Limit Linear Series in Positive Characteristic and Frobenius-Unstable Vector Bundles on Curves

by

Brian Osserman

A.B., Harvard University (1999)

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

February 2004

Author ....
Department of Mathematics
January 6, 2004

Certified by ....
Aise Johan de Jong
Professor
Thesis Supervisor

Accepted by ....
Pavel I. Etingof
Chairman, Department Committee on Graduate Students

Limit Linear Series in Positive Characteristic and Frobenius-Unstable Vector Bundles on Curves

by

Brian Osserman

Submitted to the Department of Mathematics
on January 6, 2004, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Abstract

Using limit linear series and a result controlling degeneration from separable maps to inseparable maps, we give a formula for the number of self-maps of $\mathbb{P}^1$ with ramification to order e_i at general points P_i, in the case that all e_i are less than the characteristic. We also develop a new, more functorial construction for the basic theory of limit linear series, which works transparently in positive and mixed characteristics, yielding a result on lifting linear series from characteristic p to characteristic 0, and even showing promise for generalization to higher-dimensional varieties.

Now, let C be a curve of genus 2 over a field k of positive characteristic, and V_2 the Verschiebung rational map induced by pullback under Frobenius on moduli spaces of semistable vector bundles of rank two and trivial determinant. We show that if the Frobenius-unstable vector bundles are deformation-free in a suitable sense, then they are precisely the undefined points of V_2, and may each be resolved by a single blow-up; in this setting, we are able to calculate the degree of V_2 in terms of the number of Frobenius-unstable bundles, and describe the image of the exceptional divisors.

We finally examine the Frobenius-unstable bundles on C by studying connections with vanishing p-curvature on certain unstable bundles on C. Using explicit formulas for p-curvature, we completely describe the Frobenius-unstable bundles in characteristics $3, 5, 7$. We classify logarithmic connections with vanishing p-curvature on vector bundles of rank 2 on $\mathbb{P}^1$ in terms of self-maps of $\mathbb{P}^1$ with prescribed ramification. Using our knowledge of such maps, we then glue the connections to a nodal curve and deform to a smooth curve to yield a new proof of a result of Mochizuki giving the number of Frobenius-unstable bundles for C general, and hence obtaining a self-contained proof of the resulting formula for the degree of V_2.

Thesis Supervisor: Aise Johan de Jong
Title: Professor

Acknowledgments

First and foremost, I would like to thank Johan de Jong, my thesis advisor. His tireless and enthusiastic involvement was crucial to every stage of the thesis: suggesting the original problems and potential approaches to solving them; assisting me with checking the validity of my work and filling in technical details, background material, and references; and in the end, reading the resulting drafts and making many valuable suggestions. Most importantly, in the process he guided my transition from utter mathematical dependence to some sense of being able to generate my own problems and ideas to solve them.

I would also like to thank Jason Starr and Steven Kleiman, the other members of my thesis committee. They not only provided valuable corrections and suggestions, but were also of assistance in developing the technical details of my work, as well as finding references for background results.

In addition, there are a number of people who had no official connection to the thesis, but whose many mathematical discussions were invaluable nonetheless. Joe Harris and Ezra Miller suggested appropriate tools for certain problems. Brian Conrad and Dan Laksov pointed me in the right direction on various background results. Max Lieblich was always willing to discuss a technical lemma, suggest references, and simply help me figure out what was true and what wasn't, and I took advantage of his patience again and again. David Sheppard, Astrid Giugni, and Roya Beheshti all contributed to my learning algebraic geometry, and helped me figure out various problems at various points in the thesis. Finally, David Helm has not only been a close friend and mathematical co-conspirator for many years, but was also kind enough to provide a counterexample to a lemma I was trying to prove.

I am also grateful to Alex Ghitza and Kiran Kedlaya, who gamely assisted me in feeding my illicit number theory habit while in graduate school.

Lastly, I would like to express my gratitude to the NSF, whose funding, in the form of a Graduate Research Fellowship, supported much of the research in this thesis.

Contents

This thesis is composed of a number of chapters dealing with different topics in positive-characteristic algebraic geometry. The main ideas of most of the chapters stand on their own as original work, but they also fit together to give a self-contained proof of new results on the geometry of the "Verschiebung" rational map induced by pullback under Frobenius on the coarse moduli space of semi-stable vector bundles of rank 2 and trivial determinant on general smooth curves of genus 2. A key component of this is the description of the undefined locus of this map, which is to say, of those "Frobenius-unstable" bundles which are themselves semistable, but pull back to unstable bundles under the Frobenius map. This last question had in fact already been answered, albeit in a rather different language, by Mochizuki in [42], via techniques which are substantially similar but differ in certain notable respects.

In this introduction, we outline the main results of the various chapters, and discuss their motivation and how they relate to one another. For more detailed overviews of a given chapter, including theorem statements and comparisons to results already in the literature, see the introduction to each individual chapter.

Chapter I is for the most part the most elementary chapter, although technical machinery is required from time to time. We examine the question of how many maps there are from $\mathbb{P}^1$ to itself of a given degree, and having fully prescribed ramification (counted modulo automorphism of the image). We answer this question for generally placed ramification points, and all ramification indices less than p. We also show that if any ramification index is allowed to rise above p, there need not be finitely many such maps even given tame indices, and we in fact make use of this observation in controlling degeneration of separable maps to inseparable maps, which is a key component of the proof of the main result. An appendix gives some basic representability results on functor of ramified maps between curves.

Another key idea of Chapter I is the basic theory of limit linear series; while the necessary results appear to hold even in positive characteristic using only the original construction due to Eisenbud and Harris, to be on the safe side we give a new, more transparent construction, which is the content of Chapter II. This chapter is very technical, albeit without using a tremendous amount of machinery, and gives a completely functorial theory of limit linear series which recovers the results of Eisenbud and Harris while offering a natural compactification of their scheme of limit linear series on a fixed curve of compact type, and even allowing for generalization to varieties of higher dimension. One immediate corollary is a rather general result on lifting linear series from characteristic p to characteristic 0. The dimension count which gives the theory of limit linear series its strength is substantially harder for this generalized construction, and an appendix develops a theory of "linked Grassmannian" schemes necessary to obtain the desired bounds on dimension.

If one is willing to take for granted the basic background of coarse moduli spaces of vector bundles on curves, Chapter III is nearly as elementary as Chapter I. In it, we develop explicit formulas for p-curvature of connections on vector bundles, and apply them in several ways to the case of curves of genus 2. First, as an application unrelated to the rest of our results, we derive a completely explicit formula for the strata of curves of different p-rank in a particular parameter space for curves of genus 2; this follows directly from the definition of a p-torsion line bundle, and is completely elementary. We also use our explicit formulas to compute the space of connections of vanishing p-curvature on a particular unstable bundle in characteristics $3, 5, 7$, deducing the number of Frobenius-unstable bundles in these characteristics via a standard argument. Finally, we show that the locus of connections on the same unstable bundle with nilpotent p-curvature is finite flat over our parameter space of genus 2 curves in all odd characteristics. An appendix fills in technical background on the Verschiebung map.

Chapter IV provides the necessary theory to conclude the degree of the Verschiebung map for rank 2 vector bundles on genus 2 curves from sufficient information on the Frobenius-unstable bundles on such curves. This consists of a brief examination of the degree of certain rational maps between projective spaces, followed by a more technical examination of the deformation theory of connections with their vector bundles via hypercohomology and spectral sequences. Ultimately, we are able to relate "reducedness" of the set of Frobenius-unstable bundles to reducedness of the undefined locus of the Verschiebung, and hence the degree of the map; in this situation we are further able to describe the image of the exceptional divisors. The results of Chapter III then immediately allow us to conclude the degree of the Verschiebung for general curves in characteristics $3, 5, 7$.

The subject of Chapter V is the study of connections with logarithmic poles and vanishing p-curvature on vector bundles on curves, and in particular on rank 2 vector bundles on $\mathbb{P}^1$. Techniques are largely elementary, using nothing more technical than formal local analysis. The main result is the classification of such connections in terms of self-maps of

$\mathbb{P}^1$ with prescribed ramification.

Finally, Chapter VI is the only chapter without substantial new results, and is included to pull together the other chapters to yield the desired self-contained presentation of the degree of the Verschiebung for a general genus 2 curve in any odd characteristic. The main idea is to take the relevant logarithmic connections on $\mathbb{P}^1$ of Chapter V, which are already classified in terms of the self-maps of $\mathbb{P}^1$ counted in Chapter I, glue them to obtain connections on nodal curves, and then deform those to smooth curves, applying the results of Chapter III to obtain the desired description of the Frobenius-unstable bundles, and then Chapter IV to conclude the degree of the Verschiebung.

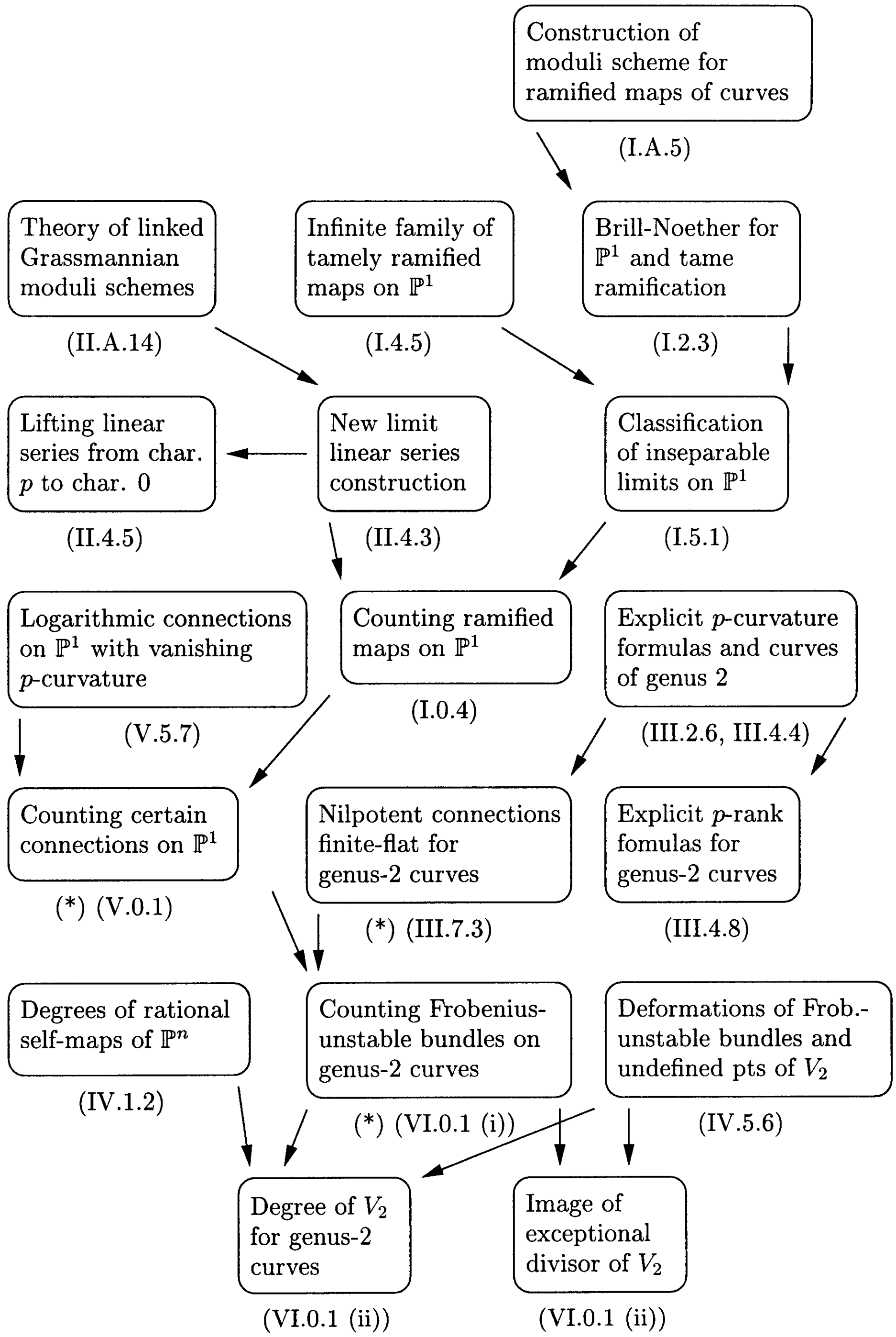
The appendix is a collection of auxiliary lemmas and "well-known" results which are hard to find (or not present) in exactly the desired form in the literature. They are included only for completeness, and are isolated largely in order to avoid distracting from the line of reasoning in the sections in which they are invoked.

Chronologically, Chapter III (with the exceptional of the determinant computations at the end) was the earliest, followed by Chapter IV. At this point, the work of Chapter V followed by much of Chapter I and parts of Chapter VI was completed, without any knowledge of Mochizuki's work. Mochizuki's work shed new light on the situation, as discussed in Section V.7, and indirectly allowed the completion of Chapter I. Chapter II was then developed to solidify the foundations of Chapter I, and finally Chapter VI was completed with simplifications adapted directly from Mochizuki's work.

Early motivation for understanding Frobenius-unstable vector bundles and the geometry of the Verschiebung map was driven by several considerations:

- First, if the base field k for our curve C is finite, the Verschiebung map is closely linked to p-adic representations of the fundamental group of C (see the introduction to [37]). In particular, A. J. de Jong observed that curves in the moduli space of vector bundles which are fixed under some iterate of the Verschiebung will correspond to p-adic representations for which the geometric fundamental group has infinite image, which he conjectures in [9] cannot happen for ℓ-adic representations.

- Next, it is known (see [27, Lem. 3.2.2]) that a semistable vector bundle on a curve cannot pull back to an unstable bundle under a separable morphism, so a good understanding of the phenomenon in the case of Frobenius gives in some sense a universal



A rough flow-chart of the various results in this thesis.

Results marked with (*) are originally due to Mochizuki.

description of bundles which become unstable after pullback under any morphisms. Gieseker and Raynaud produced some sporadic examples of Frobenius-unstable bundles in [19] and [50, p. 119], and more recently Mochizuki, Laszlo, Pauly, Joshi, Ramanan, Xia, Yu, and Lange have all contributed to our understanding of the situation; see the introduction to Chapter III.

- For vector bundles of rank one, which is the case for which our generalized Verschiebung map is named, the map induced by pullback under Frobenius acts as the dual isogeny to Frobenius on the Jacobian of the curve, and plays an important role in the study of the Jacobian and hence of the curve. One might expect that if understood better, the generalized Verschiebung could become equally important.

- Invariants such as the degree of Verschiebung nearly always seem to be given by polynomials in p, with no apparent explanation for why this should be the case. If enough examples of this phenomenon are understood, one might hope to find a general guiding principle behind it.

Motivation for understanding logarithmic connections with vanishing p-curvature on the projective line is two-fold. Our immediate motivation was of course to use degeneration arguments to conclude results on higher-genus curves, as is carried out in Chapter VI. However, such connections are interesting in their own right, as is demonstrated by a still-unsolved question of Grothendieck asking if a logarithmic connection on $\mathbb{P}^1$ in characteristic 0 which has vanishing p-curvature when reduced mod p for almost all primes p, must have "algebraic solutions", in the sense that there is an algebraic curve over $\mathbb{P}^1$ for which the pullback connection has a full set of horizontal sections. See [31] for a discussion of the problem and solution for particular connections.

Finally, our initial motivation for studying self-maps of $\mathbb{P}^1$ with prescribed ramification in characteristic p, and consequently limit linear series in characteristic p, was to understand logarithmic connections with vanishing p-curvature on $\mathbb{P}^1$. However, if one generalizes to the corresponding questions for higher-dimensional linear series and for higher-genus curves, there are a range of applications in characteristic 0, including:

- In full generality, potentially yielding an understanding of the cohomology rings of the G^r_d spaces of higher genus curves, which would generalize the well-known description of the Grassmannian cohomology ring in terms of Schubert cycles and their intersections.

- In the case of higher-dimensional series on $\mathbb{P}^1$, the study of the solutions of an A_N Bethe equation of XXX type (Mukhin and Varchenko in [43]).
- In the case of two ramification points and one-dimensional linear series, the computation of the Kodaira dimension of $\overline{\mathcal{M}}_{g,n}$ (Logan in [38]; the case where one of the two ramification points is allowed to move is particularly relevant, and solved by Logan).

Moreover, our new construction of limit linear series, giving a proper moduli scheme, offers the potential for cleaner arguments for bounding results such as the Brill-Noether theorem. Limit linear series were used to prove results on the Kodaira dimension of moduli spaces of curves, existence of curves with certain Weierstrass semigroups, monodromy actions on ordinary Weierstrass points and finite collection of linear series, and families of curves having certain special linear series. Appropriate generalization to characteristic p has the potential to yield similar results, and generalization to higher-dimensional varieties could offer tools previously unavailable even in characteristic 0, although there are new difficulties in this setting, discussed briefly in Section II.6.

Calculations were carried out using Maple, Mathematica, Macaulay, and, for automatically generating the p-curvature formulas of Section III.2 in low characteristics, simple C code.

Chapter I

Self Maps of $\mathbb{P}^1$ with Prescribed Ramification in Characteristic p

In characteristic 0, there are always finitely many rational functions on $\mathbb{P}^1$ with given ramification indices at given points, and when those points are general, the number of points is given combinatorially in terms of Schubert calculus. In characteristic p, the problem turns out to be substantially subtler, and we explore the situation in a complete range of characteristics, showing that the situation is particularly pathological in low characteristics regardless of whether the ramification is tame or wild, and ultimately solving the problem in mid range and higher characteristics by solving it in the case of three points and repeatedly letting ramification points come together to reduce inductively to this case.

The question we wish to address is simply:

Question I.0.1. Fix n points P_i on $\mathbb{P}^1$ and integers $e_i \geq 2$, with $\sum_i (e_i - 1) = 2d - 2$, and $e_i \leq d$ for all i. Then how many self maps of $\mathbb{P}^1$ of degree d are there which ramify to order e_i at the P_i, counted modulo automorphism of the image $\mathbb{P}^1$?

Notation I.0.2. When the answer to Question I.0.1 is finite, we denote it by $N(\{(P_i, e_i)\}_i)$. We denote by $N_{\text{gen}}(\{e_i\}_i)$ the value of $N(\{(P_i, e_i)\}_i)$ for general choice of P_i.

We further specify that the required ramification be the only ramification of the map, or equivalently, that the map be separable. The condition that $\sum_i (e_i - 1) = 2d - 2$ implies by the standard characteristic-p Riemann-Hurwitz formula that there are no solutions if any of the e_i is not prime to p, so we will assume throughout that all e_i are prime to p unless we specify otherwise.

Definition I.0.3. We distinguish three ranges of characteristic. We will refer to the **high characteristic** range to mean those characteristics for which $p > d$, as well as characteristic 0. The **mid characteristic** range will be characteristics for which $p \leq d$, but $e_i < p$ for all i. Finally, the **low characteristic** range will be characteristics for which $p \leq e_i$ for some i. We will see that high characteristics are uniformly well-behaved with respect to our question, while low characteristics can be extremely pathological, and the mid characteristics seem to be reasonably well-behaved, but are considerably subtler than the high characteristics.

Theorem I.0.4. *In the mid and high characteristics, we have the following complete solution to our main question:*

$$N_{\text{gen}}(\{e_i\}_i) = \sum_{\substack{d-e_{n-1}+1 \\ d-e_n+1}\leq d' \leq \substack{d \\ p+d-e_{n-1}-e_n}} N_{\text{gen}}(\{e_i\}_{i\leq n-2}, e), \text{ with } e = 2d' - 2d + e_{n-1} + e_n - 1 \quad \text{(I.0.5)}$$

$$N_{\text{gen}}(e_1, e_2, e_3) = \begin{cases} 1 & p > d \\ 0 & \text{otherwise} \end{cases} \quad \text{(I.0.6)}$$

Further, for general points P_i all of the relevant maps have no non-trivial deformations.

We make a few observations: first, since the degree d' is always no greater than d, high characteristic will remain high under recursion. Similarly, adding the two inequalities on the right, we find $e = 2d' - 2d + e_n + e_{n-1} - 1 \leq d + (p + d - e_{n-1} - e_n) - 2d + e_n + e_{n-1} - 1 = p - 1$, so mid characteristic is also preserved (or becomes high) under iteration.

Note that in the high characteristic range, we always have $p > d$, so the answer becomes independent of characteristic: this is visibly true for the second formula, and is true for the first formula because the inequality $d' \leq p + d - e_{n-1} - e_n$ is subsumed by the inequality $d' \geq e = 2d' - 2d + e_n + e_{n-1} - 1$, or equivalently $d' \leq 2d - e_{n-1} - e_n + 1$, which is necessary for the number of maps $N_{\text{gen}}(\{e_i\}_{i\leq n-2}, e)$ to be nonzero. Unsurprisingly, this characteristic-independent formula is also the answer in characteristic 0.

We begin in Section I.1 by translating the problem into a question on intersection of Schubert cycles in a Grassmannian. In Section I.2 we obtain some basic finiteness results including a ramified Brill-Noether type theorem for g^1_d's on $\mathbb{P}^1$. We then apply this in Section I.3 to solve the base case of three ramification points to derive the second equation of Theorem I.0.4. Section I.4 appears at first blush to be merely a couple of eccentric observations, including the observation that when one e_i is greater than p, the number of maps can be infinite, but these observations play key roles in Section I.5, where we give a precise analysis of when a family of separable maps can degenerate to an inseparable map, and in Section I.6, where we finally prove the first equation of Theorem I.0.4 via a degeneration argument using limit linear series. Finally, in Appendix I.A we construct a scheme representing maps between a pair of fixed curves, with at least a certain specified ramification, but at points which are allowed to move; this scheme is the key idea in the proof of the Brill-Noether-type result of Section I.2.

We remark that chronologically, the direct approach here was not the first proof discovered of our formulas. That was obtained via a correspondence with certain logarithmic connections on $\mathbb{P}^1$ together with a theorem of Mochizuki, as outlined in Section V.7. The key step of the direct argument presented here, the analysis of separable maps degenerating to inseparable ones, was derived via a careful study of the corresponding situation with connections. There is considerable literature on our main question and its natural generalizations in characteristic 0, from Eisenbud and Harris' original solution in the case of $\mathbb{P}^1$ in [14, Thm. 9.1], to combinatorial formulas in the same cases by Goldberg [21] and Scherbak [52], to formulas in the higher genus case of Logan [38, Thm. 3.1] and the author [49]. However, the present work appears to be the first attempt to approach the problem for positive characteristics.

I.1 Translation to Schubert cycles

In this section, we translate Question I.0.1 into a question on intersection of Schubert cycles, and pin down some related notation. We assume throughout that we are working over an algebraically closed field k.

Remark I.1.1. Given the results of this chapter for algebraically closed fields, one may easily argue that the same results, phrased scheme-theoretically, are true when k is not algebraically closed. However, there will be no way to conclude in this situation that all the resulting maps are actually defined over k, so there is little point in such an observation.

The main question may be easily translated into a question of the intersection of Schubert cycles on the projective Grassmannian $\mathbb{G}(1,d)$ as follows:

A rational function on $\mathbb{P}^1$ modulo automorphism of the image is precisely equivalent to a basepoint-free g^1_d, which is to say, a one-dimensional linear series of degree d, on $\mathbb{P}^1$; explicitly, this is a 2-dimensional subspace of the global sections of a line bundle $\mathscr{L}$ of degree d on $\mathbb{P}^1$ which generates $\mathscr{L}$ everywhere (the discrepency in dimension is the usual difference between affine and projective dimension). Now, $\mathscr{O}(d)$ is the unique line bundle of degree d on $\mathbb{P}^1$, and its global sections are a $(d+1)$-dimensional vector space, so our g^1_d's will be 2-dimensional subspaces in this fixed $(d+1)$-dimensional space, which is to say, points of $\mathbb{G}(1,d)$. Of course, a point of $\mathbb{G}(1,d)$ will only correspond to a map if it has no base points, but we will return to this issue shortly. Now, for each i the condition that a map ramify to order e_i at a point P_i is equivalent to requiring that our 2-dimensional subspace meet non-trivially the $(d+1-e_i)$-plane of sections of $\mathscr{O}(d)$ which vanish to order at least e_i at P_i; this is, by definition, a Schubert cycle which we will denote by $\Sigma_{e_i-1}(P_i)$, with corresponding class σ_{e_i-1}. Although the plane has codimension e_i, since we are looking at non-trivial intersections with 2-dimensional subspaces, we get a codimension e_i-1 condition in the Grassmannian; hence the notation.

Since we assumed $\sum_i(e_i-1)=2d-2$, and our Grassmannian has dimension $2d-2$, the total codimension of the Schubert cycles we are intersecting is the same as the dimension of the Grassmannian, and the expected dimension of the intersection is therefore 0. Pieri's formula will now give us the intersection product of our cycles, yielding a hypothetical formula for the answer to our question. However, there are several substantive issues to address.

The first major issue is whether or not the Schubert cycles will actually intersect transversely, even for general choice of the P_i. In characteristic 0, it follows from Kleiman's theorem [33] that general Schubert cycles (that is, Schubert cycles corresponding to general choices of flags) will intersect transversely, and in characteristic p Vakil [55, Cor. 2.7 (a)] and Belkale [2, Thm. 0.9] have recently independently shown the same to be true, but it is not the case that a general choice of points on $\mathbb{P}^1$ will correspond to a general choice of Schubert cycles in $\mathbb{G}(1,d)$, so we cannot hope to apply such general results. On the other hand, in characteristic 0 properness of the intersection (that is, having the expected dimension) for any choice of distinct P_i is straightforward, and we will reproduce the argument below in order to analyze its implications in characteristic p. Transversality for general choice of P_i in characteristic 0 is known, but more involved (see [14, Thm. 9.1]), and means that

Pieri's formula actually yields the correct number for general choice of P_i. Unfortunately, all of these statements fall apart in characteristic p, as we will see shortly.

The second issue to face is that of base points: points of $\mathbb{G}(1,d)$ with base points correspond to lower degree maps padded out by extra common factors in the defining polynomials. If these factors are away from P_i, this corresponds simply to lowering the degree without changing the ramification conditions. If, on the other hand, these factors are supported at the P_i, they will each subtract 1 from the degree while also subtracting 1 from the e_i; in particular, either way, the equality $\sum(e_i - 1) = 2d - 2$ will become a strict inequality, and Riemann-Hurwitz implies there are no such separable maps. In particular, in characteristic 0, or when $p > d$, the intersection of our Schubert cycles always actually corresponds to the desired g^1_d's. On the other hand, in general inseparable maps can and will occur, frequently contributing an excess intersection. For instance, in the case $d > p$, $e_i < p$, the Frobenius map will always contribute a $\mathbb{P}^{d-p}$ to the intersection, with one point in $\mathbb{G}(1,d)$ for every choice of a degree $d-p$ base point divisor.

These are the two issues which must be addressed in order to give an answer to the question. However, except in the base case of three points, we will not address them directly, as would be required by an intersection-theoretic approach. We will rather take a different tack, looking at moduli of g^1_d's with specified ramification for certain degenerating families.

Notation I.1.2. From this point on, there is plenty of opportunity for confusion of notation, since we will be thinking in terms of maps, g^1_d's, and polynomials almost interchangeably. Now, g^1_d's on $\mathbb{P}^1$ are equivalent to pairs of linearly independent polynomials of degree d, up to taking invertible linear combinations. However, with g^1_d's we refer to **base points**, whereas for pairs polynomials the corresponding idea is **common factors**. Given a g^1_d with $d - d'$ basepoints, we get a unique map of degree d' by getting rid of the basepoints, and similarly, given a map of degree $d' \leq d$, we get a unique g^1_d for each divisor of degree $d' - d$, simply by adding that divisor as a basepoint locus. We will refer to the locus inside $\mathbb{G}(1,d)$ of g^1_d's having base points as the **base point locus**, and the locus inside $\mathbb{G}(1,d)$ of g^1_d's for which the induced map in inseparable as the **inseparable locus**.

Warning I.1.3. The above equivalences tend to become misleading in families; in particular, if we have a linear series which develops base points in a special fiber, there is no way to remove them globally to actually produce a morphism. Linear series turn out to be the right concept for dealing with families, so whenever we are working over a base other than a field, we will always be dealing with linear series, even if we think of it as a "family of maps". For the appropriate definitions (albeit in an overly generalized context), see Chapter II.

Notation I.1.4. We will, for future reference, also specify our notation for the **different** δ of a separable map f between smooth proper curves C and D. We define δ to be the divisor on C associated to the skyscraper sheaf obtained as the cokernel of the natural map $f^*\Omega^1_D \hookrightarrow \Omega^1_C$.

Remark I.1.5. Our Schubert cycle description of the problem can also readily be described dually in terms of $(d-2)$-planes in $\mathbb{P}^d$ with prescribed intersection dimension with osculating planes at the P_i of the rational normal curve in $\mathbb{P}^d$, where our maps are given by projection from the $d-2$ planes. Thus, we are analyzing intersections of certain Schubert cycles associated to osculating flags at points of the rational normal curve. However, we will not make use of this description in our analysis.

I.2 Finiteness Results

We begin with a proposition whose argument is well-known in characteristic 0:

Proposition I.2.1. *In any characteristic, if $\sum_i(e_i - 1) = 2d - 2 - c$ for some $c \geq 0$, then the separable part of $\cap_i\Sigma_{e_i-1}(P_i)$ has dimension c if and only if it does not contain a $(c+1)$-dimensional family specializing to the inseparable locus. In particular, in high characteristics, $\cap_i\Sigma_{e_i-1}(P_i)$ always has the expected dimension c.*

Proof. Of course, c is the dimension of $\mathbb{G}(1,d)$ minus the sum of the codimensions of the cycles being intersected, so every component of the intersection must have dimension at least c, and all we need to show is that any component in the separable locus having dimension at least $c+1$ must meet the inseparable locus. By Riemann-Hurwitz, if we had $\sum_i(e_i-1) > 2d-2$, then $\cap_i\Sigma_{e_i-1}(P_i)$ must consist entirely of inseparable maps. We will therefore induct on c, with our base case (oddly, but perfectly correctly from a logical standpoint) as $c = -1$. Now, suppose we have a component of the separable locus of our intersection having dimension $c+1$. Choose any new point P distinct from the previous ones, and add the ramification index 2 at that point. Intersecting with $\Sigma_1(P)$ reduces the dimension of our family by at most 1: $\Sigma_1(P)$ is made up of lines in $\mathbb{P}^d$ intersecting a given

$(d-2)$-plane, and hence has codimension 1 in $\mathbb{G}(1,d)$, but moreover, meets any positive-dimensional (closed) subvariety of $\mathbb{G}(1,d)$, since any curve in $\mathbb{G}(1,d)$ corresponds to a surface in $\mathbb{P}^d$, which must meet every $(d-2)$-plane. But the intersection of our component with $\Sigma_1(P)$ is an (at least) c-dimensional component in the dimension $c-1$ case, so either it is entirely inseparable, or by the induction hypothesis, it must meet the inseparable locus, and in either case, we conclude our original component must have met the inseparable locus, as desired. Lastly, since the high characteristic case is precisely where the inseparable locus is empty, we conclude that no such higher-dimensional component can exist in that case. □

The case $c = 0$ is simply the full specification of a tame ramification divisor, so we restate:

Corollary I.2.2. *In high characteristics, there are finitely many self-maps of $\mathbb{P}^1$ with specified tame ramification divisor.*

The finite generation of fundamental groups of curves, together with some generalities on existence of moduli spaces of maps with certain ramification behavior, gives us a more substantive finiteness result than our first proposition:

Theorem I.2.3. *Let e_i be prime to p, and suppose $\sum_i(e_i - 1) = 2d-2$. Then for a general choice of points P_i, we have that the set of maps from $\mathbb{P}^1$ to $\mathbb{P}^1$ ramified to order e_i at P_i, modulo automorphism of the image:*

(i) is finite;

(ii) has no elements mapping any two of the P_i to the same point.

Proof. By Theorem I.A.5, we have a moduli scheme $\mathrm{MR} = \mathrm{MR}^d(\mathbb{P}^1, \mathbb{P}^1, \{e_i\}_i)$, with ramification and branching maps down to $(\mathbb{P}^1)^n$, and actions of $\mathrm{Aut}(\mathbb{P}_1)$ on both the sides, with the action on the domain being free. It is well-known that given any specified tame branch locus, up to automorphism of the cover there are only finitely many covers with the given degree and branching: this follows, for instance, from the fact (see Theorem A.1) that the tame fundamental group of $\mathbb{P}^1$ minus the branch points is a topologically finitely generated profinite group, so has only finitely many open subgroups of index d. Since each cover of degree d with the specified branching corresponds uniquely to an open subgroup of the tame fundamental group of index d, this gives the desired assertion.

Thus, each fiber of the branch morphism branch : MR $\rightarrow (\mathbb{P}^1)^n$ has only finitely many $\operatorname{Aut}(\mathbb{P}^1)$ orbits, and is therefore of dimension at most $\dim \operatorname{Aut}(\mathbb{P}^1) = 3$. We conclude that the dimension of MR is at most $n+3$. This immediately implies that a general fiber (in the sense of a fiber above a general point of $(\mathbb{P}^1)^n$, making no hypotheses on dominance) of the ramification morphism ram : MR $\rightarrow (\mathbb{P}^1)^n$ can have dimension at most 3. Since each fiber of this map is likewise composed of $\operatorname{Aut}(\mathbb{P}^1)$ orbits, and the action is free on this side, we find that a general fiber must always be empty or composed of finitely many $\operatorname{Aut}(\mathbb{P}^1)$ orbits, completing the proof of (i).

The proof of (ii) proceeds similarly: the locus MR$'$ of maps in MR sending any two ramification points to the same branch point has dimension at most $n-1+3 = n+2$, and since this property is preserved by $\operatorname{Aut}(\mathbb{P}^1)$ acting on the domain, any fiber of MR$'$ under the ramification morphism must be of dimension at least 3 if it is non-empty, so we conclude that a general fiber of the ramification morphism cannot have any points of MR$'$ in its preimage, completing the proof. □

Remark I.2.4. This finiteness theorem may be considered a first case in positive characteristic of a Brill-Noether theorem with prescribed ramification, as in [15, Thm. 4.5]. Significant generalization ought to be approachable via deformation theory of covers, as in [48]; such an approach also gives an instrinsically characteristic-p proof of the previous theorem.

I.3 The Case of Three Points

While the general problem we wish to study becomes extremely subtle in characteristic p, the special case where we only have three ramification points is, pleasantly, more tractable. This is fortuitous, since this case will form the base case of our general induction argument. We begin by observing that in this case, all three ramification points must map to distinct points: we have $\sum_i (e_i - 1) = 2d - 2$, so $\sum_i e_i = 2d + 1$. Now, any $e_j \leq d$, so $\sum_{i \neq j} e_i \geq d + 1 > d$, so we cannot have both P_i with $i \neq j$ mapping to the same point. We can also show via elementary observations that:

Lemma I.3.1. *The intersection $\cap_i \Sigma_{e_i - 1}(P_i)$ for three points is simply a $\mathbb{P}^m$ for some $m \geq 0$.*

Proof. Without loss of generality, we may assume that $P_1 = 0$, $P_2 = \infty$, and $P_3 = 1$. Now, any point in the intersection $\Sigma_{e_1-1}(P_1) \cap \Sigma_{e_2-1}(P_2)$ is a two-dimensional space of polynomials containing one vanishing to order e_1 at P_1, and one vanishing to order e_2 at P_2. We already observed that $e_1 + e_2 > d$, so these must be distinct, and they form a distinguished basis (F, G) of our space, up to scaling of F and G. If we require further that $F(1) = G(1)$, this is the same as asking that F/G map 1 to itself; since we have already, in effect, asked that it fix 0 and ∞, this determines (F, G) uniquely up to simultaneous scaling. Moreover, the vanishing conditions at 0 and ∞ are equivalent to setting a number of coefficients of F and G equal to 0. Now, with the hypothesis that F/G fix $P_3 = 1$, the last intersection says simply that $F - G$ vanishes to order e_3 at P_3. This places e_3 linear conditions on $F - G$. However, the nonzero coefficients of F and G were still completely general, and $e_1 + e_2 > d$ implies that the non-zero coefficients of F and G don't overlap, so we still get linear conditions on the coefficients of F and G. We conclude that the set of acceptable F and G is a linear space modulo simultaneous scaling of F and G, so our intersection is a $\mathbb{P}^m$, as desired. $\square$

The following lemma requires more substantial machinery, and serves as something of a substitute for Theorem I.2.3 in the case of three points. Although it is strictly speaking superfluous, we include it in order to be able to discuss its generalization (or lack thereof) later.

Lemma I.3.2. *There is a mixed-characteristic DVR A with residue field k and a triple $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3$ of points on $\mathbb{P}^1_A$ specializing to P_1, P_2, P_3 such that if p is prime to all the e_i, any rational function on $\mathbb{P}^1_k$ ramified to order e_i at the P_i can be lifted to a rational function on $\mathbb{P}^1_A$ ramified to order e_i at the $\tilde{P}_i$.*

Proof. Let A be the Witt vectors of k, a complete mixed-characteristic DVR with residue field k [54, Thm. II.5.3]. Choose any three $\tilde{P}_i$ lifting P_i to $\mathbb{P}^1_A$, and let f be any function on $\mathbb{P}^1_k$ ramified to order e_i at the P_i. Write $Q_i := f(P_i)$, and choose any (necessarily distinct) lifts $\tilde{Q}_i$ of Q_i. It is a theorem that f may be lifted to a tamely ramified $\tilde{f}$ over $\operatorname{Spec} A$, preserving the branching at $\tilde{Q}_i$ (see [4, 11, Proof of Prop. 5.1]). We note that from the particular definition of tamely ramified, $\tilde{f}$ has a ramification section over each $\tilde{Q}_i$: indeed, the ramification locus is isomorphic to the singular locus, and it follows from the definition that etale locally, the reduced induced subscheme on the singular locus is isomorphic to the union of the $\tilde{Q}_i$; since each $\tilde{Q}_i$ is a section, and the base is strictly henselian, it then follows that the (reduced induced subscheme on the) ramification locus is a union of sections, one over each $\tilde{Q}_i$, as desired. Now, these ramification sections may not be $\tilde{P}_i$, but they are certainly P_i on the special fiber, and since $\mathrm{Aut}(\mathbb{P}^1)$ is 3-transitive (in general, but this is particularly straightforward to see over a local ring), we can modify $\tilde{f}$ by an automorphism of the cover, restricting to the identity on the special fiber, and sending the ramification points of $\tilde{f}$ to $\tilde{P}_i$, giving us our desired lift. $\square$

We use these two lemmas to show:

Theorem I.3.3. *Let P_1, P_2, P_3 be three distinct points of $\mathbb{P}^1$, and e_1, e_2, e_3 positive integers. Then we have:*

(i) In any characteristic, $N(\{(P_i, e_i)\}_i)$ is finite, and is in fact always 0 or 1, being 0 if and only if there is some inseparable g^1_d of degree d with the required ramification. Moreover, when $N(\{(P_i, e_i)\}) = 1$, the intersection is actually given scheme-theoretically by a single reduced point.

(ii) $N(\{(P_i, e_i)\}_i) = 0$ whenever e_1 and e_2 are less than p, and $d \geq p$. $N(\{(P_i, e_i)\}_i) = 1$ whenever $d < p$.

Proof. We first deduce (ii) from (i): the second claim is trivial, since if $d < p$, there can be no inseparable map of degree d. For the first claim, because e_1 and e_2 are both less than p, any inseparable map will satisfy the required ramification conditions at P_1 and P_2, and if we choose our map to be Frobenius, we need to add $(e_3 - p)P_3$ to the ramification divisor via adding base points to satisfy the ramification condition at P_3. On the other hand, since Frobenius has degree p, we have $d - p$ base points to play with, and since we must have $e_3 \leq d$ by hypothesis, we will always have enough base points to create an inseparable g^1_d with the required ramification, forcing $N(\{(P_i, e_i)\}_i) = 0$, as desired.

For the proof of (i), we begin by noting that the intersection product in question is always 1: indeed, since all the Schubert cycles in question are special, this follows immediately from first applying Pieri's formula to the intersection of the first two cycles, and then the complementary-dimensional cycle intersection formula to the third (see [18, p. 271], noting that the complementary dimension formula is called the duality theorem). Next, the separable locus must be finite: we can use Lemma I.3.2, and have finiteness for the resulting lifted maps over the characteristic-0 generic point by Corollary I.2.2; alternatively, finiteness follows simply from Theorem I.2.3, using the fact that since there are no moduli for triples of points on $\mathbb{P}^1$, any triple is general. Finally, by Lemma I.3.1, our intersection is a $\mathbb{P}^m$, and is in particular connected. If it is 0-dimensional, we are done, since we get a single reduced point which must clearly correspond to either a separable or inseparable map. On the other hand, if it is positive dimensional, all but finitely many points of it must correspond to inseparable maps, and hence we cannot have any separable maps, since the inseparable locus is closed, and the entire intersection is connected. □

To rephrase a slightly special case of the second part of the theorem, we have:

Corollary I.3.4. *Suppose we are in the situation of the preceding theorem, and $e_1, e_2 < p$. Then a separable map of the specified ramification exists if and only if $d < p$.*

Remark I.3.5. This corollary certainly doesn't hold if we drop the hypothesis that at least two ramification indices be less than p. For instance, x^n is separable with exactly two ramification points of index n as long as n is prime to p, and in particular if n is larger than and prime to p, we have a separable map of degree larger than p ramified at (fewer than) three points.

Remark I.3.6. There are actually three approaches to proving the finiteness part of the above theorem, each with a different flavor: using Theorem I.2.3 and its proof via fundamental group theory, we are implicitly making use not only of lifting to characteristic 0, but of analytic methods to compare to the topological fundamental group. In contrast, Lemma I.3.2 is algebraic but still relies on lifting to characteristic 0. However, as noted in the proof, one could also prove Theorem I.2.3 instrinstically algebraically in characteristic p, via deformation theory of covers. So in fact none of what we do relies even on lifting to characteristic 0.

I.4 Some Theorems and Pathologies

We start with a fairly trivial lemma that we will want to make repeated use of:

Lemma I.4.1. *For any rational function given on the affine line by F/G, with $F, G \in k[x]$, we have:*

(i) The different is given by the order of vanishing of the differential form $(dF)G - F(dG)$. If G is non-vanishing at a point P, this is the same as the order of vanishing of the derivative of F/G.

(ii) In particular, if the order of vanishing $e-1$ of $(dF)G - F(dG)$ (equivalently, $\frac{d}{dx}(F/G)$, where G is non-vanishing) is less than p at P, then F/G is tamely ramified at P with ramification index e.

(iii) The ramification index of P where G does vanish is simply given by G's order of vanishing. Finally, at the point at infinity, if a linear combination of F and G is chosen so that $\deg F > \deg G$, the ramification index may be computed as $\deg F - \deg G$.

Proof. The second statement of part (i) is trivial. Moreover, it is clear that the order of vanishing of the derivative of F/G is the order of the different, where G is regular: the different is defined by the order of vanishing of the pullback of a regular, non-vanishing differential form. If G is regular at a point P on the affine line, F/G sends P to a point other than ∞, so the form dx is a regular non-vanishing form at both P and its image, and we find that the different is determined by the order of vanishing of $(F/G)^*dx = \frac{d}{dx}(F/G)dx$, which is the order of vanishing of F/G, as desired. To conclude the first statement, if G is regular at P we are done, but since the different is visibly invariant under automorphism of the image of a map, if G is not regular at P, we can simply interchange F and G, which will not affect the order of vanishing of $(dF)G - F(dG)$.

Part (ii) is then easily verified. Similarly, for part (iii) if G vanishes at P, by interchanging F and G we get that the order of vanishing is the ramification index. The final statement is separate and equally trivial, since F/G will vanish to order $\deg G - \deg F$ at infinity, and if that number is positive, it is by definition the ramification index at infinity. If it is negative, then by once again interchanging F and G we find that the ramification index is $\deg F - \deg G$, as desired. □

Warning I.4.2. The second part of (i) above is false if one drops the hypothesis that G is regular; while the derivative of F/G normally picks up poles of F/G, it can miss order-p poles. For instance, consider $x + x^{-p}$ at the point 0. It is clearly ramified to order p at $x = 0$ since it has an order-p pole, but its derivative on the affine line is simply equal to 1.

