z_1|+\cdots+|z_n|)^{2(n-1)}}\\ =&\int_{U_{p_i}} \frac{1}{|s(z_1,\ldots,z_n)|^{2\delta}(|z_1|+\cdots+|z_n|)^{2(n-1)(1-m\delta)}} dV(z)\end{aligned} \tag{5.5}$$

Since the right hand side of Equation 5.5 is of the form $\int \exp(-2\varphi)$ with φ being a psh function, so the L^2 extension theorem can be applied. Using the limit method and by the same trick as in the proof of Proposition 4.4, we may assume that $s(z_1,\ldots,z_n) = z_1^d$, where $d = \mathrm{mult}_0\, s \leq m(n+1)$. Noticed that

$$|z_1|^{q_1}\cdots|z_n|^{q_n} \leq C_q(|z_1|+\cdots+|z_n|)$$

for any positive real numbers $q = \{q_i\}$ satisfying $\sum_i q_i = 1$, where C_q is a positive real number depends on q, so in order to show that the integration in Equation 5.5 is finite, it suffices to show that

$$\int \frac{dV(z)}{|z_1|^{2d\delta+2q_1(n-1)(1-m\delta)}|z_2|^{2q_2(n-1)(1-m\delta)}\cdots|z_n|^{2q_n(n-1)(1-m\delta)}} < \infty \tag{5.6}$$

When $\delta < 1/(n+1)m$, we can always choose q_1 small enough, so that

$$d\delta + q_1(n-1)(1-m\delta) < 1 \quad \text{and} \quad q_i(n-1)(1-m\delta) < 1 \quad \text{for } i = 2,\ldots,n$$

So Equation 5.6 always holds for $\delta < 1/(n+1)m$, hence

$$c_{E_i}(\pi^* s) \geq \frac{1}{m(n+1)} \tag{5.7}$$

Combining Equation 5.7 and Equation 5.3, we get

Proposition 5.1. *If M is a smooth Fano manifold obtained by blownup points on $\mathbb{CP}^n$, then*

$$\alpha_{m,k}(M) \geq \frac{1}{n+1}$$

5.2 G action

In general, when $\mathrm{Aut}(M)$ is nontrivial, we should expect a better lower bound for $\alpha_{m,k}(M)$. For a G-admissible function φ, we define

$$V(\varphi) := \{ x \in M \mid c_x(\varphi) = c_M(\varphi) \}$$

By Proposition 2.3, $V(\varphi)$ is a **closed** G-invariant subvariety of M. Especially we have $\overline{G \cdot x} \subset V(\varphi)$ for any $x \in V(\varphi)$, where the closure is taken in the holomorphic Zariski topology sense.

Some notations:

- $Z = (Z_0, \ldots, Z_n)$: a fixed homogeneous coordinate for $\mathbb{CP}^n$;
- p_i : the unique point whose only nonzero coordinate appears on the i-th slot;
- $M \xrightarrow{\pi} \mathbb{CP}^n$: a blownup of $\mathbb{CP}^n$ over a subset of $\{p_0, \ldots, p_n\}$;
- G : a maximal compact subgroup of $\mathrm{Aut}(M)$;
- G_0 : the discrete subgroup of G whose action on $\mathbb{CP}^n$ fix the set $\{p_0, \ldots, p_n\}$;
- L_Λ : the hyperplane of $\mathbb{CP}^n$ defined by $Z_i = 0$ for $i \in \Lambda \subset \{0, \ldots, n\}$;
- F_i : the strict transform of the divisor $\{Z_i = 0\}$ by π;
- E_i : $\pi^{-1}(p_i)$.

5.2.1 Blownup at one point

Let M be the blownup of $\mathbb{CP}^n$ at p_0. Then G consists of element $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$ where $A \in U(n, \mathbb{C})$. Because the action of $U(n, \mathbb{C})$ on $\mathbb{CP}^{n-1}$ is transitive, so both E_0 and F_0 are $n-1$ dimensional G orbits. For $x \notin E_0 \cup F_0$, $G \cdot x$ is the sphere in $\mathbb{C}^n$ containing x whose dimension over $\mathbb{R}$ is $2n-1$. Therefore for any nonzero G-admissible φ, because of $\dim_{\mathbb{C}} V(\varphi) \leq n-1$, we have

$$V(\varphi) \subset E_0 \cup F_0$$

For any $s \in H^0(\mathbb{CP}^n, -mK_{\mathbb{CP}^n})$, if $\pi^* s \in H^0(M, -mK_M)$, by the argument in Section 5.1, we have

$$s = \sum_{\substack{i_0+\cdots+i_n=m(n+1)\\ i_1+\cdots+i_n \geq m(n-1)}} s_I Z_0^{i_0} Z_1^{i_1} \cdots Z_n^{i_n} \tag{5.8}$$