We have the following amusing and occasionally useful lemma:

Lemma I.4.3. *Given e_i all less than p, let e_i' be any integers obtained from the e_i by repeatedly replacing pairs of indices e_i, e_j with $p - e_i, p - e_j$ while holding the others fixed. Then $N_{\text{gen}}(\{e_i\}_i) = N_{\text{gen}}(\{e_i'\}_i)$*

Proof. It clearly suffices to show that the identity holds when we replace e_1, e_2 by $p - e_1, p - e_2$, and hold the rest fixed. For convenience, we also assume that F/G is unramified at infinity. We make use of the fact that by Theorem I.2.3, for P_i general, none of our maps for either of the two relevant choices of ramification indices send any two of the P_i to the same point, and in particular none send P_1 and P_2 to the same point. That is to say, for any map with ramification e_i at the P_i, we may write it (uniquely up to scalar) as F/G, where F vanishes to order e_1 at P_1, and G vanishes to order e_2 at P_2, so we can write $F/G = (x - P_1)^{e_1}F'/(x - P_2)^{e_2}G'$. If we multiply through by $(x - P_2)^p/(x - P_1)^p$, we get the new function $(x - P_2)^{p-e_2}F'/(x - P_1)^{p-e_1}G'$. Since we obtained it from the old one by multiplying by an inseparable function, it follows that the derivative is multiplied by the same function, and in particular its order of vanishing is unaffected away from P_1 and P_2, as is the order of vanishing of the denominator of the function. Since we assumed all $e_i < p$, it follows that the new function and old function have the same different and hence the same ramification away from P_1 and P_2, and *a priori* infinity. On the other hand, it is clear that the ramification at P_1 and P_2 is now $p - e_1$ and $p - e_2$, and it is easy to check that the new degree of the function allows for no new ramification at infinity. This sets up a visibly invertible, hence bijective, correspondence between our two sets of functions, and completes the proof of the corollary. □

The main usefulness of this rather eccentric fact is summarized in the following, to be applied later on:

Corollary I.4.4. *To calculate $N_{\text{gen}}(\{e_i\}_i)$ completely in the mid and high characteristic range, it suffices to do it either when all but at most one of the e_i are less than $p/2$, or, in the case p is odd, when all the e_i are odd. Moreover, it suffices to prove Theorem I.0.4 in only either of these two cases.*

Proof. The first statement follows trivially from the previous corollary. The second is simply a matter of noting that for any given number of points, the parity of the sum of the e_i is determined by the fact that d must be an integer; all the e_i being odd always gives the correct parity, and if any e_i are even, an even number of them must be, and we can replace them in pairs by $p - e_i$ to get them all to be odd.

To get the final assertion, we have to show that the formulas proposed in Theorem I.0.4 are unaffected by replacing a pair e_i and e_j with $p - e_i$ and $p - e_j$. We will show this by induction, with $n = 3$ as the base case. For convenience, we repeat the formulas in question:

$$N_{\text{gen}}(\{e_i\}_i) = \sum_{\substack{d-e_{n-1}+1 \\ d-e_n+1} \le d' \le \substack{d \\ p+d-e_{n-1}-e_n}} N_{\text{gen}}(\{e_i\}_{i \le n-2}, e), \text{ with } e = 2d' - 2d + e_{n-1} + e_n - 1$$

$$N_{\text{gen}}(e_1, e_2, e_3) = \begin{cases} 1 & p > d \\ 0 & \text{otherwise} \end{cases}$$

We begin with the three point case. Here is the main place where we have to be careful to bear in mind the condition that $e_i \le d$ for all i. Indeed, a map exists for e_i if and only if all $e_i \le d$ and $d \le p - 1$. We use $d = \frac{e_1+e_2+e_3-1}{2}$, and find that these inequalities may be rewritten as

$$\begin{matrix} e_1 - e_2 + 1 \\ e_2 - e_1 + 1 \end{matrix} \le e_3 \le \begin{matrix} e_1 + e_2 - 1 \\ 2p - 1 - e_1 - e_2 \end{matrix}$$

Replacing e_1 and e_2 by $p - e_1$ and $p - e_2$ leaves the lefthand inequalities unchanged, and switches the righthand ones, so the values of e_3 for which a map exists are precisely the same. By the visible symmetry between the e_i, this is sufficient for the three point case.

Next, since the proposed recursive equation is not visibly symmetric in the e_i, there are three cases to consider: first, $i = n - 1$, $j = n$; second, $i, j < n - 1$; and finally, $i < n - 1, j \ge n - 1$. We first note that when we perform this flip, the degree d changes to $d + p - e_i - e_j$.

Thus, for the first case, when we substitute $d+p-e_{n-1}-e_n$ for d, $p-e_{n-1}$ for e_{n-1}, and $p-e_n$ for e_n, we find that the two inequalities on the left simply switch with one another, and likewise for the inequalities on the right, so the range for d' remains unchanged. Likewise, $e = 2d' - 2d + e_n + e_{n-1} - 1$ remains unchanged, so the formula is actually precisely the same.

For the second and third cases, we will want to write the inequalities for d' as equivalent inequalities for e. We find:

$$\begin{matrix} e_n - e_{n-1} + 1 \\ e_{n-1} - e_n + 1 \end{matrix} \leq e \leq \begin{matrix} e_n + e_{n-1} - 1 \\ 2p - 1 - e_{n-1} - e_n \end{matrix}$$

In particular, these inequalities depend only on e_n and e_{n-1}, and not on d. So in the second case, since e_n and e_{n-1} are fixed, e ranges through the same values, and each term in the new recursive formula matches precisely a term in the original one, but with e_i and e_j replaced by $p - e_i$ and $p - e_j$, so we need only use the induction hypothesis to conclude the desired result.

Finally, in the third case we assume for convenience that $i < n - 1$ and $j = n$. In this case, when we substitute into our inequalities for e, we get

$$\begin{matrix} p - e_n - e_{n-1} + 1 \\ e_{n-1} + e_n + 1 - p \end{matrix} \leq e \leq \begin{matrix} p - e_n + e_{n-1} - 1 \\ p - 1 - e_{n-1} + e_n \end{matrix}$$

which then gives us that $p - e$ satisfies precisely the same inequalities that e did originally. Thus, each term in the new recursive formula matching a unique one in the old one by replacing e with $p - e$; we likewise have e_i replaced by $p - e_i$, so once again the induction hypothesis gives us the desired result. $\square$

We end with a rather surprising result illustrating that even tame ramification can have very pathological behavior when the characteristic is in the low range:

Proposition I.4.5. *Suppose $e_1 > p$ but still prime to p, and $e_i < p$ for all $i > 1$. Then if one map exists with ramification e_i at P_i, infinitely many do. In particular, if the P_i are general, no maps exist with ramification e_i at P_i.*

Proof. Without loss of generality, we may assume that P_1 is the point at infinity, and that our function maps infinity to infinity, so that if it is given by F/G, $\deg F - \deg G = e_1$. Now consider the family of functions $F/G - tx^p$, where $t \in k$. Because $e_1 > p$, both $\deg F$ and $\deg G$ are unchanged, so the ramification at infinity is unaffected. On the other hand, x^p is regular away from infinity, so it doesn't affect the order of vanishing of the denominator, and it is inseparable, so it doesn't affect the derivative of the function. Hence, the different is unchanged on the affine part, and since we assumed that all $e_i < p$ for $i > 1$, we find that the ramification is unaffected everywhere, giving us an infinite family of maps, clearly not related by automorphism, all with the same ramification. For P_i general we know that there can be at most finitely many maps with specified tame ramification by Theorem I.2.3, so we conclude that there cannot be any such maps at all. $\square$

Remark I.4.6. It will not be necessary for any of our applications, but it is trivial to generalize the above argument to show that if $e_1 > mp$ for some $m \geq 1$, and the other e_i are still less than p, there is an m-dimensional family of maps with the same ramification divisor. We can also easily show now the rather random fact that if the P_i are general, and we have a map f with $e_i < p$ for all $i > 1$, but $e_1 = mp$ wild with $m \geq 1$, the different of f at P_1 is greater than $2(m-1)p$. Indeed, if we again put P_1 at infinity and write $f = F/G$, subtracting off some multiple of x^{mp} will force the degree to drop, and leave all e_i for $i > 1$ unchanged. The index e_1 may not drop (this can only happen if the degree of F drops at least mp below the degree of G), but if it remains wild we can iterate, and since the degree drops each time, e_1 must eventually become tame. By our proposition, this new tame index, which we denote by e_1', would have to be less than p. If we denote the new degree by d', we have $2d' - 2 = e_1' - 1 + \sum_{i>1}(e_i - 1)$, and we also had $2d - 2 = \delta + \sum_{i>1}(e_i - 1)$ where δ is the different of f at P_1; thus, $\delta = e_1' - 1 + 2(d - d')$. But $d - d'$ must be greater than $(m-1)p$; if the degree of F always remained greater than the degree of G, it would have had to drop by $mp - e_1' > (m-1)p$, and would have determined d'. On the other hand, if the degree of F ever dropped below the degree of G, the degree of our map would have dropped more than mp. So $\delta > 2(m-1)p$, as desired.

Example I.4.7. To demonstrate that the statement of Proposition I.4.5 is not vacuous, we note it is not difficult to write down a concrete example. Consider, for instance, the function $x^{p+2} + x$ for any odd prime p; it has tame ramification index $p+2$ at infinity, and since its derivative is $2x^{p+1} + 1$, it has $p+1$ distinct simply ramified points on the affine line. Thus, we get from our proposition that $x^{p+2} + tx^p + x$ is an explicit example of an infinite family of distinct maps with the same fixed tame ramification divisor.

Remark I.4.8. We know that given tame ramification indices and fixed branch locus, we can have only finitely many maps with the specified branch behavior, so we must have that while the ramification points in the constructed family of maps remain fixed, the branch points move, which they visibly do, by the amount tx^p. This behavior is impossible in characteristic 0, and in high characteristics, where we already know the number of maps to be finite with either ramification or branch behavior fixed. This also holds in mid characteristics, at least for odd ramification indices, but as of yet the only proof I am aware of (see Remark V.7.2) is extremely circuitous and outside the scope of the methods of this chapter.

Remark I.4.9. Since a result much like the lifting to characteristic 0 assertion of Lemma I.3.2 holds in greater generality from the perspective of covers and specified branch behavior, one might be tempted to conjecture that the lemma itself holds at least for arbitrary numbers of points and tame ramification on $\mathbb{P}^1$. However, the preceding proposition shows that it cannot hold even in this case, since by virtue of Corollary I.2.2 only finitely many of the infinitely many constructed maps would be able to lift to characteristic 0. However, it may still be true that one can generalize Lemma I.3.2 to arbitrary numbers of points on $\mathbb{P}^1$ in the high and mid characteristic ranges. Indeed, we are able to prove this in more generality in Corollary II.4.5, subject to an expected-dimension hypothesis; in particular, in our case of self-maps of $\mathbb{P}^1$, we can conclude thanks to Theorem I.2.3 that lifting to characteristic 0 is always possible for tame ramification indices and general ramification points, or in high characteristic and any ramification points, and we have reduced the mid-characteristic case down to the conjectured finiteness of the number of maps for arbitrary distinct ramification points.

Remark I.4.10. The behavior exhibited here also appears to be fundamentally different from the existence of special linear series in characteristic 0. Standard examples of special series in characteristic 0 are situations where the expected dimension is negative, so non-existence for general curves and ramification points is mandated by the generalized Brill-Noether theorem. In such examples, the space of linear series with the prescribed ramification is supported over a maximal-dimensional subspace of $\mathcal{M}_{g,n}$, or equivalently, a general configuration where such a linear series exists has only finitely many. In particular, in none of the standard examples is the expected dimension non-negative. It is not at all clear whether or not this must always be the case in characteristic 0, but here we have an example where this fails to hold in characteristic p.

I.5 Specialization to Inseparable Maps

The ultimate goal will be to solve the map-counting problem in mid and high characteristics by repeatedly letting points come together. The main obstacle to this is understanding when a family of separable maps can have an inseparable map as its limit. We provide an answer to this question, which will seem incredibly unmotivated, and indeed arose from careful examination of the situation in the very different and at first glance totally unrelated setting of Chapter V, as discussed in Section V.7.

Our main result is:

Theorem I.5.1. *Let A be a DVR containing its residue field k and with uniformizer t, and f_t be a family of maps of degree d from $\mathbb{P}^1$ to $\mathbb{P}^1$ over* $\operatorname{Spec} A$ *(more precisely, a linear series on $\mathbb{P}^1_A$) whose generic fiber is tamely ramified along sections P_i with all $e_i < p$, and whose special fiber is inseparable. We further assume that the P_i stay away from infinity. Then if the limit of the P_i in the special fiber is denoted by $\bar{P}_i$, we have:*

(i) If the $\bar{P}_i$ are distinct, they are in a special configuration allowing the existence of separable maps of degree $d + mp - 1 + \epsilon$ ramified to order e_i at $\bar{P}_i$, and $2mp - 1 + 2\epsilon$ at infinity;

(ii) If $\bar{P}_j = \bar{P}_{j'}$ with $e_j + e_{j'} < p$, and the other $\bar{P}_i$ distinct, then the $\bar{P}_i$ are in a special configuration allowing separable maps of degree $d + mp - 1 - b + \epsilon$, ramified to order e_i at the $\bar{P}_i$ for $i \neq j, j'$, $e_j + e_{j'} - 2b - 1$ at $\bar{P}_j = \bar{P}_{j'}$, and $2mp - 1 + 2\epsilon$ at infinity;

in either case, m is some positive integer with $mp \leq d$ and ϵ is 0 or 1, and in the second case b is a non-negative integer less than $(e_j + e_{j'} - 1)/2$.

Proof. The main idea of the proof is not dissimilar to the basic operation of applying fractional linear transformations to be able to factor out a power of the uniformizer if one is given a family of maps degenerating to a constant map. However, in this case we will apply a fractional linear transformation with inseparable coefficients; this will behave similarly, but will not preserve the degree of the map, and also does not appear to work readily in nearly the generality of the constant case.

We work for the most part explicitly with pairs of polynomials and their differents, only dealing with common factors at the end to translate to rational functions and ramification indices. We can write f_t as F/G, where $F, G \in A[x]$, and have no common factors. We denote by F_0 and G_0 the polynomials obtained from F and G by setting $t = 0$, and by $\bar{F}_0$ and $\bar{G}_0$ the inseparable polynomials obtained by canceling the common factors of F_0 and G_0. Then let H_1 and H_2 be inseparable polynomials of degree strictly less than $\bar{F}_0$ and $\bar{G}_0$ respectively, such that $\bar{F}_0 H_2 - \bar{G}_0 H_1 = 1$ (this is possible by dividing the exponents

$\bar{F}_0$ and $\bar{G}_0$ by p, applying Euclid's algorithm in $k[x]$, and multiplying all exponents by p). We now construct a new family $\tilde{F}/\tilde{G}$ over $\operatorname{Spec} A$ as follows: if we denote by ν the map from $A[x]$ to itself which simply factors out common powers of t, then $\tilde{F} := \nu(F\bar{G}_0 - G\bar{F}_0)$, and $\tilde{G} := FH_2 - GH_1$. It is easy to check that applying an inseparable fractional linear transformation to F/G will change $(dF)G - F(dG)$ by the determinant of the transformation; in our case, by construction the determinant is 1, and it follows that $(d\tilde{F})\tilde{G} - \tilde{F}(d\tilde{G})$ is the same as $(dF)G - F(dG)$, but with a positive power of t factored out.

At $t = 0$, we note that since we had $\bar{F}_0 H_2 - \bar{G}_0 H_1 = 1$, $\tilde{G}$ is made up precisely of the common factors of F_0 and G_0, of which there can be at most $d - \deg f_0$. Since we removed a positive power of t from $(dF)G - F(dG)$, if we still have an inseparable limit, we can repeat the process as many times as necessary to remove all the powers of t. Each time we do, the degree of the denominator at $t = 0$ is reduced by at least the degree of the limit, and the degree of the numerator for any t increases by at most the degree of the limit. We denote the sum of the degrees of the inseparable limits as mp. We thus end up with a family $\tilde{F}/\tilde{G}$ which over the generic fiber has the same different as F/G away from infinity. If we let K be the fraction field of A, we also note that since t is a unit in K and we transformed by an invertible matrix of polynomials, the ideal generated by $\tilde{F}, \tilde{G}$ in $K[x]$ is the same as that generated by F, G. Since F, G had no common factors over K, it follows that $\tilde{F}, \tilde{G}$ have no common factors either. Now, since we have no common factors, we find that away from $t = 0$ (that is, at the generic fiber), Lemma I.4.1 implies that $\tilde{F}/\tilde{G}$ has the same ramification as F/G except possibly at infinity, since all the e_i were specified to be less than p. If we denote the generic degree of F/G by d, the greater of the degrees of $\tilde{F}, \tilde{G}$ at $t = 0$ by $\tilde{d}$, and the degree of $\tilde{G}$ at $t = 0$ by d_0, we have $\tilde{d} \leq d + mp$, and $d_0 \leq d - mp$, and we must have in particular $mp \leq d$.

The main idea of the rest of the proof is to show that our construction creates enough new ramification at infinity to bound the degree $\tilde{d}$ strongly from below, essentially determining the situation. We claim that at $t = 0$, the degree of the different of $(\tilde{F}, \tilde{G})$ away from infinity is the same as the generic degree of the different of f_t. Indeed, this follows from our hypothesis that the P_i stay away from infinity, because when the limit is separable, the limit of the different is the different of the limit, with orders adding when points come together. It follows that

$$2\tilde{d}-2=2d-2+\delta_\infty \geq 2d-2+e_\infty-1,$$

where δ_∞ and e_∞ are the order of the different and the ramification index at infinity of $\tilde{F}/\tilde{G}$ at $t=0$, respectively. Note that we do not need to consider base points at infinity, since we are working with polynomials on the affine part, and $\tilde{d}$ is the degree obtained after substituting in $t=0$. The above inequality gives in particular that $\tilde{d} \geq d$, so since the degree of $\tilde{G}$ at $t=0$ was strictly less than d, $\tilde{d}$ is simply the degree of $\tilde{F}$ at $t=0$. Thus, the ramification index at infinity is simply $\tilde{d}-d_0 \geq \tilde{d}-d+mp$, and it is at least mp. Substituting back into the earlier inequality, we find

$$2\tilde{d}-2 \geq 2d-2+(\tilde{d}-d+mp)-1=\tilde{d}+d-3+mp,$$

which immediately gives $\tilde{d} \geq d+mp-1$. We then also have

$$e_\infty \geq \tilde{d}-d+mp \geq d+mp-1-d+mp=2mp-1.$$

We now translate back into the language of maps, by removing common factors. In the situation with $\bar{P}_j=\bar{P}_{j'}$, we let b be the number of common factors of $\tilde{F}|_{t=0}, \tilde{G}_{t=0}$ at $\bar{P}_j=\bar{P}_{j'}$. For notational convenience, we will call the maximal order of vanishing at P of a (non-zero) linear combination of polynomials F', G' the **vanishing index** of F', G' at P; when there are no common factors, this is simply the ramification index of the corresponding rational function. Considering $(\tilde{F}_{t=0}, \tilde{G}_{t=0})$, we have two cases to address. If $\tilde{d}=d+mp-1$, then substituting back into our first inequality we must have $e_\infty=2mp-1$, and we see that the entire different is accounted for. Similarly, if $\tilde{d}=d+mp$ (we already noted it cannot be greater), we see that $e_\infty=2mp$ or $2mp+1$, and in either case the entire different is again accounted for. We now argue that away from infinity we have no common factors except at $\bar{P}_j=\bar{P}_{j'}$ in the situation that they were equal. Because we assumed that $e_j+e_{j'}<p$ in that case, the different in the limit is everywhere less than p, and we need not worry about wild ramification. It then follows that at any point P, the different is the sum of the common factors and the vanishing index (minus one) at that point, so to achieve a fixed different, common factors force the vanishing index down. It is clear that vanishing index

cannot drop under specialization, so this can only happen at points in the $t = 0$ fiber where the limit of the different is greater than the limits of the differents approaching the point, which can only happen where the ramification sections converge, where their differents add in the limit. This shows that there are no common factors except possibly where $\bar{P}_j = \bar{P}_{j'}$. When we factor out the b common factors, the degree drops by b, and the vanishing index drops from $e_j + e_{j'} - b - 1$ to $e_j + e_{j'} - 2b - 1$. is exactly e_i at all $\bar{P}_i$ for $i \neq j, j'$.

In the cases that e_∞ is tame, we are done. The only other possibility was that $\tilde{d} = d+mp$, and $e_\infty = 2mp$. In this case, if we take $\tilde{F}|_{t=0}/\tilde{G}|_{t=0}$ and subtract off an appropriate multiple of x^{2mp}, we can reduce the degree of the numerator by at least one, and we claim we end up in the first case again: as in the construction in the proof of Proposition I.4.5, we will not change the ramification away from ∞, nor have we changed the denominator at all, but we have visibly forced the numerator to have degree at most $d + mp - 1$, so all our inequalities still hold and we are in fact back to the first case, completing the construction and the proof of the theorem. □

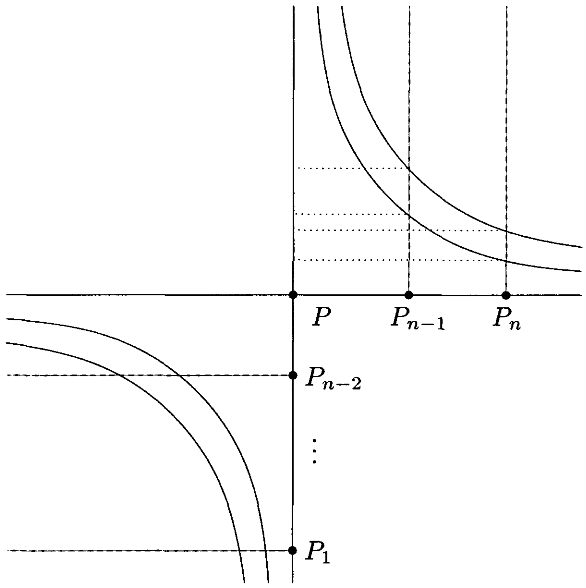
Putting the theorem together with Proposition I.4.5, and noting that there are only finitely many possibilities for m, b, ϵ, we conclude:

Corollary I.5.2. *In the situation of the preceding theorem, if the $\bar{P}_i$ are general, there cannot be any f_t as described, having an inseparable limit.*

Remark I.5.3. In the iterated operation on F/G described in the argument for the theorem, it is possible that on some step, the explicitly-presented map on the special fiber would become not merely inseparable, but constant. This is actually handled implicitly in the iterated process, simply by a change of basis, with m remaining unchanged and the power of t decreasing. This change of basis procedure is, of course, exactly why the space of g^r_d's of a given dimension is proper, so in some sense the procedure of the above argument may be viewed as a generalization of that fact for our particular situation.

I.6 The Degeneration Argument

We complete the proof of Theorem I.0.4 in this section via a degeneration argument. The basic situation we will consider is the family arising as follows:



An explicit family of smooth $\mathbb{P}^1$'s degenerating to a node, with n sections.

Inside of $\mathbb{P}^1 \times \mathbb{P}^1$, take the family of hyperbolas given in affine coordinates by $xy = t$, degenerating at $t = 0$ to the union of the x-axis and y-axis. For each $t \neq 0$, we get a smooth $\mathbb{P}^1$, and fix isomorphisms between them by projecting to the y-axis. Choose an isomorphism between our abstract $\mathbb{P}^1$ and the y-axis sending P to the node; we can now speak of $P_1, \dots P_{n-2}$ as well as P as fixed points on the y-axis and simultaneously on all the $\mathbb{P}^1$'s in our family; they are (constant) sections of our family. Now, choose any two points P^0_{n-1} and P^0_n on the x-axis away from 0, and define sections P^t_{n-1} and P^t_n similarly via projection from our family to the x-axis rather than the y-axis. Under our fixed trivialization of the smooth fibers of the family, these sections both tend towards the section defined by P (see figure). We will consider this as a family X over $\operatorname{Spec} k[t]$, and write X_t for the associated local family over $\operatorname{Spec} k[t]_{(t)}$.

We briefly review the main concepts of the theory of limit linear series as it relates to our situation. See Chapter II for general definitions and, where applicable, proofs. On any non-singular fiber of our family, we know that a map to $\mathbb{P}^1$ (modulo automorphism of the image) corresponds to a g^1_d on that fiber; we see that given a g^1_d on the family away from the special fiber, we can obtain a g^1_d on either the x- or y-axis simply by projecting all fibers to the appropriate choice of axis. This pair gives the associated Eisenbud-Harris limit series on the nodal fiber; we have vanishing sequences at a^x_i and a^y_i for $i = 0, 1$ at the node, and the degree of the induced map on the x-axis (respectively, y-axis) is at most $d - a_0^x$ (respectively, $d - a_0^y$), with the ramification index of the map at the node given by $a_1^x - a_0^x$ (respectively, $a_1^y - a_0^y$). We have the inequalities $a_0^x + a_1^y \geq d$, $a_1^x + a_0^y \geq d$; the data of a pair of g_d^1's on the components with vanishing sequences satisfying these inequalities is in fact the definition of an Eisenbud-Harris limit series, and we say that a given limit series is refined if these are both equalities.

It is easy to see that for arbitrary a_i^x and a_i^y satisfying the necessary inequality, the space of g_d^1's on the x-axis will have total required ramification at least $2(d - a_0^x) - 2$, and similarly for the y-axis, so in particular by Riemann-Hurwitz if the limits are separable, we immediately conclude that they must form a refined limit series, and they cannot have any additional base points, so the corresponding maps must have degrees precisely $d - a_0^x$ and $d - a_0^y$, with ramification index $a_1^x - a_0^x = a_1^y - a_0^y$ at the node.

Given these observations, our general theory, and specifically Theorem II.4.3, gives us:

Theorem I.6.1. *Associated to our families X and X_t, and any choice of ramification indices e_i such that $\sum_i (e_i - 1) = 2d - 2$, are schemes $G_d^1 := G_d^1(X, \{(P_i, e_i)\}_i)$ and $G_d'^1 := G_d^1(X_t, \{(P_i, e_i)\}_i)$, with the latter obtained from the former by base change, and the fibers parametrizing (limit) linear series with the required ramification on the fibers of X and X_t. We also have open subschemes $G_d^{1,\mathrm{sep}}$ and $G_d'^{1,\mathrm{sep}}$ parametrizing limit series which are separable when restricted to every component of every fiber; over $t = 0$, $G_d^{1,\mathrm{sep}}$ (equivalently, $G_d'^{1,\mathrm{sep}}$) parametrizes simply Eisenbud-Harris limit series, and contains only refined limit series.*

Proof. Most of this is immediate from Theorem II.4.3. The fact that $G_d^{1,\mathrm{sep}}$ parametrizes Eisenbud-Harris series on the special fiber follows from Corollary II.5.9 together with the assertion that the only separable Eisenbud-Harris limit series are refined, which we observed above. □

Given this language, we can readily apply Corollary II.5.12 to obtain:

Corollary I.6.2. *With the notation of the above theorem, if $P_1, \dots P_{n-2}$ and P are chosen generally, and $e_{n-1} + e_n < p$, then $G_d'^{1,\mathrm{sep}}$ is finite etale over $\operatorname{Spec} k[t]_{(t)}$. In particular, it has the same number of points, all reduced, in the geometric generic and special fibers, and the fibers of $G_d^{1,\mathrm{sep}}$ have the same number of points for t general as at $t = 0$.*

Proof. First, the assertion on the fibers of $G_d^{1,\mathrm{sep}}$ for general d follows immediately from the statement on $G_d'^{1,\mathrm{sep}}$, together with the fact that $G_d'^{1,\mathrm{sep}}$ is obtained from $G_d^{1,\mathrm{sep}}$ simply by localization of the base around $t = 0$.

Next, to obtain the desired statement on $G_d'^{1,\mathrm{sep}}$, we need only verify that the three conditions of Corollary II.5.12 are satisfied: first, that every separable Eisenbud-Harris limit series on the special fiber is refined; second, that the scheme of separable Eisenbud-Harris limit series on the special fiber consists of a finite number of reduced points; and third, that if A is a DVR, any A-valued point of G_d^1 mapping flatly to $\operatorname{Spec} k[t]_{(t)}$ and being separable at the generic point is also separable on the closed point. Condition (I) is satisfied even without the generality hypothesis, as stated in the above theorem.

Condition (III) is for the most part simply an application of Corollary I.5.2; indeed, given an A-valued point of $G_d'^1$ flat over $\operatorname{Spec} k[t]_{(t)}$, projection to the y-axis would give a family of g_d^1's on $\mathbb{P}^1$ with ramification sections specializing to the $P_1, \dots P_{n-2}, P$, which are general by hypothesis. Then Corollary I.5.2 says that if the family is generically separable, it must remain separable on the special fiber. It remains to see that the same holds if we project to the x-axis. For this, considering the different we note that the vanishing sequence on the y-axis at the node will satisfy $a_0^y + a_1^y - 1 = e_{n-1} + e_n - 2$, and in particular $a_1^y < p$. On the other hand, $a_0^x + a_1^y \geq d$, so since a_0^x is the number of base points acquired on the x-axis, the degree on the x-axis is less than or equal to $d - a_0^x \leq a_1^y < p$, and we also cannot have an inseparable limit along the x-axis, giving condition (III).

Lastly, we prove the validity of condition (II) by induction on n. The basic observation is because the space of refined Eisenbud-Harris limit series may be viewed simply as a disjoint union over all vanishing sequences satisfying $a_i^x + a_{1-i}^y = d$ of the products of the schemes parametrizing g_d^1's with appropriate ramification on each component, it suffices to see that these latter are made up of reduced points. It is easy to see that as the vanishing sequences vary, if we simply remove the base points a_0^x and a_0^y, we will have the same ramification index e at the node on each component, the degrees on each component will be such that the expected dimension (taking e into account as well as the e_i) will be zero, and e will vary arbitrarily given this constraint, together with the constraint that the degrees on each component be at most d. In particular, it suffices to see that for points chosen generally, the scheme of separable g_d^1's in the $(n-1)$-point and 3-point cases always consist of a finite number of reduced points, and by induction on the statement of our corollary, it is enough to see this in the 3-point case, which we have conveniently already handled in Theorem I.3.3. □

We are now ready for:

Proof of Theorem I.0.4. First, we may assume that $e_{n-1}+e_n < p$, thanks to Corollary I.4.4. By Corollary I.6.2, for all our points chosen generally, and a general choice t, $G_d^{1,\text{sep}}$ has the same number of points over that particular t as it does over $t=0$. This sets up a simple recursion formula to calculate $N_{\text{gen}}(\{e_i\}_i)$: the number will be given by the number over the special fiber, which is the sum over all choices e of ramification index at the node of $N_{\text{gen}}(e, e_{n-1}, e_n)N_{\text{gen}}(\{e_i\}_{i<n-1}, e)$.

We recall that the formula we wanted to prove for Theorem I.0.4 (the second formula having already been handled by Theorem I.3.3) was

$$N_{\text{gen}}(\{e_i\}_i) = \sum_{\substack{d-e_{n-1}+1 \\ d-e_n+1}\le d' \le \substack{d \\ p+d-e_{n-1}-e_n}} N_{\text{gen}}(\{e_i\}_{i\le n-2}, e), \text{ with } e = 2d'-2d+e_{n-1}+e_n-1$$

and that in the proof of Corollary I.4.4 we showed that the above inequalities for d' were equivalent to the following inequalities on e:

$$\begin{matrix} e_n - e_{n-1} + 1 \\ e_{n-1} - e_n + 1 \end{matrix} \le e \le \begin{matrix} e_n + e_{n-1} - 1 \\ 2p - 1 - e_{n-1} - e_n \end{matrix}$$

We begin by showing that the above inequalities for e give precisely the range for which $N_{\text{gen}}(e, e_{n-1}, e_n) = 1$. But with Theorem I.3.3 at our disposal, this is a trivial observation, since $e_n - e_{n-1} + 1 \le e$, $e_{n-1} - e_n + 1 \le e$ and $e \le e_n + e_{n-1} - 1$ are precisely the inequalities insuring that the ramification indices are less than the degree of the map, and $e \le 2p - 1 - e_{n-1} - e_n$ insures that the degree is less than p. Finally, we need to know that the degree on the three-point component will be less than d. This degree will be given by $\frac{e+e_{n-1}+e_n-1}{2}$, so we find that *a priori*, we need $e \le 2d - e_{n-1} - e_n + 1$. However, we note that the right hand side is actually $2d' - e$, so this inequality is equivalent to $e \le d'$, and we needn't include it with the conditions, as if it is violated we will have $N_{\text{gen}}(\{e_i\}_{i\le n-2}, e) = 0$, and there will be no contribution to the sum. This completes the proof of our main theorem. □

As an application, we note that in the case of four points, for a given d' as in Theorem

I.0.4, $N_{\text{gen}}(e_1, e_2, e) = 1$ if $e_1, e_2, e \leq d'$ and $p > d'$, and $N_{\text{gen}}(e_1, e_2, e) = 0$ otherwise. Rewriting this condition in terms of d', we get the bounds $e_1 \leq d'$, $e_2 \leq d'$, $d' \leq 2d - e_3 - e_4 + 1$, $d' \leq p - 1$, and including these bounds for d' along with those of Theorem I.0.4, simply by substracting the various bounds for possible values of d' we obtain:

Corollary I.6.3. *The number $N_{\text{gen}}(\{e_i\}_i)$ of self-maps of $\mathbb{P}^1$ of degree d in characteristic p, ramified to orders $e_1, \ldots e_4$ at four general points, with each $e_i < p$ and $2d-2 = \sum_i (e_i - 1)$, and counted modulo automorphism of the image, is given by the formula*

$$N_{\text{gen}}(\{e_i\}_i) = \min\{\{e_i\}_i, \{d+1-e_i\}_i, \{p-e_i\}_i, \{p-d-1+e_i\}_i\},$$

or equivalently,

$$N_{\text{gen}}(\{e_i\}_i) = \min\{\{e_i\}_i, \{d+1-e_i\}_i\} - \max\{0, d+1-p\}.$$

Further, all of these maps are without any nontrivial deformations.

Example I.6.4. We explore an example which demonstrates all the basic behaviors we have described so far, and may be solved explicitly: maps of degree 3, with four simple ramification points. We may assume without loss of generality that $P_1 = 0, P_2 = \infty, P_3 = 1$, and we let P_4 be a general parameter λ. We see immediately that our four ramification points must have distinct images, so we may further specify that our maps fix P_1 and P_2, from which we deduce that they are of the form $f = \frac{x^2(ax+b)}{x+c}$, with a, b, c all nonzero. Since we did not specify that P_3 be fixed, we have one remaining degree of freedom, and may set $b = 1$. Now, if we consider the zeroes and poles of df, we can calculate directly that our possible maps satisfy $2c = 2a\lambda$ and $1 + 3ac = -(1+\lambda)2a$, which in characteristic $\neq 2$ means c is determined by a and λ, and a satisfies $3\lambda a^2 + 2(1+\lambda)a + 1 = 0$. In characteristic 3, we get a unique (separable) solution, while in characteristics 0 or $p > 3$, we get two solutions for general λ. We find that these solutions come together when $1 - \lambda + \lambda^2 = 0$. Finally, in characteristic 3, we also see that the unique solution $f = \frac{x^3+(1+\lambda)x^2}{(1+\lambda)x+\lambda}$ specializes to an inseparable one when λ goes to -1.

We conclude with some further questions. We could reasonably start with remaining questions about the case of $\mathbb{P}^1$, including:

Question I.6.5. Is it true that for a given d and e_i, the number of maps is either always finite or always infinite as the P_i are allowed to move? Can we prove that it is always finite in the mid-characteristic case?

Question I.6.6. What happens in low characteristic when more than one ramification index is greater than p? Does the number of maps become finite again? If so, can we give a formula for it?

Question I.6.7. What can we say about the dimension of spaces of wildly ramified maps? When do wildly ramified maps exist for general ramification points?

This last question is explored further in [48].

Of course, one could ask the same questions about maps from higher-genus curves to $\mathbb{P}^1$. These have been answered in the case of characteristic 0 in [49], and the argument there would also apply in characteristic p given an appropriate generalization of Theorem I.5.1 to control the possibility of separable maps specializing to inseparable maps. The case of higher-dimensional linear series is still open, but may not be any harder than the one-dimensional case as far as controlling inseparable maps is concerned.

I.A Appendix: Moduli Schemes of Ramified Maps

The goal of this appendix is to construct moduli schemes of maps of curves required to have at least given ramification, but at unspecified points. Before we begin, we recall the well-known corollary of Grothendieck's work on the Hilbert scheme:

Theorem I.A.1. *Given X and Y two smooth, projective, geometrically connected curves over a locally Noetherian scheme S and a positive integer d, then the functor $\mathcal{M}or^d_S(X,Y)$ parametrizing degree d morphisms from X to Y over S is representable by a quasi-projective scheme. In particular, $\mathcal{A}ut_S(X) = \mathcal{M}or^1_S(X,X)$ is representable.*

Proof. Without the degree hypothesis, the functor is constructed in [23, p. 221-20] (where it is called Hom) as an open subscheme of the Hilbert scheme via the graph associated to a morphism. Now, if $\mathscr{L}$ and $\mathscr{M}$ are ample line bundles on X and Y, and f a morphism of degree d, the degree of the graph under the projective imbedding of $X \times_S Y$ induced by $\pi_1^*\mathscr{L} \otimes \pi_2^*\mathscr{M}$ will be $\deg(\mathscr{L} \otimes f^*\mathscr{M}) = \deg \mathscr{L} + d \deg \mathscr{M}$; in particular, this is different for each d, so gives a different Hilbert polynomial for each d, so the Mor scheme is naturally a disjoint union over all d of schemes representing Mor^d, each of which is quasi-projective, being an open subscheme of a Hilbert scheme for a fixed polynomial.

Note that a corollary of this argument is that degree really is well-defined in this context: the Hilbert polynomial and the degrees of $\mathscr{L}$ and $\mathscr{M}$ will all be preserved under base change, so the Hilbert polynomial of a map is determined by the degree on any fiber, which in turn determines the degree on any other fiber.

In order to see that $\mathcal{A}ut_S(X) = \mathcal{M}or^1_S(X,X)$, we first note that for any $d > 0$, $\mathcal{M}or^d_S(X,Y)$ consists entirely of scheme-theoretically surjective morphisms (in the sense that they do not factor through any proper closed subscheme of Y; since all relevant morphisms are proper, this is equivalent to set-theoretic surjectivity together with injectivity of the induced map on structure sheaves). Indeed, given $f \in \mathcal{M}or^d$, since X is flat over S, and for any $s \in S$, the map $f_s : X_s \to Y_s$ is a non-constant map between smooth curves and hence flat; by the criterion on flatness and fibers (see [63, Thm. 11.3.10]), f is also flat. But f is set-theoretically surjective because every f_s is, so we conclude that f is faithfully flat, and it is scheme-theoretically surjective onto Y. Now, to see that $\mathcal{A}ut_S(X) = \mathcal{M}or^1_S(X,X)$, it suffices to note that in our situation, one can check whether f is a closed immersion on each fiber f_s (see [59, Prop. 4.6.7]), so the desired assertion follows from the well-known case of smooth curves over $S = \operatorname{Spec} k$ (see, for instance, [59, Cor. 4.4.9]). □

We also have:

Proposition I.A.2. *With the notation of the preceding theorem, there exists an open subscheme* $\mathrm{Mor}^{d,\mathrm{sep}}_S(X,Y)$ *of* $\mathrm{Mor}^d_S(X,Y)$ *parametrizing morphisms which are separable on every fiber.*

Proof. Let $M := \mathrm{Mor}^d_S(X,Y)$, X_M and Y_M be the pullbacks of X and Y to M, $\tilde{f} : X_M \to Y_M$ be the universal morphism of degree d, defined over M; we get an induced map $\tilde{f}^*\Omega^1_{Y_M/M} \to \Omega^1_{X_M/M}$ of line bundles on X_M, with the kernel giving the locus on X_M where $\tilde{f}$ is ramified. The complement is an open set, and its image in M is clearly the locus of separable maps; since X is flat and of finite type over S, X_M is flat and of finite type over M, and in particular open, so we have constructed an open subscheme of M corresponding to separable maps, as desired. □

We also recall a standard construction involving the jet bundle, or bundle of principal parts $\mathscr{P}^n_{X/S}$, associated to an S-scheme X. The terminology and notation is not standard, however.

Definition I.A.3. We define the **nth cotangent bundle** $\Upsilon^n_{X/S}$ to be the kernel of the natural map $\mathscr{P}^n_{X/S} \to \mathscr{O}_X$; explicitly, consider $\mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{O}_X$ as an $\mathscr{O}_X$-module via left multiplication, and consider the natural map to $\mathscr{O}_X$ sending $a \otimes b$ to ab. Then if we denote the kernel of this map by $\mathscr{I}_{X/S}$, with the induced $\mathscr{O}_X$-module structure, $\Upsilon^n_{X/S} := \mathscr{I}_{X/S}/\mathscr{I}^{n+1}_{X/S}$.

We recall:

Proposition I.A.4. *With notation as in the preceding definition,*

(i) $\Upsilon^n_{X/S}$ *is compatible with base change.*

(ii) On affine opens, $\mathscr{I}_{X/S}$ *is generated by elements of the form* $a \otimes 1 - 1 \otimes a$*, for* $a \in \mathscr{O}_X$*.*

(iii) If X *is smooth over* S*,* $\Upsilon^n_{X/S}$ *is locally free.*

Proof. Compatibility with base change for $\mathscr{P}^n_{X/S}$ is [64, Prop. 16.4.5]; because $\Upsilon^n_{X/S}$ is the kernel of a map (clearly compatible with base change) to $\mathscr{O}_X$, and $\mathscr{O}_X$ is free, it follows that $\Upsilon^n_{X/S}$ is compatible with base change. (ii) is [61, Lem. 0.20.4.4]. Finally, (iii) follows from the same statement for $\mathscr{P}^n_{X/S}$, which is [64, Prop. 17.12.4], since $\Upsilon^n_{X/S}$ is the kernel of a surjective map from $\mathscr{P}^n_{X/S}$ to $\mathscr{O}_X$ (in fact, this is somewhat gratuitous, since the argument for $\mathscr{P}^n_{X/S}$ works without modification for $\Upsilon^n_{X/S}$). $\square$

We now specify in full detail the functor we wish to represent: for a pair of smooth, projective, geometrically connected curves X, Y over a locally Noetherian base S, n integers e_i, and $d \geq 1$, we consider the functor $\mathcal{MR}^d_S(X, Y, \{e_i\}_i)$ given by, for any scheme T over S:

$\mathcal{MR}^d_S(X, Y, \{e_i\}_i)$ is the set of separable morphisms f from X_T to Y_T over T of degree d, together with a choice of n disjoint T-valued points P_i of X_T, such that the fiber of $f(P_i)$ contains an e_ith-order thickening of P_i inside of X_T for each i.

Conceptually, this functor is the functor of maps f of degree d between X and Y, together with points P_i on X which are (at least) e_ith-order ramification points of f.

Our main result is:

Theorem I.A.5. *The functor $\mathcal{MR} = \mathcal{MR}_S^d(X, Y, \{e_i\}_i)$ is representable by a scheme* MR. *We also have the natural data of morphisms* ram : MR $\to X^n$ *and* branch : MR $\to Y^n$ *and actions of the group schemes* $\mathrm{Aut}(X)$ *and* $\mathrm{Aut}(Y)$ *on* MR *over* Y^n *and* X^n *respectively. Furthermore,* $\mathrm{Aut}(Y)$ *acts freely on* MR.

Proof. First we note that all the assertions other than representability can be verified simply on the functor level: the morphism ram is the forgetful transformation which takes a point of MR and remembers only the P_i; similarly, the morphism branch remembers the $f(P_i)$, which are sections of Y_T. A point g of $\mathrm{Aut}(Y)$ act on points of MR by sending f to $g \circ f$ and leaving the P_i fixed, and similarly $g \in \mathrm{Aut}(X)$ acts on MR by sending f to $f \circ g$ and the P_i to $g^{-1}P_i$, which fixes $f(P_i)$. The freeness of the $\mathrm{Aut}(Y)$ action follows easily from the statement that any point of MR corresponds to a scheme-theoretically surjective map, noted in the proof of Theorem I.A.5.

Clearly, we have a forgetful map from MR to $M = \mathrm{Mor}^{d,\mathrm{sep}}(X, Y)$; since the latter is representable, it will enough to show that the map of functors is also representable. In fact, if we use the convention that X_M^n denotes the product of n copies of X_M over M, the sections in the definition of our functor will allow us to describe MR as a closed subscheme of X_M^n with the pairwise diagonals removed. We claim that it is enough to handle the case $n = 1$: suppose we have done this case, and for each i let MR_i be the resulting scheme; then the product of the MR_i over M will nearly represent our functor, lacking only the disjointness hypothesis on the sections. However, we can consider this product as a closed subscheme of X_M^n, and simply removing the pairwise diagonals of this product will clearly give us the desired disjointness.

Since X and Y are smooth over S by hypothesis, $\Upsilon_{X/S}^{e-1}$ and $\Upsilon_{Y/S}^{e-1}$ are locally free, so the kernel of any morphism $f^*\Upsilon_{Y/S}^{e-1} \to \Upsilon_{X/S}^{e-1}$ is representable by a closed subscheme of Y over S, and the following lemma completes the proof of our theorem: $\square$

Lemma I.A.6. *Let $f : X \to Y$ be a morphism of separated S-schemes. Then there is a natural map $f^*\Upsilon_{Y/S}^{e-1} \to \Upsilon_{X/S}^{e-1}$ such that for any T over S, and any section $\sigma : T \to X_T$ we have:*

$(f^*\Upsilon_{Y/S}^{e-1} \to \Upsilon_{X/S}^{e-1})_{\sigma(T)} = 0$ if and only if the fiber of f_T over $f_T(\sigma(T))$ contains an eth order thickening of $\sigma(T)$ inside X_T.

Proof. The map from $f^*\Upsilon_{Y/S}^{e-1} \to \Upsilon_{X/S}^{e-1}$ is simply the one induced by $f^* \otimes f^* : f^{-1}\mathscr{O}_Y \otimes_{\mathscr{O}_S}$

$f^{-1}\mathscr{O}_Y \to \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{O}_X$. Our assertion is local, so we immediately reduce to affines, and consider the situation that $X_T = \operatorname{Spec} A$, $Y_T = \operatorname{Spec} B$, and $T = \operatorname{Spec} R$. Since X and Y are separated over S, a section is a closed immersion, so we also denote by I_σ the ideal corresponding to $\sigma(T)$ in X_T, and I'_σ the ideal of $A \otimes_R A$ given by $I_\sigma \otimes_R A$. In this situation, $(f^*\Upsilon^{e-1}_{Y/S} \to \Upsilon^{e-1}_{X/S})_{\sigma(T)} = 0$ if and only if $(f_T^*\Upsilon^{e-1}_{Y_T/T} \to \Upsilon^{e-1}_{X_T/T})_{\sigma(T)} = 0$, by Proposition I.A.4 (i), and this is equivalent to the assertion that for all $v \in \mathscr{I}_{Y_T/T}$, the image of f_T^* in $\mathscr{I}_{X_T/T}$ is actually in the ideal generated by I'_σ and $\mathscr{I}^e_{X_T/T}$. By Proposition I.A.4 (ii), $\mathscr{I}_{Y_T/T}$ is generated by elements of the form $b \otimes 1 - 1 \otimes b$ with $b \in B$, so the preceding is equivalent to the statement that for all $b \in B$, $f_T^*b \otimes 1 - 1 \otimes f_T^*b$ is in $(I'_\sigma, \mathscr{I}^e_{X_T/T})$. Now, this will be true if and only if it is true for all b with $f_T^*b \in I_\sigma$: the only observation is that since I_σ is the ideal of a section, we can write any $a \in A$, hence any f_T^*b, as $r + i$ where r is a pullback of an element of R, and $i \in I_\sigma$; then $f_T^*b \otimes 1 - 1 \otimes f_T^*b = i \otimes 1 - 1 \otimes i$, and $i = f_T^*(b - r)$.

We claim that our previous reduction is in turn is equivalent to the property that for all $b \in B$ with $f_T^*b \in I_\sigma$, $1 \otimes f_T^*b$ is in $(I'_\sigma, A \otimes I^e_\sigma)$; clearly, $f_T^*b \otimes 1 \in I'_\sigma$, so it suffices to show that $(I'_\sigma, A \otimes I^e_\sigma) = (I'_\sigma, \mathscr{I}^e_{X_T/T})$. The main observation is that once again using the fact that I_σ is the ideal of a section, $\mathscr{I}_{X_T/T}$ is in fact generated by elements of the form $i \otimes 1 - 1 \otimes i$, where $i \in I_\sigma$. Thus, $\mathscr{I}^e_{X_T/T}$ is generated by products $\prod_{j \leq e}(i_j \otimes 1 - 1 \otimes i_j)$, which are equivalent modulo I'_σ to elements of the form $1 \otimes (\prod_{j \leq e} i_j)$ with each $i_j \in I_\sigma$, proving the previous claim. But now we are nearly done: $A \otimes_R A$ modulo $(I'_\sigma, A \otimes I^e_\sigma)$, recalling that $I'_\sigma := I_\sigma \otimes A$, is isomorphic to $A/I_\sigma \otimes_R A/I^e_\sigma$, and since I_σ is the ideal of a section, $A/I_\sigma \cong R$, so $A/I_\sigma \otimes_R A/I^e_\sigma \cong A/I^e_\sigma$. Thus, $1 \otimes f_T^*b \in (I'_\sigma, A \otimes I^e_\sigma)$ if and only if $f_T^*b \in I^e_\sigma$, and we conclude that our original condition is equivalent to the statement that for all $b \in B$ with $f_T^*b \in I_\sigma$, we actually have $f_T^*b \in I^e_\sigma$. Finally, the fiber of $f(\sigma(T))$ is given by $\operatorname{Spec} A/I_\sigma \otimes_B A$, cut out in X_T by the ideal of $a \in A$ generated by f_T^*b, with $f_T^*b \in I_\sigma$. The fiber contains an eth order thickening of σ_T if and only if this ideal is contained in I^e_σ, if and only if for all b with $f_T^*b \in I_\sigma$, we actually have $f_T^*b \in I^e_\sigma$. This completely the proof of the lemma. $\square$

Remark I.A.7. The $\operatorname{Aut}(X)$ action is not free in general, often having a non-trivial finite sub-group scheme stabilizing any given morphism. However, it is easy enough to see that the stabilizer of any k-valued point $f \in \mathrm{MR}$ is in fact a finite group scheme; indeed, in this case, we may as well set $S = \operatorname{Spec}(k)$. Since $\operatorname{Aut}(X)$ is a finite-type group scheme, the stabilizer will likewise be a finite-type group scheme over k, and it thus suffices to show that it consists of only finitely many $\bar{k}$-valued points to see that it is in fact a finite group scheme. Now, an automorphism of $X_{\bar{k}}$ is determined on the generic point, and will have to fix $K(Y_{\bar{k}})$ inside $K(X_{\bar{k}})$ in order to fix f; since $K(Y_{\bar{k}})$ is a finite subfield of $K(X_{\bar{k}})$, the the relevant automorphism group is finite, and we conclude that we have finitely many $\bar{k}$-valued automorphisms of X over Y, as desired.

Remark I.A.8. The main arguments of this section immediately generalize to smooth, projective, geometrically connected schemes of higher relative dimension over the base S, with the only modification being that not all morphisms will be finite of a given degree. However, we restricted ourselves primarily to curves because only that case corresponds to a clearly useful notion of ramification.

Chapter II

The Limit G^r_d Moduli Scheme

In this chapter, we give a new construction for limit linear series, very much in the spirit of Eisenbud and Harris' theory in [15], but more functorial in nature, and involving a substantially new approach which appears better suited to generalization. This new construction also has the desirable properties that the resulting moduli scheme is always proper, and appears likely to be connected in at least some cases for reducible curves, which the Eisenbud-Harris construction never was. We should remark that we do not actually see any obstructions to their original construction working in characteristic p, but the independence of characteristic is more transparent to us in our construction. We begin with an overview of the basic ideas of limit linear series; for those unfamiliar with linear series, the actual definitions and notation are all recalled below.

While our main theorem is too technical to state in an introduction, we can outline the main concepts involved. The basic idea of limit linear series is to analyze how linear series behave as a family of smooth curves X/B degenerates to a nodal curve X_0; a key distinguishing feature of the theory is that rather than standard deformation-theoretic techniques to obtain results from the degeneration, a simple dimension count on the special fiber produces results immediately.

More specifically, recall that a proper, geometrically reduced and connected nodal curve with smooth components is said to be of **compact type** if the dual graph is a tree, or equivalently if the (connected component of the) Picard scheme is proper. Now, if X_0 is not of compact type, line bundles on the smooth curves may not limit to a line bundle on the nodal fiber, as the Picard scheme of the family (and specifically of the nodal fiber) will not be proper. On the other hand, if the nodal fiber is reducible, limiting line bundles will exist, but will not be unique, as one can always twist by one of the components of the reducible fiber to get a new line bundle, isomorphic away from the nodal fiber to the original one. However, this turns out to be the only ambiguity. To explain the approach to this issue, we consider for simplicity the case of g^r_d's where the family X/B has smooth general fiber, and X_0 consists of two smooth components Y and Z, meeting at a single node P. Given a line bundle of degree d on X, we will say it has degree $(i, d-i)$ on X_0 if it restricts to a line bundle of degree i on Y and degree $d-i$ on Z. Eisenbud and Harris approached the problem by considering the linear series obtained by looking at the two possible limit line bundles obtained by requiring degrees $(d, 0)$ and $(0, d)$ on X_0. Since the degree 0 components cannot contribute anything to the space of global sections chosen for the g^r_d, this is equivalent to specifying a g^r_d on each of Y and Z; they showed that if such a pair arises as a limit of g^r_d's from the smooth fibers, it will satisfy the ramification condition

$$a_i^Y(P) + a_{r-i}^Z(P) \geq d. \tag{II.0.1}$$

They refer to such pairs on a nodal fiber as **crude limit series**, and when the inequality is replaced by an equality, as **refined limit series**.

Eisenbud and Harris' moduli scheme construction requires restriction to refined limit series, and as such is not generally proper, and is also necessarily disconnected, being constructed as a disjoint union over the different possible ramification indices at the nodes. Moreover, the necessity to specify ramification indices makes it unsuitable for generalizing from curves to higher-dimensional varieties. The basic idea of our construction is to remember not just the line bundles of degree $(d, 0)$ and $(0, d)$ on X_0, but also the $d-1$ line bundles of degree $(i, d-i)$ that lie in between. One can then replace the ramification condition with a simpler compatibility condition on the corresponding spaces of global sections, yielding a very functorial approach to constructing the moduli scheme. Further, one can show a high degree of compatibility with Eisenbud and Harris' construction: in particular, for a curve (of compact type) over a field, our construction contains the Eisenbud-Harris version as an open subscheme. One interesting difference between the constructions is that when the Eisenbud-Harris scheme has the expected dimension, it is determinantal, which makes proving the required lower bound on the dimension very straightforward. Our construction does not have any obvious determinantal interpretation, so the dimension count is more difficult, and it is also possible that the resulting moduli scheme may be somewhat more pathological in general as a result.

We begin in Section II.1 with a review of the basics of linear series, but in arbitrary characteristic. In Section II.2 we give the precise conditions on the families of curves we will consider, and show that such families may be contructed as necessary. In Section II.3 we define the limit linear series functors we will consider, and our main theorem, the representability of these functors, is proved in Section II.4; we conclude with corollaries as in Eisenbud and Harris on smoothing linear series from the special fiber when the dimension is as expected, including in the cases of positive and mixed characteristic. We compare our theory to that of Eisenbud and Harris in Section II.5, and conclude with some further questions in Section II.6. Finally, in Appendix II.A we develop a theory of linked Grassmannian schemes, which parametrize collections of sub-bundles of a sequence of vector bundles linked together by maps between the bundles; this is used in the construction of the limit linear series scheme in the main theorem, and in particular to obtain the necessary lower bound on its dimension.

The work here is of course entirely inspired by Eisenbud and Harris' original construction in [15]. Attempts to generalize this theory have thus far been sparse, but include for instance work of Esteves [16] to generalize to certain curves not of compact type.

II.1 Linear Series in Arbitrary Characteristic

Before getting into the technical definitions related to the central construction, we begin with a few preliminary definitions and lemmas in the case of a smooth proper geometrically integral curve C of genus g, over a field k of any characteristic.

First, recall:

Definition II.1.1. If $\mathscr{L}$ is a line bundle of degree d on C, and V an $(r+1)$-dimensional subspace of $H^0(C, \mathscr{L})$, we call the pair $(\mathscr{L}, V)$ a g^r_d or a **linear series** of degree d and dimension r on C. Given $(\mathscr{L}, V)$ a g^r_d on C, and a point P of C, there is a unique sequence of $r+1$ increasing integers $a_i^{(\mathscr{L},V)}(P)$ called the **vanishing sequence** of $(\mathscr{L}, V)$ at P, given by the orders of vanishing at P of sections in V. We also define $\alpha_i^{(\mathscr{L},V)}(P) := a_i^{(\mathscr{L},V)}(P) - i$, the **ramification sequence** of $(\mathscr{L}, V)$ at P. $(\mathscr{L}, V)$ is said to be **unramified** at P if all

$\alpha_i^{(\mathscr{L},V)}(P)$ are zero; otherwise, it is **ramified** at P.

Warning II.1.2. Since the ramification and vanishing sequences are equivalent data, we tend to refer to conditions stated in terms of either one simply as "ramification conditions." We will also drop the $(\mathscr{L},V)$ superscript or replace it as appropriate, particularly when we have a linear series on each component of a reducible curve, when we will tend to simply use the component to indicate which series we are referring to.

The following definitions, being tailored to characteristic p, may be less standard:

Definition II.1.3. We say a linear series $(\mathscr{L},V)$ on C is **separable** if it is not everywhere ramified. Otherwise, it is **inseparable**. At a point P, we say that $(\mathscr{L},V)$ is **tamely ramified** if the characteristic is 0 or if the vanishing orders $a_i(P)$ are maximally distributed mod p (in particular, this holds at any unramified point). Otherwise, we say that $(\mathscr{L},V)$ is **wildly ramified** at P.

The following result is a characteristic-p version of a standard Plücker formula, whose proof simply adapts standard techniques:

Proposition II.1.4. *Let C be a smooth proper geometrically integral curve of genus g over a field k, and $(\mathscr{L},V)$ a g^r_d on C. Then either $(\mathscr{L},V)$ is inseparable, or we have the inequality*

$$\sum_{P\in C}\sum_i \alpha_i(P) \leq (r+1)d + \binom{r+1}{2}(2g-2).$$

Furthermore, this will be an equality if and only if $(\mathscr{L},V)$ is everywhere tamely ramified; in particular, in this case inseparability is impossible.

Proof. We simply use the argument of [14, Prop. 1.1]. Even though it is intended for characteristic 0, the proof follows through equally well in characteristic p for our modified statement, noting that their "Taylor expansion" map is defined independent of characteristic, and their formulas then hold on a formal level. Indeed, their argument shows that if $(\mathscr{L},V)$ induces a non-zero section $s(\mathscr{L},V)$ of $\mathscr{L}^{\otimes r+1}\otimes(\Omega^1_C)^{\otimes\binom{r+1}{2}}$, we get the desired inequality, with equality if and only if the determinant of their Lemma 1.2 is non-zero at all P (where, as in the proof of the proposition, $X_j := \alpha_j^{(\mathscr{L},V)}(P)$). In fact, if this determinant is non-zero anywhere, we see also that $s(\mathscr{L},V)$ has finite order of vanishing at that point, and cannot be the zero section. Next, their same lemma shows that their determinant will be non-zero at a point P if and only if $(\mathscr{L}, V)$ is tamely ramified at P. This means that if we show that inseparability corresponds precisely to having $s(\mathscr{L}, V) = 0$, we are done. But this also follows trivially, since on the one hand any unramified point is in particular tamely ramified, and will in fact give a non-vanishing point of $s(\mathscr{L}, V)$, and on the other hand, if $s(\mathscr{L}, V)$ is non-zero, we have seen that we can get only finitely many ramification points. $\square$

Note that because vanishing sequences are bounded by d, if $d < p$ wild ramification is not possible, so the previous proposition immediately implies:

Corollary II.1.5. *Wildly ramified or inseparable linear series of degree d are only possible when $d \geq p$.*

Finally, we have the notation:

Definition II.1.6. Given n points P_i and n ramification sequences $\alpha^i = \{\alpha^i_j\}_j$, we write $\rho := \rho(g, r, d; \alpha^i) := (r+1)(d-r) - rg - \sum_{i,j} \alpha^i_j$. This is the **expected dimension** of linear series of degree d and dimension r on a curve of genus g, with at least the specified ramification at the P_i.

II.2 Smoothing Families

In this section we describe the families of curves whose limit linear series we will study, called "smoothing families", and then give some basic existence results. While the definition of a smoothing family is rather technical, we expect that most applications will involve smoothing a given reducible curve over a one-dimensional base, so we conclude with a theorem giving the existence of such families satisfying all our technical conditions, given the desired reducible fiber. However, we work over a fairly arbitrary base, largely because this simultaneously handles another case that substantially precedes the theory of limit linear series, but tends to come up nonethless: a universal family of smooth curves, where we might want to show that a given property will hold for a general curve if it holds for a single one.

Our central technical definition is:

Definition II.2.1. A morphism of schemes $\pi : X \to B$, together with sections $P_1, \dots P_n : B \to X$ constitutes a **smoothing family** if:

(I) B is regular and connected;

(II) π is flat and proper;

(III) The fibers of π are genus-g curves of compact type;

(IV) The images of the P_i are disjoint and contained in the smooth locus of π;

(V) Each connected component Δ' of the singular locus of π maps isomorphically onto its scheme-theoretic image Δ in B, and furthermore $X|_{\pi^{-1}\Delta}$ breaks into two (not necessarily irreducible) components intersecting along Δ';

(VI) Any point in the singular locus of π which is smoothed in the generic fiber is regular in the total space of X;

(VII) There exist sections D_i contained in the smooth locus of π such that every irreducible component of any geometric fiber of π meets at least one of the D_i.

We begin with a lemma on two methods of obtaining new smoothing families from a given one:

Lemma II.2.2. *Let X/B, P_i be a smoothing family. Then*

(i) If $B' \to B$ is either a k-valued point of B for any field k, a localization of B, or a smooth morphism with B' connected, then base change to B' gives a new smoothing family.

(ii) If Δ' is a node of X/B which is not smoothed in the generic fiber, let Y, Z be the components of X with $Y \cup Z = X$, $Y \cap Z = \Delta'$. Then restriction to Y or Z gives a new smoothing family.

Proof. For (i), the only properties of a smoothing family not preserved under arbitrary base change are (I) and (VI); these are visibly automatic in the cases $B' = \operatorname{Spec} k$ or B' a localization of B, while the case of B' smooth over B and connected follows from the fact a scheme smooth over a regular scheme is regular (see [3, Prop. 2.3.9], for instance).

For (ii), the only condition that isn't immediately clear is that flatness is preserved. However, we have an exact sequence of sheaves (on X) $0 \to \mathscr{O}_X \to \mathscr{O}_Y \oplus \mathscr{O}_Z \to \mathscr{O}_{Y\cap Z} \to 0$; $\mathscr{O}_X$ is flat over $\mathscr{O}_B$ by hypothesis, and $\mathscr{O}_{Y\cap Z}$ is actually assumed to be isomorphic to $\mathscr{O}_B$, so we conclude that $\mathscr{O}_Y \oplus \mathscr{O}_Z$ and hence $\mathscr{O}_Y$ and $\mathscr{O}_Z$ are flat over $\mathscr{O}_B$, and restriction to either Y or Z does in fact yield a new smoothing family. $\square$

We now proceed to develop some results on construction of smoothing families.

Lemma II.2.3. *Let $\pi : X \to B$ be a family satisfying conditions (I)-(III) and (VI) of a smoothing family, $\bar{X}_0$ a chosen geometric fiber of π mapping to a point $P \in B$, and $\bar{P}_i$ smooth closed points on $\bar{X}_0$ with images in X_P having residue fields separable extensions of $\kappa(P)$. Suppose further that each component Δ' of the singular locus of X/B is flat over its image Δ in B. Then there is an etale base change of π and sections P_i specializing to the $\bar{P}_i$ which yield a smoothing family still containing $\bar{X}_0$ as a geometric fiber, and with the same geometric generic fiber as π.*

Proof. First, localize B so as to avoid any components of the singular locus not occurring in X_0. Then, in addition to the $\bar{P}_i$, choose one smooth closed point $\bar{D}_i$ on each component of X_0, each having field of definition a separable extension of $\kappa(P)$ (this is possible because X_0 is geometrically reduced by hypothesis, so smooth on a non-empty open set, and then the set of points with separable fields of definition is dense, see [3, Prop. 2.2.16 and Cor. 2.2.13]). Next, by [3, Prop. 2.2.14], after possible etale base change we can find the desired sections P_i and D_i of π, each going through the corresponding $\bar{P}_i$ or $\bar{D}_i$; we can (Zariski) localize the base once more to obtain the desired disjointness of the P_i and to ensure that all P_i and D_i remain in the smooth locus (the latter is automatically satisfied for nodes which are smoothed generically, but not for those which are not).

All that remains is to show that we can obtain condition (V) as well. We begin with the hypothesis that all the nodes map isomorphically to their images. By Theorem A.2 the singular locus of a family of nodal curves is finite and unramified over the base, so our hypothesis that each connected component Δ' of the singular locus is also flat over its image Δ implies that Δ' is etale over Δ, and by [3, Prop. 2.3.8 b)], after an etale base change of Δ, Δ' will map isomorphically to Δ. In the case that $\Delta = B$ we will be done, while in the case $\Delta \neq B$ we simply apply [51, Cor. V.1, p. 52] to lift our etale cover of Δ to an etale cover of B, at least locally around the image of X_0.

Finally, we need to make sure that X breaks into components around each node. For each connected component Δ' of the singular locus of π, if we can produce an etale base change which causes the generic fiber X_1^{Δ} of $X|_{\Delta}$ to break, $X|_{\Delta}$ will break as well. By hypothesis,

X_1^Δ breaks geometrically, and it will break into components over some given intermediate field K if and only if the geometric components are $\mathrm{Gal}(\bar{K}/K)$-invariant. But by [10, Lem. 4.2], there is in fact a finite etale base change of Δ such that the geometric generic fiber of the base change is in natural correspondence as a Galois set with the geometric components of X_1^Δ. In particular, after this base change, any generic fiber lying over X_1^Δ will have all its geometric components Galois-invariant, and hence defined over its field of definition. If $\Delta = B$, we are done; if not, we apply the same result as above to obtain an etale base change of B specializing to our given one on Δ. $\square$

For typical applications of limit linear series, we expect that the following theorem, which follows fairly easily from a theorem of Winters, will render irrelevant the technical hypotheses of our smoothing families:

Theorem II.2.4. *Let X_0 be any curve of compact type over an algebraically closed field k, and $\bar{P}_1, \dots \bar{P}_n$ distinct smooth closed points. Then X_0 may be placed into a smoothing family X/B with sections P_i specializing to the $\bar{P}_i$, where B is a curve over k, and where the generic (and hence general) fiber of X over B is smooth.*

Proof. Since the compact type hypotheses include that X_0 is reduced, has smooth components, and only nodes for fibers, we find that X_0 is locally planar, and has normal crossings. Then setting all $m_i = 1$, we can apply [56, Prop. 4.2] to obtain a proper map over $\operatorname{Spec} k$ from some regular surface $\tilde{X}$ to some regular curve $\tilde{B}$, having X_0 as a fiber. This must automatically be flat, for instance by [13, Thm. 18.16b], and noting that in this case the fiber dimension must be constant. Now, we localize $\tilde{B}$ if necessary so that all fibers are at most nodal. We then claim that the generic fiber X_1 must be smooth: the locus of nodes in fibers of $\tilde{X}/\tilde{B}$ must be finite and unramified over $\tilde{B}$ by Theorem A.2, so if they exist in X_1, they correspond to a finite set of points, which, being unramified, must have residue fields given by separable extensions of $K(B)$. However, since we are looking at the generic fiber, the local rings in the fiber are the same as in all of $\tilde{X}$, which is regular by hypothesis, so [13, Cor. 16.21] implies that any such points would be smooth points, and could not be nodes by definition, yielding the smoothness of X_1. Now note that the algebraically closed hypothesis renders any issues of separability of residue extension field moot. Further, our nodes are all isolated points, so we claim the flatness of each connected component of the singular locus is automatic: in fact the nodes are reduced by the final remark below, but even if they weren't, they would each map isomorphically onto their image, since the map is given by a finite, unramified map of local schemes with algebraically closed residue field. Therefore, we can apply the preceding lemma to obtain our desired smoothing family. $\square$

Remark II.2.5. There are a number of differences between our definition of a smoothing family, and the one used in Eisenbud and Harris' original construction in [15]. None of these are due to the different construction. Extra conditions such as the reducedness of B and the regularity of X at smoothed nodes are in fact necessary to ensure that certain closed subschemes are actually Cartier divisors, and the condition that X break into distinct components above the nodes is likewise tacitly assumed, but not automatic. The regularity of B is necessary to make the sort of dimension-count arguments employed in the construction. Conversely, the hypotheses on the characteristic (or even existence) of a base field appears to be unnecessary in their construction, as does the hypothesis that the relatively ample divisor be disjoint from the ramification sections. The only hypothesis we include here that may be truly gratuitous is that the relatively ample divisor be composed of global sections, but it is convenient and, as we have shown, not difficult to achieve. Presumably, we could also relax the compact type hypothesis and still achieve smoothing results, as long as we did not require properness for our limit series moduli scheme.

Remark II.2.6. We do not claim that the moduli scheme could not be constructed under weaker hypotheses, but merely that our hypotheses are those which are necessary for our particular argument. It seems quite likely that one could drop many of the hypotheses on both X and B if, for instance, one were to first make the construction over a universal deformation space, or even the moduli space of curves itself, and then obtain it for any family via base change.

Remark II.2.7. It is not true that condition (VI) of a smoothing family is preserved under base change by arbitrary closed immersions $B' \to B$, even when B' is regular and connected. For example, consider any smoothing family with $B = \mathbb{A}^2_k$, and having a node Δ' with Δ given by the x-axis. Then if B' is the parabola $y = x^2$, base change to B' will create a singularity in X above the origin.

Remark II.2.8. In fact, the hypotheses for a smoothing family π imply that every connected component of the singular locus of π is regular, and in particular irreducible and reduced. However, we will not need this, so we do not pursue it.

II.3 The Relative $\mathcal{G}^r_d$ Functor

Given, in addition to a smoothing family, integers r, d, and ramification sequences $\alpha^i := \{\alpha^i_j\}_j$ for each of our P_i, we will associate a $\mathcal{G}^r_d$ functor to our smoothing family; this functor will initially appear to include a lot of extraneous data, but we will show that it actually gives the "right" functor, at least in the sense that it associates a reasonable set to any geometric point of B.

We will work with a very simple smoothing family, in order to avoid drowning in notation, in essence restricting our families to reducible curves with only two components. Although this may seem like a severe restriction, in practice any argument which could be made with more components ought to be approachable by instead inductively using degenerations to curves with only two components.

Situation II.3.1. We assume that X/B is a smoothing family with at most one node (in the sense that the singular locus of π is irreducible). If there is a node, we introduce some notation: denote by Δ' the singular locus of π, and Δ its image in B; by hypothesis, π maps Δ' isomorphically to Δ. We now distinguish three cases: case (1) is that there is no node; case (2) is that Δ is all of B; and case (3) is that Δ is a Cartier divisor on B. In case (2), we denote by Y and Z the components of X intersecting along Δ'.

We claim that with the specified hypotheses, these three cases are all the possibilities: by Theorem A.2, if Δ is non-empty, it is locally generated principally; if it is 0, Δ is all of B, and X is reducible. Otherwise, the integrality of B ensures that Δ is a Cartier divisor.

Now, let Y and Z be the components of $X|_{\pi^{-1}\Delta}$, necessarily irreducible and intersecting along Δ'. In case (2), we will make use of the morphism (actually an isomorphism onto a connected component) $\mathrm{Pic}^{d-i}(Y_T/T) \times \mathrm{Pic}^i(Z_T/T) \to \mathrm{Pic}^d(X_T/T)$ for any i and any T over B, obtained from Lemma A.9, in order to think of a pair of line bundles $\mathscr{L}_Y, \mathscr{L}_Z$ on Y_T and Z_T as a line bundle on X_T, which we will denote $(\mathscr{L}_Y, \mathscr{L}_Z)$. In case (3), by the nonsingularity hypothesis, Y and Z are Cartier divisors in X, so we have associated line bundles on X, $\mathscr{O}_X(Y)$ and $\mathscr{O}_X(Z)$. Moreover, because Δ is a Cartier divisor on B, and $\mathscr{O}_X(Y+Z) \cong \mathscr{O}_X(\pi^*\Delta) \cong \pi^*\mathscr{O}_B(\Delta)$, we have that locally on B (that is, when X is restricted to sufficiently small open sets in B), $\mathscr{O}_X(Y+Z) \cong \mathscr{O}_X$. In order to ensure that our functor is globally well-defined, we will need the following lemma:

Lemma II.3.2. *Let $\pi : X \to B$ be a proper morphism with geometrically reduced and connected fibers, $\mathscr{L}$ and $\mathscr{L}'$ two isomorphic line bundles on X, and V and V' sub-modules of $\pi_*\mathscr{L}$ and $\pi_*\mathscr{L}'$ respectively. Then the property that "V maps into V'" is independent of the choice of isomorphism between $\mathscr{L}$ and $\mathscr{L}'$.*

Proof. $\mathrm{Hom}(\mathscr{L}, \mathscr{L}') \cong \mathscr{L}^{-1} \otimes \mathscr{L}' \cong \mathscr{O}_X$, by hypothesis. Thus, any two choices of isomorphism must differ by a unit in $H^0(X, \mathscr{O}_X)$, or equivalently, $H^0(B, \pi_*\mathscr{O}_X)$. By Lemma A.24, $\pi_*\mathscr{O}_X \cong \mathscr{O}_B$, so any two isomorphisms between $\mathscr{L}$ and $\mathscr{L}'$ must differ by (the pullback of) a unit in $\mathscr{O}_B$. Since a choice of sub-module V' of $\pi_*\mathscr{L}'$ is invariant under multiplication by a unit in $\mathscr{O}_B$, we find that whether or not V is mapped into V' under the isomorphism $\pi_*\mathscr{L} \to \pi_*\mathscr{L}'$ induced by a choice of isomorphism between $\mathscr{L}$ and $\mathscr{L}'$ is in fact independent of choice of isomorphism, as asserted. □

Given a morphism $f : T \to B$, despite the notational headache, for the sake of preciseness we denote by a subscript T the various pullbacks under f; we now describe the set of objects our functor $\mathcal{G}^r_d(X/B, \{(P_i, \alpha^i)\})$ associates to T, first giving a conceptual overview of each case, and then pinning down the technical details. The set associated to T will consist of all objects of the form:

case (1): a line bundle $\mathscr{L}$ of degree d on X_T, and a rank r sub-bundle V of $\pi_{T*}\mathscr{L}$, with the desired ramification along the $P_{i,T}$.

case (2): a line bundle $\mathscr{L}$ of degree d on X_T, which has degree d when restricted to Y_T, and degree 0 when restriced to Z_T, together with rank r sub-bundles V_i of $\pi_{T*}\mathscr{L}^i$, where $\mathscr{L}^i := (\mathscr{L}|_{Y_T}(-i\Delta'_T), \mathscr{L}|_{Z_T}(i\Delta'_T))$. Each V_i must map to V_{i+1} under the natural map given by inclusion on Z_T and 0 on Y_T, and each V_i must map to V_{i-1} under inclusion on Y_T and 0 on Z_T. Finally, we impose the desired ramification along the $P_{i,T}$ as in the first case, with the caveat that we impose it only on V_0 if P_i is on Y, and only on V_d if P_i is on Z.

case (3): a line bundle $\mathscr{L}$ of degree d on X_T, which has degree d when restricted to Y_T, and degree 0 on Z_T, and rank r sub-bundles V_i of $\pi_{T*}(\mathscr{L}^i)$, where $\mathscr{L}^i := (\mathscr{L} \otimes \mathscr{O}_X(Y)_T^{\otimes i})$, for $0 \leq i \leq d$. Each V_i must map to V_{i+1} under the natural map $\pi_{T*}(\mathscr{L}^i) \to \pi_{T*}(\mathscr{L}^{i+1})$. Further, locally on T, Δ_T will be principal, so that $\mathscr{O}_T(\Delta_T) \cong \mathscr{O}_T$, and $\mathscr{O}_X(Y+Z)_T \cong \mathscr{O}_{X,T}$. On any such open of T, we require that V_i map to V_{i-1} under the natural map $\pi_{T*}(\mathscr{L}^i) \to \pi_{T*}(\mathscr{L}^i \otimes \mathscr{O}_X(Z)_T)$, composed with the map to $\pi_{T*}(\mathscr{L}^{i-1})$ induced by some choice of isomorphism $\mathscr{O}_X(Y+Z)_T \cong \mathscr{O}_{X,T}$. The previous lemma insures that this condition is independent of the choice of isomorphism, and therefore makes sense globally on T. Finally, we impose the desired ramification along the $P_{i,T}$ as in the first two cases, imposing it only on V_0 if P_i specializes to Y, and only on V_d if P_i specializes to Z.

Remark II.3.3. The definition of $\mathcal{G}^r_d$ in cases (2) and (3) depends *a priori* on the choice of Y and Z, but it isn't hard to see that in fact it doesn't. This is trivial in case (2), since the definition is visibly symmetric, while in case (3), we seem to run afoul of the fact that $\mathscr{O}_X(Y)$ need not be globally isomorphic to $\mathscr{O}_X(-Z)$. However, they are locally isomorphic on the base, and by virtue of Lemma II.3.2 and the equivalence on the Picard functor specified immediately below, this will be enough to establish a correspondence between $\mathcal{G}^r_d$ as defined, and $\mathcal{G}^r_d$ with the roles of Y and Z switched.

We now specify the technical details:

First of all, there is a necessary equivalence condition on line bundles in order to obtain representability of the Picard functor. We will consider two relative line bundles to be equivalent if they become isomorphic Zariski-locally on T. In particular, the $\mathscr{L}^i$ in case (2) are defined only up to isomorphism locally on T. However, this isn't a problem because by Lemma II.3.2, we get a well-defined equivalence relation on our functor, since the particular choice of local isomorphisms doesn't affect anything.

By sub-bundle, since $\pi_{T*}\mathscr{L}$ is not necessarily locally free, we cannot require that the quotient be locally free. Rather, we require:

Definition II.3.4. We define V to be a **sub-bundle** of $\pi_{T*}\mathscr{L}$ if in addition to V being a locally free sheaf, for any $S \to T$, the map $V_S \to \pi_{S*}\mathscr{L}_S$ remains injective.

Note that in this definition, we are pushing forward the pullback of $\mathscr{L}$, and not the other way around. The required sheaf map is gotten by composing the induced map $V_S \to (\pi_{T*}\mathscr{L})_S$ with the natural map $(\pi_{T*}\mathscr{L})_S \to \pi_{S*}\mathscr{L}_S$.

Finally, ramification is imposed by considering the sequence of maps

$$V \to \pi_{T*}\mathscr{L}|_{(d+1)P_{i,T}} \to \pi_{T*}\mathscr{L}|_{dP_{i,T}} \to \dots \pi_{T*}\mathscr{L}|_{P_{i,T}} \to 0$$

We denote by β_m the composition map $V \to \pi_{T*}\mathscr{L}|_{mP_{i,T}}$. Then β_{d+1} automatically has rank $r+1$, and β_0 has rank 0, and to get the desired vanishing sequence at $P_{i,T}$, we bound the ranks of the β_m from above, increasing precisely at the a^i_j; that is, $\operatorname{rk}\beta_m \leq j$ for all

$m \leq a^i_j$. As before, $a^i_j := \alpha^i_j + j$. Of course, this condition only has to be imposed once per a^i_j, at $m = a^i_j$, and will then be automatic for all m less than a^i_j but greater than a^i_{j-1}.

Intuitively and in applications, $\mathcal{G}^r_d$ should be thought of primarily as parametrizing, for any closed point x of B, the limit series on the fiber of X over x. We also have the subfunctor of $\mathcal{G}^r_d$, which we denote by $\mathcal{G}^{r,\mathrm{sep}}_d$, consisting of those linear series which are separable in every fiber: it is clear what this means for smooth curves, while for reducible curves we require both $V_0|_Y$ and $V_d|_Z$ to be separable.

One can verify quite directly that the $\mathcal{G}^r_d$ we have defined is in fact a functor. However, since we defined it differently in three separate cases, we also need to check:

Lemma II.3.5. *$\mathcal{G}^r_d$ and $\mathcal{G}^{r,\mathrm{sep}}_d$ are compatible with base change.*

Proof. Suppose we pull back from X/B to a new smoothing family X'/B'. The first and second cases are preserved under pullback, so here it really is trivial to verify that our construction is compatible with base change. However, the third case can pull back to any of the three. If X'/B' is still in the third case, it is no subtler than the first two cases. If B' misses Δ, X'/B' will be in the first case, and we have to verify that we still get the right functor. The only real observation here is that $\mathscr{O}_X(Y)$ and $\mathscr{O}_X(Z)$ pull back to the trivial bundle bundle in this case, so in fact on X'/B' all the maps between our $\mathscr{L}^i$ will be isomorphisms, and any V_i (along with its ramification conditions) completely determine the others, so we do in fact get the right functor. On the other hand, if B' is contained in Δ, X'/B' ends up in the second case. Here, the point is that $\mathscr{O}_X(Y)$ clearly pulls back to $\mathscr{O}_{Z'}(\Delta')$ on Z'. Then since $\mathscr{O}_X(Z)$ likewise pulls back to $\mathscr{O}_{Y'}(\Delta')$ on Y', and $\mathscr{O}_X(Z)$ is locally on B the inverse of $\mathscr{O}_X(Y)$, we find that up to the equivalence in the definition of the Picard functor, $\mathscr{O}_X(Y)$ must pull back to $\mathscr{O}_{Y'}(-\Delta')$, and we once again get the correct functor, completing the proof of compatibility with base change.

Finally, because $\mathcal{G}^{r,\mathrm{sep}}_d$ was defined as a sub-functor of $\mathcal{G}^r_d$ in terms of behavior on fibers, it immediately follows that it too is compatible with base change. □

Warning II.3.6. While we will later show compatibility with Eisenbud-Harris refined limit series over a field, note that in fact the compactification we obtain will have many points for each Eisenbud-Harris crude limit series, as there may be a positive-dimensional space of ways to fill in $V_1, \dots V_{d-1}$ when a given V_0 and V_d correspond to a crude limit series.

II.4 Representability

The main theorem is the representability of our $\mathcal{G}^r_d$ functors. However, to ease the pain of the proof, we begin with some technical lemmas before proceeding to the statement and proof of the main theorem.

We begin with some compatibility checks on our notion of sub-bundle:

Lemma II.4.1. *Our notion of sub-bundle has the following desireable properties:*

(i) Suppose we have $\mathscr{L}$ such that $\pi_*\mathscr{L}$ is locally free, and the higher derived pushforward functors vanish. Then our definition of sub-bundle of $\pi_*\mathscr{L}$ is equivalent to the usual one (that is, a locally free sub-sheaf with locally free quotient).

(ii) If D is an effective Cartier divisor on X, flat over B, and $\mathscr{L}$ any line bundle on X, then a sub-sheaf V of $\pi_*\mathscr{L}$ is a sub-bundle of $\pi_*\mathscr{L}$ if and only if it is a sub-bundle of $\pi_*\mathscr{L}(D)$ under the natural inclusion.

(iii) Let V_1, V_2 be sub-bundles of rank r of $\pi_*\mathscr{L}$ in our sense, and suppose $V_1 \subset V_2$. Then $V_1 = V_2$.

Proof. For (i), we make use of the facts that for a locally finitely-presented $\mathscr{O}_X$-module, flatness is equivalent to being locally free, (see [13, Exer. 6.2]), and that flatness is also equivalent to the vanishing of $\mathrm{Tor}^1_S(\mathscr{O}_S, Q_S)$ for all base changes S (in fact, it suffices to consider finitely presented closed immersions; see [13, Prop. 6.1]). We will also use that under the hypotheses on $\mathscr{L}$, for any base change S/B, the natural map $(\pi_*\mathscr{L})_S \to \pi_*\mathscr{L}_S$ is an isomorphism (see [60, Cor. 6.9.9] and the comments [60, 6.2.1]). Now, if the quotient $Q := \pi_*\mathscr{L}/V$ is locally free, it is flat, and the injectivity of $V \to \pi_*\mathscr{L}$ is preserved under base change, so applying $(\pi_*\mathscr{L})_S \cong \pi_*\mathscr{L}_S$, we get $V_S \hookrightarrow \pi_*\mathscr{L}_S$, as desired. Conversely, once again using $(\pi_*\mathscr{L})_S \cong \pi_*\mathscr{L}_S$, $V_S \hookrightarrow \pi_*\mathscr{L}_S$ gives us that the injectivity of $V \to \pi_*\mathscr{L}$ is preserved under base change; this in turn implies that $\mathrm{Tor}^1_S(\mathscr{O}_S, Q_S) = 0$ for all S, since $\pi_*\mathscr{L}$, being locally free and hence flat, has vanishing Tor. This implies that Q is flat, and since Q will be finitely presented, this gives us that Q is locally free, completing the proof of the assertion.

(ii) will follow immediately if show that $\pi_*\mathscr{L}_S \to \pi_*\mathscr{L}(D)_S$ is injective for all S. However, $\mathscr{L}_S \to \mathscr{L}(D)_S$ is injective for all S thanks to the flatness of D over B, since the cokernel $\mathscr{L}(D)|_D$ will be flat over B, and hence has vanishing Tor^1 for any $\mathscr{O}_B$-module. Then injectivity is trivially preserved under pushforward, so we are done.

(iii) is straightforward: let $Q = V_2/V_1$, so that we have

$$0 \to V_1 \to V_2 \to Q \to 0$$

on B, and let $b \in B$ be any point of B. If we base change to $\operatorname{Spec}\kappa(b)$, we get

$$V_{1b} \to V_{2b} \to Q_b \to 0.$$

Now, $V_{1b} \to \pi_* \mathscr{L}_b$ factors through V_{2b}, so by the definition of V_{1b} being a sub-bundle, we must have $V_{1b} \hookrightarrow V_{2b}$. But both V_{1b} and V_{2b} are r-dimensional vector spaces over $\kappa(b)$, so injectivity implies surjectivity, and we find that $Q_b = 0$, and hence the stalk of Q at b must be zero by Nakayama's lemma. Since b was arbitrary in B, we have $Q = 0$, and $V_1 = V_2$, as asserted. $\square$

We also have a lemma illustrating how we will use our sections D_i:

Lemma II.4.2. *Let X/B be a smoothing family, and $\mathscr{L}^i$ any finite collection of line bundles on X, of degree d. Then locally on B, there exists an effective divisor D on X satisfying:*

(i) D is supported in the smooth locus of π, and is disjoint from the P_i.

(ii) $\pi_* \mathscr{L}(D)$ is locally free and $R^i \pi_* \mathscr{L}(D) = 0$ for $i > 0$.

(iii) $\pi_* \mathscr{L}(D) \to \pi_* \mathscr{L}(D)|_{jP_i}$ is surjective for all i and all $j \leq d+1$.

Proof. With D_i any collection of sections as in the definition of a smoothing family, let $D' = \sum_i D_i$; then locally on B, for m sufficiently large, mD' will have the following properties:

(i) $\mathscr{O}_X(mD')$ is very ample;

(ii) $H^i(X, \mathscr{L}^\ell(mD')) = 0$ for all $i > 0$ and all ℓ on every fiber of π;

(iii) $H^1(X, \mathscr{L}^\ell(mD' - jP_i)) = 0$ for all $j \leq d+1$ and all ℓ on every fiber of π.