Noticed that by the condition $i_1 + \cdots + i_n \geq m(n-1)$, we have $i_0 \leq 2m$. So for a general point $x \in F_0$, $\mathrm{mult}_x \pi^*(s) \leq 2m$, hence by Proposition 4.4,

$$c_x(s) \geq \frac{1}{2m}$$

Therefore for any G-admissible $s = \{s_1, \dots, s_k\}$,

$$c_x(s_1, \dots, s_k) \geq c_x(s_1) \geq \frac{1}{2m}$$

Since F_0 is a G-orbit, so;

$$c_{F_0}(s_1, \dots, s_k) \geq \frac{1}{2m}$$

Around E_0, since for any G-admissible $s = \{s_1, \dots, s_k\}$, we have

$$\sum_{\sigma \in G_0} ||\sigma(s_1)||_g \leq \frac{1}{|G_0|} \sum_{i=1}^{k} ||s_i||_g$$

therefore

$$c_{E_0}(s) \geq c_{E_0}(\sum_{\sigma} ||\sigma(s_1)||_g)$$

so it suffices to estimate $c_{E_0}(\sum ||\sigma(s_1)||_g)$.

Let s be as in Equation 5.8. We say that an index $I = (i_0, i_1, \dots, i_n)$ appears in s if $s_I \neq 0$ in Equation 5.8. Let $\mathrm{Ind}(s)$ be the set of all indexes which appears in $\sigma(s)$ for some $\sigma \in G_0$. Let $J = (j_0, \dots, j_n)$ be the unique index in $\mathrm{Ind}(s)$ define by

$$\begin{gathered} j_1 + \ldots + j_n = \min\{i_1 + \cdots + i_n \mid I \in \mathrm{Ind}(s)\} \\ j_1 \leq j_2 \leq \cdots \leq j_n \end{gathered} \tag{5.9}$$

Assume that $(j_0, \dots, j_n)$ appears in s. Let p be the point on E_0 corresponding to the line $L_{\{1,\ldots,n-1\}}$, then around p, $\pi^*(s)$ can be represented as

$$\begin{aligned}\pi^*(s)(w_1,\ldots,w_n) &= \frac{s(w_1w_n,\ldots,w_{n-1}w_n,w_n)}{w_n^{m(n-1)}}\\ &= \sum_{\substack{i_0+\cdots+i_n=m(n+1)\\ i_1+\cdots+i_n\geq m(n-1)}} s_I w_1^{i_1}\cdots w_{n-1}^{i_{n-1}} w_n^{i_1+\cdots+i_n-m(n-1)}\end{aligned}$$

Now using limit method, consider the family $s(w,t,u)$

$$(t,u)\mapsto s(w,t,u)=\frac{1}{t^du^e}\,\pi^*s(tuw_1,tu^2w_2\ldots,tu^nw_n)$$

where $d=\mathrm{mult}_t\pi^*s(tuw_1,\ldots,tu^nw_n)$ and $e=\mathrm{mult}_u\pi^*s(tuw_1,\ldots,tu^nw_n)$, then by the same argument of Proposition 4.4, $c_p(s(\cdot,t,u))=c_p(s)$ for $tu\neq 0$. And by the choice of $(j_0,\ldots,j_n)$, when $t=0$ and $u=0$,

$$s(w,0,0)=s_Jw_1^{j_1}\cdots w_{n-1}^{j_{n-1}}w_n^{j_1+\cdots+j_n-m(n-1)}$$

By the fact that $j_0+\cdots+j_n=m(n+1)$ and $j_{n-1}\leq j_n$, we have $j_{n-1}\leq m(n+1)/2$ and $j_1+\cdots+j_n-m(n-1)\leq 2m$, so