Indeed, Proposition A.5 implies that D' is π-ample. Then, locally on B we have that D' is ample, so sufficiently large multiples are very ample by [26, Prop. III.7.5]. Noting that (iii) may be replaced by the condition that $H^1(X, \mathscr{L}^\ell(mD' - jP_i)) = 0$ on fibers, the usual Riemann-Roch/Serre duality arguments (using the nodal version of these versions, as in Theorem A.4) imply that the lower bound on m for (ii) and (iii) depends only on d and the genus of the curves in the family, so may be chosen independent of fiber. Next, again locally on the base, we can choose D to be any divisor linearly equivalent to $m'D$ which is disjoint from the P_i and singular locus π; this is possible because $m'D$ is very ample. D then satisfies assertion (i) of the lemma, and conditions (ii) and (iii) above are still satisfied with D in place of mD', as they only depend on the isomorphism class of the associated line bundle. Finally, it follows from invariance of the Euler characteristic [60, Thm. 7.9.4] that $H^0(X, \mathscr{L}^\ell(D))$ is constant on fibers, and then cohomology and base change (see Theorem A.32) yields the desired assertions (ii) and (iii) of the lemma, noting that $R^1\pi_*(\mathscr{L}^\ell(D - jP_i)) = 0$ implies (iii). $\square$

We can now prove our central result:

Theorem II.4.3. *If $\pi : X \to B$, $P_1, \dots P_n : B \to X$ is a smoothing family satisfying the two-component hypothesis of Situation II.3.1, and $\alpha^i := \{\alpha^i_j\}_j$ ramification sequences, then $\mathcal{G}^r_d = \mathcal{G}^r_d(X/B; \{(P_i, \alpha^i)\}_i)$ is represented by a scheme G^r_d, compatible with base change to any other smoothing family. This scheme is projective, and if it is non-empty, the local ring at any point $x \in G^r_d$ closed in its fiber over $b \in B$ has dimension at least $\dim \mathscr{O}_{B,b} + \rho$, where $\rho = \rho(g, r, d; \alpha^i)$ as in Definition II.1.6. Furthermore, $\mathcal{G}^{r,\mathrm{sep}}_d$ is also representable, and is naturally an open subscheme of G^r_d.*

Proof. Once the $\mathcal{G}^r_d$ functor has been defined, the proof of its representability is long but for the most part extremely straightforward, using nothing more than the well-known representability of the various functors in terms of which we have described $\mathcal{G}^r_d$. The one trick, which was also used by Eisenbud and Harris but which apparently goes back to classical constructions, is to twist a universal line bundle $\mathscr{L}$ by a high power of an ample divisor so that its pushforward will be locally free, and its sub-bundles will be parametrized simply by a standard Grassmannian scheme. The dimension count is an altogether different story; it is harder than in Eisenbud and Harris' construction, and is essentially the subject of the appendix to this chapter. Finally, the representability of the subscheme of separable limit series does require a few tricks, which are described below.

We begin by remarking that, given a functor F from schemes over B to sets, if we wish to construct a scheme over B representing F, it will suffice to make the construction locally on B if and only if we can show that when F is restricted to the category of Zariski open subschemes of B, it gives a sheaf of sets on B. Because line bundles and vanishing conditions are determined locally on the base (at least, given our equivalence condition on line bundles), this condition is satisfied by our $\mathcal{G}^r_d$ functor, and in constructing the representing scheme, we can restrict to open subsets of B whenever desired.

As in defining the functor, we have three cases to consider. The first is the simplest. We start in this case with the relative Picard scheme $P = \mathrm{Pic}^d(X/B)$ (see Theorem A.7), and the universal line bundle $\tilde{\mathscr{L}}$ on $X \times_B P$. Working locally on B, let D be the divisor provided by Lemma II.4.2 for $\tilde{\mathscr{L}}$, viewing $X \times_B P$ as a smoothing family over P. Now, if d were large relative to the genus g of the curves in the family, we would have that any line bundle of degree d would have a $(d+1-g)$-dimensional space of global sections, and we could construct a G^r_d scheme simply by taking a relative Grassmannian scheme of $\pi_{P*}\tilde{\mathscr{L}}$ over P. Of course, d need not be large, so we simply cheat by artificially raising the degree of $\tilde{\mathscr{L}}$. Specifically, we let G be the relative Grassmannian scheme of $\pi_{P*}(\tilde{\mathscr{L}}(D))$; by construction, this is locally free, so the relative Grassmannian scheme G/P exists (Theorem A.11). We define our G^r_d scheme to be the closed subscheme of G cut out by two conditions: first, that any sub-bundle V of $\pi_{P*}(\tilde{\mathscr{L}}(D))$ vanishes on D, and second, that each of our vanishing sequence conditions is satisfied at the P_i, in the obvious way from our explicit definition of the ramification conditions on the functor. The first insures that we actually only get sub-bundles which actually sit inside of $\pi_{P*}\tilde{\mathscr{L}}$ and the second gives us the desired ramification behavior at the P_i; both can be described as rank restrictions on maps between vector bundles, so naturally give a closed subscheme cut out locally by minors. This completes the construction in the first case.

Now, in the second case, we use the Picard schemes $P^i := \mathrm{Pic}^{d-i,i}(X/B)$ of Theorem A.7, the schemes parametrizing line bundles on X with degrees $d-i$ and i when restricted to Y and Z respectively. These are all naturally isomorphic to one another, making use of the decomposition of both Picard schemes as products of Picard schemes for Y and Z (Lemma A.9), and tensoring as many times as necessary by $\mathscr{O}_Y(\Delta')$ on Y and $\mathscr{O}_Z(-\Delta')$ on Z; in particular, we can identify all of them with a fixed P over B. On each P^i, we have a universal line bundle $\tilde{\mathscr{L}}^i$, and just as in the first case, we take a very ample divisor D obtained from Lemma II.4.2 for the $\tilde{\mathscr{L}}^i$, twist $\tilde{\mathscr{L}}^i$ by D, and then construct Grassmannian bundles G^i, this time one for each $\tilde{\mathscr{L}}^i$. Denoting by G the product of all these Grassmannians over P, we take the closed subscheme inside G cut out by, as in the first case, vanishing on D and the required ramification conditions along the P_i. Here, we actually write $D = D^Y + D^Z$, where D^Y and D^Z are supported on Y and Z respectively; we can do this because D is disjoint from $\Delta' = Y \cap Z$. We then impose vanishing along D^Y only in G^0, and along D^Z only in G^d. Also, as in the definition of the $\mathcal{G}^r_d$ functor, ramification conditions will be imposed only in G^0 or G^d as appropriate. Finally, we make use of the construction Lemma II.A.3 in the following appendix to add the requirement that the V_i each map into V_{i+1} on Z and V_{i-1} on Y under the natural maps, also as in the definition of the functor. This completes the construction in the second case.

In the final case, the first step is to work sufficiently locally on B that Δ is principal, so that $\mathscr{O}_B(\Delta) \cong \mathscr{O}_B$ and $\mathscr{O}_X(Y+Z) \cong \mathscr{O}_X$. We then fix an isomorphism $\mathscr{O}_X(Y+Z) \to \mathscr{O}_X$, although as we saw when defining the $\mathcal{G}^r_d$ functor, the choice of isomorphism will not actually affect anything. The rest proceeds very similarly to the second case: our Picard schemes P^i are described identically, but now to describe isomorphisms between the P^i, we tensor as necessary by $\mathscr{O}_X(Y)$; this clearly has degree 1 when restricted to Z, but since $\mathscr{O}_X(Y+Z)$ is trivial, it must have degree -1 when restricted to Y. Replacing the maps between the $\tilde{\mathscr{L}}^i$ with the appropriate maps for this case, the rest of the construction then proceeds identically to the previous case, with the exception that we cannot decompose D, so we impose vanishing along D only on G^0.

Because in each case the construction used only Picard schemes, Grassmannians, fiber products, and closed subschemes obtained by bounding the rank of maps between vector bundles, it nearly follows from the standard representability theorems for these functors that the G^r_d scheme we have constructed represents the $\mathcal{G}^r_d$ functor. We do need to note that in the second and third cases, our conditions for vanishing along D actually imply that all V_i vanish along D: in the second case, this follows because $\tilde{\mathscr{L}}^i|_{D^Y} \cong \tilde{\mathscr{L}}^0|_{D^Y}$ and $\tilde{\mathscr{L}}^i|_{D^Z} \cong \tilde{\mathscr{L}}^d_{D^Z}$ for all i (this is clear conceptually from the disjointness of D^Y and D^Z with Δ', but it follows formally from, for instance, [13, Exer. A3.16]); in the third case, we have by the same argument that the kernel of $\tilde{\mathscr{L}}^i|_D \to \tilde{\mathscr{L}}^0|_D$ is supported on $iZ \cap D$, and in particular above Δ, so it vanishes under push-forward, and we have $\pi_{P*}\tilde{\mathscr{L}}^i|_D \hookrightarrow \pi_{P*}\tilde{\mathscr{L}}^0|_D$. Now, the only remaining parts that need to be checked are that our definition of sub-bundle is compatible with the usual definition for the Grassmannian functor, which follows from Lemma II.4.1, and that ramification conditions imposed on sub-bundles of $\mathscr{L}(D)$ are equivalent to the desired ramification for sub-bundles of $\mathscr{L}$. The latter follows from the disjointness of D and the P_i, since then we get $\mathscr{L}|_{jP_i} \cong \mathscr{L}(D)|_{jP_i}$, so the pushforwards are isomorphic, and the rank conditions for ramification are trivially equivalent. It is automatically projective over P, having been constructed as a closed subscheme of a product of Grassmannians, which are themselves projective (Theorem A.11). But P is projective over B by Theorem A.7, so we find that G^r_d is projective over B, as asserted. Compatibility with base change has already been proven in Lemma II.3.5.

We now verify that the moduli scheme we have constructed has the desired lower bound on its dimension. In all cases, we will be making use of codimension of intersection arguments which require the ambient scheme to be regular: however, since we assumed that B is regular, and P is smooth over B, it is regular. Our ambient scheme G is a product of Grassmannians over P, each of which is smooth over P of fiber dimension $(r+1)(d''-r-1)$, where d'' is the rank of $\pi_{P*}\tilde{\mathscr{L}}^i(D)$ (Theorem A.11). Hence, G is itself regular. We can thus argue that codimensions of intersections are bounded by the sum of the individual codimensions, for instance by [41, p. 261], and noting that codimensions are preserved under localization. We therefore begin by bounding the codimension of any component of G^r_d inside G. Now, in the first case, vanishing along D imposes $(\operatorname{rk} V)(\operatorname{rk} \pi_{P*}\tilde{\mathscr{L}}|_D) = (r+1)(\deg D)$ conditions. Next, we count the number of conditions imposed by ramification, expressed in terms of the vanishing sequences $a^i_j := \alpha^i_j + j$. This defines a Schubert cycle; the only observations to make are that by hypothesis the evaluation maps $\pi_{P*}\tilde{\mathscr{L}}(D) \to \pi_{P*}(\tilde{\mathscr{L}}(D)|_{jP_i})$ are surjective, and that although the $\pi_{P*}(\tilde{\mathscr{L}}(D)|_{jP_i})$ for $j \leq d+1$ do not give a sequence of quotients of every rank of $\pi_{P*}\mathscr{L}(D)$, we can extend to any sequence containing quotients of every rank, and since the a^i_j are all less than $d+1$, we will always get the same Schubert cycle regardless. In particular, by Theorem A.11 the imposition of ramification at P_i gives an integral subscheme of codimension $\sum_j(a^i_j - j) = \sum_j(\alpha^i_j)$ inside G. Thus the total codimension of any component of G^r_d inside G is at most $(r+1)(\deg D) + \sum_j(\alpha^i_j)$.

In the second and third cases, the only real difference is that we replace the Grassmannian with the linked Grassmannian of the following section; it is easily verified that because the maps on $\pi_{P*}\mathscr{L}^i$ are induced from maps on the $\mathscr{L}^i$, they satisfy the conditions of a linked Grassmannian (Definition II.A.4): condition (I) follows from Lemma A.24, while conditions (II) and (III), being expressed in terms of the emptyness of subschemes whose definitions are compatible with base change, can be checked fiber by fiber, where they are clearly satisfied. The only non-tautological part is that in the case of a reducible fiber, everything in the kernel of f_i really is in the image of g_i and vice versa, but this will be all right because all the $\mathscr{L}^i$ are sufficiently ample, so in particular given any global section on Y, we can find a global section on Z that agrees at the node, and vice versa. Then it follows from Theorem II.A.14 that every component of the linked Grassmannian has codimension $d(r+1)(d''-r-1)$, and the rest of the calculation proceeds the same way, with the minor exception in the second case that we have to compute vanishing on D^Y and D^Z separately and use $\deg D^Y + \deg D^Z = \deg D$.

Now let x be any point of G^r_d, closed in its fiber over $b \in B$. The point here is that since G is regular, it is catenary, and we can compute the dimension of the local ring of an irreducible closed subscheme containing x (and in particular, of a component of G^r_d at x) as the dimension of $\mathscr{O}_{G,x}$ minus the codimension of the component. That is to say, we can prove the desired statement via naive substraction of codimension from dimension. Our Picard scheme is smooth of relative dimension g over B, and $\pi_{P*}\mathscr{L}^i(D)$ is of rank $d+\deg D+1-g$, so our Grassmannian schemes are (smooth) of relative dimension $(r+1)(d+\deg D-r-g)$ over P. It follows by [64, Prop. 17.5.8] that $\dim \mathscr{O}_{G,x} = \dim \mathscr{O}_{B,b}+g+(r+1)(d+\deg D-r-g)$ in the first case, and $\dim \mathscr{O}_{B,b} + g + (d+1)(r+1)(d+\deg D - r - g)$ in the second. Our desired lower bound for $\dim \mathscr{O}_{G^r_d,x}$ now follows from our above codimension bounds.

Finally, we need to show that the sub-functor of separable limit series is representable by an open subscheme, or equivalently, that its complement is representable by a closed subscheme. Since we are interested in the open subscheme, we needn't concern ourselves overly with the particular choice of scheme-theoretic definition of the inseparable locus, and can in particular define it fiber by fiber, as long as we show that the result is closed. One approach would be to observe that as we are looking at a sub-functor of a functor already known to be representable, it would suffice to check the property of being a closed subfunctor etale locally, and etale locally we could produce sufficiently many sections of π specializing to Y and Z that we could invoke Proposition II.1.4 to cut out the inseparable subscheme as the union of the two closed subschemes obtained by imposing (the mildest possible) ramification along all the sections specializing to Y, and all the sections specializing to Z. However, we will instead give the following more functorial approach:

First, denote by $\tilde{\mathscr{F}}^i$ the universal sub-bundles of $\pi_{P*}\tilde{\mathscr{L}}$ on our G^r_d scheme. As in the proof of Proposition II.1.4, we can construct a map $\tilde{\mathscr{F}}^i \otimes \mathscr{O}_{X\times_B G^r_d} \to \mathscr{P}^r(\tilde{\mathscr{L}})$ where $\mathscr{P}^r$

denotes the bundle of principal parts of order r; taking $(r+1)$st exterior powers gives a map $s^{\text{univ}} : \det(\tilde{\mathscr{F}}^i) \to \tilde{\mathscr{L}}^{\otimes r+1} \otimes (\Omega^1_{X/B})^{\otimes\binom{r+1}{2}}$; we noted that in the smooth case, the inseparable series are precisely the ones where this map vanishes on X; that is, the kernel defines a closed subscheme of $X \times_B G^r_d$, and our separable subscheme is the image under π_P in G^r_d of its complement; since π_P is flat and finite type, this will be open in G^r_d as desired. In general, there are two additional points to check, but it does not end up being any harder. We first claim that we still obtain inseparable subschemes by looking at the kernel of the pushforward of s^{univ}, with the subschemes of limit series inseparable on Y and Z respectively coming from $\mathscr{F}^0$ and $\mathscr{F}^d$. For this, we need only check that s^{univ} for $\mathscr{F}^0$ will always vanish on Z, as the situation is clearly the same for $\mathscr{F}^d$. But since $\mathscr{F}^0$ comes from a line bundle of degree zero on Z, it can have rank at most one when restricted to Z, so its $(r+1)$st exterior power always vanishes on Z, as desired. Next, since π_P is no longer smooth, $\Omega^1_{X/B}$ is not locally free, so the kernel of s^{univ} does not *a priori* naturally describe a closed subscheme. However, this is easily solved: it certainly suffices to test inseparability on the open dense subset of each fiber which is smooth, so we simply restrict to the smooth locus of X over B before taking the vanishing scheme of s^{univ} and the image of its complement. $\square$

Our first application is the same regeneration/smoothing theorem due to Eisenbud-Harris, except that now it *a priori* gives results on smoothings of crude limit series as well, and we are also able to include upper bounds of dimensions of general fibers in certain cases. We state it first generally, for any desired fixed choice of smoothing of the special fiber and its sections (that is, for any given smoothing family), and then apply our result on existence of smoothing families to conclude that appropriately well-behaved limit series on a special curve of compact type will yield information about the linear series on some unspecified nearby smooth curve; this last is enough for most results involving a general curve, where one can show that it suffices to produce a single smooth curve with a given desired property.

We have:

Corollary II.4.4. *In the situation of Theorem II.4.3, suppose that $\rho \geq 0$, that U is any open subscheme of our G^r_d scheme, and that for some point $b \in B$, the fiber of U over b has the expected dimension ρ. Then every point of the fiber may be smoothed to nearby points.*

Specifically:

(i) *The map from U to B is open at any point in the fiber over b, and for any component Z of U whose image contains b, the generic fiber of Z over U has dimension ρ.*

(ii) *If further U is closed in G^r_d, then there is a neighborhood V of b such that the preimage of V in U is open over V, and for each component Z of U, every component of every fiber of Z over V has dimension precisely ρ.*

In particular, if X_0 is a curve of compact type (with two components) over an algebraically closed field, with $\bar{P}_1, \dots \bar{P}_n$ distinct smooth closed points of X_0, α^i any collection of ramification sequences, and U_0 any open subset of $G^r_d(X_0/k; \{(P_i, \alpha^i)\}_i)$ having expected dimension ρ, then there exists a smooth curve X_1 over a one-dimensional function field k' over k, specializing to X_0, with points P_i specializing to the $\bar{P}_i$, and such that every point of U_0 smooths to X_1; if further $U_0 = G^r_d(X_0/k; \{(\bar{P}_i, \alpha^i)\}_i)$, then $G^r_d(X_1/k'; \{(P_i, \alpha^i)\}_i)$ also has dimension ρ.

Proof. For (i), first note that since U is quasi-projective over a Noetherian base, it is of finite type over B. Now, let $x \in Z$ be any closed point in the fiber of Z over b, and η the generic point of Z. Say η maps to ξ; then the dimension of the fiber of Z over ξ is at most ρ, by Chevalley's theorem on semi-continuity of fiber dimension [63, Thm. 13.1.3], and at least ρ by Theorem II.4.3, after base change to ξ. We then claim that ξ is the generic point of B. Certainly, ξ is the generic point of the scheme-theoretic image of Z, a closed irreducible subscheme B' of B containing b; if $B' \neq B$, by the regularity of B, $\dim \mathscr{O}_{B',b} < \dim \mathscr{O}_{B,b}$, and since Z factors through B', we have by the most elementary fiber dimension theorem [62, Prop. 5.5.2] that $\dim \mathscr{O}_{Z,x} \leq \dim \mathscr{O}_{B',b} + \rho < \dim \mathscr{O}_{B,b} + \rho$, contradicting Theorem II.4.3. Thus $B' = B$ as desired.

For the openness assertion, it suffices to prove that the image of U contains a neighborhood of b, since if we replace U by any neighborhood of a point of the fiber of U over b, the hypotheses of our corollary are still satisfied. Let b_1 be a point of B, specializing to b; let B_1 be the closure of b_1 in B, and consider the base change $U_1 \to B_1$. If B_1 has codimension c in B, then every component of U_1 would have codimension at most c in U (this follows by applying [13, Thm. 10.10] to generic points of the relevant components), so since U, being of finite type over a universally catenary scheme, is catenary, if we restrict to a component Z of U_1 passing through x, we have $\dim \mathscr{O}_{Z,x} \geq \dim \mathscr{O}_{U,x} - c = \dim \mathscr{O}_{B,b} + \dim U_b - c =$

$\dim \mathscr{O}_{B_1,b} + \dim U_b$, so applying our earlier argument with Z in place of U, we see that Z maps dominantly to B_1, so that b_1 is in the image of U. Now, by constructibility of the image [61, Thm. 1.8.4], since $f(U)$ contains every point of B specializing to b, it must contain some neighborhood U of b, as desired.

For (ii), if U is closed in G^r_d we have that it is proper over B, and every component Z of U either contains b in its image, or is supported on a closed subset of B away from b. In the first case, we can apply (i) to conclude that Z maps surjectively to B, and upper semi-continuity of fiber dimension for closed morphisms of finite type [63, Cor. 13.1.5] to conclude that the locus on B of fibers of Z having a component of dimension greater than ρ is closed, and once again doesn't contain b, giving by its complement a V of the desired form. Of course, in the second case, we simply choose V to be disjoint from the image of Z. We have therefore constructed a V for each Z, and since there are only finitely many components of U, we may simply take their intersection. Openness now follows from (i) and the fact that all fibers over V have dimension ρ.

Finally, given an X_0 as described, we can apply Theorem II.2.4 to place X_0 into a smoothing family X/B with generic fiber X_1; the desired assertions then follow immediately from the main assertions of the corollary. □

The finite case is particularly nice, but we put off any discussion of it until after we have introduced the language of Eisenbud-Harris limit series in the next section.

Even without knowing anything about the separable locus being closed, which in general seems to be a subtle issue, we can still obtain results on lifting from characteristic p to characteristic 0. However, note that the expected dimension hypothesis in the following corollary is not only key to the argument, but at least in some cases both non-vacuous and necessary for the validity of the conclusion. See in particular Proposition I.4.5 and the following discussion. In any case, our machinery now easily yields:

Corollary II.4.5. *In the situation of Theorem II.4.3, suppose that $\rho \geq 0$, that B is a mixed-characteristic DVR, and that the special fiber of some U open inside $G^{r,\mathrm{sep}}_d$ has the expected dimension ρ. Then every point x_0 of U in the special fiber may be lifted to characteristic 0, in the sense that there will be a point x_1 of the generic fiber of U (and in particular of $G^{r,\mathrm{sep}}_d$) specializing to x_0.*

In particular, suppose that X_0 is a smooth, proper curve over a perfect field k of char-

acteristic p, with $\bar{P}_1, \dots \bar{P}_n$ distinct closed points of X_0, α^i any collection of ramification sequences, and U_0 any open subset of $G^{r,\mathrm{sep}}_d(X_0/k; \{(P_i, \alpha^i)\}_i)$ having expected dimension ρ; then there exists a smooth curve X_1 over the fraction field of the Witt vectors of k, specializing to X_0, with points P_i specializing to the $\bar{P}_i$, and such that every point of U_0 may be lifted to a point of X_1.

Proof. The first assertion follows immediately from the openness proven in Corollary II.4.4.

For the second assertion, let A be the Witt vectors of k [54, Thm. II.5.3], then by [4, 11, Thm 1.1] we can find an X over $\operatorname{Spec} A$ whose special fiber is X_0, and since A is complete and the $\bar{P}_i$ smooth points we can lift them to sections P_i of X (for instance, by definition of formal smoothness [3, Prop. 2.2.6]). Applying the first assertion then gives the desired result. $\square$

Remark II.4.6. It seems likely that due to the smoothness of the exact locus of the linked Grassmannian (see Definition II.A.9 and Proposition II.A.11), and the determinantal description of the G^r_d space inside it, that at least on the (possibly empty) exact locus, if the dimension is the expected dimension, the G^r_d scheme ought to be Cohen-Macaulay. If then the dimension of a special fiber is correct, the exact locus of the G^r_d scheme will be flat over the base at points of this fiber.

Remark II.4.7. In final assertion of Corollary II.4.4, note that we could not replace the hypothesis that $U_0 = G^r_d(X_0/k; \{(\bar{P}_i, \alpha_i)\}_i)$ by the weaker hypothesis that U_0 is closed in the G^r_d scheme over X_0. The problem is that there is no obviously meaningful way of extending such a U_0 to an open and closed subscheme of G^r_d over all of X. This comes up in particular in the case that U_0 is the separable series subscheme, when the expected dimension is 0; one could try to extend U_0 to "separable series on X_1 which specialize to separable series on X_0", but this is not a particularly satisfactory class of objects to study.

Remark II.4.8. Corollary II.4.4 is more than enough for our purposes, but one might be tempted to ask whether the assertion of (ii) remains true without the closedness hypothesis on U, if restricted to components Z of U whose images contain b. In view of (i) and the standard constructibility theorems [63, Prop. 9.3.2], the only way this could fail would be if there were a locally closed (and without loss of generality, irreducible) subscheme B' of B specializing to b on which the fiber dimension was strictly greater than ρ. Unfortunately, there is no reason to think that this couldn't happen, as there are certainly quasi-projective morphisms even of schemes of finite type over a field, with the base regular, exhibiting this behavior: consider for instance the blowup of $\mathbb{P}^3 \times \mathbb{A}^3$ along a line lying above a line in $\mathbb{A}^3$, considered as a scheme over $\mathbb{A}^3$, with the four-dimensional component of the fiber over the origin removed. Indeed, in this case both schemes in question are regular, so the fiber over the origin having the "correct" dimension implies that the map is flat at every point in the fiber; yet we see that this does imply that there is a neighborhood of the origin on which all fibers have the same dimension.

II.5 Comparison to Eisenbud-Harris Theory

This is all well and good, but as of yet there is little apparent connection to Eisenbud and Harris' limit series. After all, a substantial part of the point of their theory was that on a reducible curve, it ought to suffice to consider only a single linear series per component, so that their construction occurs inside the product of only two Grassmannians. Furthermore, they impose additional ramification conditions at the nodes that are not readily apparent from our construction. Our next task is therefore to conduct a more thorough comparison between what our $\mathcal{G}^r_d$ functor yields for a reducible curve over a field, and the Eisenbud-Harris functor for this case. Throughout this section we therefore assume we are in:

Situation II.5.1. X/B is a smoothing family with X reducible; specifically, falling into case (2) of Situation II.3.1.

Remark II.5.2. For notational convenience, all statements of this section are given in terms of G^r_d schemes without specified ramification. However, it will follow from the brief discussion in the proof of the lemma which follows that ramification conditions are completely compatible from our perspective and from the Eisenbud-Harris perspective, so all the results we give in this section are immediately valid for G^r_d schemes with specified ramification as well.

The first step is to consider the "forgetful" map we obtain from our $\mathcal{G}^r_d(X/B)$ functor into the product of $\mathcal{G}^r_d(Y/B)$ and $\mathcal{G}^r_d(Z/B)$, simply by forgetting all the intermediate $(\mathscr{L}^i, V_i)$, keeping only $(\mathscr{L}^0, V_0)$ and $(\mathscr{L}^d, V_d)$, and restricting these to Y and Z respectively. When we start with a point $\{(\mathscr{L}^i, V_i)\}_i$ of $G^r_d(X/B)$, we denote the linear series on each component obtained this way by $(\mathscr{L}^Y, V^Y)$ and $(\mathscr{L}^Z, V^Z)$.

We have:

Lemma II.5.3. *On a reducible curve X/B (i.e., in case (2) of Situation II.3.1), given any T-valued point $\{(\mathscr{L}^i, V_i)\}_i$ of $G^r_d(X/B)$, the forgetful map composed with restriction to Y and Z gives a T-valued point of $G^r_d(Y/B) \times G^r_d(Z/B)$. In particular, this defines a morphism $FR : G^r_d(X/B) \to G^r_d(Y/B) \times_B G^r_d(Z/B)$. A limit series in $G^r_d(X/B)$ is separable if and only if its image under FR is separable in both $G^r_d(Y/B)$ and $G^r_d(Z/B)$.*

Proof. It is clearly enough to show that given T/B, and $\{V_i \subset \pi_{T*}\mathscr{L}^i\}$ a T-valued point of $G^r_d(X/B)$, then $V_0|_{Y_T}$ is a sub-bundle of $L^0|_{Y_T}$ and correspondingly for Z_T and V_d. This would immediately give the result for G^r_d schemes without specified ramification, but since we specified ramification solely on V_0 or V_d depending on whether the relevant section was on Y or Z, the ramification conditions will certainly be preserved as well. Now, the point here is simply that because $\mathscr{L}^0$ has degree d on Y_T and 0 on Z_T, injectivity is determined entirely on Y_T. Indeed, let S be any T-scheme. By the definition of sub-bundle, we have $V_{0S} \hookrightarrow \pi_{S*}\mathscr{L}^i_S$, and we need only show that $V^{0S}|_{Y_S}$ injects into $\pi_{S*}L^0_S|_{Y_S}$. But this is equivalent to the statement that for any U open in S, and any section s of $\pi_{S*}\mathscr{L}^i_S(U) = \Gamma\mathscr{L}^i_U$, if s vanishes on Y_U it must vanish on all of X_U. It suffices to see that it vanishes on Z_U, but this is clear, since the vanishing along Y_U means it vanishes along Δ', and would therefore have to be a global section of $\mathscr{L}^0|_{Z_U}(-\Delta')$, which has negative degree and therefore can't have any non-zero global sections, for instance by considering each fiber, and then applying Nakayama's lemma. The case of $\mathscr{L}^d$ and Z is clearly symmetric, so we can conclude the desired result.

The statement on separability is immediate from the definition of separability of a limit series on a reducible curve. □

Notation II.5.4. In the same situation as the previous lemma, we denote by V^Y_i the image of V_i inside $\pi_*(\mathscr{L}^Y(-i\Delta')))$, and similarly for Z.

Lemma II.5.5. *In the same situation as the previous lemma, we have the following additional observations (and consequent notation):*

(i) V^Y_i injects naturally into V^Y, and similarly for Z

(ii) V^Y_i will be contained in $\ker \beta^Y_i \subset V^Y$, where $\beta^Y_i : V^Y \to \pi_{T*}\mathscr{L}^Y|_{i\Delta'}$ is the natural ith order evaluation map at Δ', and V^Z_i will similarly be contained in $\ker \beta^Z_{d-i} \subset V^Z$

(iii) The induced map $V_i \to V^Y \oplus V^Z$ in fact exhibits V_i as a sub-bundle of $V^Y \oplus V^Z$

Proof. Assertions (i) and (ii) are clear, as the map $V_i^Y \to V_Y := V_0|_{Y_T}$ is by definition the map induced by the inclusion $\mathscr{L}^Y(-i\Delta') \hookrightarrow \mathscr{L}^Y$. For (iii), it suffices to show that $V_i \to V^Y \oplus V^Z$ is injective after any base change $S \to T$; on the other hand, we have by hypothesis that $V_{iS} \hookrightarrow \pi_{S*}\mathscr{L}_S^i$, and it is certainly true since restriction commutes with base change that $\pi_{S*}\mathscr{L}_S^i \hookrightarrow \pi_{S*}\mathscr{L}_S^{iY} \oplus \pi_{S*}\mathscr{L}_S^{iZ} \hookrightarrow \pi_{S*}\mathscr{L}_S^Y \oplus \pi_{S*}\mathscr{L}_S^Z$, and since the map to $V^Y \oplus V^Z$ factors through this, it must also remain injective. □

Definition II.5.6. In case (2) of Situation II.3.1, we define an **Eisenbud-Harris (crude) limit series** on X to be a pair $((\mathscr{L}^Y, V^Y), (\mathscr{L}^Z, V^Z))$ in $G_d^r(Y) \times_B G_d^r(Z)$ satisfying $a_i^Y(\Delta') + a_{r-i}^Z(\Delta') \geq d$ (see below). The closed subscheme of $G_d^r(Y) \times_B G_d^r(Z)$ obtained by these ramifications conditions will be denoted $G_{d,\mathrm{EH}}^r(X/B)$. We also define $G_{d,\mathrm{EH}}^{r,\mathrm{sep}}(X/B) \subset G_{d,\mathrm{EH}}^r(X/B)$ to be the open subscheme of limit series which are separable on each component, and $G_{d,\mathrm{EH}}^{r,\mathrm{ref}}(X)$ to be the open subscheme of **refined** Eisenbud-Harris limit series satisfying $a_i^Y(\Delta') + a_{r-i}^Z(\Delta') = d$, or more precisely, the complement of the closed subscheme satisfying $a_i^Y(\Delta') + a_{r-i}^Z(\Delta') > d$.

We remark that these ramification conditions do in fact give a canonical closed subscheme structure: for each sequence of $r+1$ non-decreasing integers $0 \leq a_i \leq d$, we get a closed subscheme defined by the conditions $a_i^Y(\Delta') \geq a_i$, $a_{r-i}^Z(\Delta') \geq d - a_i$; there are only finitely many such sequences, so the union of the closed subschemes obtained over each of them is again a closed subscheme. However, this definition gives us trouble when we attempt to show that our $G_d^r(X/B)$ maps into $G_{d,\mathrm{EH}}^r(X/B)$, as it is difficult to describe the T-valued points of a union of schemes in terms of the T-valued points of the individual schemes. As a result, we settle for the somewhat weaker:

Proposition II.5.7. *We have the following facts about the image of $FR : G_d^r(X/B) \to G_d^r(Y/B) \times_B G_d^r(Z/B)$:*

(i) FR has set-theoretic image precisely $G_{d,\mathrm{EH}}^r(X/B)$;

(ii) Scheme-theoretically, $G_d^r(X/B)$ maps into the closed subscheme satisfying a slightly weakened form of Eisenbud and Harris's ramification conditions at the node:

$$a_i^Y(\Delta') + a_{r-i}^Z(\Delta') \geq d - 1.$$

(iii) *The open subscheme of $G^r_d(X/B)$ mapping set-theoretically into $G^{r,\mathrm{ref}}_{d,\mathrm{EH}}(X)$ actually maps scheme-theoretically into $G^{r,\mathrm{ref}}_{d,\mathrm{EH}}(X) \subset G^r_{d,\mathrm{EH}}(X/B)$.*

Proof. In general, for a T-valued pair $((\mathscr{L}^Y, V^Y), (\mathscr{L}^Z, V^Z))$, define a^Y_j to be the largest integer i with $\operatorname{rk} \beta^Y_i \leq j$ everywhere on T, and similarly for Z. The set-theoretic statement may be checked point by point, and is equivalent to saying that when $T = \operatorname{Spec} k$ for some k, $((\mathscr{L}^Y, V^Y), (\mathscr{L}^Z, V^Z))$ is in the image of FR if and only if $a^Y_j + a^Z_{r-j} \geq d$ for all j. For (ii), it is enough to check that for arbitrary local T, $a^Y_j + a^Z_{r-j} \geq d-1$ for all j, and for (iii), we want to show in this case that if the point obtained by restriction to the closed point of T satisfies $a^Y_j + a^Z_{r-j} = d$ for all j, then the entire T-valued point does. In all cases, we make use of the fact from Lemma II.5.5 that V^Y_i may be considered as lying inside $\ker \beta^Y_i$, and V^Z_i in $\ker \beta^Z_{d-i}$. Conceptually, the basic idea is that for V_i to maintain rank $r+1$ at each i, the ranks of $\ker \beta^Y_i$ and $\ker \beta^Z_{d-i}$ must add up to at least $r+1$, and looking at $i = a^Y_j$ for different j should yield the desired inequalities. As we will see, this works over a field, but is not quite so nice for a more general T.

Now, for the set-theoretic statement of part (i), suppose we have a T-valued point $\{(\mathscr{L}^i, V_i)\}_i$ of $G^r_d(X/B)$, with $T = \operatorname{Spec} k$; we want to show that it maps into $G^r_{d,\mathrm{EH}}(X/B)$. Since V_i is glued from subspaces of $\ker \beta^Y_i$ and $\ker \beta^Z_{d-i}$ and has dimension $r+1$, we conclude that the dimensions of $\ker \beta^Y_i$ and $\ker \beta^Z_{d-i}$ add up to at least $r+1$, so $\operatorname{rk} \beta^Y_i + \operatorname{rk} \beta^Z_{d-i} \leq r+1$. It immediately follows that $a^Y_j + a^Z_{r+1-j} \geq d$ for all j. On the other hand, for a given j, set $i = a^Y_j$; we know that $\operatorname{rk} \beta^Y_{i+1} > j$, so one of the sections in V^Y_i is non-vanishing at Δ' when considered as a section of $\mathscr{L}^i(-i\Delta')$, and to use it in V_i, it must be glued to a section of V^Z_{d-i} similarly non-vanishing at Δ'. Thus, the dimension of V_i is strictly less than the sum of the dimensions of $\ker \beta^Y_i$ and $\ker \beta^Z_{d-i}$, so our earlier argument gives $\operatorname{rk} \beta^Y_i + \operatorname{rk} \beta^Z_{d-i} < r+1$, and we actually conclude $a^Y_j + a^Z_{r-j} \geq d$, as desired. For later use, note that when $a^Y_j + a^Z_{r-j} = d$ for all j, this argument shows that we have $\ker \beta^Y_i = V^Y_i$ and $\ker \beta^Z_{d-i} = V^Z_i$ for all i, and in particular when $i = a^Y_j = d - a^Z_{r-j}$, $\dim V^Y_i + \dim V^Z_{r-i} = r+2$, since we will have had to glue a section from each component non-vanishing at Δ'.

Conversely, given a $((\mathscr{L}^Y, V^Y), (\mathscr{L}^Z, V^Z))$ satisfying the Eisenbud-Harris inequalities, we construct the $\mathscr{L}^i$ by gluing $\mathscr{L}^Y(-i\Delta')$ and $\mathscr{L}^Z((i-d)\Delta')$, and set $V_0 = V^Y$, $V_d = V^Z$. Note that sections in V^Y which vanish at Δ' are extended by 0 along Z. If there is a non-vanishing section, $a^Y_0 = 0$, so $a^Z_r \geq d$, and $\mathscr{L}^Z$, being a degree d line bundle with a non-zero section vanishing to order at least d at Δ', must be $\mathscr{O}_{Z_T}(d\Delta')$, so we get the trivial bundle on Z for V_0, and can (uniquely) extend sections not vanishing at Δ', also. We also note that this implies that we have V_0 mapping into V^Z under iterations of f_i; indeed, this map is uniformly zero unless there was a section of V_0 non-vanishing at Δ', and such a section had to be constant on Z, hence mapping to the (unique up to scaling) section in V^Z vanishing to order d at Δ'. By symmetry, we can make the same arguments for V^Z to get our V_d. Now we inductively construct each V_i for $i = 1, 2, \dots, d-1$ in terms of V_{i-1} and V_d; it is not clear how to induct on the weakest possible statement, which is that given V_{i-1} and V_d linked to one another under iterates of f_j and g_j, there is a V_i linked to V_{i-1} and to V_d under iterates of f_i and g_i. The key additional condition to add to our induction hypothesis, already satisfied by our V_0 and V_d, will be to require that each V_i have a basis of sections each of which is either non-vanishing at Δ', or vanishes uniformly on either Y or Z, with at most one basis element in the first category. We denote the number of each of these by r_i^1, r_i^2, and r_i^3 respectively, where we have r_i^1 always 0 or 1, and $r_i^1 + r_i^2 + r_i^3 = r + 1$ for all i. The last condition of our induction hypothesis is that r_i^3 is always the maximal possible value, which is $\dim \ker \beta_{i+1}^Y$. Note that since this is non-decreasing, if we construct a V_i with $r_i^3 = r_{i-1}^3$, maximality is automatically satisfied. Of course, we still require that each V_i be linked to V_{i-1}, and to V_d under iterated composition of the f_j and g_j.

Now, for general i, suppose we have constructed the V_j up to V_{i-1} satisfying our induction hypothesis. To construct V_i, the basis elements vanishing on Y must contain $f_{i-1}(V_{i-1})$, which is an $(r_{i-1}^1 + r_{i-1}^2)$-dimensional space, and of course they must map into V^Z. Since $f_{i-1}(V_{i-1})$ maps into V^Z, we can choose $r_{i-1}^1 + r_{i-1}^2$ such sections, by taking any basis of $f_{i-1}(V_{i-1})$. Next, the basis elements vanishing on Z must be contained in $g_{i-1}^{-1}(V_{i-1})$, and we choose them to be a basis of the intersection of $g_{i-1}^{-1}(V_{i-1})$ with the sections vanishing on Z. This is at most an r_{i-1}^3-dimensional space, with equality if all of the r_{i-1}^3 basis elements of V_{i-1} vanish to order greater than one at Δ'. If there was a one-dimensional subspace of sections vanishing to order exactly one at Δ', g_{i-1}^{-1} will instead be $(r_{i-1}^3 - 1)$-dimensional. Now, by our induction hypothesis, the iterated image of V_d under the g_j is contained in the span of the basis elements vanishing on Z in V_{i-1}, and automatically vanishes to order at least 2 at Δ' in V_{i-1} (using that $i < d$), so it is automatically contained in the span of the basis elements we have chosen for V_i which vanish on Z. Now, if we had r_{i-1}^3 such basis elements, we are done. If not, we had a section of V_{i-1} vanishing on Z and vanishing to first order at Δ', whose preimage on Y under g_{i-1} will therefore be non-vanishing at

Δ', and it follows that $\dim\ker\beta_i^Y = \dim\ker\beta_{i+1}^Y + 1$, and therefore that $a^Y_{r+1-r^3_{i-1}} = i$; in particular, the required maximality of r_i^3 is satisfied. It also follows that $a^Z_{r^3_{i-1}-1} \geq d-i$; if it is equal, we can find a section of V^Z vanishing to order precisely $d-i$ at Δ', which we could glue to our final section of $g_{i-1}^{-1}(V_{i-1})$ to obtain our $(r+1)$st generator for V_i, which will be non-vanishing at Δ'. Otherwise, we have $a^Z_{r^3_{i-1}-1} > d-i$, so following through the definitions, $\dim\ker\beta^Z_{d-i+1} \geq r+2-r^3_{i-1} = 1 + r^1_{i-1} + r^2_{i-1}$; we chose $r^1_{i-1}+r^2_{i-1}$ generators vanishing on Y already, but this means we can choose another one in $\ker\beta^Z_{d-i+1}$ to be our $(r+1)$st generator for V_i. This completes the proof of the set-theoretic surjectivity of FR onto $G^r_{d,\mathrm{EH}}(X/B)$.

For the scheme-theoretic statements, let $T = \operatorname{Spec} A$ where A is any local ring with maximal ideal $\mathfrak{m}$; our main assertion is that for any i, V_i^Y and V_i^Z, considered inside $V^Y \cong V_0$ and $V^Z \cong V_d$ respectively, must contain sub-bundles of ranks adding up to at least $r+1$. By Lemma II.5.5, we have $0 \to V_i \to V^Y \oplus V^Z \to Q \to 0$ for some locally free Q, and since A is local, this is actually a sequence of free modules. Modulo $\mathfrak{m}$, this is just a sequence of vector spaces, and since V_i has dimension $r+1$ and injects into $V_i^Y \oplus V_i^Z$, for some j we can take a basis of V_i whose first $r+1-j$ terms are linearly independent in V_i^Y, and whose remaining j are linearly independent in V_i^Z. Lifting these to a basis of V_i over A, we find that by Nakayama's lemma, we have constructed sub-bundles of V_i of rank $r+1-j$ and j whose images in V_i^Y and V_i^Z are sub-bundles of V^Y and V^Z of rank $r+1-j$ and j, as desired. Since these come from V_i, they are contained in $\ker\beta_i^Y$ and $\ker\beta^Z_{d-i}$, and we conclude that $\operatorname{rk}\beta_i^Y \leq j$, $\operatorname{rk}\beta^Z_{d-i} \leq r+1-j$ on T. This gives us $a_j^Y + a^Z_{r+1-j} \geq d$. As in the fields case, if we set $i = a_j^Y$, then for $i+1$, by hypothesis $\operatorname{rk}\beta^Y_{i+1}$ is not less than or equal to j on all of T, so our constructed sub-bundle of V^Y could have rank at most $r-j$, and the sub-bundle of V^Z would have to have rank at least $j+1$, giving $\operatorname{rk}\beta^Z_{d-i-1} \leq r-j$ on T, and yielding $a_j^Y + a^Z_{r-j} \geq d-1$, and giving statement (ii).

Finally, for statement (iii), we need only combine this argument with our earlier observation that at the closed point, where by hypothesis we had $a_j^Y + a^Z_{r-j} = d$ for all j, we necessarily have $\dim V_i^Y + \dim V_i^Z = r+2$; thus we can in fact choose a basis of V_i (still modulo $\mathfrak{m}$) which has rank $r+1-j$ in V_i^Y and rank $j+1$ (rather than j) in V_i^Z, and we then get the desired inequality $a_j^Y + a^Z_{r+1-j} \geq d$ for the entire T-valued point, as desired. Note that the subscheme of G^r_d in question is open simply because G^r_d is known to map set-theoretically into $G^r_{d,\mathrm{EH}}$, and $G^{r,\mathrm{ref}}_{d,\mathrm{EH}}(X)$ is open inside $G^r_{d,\mathrm{EH}}$. $\square$

Luckily, the situation is easier to get a handle on for the open subset of refined series which Eisenbud and Harris actually used in their construction. We will show that the space of refined limit series is actually isomorphic to an open subscheme of our G^r_d scheme. Note that there is a slight difference in indexing between this section and Appendix II.A, in that here indices for bundles and subspaces start with 0, whereas in the appendix they start at 1. However, this difference will not be at all relevant to any of our arguments.