$$c_p(s(\cdot,0,0))\geq \min\Big(\frac{1}{2m},\frac{2}{m(n+1)}\Big)$$

Hence for G-admissible $s=\{s_1,\ldots,s_k\}$,

$$c_p(s)\geq \min\Big(\frac{1}{2m},\frac{2}{m(n+1)}\Big)$$

Therefor by the fact that E_0 is a G orbit,

$$c_{E_0}(s)\geq \min\Big(\frac{1}{2m},\frac{2}{m(n+1)}\Big)$$

So we get

Proposition 5.2. *Let M be the blowup of $\mathbb{CP}^n$ at one point, then*

$$\alpha_{m,k}(M)\geq \min\Big(\frac{1}{2},\frac{2}{n+1}\Big)$$

When M is the blownup of $\mathbb{CP}^2$, consider the function

$$\varphi = \frac{(|Z_1|^2 + |Z_2|^2)^{3/2}}{(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|Z_1|^2 + |Z_2|^2)^{1/2}}$$

then easy to see that φ is G-admissible, and

$$c_M(\varphi) = \frac{1}{2}$$

so we get

$$\alpha(M) \leq \frac{1}{2}$$

Corollary 5.3. *If M is the blownup of $\mathbb{CP}^2$ at one point, then*

$$\alpha(M) = \frac{1}{2}$$

5.2.2 Blownup at n points

Let M be the blownup of $\mathbb{CP}^n$ at $p_1, \ldots, p_n$. We have already showed in Proposition 5.1 that $\alpha_{m,k}(M) \geq 1/(n+1)$. If we take $s = Z_0^{m(n+1)} \in H^0(\mathbb{CP}^2, -mK_{\mathbb{CP}^2})$. Then it is easy to check that $\pi^*(s)$ is G-admissible, and

$$c_M(\pi^* s) = \frac{1}{m(n+1)}$$

so we get

Proposition 5.4. *Let M be the blownup of $\mathbb{CP}^n$ at n points, then*

$$\alpha_{m,1}(M) = \frac{1}{n+1}$$

Remark. The surface case of Proposition 5.4 has also been proved by Song [19].

5.2.3 Blownup at $n+1$ points

Let M be the blownup of $\mathbb{CP}^n$ at $p_0, \ldots, p_n$. Then G is generated by G_0 and G_1 where G_1 is the continuous group consisting of element of the form

$$\operatorname{diag}(e_0, \ldots, e_n) \in PSL(n), \quad |e_0| = \cdots = |e_n| = 1 \tag{5.10}$$

For any $\Lambda \subsetneq \{0, \ldots, n\}$, we define

$$F_\Lambda := \pi_*^{-1}(L_\Lambda)$$
$$F_\Lambda^o := F_\Lambda \setminus \cup_{\Theta \subsetneq \Lambda} F_\Theta$$

Let $\Omega = (\cup_{|\Lambda|=n-2} F_\Lambda) \cap (\cup_{i=0}^n E_i)$ then easy to check that Ω is the unique zero dimension G-orbit.

Claim. For any G-admissible function φ and any $p \in \Omega$, we have

$$c_M(\varphi) = c_p(\varphi)$$

Proof. It suffices to show that for any $x \in M$, $\Omega \subset \overline{G \cdot x}$, and the closure is taken in holomorphic Zariski topology sense.

Over $\{Z_0 \neq 0\}$, a local coordinate is given by $z_i = Z_i / Z_0$, $i = 1, \ldots, n$ for point $x \neq p_0$, and the action of G_1 is given by

$$\operatorname{diag}(1, e_1, \ldots, e_n)(z_1, \ldots, z_n) = (e_1 z_1, \ldots, e_n z_n)$$

So suppose the coordinate of x is $(x_1, \ldots, x_l, 0, \ldots, 0)$, with $x_i \neq 0$, for $i = 1, \ldots, l$, then $G \cdot x$ is a torus in $\mathbb{C}^l$.

More general, since $F_\Lambda^o \setminus (\cup_{i=0}^n E_i)$ and $F_\Lambda^o \cap E_j$ for $j \notin \Lambda$ form a disjoint cover of M. So $\forall x \in M$, either $x \in F_\Lambda^o \setminus (\cup_0^n E_i)$ or $x \in F_\Lambda^o \cap E_j$ for some Λ. In either cases, there is local coordinate system $z_1, z_2, \ldots, z_n$ for M, such that

- $F_\Lambda^o \cap E_j$ or $F_\Lambda^o \setminus (\cup_1^n E_i)$ is defined by

$$z_{k+1} = \cdots = z_n = 0$$

- the G_1 action is given by

$$\mathrm{diag}(1, e_1, \dots, e_n)z = (e_1 z_1, \dots, e_n z_n)$$

- the coordinate of x is $(x_1, \dots, x_n)$ with

$$x_i \neq 0 \quad \text{for } i = 1, \dots, k$$

Then by Cauchy Integral Formula, we know that any holomorphic function vanishing on $G_1 \cdot x$ must also vanish on either $F_\Lambda^o \cap E_j$ or $F_\Lambda^o \setminus (\cup_1^n E_i)$. So the closure of $G \cdot x$ must contains either F_Λ or $F_\Lambda \cap E_j$. Therefore $\Omega \subset \overline{G \cdot x}$. □