Proposition II.5.8. *Suppose that $(\mathscr{L}^Y, V^Y)$ and $(\mathscr{L}^Z, V^Z)$ form a T-valued point of $G^{r,\mathrm{ref}}_{d,\mathrm{EH}}(X/B)$. Then we have that $(\mathscr{L}^Y, V^Y)$ and $(\mathscr{L}^Z, V^Z)$ are the image of a unique T-valued point under FR.*

Proof. It clearly suffices to handle the case that T is connected, so we make this hypothesis. In this case, the basic observation is that we get a unique vanishing sequence at Δ' for V^Y and V^Z, in the sense that the subscheme of each such connected component satisfying a stronger ramification condition is empty. Indeed, by hypothesis, the closed subscheme of T satisfying $a^Y_i(\Delta') + a^Z_{r-i}(\Delta') > d$ is empty, and we get a ramification sequence at Δ' for the generic points of each component of T; now, because ramification conditions are closed, if the two sequences differed at different generic points, any point in the intersection of the corresponding components would have to be strictly stronger than either alone, and would therefore satisfy $a^Y_i(\Delta') + a^Z_{r-i}(\Delta') > d$, a contradiction. Now, if β^Y_i is the evaluation map $V^Y \to \pi_{T*}\mathscr{L}^Y|_{i\Delta'}$ and similarly for β^Z_i, this immediately implies that each β^Y_i and β^Z_i has rank determined exactly by the vanishing sequences, in the strong sense that for some r, the closed subscheme where the rank is less than or equal to r is all of T, but the closed subscheme where the rank is strictly less than r is empty. We conclude from [13, Prop. 20.8] (see also the related comments on p. 407) that the images of the β^Y_i and β^Z_i are locally free, with locally free quotients. If we denote by a_j the vanishing sequence for V^Y, we also note that $\ker \beta^Y_i$ will have rank $r+1-j$ if $a_{j-1} < i \leq a_j$, and following through the definitions we see that $\ker \beta^Z_i$ will have rank $r+1-j$ if $d - a_{r-j+1} < i \leq d - a_{r-j}$, or equivalently, $\ker \beta^Z_{d-i}$ will have rank j if $a_{j-1} \leq i < a_j$, so we find that $\ker \beta^Y_i = \ker \beta^Y_{i+1}$ if and only if $\ker \beta^Z_{d-i} = \ker \beta^Z_{d-i+1}$, and $\operatorname{rk} \ker \beta^Y_i + \operatorname{rk} \ker \beta^Z_{d-i+1} = r+1$ for all i.

The main idea is to construct the V_i as the subspace of $\ker \beta^Y_i \oplus \ker \beta^Z_{d-i}$ which agree on the two maps given by evaluation at Δ'. This would then be unique, as it follows from Lemma II.5.5 that any possible V_i would have to be contained in the constructed one, and by Lemma II.4.1 any two sub-bundles of a given rank with one contained in the other must be the same. Now we want to show existence. We work locally on the base, so that $\mathscr{L}^Y|_{\Delta'} \cong \mathscr{L}^Z|_{\Delta'} \cong \mathscr{O}_{\Delta'} = \mathscr{O}_T$, and fix a choice of these isomorphisms. As prescribed for gluing together line bundles defined on components, we define $\mathscr{L}^i$ by the short exact sequence

$$0 \to \mathscr{L}^i \to \mathscr{L}^Y(-i\Delta') \oplus \mathscr{L}^Z((i-d)\Delta') \to \mathscr{O}_{\Delta'} \to 0,$$

and pushforward gives us

$$0 \to \pi_{T*}\mathscr{L}^i \to \pi_{T*}\mathscr{L}^Y(-i\Delta') \oplus \pi_{T*}\mathscr{L}^Z((i-d)\Delta') \to \mathscr{O}_T.$$

We then define V_i to be the kernel of the induced map, so that:

$$0 \to V_i \to \ker \beta_i^Y \oplus \ker \beta_{d-i}^Z \to \mathscr{O}_T.$$

We have to show that V_i is a sub-bundle of $\pi_{T*}\mathscr{L}^i$ of the correct rank. We first observe that the image $\beta_{i+1}^Y(\ker \beta_i^Y)$ has locally free quotient in $\pi_{T*}\mathscr{L}^Y|_{(i+1)\Delta'}$, and similarly for Z: this image is inside $\operatorname{im} \beta_{i+1}^Y$ by definition, and the quotient is easily seen to be isomorphic to $\operatorname{im} \beta_i^Y$, via the map $\pi_{T*}\mathscr{L}^Y|_{(i+1)\Delta'} \to \pi_{T*}\mathscr{L}^Y|_{i\Delta'}$. Thus $\beta_{i+1}^Y(\ker \beta_i^Y)$ is a sub-bundle of a sub-bundle, and must itself be a sub-bundle of $\pi_{T*}\mathscr{L}^Y|_{(i+1)\Delta'}$. Now, we can factor β_{i+1}^Y restricted to $\ker \beta_i^Y$ as

$$\ker \beta_i^Y \to \mathscr{L}^Y(-i\Delta')|_{\Delta'} \hookrightarrow \mathscr{L}^Y|_{(i+1)\Delta'}$$

and we just showed that the cokernel of the composition is locally free; in particular, it is torsion-free, and the cokernel of the first map must also be torsion-free. But since $\mathscr{L}^Y(-i\Delta')|_{\Delta'}$ is a line bundle, this means the first map must be either zero or surjective, with surjectivity precisely when $\operatorname{rk}\ker \beta_i^Y = \operatorname{rk}\ker \beta_{i+1}^Y + 1$, and the ranks equal otherwise. We obtain the corresponding result for Z, and immediately conclude that V_i is a sub-bundle of $\ker \beta_i^Y \oplus \ker \beta_{d-i}^Z$, with equality if and only if both $\ker \beta_i^Y = \ker \beta_{i+1}^Y$ and $\ker \beta_{d-i}^Z = \ker \beta_{d-i+1}^Z$, and corank one otherwise. Thus, our hypotheses imply that V_i has rank $r+1$. The last observation is that V_i being a sub-bundle of $V^Y \oplus V^Z$ implies that it is a sub-bundle (in our generalized sense) of $\pi_{T*}\mathscr{L}^i$, but this follows easily from the fact that V^Y and V^Z are sub-bundles of $\pi_{T*}\mathscr{L}^Y$ and $\pi_{T*}\mathscr{L}^Z$, since we then obtain for any S over T

an injection $V_{iS} \hookrightarrow \pi_{S*}\mathscr{L}_S^Y \oplus \pi_{S*}\mathscr{L}_S^Z$ through which the map $V_{iS} \to \pi_{S*}\mathscr{L}_S^i$ factors. $\square$

We immediately conclude:

Corollary II.5.9. *The map $FR : G^r_d(X/B) \to G^r_d(Y/B) \times G^r_d(Z/B)$ induces an isomorphism from an open subscheme $G^r_d(X/B)$ onto $G^{r,\mathrm{ref}}_{d,\mathrm{EH}}(X/B)$, and on the corresponding separable subschemes of these.*

We may therefore think of the scheme of refined Eisenbud-Harris limit series as forming an open subscheme of our G^r_d scheme itself:

Definition II.5.10. We say that a point of G^r_d is a **refined** limit series if it maps under FR to $G^{r,\mathrm{ref}}_{d,\mathrm{EH}}$, and we denote the open subscheme of refined limit series by $G^{r,\mathrm{ref}}_d \subset G^r_d$.

Since in practice it is less cumbersome to work with Eisenbud-Harris series on a given reducible curve, we state our main corollary for the finite case of Theorem II.4.3 in a situation where one can (nearly) restrict attention entirely to the Eisenbud-Harris series. We now drop the hypothesis that we are in case (2), and for notational convenience define:

Definition II.5.11. In case (1), we simply define $G^r_{d,\mathrm{EH}}(X/B)$ to be equal to $G^r_d(X/B)$ and similarly for $G^{r,\mathrm{sep}}_{d,\mathrm{EH}}(X/B)$ and $G^{r,\mathrm{ref}}_{d,\mathrm{EH}}(X/B)$.

Corollary II.5.12. *In the situation of Theorem II.4.3, suppose that $B = \operatorname{Spec} A$ with A a DVR having algebraically closed residue field, and $\rho = 0$, and denote by X_0 and X_1 the special and generic fibers of X/B. Then consider the following conditions:*

(I) $G^{r,\mathrm{sep}}_{d,\mathrm{EH}}(X_0) \subset G^{r,\mathrm{ref}}_{d,\mathrm{EH}}(X_0)$

(II) $G^{r,\mathrm{sep}}_{d,\mathrm{EH}}(X_0)$ *consists of m reduced points for some $m > 0$.*

(III) For any DVR A', and any A'-valued point of $G^r_d(X)$ such that the induced map $\operatorname{Spec} A' \to \operatorname{Spec} A$ is flat and the generic point of $\operatorname{Spec} A'$ maps into $G^{r,\mathrm{sep}}_d(X)$, then the special point of $\operatorname{Spec} A'$ maps into $G^{r,\mathrm{sep}}_d(X)$ as well.

If (I) and (II) hold, we have that the $G^{r,\mathrm{sep}}_d(X_1)$ geometrically contains at least m points; if further (III) holds, then $G^{r,\mathrm{sep}}_d(X)$ is finite etale over B, and the geometric generic fiber also consists of exactly m reduced points.

Proof. First, we have by virtue of (I) that $G^{r,\mathrm{sep}}_{d,\mathrm{EH}}(X_0) \cong G^{r,\mathrm{sep}}_d(X_0)$. Setting $U = G^{r,\mathrm{sep}}_d(X/B)$, if we choose any point x in the special fiber, applying Corollary II.4.4 we find that any component Z of U passing through x maps dominantly to B with 0-dimensional generic fiber. To count the number of points, we can take Z to be reduced, in which case by [26, Prop. III.9.7], Z is flat over B, and the number of points in the geometric generic fiber must be at least the number in the special fiber, giving the desired assertion.

In the case that (III) holds, we claim that U is in fact proper over B. It suffices to show that U is closed in $G^r_d(X)$, so choose $y \in U$, $y' \in G^r_d(X)$ distinct points with y specializing to y'; by [58, Prop. 7.1.9] we can find a DVR A' and a map $\operatorname{Spec} A' \to G^r_d(X)$ with the generic point mapping to y and the special point mapping to y'. The image of $\operatorname{Spec} A'$ cannot be contained in the special fiber by 0-dimensionality. If it is not contained in the generic fiber, it gives a flat map $\operatorname{Spec} A' \to \operatorname{Spec} A$, and the hypothesis of (III) implies $y' \in U$ as well. The only case left is when the image of $\operatorname{Spec} A'$ is in the generic fiber; for this, it suffices to show that the generic fiber of U is likewise 0-dimensional, which by the preceding arguments will follow if we show that every component Z of U meets the special fiber of U. But this is now clear: let y be the generic point of Z, then by properness of $G^r_d(X/B)$ there is a y' in the special fiber of $G^r_d(X/B)$ which is a specialization of y; we are therefore back in the flat case of our argument, and $y' \in U$, completing the proof of the assertion that U is closed in $G^r_d(X/B)$, and hence that U is proper over B.

Given properness, since U is unramified in the special fiber, it must be unramified over B; thus, the fibers are reduced, and by Lemma A.22, U is reduced and flat over B, so we conclude the desired finite etaleness, and the geometric generic fiber must then consist of the same number of reduced points as the special fiber, as desired. $\square$

Remark II.5.13. Note that even if G^r_d maps scheme-theoretically into $G^r_{d,\mathrm{EH}}$, the statement of Proposition II.5.7 would be false if we replaced $d-1$ in the inequality by d: we would expect it to fail precise at the intersection of the closed subschemes defined by different choices of a^Y_j and a^Z_j with $a^Y_j + a^Z_{r-j} = d$; recall that $G^r_{d,\mathrm{EH}}$ was defined as a union over all such closed subschemes. This would be asking for any scheme-valued point mapping to the union of these components to map into one component or the other, and this is essentially never the case. For instance, consider $T = \operatorname{Spec} k[\epsilon]/(\epsilon^2)$; the tangent space at a union of components is always expected to be strictly larger than the union of the individual tangent spaces. Of course, we do not *a priori* know when these components of $G^r_{d,\mathrm{EH}}$ might be separated from one another in G^r_d, but it is easy enough to write down examples where they are not, and thereby where one does not have the desired inequality.

The simplest case is $Y \cong Z \cong \mathbb{P}^1$, with affine coordinate functions y and z vanishing at the node, $d = 2$, $r = 0$, and a $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$-valued limit series given by

$$(y^2 + \epsilon y, 0), (y + \epsilon, \epsilon), (\epsilon, z + \epsilon).$$

Here we are identifying sections of $\mathscr{O}(d')$ with polynomials of degree d', and for each i specifying pairs of sections of degree $2 - i$ and i on the components, required to agree at $y = z = 0$; the inclusion maps are then given by multiplication by y or z on the appropriate component, and 0 on the other component. This has $a_0^Y = 1$, $a_0^Z = 0$; note that modulo ϵ, it has $a_0^Y = 2$, $a_0^Z = 1$, so it does not correspond to a refined series, as required by the above proposition.

Remark II.5.14. Continuing along the same line of reasoning, we see that the scheme-theoretic statement of Proposition II.5.7 is actually surprisingly strong; indeed, it implies that if there are two components of the locus of refined series which meet at a point of G^r_d, then if the components have vanishing sequences at the node given by a_j^Y, a_j^Z, and $a_j'^Y, a_j'^Z$, we must have $|a_j^Y - a_j'^Y| \leq 1, |a_j^Z - a_j'^Z| \leq 1$ for all j, as otherwise the scheme-valued point induced by the local ring of G^r_d at our point would fail to satisfy the inequality of our proposition. This is not at all the case for $G^r_{d,\mathrm{EH}}$, so we see that in point of fact, G^r_d does separate out many components of refined series which meet in $G^r_{d,\mathrm{EH}}$. Heuristically, we then have a picture of the geometry of G^r_d as being that of $G^r_{d,\mathrm{EH}}$ on the open subscheme of refined limit series, and separating out certain intersections of components above the boundary locus of crude limit series. Of course, this is a description only of "expected behavior", as either scheme could have components supported only in the boundary, or could even fail to have a non-empty locus of refined limit series.

Remark II.5.15. First, we note that it is not hard to deduce from the proof of Proposition II.5.7 than any refined point is the image of an exact point of the linked Grassmannian used in the construction (see Definition II.A.9): over a field, we noted that we had for a refined point $V_i^Y = \ker \beta_i^Y$, and $V_i^Z = \ker \beta_i^Z$, and one easily checks that the kernel and image of the relevant inclusion maps at the index i are therefore both described as $\ker \beta_{d-i+1}^Z$ in one direction, and $\ker \beta^Y_{i+1}$ in the other, so we conclude that the relevant kernels and images are the same in both directions, as required. Note that the twisting up of the line bundles by an ample divisor doesn't affect whether the point is exact, which is determined on the level of the sub-bundles, independent of the ambient vector bundles.

However, the converse is false. Indeed, there exist non-refined points for which there is an exact point above them, and there exist others for which there isn't. We see both already in the simplest case of $Y \cong Z \cong \mathbb{P}^1$, and $d = 2, r = 0$. In this case, if we choose affine coordinates y, z so that the node corresponds to 0 on each component, we can represent our g^r_d's as three pairs of polynomials in y and in z, of degree $(2,0)$, $(1,1)$, $(0,2)$, with each pair having value agreeing at 0, related to one another by multiplication by y and z, and not uniformly vanishing. For instance, we could consider $(y^2, 0)$, (y, z), $(0, z^2)$. If we look at the first and last pair, we see this has $a^Y_0 = a^Z_0 = 2$, so is certainly not refined, but is in fact an exact point, as (y,z) maps to $(y^2, 0)$ to the left, and $(0, z^2)$ to the right, so surjects onto the kernels of the maps going in the other direction. On the other hand, if we start with the two pairs $(y^2, 0)$, $(0, z^2 + z)$, it is easy to see that essentially the only way to fill in the middle pair is with $(y, 0)$, and this point is not exact, as the maps between the second and third pair are zero is both directions.

II.6 Further Questions

This construction brings up a number of natural further questions, and we briefly set out a few of them. First, as mentioned earlier, the Eisenbud-Harris limit series scheme on a reducible curve was never connected. However, in our case it seems as though the crude limit series ought to serve as bridges between components of refined limit series with differing ramification sequences at the node. In fact, at first blush it may appear based on dimension-counting that crude limit series should simply be the closure of the refined series in many cases, and this may well be true in the Eisenbud-Harris scenario of only looking at a g^r_d on each component, but because our crude series will often map with positive-dimensional fibers to the Eisenbud-Harris crude series, the geometry is not entirely clear. For similar reasons, even though our construction *a priori* gives results on smoothing of crude series, the expected dimension hypothesis for all limit series will not follow immediately from having the expected dimension of refined series. We can therefore reasonably ask:

Question II.6.1. When is the space of limit series on a reducible curve connected?

Question II.6.2. When is the space of refined limit series dense in the space of all limit series?

Question II.6.3. In characteristic 0, what can we say about the dimension of spaces of crude limit series, and by extension their smoothability? In particular, can we smooth a "general" crude series, as we can in the case of refined series (the case of refined series follows from [15, Thm. 4.5])?

We remark that bounding the dimension of crude series on a reducible curve, given an understanding of dimensions of linear series on each component, should be a combinatorial problem, and if the bound is restrictive enough to imply that on a general curve the crude series have dimension at most as large as the dimension of refined series, it will follow that for a general reducible curve, we can always apply the strong form (that is, part (ii)) of Corollary II.4.4 to our entire G^r_d space. In particular, we can actually make use of the properness of the constructed G^r_d scheme to obtain direct arguments for theorems such as Brill-Noether, without requiring arguments involving blowing up the family, as used in [25, p. 261].

Given our inability to adequately describe the T-valued points of $G^r_{d,\mathrm{EH}}(X/B)$, we can also ask:

Question II.6.4. Does $G^r_d(X/B)$ actually map scheme-theoretically into $G^r_{d,\mathrm{EH}}(X/B)$? Is it scheme-theoretically surjective?

Finally, thinking in terms of generalizations, replacing curves by higher-dimensional varieties in our main theorem seems at this point merely a formality, and presents an intriguing array of possibilities and complications. On the one hand, the ability to generalize the theory of limit linear series to higher-dimensional varieties potentially provides a powerful new tool to approach a range of questions on linear series. On the other hand, the "expected dimension" hypothesis of our main theorem is suddenly more of a burden in dimension higher than one. This is amply demonstrated by the interpolation problem (see [7] and [20]), where one sees first that expected dimension for general ramification points need not hold, even for zero-dimensional linear series on $\mathbb{P}^2$, and second, that standard degeneration arguments have thus far failed to succeed in describing when exactly the expected dimension is in fact correct.

In applications, another important direction of generalization is specified ramification along one or more unspecified smooth sections; this may now be accomplished just as with the case of the Eisenbud-Harris theory, by looking at positive-dimensional "special fibers" and allowing ρ to become negative; see [25, p. 270] for an example.

II.A Appendix: The Linked Grassmannian Scheme

In this appendix, we develop of a theory of a moduli scheme parametrizing collections of sub-bundles of vector bundles on a base scheme, linked together via maps between the vector bundles. Representability by a proper scheme is easy and true quite generally; however, to obtain dimension formulas will require more hypotheses and more work. These hypotheses, while reasonably natural and easy to state, are motivated by the idea that the vector bundle maps are induced as pushforwards of certain maps of sufficiently ample line bundles on a scheme proper over the base scheme, as in the situation of Chapter II and its natural generalization to higher-dimensional varieties.

We first specify the objects we will study in more detail; for the remainder of this section, we will be in:

Situation II.A.1. S is any base scheme, and $\mathscr{E}_1, \dots \mathscr{E}_n$ are vector bundles on S, each of rank d. We have maps $f_i : \mathscr{E}_i \to \mathscr{E}_{i+1}$ and $g_i : \mathscr{E}_{i+1} \to \mathscr{E}_i$, and a positive integer $r < d$.

The functor we wish to study may now be easily described:

Definition II.A.2. In this situation, we have the functor $\mathcal{LG}(r, \{E_i\}, \{f_i, g_i\})$, associating to each S-scheme T, the set of sub-bundles $V_1, \dots V_n$ of $\mathscr{E}_{1,T}, \dots \mathscr{E}_{n,T}$ of rank r and satisfying $f_{i,T}(V_i) \subset V_{i+1}$, $g_{i,T}(V_{i+1}) \subset V_i$.

Then without any further hypotheses, we have:

Lemma II.A.3. *$\mathcal{LG}(r, \{E_i\}, \{f_i, g_i\})$ is representable by a projective scheme* LG *over S, which is naturally a closed subscheme of a product of Grassmannian schemes over S; this product is smooth and projective over S of relative dimension $nr(d-r)$.*

Proof. Let G_i be the schemes of Grassmannians of rank r sub-bundles of $\mathscr{E}_i$, as in Theorem A.11. The f_i do not induce morphisms from G_i to G_{i+1}, because a sub-bundle of $\mathscr{E}_i$ may very well not map to a sub-bundle of $\mathscr{E}_{i+1}$, thanks to the condition in the definition of a sub-bundle that the quotient sheaf be locally free. We can, however, construct a closed subscheme of $G := G_1 \times_S \cdots \times_S G_n$ which represents our functor. To construct this subscheme cut out by the inclusion conditions, we denote our projection maps from G to each G_i by π_i, and the maps from each G_i to S by ϕ_i. Let $\mathscr{F}_i$ be the universal sub-bundles on each G_i.

Then each f_i induces a map

$$\pi_i^* \mathscr{F}_i \to \pi_i^* \phi_i^* \mathscr{E}_i = \pi_{i+1}^* \phi_{i+1}^* \mathscr{E}_i \xrightarrow{f_i} \pi_{i+1}^* \phi_{i+1}^* \mathscr{E}_{i+1} \to \pi_{i+1}^* \phi_{i+1}^* \mathscr{E}_{i+1} / \pi_{i+1}^* \mathscr{F}_{i+1}$$

on G, and the kernel of this map is a closed subscheme which imposes precisely the condition that $f_i(V_i) \subset V_{i+1}$. Similarly, g_i induces a map $\pi_{i+1}^* \mathscr{F}_{i+1} \to \pi_i^* \phi_i^* \mathscr{E}_i / \pi_i^* \mathscr{F}_i$ on G whose kernel imposes the condition $g_i(V_{i+1}) \subset V_i$. Taking the intersection of these closed subschemes for all f_i and g_i thus gives a scheme representing $\mathcal{LG}(r, \{E_i\}, \{f_i, g_i\})$, which as a closed subscheme of G is projective over S. $\square$

However, in order to say anything of substance about the scheme representing our functor, and in particular to get the necessary lower bound on dimension, we need to make a number of additional hypotheses. We define:

Definition II.A.4. In Situation II.A.1, we say that $\mathrm{LG}(r, \{E_i\}, \{f_i, g_i\})$ is a **linked Grassmannian** of **length** n if S is integral and Cohen-Macaulay, and the following additional conditions on the f_i and g_i are satisfied:

(I) There exists some $s \in \mathscr{O}_S$ such that $f_i g_i = g_i f_i$ is scalar multiplication by s for all i.

(II) Wherever s vanishes, the kernel of f_i is precisely equal to the image of g_i, and vice versa. More precisely, for any i and given any two integers r_1 and r_2 such that $r_1 + r_2 < d$, then the closed subscheme of S obtained as the locus where f_i has rank less than or equal to r_1 and g_i has rank less than or equal to r_2 is empty.

(III) At any point of S, $\operatorname{im} f_i \cap \ker f_{i+1} = 0$, and $\operatorname{im} g_{i+1} \cap \ker g_i = 0$. More precisely, for any integer r_1, and any i, we have locally closed subschemes of S corresponding to the locus where f_i has rank exactly r_1, and $f_{i+1} f_i$ has rank less than or equal to $r_1 - 1$, and similarly for the g_i. Then we require simply that all of these subschemes be empty.

Remark II.A.5. The hypothesis that S is integral and Cohen-Macaulay is unnecessary for most of our analysis. We use it only in the dimension-theory portion of the argument, to ensure that LG is catenary.

From this point on, we assume that LG is a linked Grassmannian, and we denote its map to S by π.

The following lemma will be convenient for constructing and manipulating points of LG:

Lemma II.A.6. *Let $\{V_i\}_i$ be a k-valued point of* LG, *and suppose $s = 0$ in k. Then for any i, we can decompose V_i as $f_{i-1}(V_{i-1}) \oplus \ker f_i|_{V_i} \oplus C$ for some complementary subspace $C \subset V_i$. Indeed, if we specify any $C' \subset \ker g_{i-1}|_{V_i}$ which intersects $f_{i-1}(V_{i-1})$ trivially, we may choose $C = C' \oplus C''$ for some C''.*

Proof. Clearly, it suffices to show that for any C' as in the statement, we have that $f_{i-1}(V_{i-1}) \oplus \ker f_i|_{V_i} \oplus C'$ injects into V_i. But suppose we have $v_1 \in f_{i-1}(V_{i-1})$, $v_2 \in \ker f_i|_{V_i}$, and $v_3 \in C'$, such that $v_1 + v_2 + v_3 = 0$. If we apply g_{i-1}, by hypothesis $g_{i-1}(v_3) = 0$, and $g_{i-1}(v_1) = 0$ because v_1 is in the image of f_{i-1} and we assumed $s = 0$. So we find that $g_{i-1}(v_2) = 0$, which we claim implies $v_2 = 0$: indeed, $v_2 \in \ker f_i$ by hypothesis, so by the second condition of a linked Grassmannian it is in the image of g_i, and by the third condition, it cannot map to 0 under g_{i-1} unless it is 0. Hence $v_2 = 0$, so $v_1 + v_3 = 0$, and since we assumed that C' was disjoint from $f_{i-1}(V_{i-1})$, we get $v_1 = v_3 = 0$ as well. $\square$

In order to make inductive arguments convenient, we define:

Definition II.A.7. If LG is a linked Grassmannian of length n, and n' any positive integer less than n, we have the **truncation map** from LG to the linked Grassmannian of length n' obtained by forgetting all $\mathscr{E}_i$, f_i, and g_i for all $i > n'$.

We will want to know that the truncation map is always surjective, even on certain classes of families:

Lemma II.A.8. *The truncation map is surjective for all n'. Further, in the case that the base is a point, let $x = \{V_i\}_i$ be any point of* LG, *and suppose we have a family $\tilde{x}_{n'} = \tilde{V}_i|_{i \le n'}$ (that is, a scheme-valued point of the restricted LG scheme) specializing to the truncation of x to length n', and such that $\tilde{V}_{n'}$ may be written as $\tilde{C}_{n'} \oplus \ker f_{n'}|_{V_{n'}}$ for some family $\tilde{C}_{n'}$. Then $\tilde{x}_{n'}$ may be lifted a family $\tilde{x}$ of length n, specializing to x, possibly after a Zariski localization of the base of the family.*

Proof. Surjectivity may be checked on points, and given the description of the $\mathcal{LG}$ functor, it clearly suffices to handle the case $n = 2$, $n' = 1$, since the only issue in lifting a point of a linked Grassmannian of length n' to one of length $n'+1$ is the vector space $V_{n'}$ corresponding to that point, and once we have handled $n' = n - 1$, we can simply project inductively to get the general case.

Now, over a point we may consider $\mathscr{E}_1 = \mathscr{E}_2 = E$ to be a single d-dimensional vector space, and f_1 and g_1 to be self-maps of E. Let V_1 be a vector space of dimension r inside E; we just need to show that there exists a V_2 of dimension r inside E such that $f_1(V_1) \subset V_2$, and $g_1(V_2) \subset V_1$, or equivalently, such that V_2 is contained in $g_1^{-1}(V_1)$ and contains $f_1(V_1)$. $f_1(V_1)$ certainly has dimension less than or equal to r, and is contained in $g_1^{-1}(V_1)$ by hypothesis, so it suffices to observe that $g_1^{-1}(V_1)$ must have dimension at least r, since its dimension is obtained as $\dim \ker g_1 + \dim(V_1 \cap \operatorname{im} g_1)$, and the codimension of $\operatorname{im} g_1$ in E and therefore in V_1 is bounded by the dimension of $\ker g_1$.

For the second assertion, it clearly suffices to show that we can lift to a $\tilde{V}_{n'+1}$ of the form $\tilde{C}_{n'+1} \oplus \ker f_{n'+1}|_{V_{n'+1}}$ and specializing to the truncation of x to length $n' + 1$, since then we can iterate until we have lifted all the way to length n. Thanks to Lemma II.A.6, we can write $V_{n'} = C_{n'} \oplus \ker f_{n'}|_{V_{n'}}$, and $V_{n'+1} = f_{n'}(V_{n'}) \oplus \ker f_{n'+1}|_{V_{n'+1}} \oplus C_{n'+1}$ for some $C_{n'}$ and $C_{n'+1}$, with $\tilde{C}_{n'}$ specializing to $C_{n'}$, and in particular, having full rank under $f_{n'}$ except possibly on a closed subset of the base supported away from x, where the rank could drop. Away from this locus on the base, if we replace $V_{n'+1}$ by $f_{n'}\tilde{V}_{n'} \oplus \ker f_{n'+1}|_{V_{n'+1}} \oplus C_{n'+1}$ (that is, if we set $\tilde{C}_{n+1} = f_{n'}\tilde{V}_{n'} \oplus C_{n'+1}$), noting that $f_{n'}\tilde{V}_{n'} = f_{n'}\tilde{C}_{n'}$, we clearly obtain a lifting with the desired properties. $\square$

The key notion for getting a handle on the LG scheme is the following:

Definition II.A.9. We say that a point of a linked Grassmannian scheme is **exact** if the corresponding collection of vector spaces V_i satisfy the conditions that $\ker g_i|_{V_{i+1}} \subset f_i(V_i)$ and $\ker f_i|_{V_i} \subset g_i(V_{i+1})$ for all i.

The last part of assertion (ii) of the following lemma is gratuitous, but it follows immediately from the argument for the rest, and may perhaps shed some little light on the overall situation.

Lemma II.A.10. *We have the following description of exact points:*

(i) The exact points form an open subscheme of LG*, and are naturally described as the complement of the closed subscheme on which* $\operatorname{rk} f_i|_{V_i} + \operatorname{rk} g_i|_{V_{i+1}} < r$ *for some* i*.*

(ii) In the case $s = 0$*, we find that we can describe exact points as those with* $\operatorname{rk} f_i(V_i) + \operatorname{rk} g_i(V_{i+1}) = r$ *for all* i*, even for arbitrary scheme-valued points, and we also find that an exact point has* $f_i(V_i)$ *a sub-bundle of* V_{i+1}*, and* $g_i(V_{i+1})$ *a sub-bundle of* V_i *for all* i*.*

Proof. We certainly get a closed subscheme as described, simply by taking the union over all i and r_1, r_2 with $r_1 + r_2 < r$ of the loci described by $\operatorname{rk} f_i|_{V_i} \leq r_1$ and $\operatorname{rk} g_i|_{V_{i+1}} \leq r_2$. We immediately see that the points of this set are precisely the complement of the exact points, since outside the locus where s vanishes, both f_i and g_i are invertible, and correspondingly all points are exact; on the other hand, if s vanishes at our point, we have $\operatorname{im} f_i \subset \ker g_i$ and $\operatorname{im} g_i \subset \ker f_i$ for all i, so we already have that $\dim f_i(V_i) + \dim g_i(V_{i+1}) \leq r$ and we get strict inequality if and only if these containments are strict.

For the second part, if a T-valued point satisfies $\operatorname{rk} f_i|_{V_i} + \operatorname{rk} g_i|_{V_{i+1}} = r$ for all i on all of T, by definition it is in the complement of the closed subscheme of non-exact points defined above, so it is certainly exact. Conversely, for the other direction it suffices to work over local rings, so suppose we have a T-valued point (V_i) of LG where T is local, and the point $(\bar{V}_i)$ of LG at the closed point of T is exact. This then is a T-valued point of our open subscheme of exact points, and we want to show it has the asserted properties. Since s is zero on T, then for any given i we have $\ker \bar{f}_i|_{\bar{V}_i} = \bar{g}_i(\bar{V}_{i+1})$ and vice versa; in particular, if we choose $\bar{v}_1, \dots \bar{v}_{r_1}$ spanning $\bar{g}_i(\bar{V}_{i+1})$ and $\bar{v}'_1, \dots \bar{v}'_{r_2}$ spanning $\bar{f}_i(\bar{V}_i)$, we find that $r_1 + r_2 = r$, and we obtain a basis $\bar{e}_i$ (respectively, $\bar{e}'_i$) for $\bar{V}_i$ (respectively, $\bar{V}_{i+1}$) given by the $\bar{v}_j$ and $g_i(\bar{v}'_j)$ (respectively, $\bar{v}'_j$ and $f_i(\bar{v}_j)$). Choosing any lifts v_j and v'_j of the $\bar{v}_j$ and $\bar{v}'_j$, we obtain lifts e_j and e'_j as well, which by Nakayama's lemma freely generate V_i and V_{i+1}. But because $s = 0$, $g_i(V_{i+1}) \subset \ker f_i$ and vice versa; the e_j are all either $v_{j'}$ or $g_i(v'_{j'})$ for some j', so only the former have non-zero images under f_i, and we get that $f_i(V_i)$ is generated by the $f_i(v_j)$, with quotient generated (again by Nakayama) by the v'_j, so both must be freely generated. The same holds for g_i, so we find that $\operatorname{rk} f_i|_{V_i} = r_1$, $\operatorname{rk} g_i|_{V_{i+1}} = r_2$, and $f_i(V_i)$ and $g_i(V_{i+1})$ are both sub-bundles, as desired. □

Our main technical lemma for this section is:

Lemma II.A.11. *We have the following statements on exact points:*

(i) The exact points are dense in LG, *and indeed dense in every fiber.*

(ii) Given any exact point $x \in$ LG, let y be its image in S, suppose A is a local ring, and A' a quotient of A. Let $T = \operatorname{Spec} A$, and $T' = \operatorname{Spec} A'$. Then given any diagram containing the solid arrows of

$$\begin{tikzcd} T' \arrow[r, "f"] \arrow[d] & \mathrm{LG} \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

with the closed point of T' mapping to x, the dashed arrow may also be filled in. In particular, x is a smooth point of LG *over S.*

Proof. For (i), To see that the exact points are dense in every fiber, suppose we have a non-exact point; we just observed that this corresponds to a set of V_i such that for at least one i, we have $\dim f_i(V_i) + \dim g_i(V_{i+1}) < r$. In particular, we are in the situation where $f_i g_i = g_i f_i = 0$, and not the situation where the f_i and g_i are invertible. Now, choose the smallest i such that $\dim f_i(V_i) + \dim g_i(V_{i+1}) < r$, and truncate our linked Grassmannian to $i+1$; here, we show that there are nearby points in the fiber such that the condition $\dim f_i(V_i)+\dim g_i(V_{i+1}) = r$ is satisfied. We leave V_1 through V_i unmodified. By hypothesis, there are vectors in V_{i+1} in the kernel of g_i which are not in $f_i(V_i)$, and vice versa; indeed, we see that $r' := \dim \ker g_i|_{V_{i+1}} - \dim f_i(V_i) = \dim \ker f_i|_{V_i} - \dim g_i(V_{i+1}) = r - \dim f_i(V_i) - \dim g_i(V_{i+1})$. Choose C_i and C_{i+1} in $\ker f_i|_{V_i}$ and $\ker g_i|_{V_{i+1}}$ of dimension r', intersecting $g_i(V_{i+1})$ and $f_i(V_i)$ trivially; we have that together with these spaces, they must complete the span of $\ker f_i|_{V_i}$ and $\ker g_i|_{V_{i+1}}$ respectively. Since $C_i \subset \ker f_i|_{V_i}$, it is in $\operatorname{im} g_i$, and we can find $e_1, \dots e_{r'} \in E_{i+1}$, whose span is necessarily disjoint from V_{i+1}, and which map to a basis of C_i under g_i. By Lemma II.A.6, we can write $V_{i+1} = f_i(V_i) \oplus \ker f_{i+1}|_{V_{i+1}} \oplus C_{i+1} \oplus C''$ for some C''. If we take any basis $e'_1, \dots e'_{r'}$ for C_{i+1}, we can make a family $\tilde{V}_{i+1}$ over $\mathbb{A}^1$ by replacing C_{i+1} with the span of $e'_i + te_i$ for all i, as t varies.

Now, $\tilde{V}_{i+1}$ specializes to V_{i+1} at $t = 0$, and we see that it always remains linked to $V_1, \dots V_i$, left unmodified: it certainly maps into V_i under g_i, since we are modifying basis elements by the e_i, which were chosen to map into V_i; on the other hand, our construction leaves the summand $f_i(V_i)$ unmodified, so f_i certainly maps V_i into any member of $\tilde{V}_{i+1}$. On the other hand, we also observe that we now have $\dim f_i(V_i) + \dim g_i(\tilde{V}_{i+1}) = r$ whenever $t \neq 0$: indeed, C_{i+1} was in the kernel of g_i for $t = 0$, so we still have $g_i(\tilde{V}_{i+1}) \supset g_i(V_{i+1})$; for any $t \neq 0$, C_{i+1} maps isomorphically to C_i under g_i; finally, since we chose C_i to, together with $g_i(V_{i+1})$, span $\ker f_i|_{V_i}$, we find that for any $t \neq 0$, $g_i(\tilde{V}_{i+1}) = \ker f_i|_{V_i}$, giving the desired exactness at i. Finally, by Lemma II.A.8, we can lift this family to a family $\tilde{V}_j$ for all j, specializing to our given point, but now satisfying $\dim f_i(V_i) + \dim g_i(V_{i+1}) = r$ for a general point in the family; we conclude that the points which are non-exact at the ith step (but exact for $j < i$) are in the closure of those which are exact through the ith step, and by induction are actually in the closure of the points which are exact at all steps.

For assertion (ii), $f(T')$ corresponds to a collection $\{V_i\}_i$ over A'; Our $\mathscr{E}_i$ are now all free modules of rank d over A, and we simply want to produce free A-submodules $\tilde{V}_i$ linked by the f_i and g_i and restricting to the given V_i in the quotient ring A'. To do this, denote by $\bar{V}_i$ the collection of subspaces over $\kappa(f^{-1}(x))$ corresponding to x, and let r_i be the dimension of $f_i(\bar{V}_i)$ for each i. Begin by choosing a set of r_1 elements $e_i^1 \in V_1$ whose images under f_1 modulo the maximal ideal form a basis of $f_1(\bar{V}_1)$. Then choose $\tilde{e}_i^1$ any lifts of these elements to $\mathscr{E}_1$, and define $\tilde{e}_j^2 = f_1(\tilde{e}_j^1)$ for all $j \leq r_1$. Now, by condition (III) for a linked Grassmannian, the images of e_j^2 under f_2 remain linearly independent modulo the maximal ideal, but they may not span $f_2(\bar{V}_2)$; if not, choose $r_2 - r_1$ additional elements $e_j^2 \in V_2$ for $r_1 < j \leq r_2$ so that $\{f_2(e_j^2)\}_{j \leq r_2}$ span $f_2(\bar{V}_2)$, and choose any lifts $\tilde{e}_j^2$ of these. Now define $\tilde{e}_j^3 = f_2(\tilde{e}_j^2)$ for all $j \leq r_2$, and proceed similarly until we have defined $\tilde{e}_j^i$ for all j and all $j \leq r_i$, where we set r_n, not defined *a priori*, to be equal to r_{n-1}. Now, if our f_i and g_i are invertible at the special point, which is to say, if the s from condition (I) of a linked Grassmannian is non-zero in $\kappa(y)$, we will have $r_i = r$ for all i, and we will have already defined $\tilde{e}_j^i$ for all i and j.

However, if s is zero in $\kappa(y)$, there is more work to be done. In this case, let r_i' be the dimension of $g_i(\bar{V}_{i+1})$; then replacing j by $r + 1 - j$ in the indices for the basis vectors, and moving back from V_n to V_1, we may use the g_i to similarly define $\tilde{e}_j^i$ for all i and all $j \geq r + 1 - r_{i+1}'$, where we use the convention $r_1' = r_2'$. We claim that for any i, we will have defined $\tilde{e}_j^i$ for all j, and that we have not created conflicting definitions in this manner. For the first assertion, we need to know that $r_{i-1}' + r_i \geq r$, but by the third condition of a linked Grassmannian, it suffices to check that $r_i' + r_i = r$, and we observe that this is precisely the condition for a point to be exact, since then we have $r_i = \dim \ker g_i = r - r_i'$. To see that we don't get conflicting definitions, we first note that if $\tilde{e}_j^i$ was defined as the image of $\tilde{e}_j^{i-1}$ under f_{i-1}, this implies that $j \leq r_{i-1}$, whereas for $\tilde{e}_j^i$ to have been defined in the g_i process, we would have had $j \geq r + 1 - r'_i = r_i + 1$, and $r_i \geq r_{i-1}$. Therefore, it suffices to show that all $\tilde{e}^i_j$ not defined as either $f_{i-1}(\tilde{e}^{i-1}_j)$ or $g_i(\tilde{e}^{i+1}_j)$ could be chosen compatibly for both the f_i and g_i definition process. But this is clear: we chose such intermediate e^i_j to fill out bases under f_i and g_{i-1} for $f_i(\bar{V}_i)$ and $g_{i-1}(\bar{V}_i)$, but otherwise arbitrarily, and then the $\tilde{e}^i_j$ were just arbitrary lifts of the e^i_j. There were $r_i - r_{i-1} = r'_{i-1} - r'_i$ such elements in both the f_i and g_i definition process, so as long as we specify that these intermediate e^i_j should always be chosen to be linearly independent modulo the maximal ideal from the span of $f_{i-1}(\bar{V}_i)$ and $g_i(\bar{V}_{i+1})$ together, we get $r - r_{i-1} - r'_i = r_i - r_{i-1}$ elements which (again, using that exactness means that $f_{i-1}(\bar{V}_i) = \ker g_i|_{\bar{V}_{i+1}}$ and $g_i(\bar{V}_{i+1}) = \ker f_i|_{\bar{V}_i}$) are easily seen to simultaneously satisfy the requirements for both the f_i and g_i definition process.

Finally, we claim that the $\tilde{e}^i_j$ constructed in this manner define sub-bundles $\tilde{V}_i$ of $\mathscr{E}_i$ of rank r, restricting in A' to the V_i, and linked by the f_i and g_i, as desired. To see that the $\tilde{V}_i$ are sub-bundles of rank r which restrict to the V_i in A', it suffices to show that the e^i_j are bases of $\bar{V}_i$ modulo the maximal ideal: given this, Nakayama's lemma immediately implies that they generate the V_i, so the restriction to A' is correct; it also shows that any submodule of a free module over a local ring generated by the lifts of linearly independent elements at the special fiber is necessarily free, with free quotient, giving the sub-bundle assertion as well. On the other hand, it is clear from the construction that the reduction of the e^i_j must be linearly independent in $\bar{V}_i$, and since there are exactly r of them, and V_i has dimension r, they form a basis. In the case that s was non-zero in $\kappa(y)$, the $\tilde{V}_i$ are linked under the f_i by construction, and must likewise be linked under the g_i, since g_i is a unit times the inverse of f_i. In the case where s was zero in $\kappa(y)$, take any $\tilde{e}^i_j$ for $i < n$; we show that its image under f_i is a scalar multiple of $\tilde{e}^{i+1}_j$. Indeed, in the case that $f_i(\tilde{e}^i_j)$ was non-zero, we defined $\tilde{e}^{i+1}_j = f_i(\tilde{e}^i_j)$. But if $f_i(\tilde{e}^i_j)$ was zero, then in our construction e^i_j was necessarily chosen as $g_i(e^{i+1}_j)$, and we defined $\tilde{e}^i_j = g_i(\tilde{e}^{i+1}_j)$, so $f_i(\tilde{e}^i_j) = s(\tilde{e}^{i+1}_j)$, as desired. Thus we have constructed a map from T to $\mathcal{LG}$ lifting f, which by [64, Prop. 17.14.2] completes the proof of part (ii). □

The following proposition provides a strong converse to part (ii) of the above lemma:

Proposition II.A.12. *The non-exact points of a fiber are precisely the intersections of the components of that fiber.*

Proof. Since the exact points are smooth, they are certainly not in any intersection of components. For the other direction, we first observe that if we have two exact points given by V_i and V_i', and denote by r_i and r_i' the dimensions of $f_i(V_i)$ and $f_i(V_i')$, if some $r_i \neq r_i'$, the two points must lie on distinct components of LG. For this, it suffices to show that any exact point must have the same r_i as the generic point of its component. But the $\dim f_i(V_i)$ could only drop under specialization, and by exactness we also have $r_i = r - \dim g_i(V_{i+1})$, which can only increase under specialization. Thus, to show that any non-exact point is in the intersection of components, it suffices to exhibit it as the specialization of two different exact points with distinct r_i.

Looking at the proof of Lemma II.A.11 part (i), we see that any point which is non-exact at i_0, with i_0 minimal, can expressed as the specialization of an exact point with r_i unchanged for all $i \leq i_0$; however, upon closer examination, we see that in fact the process leaves all the r_i unchanged, simply increasing the dimensions of the $g_i(V_{i+1})$ as necessary to make the points exact. On the other hand, we note that the linked Grassmannian situation is completely symmetric in the f_i and g_i, so now that we have shown that any point can be written as the specialization of an exact point with the $\dim f_i(V_i)$ unchanged, it follows by symmetry that there is another exact point specializing to our given point, leaving the dimensions of the $g_i(V_{i+1})$ intact, and therefore necessarily increasing at least some of the r_i. This then expresses our non-exact point as lying in the intersection of two components, as desired. □

We can also use the smoothness at exact points to compute the dimension of fibers of LG:

Lemma II.A.13. *The fibers of* LG *over* S *have every component of dimension precisely* $r(d-r)$.

Proof. In view of Lemma II.A.11, the exact points are dense in every fiber, and smooth, and in particular dense and smooth in any fiber. We can therefore compute the dimension of any component of the fiber by showing that its tangent space at any exact point has the desired dimension. Since we are only looking at a fiber, we set $S = \operatorname{Spec} k$. If $s \neq 0$ in k, LG $\cong \mathbb{G}(r,d)$, and is smooth of dimension $r(d-r)$, so there is nothing to show. Otherwise, suppose we have a collection of V_i corresponding to an exact point. Then $\ker f_i|_{V_i} = g_i(V_{i+1})$ for all i, so we use Lemma II.A.6 to write each V_i as $f_{i-1}(V_{i-1}) \oplus g_i(V_{i+1}) \oplus C_i$ for some complementary space C_i. Our first assertion is that the dimensions d_i of the C_i add up to r.

Indeed, if we let $r_i = \dim f_i(V_i)$, and $r_i' = \dim g_i(V_{i+1})$, we have $r_i = r - r_i'$ from exactness, and for $1 < i < n$, $d_i = r - r_{i-1} - r_i' = r_i - r_{i-1}$, with $d_1 = r - r_1' = r_1$ and $d_n = r - r_n$, so we see we indeed have $\sum_i d_i = r$.

The next claim is that first-order deformations of the V_i inside of LG correspond precisely to first-order deformations of each C_i individually inside E_i, taken modulo deformation of C_i which remain inside V_i. Any deformation of the C_i together will yield a deformation of the V_i: we use our direct sum decomposition to inductively define the induced deformation, obtaining deformations of $f_i(V_i)$ as the image of the deformation of C_{i-1} together with the (inductively obtained) deformation of $f_{i-1}(V_{i-1})$, and similarly for the $g_i(V_{i+1})$. Moreover, since each $f_i(V_i)$ is spanned by $f_{i-1}(V_{i-1})$ together with C_{i-1}, this is the only possible way to obtain a deformation of the V_i given deformations of the C_i. Clearly, two deformations of the C_i will yield equivalent deformations of V_i if and only if their difference is a deformation of the C_i inside of its V_i. Finally, any deformation of the V_i may be expressed (non-uniquely) as a deformation of its summands, and in particular gives a deformation of the C_i, at least up to the same equivalence relation. Since the deformation of the V_i induced by the deformations of the C_i was unique, this must invert our first construction, completing the proof of the claim.

Now we are done: first-order deformations of any given C_i are given by the tangent space to $\mathbb{G}(d_i, d)$, which is a variety smooth of dimension $d_i(d - d_i)$, so has $d_i(d - d_i)$-dimensional tangent space at any point. Similarly, the space of deformations of C_i inside of V_i has dimension $d_i(r - d_i)$; the difference is $d_i(d - r)$. Thus, the total dimension of our tangent space is $\sum_i d_i(d - r) = r(d - r)$, as asserted. $\square$

We now have all the tools to prove our main result:

Theorem II.A.14. *A linked Grassmannian scheme is a closed subscheme of the obvious product of Grassmannian schemes over S; it is projective over S, and each component has codimension $(n-1)r(d-r)$ inside the product, and maps surjectively to S. If s is non-zero, then* LG *is also irreducible.*

Proof. We already have that the linked Grassmannian is projective over S, and lies inside the obvious product of Grassmannians, which we denote by G. It is easy to see each component maps dominantly onto S, since the exact points are both smooth and dense by Lemma II.A.11.

For the dimension statement, given any component of LG, let x be an exact point of LG on the specified component, and not on any other component, and s the image of x in S. Since $\mathscr{O}_{S,s}$ is Cohen-Macaulay, it is Noetherian, and of finite dimension (it follows from [41, Thm. 13.5] that the height of the maximal ideal of a Noetherian local ring, and hence the dimension of the ring, is finite). By Lemma II.A.13, we have that $\mathscr{O}_{\mathrm{LG},x}$ is smooth over $\mathscr{O}_{S,s}$ of relative dimension $r(d-r)$, and in particular reduced, by [3, Prop. 2.3.9]. But by the choice of x, $\mathscr{O}_{\mathrm{LG},x}$ is also irreducible, hence integral. Similarly, by Theorem A.11, we have that $\mathscr{O}_{G,x}$ is locally affine over $\mathscr{O}_{S,s}$, hence integral and smooth of relative dimension $nr(d-r)$. Denote by P the prime ideal of $\mathscr{O}_{G,x}$ corresponding to our component; since $\mathscr{O}_{\mathrm{LG},x}$ is integral, we have $\mathscr{O}_{\mathrm{LG},x} = \mathscr{O}_{G,x}/P$. We simply want to show that the maximal chain of prime ideals between P and (0) has length $(n-1)d(d-r)$. Since S is Cohen-Macaulay, it is universally catenary by [41, Thm. 17.9], so G is catenary, and it suffices to find the maximal length of a chain from $\mathfrak{m}_x$ to (0), and subtract the maximal length of a chain from $\mathfrak{m}_x$ to P, which is to say, to subtract the dimension of $\mathscr{O}_{\mathrm{LG},x}$ from $\mathscr{O}_{G,x}$. But by [64, Prop. 17.5.8 (i)], we have that this difference is precisely the difference of the fiber dimensions, which is $(n-1)r(d-r)$, as desired.

Finally, when s is non-zero, over the open subset of S where s is invertible, the fibers are all simply Grassmannians of dimension $r(d-r)$; since the map is proper, we conclude that LG is irreducible over this locus, of dimension $r(d-r)$. On the other hand, since every component maps dominantly to S, there cannot be any component of LG contained in the locus where s vanishes, yielding the desired irreducibility. □

Remark II.A.15. We have stated our results in terms of codimension rather than dimension because for a catenary scheme, the notion of codimension is global, while the notion of dimension only makes sense locally, unless we add hypotheses along the lines of being of finite type over a base field.

Warning II.A.16. Lemma II.A.10 sounds quite innocuous, but there are some real pitfalls to be aware of. Consider the simple example of $n = d = 2$, $r = 1$, $S = \operatorname{Spec} k$, $E_1 = E_2 = k^2$, $f_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $f_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. In this case, if V_1 is generated by $v_1 = \begin{bmatrix} X_0 \\ X_1 \end{bmatrix}$ and V_2 by $v_2 = \begin{bmatrix} Y_0 \\ Y_1 \end{bmatrix}$, we find the condition for them to be linked is simply that $X_0 Y_1 = 0$, and it is easy enough to check that we actually get that LG is scheme-theoretically cut out by this equation inside $\mathbb{P}^1 \times \mathbb{P}^1$, giving a pair of $\mathbb{P}^1$'s attached at $X_0 = Y_1 = 0$, which is the only non-exact point. Our lemma has shown that deformations have to behave well at the exact points, but if we consider the T-valued point for $T = \operatorname{Spec} k[\epsilon]/(\epsilon^2)$ with V_1 generated by $v_1 = \begin{bmatrix} \epsilon \\ 1 \end{bmatrix}$ and V_2 generated by $v_2 = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$, we note two pathologies:

First, this point actually satisfies our initial set-theoretic description of an exact point, that $\ker g_1|_{V_2} \subset f_1(V_1)$ and vice versa, as both images and kernels will be given precisely by ϵv_i. So this description, while clean for dealing with both $s = 0$ and s invertible simultaneously, is only valid from a set-theoretic point of view.

Second, while we have shown that at any (scheme-valued) exact point, there will be an r_1 and r_2 with $r_1 + r_2 = r$ and $\operatorname{rk} f_1|_{V_1} \leq r_1$, $\operatorname{rk} g_1|_{V_2} \leq r_2$, we see that by allowing the ranks to drop at the closed point, we actually allow them to increase on the local ring level. Specifically, in our case $r = 1$, so either r_1 or r_2 would have to be 0, but neither f_1 nor g_1 is the zero map. Of course, this makes perfect sense geometrically, as the node will necessarily have tangent vectors which don't point along either branch, but it underscores the fact that the T-valued points of a union of schemes is not simply the union of the T-valued points of the individual schemes.

We conclude with an example and some further questions which we have not pursued here because they are not necessary for our applications.

Example II.A.17. We consider the situation of $S = \operatorname{Spec} k$, $n = 2$. In this case, it is easy to describe the components explicitly, as well as to see their dimensions without invoking any deformation theory. We already know that if $s \neq 0$, we just get a Grassmannian, so we assume that $s = 0$. If we write $d_1 = \operatorname{rk} f_1$, $d_2 = \operatorname{rk} g_1$ (on the entire vector space), we have $d_1 + d_2 = d$ by condition (II) of a linked Grassmannian. We will see that there are $\min\{r+1, d-r+1, d_1+1, d_2+1\}$ components, each of dimension $r(d-r)$, and indexed by the dimension of $f_1(V_1)$ on general points.

Indeed, we saw in the proof of Lemma II.A.8 that the fiber of any point V_1 of G_1 under truncation is simply the Grassmannian of vector spaces V_2 containing $f_1(V_1)$ and contained in $g_1^{-1}(V_1)$, which had dimension $\dim \ker g_1 + \dim(V_1 \cap \operatorname{im} g_1)$. We need to see that this dimension depends only on the dimension of $f_1(V_1)$, which we will denote by r_1. By condition (II) of a linked Grassmannian, $\ker g_1 = \operatorname{im} f_1$, and $\operatorname{im} g_1 = \ker f_1$, so we may write this as $\dim \operatorname{im} f_1 + \dim(V_1 \cap \ker f_1)$. Now, since we are over a point, $\dim \operatorname{im} f_1$ is invariant, and on the other hand, $\dim(V_1 \cap \ker f_1) = r - r_1$, so we can write everything in terms of r_1, as desired. Specifically, we have a Grassmannian of r-dimensional subspaces of a $(d_1 + r - r_1)$-dimensional space, containing an r_1-dimensional space, and this has dimension $(r - r_1)(d_1 - r_1)$.

We now obtain our assertions without too much trouble: fix an $r_1 \leq \min\{r, d_1\}$ also satisfying $r_1 \geq \max\{0, r - d_2\}$, and consider the locally closed subset $G_1^{r_1}$ in G_1 with $\dim f_1(V_1) = r_1$. Note that the specified range is precisely the range for which this will be non-empty. Now, $G_1^{r_1}$ is irreducible of codimension $(r-r_1)(d_1-r_1)$, since it is an open subset of the locus in G_1 with $\dim f_1(V_1) \leq r_1$, which corresponds simply to a Schubert cycle, which is irreducible of codimension $(r - r_1)(d_1 - r_1)$ (Theorem A.11). If we base change LG to $G_1^{r_1}$, we get a proper map with irreducible equidimensional fibers, mapping surjectively to an irreducible base, so in fact LG becomes irreducible, and has dimension precisely $r(d-r)$. Since this dimension remains constant as r_1 decreases, and the codimension of $G_1^{r_1}$ increases as r_1 decreases, we find we must have exactly one irreducible component of LG for each choice of r_1.

Question II.A.18. Can we show that LG is flat over S? That it is reduced?

Question II.A.19. Can we describe the components of LG for $n > 2$?

Chapter III

Explicit Formulas and Frobenius-Unstable Bundles

The primary goal of this chapter is to develop very explicit formulas for p-curvature potentially applicable to a wide range of situations, and to apply them to the study of semi-stable vector bundles of rank-2 vector bundles on genus-2 curves which pull back to unstable bundles under the relative Frobenius morphism. Using a combination of characteristic-specific and characteristic-independent techniques, we apply our explicit p-curvature formulas to connections on certain unstable vector bundles to classify such "Frobenius-unstable" bundles in characteristics 3, 5, and 7. Unlike the degeneration approaches to the same results, used first by Mochizuki in [42], and subsequently in Chapter VI of this work, which have the advantage of being characteristic-independent and more suitable to generalization, the p-curvature formulas obtained here may be used to study arbitrary smooth curves, and do not give results only for general curves. This distinction is underscored by an algorithm derived via the same techniques to explicitly describe the loci of curves of genus 2 and p-ranks 0 or 1 in any specified characteristic.

Our main theorem is:

Theorem III.0.1. *Let C be a smooth, proper curve of genus 2 over an algebraically closed field k of characteristic p; it may be described on an affine part by $y^2 = g(x)$ for some quintic g. Then the number of semistable vector bundles on C with trivial determinant which pull back to unstable vector bundles under the relative Frobenius morphism is:*

$p = 3$*:* $16 \cdot 1$*;*

$p = 5$: $16 \cdot e_5$, *where* $e_5 = 5$ *for a general* C, *and is given for an arbitrary* C *as the number of distinct roots of a quintic polynomial with coefficients in terms of the coefficients of* g*;*

$p = 7$: $16 \cdot e_7$, *where* $e_7 = 14$ *for a general* C, *and is given for an arbitrary* C *as the number of points in the intersection of four curves in* $\mathbb{A}^2$ *whose coefficients are in terms of the coefficients of* g.

Together with prior results (see, for instance, [37]) in characteristic 2, we see that although the answer is the same for all curves in characteristics 2 and 3, already by characteristic 5 there is variation, albeit easily analyzed, and by characteristic 7 it has arrived at what is essentially the worst case scenario: an (apparently) incomplete intersection of hypersurfaces in affine space, with no obvious means even of showing, for instance, that it is always non-empty. It becomes apparent that the good behavior of lower characteristics is simply due to the p-curvature formulas yielding polynomials of sufficiently low degree to allow isolating variables as much as needed, but that this does not occur in higher degree, and for characteristics 7 and up it seems unlikely that this sort of explicit computation will have much luck in analyzing arbitary curves.

Section III.1 recalls the necessary background and notation. Section III.2 is devoted to developing explicit and completely general combinatorial formulas for p-curvature, and applying them to derive a much simpler formula in the rank 1 case. Section III.3 uses some more abstract observations to reduce finding the Frobenius-unstable bundles in our case (indeed, in any odd characteristic) to analysis of a single vector bundle $\mathscr{E}$. Section III.4 applies the rank 1 p-curvature formulas of Section III.2 to the case of genus 2, giving explicit formulas for a function we will need, and also applying them to give an explicit algorithm for generating p-rank formulas in any specific odd characteristic. Section III.5 carries out the characteristic-independent part of the desired computation for a single $\mathscr{E}$, namely computing the space of transport-equivalence classes of connections on $\mathscr{E}$, and Section III.6 concludes the computation, giving definitive answers in characteristics 3 and 5, as well as a look at what happens in characteristic 7. Section III.7 explicitly recovers in the genus 2 case a finite-flatness result of Mochizuki, which provides for statements about general curves and completes the proof of Theorem III.0.1. Section III.8 then discusses some further calculations and questions arising from the chapter.

The answer has already been obtained in characteristic 2 by Laszlo and Pauly: there is always a single Frobenius-unstable bundle (see [37], argument for Prop. 6.1 2.; the equations for an ordinary curve are not used). Joshi, Ramanan, Xia and Yu obtain results on the Frobenius-unstable locus in characteristic 2 for higher-genus curves in [28]. Mochizuki [42] has obtained the answer in the language of projective line bundles for a general curve of genus 2 in any odd characteristic, via techniques quite similar to the degeneration techniques which we pursue in Chapter VI. However, such degeneration techniques are typically only suited for obtaining results on a general curve, with the caveat that a certain finite flatness result [42, II, Thm. 2.8, p. 153] for arbitrary curves does show that the number of Frobenius-unstable bundles on an arbitrary smooth curve is at most the number on a general curve. See the introduction to Chapter VI for a more detailed discussion. Finally, Lange and Pauly [34] have, concurrently with the preparation of the present work, recovered Mochizuki's formula in the case of ordinary curves via a completely different approach, although they obtain only an inequality, rather than an equality.

III.1 Background: Definitions and Notation

We begin by reviewing the relevant background definitions and terminology. We work over an arbitrary base scheme S for the sake of a few technical points which arise, but for conceptual purposes it will suffice to take $S = \operatorname{Spec}(k)$ for some field k. Throughout this section, X will be a smooth, proper scheme over S. However, properness is only necessary for statements involving connections with trivial determinant or degrees of line bundles.

We start by recalling the language of connections on vector bundles and associated operations, which are independent of characteristic. We have the rank-n vector bundle $\Omega^1_{X/S}$ of 1-forms on X over S; given any other vector bundle $\mathscr{E}$ on X, a **connection** on $\mathscr{E}$ is an $\mathscr{O}_S$-linear map $\nabla : \mathscr{E} \to \Omega^1_{X/S} \otimes \mathscr{E}$ satisfying $\nabla(fs) = f\nabla(s) + df \otimes s$. Now, given an automorphism ϕ of $\mathscr{E}$, we can **transport** any connection ∇ along ϕ to get a new connection, given as $\phi \otimes 1 \circ \nabla \circ \phi^{-1}$. We also recall that given an affine open $U \subset X$, θ is called a **derivation** on $\mathscr{O}_X(U)$ if it is an $\mathscr{O}_S$-linear map from $\mathscr{O}_X(U)$ to itself satisfying the Leibniz rule $\theta(fg) = \theta(f)g + f\theta(g)$; associated to θ is a unique $\mathscr{O}_X(U)$-linear homomorphism $\hat{\theta}$ from $\Omega^1_{X/S}(U)$ to $\mathscr{O}_X(U)$, which gives θ upon precomposition with $d : \mathscr{O}_X(U) \to \Omega^1_{X/S}(U)$ (see [13, p. 386]).

We now recall some standard vector bundle constructions, and note that they extend to connections. If $\mathscr{E}$ is a vector bundle of rank n on X, there is the associated **determinant** line bundle, obtained simply as $\bigwedge^n \mathscr{E}$. We remark that the operations of tensor product, wedge product, and dualization (and in particular, of taking the determinant bundle or a homomorphism bundle) extend naturally to vector bundles with connection: indeed, given connections ∇_i on $\mathscr{E}_i$ for $i = 1, \dots, m$, one can define a connection on $\mathscr{E}_1 \otimes \cdots \otimes \mathscr{E}_m$ by the formula $\nabla_1 \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \nabla_2 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes \nabla_m$; moreover, in the case that all the $\mathscr{E}_i = \mathscr{E}$ and $\nabla_i = \nabla$ are equal, this descends to the quotient bundle defining the mth wedge product of $\mathscr{E}$. Now suppose ∇ is a connection on some $\mathscr{E}$; we construct a canonical connection $\nabla^\vee$ on $\mathscr{E}^\vee$ by associating to every functional $v \in \mathscr{E}^\vee(U)$ the element $\nabla^\vee(v)$ of $\mathscr{E}^\vee \otimes \Omega^1_X(U)$ described by $s \mapsto -v(\nabla(s))$; we then see that $\nabla^\vee(fv)(s) = -fv(\nabla(s)) = v(-f\nabla(s)) = v(s \otimes df - \nabla(fs)) = v(s) \otimes df - v(\nabla(fs)) = v(s) \otimes df + \nabla^\vee(v)(fs)$, so $\nabla^\vee$ is indeed a connection. Using these associated connections, we immediately find that for vector bundles $\mathscr{E}, \mathscr{F}$ with connections ∇, ∇', we get canonical induced connections on $\mathcal{H}om(\mathscr{E}, \mathscr{F})$ and $\det \mathscr{E}$. The former is given explicitly by $\varphi \mapsto \nabla' \circ \varphi - (\varphi \otimes 1) \circ \nabla$, and therefore its horizontal sections are precisely homomorphisms from $\mathscr{E}$ to $\mathscr{F}$ which commute with ∇, ∇'. We call the latter the **determinant connection**. We can therefore say that a vector bundle $\mathscr{E}$ has **trivial determinant** if its determinant bundle is isomorphic to $\mathscr{O}_X$; given a connection ∇ on such an $\mathscr{E}$, we say that ∇ has **trivial determinant** if the induced connection on $\det \mathscr{E}$ corresponds simply to d under a chosen isomorphism $\det \mathscr{E} \cong \mathscr{O}_X$. It is easy to check that scalar automorphisms won't affect connections, so when X is proper the notion of trivial determinant for a connection is independent of the choice of automorphism. Finally, we note that given a connection ∇ on $\mathscr{E}$, we get an induced connection not only on $\mathcal{E}nd(\mathscr{E})$, but also on the sub-bundle $\mathcal{E}nd^0(\mathscr{E})$ of traceless endomorphisms; indeed, this is checked easily from the definition of the induced connection on $\mathcal{E}nd(\mathscr{E})$, if one thinks of $\mathcal{E}nd(\mathscr{E})$ as $\mathscr{E}^\vee \otimes \mathscr{E}$ and the trace as being given by evaluation.

Now suppose that S has characteristic p. We review the associated Frobenius morphisms, and then discuss the notion of p-curvature which is associated to an integrable connection. There is a canonical variety $X^{(p)}$, the **p-twist** of X over S, and the **relative Frobenius morphism** $F_{X/S} : X \to X^{(p)}$ over S; if we denote by F_X and F_S the absolute Frobenius morphisms of X and S, $X^{(p)} := X \times_S S$ under the map $F_S : S \to S$, and $F_{X/S}$ is characterized by $F_X = \pi_{X/S} \circ F_{X/S}$, where $\pi_{X/S}$ is the change of base of F_S to X. Since we work almost entirely with the relative Frobenius morphism, we will also denote it simply by F when there is no possibility of confusion on what X and S are. If $\mathscr{F}, \mathcal{G}$ are $\mathscr{O}_X$-modules, we also define a map $\varphi : \mathscr{F} \to \mathcal{G}$ to be **p-linear** if it is additive, and satisfies $\varphi(fs) = f^p\varphi(s)$ for $s \in \mathscr{F}(U)$, $f \in \mathscr{O}_X(U)$, and all open $U \subset X$. Now, for any $\mathscr{O}_X$-module $\mathscr{F}$, it is easy to see that the natural map $\pi_{\mathscr{F}} : \mathscr{F} \to F_X^*\mathscr{F}$ gives a "universal p-linear map", which is to say, any p-linear map $\mathscr{F} \to \mathcal{G}$ factors through $\pi_{\mathscr{F}}$ to give a unique $\mathscr{O}_X$-linear map $F_X^*\mathscr{F} \to \mathcal{G}$.

If ∇ is an **integrable** connection on $\mathscr{E}$, which is to say a connection such that the usual curvature ∇^2 vanishes, then there is a notion of p-curvature associated to ∇ defined as follows: for any affine open U of X, and any derivation θ on U, we define $\nabla_\theta : \mathscr{E} \to \mathscr{E}$ by composing ∇ with $1 \otimes \hat{\theta}$. Now, since we are in characteristic p, θ^p is another derivation, and we define the **p-curvature** on U to be the map from $\mathrm{Der}(\mathscr{O}_X(U))$ to $\mathcal{E}nd_{\mathscr{O}_S}(\mathscr{E})$ given by $\psi_\nabla(\theta) = (\nabla_\theta)^p - \nabla_{(\theta^p)}$. We extend this to a sheaf morphism $\psi_\nabla : \mathrm{Der}(\mathscr{O}_X) \to \mathcal{E}nd_{\mathscr{O}_S}(\mathscr{E})$, and it turns out that it actually takes values in $\mathcal{E}nd_{\mathscr{O}_X}(\mathscr{E})$, and moreover is p-linear on $\mathrm{Der}(\mathscr{O}_X)$ (that is, $\psi_\nabla(f\theta) = f^p\psi_\nabla(\theta)$) (see [30, 5.0.5, 5.2.0]). Finally, $\psi_\nabla(\theta)$ commutes with $\nabla_{\theta'}$ for any θ' [30, 5.2.3]. Finally, if we pull back a vector bundle $\mathscr{E}$ on $X^{(p)}$ under F, the sections of $F^*\mathscr{E}$ on $U \subset X$ will be described as $f \otimes s$, where $f \in \mathscr{O}_X(U), s \in \mathscr{E}(U)$, and we have $(F^*f) \otimes s = 1 \otimes fs$. We then see that we can have a **canonical connection** ∇^{can} on $F^*\mathscr{E}$ defined by $f \otimes s \mapsto s \otimes df$. The main important of p-curvature for us is Theorem III.1.4 below, but we will first explore the formal properties of the p-curvature map.

Since, $\mathrm{Der}(\mathscr{O}_X) \cong (\Omega^1_{X/S})^\vee$, the p-linearity means we can consider ϕ_∇ as an $\mathscr{O}_X$-linear map $F_X^*(\Omega^1_{X/S})^\vee \to \mathcal{E}nd(\mathscr{E})$; compatibility of $\Omega^1_{X/S}$ with base change yields $\pi^*_{X/S}\Omega^1_{X/S} \cong \Omega^1_{X^{(p)}/S}$, so $F_X^*\Omega^1_{X/S} \cong F^*\Omega^1_{X^{(p)}/S}$, and we finally find we can consider p-curvature as giving a global section

$$\psi_\nabla \in \Gamma(X, \mathcal{E}nd(\mathscr{E}) \otimes F^*\Omega^1_{X^{(p)}/S}).$$

We claim that in fact, ψ_∇ lies in the kernel of the connection ∇^{ind} induced on $\mathcal{E}nd(\mathscr{E}) \otimes F^*\Omega^1_{X^{(p)}}$ by ∇ on $\mathscr{E}$ (inducing $\nabla^{\mathcal{E}nd}$ on $\mathcal{E}nd(\mathscr{E})$) and ∇^{can} on $F^*\Omega^1_{X^{(p)}}$. This follows formally from the fact that $\psi_\nabla(\theta)$ commutes with $\nabla_{\theta'}$ for all θ, θ'; if one thinks of ψ_∇ as a linear map $F^*\mathrm{Der}_{X^{(p)}} \to \mathcal{E}nd(\mathscr{E})$, one may use this commutativity to explicitly write down the actions of ∇^{can} and $\nabla^{\mathcal{E}nd}$, and see that they commute with ψ_∇. We therefore obtain the strengthened statement:

$$\psi_\nabla \in \Gamma(X, \mathcal{E}nd(\mathscr{E}) \otimes F^*\Omega^1_{X^{(p)}/S})^{\nabla^{\mathrm{ind}}}, \qquad \text{(III.1.1)}$$

where ∇^{ind} is the connection induced by ∇ and ∇^{can}. Last, we also note that because $\phi \otimes 1$ commutes with $1 \otimes \theta$, transport by ϕ simply conjugates the p-curvature map by ϕ; in particular, vanishing p-curvature is transport-invariant.

If $\mathscr{E}$ has trivial determinant, we may restrict to connections with trivial determinant, and then obtain a map to endomorphisms with 0 trace, by Proposition A.30 (ii). Because we have worked over an arbitrary base scheme S of characteristic p, in particular our construction works after arbitrary base change; we can therefore think of ψ as giving algebraic maps between affine spaces, determined on their T-valued points for all T over S:

$$\begin{aligned}\psi &: \Gamma(X, \mathrm{Conn}(\mathscr{E})) \to \Gamma(X, \mathcal{E}nd(\mathscr{E}) \otimes F^*\Omega^1_{X^{(p)}})\\ \psi^0 &: \Gamma(X, \mathrm{Conn}^0(\mathscr{E})) \to \Gamma(X, \mathcal{E}nd^0(\mathscr{E}) \otimes F^*\Omega^1_{X^{(p)}})\end{aligned} \tag{III.1.2}$$

We can then take the determinant of the resulting endomorphisms to get maps

$$\det \psi : \Gamma(X, \mathrm{Conn}(\mathscr{E})) \to \Gamma(X, (F^*\Omega^1_{X^{(p)}})^{\otimes n})$$

$$\det \psi^0 : \Gamma(X, \mathrm{Conn}^0(\mathscr{E})) \to \Gamma(X, (F^*\Omega^1_{X^{(p)}})^{\otimes n})$$

where $\mathscr{E}$ has rank n. In the case of $\mathrm{Conn}^0(\mathscr{E})$ on an $\mathscr{E}$ of trivial determinant, it is not hard to check that ∇^{can} is the induced connection on $(F^*\Omega^1_{X^{(p)}})^{\otimes n}$, and that (III.1.1) then implies that the image of $\det \psi$ is in the kernel of ∇^{can}, and may therefore be considered an element of $\Gamma(C^{(p)}, (\Omega^1_{C^{(p)}})^{\otimes n})$. We conclude that we have in this case:

$$\det \psi^0 : \Gamma(X, \mathrm{Conn}^0(\mathscr{E})) \to \Gamma(X^{(p)}, (\Omega^1_{X^{(p)}})^{\otimes n}) \tag{III.1.3}$$

We now begin to explore the importance of p-curvature, leading up to Theorem III.1.4. The main point is that p-curvature is extremely useful for studying pullbacks of vector bundles under Frobenius: given $\mathscr{E}$ on X, with a connection ∇, ∇ is by definition linear on functions f with $df = 0$, so the kernel of ∇ is naturally a $\mathscr{O}_{X^{(p)}}$-module. Moreover, if we have $\mathscr{F}$ on $X^{(p)}$, it is easy to see from the definition that the kernel of ∇^{can} on $F^*\mathscr{F}$ recovers $\mathscr{F}$. We also see that given a derivation θ, $\nabla^{\mathrm{can}}_\theta$ is given by $f \otimes s \mapsto (\theta f) \otimes s$, so that the p-curvature associated to ∇^{can} is visibly always 0. This may seem suggestive, and indeed the Cartier theorem states that given a vector bundle $\mathscr{E}$ with a connection ∇ whose p-curvature vanishes, then $\mathscr{E}$ is the pullback of a vector bundle on $X^{(p)}$ under Frobenius, with ∇ being the corresponding canonical connection. One can even construct an appropriate categorical equivalence in this manner, known as the **Cartier isomorphism**:

Theorem III.1.4. *(Essentially [30, 5.1]) Let X be a smooth S-scheme, with S having characteristic p, and let $F : X \to X^{(p)}$ be the relative Frobenius morphism. Then pullback under Frobenius (together with the associated canonical connection) and taking kernels of connections are mutually inverse functors, giving an equivalence of categories between the category of vector bundles of rank n on $X^{(p)}$ and the full subcategory of the category of vector bundles of rank n with integrable connection on X consisting of objects whose connection has p-curvature zero.*

Furthermore, the same statement holds when restricted to the full subcategories of vector bundles with trivial determinant on $X^{(p)}$, and vector bundles with connection both having trivial determinant on X.

Proof. The main ingredient is the Cartier isomorphism theorem, giving the same statement in the case of coherent sheaves; see [30, 5.1]. It only remains to check that vector bundles are mapped to vector bundles of the same rank, in either direction. This is of course trivial for the functor given by F^*; for the other direction, let $\mathscr{E}$ be a vector bundle with integrable connection ∇, and suppose the p-curvature of ∇ is zero. Now, $\mathscr{E}^{\nabla}$ is a sub-module of $F_*\mathscr{E}$; since F is finite, $F_*\mathscr{E}$ is coherent, and $X^{(p)}$ being Noetherian, $\mathscr{E}^{\nabla}$ is also coherent. Now, we already know that $F^*\mathscr{E}^{\nabla} \cong \mathscr{E}$, so $\mathscr{E}^{\nabla}$ is locally free of rank n if and only if it is locally free, if and only if it is flat over $X^{(p)}$. But since X is smooth over S, by Proposition A.25 F is (faithfully) flat, and we know that $F^*\mathscr{E}^{\nabla}$ is flat on X, so we conclude that $\mathscr{E}^{\nabla}$ is flat on $X^{(p)}$, as desired.

Lastly, the correspondence in the case of trivial determinant follows immediately from Proposition A.30 (i). □

Using this functoriality, two bundles pulling back to $\mathscr{E}$ are isomorphic if and only if their corresponding canonical connections on $\mathscr{E}$ are related by transport under some automorphism, so classifying isomorphism classes of vector bundles pulling back to $\mathscr{E}$ under Frobenius is equivalent to classifying transport equivalence classes of connections on $\mathscr{E}$ with vanishing p-curvature.

We see from the Cartier isomorphism that p-curvature is an important tool for studying, among other things, the action of pullback by Frobenius on moduli spaces of vector bundles on X. The moduli perspective will be deferred until Chapter IV, but we review the appropriate vector bundle terminology for this chapter in the case where $X = C$ is a proper smooth curve, noting that in this case, all connections are automatically integrable, so have well-defined p-curvature. The **degree** of $\mathscr{E}$ is the degree of the determinant, as a line bundle. The **slope** of $\mathscr{E}$ is defined to be $\frac{1}{n}$ times the degree of $\mathscr{E}$, and $\mathscr{E}$ is said to be **stable** if every proper, non-zero sub-bundle has slope less than the slope of $\mathscr{E}$, and **semistable** if every sub-bundle has slope less than or equal to the slope of $\mathscr{E}$. If k has positive characteristic, and F denotes the relative Frobenius morphism from $C^{(p)}$ to C, we say that a semistable $\mathscr{E}$ on C is **Frobenius unstable** if $F^*\mathscr{E}$ is unstable on $C^{(p)}$.

Finally, we recall the tools of trivializations and matrix expressions that will form the underpinnings of our explicit calculations.

If we have any vector bundle $\mathscr{E}$ on C which is trivialized on open sets U_1 and U_2, we shall say E is the **transition matrix** for $\mathscr{E}$ for the given trivialization if sections s_i on U_i, written in terms of the trivialization, are related by

$$s_1 = Es_2. \tag{III.1.5}$$

Then, a map between two such bundles with transition matrices E_1 and E_2 is given by matrices S_i regular on their respective U_i, which send s_i to $S_i s_i$ and must therefore satisfy the relationship

$$S_1 E_1 = E_2 S_2. \tag{III.1.6}$$

Similarly, a connection on $\mathscr{E}$ is equivalent to one-form-valued matrices T_i regular on their respective U_i satisfying

$$\nabla(s_i) = T_i s_i + ds_i. \tag{III.1.7}$$

As result, we find that the T_i must satisfy

$$T_1 = ET_2E^{-1} + E(dE^{-1}). \tag{III.1.8}$$

For the convenience of dealing with functions rather than forms, if we have ω_i trivializing the canonical bundle on the U_i, and we take the naturally induced trivialization of $\mathscr{E} \otimes \Omega^1_C$

on the U_i, we can write connection matrixes of functions $\bar{T}_i = \frac{T_i}{\omega_i}$, so that

$$\nabla(s_i) = \bar{T}_i s_i \omega_i + ds_i, \tag{III.1.9}$$

and obtain in this case the formula

$$\bar{T}_1 = \frac{\omega_2}{\omega_1} E \bar{T}_2 E^{-1} + E \frac{dE^{-1}}{\omega_1}. \tag{III.1.10}$$

Lastly, transport of a connection given by T_i along an automorphism given by S_i is determined by the formula:

$$T_i \mapsto S_i^{-1} T_i S_i + S_i^{-1}(dS_i), \tag{III.1.11}$$

or equivalently,

$$\bar{T}_i \mapsto S_i^{-1} \bar{T}_i S_i + S_i^{-1} \frac{dS_i}{\omega_i}. \tag{III.1.12}$$

Finally, we remark that it is not hard to check that given a trivialization of $\mathscr{E}$ on U_1 and U_2 with transition matrix E, the line bundle $\det \mathscr{E}$ will be trivialized on U_1 and U_2, with transition function given by $\det E$. If further we have a connection matrix T_i on U_i, the determinant connection on $\det \mathscr{E}$ is given simply by the function $\operatorname{Tr} T_i$ on U_i.

III.2 Explicit p-curvature Formulas

The formulas developed in this section should be of use in any dimension, but for the sake of simplicity of notation, we will restrict ourselves to the case where $X = C$ is an arbitrary smooth curve. We pin down our notation further:

Situation III.2.1. U denotes an affine open on a smooth curve C. We are given a vector bundle $\mathscr{E}$ trivialized on U, and Ω^1_C is trivialized by a one-form ω on U. Under these trivializations and the corresponding tensor trivialization of $\mathscr{E} \otimes \Omega^1_C$, we get as in III.1.9 a connection matrix $\bar{T}$ on U associated to any connection ∇ on $\mathscr{E}$. Finally, for a derivation θ we denote $\hat{\theta}(\omega)$ by f_θ, and set θ_0 as the derivation on U with $f_{\theta_0} = 1$.

We first claim:

Lemma III.2.2. *Given a connection ∇ described on an affine open U by a connection matrix $\bar{T}$, a derivation θ on U, and a section s of $\mathscr{E}$ on U, we have $\nabla_\theta(s) = f_\theta \bar{T} s + \theta s$,* *where f_θ is defined as $\hat{\theta}(\omega)$, and θ acts on s by application to each coefficient.*

Proof. Under our choice of trivializations for $\mathscr{E}$ and $\mathscr{E} \otimes \Omega^1_C$, we had ∇ given by

$$s \mapsto \bar{T}s + \frac{ds}{\omega}$$

and the trivialization for $\mathscr{E} \otimes \Omega^1_C$ was gotten on U by tensoring with ω, so we find that ∇_θ sends

$$s \mapsto f_\theta \bar{T}s + f_\theta \frac{ds}{\omega}.$$

Lastly, we recall that by the construction of how to get between $\hat{\theta}$ and θ, $f_\theta \frac{ds}{\omega}$ is precisely θs, giving the desired result. $\square$

We thus obtain:

Corollary III.2.3. $\psi_\nabla(\theta) = (f_\theta \bar{T} + \theta)^p - f_{\theta^p}\bar{T} - \theta^p$

We next observe that because p-curvature is p-linear on derivations, and because every derivation on U can be written as some function on U times the derivation θ_0, we conclude:

Lemma III.2.4. *The p-curvature of a connection ∇ is identically 0 if and only if $\psi_\nabla(\theta_0) = 0$ for the particular derivation θ_0, which is to say for the derivation given by $f_\theta = 1$.*

In this situation, we have $\psi_\nabla(\theta_0) = (\bar{T} + \theta_0)^p - f_{\theta_0^p}\bar{T} - \theta_0^p$. However, from this formula it is not at all clear that the result is an endomorphism, nor how to compute the corresponding matrix for the p-curvature on U. In order to be able to work this out explicitly, we develop explicit formulas for $(\bar{T} + \theta_0)^n$, using the commutation relation $\theta_0 \bar{T} = (\theta_0 \bar{T}) + \bar{T}\theta_0$, where $(\theta_0 \bar{T})$ denotes the application of θ_0 to the coordinates of $\bar{T}$ (it might perhaps be preferable to denote this by $\theta_0(\bar{T})$, but I find the former notation will make formulas easier to parse).

Proposition III.2.5. *Given $\mathfrak{i} = (i_1, \ldots, i_\ell) \in \mathbb{N}^{\ell-1} \times (\mathbb{N} \cup \{0\})$ with $\sum_{j=1}^{\ell} i_j = n$, denote by $\hat{n}_\mathfrak{i}$ the coefficient of $\bar{T}_\mathfrak{i} = (\theta_0^{i_1-1}\bar{T}) \ldots (\theta_0^{i_{\ell-1}-1}\bar{T})\theta_0^{i_\ell}$ in the full expansion of $(\bar{T} + \theta_0)^n$. Also denote by $\mathfrak{i}_0$ the vector $(i_1, \ldots, i_{\ell-1}, 0)$. Then we have:*

$$\hat{n}_\mathfrak{i} = \binom{n}{i_\ell} \hat{n}_{\mathfrak{i}_0}$$

Proof. This proof and the following may be carried out by induction purely numerically as follows: by definition, we have

$$(\bar{T}+\theta_0)^n = (\bar{T}+\theta_0)(\sum_{\ell'}\sum_{|\mathrm{i}'|=n-1}\hat{n}_{\mathrm{i}'}\bar{T}_{\mathrm{i}'}),$$

where $\mathrm{i}' = (i'_1,\ldots,i'_{\ell'})$, and $|\mathrm{i}'| := \sum_j i'_j$. Multiplying out and commuting the θ_0 from left to right until we obtain another such expression (using the relation that $\theta_0(\theta_0^i\bar{T}) = (\theta_0^{i+1}\bar{T})+(\theta_0^i\bar{T})\theta_0)$, we find two cases: $i_1 = 1$ and $i_1 > 1$; we will illustrate the case $i_1 > 1$, the other being essentially the same. In this case, we obtain the inductive formula $\hat{n}_{\mathrm{i}} = \sum_j \hat{n}_{\mathrm{i}-1_j}$, where 1_j denotes the vector which is 1 in the jth position and 0 elsewhere, and where j is allowed to range only over values where $i_j > 1$. We then have also that $\hat{n}_{\mathrm{i}_0} = \sum_{j<\ell}\hat{n}_{\mathrm{i}_0-i_j}$, so that if we induct on n, we have $\hat{n}_i = \sum_j \hat{n}_{i-1_j} = \sum j < \ell\binom{n-1}{i_\ell}\hat{n}_{(\mathrm{i}-1_j)_0} + \binom{n-1}{i_\ell-1}\hat{n}_{(\mathrm{i}-1_\ell)_0} = (\binom{n-1}{i_\ell}+\binom{n-1}{i_\ell-1})\hat{n}_{\mathrm{i}_0}$, where the last identity makes use of the observation that $\mathrm{i}_0 = (\mathrm{i}-1_\ell)_0$. Then the identity $\binom{n-1}{r}+\binom{n-1}{r-1} = \binom{n}{r}$ completes the proof.