Let p be the unique point on E_0 corresponding to the line $L_{\{2,\dots,n\}}$. By the same argument as in Section 5.2.1, we only need to estimate

$$c_p(\sum_{\sigma \in G_0} |\sigma(\pi^* s)|)$$

for $s \in H^0(\mathbb{CP}^n, -mK_{\mathbb{CP}^n})$ with $\pi^* s \in H^0(M, -mK_M)$. By the argument in Section 5.1, we have

$$s = \sum_{\substack{i_0+\cdots+i_n=m(n+1) \\ i_k \leq 2m \text{ for } k=0,\dots,n}} s_I Z_0^{i_0} Z_1^{i_1} \cdots Z_n^{i_n} \tag{5.11}$$

Then around p, $\pi^* s$ can be written as

$$\begin{aligned} \pi^* s(w_1, \dots, w_n) &= \frac{s(w_1, w_1 w_2, \dots, w_1 w_n)}{w_1^{m(n-1)}} \\ &= \sum_{\substack{i_0+\cdots+i_n=m(n+1) \\ i_k \leq 2m \text{ for } k=0,\dots,n}} s_I w_1^{i_1+\cdots+i_n-m(n-1)} w_2^{i_2} \cdots w_n^{i_n} \end{aligned} \tag{5.12}$$

Define $(j_0, \dots, j_n)$ by the same condition as in Equation 5.9 except that in this case we require that $j_0 \geq j_1 \geq \cdots \geq j_n$, and also assume that $(j_0, j_1, \dots, j_n)$ appears in s, then by the same argument as in Section 5.2.1, it reduces to the case when

$$\pi^* s = w_1^{j_1+\cdots+j_n-m(n-1)} w_2^{j_2} \cdots w_n^{j_n}$$

Now that $j_0 \geq j_1 \geq \cdots \geq j_n$, we have

$$j_1 + \cdots + j_n \leq \frac{n}{n+1} m(n+1) \leq nm$$

hence $j_1 + \cdots + j_n - m(n-1) \leq m$, and also we have

$$j_2 \leq \frac{j_0 + j_1 + j_2}{3} \leq \frac{m(n+1)}{3}$$

So

$$c_p(\pi^* s) \geq \frac{3}{m(n+1)}$$

Hence we get

$$\alpha_{k,1}(M) \geq \frac{3}{n+1}$$

If $s = \{s_1, s_2\}$ is G-admissible, then since s_1 and s_2 are linear independent, so we may assume that the term

$$\left(\frac{m(n+1)}{3}, \frac{m(n+1)}{3}, \frac{m(n+1)}{3}, 0, \ldots, 0\right)$$

will not appear in either s_1 or s_2. Let's assume that it will not appear in s_1. Then, when we compute $c_p\big(\sum_\sigma |\sigma(s_1)|\big)$, the $(j_0, \ldots, j_n)$ must satisfy that $j_2 < m(n+1)/3$. So

$$\max c_p\Big(\sum_\sigma |\sigma(s_1)|\Big) > \frac{3}{m(n+1)}$$

But we also have

$$c_p(s_1, s_2) \geq c_p\Big(\sum_\sigma |\sigma(s_1)|\Big)$$

Thus

$$\alpha_{k,2} > \frac{3}{n+1}$$

So we get

Proposition 5.5. *Let M be the blownup of $\mathbb{CP}^n$ at $n+1$ points in general position. Then we have*

$$\alpha_{m,k}(M) \geq \frac{3}{n+1}$$

and when $k \geq 2$

$$\alpha_{m,k}(M) > \frac{3}{n+1}$$

Let M be the blownup of $\mathbb{CP}^2$ at 3 points, consider the section $s(Z_0, Z_1, Z_2) = Z_0^m Z_1^m Z_2^m \in H^0(\mathbb{CP}^2, -mK_{\mathbb{CP}^2})$. then easy to check that $\pi^* s \in H^0(M, -mK_M)$ and $\pi^* s$ is G-admissible, and also $c_M(\pi^* s) = 1$, so we get

Corollary 5.6 ([22]). *Let M be the blownup of $\mathbb{CP}^2$ at 3 points, then*

$$\alpha_{m,1}(M) = 1$$

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