However, such inductive arguments to verify pre-supplied formulas are frequently not enlightening, so we attempt to give arguments for this and the next result to explain directly how the formulas in question arise. Considering as above what happens when one takes an expression in $\bar{T}$ and θ_0, multiplies on the left by $(\bar{T}+\theta_0)$, and commutes the θ_0 to the right, one sees that the coefficient of $\bar{T}_{\mathrm{i}} = (\theta_0^{i_1-1}\bar{T})\ldots(\theta_0^{i_{\ell-1}-1}\bar{T})\theta_0^{i_\ell}$ in the expansion of $(\bar{T}+\theta_0)^n$ is obtained by counting the number of ways of starting with the empty expression θ_0^0, and getting to the given expression $\bar{T}_{\mathrm{i}}$, with a valid step being to either add a $\bar{T} = (\theta_0^0\bar{T})$ on the left, or to increment any existing power of θ_0. Since each step increases the sum of the i_j by one, the number of steps is in fact always determined, but I will include it nonetheless to make calculations more transparent. We immediately see that since we start with θ_0^0, and need to end up with $\theta_0^{i_\ell}$ on the right, each way of getting to $\bar{T}_{\mathrm{i}}$ in n steps yields a unique path to $\bar{T}_{\mathrm{i}_0}$ in $n-i_\ell$ steps, simply by skipping the steps which increment the final power of θ_0, and conversely, for each way of getting to $\bar{T}_{\mathrm{i}_0}$ (taking $n-i_\ell$ steps), we obtain a way of getting to $\bar{T}_{\mathrm{i}}$ by inserting i_ℓ steps incrementing the trailing power of θ_0, which may occur at any time; thus, there are $\binom{n}{i_\ell}$ ways of doing this, yielding the desired result. □

It follows that if $n = p$, $\hat{n}_{\mathrm{i}}$ is nonzero mod n only if $i_\ell = 0$ or $i_\ell = n$, and in the latter case, we have $\ell = 1$, $i_1 = p$, and $\hat{n}_{\mathrm{i}} = 1$, which precisely cancels the θ_0^p subtracted off in the formula for $\psi_\nabla(\theta_0)$. We immediately see that $\psi_\nabla(\theta_0)$ is in fact given entirely by linear terms, explicitly recovering the statement we already knew to be true that p-curvature takes values in the space of $\mathscr{O}_C$-linear endomorphisms of $\mathscr{E}$. We may now restrict our attention to the linear terms in the expansion, and will shift our notation accordingly:

Proposition III.2.6. *Given* $\mathfrak{i} = (i_1, \dots, i_\ell) \in \mathbb{N}^\ell$ *with* $\sum_{j=1}^{\ell} i_j = n$*, denote by* $n_\mathfrak{i}$ *the coefficient of* $\bar{T}_\mathfrak{i} = (\theta_0^{i_1-1}\bar{T}) \dots (\theta_0^{i_\ell-1}\bar{T})$ *in the full expansion of* $(\bar{T} + \theta_0)^n$*. Also denote by* $\hat{\mathfrak{i}}$ *the truncated vector* $(i_1, \dots, i_{\ell-1})$*. Then we have:*

$$n_\mathfrak{i} = \binom{n-1}{i_\ell - 1} n_{\hat{\mathfrak{i}}}$$

We thus get

$$n_\mathfrak{i} = \prod_{j=1}^{\ell} \binom{n - 1 - \sum_{m=j+1}^{\ell} i_m}{i_j - 1} = \frac{(n-1)!}{(\prod_{j=1}^{\ell}(i_j - 1)!)(\prod_{j=1}^{\ell-1}(\sum_{m=1}^{j} i_m))}$$

Proof. We make the same analysis as in the proof of the previous lemma; given a path to the given expression $\bar{T}_\mathfrak{i}$ in n steps, we get a unique path to the truncated expression $\bar{T}_{\hat{\mathfrak{i}}}$ in $n - i_\ell$ steps by skipping the creation of the final term as well as all steps incrementing its power of θ_0. Conversely, given a path to $\bar{T}_{\hat{\mathfrak{i}}}$ in $n - i_\ell$ steps, to obtain a path to $\bar{T}_\mathfrak{i}$ we must add in the creation of the final term, and the incrementation of the powers of θ_0 for that term. However, the former is entirely fixed: since new terms can be added only on the left, and the extra term must end up on the right, its creation must occur on the very first step, and there is therefore no choice involved. The only choices that arise are where to add in the $i_\ell - 1$ incrementations of the power of θ_0 in that final term, which must fall somewhere in the remaining $n - 1$ steps, giving the desired result. $\square$

We note that this actually implies that every such term in the expansion of $(\bar{T} + \theta_0)^n$ is nonzero mod n when $n = p$, since the numerator in the resulting formula is simply $(n-1)!$. Thus, the p-curvature formula is actually always 'maximally' complex in some sense, having $2^{p-1} + 1$ terms (a choice of $\mathfrak{i}$ is equivalent to choosing the resulting partial sums, which can be any subset of $\{1, \dots, p-1\}$; but there is also the $f_{\theta_0^p}\bar{T}$ term). However, when some of the terms commute, the formulas tend to simplify considerably. In this chapter we will only apply the following proposition to the particular case of rank one, where everything commutes, but we prove a more general form for later use.

Proposition III.2.7. *Given* $\mathrm{i} = (i_1, \ldots, i_\ell) \in \mathbb{N}^\ell$ *with* $\sum_{j=1}^{\ell} i_j = n$*, and given a subset* $\Lambda \subset \{1, \ldots, \ell\}$*, denote by* S_ℓ^Λ *the subset of* S_ℓ *which preserves the order of the elements of* Λ*. We also denote by* n_{i}^Λ *the sum over all* $\sigma \in S^\Lambda \ell$ *of* $n_{\sigma(\mathrm{i})}$*, where* $\sigma(\mathrm{i})$ *denotes the vector obtained from* i *by permuting the indices under* σ*. Then we have:*

$$n_{\mathrm{i}}^{\Sigma} = \frac{n!}{\prod_{j=1}^{\ell}(i_j - 1)! \prod_{j=1}^{\ell}(i_j + \sum_{m<j}^{m,j\in\Lambda} i_m)}.$$

Note that the last sum in the denominator is non-empty only for $j \in \Lambda$*.*

Proof. Applying our previous formula, we really just want to show that

$$\sum_{\sigma\in S_\ell^\Lambda} \prod_{j=1}^{\ell-1} \frac{1}{\sum_{m=1}^{j} i_{\sigma(m)}} = \frac{n}{\prod_{j=1}^{\ell}(i_j + \sum_{m<j}^{m,j\in\Lambda} i_m)} = \frac{\sum_{j=1}^{\ell} i_j}{\prod_{j=1}^{\ell}(i_j + \sum_{m<j}^{m,j\in\Lambda} i_m)},$$

and dividing through by $n = \sum_{j=1}^{\ell} i_j$ reduces the identity to

$$\sum_{\sigma\in S_\ell^\Lambda} \prod_{j=1}^{\ell} \frac{1}{\sum_{m=1}^{j} i_{\sigma(m)}} = \frac{1}{\prod_{j=1}^{\ell}(i_j + \sum_{m<j}^{m,j\in\Lambda} i_m)}.$$

We show this by induction on ℓ (noting that it is rather trivial in the case $\ell = 1$, whether or not Λ is empty), breaking up the first sum over S_ℓ into $\ell - |\Lambda| + 1$ pieces, depending on which i_r ends up in the final place. There are two cases to consider: $r \notin \Lambda$, or $r = \Lambda_{\max}$. In either case, the relevant part of the sum on the left hand side becomes $\sum_{\sigma\in S_\ell^{\Lambda,r}} \prod_{j=1}^{\ell} \frac{1}{\sum_{m=1}^{j} i_{\sigma(m)}}$, where $S_\ell^{\Lambda,r}$ denotes the subset of S_ℓ^Λ sending r to ℓ. Now, the point is that this may be considered as a subset of the symmetric group acting on a set of $\ell - 1$ elements, allowing us to apply induction; in the case that $r \notin \Lambda$, Λ is in essence unaffected, and we find that

$$\sum_{\sigma\in S_\ell^{\Lambda,r}} \prod_{j=1}^{\ell} \frac{1}{\sum_{m=1}^{j} i_{\sigma(m)}} = \frac{1}{n} \sum_{\sigma\in S_\ell^{\Lambda,r}} \prod_{j\neq r} \frac{1}{\sum_{m\leq j}^{m\neq r} i_{\sigma(m)}} = \frac{1}{n} \frac{1}{\prod_{j\neq r}(i_j + \sum_{m<j}^{m,j\in\Lambda} i_m)}$$

$$= \frac{i_r + \sum_{m<r}^{m,r\in\Lambda} i_m}{n \prod_{j=r}^{\ell}(i_j + \sum_{m<j}^{m,j\in\Lambda} i_m)} = \frac{i_r}{n \prod_{j=r}^{\ell}(i_j + \sum_{m<j}^{m,j\in\Lambda} i_m)},$$

since $r \notin \Lambda$. In the case that $r = \Lambda_{\max}$, we effectively reduce the size of Λ by one, but because r is maximal in Λ, for $j \neq r$ the term $\Lambda_{m<j}^{m,j\Lambda} i_m$ is unaffected by omitting r from Λ.

We thus find

$$\sum_{\sigma\in S_\ell^{\Lambda,r}}\prod_{j=1}^{\ell}\frac{1}{\sum_{m=1}^{j} i_{\sigma(m)}}=\frac{i_r+\sum_{m<j}^{m,j\in\Lambda} i_m}{n\prod_{j=r}^{\ell}(i_j+\sum_{m<j}^{m,j\in\Lambda} i_m)}=\frac{\sum_{j\in\Lambda} i_j}{n\prod_{j=r}^{\ell}(i_j+\sum_{m<j}^{m,j\in\Lambda} i_m)}.$$

Adding these up as r ranges over $\Lambda_{\max}$ and all values not in Λ, and using $n=\sum_j i_j$, we get the desired identity. □

From this proposition, we see that when $n=p$ and Λ is empty, so that all the involved matrices commute, we have

$$n_{\mathfrak{i}}^{\varnothing}=\frac{n!}{\prod_{j=1}^{\ell}(i_j)!},$$

but the actual coefficient will be $n_{\mathfrak{i}}^{\varnothing}/P_{\mathfrak{i}}$, where $P_{\mathfrak{i}}$ is the number of permutations fixing the vector $\mathfrak{i}$, since summing up over all permutations will count each term $P_{\mathfrak{i}}$ times. We see that this expression can be non-zero mod n only if either $P_{\mathfrak{i}}$ is a multiple of n, or some i_j is. Since $P_{\mathfrak{i}}$ is the order of a subgroup of S_ℓ, it can be a multiple of n if and only if $\ell=n$ and each $i_j=1$. On the other hand, an i_j can be a multiple of n if and only if $\ell=1$ and $i_1=n$; these two terms simply reiterate that the coefficients of $\bar{T}^n$ and $(\theta_0^{n-1}\bar{T})$ are both 1, and we see that every other coefficient vanishes mod n, from which we obtain the vastly friendlier formula:

Corollary III.2.8. *In rank 1, p-curvature is given by:*

$$\psi_\nabla(\theta_0)=\bar{T}^p+(\theta_0^{p-1}\bar{T})-f_{\theta_0^p}\bar{T}.$$

We also record the general p-curvature formulas in characteristics 3, 5, and 7, for later use.

Characteristic 3:

$$\psi_\nabla(\theta_0)=\bar{T}^3+(\theta_0\bar{T})\bar{T}+2\bar{T}(\theta_0\bar{T})+(\theta_0^2\bar{T})-f_{\theta_0^3}\bar{T} \tag{III.2.9}$$

Characteristic 5:

$$\begin{aligned}\psi_\nabla(\theta_0) &= \bar{T}^5 + 4\bar{T}^3(\theta_0^1\bar{T}) + 3\bar{T}^2(\theta_0^1\bar{T})\bar{T} + \bar{T}^2(\theta_0^2\bar{T}) + 2\bar{T}(\theta_0^1\bar{T})\bar{T}^2 \\ &+3\bar{T}(\theta_0^1\bar{T})^2 + 3\bar{T}(\theta_0^2\bar{T})\bar{T} + 4\bar{T}(\theta_0^3\bar{T}) + (\theta_0^1\bar{T})\bar{T}^3 + 4(\theta_0^1\bar{T})\bar{T}(\theta_0^1\bar{T}) \\ &+3(\theta_0^1\bar{T})^2\bar{T} + (\theta_0^1\bar{T})(\theta_0^2\bar{T}) + (\theta_0^2\bar{T})\bar{T}^2 + 4(\theta_0^2\bar{T})(\theta_0^1\bar{T}) + (\theta_0^3\bar{T})\bar{T} \\ &+(\theta_0^4\bar{T}) - f_{\theta_0^5}\bar{T}\end{aligned} \tag{III.2.10}$$

Characteristic 7:

$$\begin{aligned}\psi_\nabla(\theta_0) &= \bar{T}^7 + 6\bar{T}^5(\theta_0^1\bar{T}) + 5\bar{T}^4(\theta_0^1\bar{T})\bar{T} + \bar{T}^4(\theta_0^2\bar{T}) + 4\bar{T}^3(\theta_0^1\bar{T})\bar{T}^2 \\ &+3\bar{T}^3(\theta_0^1\bar{T})^2 + 3\bar{T}^3(\theta_0^2\bar{T})\bar{T} + 6\bar{T}^3(\theta_0^3\bar{T}) + 3\bar{T}^2(\theta_0^1\bar{T})\bar{T}^3 \\ &+4\bar{T}^2(\theta_0^1\bar{T})\bar{T}(\theta_0^1\bar{T}) + \bar{T}^2(\theta_0^1\bar{T})^2\bar{T} + 3\bar{T}^2(\theta_0^1\bar{T})(\theta_0^2\bar{T}) + 6\bar{T}^2(\theta_0^2\bar{T})\bar{T}^2 \\ &+\bar{T}^2(\theta_0^2\bar{T})(\theta_0^1\bar{T}) + 3\bar{T}^2(\theta_0^3\bar{T})\bar{T} + \bar{T}^2(\theta_0^4\bar{T}) + 2\bar{T}(\theta_0^1\bar{T})\bar{T}^4 \\ &+5\bar{T}(\theta_0^1\bar{T})\bar{T}^2(\theta_0^1\bar{T}) + 3\bar{T}(\theta_0^1\bar{T})\bar{T}(\theta_0^1\bar{T})\bar{T} + 2\bar{T}(\theta_0^1\bar{T})\bar{T}(\theta_0^2\bar{T}) \\ &+\bar{T}(\theta_0^1\bar{T})^2\bar{T}^2 + 6\bar{T}(\theta_0^1\bar{T})^3 + 6\bar{T}(\theta_0^1\bar{T})(\theta_0^2\bar{T})\bar{T} + 5\bar{T}(\theta_0^1\bar{T})(\theta_0^3\bar{T}) \\ &+3\bar{T}(\theta_0^2\bar{T})\bar{T}^3 + 4\bar{T}(\theta_0^2\bar{T})\bar{T}(\theta_0^1\bar{T}) + \bar{T}(\theta_0^2\bar{T})(\theta_0^1\bar{T})\bar{T} + 3\bar{T}(\theta_0^2\bar{T})^2 \\ &+4\bar{T}(\theta_0^3\bar{T})\bar{T}^2 + 3\bar{T}(\theta_0^3\bar{T})(\theta_0^1\bar{T}) + 5\bar{T}(\theta_0^4\bar{T})\bar{T} + 6\bar{T}(\theta_0^5\bar{T}) + (\theta_0^1\bar{T})\bar{T}^5 \\ &+6(\theta_0^1\bar{T})\bar{T}^3(\theta_0^1\bar{T}) + 5(\theta_0^1\bar{T})\bar{T}^2(\theta_0^1\bar{T})\bar{T} + (\theta_0^1\bar{T})\bar{T}^2(\theta_0^2\bar{T}) \\ &+4(\theta_0^1\bar{T})\bar{T}(\theta_0^1\bar{T})\bar{T}^2 + 3(\theta_0^1\bar{T})\bar{T}(\theta_0^1\bar{T})^2 + 3(\theta_0^1\bar{T})\bar{T}(\theta_0^2\bar{T})\bar{T} \\ &+6(\theta_0^1\bar{T})\bar{T}(\theta_0^3\bar{T}) + 3(\theta_0^1\bar{T})^2\bar{T}^3 + 4(\theta_0^1\bar{T})^2\bar{T}(\theta_0^1\bar{T}) + (\theta_0^1\bar{T})^3\bar{T} \\ &+3(\theta_0^1\bar{T})^2(\theta_0^2\bar{T}) + 6(\theta_0^1\bar{T})(\theta_0^2\bar{T})\bar{T}^2 + (\theta_0^1\bar{T})(\theta_0^2\bar{T})(\theta_0^1\bar{T}) \\ &+3(\theta_0^1\bar{T})(\theta_0^3\bar{T})\bar{T} + (\theta_0^1\bar{T})(\theta_0^4\bar{T}) + (\theta_0^2\bar{T})\bar{T}^4 + 6(\theta_0^2\bar{T})\bar{T}^2(\theta_0^1\bar{T}) \\ &+5(\theta_0^2\bar{T})\bar{T}(\theta_0^1\bar{T})\bar{T} + (\theta_0^2\bar{T})\bar{T}(\theta_0^2\bar{T}) + 4(\theta_0^2\bar{T})(\theta_0^1\bar{T})\bar{T}^2 \\ &+3(\theta_0^2\bar{T})(\theta_0^1\bar{T})^2 + 3(\theta_0^2\bar{T})^2\bar{T} + 6(\theta_0^2\bar{T})(\theta_0^3\bar{T}) + (\theta_0^3\bar{T})\bar{T}^3 \\ &+6(\theta_0^3\bar{T})\bar{T}(\theta_0^1\bar{T}) + 5(\theta_0^3\bar{T})(\theta_0^1\bar{T})\bar{T} + (\theta_0^3\bar{T})(\theta_0^2\bar{T}) + (\theta_0^4\bar{T})\bar{T}^2 \\ &+6(\theta_0^4\bar{T})(\theta_0^1\bar{T}) + (\theta_0^5\bar{T})\bar{T} + (\theta_0^6\bar{T}) - f_{\theta_0^7}\bar{T}\end{aligned} \tag{III.2.11}$$

III.3 Preparations in Genus 2

Having derived these entirely general p-curvature formulas, we now make use of some less explicit observations in preparation for applying this to the specific case we examine for the rest of the chapter. This case is:

Situation III.3.1. C is a smooth, proper curve of genus 2, over an algebraically closed field k of characteristic p.

In this situation, A. J. de Jong observed that there are only finitely many Frobenius-unstable vector bundles of rank 2 and trivial determinant on C (see Lemma IV.A.10 for finiteness, although we will also obtain a more direct proof from Corollary III.7.4), and we have the following description of them (given previously in [29, Prop. 3.5]):

Proposition III.3.2. *Let $\mathscr{F}$ be a semistable rank 2 vector bundle on C with trivial determinant, and suppose $\mathscr{E} = F^*\mathscr{F}$ is unstable. Then there is a non-split exact sequence*

$$0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{L}^{-1} \to 0$$

where $\mathscr{L}$ is a **theta characteristic***, that is, $\mathscr{L}^{\otimes 2} \cong \Omega^1_C$.*

Proof. Suppose $\mathscr{E}$ is unstable, and the Frobenius pullback of some $\mathscr{F}$ which is (semi)stable of trivial determinant. Then $\mathscr{E}$ also has trivial determinant, and in particular degree 0, so for it to be unstable is equivalent to having a subsheaf locally free of rank 1, and of positive degree, say $\mathscr{L}$. If we choose $\mathscr{L}$ of maximal degree, it will automatically be saturated inside $\mathscr{E}$, and since C is a smooth curve, it will have locally free quotient, and we will have an exact sequence

$$0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{L}^{-1} \to 0$$

(where we know the quotient bundle is $\mathscr{L}^{-1}$ because the determinant of $\mathscr{E}$ will be the tensor product of $\mathscr{L}$ with the quotient bundle, and has to be trivial by hypothesis).

But since $\mathscr{E} = F^*\mathscr{F}$, we have ∇^{can} on $\mathscr{E}$, a connection with vanishing p-curvature, and if we tensor our exact sequence by Ω^1_C, we get the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{L} \arrow[r] & \mathscr{E} \arrow[r] \arrow[d, "\nabla^{\text{can}}"] & \mathscr{L}^{-1} \arrow[r] & 0 \\
0 \arrow[r] & \Omega^1_C \otimes \mathscr{L} \arrow[r] & \Omega^1_C \otimes \mathscr{E} \arrow[r] & \Omega^1_C \otimes \mathscr{L}^{-1} \arrow[r] & 0
\end{tikzcd}$$

If we start with $\mathscr{L}$, and follow it into $\mathscr{E}$, down along ∇^{can}, and then over to $\Omega^1_C \otimes \mathscr{L}^{-1}$, we will get a map from $\mathscr{L}$ to $\Omega^1_C \otimes \mathscr{L}^{-1}$ which one can verify directly is in fact linear (even though ∇^{can} was not). Now, we note that this map is non-zero: if it were 0, we would find that the image of $\mathscr{L}$ under ∇^{can} was actually contained in $\Omega^1_C \otimes \mathscr{L}$, meaning we would have a connection with vanishing p-curvature on $\mathscr{L}$, compatible with ∇^{can}. But this would mean that $\mathscr{L}$ was the Frobenius pullback of some $\mathscr{L}'$, necessarily of positive degree, and mapping into $\mathscr{F}$, contradicting the hypothesis that $\mathscr{F}$ was semistable. But now we have a nonzero map from $\mathscr{L}$ to $\Omega^1_C \otimes \mathscr{L}^{-1}$, meaning that the latter has to have higher degree. This gives us $2 - \deg(\mathscr{L}) \geq \deg(\mathscr{L})$, so $\deg(\mathscr{L}) \leq 1$. Since we assumed initially that $\deg(\mathscr{L}) > 0$, we find $\deg(\mathscr{L}) = 1$, and since any nonzero map between line bundles of the same degree must be an isomorphism, $\mathscr{L} \cong \Omega^1_C \otimes \mathscr{L}^{-1}$, and $\mathscr{L}^{\otimes 2} \cong \Omega^1_C$. □

In fact, the converse of this proposition is almost true. Namely, given a rank 2 vector bundle $\mathscr{E}$ fitting into such an exact sequence, with a connection whose p-curvature vanishes, then the corresponding vector bundle $\mathscr{F}$ which pulls back to $\mathscr{E}$ under Frobenius is stable, and has determinant which may not be trivial, but whose pth tensor power is trivial: the determinant statement follows simply because taking determinants commutes with pulling back under Frobenius, and p-torsion line bundles are visibly the kernel of F^* on the space of line bundles (note that this is true as a set-theoretic statement for either the relative or absolute Frobenius morphism). We see that $\mathscr{F}$ is stable because if $\mathscr{M}$ is a nonnegative line bundle mapping into $\mathscr{F}$, then $F^*\mathscr{M}$ is a line bundle of degree either 0 or at least p, mapping into $\mathscr{E}$, and the following lemma completes the proof:

Lemma III.3.3. *Let $\mathscr{E}$ be a rank 2 vector bundle of degree 0, and suppose $\mathscr{L}$ is a positive line bundle giving an exact sequence*

$$0 \to \mathscr{L} \to \mathscr{E} \to \det \mathscr{E} \otimes \mathscr{L}^{-1} \to 0$$

Then $\mathscr{L}$ is unique, and is the maximal degree line bundle inside $\mathscr{E}$, and $\mathscr{E}$ is not an extension of any two degree 0 line bundles.

Proof. Suppose we have $\mathscr{M}$ another positive line bundle mapping into $\mathscr{E}$. Then since $\det \mathscr{E} \otimes \mathscr{L}^{-1}$ is negative, the composition with that quotient map must be 0, meaning that the image of $\mathscr{M}$ is contained in the image of $\mathscr{L}$, and therefore $\mathscr{M}$ maps into $\mathscr{L}$. Thus, $\mathscr{M}$ has degree less than or equal to $\mathscr{L}$, with equality if and only if they are isomorphic, which also gives uniqueness of $\mathscr{L}$. Similarly, if $\mathscr{E}$ were an extension of $\mathscr{M}$ and $\det \mathscr{E} \otimes \mathscr{M}^{-1}$ for some $\mathscr{M}$ of degree 0, the composition of the inclusion of $\mathscr{L}$ into $\mathscr{E}$ with the quotient to $\det \mathscr{E} \otimes \mathscr{M}^{-1}$ would have to be 0, so $\mathscr{L}$ would include into $\mathscr{M}$, which isn't possible. □

Moreover, there are only finitely many $\mathscr{E}$ to examine:

Proposition III.3.4. *There are only* 16 *choices for* $\mathscr{E}$ *as described in Proposition III.3.2, one for each choice of* $\mathscr{L}$.

Proof. Any two choices of $\mathscr{L}$ differ by a line bundle of order 2, which is to say a 2-torsion element of the Jacobian of C. Since C has genus 2, the Jacobian is 2-dimensional, and there are $2^{2\cdot 2} = 16$ such line bundles (using the fact that p is odd). With $\mathscr{L}$ chosen, we see:

Since the choices for $\mathscr{E}$ are parametrized by $\mathrm{Ext}^1(\mathscr{L}^{-1}, \mathscr{L})$ [26, Ex III.6.1]. We have:

$$\begin{aligned} \mathrm{Ext}^1(\mathscr{L}^{-1}, \mathscr{L}) &\cong \mathrm{Ext}^1(\mathscr{O}_C, \mathscr{L}^{\otimes 2}) = \mathrm{Ext}^1(\mathscr{O}_C, \Omega^1_C) \text{ [26, Prop III.6.7]} \\ &\cong H^1(C, \Omega^1_C) \text{ [26, Prop III.6.3]} \\ &\cong H^0(C, \mathscr{O}_C) \cong k \text{ [26, Cor III.7.7 (Serre duality)]} \end{aligned}$$

It should be noted that although the space of extensions is one dimensional, there will be a unique isomorphism class for $\mathscr{E}$ among non-trivial extensions, since scaling an extension by an element of k^* only changes the isomorphism class of the extension, and not of the vector bundle. Thus, we have one $\mathscr{E}$ for each $\mathscr{L}$, and a total of 16 choices. $\square$

Lastly, the bijectivity of F^* on 2-torsion line bundles of Pic shows that starting with one choice of $\mathscr{E}$, and an $\mathscr{F}$ pulling back to $\mathscr{E}$ under Frobenius, tensoring by line bundles of order 2 will give $\mathscr{F}'$ pulling back to each $\mathscr{E}'$ in a way that induces bijections between the sets of bundles pulling back to any given pair of $\mathscr{E}$ and $\mathscr{E}'$. Thus, it actually suffices to handle the problem for a single choice of $\mathscr{E}$, at least set-theoretically. But the same argument holds over an arbitrary base, so we actually find that from a functorial perspective as well, the different $\mathscr{F}$ pulling back to given choices of $\mathscr{E}$ are (canonically) identified between the 16 choices of $\mathscr{E}$. We thus have:

Corollary III.3.5. *There are canonical isomorphisms of functors* $F_{\mathscr{E}}$ *for the different* $\mathscr{E}$ *of Proposition III.3.4, where* $F_{\mathscr{E}}(T)$ *for* T/k *is defined as the set of isomorphism classes of* $\mathscr{F}$ *on* $C^{(p)} \times_k T$ *such that* $F^*\mathscr{F} \cong \mathscr{E}_T$. *The same statement also holds for the sub-functors* $F^0_{\mathscr{E}}$, *defined by requiring the* $\mathscr{F}$ *of* $F_{\mathscr{E}}$ *to have trivial determinant.*

By our earlier comments on p-curvature, it will suffice to classify transport equivalence classes of connections on $\mathscr{E}$ for which the p-curvature vanishes, a condition which may be checked on any Zariski open of C. Lastly, we want to only find connections on $\mathscr{E}$ corresponding to pullbacks of vector bundles with trivial determinant. This is almost, but not quite automatic: as remarked earlier, any $\mathscr{F}$ pulling back to $\mathscr{E}$ will have to have a determinant which is p-torsion in $J(C)$, but tensoring any given $\mathscr{F}$ with such lines bundles will achieve any p-torsion line bundle as the determinant, so we will have to explicitly restrict our consideration to connections having trivial associated determinant connections; by Proposition III.1.4, this will achieve the desired result. Because the determinant of a connection is found by taking the trace of the corresponding matrix, this will actually simplify the resulting formulas considerably.

III.4 On f_{θ^p} and p-rank in Genus 2

In this section, we give an explicit formula for f_{θ^p} on a genus 2 curve C, and note that we can use these ideas to derive explicit formulas for the p-rank of the Jacobian of C. Throughout, we work under the hypotheses and notation of Situations III.3.1 and III.2.1.

We first note that (irrespective of the genus of C), recalling that θ_0 is the derivation given by $f_{\theta_0} = 1$, although $f_{\theta_0^p}$ will be 0 only if ω is df for some f on U, we will always have:

Lemma III.4.1. $\theta_0 f_{\theta^p} = 0$.

Proof. Given any f, $\theta_0^p f = f_{\theta_0^p}\frac{df}{\omega} = f_{\theta_0^p}\theta_0(f)$, so $\theta_0^{p+1} f = \theta_0(f_{\theta_0^p}\theta_0(f)) = \theta_0(f_{\theta_0^p})\theta_0(f) + f_{\theta_0^p}\theta_0^2(f) = \theta_0(f_{\theta_0^p})\frac{df}{\omega} + \theta_0^{p+1}(f)$. Since this is true for all f, we must have $\theta_0(f_{\theta_0^p}) = 0$, as desired. $\square$

We now specify some normalizations and notational conventions special to genus 2 which we will follow throughout the remainder of the chapter.

First, we choose one of the six Weierstrass points of C, which we will denote by w. We set $U_2 = C \smallsetminus w$. Let x be a function with a pole of order 2 at w and regular elsewhere, and y a function with a pole of order 5 at w and regular elsewhere. Note that $1, x, x^2, y, x^3, xy, x^4, x^2y, x^5$ form a basis of $\Gamma(\mathscr{O}_C(10[w]))$, in which y^2 is an element. Thus, we can write y^2 in terms of this basis, and assuming that the characteristic of k is not 2, then translating y by appropriate multiples of $1, x, x^2$ will eliminate the y, xy, x^2y terms, leaving only a quintic in x. We note that the x^5 coefficient is necessarily non-zero, as this is the only other term with a pole of order 10 at w, which means that we can scale x and y (without even any field extension) to make our quintic monic. Having thus normalized, we get $y^2 = x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5$, and we will write this quintic, which gives an explicit equation for (an affine part of) C, as $g(x)$. We now set $\omega_2 = y^{-1}dx$, noting that y has simple zeroes at the roots of g (which correspond to the Weierstrass points other than w), and a pole of order 5 at w, while dx has simple zeroes at the same points, and a pole of order 3 at w, so that ω_2 is everywhere regular with a double zero at w, and in particular trivializes Ω^1_C on U_2. To summarize, we have put ourselves in the following situation:

Situation III.4.2. C is a smooth, proper genus 2 curve over an algebraically closed field k. It is presented explicitly on an affine open set U_2 by

$$y^2 = g(x) = x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5,$$

with the complement of U_2 being a single, smooth, Weierstrass point w at infinity. We also have the form $\omega_2 = y^{-1}dx$ a form trivializing Ω^1_C on U_2, and the derivation θ_0 on U_2 determined by $\hat{\theta}_0(\omega_2) = 1$, which is to say, $\theta_0 f = y\frac{df}{dx}$.

For this section only, we set $U = U_2$ and $\omega = \omega_2$. We set $g_k(x) = \theta_0^{k-1}x$, and we will see that this is indeed a polynomial in x for k odd. Noting that $\theta_0(x) = y$ and $\theta_0(y) = \frac{1}{2}g'(x)$, we have that for k odd, $g_k(x) = \theta_0^2(g_{k-2}(x)) = \theta_0(yg'_{k-2}(x)$, and we get the recursive formula:

$$g_k(x) = g''_{k-2}(x)g(x) + \frac{1}{2}g'_{k-2}(x)g'(x) \tag{III.4.3}$$

for k odd.

But $f_{\theta_0^p} = \hat{\theta}_0^p(y^{-1}dx)$ by definition, which is just $y^{-1}\theta_0^p(x)$ by the definition of $\hat{\theta}_0$, so we also find

$$f_{\theta_0^p} = y^{-1}\theta_0 g_p(x) = g'_p(x) \tag{III.4.4}$$

In particular, $f_{\theta_0^p}$ is a polynomial in x, and can therefore only have nonzero terms mod p in degrees which are multiples of p. However, the degree of $g_p(x)$ goes up by 3 every time p goes up 2, and $g_3(x)$ has degree 4, so we see that the degree of $g_p(x)$ is always less than $2p$. Hence the only nonzero terms of $f_{\theta_0^p}$ are the constant term and the pth power term (from which it follows that the only nonzero terms of $g_p(x)$ are the constant, linear, pth power, and $(p+1)$st power terms).

For later use, we note the formulas for characteristics 3, 5, and 7 obtained by combining equations III.4.3 and III.4.4:

Characteristic 3:

$$f_{\theta_0^3} = x^3 + a_3 \tag{III.4.5}$$

Characteristic 5:

$$f_{\theta_0^5} = 2a_1x^5 + a_3^2 + 2a_2a_4 + 2a_1a_5 \tag{III.4.6}$$

Characteristic 7:

$$f_{\theta_0^7} = (3a_1^2 + 3a_2)x^7 + a_3^3 + 6a_2a_3a_4 + 3a_1a_4^2 + 3a2^2a_5 + 6a_1a_3a_5 + 6a_4a_5 \tag{III.4.7}$$

As a final note, we can use this to derive explicit formulas for the p-rank of the Jacobian of C in terms of the coefficients of $g(x)$. Indeed, the p-torsion of the Jacobian is simply the number of (transport equivalence classes of, but endomorphisms of a line bundle are only scalars, and hold connections fixed) connections with p-curvature 0 on the trivial bundle. We note that the space of connections on $\mathscr{O}_C$ can be written explicitly as $f \mapsto df + f(c_1 + c_2x)\omega$, meaning the connection matrix on U is given simply by the scalar $T = c_1 + c_2x$. Using the rank-1 p-curvature formula of Equation III.2.8, $\psi_\nabla(\theta_0) = (c_1 + c_2x)^p + \theta_0^{p-1}(c_1 + c_2x) - f_{\theta_0^p}(c_1 + c_2x) = c_1^p + c_2^px^p + c_2g_p(x) - g_p'(x)(c_1 + c_2x)$. If we denote by h_1, h_2, h_3, h_4 the polynomials in the coefficients of $g(x)$ giving the constant, linear, pth power, and $(p+1)$st power terms of $g_p(x)$, we get: $\psi_\nabla(\theta_0) = (c_1^p + c_2h_1 - c_1h_2) + (c_2h_2 - c_2h_2)x + (c_2^p + c_2h_3 - (p+1)c_1h_4)x^p + (c_2h_4 - (p+1)c_2h_4)x^{p+1} = (c_1^p + c_2h_1 - c_1h_2) + (c_2^p + c_2h_3 - c_1h_4)x^p$. Setting the p-curvature to zero, we obtain:

$$0 = (c_1^p + c_2h_1 - c_1h_2) + (c_2^p + c_2h_3 - c_1h_4)x^p.$$

We first consider this equation in the case that $h_4 \neq 0$. In this case, we find that we can write $c_1 = \frac{c_2^p + c_2h_3}{h_4}$, and see that we need $\frac{(c_2^p + c_2h_3)^p}{h_4^p} + c_2h_1 - \frac{c_2^p + c_2h_3}{h_4}h_2 = 0$; multiplying through by h_4^p and collecting terms, this becomes $c_2^{p^2} + (h_3^p - h_2h_4^{p-1})c_2^p + (h_1h_4^p - h_2h_3h_4^{p-1})c_2 = 0$. This is separable if and only if the linear term is nonzero, that is, if $h_1h_4^p - h_2h_3h_4^{p-1} \neq 0$. Otherwise, it has p solutions if $h_3^p - h_2h_4^{p-1} \neq 0$, and 1 solution in the final case.

On the other hand, in the case that $h_4 = 0$, c_2 becomes independent of c_1, and has p solutions if $h_3 \neq 0$, and 1 solution otherwise; similarly, the number of solutions for c_1 becomes independent of c_2, being p if $h_2 \neq 0$, and 1 otherwise. In this case, we see we get

p^2 solutions if and only if both h_2 and h_3 are nonzero; p solutions if either but not both are nonzero, and 1 solution if they are both 0.

We conclude:

Proposition III.4.8. *With notation as above, the p-rank of the Jacobian of C is:*

2 *if:* $h_1h_4 - h_2h_3 \neq 0$;

1 *if:* $h_1h_4 - h_2h_3 = 0$ *but either* $h_3^p - h_2h_4^{p-1} \neq 0$ *or* $h_1^ph_4 - h_2^{p+1} \neq 0$;

0 *if:* $h_1h_4 - h_2h_3 = h_3^p - h_2h_4^{p-1} = h_1^ph_4 - h_2^{p+1} = 0$.

Proof. We need only check that the polynomial conditions given are equivalent to those derived above. The case that $h_4 = 0$ is the easy case, since then $h_1h_4 - h_2h_3 = 0$ is clearly equivalent to either $h_2 = 0$ or $h_3 = 0$, and $h_3^p - h_2h_4^{p+1} = h_1^ph_4 - h_2^{p+1} = 0$ is clearly equivalent to $h_2 = h_3 = 0$. If $h_4 \neq 0$, we certainly have that $h_1h_4 - h_2h_3 = 0$ is equivalent to $h_1h_4^p - h_2h_3h_4^{p-1} = 0$, and if $h_3^p - h_2h_4^{p-1} = h_1^ph_4 - h_2^{p+1}$ then in particular $h_3^p - h_2h_4^{p-1} = 0$. So we just need to check that given $h_1h_4 - h_2h_3 = 0$ and $h_4 \neq 0$, then $h_3^p - h_2h_4^{p-1} = 0$ implies that $h_1^ph_4 - h_2^{p+1} = 0$. But this is easy enough; just multiply through $h_3^p - h_2h_4^{p-1} = 0$ by h_2^p, substitute $h_1^ph_4^p$ for $h_2^ph_3^p$, and cancel h_4^{p-1}. □

For $p = 3$, $g_p(x) = 1x^4 - a_1x^3 + a_3x - a_4$, so h_4 is always nonzero, and we find that the p-rank of C is 2 when $a_4 - a_1a_3 \neq 0$, 1 when $a_4 - a_1a_3 = 0$ but $a_1^3 - a_3 \neq 0$, and 0 when $a_4 - a_1a_3 = a_1^3 - a_3 = 0$.

For $p = 5$, $g_p(x) = 2a_1x_6 + (4a_1^2 + 3a_2)x^5 + (a_3^2 + 2a_2a_4 + 2a_1a_5)x + (3a_3a_4 + 3a_2a_5)$, so the p-rank of C is 2 when $a_1(a_3a_4 + a_2a_5) - (4a_1^2 + 3a_2)(a_3^2 + 2a_2a_4 + 2a_1a_5) \neq 0$. The p-rank is 1 when $a_1(a_3a_4 + a_2a_5) - (4a_1^2 + 3a_2)(a_3^2 + 2a_2a_4 + 2a_1a_5) = 0$ but either $4a_1^{10} + 3a_2^5 - (a_3^2 + 2a_2a_4 + 2a_1a_5)a_1^4 \neq 0$ or $(3a_3^5a_4^5 + 3a_2^5a_5^5)2a_1 - (a_3^2 + 2a_2a_4 + 2a_1a_5)^6 \neq 0$. Lastly, the p-rank is 0 when $a_1(a_3a_4 + a_2a_5) - (4a_1^2 + 3a_2)(a_3^2 + 2a_2a_4 + 2a_1a_5) = 4a_1^{10} + 3a_2^5 - (a_3^2 + 2a_2a_4 + 2a_1a_5)a_1^4 = (3a_3^5a_4^5 + 3a_2^5a_5^5)2a_1 - (a_3^2 + 2a_2a_4 + 2a_1a_5)^6 = 0$.

While explicit computations of the p-rank of the Jacobian of a curve are not hard in general, it is perhaps worth mentioning that this method, aside from providing a complete and explicit solution for genus 2 curves, does so in a sufficiently elementary way that it can be presented as a calculation of the p-torsion of $\mathrm{Pic}(C)$ without knowing any properties of the Jacobian, or even that it exists.

III.5 The Space of Connections

In this section we carry out the first portion of the necessary computations for Theorem III.0.1, by expliciting calculating the space of transport-equivalence classes of connections on a particular vector bundle $\mathscr{E}$. We suppose:

Situation III.5.1. With the notation and hypotheses of Situation III.4.2, we further declare that $\mathscr{E}$ is the bundle determined by Propositions III.3.2 and III.3.4 for the choice $\mathscr{L} = \mathscr{O}_C([w])$

Of course, since w was chosen arbitrarily among the Weierstrass points of C, this will handle the calculation simultaneously for 6 different choices of $\mathscr{E}$, but since the answer will be in terms of the a_i, which depended on w, even for these six it is still helpful to know *a priori* that the answer is independent of the choice of $\mathscr{E}$.

In this situation, if U_1, U_2 are a trivializing cover for $\mathscr{L}$, with transition function φ_{12}, then $\mathscr{L}^{-1}, \mathscr{L}^{\otimes 2} = \Omega^1_C$, and $\mathscr{E}$ are all trivialized by this cover as well, and $\mathscr{E}$ can be represented with a transition matrix of the form

$$E = \begin{bmatrix} \varphi_{12} & \varphi_{\mathscr{E}} \\ 0 & \varphi_{12}^{-1} \end{bmatrix}$$

for some $\varphi_{\mathscr{E}}$ regular on $U_1 \cap U_2$.

We see immediately that we can choose φ_{12} to be regular on U_1 with a simple zero at w, and non-vanishing elsewhere, just by choosing any function with a simple zero at w, and setting U_1 to be the complement of any other zeroes or poles it has. For compatibility of trivializations of $\mathscr{L}$ and Ω^1_C, we must then set $\omega_1 = \varphi_{12}^{-2}\omega_2$. Beyond these properties, our specific choice of φ_{12} will be completely irrelevant, but we note that it is possible to choose φ_{12} to vary algebraically (in fact, to be invariant) as our a_i and the corresponding curves vary: we can simply set $\varphi_{12} = \frac{x^2}{y}$.

Proposition III.5.2. *The unique non-trivial isomorphism class for $\mathscr{E}$ may be realized by setting $\varphi_{\mathscr{E}} = \varphi_{12}^{-2}$.*

Proof. We need only show that there cannot be a splitting map from $\mathscr{E}$ back to $\mathscr{L}$. Under our trivializations, such a map would have to be given by vectors of the form $\begin{bmatrix} 1 & f \end{bmatrix}$ on U_2

and $\begin{bmatrix}1 & g\end{bmatrix}$ on U_1, with

$$\varphi_{12}\begin{bmatrix}1 & f\end{bmatrix} = \begin{bmatrix}1 & g\end{bmatrix}\begin{bmatrix}\varphi_{12} & \varphi_{\mathscr{E}} \\ 0 & \varphi_{12}^{-1}\end{bmatrix} = \begin{bmatrix}\varphi_{12} & \varphi_{12}^{-2}+\varphi_{12}^{-1}g\end{bmatrix}$$

Dividing through by φ_{12}, we find that $f = \varphi_{12}^{-3} + \varphi_{12}^{-2}g$, and solving for g we see that $g = \varphi_{12}^2 f - \varphi_{12}^{-1}$ would have to be regular on U_1. Thus, f would have to have a triple pole at w, and would have to be regular everywhere else (it is regular on U_2 by hypothesis, and away from w on U_1 because φ_{12} is invertible everywhere else on U_1). There is no such f, since w is a Weierstrass point on a hyperelliptic curve. □

We now note that since φ_{12} has a simple zero at w, and is invertible elsewhere on U_1, and ω_1 is invertible everywhere on U_1, $\frac{d\varphi_{12}}{\omega_1}$ is likewise everywhere invertible on U_1. In addition, $\varphi_{\mathscr{E}} = \varphi_{12}^{-2}$, so $d\varphi_{\mathscr{E}} = -2\varphi_{12}^{-3}d\varphi_{12}$, and $\frac{d\varphi_{\mathscr{E}}}{\omega_1}$ is regular and nonvanishing on U_1 except for a pole of order 3 at w.

Now, if ω_1 and ω_2 trivialize Ω_C^1 on U_1 and U_2 and satisfy $\frac{\omega_1}{\omega_2} = \varphi_{12}^{-2}$, we can trivialize $\mathscr{E} \otimes \Omega_C^1$ by tensoring with ω_i on U_i, which will give it transition matrix $\varphi_{12}^2 E$. We can then represent a connection $\nabla : \mathscr{E} \to \mathscr{E} \otimes \Omega_C^1$ by 2×2 transition matrices $\bar{T}_1$ and $\bar{T}_2$ of functions regular on U_1 and U_2 respectively.

These act by sending $s_i \mapsto \bar{T}_i s_i + \frac{ds_i}{\omega_i}$ on U_i, where the s_i are given as vectors under the trivialization, and by Equation III.1.10 we have that $\bar{T}_1$ and $\bar{T}_2$ are related by:

$$\bar{T}_1 = \varphi_{12}^2 E\bar{T}_2E^{-1} + E\frac{dE^{-1}}{\omega_1}$$

We now explicitly compute $\bar{T}_2$ in terms of $\bar{T}_1$ in preparation for computing the space of connections. If $\bar{T}_2 = \begin{bmatrix}f_{11} & f_{12} \\ f_{21} & f_{22}\end{bmatrix}$, then:

$$\bar{T}_1 = \begin{bmatrix}\varphi_{12}^2 f_{11} + f_T & \varphi_{12}^4 f_{12} + \varphi_{12}^3\varphi_{\mathscr{E}}(f_{22} - f_{11}) - \varphi_{12}\varphi_{\mathscr{E}} f_T - \varphi_{12}\frac{d\varphi_{\mathscr{E}}}{\omega_1} \\ f_{21} & \varphi_{12}^2 f_{22} - f_T\end{bmatrix} \tag{III.5.3}$$

where $f_T = \varphi_{12}\varphi_{\mathscr{E}} f_{21} - \varphi_{12}^{-1}\frac{d\varphi_{12}}{\omega_1}$

Note that this implies f_{21} is everywhere regular and hence constant.

We now show:

Proposition III.5.4. *The space of connections on $\mathscr{E}$ is given by $f_{21} = C_1$, $f_{11} = c_1 + c_2x$, $f_{22} = c_3 + c_4x$, and $f_{12} = c_5 + c_6x + c_7x^2 + c_8y + C_2x^3$, where the c_i are arbitrary constants subject to the single linear relation $c_8 = C_2(c_2 - c_4)$, and C_1 and C_2 are predetermined nonzero constants satisfying $C_1C_2 = \frac{-1}{2}$.*

Proof. We begin by looking at the lower right entry of the matrix for $\bar{T}_1$ in Equation III.5.3, and note the $\varphi_{12}^{-1}\frac{d\varphi_{12}}{\omega_1}$ has a simple pole at w which must be cancelled by one of the other terms. We also note that since $\varphi_{\mathscr{E}} = \varphi_{12}^{-2}$, and f_{21} must be constant, the $\varphi_{12}\varphi_{\mathscr{E}}f_{21} = \varphi_{12}^{-1}f_{21}$ is regular on U_1 outside of w, where it can have at most a simple pole. Thus the $\varphi_{12}^2 f_{22}$ term must likewise be regular on U_1 away from w, with at most a simple pole at w. Since f_{22} must be regular on U_2 by hypothesis, we conclude it is regular on C except possibly for a pole of order at most 3 at w. Now, as we just observed in the proof of Proposition III.5.2, a pole of order 3 isn't possible, so $f_{22} \in \Gamma(\mathscr{O}_C(2[w]))$. This means that the simple poles of the other two terms must cancel, and f_{21} is determined as a (nonzero) constant C_1: explicitly, $C_1 = \frac{d\varphi_{12}}{\omega_1}(w)$. Precisely the same argument applies to the upper right entry, placing $f_{11} \in \Gamma(\mathscr{O}_C(2[w]))$ and determining f_{21} (luckily, as the same constant!), so it only remains to analyze the upper right entry of the matrix.

We immediately observe that on U_1, each term (excluding the $\varphi_{12}^4 f_{12}$ term) is regular except possibly for a pole of order at most 2 at w, which of course implies that $\varphi_{12}^4 f_{12}$ is also, and we can conclude that f_{12} is regular on C except for a pole of order at most 6 at w. Then we have $f_{21} = C_1 \in k^*$, $f_{11} = c_1 + c_2x$, $f_{22} = c_3 + c_4x$, and $f_{12} = c_5 + c_6x + c_7x^2 + c_8y + C_2x^3$, and we claim that C_2 is also determined: the only other terms which can have double poles are $-\varphi_{12}^2\varphi_{\mathscr{E}}^2 f_{21} + \varphi_{\mathscr{E}}\frac{d\varphi_{12}}{\omega_1} - \varphi_{12}\frac{d\varphi_{\mathscr{E}}}{\omega_1} = -\varphi_{12}^{-2}f_{21} + 3\varphi_{12}^{-2}\frac{d\varphi_{12}}{\omega_1}$ which are now predetermined, so C_2 is also determined, explicitly as $-2(\varphi_{12}^{-6}x^{-3}\frac{d\varphi_{12}}{\omega_1})(w) = -2(\varphi_{12}^{-6}x^{-3})(w)C_1$. Lastly, we note that there is a linear relation on c_2, c_4, and c_8 to insure that the simple poles cancel.

To conclude the proof, we give a more explicit description of this linear relation, and use formal local analysis at w to obtain the desired statements on it and C_1 and C_2. Explicitly, our linear relation is given as $c_8 = ((\varphi_{12}^{-3}y^{-1}x)(w))(c_2 - c_4) + ((y^{-1}\varphi_{12}^{-5})(w))((\varphi_{12}^{-1}(C_1 - 3\frac{d\varphi_{12}}{\omega_1}) - C_2x^3\varphi_{12}^5)(w))$. Now, choose a local coordinate z at w; we will denote by $\ell_z(f)$ the leading term of the Laurent series expansion for f in terms of z, and by abuse of notation, $\ell_z(\omega)$ for $\ell_z(\omega/dz)$ when ω is a 1-form. Recalling that $\omega_1 = \varphi_{12}^{-2}y^{-1}dx$, we see that since φ_{12} has a simple zero at w, and x has a double pole, $C_1 = \ell_z(\frac{d\varphi_{12}}{\omega_1}) = \frac{\ell_z(\varphi_{12})}{\ell_z(\varphi_{12})^{-2}\ell_z(y)^{-1}(-2\ell_z(x))} = \frac{-\ell_z(\varphi_{12})^3\ell_z(y)}{2\ell_z(x)}$. We then get that $C_2 = \frac{-2}{\ell_z(\varphi_{12})^6\ell_z(x)^3}\frac{-\ell_z(\varphi_{12})^3\ell_z(y)}{2\ell_z(x)} =$

$\frac{\ell_z(y)}{\ell_z(\varphi_{12})^3\ell_z(x)^4} = \frac{\ell_z(x)}{\ell_z(\varphi_{12})^3\ell_z(y)}$, since from our normalization we see $\ell_z(x)^5 = \ell_z(y)^2$. It follows that $C_1C_2 = \frac{-1}{2}$, and we also see immediately that C_2 is indeed the coefficient of $(c_2 - c_4)$ in our linear relation.

It only remains to show that the constant term in that relation is in fact 0. We may write it as $((y^{-1}\varphi_{12}^{-5})(w))((\varphi_{12}^{-1}(C_1 - 3\frac{d\varphi_{12}}{\omega_1} - C_2x^3\varphi_{12}^6))(w)$, so it suffices to show that $C_1 - 3\frac{d\varphi_{12}}{\omega_1} - C_2x^3\varphi_{12}^6$ (which we know vanishes at w from how we chose C_2) vanishes to order at least 2 at w. In characteristic 3 this is immediate, because the middle term drops out, while neither of the other terms can have a non-zero linear term in their z expansion. To work out the general case, we note that since φ_{12} has a simple zero at w, we may as well choose our local coordinate $z = \varphi_{12}$. We denote by ℓ'_z the second term in a z-expansion. Then we have $\varphi_{12} = z$, $x = \ell_z(x)z^{-2} + \ell'_z(x)z^{-1} + \dots$, and $y = \ell_z(y)z^{-5} + \ell'_z(y)z^{-4} + \dots$. From our quintic relation on x, y, we know that $\ell_z(x)^5 = \ell_z(y)^2$, but looking at the next term we see that we also have $2\ell_z(y)\ell'_z(y) = 5\ell_z(x)^4\ell'_z(x)$. We also have the relation $\ell_z(y) = C_2^{-1}\ell_z(x)\ell_z(\varphi_{12})^{-3} = C_2^{-1}\ell_z(x)$. Together, we get that we can write $\ell_z(y)$ in terms of $\ell_z(x)$, and then we find $\ell_z(x)^5 = C_2^{-2}\ell_z(x)^2$, so $\ell_z(x)^3 = C_2^{-2}$, and then $5\ell_z(x)^4\ell'_z(x) = 2\ell_z(y)\ell'_z(y) = 2C_2^{-1}\ell_z(x)\ell'_z(y)$, so we can also write $\ell'_z(y) = \frac{5}{2}C_2^{-1}\ell'_z(x)$. We can now compute all the desired terms solely in terms of $\ell_z(x), \ell'_z(x)$ and C_2 (substituting $C_1 = \frac{-1}{2}C_2$) and we see that they do indeed cancel to order 2, as desired. $\square$

We also consider the endomorphisms of $\mathscr{E}$, so that we can normalize our connections via transport to simplify calculations. An endomorphism is given by matrices S_i regular on U_i, satisfying the relationship (Equation III.1.6) $S_1 = ES_2E^{-1}$. If we write $S_2 = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$ we find that

$$S_1 = \begin{bmatrix} g_{11} + \varphi_{12}^{-1}\varphi_{\mathscr{E}}g_{21} & \varphi_{12}^2g_{12} + \varphi_{12}\varphi_{\mathscr{E}}g_{22} - \varphi_{12}\varphi_{\mathscr{E}}g_{11} - \varphi_{\mathscr{E}}^2g_{21} \\ \varphi_{12}^{-2}g_{21} & g_{22} - \varphi_{12}^{-1}\varphi_{\mathscr{E}}g_{21} \end{bmatrix} \tag{III.5.5}$$

We now compute the space of endomorphisms of $\mathscr{E}$, and consequently the space of transport-equivalence classes of connections:

Proposition III.5.6. *The space of endomorphisms of $\mathscr{E}$ is given by $g_{21} = 0$, $g_{11} = g_{22} \in k$, and $g_{12} = g_{12}^0 + g_{12}^1x \in \Gamma(\mathscr{O}_C(2[w]))$. Every connection on $\mathscr{E}$ has a unique transport-equivalent connection with $f_{11} = 0$.*

Proof. Noting that the lower left entry for S_1 in equation III.5.5 is $\varphi_{12}^{-2}g_{21}$, we see that g_{21} has to be regular everywhere on C, and vanishes to order at least 2 at w; hence, it is 0. We then see that the upper left and lower right entries are just g_{11} and g_{22} respectively, meaning that these are both everywhere regular and hence constant. Finally, the upper right term is then $\varphi_{12}^2 g_{12} + \varphi_{12}^{-1}(g_{22} - g_{11})$; the second term will have a simple pole at w if and only if $g_{22} \neq g_{11}$, and since g_{12} cannot have a triple pole at w, we conclude that $g_{22} = g_{11}$, and finally that $g_{12} \in \Gamma(\mathscr{O}_C(2[w]))$. However, we are primarily interested in transporting connections along automorphisms, and this determines an automorphism if and only if $g_{11} \neq 0$; noting that transport along an automorphism is invariant under scaling the automorphism, we can set $g_{11} = g_{22} = 1$ without loss of generality. Now, since S_2 is upper triangular, with constant diagonal coefficients, $S_2^{-1}\frac{dS_2}{\omega_2}$ has only its upper right coefficient non-zero. Moreover, conjugating $\bar{T}_2$ by S_2 will simply substract $f_{21}g_{12}$ from the upper left coefficient of $\bar{T}_2$. Since we know f_{21} is a determined nonzero constant, and g_{12} and f_{11} can both be arbitrary in $\Gamma(\mathscr{O}_C(2[w]))$, this means that each connection has a unique transport class with $f_{11} = 0$, as desired. □

Thus, from now on we will normalize our calculations as follows: $f_{11} = 0$ by transport; $f_{22} = 0$ as we want the determinant connection (given as the trace) to be 0; $f_{21} = 1$, which we accomplish by scaling φ_{12} appropriately: we saw that $f_{21} = \frac{d\varphi_{12}}{\omega_1}(w)$, and recalling that $\omega_1 = \varphi_{12}^{-2}y^{-1}dx$, scaling φ_{12} by a cube root of f_{21} will do the trick. We also note that this does not pose any problems for our prior choice of $\varphi_{12} = \frac{x^2}{y}$; one can check that for this choice, $\frac{d\varphi_{12}}{\omega_1}$ may be written as a rational function in x, of the form $\frac{-x^{10}+\text{lower-order terms}}{2x^{10}+\text{lower-order terms}}$; since w is the "point at infinity", it follows that $f_{21} = \frac{-1}{2}$, and the scaling factor for φ_{12} is independent of the a_i. Lastly, since $c_8 = 0$ now that $c_2 = c_4 = 0$, we conclude that we are reduced to considering the case:

Situation III.5.7. Our connection matrix T_2 on U_2 is of the form $T_2 = \begin{bmatrix} 0 & f_{12} \\ 1 & 0 \end{bmatrix}$, with $f_{12} = c_5 + c_6 x + c_7 x^2 - \frac{1}{2}x^3$.

Remark III.5.8. We make some remarks on the validity of Propositions III.5.4 and III.5.6 in more general settings, which will be of use in later chapters. First, for use in Chapter IV, we note that both statements hold after base change to an arbitrary k-algebra A if one simply replaces the k-valued constants by A-valued constants. Indeed, if we denote by f the map $\operatorname{Spec} A \to \operatorname{Spec} k$, and π the structure map $C \to \operatorname{Spec} k$, this replacement corresponds to the natural map $f^*\pi_*\mathscr{F} \to \pi_{f*} f_\pi^*\mathscr{F}$ for the sheaves $\mathcal{E}nd(\mathscr{E}) \otimes \Omega^1_C$ and $\mathcal{E}nd(\mathscr{E})$. But since the base is a point, every base change is flat, and it immediately follows [26, Prop. III.9.3] that this natural map is always an isomorphism, giving the desired statement.

Next, for use in Chapter VI we also note that both propositions hold when our defining polynomial is allowed to degenerate to introduce nodes in the affine part of C away from w (see Section VI.1 for appropriate definitions in this context). We need only replace Ω^1_C by ω_C, which is still isomorphic to $\mathscr{O}(2[w])$: indeed, since the nodal fibers are irreducible, limits of line bundles are unique, and the existence of (and compatibility with base change of) a relative dualizing sheaf in this situation (see [8, p. 157, Thm. 3.6.1] implies that the dualizing bundle on any nodal fiber is the limit of the dualizing bundles on the nearby smooth fibers, which is $\mathscr{O}(2[w])$. It then suffices to note that it is still true that there can be no function in $\Gamma(\mathscr{O}(3[w])) \smallsetminus \Gamma(\mathscr{O}(2[w]))$, by the same Riemann-Roch argument as in the smooth case (using Riemann-Roch for nodal curves; see Theorem A.4).

Finally, we will also want to see that in fact our propositions hold if we look at families of curves obtained from maps $k[a_1, \dots a_5] \to A$ taking values in the open subset $U_{\text{nod}} \subset \mathbb{A}^5$. For this, we make use of the fact that, as remarked immediately above, we can choose φ_{12} to be a specific function for the whole family. Once again, if we denote by $\mathscr{F}$ the sheaf $\mathcal{E}nd(\mathscr{E}) \otimes \omega_C$ or $\mathcal{E}nd(\mathscr{E})$ as appropriate, but this time in the universal setting over U_{nod}, the theory of cohomology and base change (see Theorem A.32) gives that since $h^0(C, \mathscr{F})$ is constant on fibers, $\pi_*\mathscr{F}$ is locally free of the same rank, and pushforward commutes with base change. Now, if we let our constants describing sections of $\mathscr{F}$ lie in $k[a_1, \dots, a_5]$, we clearly obtain a subsheaf of $\mathscr{F}$ of the correct rank; further, the inclusion map is an isomorphism when restricted to every fiber, so it must in fact be an isomorphism. We then obtain the desired statement for all A because we had that pushforward commutes with base change.

It follows formally that the closed subschemes we describe explicitly corresponding to vanishing p-curvature in Section III.6 and nilpotent p-curvature in Section III.7 are also functorial descriptions which hold for nodal curves.

III.6 Calculations of p-curvature

Continuing with the situation and notations of the previous section, and in particular that of Situation III.4.2, we conclude with the p-curvature calculations to complete the proof of Theorem III.0.1, except for the statement on the general curve in characteristic 7, which depends on the results of the subsequent section.

We write:

$$\psi_{\nabla}(\theta_0) = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$

The first case we handle is $p = 3$. Equation III.4.5 gave us $f_{\theta_0^3} = x^3 + a_3$. We show:

Proposition III.6.1. *In characteristic* 3, $\mathscr{E}$ *has a unique transport equivalence class of connections with p-curvature zero and trivial determinant.*

Proof. With all of our normalizations from Situation III.5.7, the p-curvature matrix given by Equation III.2.9 becomes rather tame:

$$\psi_{\nabla}(\theta_0) = \begin{bmatrix} \theta_0 f_{12} & f_{12}^2 + \theta_0^2 f_{12} - f_{\theta_0^3} f_{12} \\ f_{12} - f_{\theta_0^3} & -\theta_0 f_{12} \end{bmatrix}$$

Even better, we note that we have

$$h_{12} = \theta_0(h_{11}) + f_{12}h_{21},$$

so h_{12} vanishes if h_{11} and h_{21} do. Similarly, recalling that by Lemma III.4.1, $\theta_0 f_{\theta_0^3} = 0$, we see that

$$h_{11} = \theta_0(h_{21}),$$

and

$$h_{22} = -h_{11}.$$

Hence, to check if the p-curvature vanishes, it suffices to check that h_{21} vanishes.

But this is a triviality, as we simply get that $h_{21} = 0$ if and only if $f_{12} = a_3 + x^3$, which, recalling that after normalization f_{12} was given by $c_5 + c_6 x + c_7 x^2 - \frac{1}{2}x^3$, and noting that in characteristic 3, $\frac{-1}{2} = 1$, means we get the unique solution $c_5 = a_3, c_6 = c_7 = 0$. □

We now handle the case $p = 5$. We had from Equation III.4.6 that $f_{\theta_0^5} = 2a_1x^5 + a_3^2 +$

$2a_2a_4 + 2a_1a_5$.

Proposition III.6.2. *In characteristic* 5*, the number of transport equivalence classes of connections with p-curvature zero and trivial determinant is given as the number of roots of the quintic polynomial:*

$$\begin{aligned}&(3a_1a_2^2 + 3a_2a_3 + a_5) + (a_1^2a_2 + a_2^2 + 3a_1a_3 + 4a_4)c_5 +\\&(3a_1^3 + 4a_1a_2 + a_3)c_5^2 + (3a_1^2 + 4a_2)c_5^3 + a_1c_5^4 + 4c_5^5\end{aligned} \tag{III.6.3}$$

Proof. With our normalizations as above, in terms of f_{12} and $f_{\theta_0^5}$, the p-curvature matrix obtained from Equation III.2.10 is

$$\psi_\nabla(\theta_0) = \begin{bmatrix} 4f_{12}\theta_0(f_{12}) + \theta_0^3(f_{12}) & f_{12}^3 + 4(\theta_0(f_{12}))^2 + 2f_{12}\theta_0^2(f_{12}) + \theta_0^4(f_{12}) + 4f_{12}f_{\theta_0^5} \\ f_{12}^2 + 3\theta_0^2(f_{12}) + 4f_{\theta_0^5} & f_{12}\theta_0(f_{12}) + 4\theta_0^3(f_{12}) \end{bmatrix}$$

Conveniently, we note that as before it actually suffices to check that h_{21} is 0: recalling again that by Lemma III.4.1, $\theta_0(f_{\theta_0^5}) = 0$, we see that

$$h_{22} = 3\theta_0(h_{21});$$

$$h_{11} = -h_{22};$$

and

$$h_{12} = f_{12}h_{21} + 2\theta_0^2(h_{21}).$$

Substituting in for f_{12} and $f_{\theta_0^5}$, we get that the remaining (lower left) term is given by

$$\begin{aligned}&(4a_3^2 + 3a_2a_4 + 3a_1a_5 + c_3^2 + a_5c_5) + (a_5 + 3a_3c_4 + 2c_3c_4 + 4a_4c_5)x +\\&(2a_2c_4 + c_4^2 + 2a_3c_5 + 2c_3c_5)x^2 + (4a_3 + 4c_3 + a_1c_4 + 2c_4c_5)x^3 +\\&(3a_2 + 4c_4 + 3a_1c_5 + c_5^2)x^4\end{aligned}$$

The x^3 term gives us $c_3 = 4a_3 + a_1c_4 + 2c_4c_5$, while the x^4 term gives us $c_4 = 3a_2 + 3a_1c_5 + c_5^2$. Making these substitutions into the other terms, we find that the x^2 term drops out, while the coefficient of x is:

$$(3a_1a_2^2 + 3a_2a_3 + a_5) + (a_1^2a_2 + a_2^2 + 3a_1a_3 + 4a_4)c_5+$$
$$(3a_1^3 + 4a_1a_2 + a_3)c_5^2 + (3a_1^2 + 4a_2)c_5^3 + a_1c_5^4 + 4c_5^5$$

The constant coefficient is $c_5 + 3a_1$ times the x coefficient, so we get that the connections with p-curvature 0 correspond precisely to the roots of the above polynomial, as desired. □

In particular, there are always between 1 and 5 connections with vanishing p-curvature on our $\mathscr{E}$, and the number clearly does actually depend on the curve. This is the first case in which this quantity can actually vary with the curve, and it appears to provide a new non-trivial invariant for curves of genus 2. It might be interesting to explicitly work out the strata, and determine, for instance, if all possible values between 1 and 5 actually occur, despite the fact that the moduli space of curves is only 3-dimensional. It is also worth noting that scheme-theoretically, the number of solutions still remains constant in this case. We will see that in higher characteristics, it will be far from obvious whether or not even this much remains true.

Lastly, we take a look at the case $p = 7$. Equation III.4.7 gave us:

$$f_{\theta_0^7} = a_3^3 + 6a_2a_3a_4 + 3a_1a_4^2 + 3a_2^2a_5 + 6a_1a_3a_5 + 6a_4a_5 + (3a_1^2 + 3a_2)x^7.$$

We will show:

Proposition III.6.4. *In characteristic 7, the number of transport equivalence classes of connections on $\mathscr{E}$ with p-curvature 0 and trivial determinant is given as the intersection of four plane curves in $\mathbb{A}^2$. For a general curve, it is positive. The locus $F_{2,7}$ of transport equivalence classes of connections on $\mathscr{E}$ with p-curvature 0 and trivial determinant considered over the $\mathbb{A}^5$ with which we parametrize genus 2 curves is cut out by 4 hypersurfaces in $\mathbb{A}^5 \times \mathbb{A}^2$.*

Proof. Here, even with our normalizations the p-curvature matrix obtained from Equation III.2.11 is rather messy, but we find its coefficients are given by:

$$h_{11} = 2f_{12}^2\theta_0(f_{12}) + \theta_0(f_{12})\theta_0^2(f_{12}) - 3f_{12}\theta_0^3(f_{12}) + \theta_0^5(f_{12})$$

$$h_{21} = -f_{\theta_0^7} + f_{12}^3 + 3(\theta_0(f_{12}))^2 - f_{12}\theta_0^2(f_{12}) - 2\theta_0^4(f_{12})$$

$$h_{12} = -f_{\theta_0^7} f_{12} + f_{12}^4 + f_{12}^2\theta_0^2(f_{12}) + (\theta_0^2(f_{12}))^2 - 2\theta_0(f_{12})\theta_0^3(f_{12}) + 2f_{12}\theta_0^4(f_{12}) + \theta_0^6(f_{12})$$

$$h_{22} = -2f_{12}^2\theta_0(f_{12}) - \theta_0(f_{12})\theta_0^2(f_{12}) + 3f_{12}\theta_0^3(f_{12}) - \theta_0^5 f_{12}$$

Once again, it is enough to consider a single one of these coefficients, as:

$$h_{11} = 3\theta_0(h_{21})$$

$$h_{12} = f_{12}h_{21} + 3\theta_0^2(h_{21})$$

$$h_{22} = -h_{11}$$

Looking then at the formula for h_{21}, substituting in for f_{12} and $f_{\theta_0^7}$ gives a polynomial of degree 6 in x. The x^6 term lets us solve for c_3:

$$c_3 = 5a_1a_2 + a_3 + 4a_1c_4 + 4a_1^2c_5 + c_4c_5 + 2a_1c_5^2 + 5c_5^3$$

The x^5 term is then

$h_{7,1} = 2a_1^2a_2 + a_1a_3 + 5a_4 + 4a_1^2c_4 + 5a_2c_4 + 6c_4^2 + 3a_1^3c_5 + 6a_1a_2c_5 + 3a_3c_5 + 5a_1c_4c_5 + 3a_1c_5^3 + 6c_5^4$

while the x^4 term is $-c_5$ times the x^5 term, and the x^3 term is $-(c_5^2 + a_1c_5 + 3a_2 + c_4)$ times the x^5 term. Taking the x^2 term minus $-(5c_5^3 + 5a_1c_5^2 + 2c_4c_5 + 5a_1a_2 + 4a_3 + 2a_1c_4)$ times the x^5 term leaves:

$h_{7,2} = 3a_1^3a_2^2 + 6a_1^2a_2a_3 + 4a_1a_3^2 + 4a_3a_4 + 2a_2a_5 + 3a_1^3a_2c_4 + 4a_1^2a_3c_4 + 2a_1a_4c_4 + 4a_5c_4 + a_1^3c_4^2 + a_3c_4^2 + 3a_1c_4^3 + a_1^4a_2c_5 + 5a_1^3a_3c_5 + a_1^2a_4c_5 + 3a_1a_5c_5 + 6a_1^4c_4c_5 + a_1^2a_2c_4c_5 + a_1a_3c_4c_5 + 3a_4c_4c_5 + a_1^2c_4^2c_5 + 5a_2c_4^2c_5 + c_4^3c_5 + 4a_1^3a_2c_5^2 + 6a_1^2a_3c_5^2 + a_1a_4c_5^2 + 3a_5c_5^2 + 3a_1^3c_4c_5^2 + a_1a_2c_4c_5^2 + a_3c_4c_5^2 + 3a_1^2a_2c_5^3 + a_1a_3c_5^3 + 4a_1^2c_4c_5^3 + 6c_4^2c_5^3$

Similarly, taking the x term minus $-(5c_4c_5^2 + 5a_1c_4c_5 + 6a_4 + 2c_4^2)$ times the x^5 term leaves:

$h_{7,3} = 5a_1^2a_2a_4 + 6a_1a_3a_4 + a_1a_2a_5 + 5a_1^2a_2^2c_4 + 4a_1a_2a_3c_4 + 3a_1^2a_4c_4 + 2a_1a_5c_4 + 5a_1^2a_2c_4^2 + a_1a_3c_4^2 + 3a_4c_4^2 + 3a_2c_4^3 + 5c_4^4 + 4a_1^3a_4c_5 + 6a_3a_4c_5 + 5a_1^2a_5c_5 + 3a_2a_5c_5 + 4a_1^3a_2c_4c_5 + 4a_1^2a_3c_4c_5 + a_1a_4c_4c_5 + 6a_5c_4c_5 + 3a_1^3c_4^2c_5 + 4a_1a_2c_4^2c_5 + 4a_3c_4^2c_5 + a_1c_4^3c_5 + 2a_1^2a_4c_5^2 + 6a_1a_5c_5^2 + 2a_1^2a_2c_4c_5^2 +$

$2a_1a_3c_4c_5^2+a_4c_4c_5^2+5a_1^2c_4^2c_5^2+4a_2c_4^2c_5^2+5c_4^3c_5^2+5a_1a_4c_5^3+a_5c_5^3+5a_1a_2c_4c_5^3+5a_3c_4c_5^3+2a_1c_4^2c_5^3$

Lastly, taking the constant term minus $-(6c_5^5+5c_4c_5^3+3a_1^2c_5^3+2a_1^3c_5^2+5a_1a_2c_5^2+2a_3c_5^2+6a_1c_4c_5^2+5a_1^2a_2c_5+2a_1a_3c_5+2a_4c_5+a_1^2c_4c_5+2a_2c_4c_5+2c_4^2c_5+6a_1a_4+4a_5+4a_1a_2c_4+3a_3c_4+3a_1c_4^2)$ times the x^5 term leaves:

$h_{7,4}=6a_1^3a_2^3+5a_1^2a_2^2a_3+a_1a_2a_3^2+5a_1^3a_2a_4+6a_1^2a_3a_4+a_2a_3a_4+6a_1a_4^2+a_1^2a_2a_5+4a_2^2a_5+5a_1a_3a_5+4a_4a_5+4a_1^2a_2a_3c_4+a_1a_3^2c_4+3a_1^3a_4c_4+2a_1a_2a_4c_4+3a_3a_4c_4+2a_1^2a_5c_4+3a_1^3a_2c_4^2+6a_1a_2^2c_4^2+a_2a_3c_4^2+6a_5c_4^2+6a_1^3c_4^3+4a_1a_2c_4^3+4a_3c_4^3+4a_1c_4^4+2a_1^4a_2^2c_5+3a_1^3a_2a_3c_5+4a_1^4a_4c_5+2a_1^2a_2a_4c_5+2a_1a_3a_4c_5+5a_1^3a_5c_5+a_3a_5c_5+3a_1^4a_2c_4c_5+2a_1^2a_2^2c_4c_5+2a_1^3a_3c_4c_5+2a_1a_2a_3c_4c_5+5a_3^2c_4c_5+6a_1^2a_4c_4c_5+6a_2a_4c_4c_5+5a_1a_5c_4c_5+2a_1^4c_4^2c_5+2a_1^2a_2c_4^2c_5+3a_2^2c_4^2c_5+6a_1a_3c_4^2c_5+4a_4c_4^2c_5+a_2c_4^3c_5+5c_4^4c_5+a_1^3a_2^2c_5^2+5a_1^2a_2a_3c_5^2+2a_1^3a_4c_5^2+2a_1a_2a_4c_5^2+2a_3a_4c_5^2+6a_1^2a_5c_5^2+5a_1^3a_2c_4c_5^2+2a_1a_2^2c_4c_5^2+a_1^2a_3c_4c_5^2+2a_2a_3c_4c_5^2+4a_1a_4c_4c_5^2+5a_5c_4c_5^2+a_1^3c_4^2c_5^2+6a_1a_2c_4^2c_5^2+2a_1c_4^3c_5^2+6a_1^2a_2^2c_5^3+2a_1a_2a_3c_5^3+5a_1^2a_4c_5^3+a_1a_5c_5^3+2a_1^2a_2c_4c_5^3+6a_1a_3c_4c_5^3+5a_4c_4c_5^3+6a_1^2c_4^2c_5^3+4a_2c_4^2c_5^3+3c_4^3c_5^3$

These four polynomials are then the defining equations in characteristic 7, describing the locus as an intersection of 4 affine plane curves, as desired. Without developing the theory further, we will not be able to say a tremendous amount, but we can make certain statements. First, by direct computation in Macaulay 2, the coordinate ring of the affine algebraic set cut out by these equations has dimension 5. Since we know that it can only have dimension 0 over any given choice for the a_i, this implies that it has a non-empty fiber for a general choice of a_i. That is to say, a general curve of genus 2 does in fact have at least one connection with p-curvature 0 on $\mathscr{E}$. □

Finally, we compute an example which will be of theoretical interest later.

Lemma III.6.5. *For the curve given by $a_1=a_2=a_3=0, a_4=1$, and $a_5=3$, there are 14 solutions to our equations, all reduced. Further, the local rings of $F_{2,7}$ at each of these points are all isomorphic.*

Proof. First, we set $a_1=a_2=a_3=0$, $a_4=1$ and $a_5=3$, and our defining equations become considerably simpler:

$$h_{7,1}=5+6c_4^2+6c_5^4$$

$$h_{7,2}=5c_4+3c_4c_5+c_4^3c_5+2c_5^2+6c_4^2c_5^3$$

$$h_{7,3} = 3c_4^2 + 5c_4^4 + 4c_4c_5 + c_4c_5^2 + 5c_4^3c_5^2 + 3c_5^3$$

$$h_{7,4} = 5 + 4c_4^2 + 4c_4^2c_5 + 5c_4^4c_5 + c_4c_5^2 + 5c_4c_5^3 + 3c_4^3c_5^3$$

If we use $h_{7,1}$ to substitute for c_4^2 in $h_{7,2}$, we get:

$$c_4(5 + c_5 + 6c_5^5) + c_5^2(2 + 2c_5 + c_5^5)$$

We show that we cannot have $5 + c_5 + 6c_5^5 = 0$: if so, the second half of the above equation would also have to be 0, so either $c_5 = 0$ or $5 + c_5 + 6c_5^5 = 0$; the former case immediately contradicts the assumed quintic expression for c_5, while the latter case would give, upon adding the two quintics for c_5, $3c_5 = 0$, also a contradiction. Thus, we can localize away from $5 + c_5 + 6c_5^5$, setting $c_4 = \frac{c_5^2(2+2c_5+c_5^5)}{5+c_5+6c_5^5}$. Making this substitution and taking numerators, the $h_{7,i}$ give four polynomials in c_5. However, they are multiples of the polynomial given by $h_{7,1}$, which is:

$$6 + c_5 + 5c_5^2 + 6c_5^4 + 2c_5^5 + 6c_5^6 + 6c_5^9 + 3c_5^{10} + 5c_5^{14}$$

This then gives the 14 reduced solutions, and the fact that the local rings of $F_{2,7}$ at each of these points are isomorphic follows from the fact that this degree 14 polynomial is irreducible over $\mathbb{F}_7$: the local ring is given explicitly by inverting everything away from the maximal ideals, these ideals being determined by values of the a_i and c_i. The a_i are the same at all the points by hypothesis, we saw above that c_4 is given as a rational function in c_5, and the possible values of c_5 are given as roots of an irreducible polynomial over $\mathbb{F}_7$, so the Galois group of $\bar{\mathbb{F}}_7$ over $\mathbb{F}_7$ permutes these values for c_5 transitively, and gives explicit isomorphisms between the different local rings. □

While we could do further calculations to try to say something about the behavior of a general curve in characteristic 7, once we have developed the theory of the following section we will be able to make a number of statements with nothing more than what we already have, so we will not bother with any further computations.

III.7 On The Determinant of the p-Curvature Map

In this section we explicitly calculate the highest degree terms of $\det \psi$, the determinant of the p-curvature map in the case of a genus 2 curve and the specific unstable vector bundle of Situation III.5.1. We use the calculation to prove that $\det \psi$ is finite flat, of degree p^3, and therefore conclude that in families of curves, the kernel of $\det \psi$ is finite flat. This has immediate implications for the p-curvature zero connections on $\mathscr{E}$ as well.

We wish to compute in our specific situation the morphism $\det \psi^0$ (III.1.3), which is to say, the morphism obtained from ψ^0 (III.1.2) by taking the determinant. In fact, we take ψ^0 to be the induced map on transport-equivalence classes of connections; since p-curvature is conjugated by an automorphism under transport, the determinant is not affected, so the map of (III.1.3) descends to the quotient. We remark that in the situation of rank 2 vector bundles with trivial determinant, and after restricting to connections with trivial determinant, because the image of ψ^0 is contained among the traceless endomorphisms, the vanishing of the determinant of the p-curvature is then equivalent to nilpotence of the endomorphisms given by the p-curvature map. Such connections are frequently called **nilpotent** in the literature (see, for instance, [30] or [42]).

We now take our curve C of genus 2 from before, with $\mathscr{E}$ the particular unstable bundle of rank 2 we had been studying, as in Situations III.4.2, III.5.1, and III.5.7. We also take the particular θ_0 from before, sending ω_2 to 1. Since ω_2 has a double zero at w, we see that θ_0 has a double pole there, so that our explicit identification of Ω^1_C is as $\mathscr{O}(2[w])$. We know that our space of connections with trivial determinant on $\mathscr{E}$ is (modulo transport) 3-dimensional, and of course $h^0(C^{(p)}, (\Omega^1_{C^{(p)}})^{\otimes 2}) = \deg(\Omega^1_{C^{(p)}})^{\otimes 2} + 1 - g = 4g - 4 + 1 - g = 3g - 3 = 3$, so we have a map from $\mathbb{A}^3$ to $\mathbb{A}^3$. We choose coordinates on the first space to be given by the c_5, c_6, c_7 determining f_{12}, while the function we will get will be of the form $f_1(c_5, c_6, c_7) + f_2(c_5, c_6, c_7)x^p + f_3(c_5, c_6, c_7)x^{2p}$, and we obtain coordinates on the image space as the monomials in x (that is to say, we take the map from $\mathbb{A}^3$ to itself given by the polynomials f_1, f_2, f_3).

We will use our earlier calculations to show:

Lemma III.7.1. *The leading term of f_i is $-c_{i+4}^p$, with all other terms of strictly lesser total degree in the c_i.*

Proof. If $T = \begin{bmatrix} 0 & f_{12} \\ 1 & 0 \end{bmatrix}$ is the connection matrix for ∇, we claim that the leading term will come from the T^p term in the p-curvature formula. Now, $T^2 = \begin{bmatrix} f_{12} & 0 \\ 0 & f_{12} \end{bmatrix}$, so we find

$$T^p = \begin{bmatrix} 0 & (f_{12})^{\frac{p+1}{2}} \\ (f_{12})^{\frac{p-1}{2}} & 1 \end{bmatrix}$$

Now, f_{12} is linear in the c_i, as are $\theta_0^i f_{12}$ for all i. Considering the p-curvature formula coefficients as polynomials in $\theta_0^i f_{12}$, we will show that the degree of the remaining terms are all less than or equal to $\frac{p-1}{2}$, with the degree of the terms in the lower left strictly less. This will imply that the leading term of the determinant is given by

$$-(f_{12})^p = -(c_5 + c_6 x + c_7 x^2 - \frac{1}{2}x^3)^p = -c_5^p - c_6^p x^p - c_7^p x^{2p} + \frac{1}{2^p} x^{3p}$$

giving the desired formula for the leading terms of the constant, x^p, and x^{2p} terms.

We observe that since $\theta_0^i T = \begin{bmatrix} 0 & \theta_0^i f_{12} \\ 0 & 0 \end{bmatrix}$ for all $i > 0$, $(\theta_0^i T)(\theta_0^j T) = 0$ for any $i, j > 0$. We use this and the fact that T^2 is diagonal to write any term in the p-curvature as one of the following:

(1) $T^{2i_0}(\theta_0^{i_1} T)T \ldots (\theta_0^{i_k} T)$

(2) $T^{2i_0}T(\theta_0^{i_1} T)T \ldots (\theta_0^{i_k} T)$

(3) $T^{2i_0}(\theta_0^{i_1} T)T \ldots (\theta_0^{i_k} T)T$

(4) $T^{2i_0}T(\theta_0^{i_1} T)T \ldots (\theta_0^{i_k} T)T$

where $2i_0 + \sum_{j>0}(i_j + 2) = p+1, p, p, p-1$ respectively.

We observe that these correspond to non-negative upper right, lower right, upper left, and lower left coefficients, respectively (in particular, at most one is non-zero). We know that the first term is a scalar matrix of degree i_0 in f_{12}. We see that $T(\theta_0^{i_j} T) = \begin{bmatrix} 0 & 0 \\ 0 & \theta_0^i f_{12} \end{bmatrix}$, so a product of $k-1$ such terms has total degree $k-1$ in the $\theta_0^i f_{12}$. Lastly, multiplying on the left by $(\theta_0^{i_1} T)$ raises the degree by one and moves the nonzero coefficient back to the upper right. Thus, in the first case, we get total degree $i_0 + k$. But we see that this is actually the same in the other cases, as multiplying on the left or right by T just moves the nonzero coefficient, without changing it. Finally, with $k > 0$, we have $i_0 + k < \frac{1}{2}(2i_0 + \sum_{j>0}(i_j + 2))$, which is $\frac{1}{2}$ times $p+1, p, p$ or $p-1$ depending on the case. But this is precisely what we wanted to show, since it forces the degree to be less than or equal to $\frac{p-1}{2}$ in the first three cases, and strictly less in the fourth.

Lastly, $-f_{\theta_0^p}T$ is linear in the c_i in the upper right term, and constant in the rest, so doesn't cause any problems for $p \geq 3$. $\square$

Remark III.7.2. Note that it is important in the proof of the preceding lemma that we are looking for the leading terms in the degrees of the c_i, and not of x. In particular, our general theory tells us that the x^{3p} term showing up in the proof must be cancelled elsewhere, and with some effort, one can see that this is true. Strangely, it is cancelled by the $-f_{\theta_0^p}$ term in characteristic 3, but by the $\theta_0^{p-1}T$ term in higher characteristics.

This lemma allows us to recover, in a completely explicit and elementary fashion, the genus 2 case of a theorem of Mochizuki:

Theorem III.7.3. *On the unstable vector bundle $\mathscr{E}$ described by Situation III.5.1 for a smooth proper genus* 2 *curve C as in Situation III.4.2, the map* $\det\psi$ *is a finite flat morphism from $\mathbb{A}^3$ to $\mathbb{A}^3$, of degree p^3. Further,* $\det\psi$ *remains finite flat when considered as a family of maps over the open subset $U_{\mathrm{ns}} \subset \mathbb{A}^5$ corresponding to nonsingular curves. Lastly, the induced map from* $\ker\det\psi$ *to U_{ns} is finite flat.*

Proof. It suffices to prove the asserted finite flatness for the family of maps $\mathbb{A}^3 \times U_{\mathrm{ns}} \to \mathbb{A}^3 \times U_{\mathrm{ns}}$ over U_{ns}, since the statements on individual curves and on the kernel of $\det\psi$ both follow from restriction to fibers. However, we have functions which are regular not only on U_{ns}, but on all of $\mathbb{A}^5$, so we will simply work over the polynomial ring to prove the assertion. However, Lemma III.7.1 makes the assertion clear: indeed, if we denote the coordinates on the image $\mathbb{A}^3$ by x_1, x_2, x_3, we merely need to see that the algebra

$$k[a_1, \dots a_5, x_1, x_2, x_3, c_5, c_6, c_7]/(x_i - f_i(c_5, c_6, c_7))_i$$

is a finite flat module of degree p^3 over $k[a_1, \dots, a_5, x_1, x_2, x_3]$. But we know that the highest degree terms of f_i in the c_i are $-c_{i+4}^p$, so we see that it is in fact free of rank p^3, since it is clearly freely generated by the monomials $c_5^{e_1}c_6^{e_2}c_7^{e_3}$ where the e_i are all less than p. $\square$

Corollary III.7.4. *The subscheme of* $U_{\mathrm{ns}} \times \mathbb{A}^3$ *giving connections with* p*-curvature* 0 *is finite over* U_{ns}.

Proof. This is a closed subscheme of $\ker \det \psi$, so the result follows immediately from the preceding theorem; although flatness need not be preserved under restriction to a closed subset, finiteness is, directly from the definition. $\square$

Remark III.7.5. This is, in effect, saying that even though the p-curvature 0 connections are given as a variety in affine space, no component can go off to infinity. From the perspective of Chapter IV, we would expect this heuristically based on the fact that by Theorem IV.0.1, the number of p-curvature connections dropping at a curve ought to force the degree of the Verschiebung map up, which we know cannot happen (see Corollary A.27). However, since it is possible to have new isolated components of the p-curvature 0 variety at these curves to balance the degree, such an approach appears incapable of giving an actual proof.

In particular, any subscheme which must be closed in the scheme of p-curvature 0 connections has closed image in U_{ns}. We conclude, for instance:

Corollary III.7.6. *The locus* $U_{\mathrm{unr}} \subset U_{\mathrm{ns}}$*, consisting of curves for which the* p*-curvature* 0 *connections on* $\mathscr{E}$ *is a reduced scheme, is open in* U_{ns}. *In particular, if a single curve has a reduced scheme of* p*-curvature* 0 *connections on* $\mathscr{E}$*, a general curve does as well.*

Proof. Since we are over an algebraically closed field, and the scheme of connections with vanishing p-curvature is finite from the preceding corollary, its reducedness is equivalent to unramifiedness, which is an open condition. $\square$

We are now ready to put together previous results to finish the proof of our main theorem:

Proof of Theorem III.0.1. Our explicit example of a single curve in characteristic 7 having a scheme of transport equivalence classes of connections on $\mathscr{E}$ with vanishing p-curvature which consisted of 14 reduced points, each of whose local rings on $F_{2,7}$ were isomorphic, will now turn out to be more powerful. By Corollary III.7.6, we immediately have that a general curve has its scheme of connections with vanishing p-curvature consisting only of reduced points. We next note that by the properness provided by Corollary III.7.4, any component of $F_{2,7}$ of dimension 5 must actually occur on every curve; we calculated that $F_{2,7}$ had dimension 5, so there is some such component, which must go through one of the 14 points on our example curve. This means that the local ring of $F_{2,7}$ at that point has dimension 5, which implies that the local rings at all 14 do, meaning that all 14 lie on 5-dimensional components of $F_{2,7}$. On a curve above which $F_{2,7}$ is reduced (which will be satisfied, as we've noted, for a general curve), we will find each of these 14 points occurs. All that remain to show is that a general curve cannot have more than 14 reduced points over it on $F_{2,7}$. But if this occurred, there would be an additional 5-dimensional component of $F_{2,7}$ which would not occur above our example curve, contradicting properness. $\square$

III.8 Further Remarks and Questions

We conclude with some final remarks on questions arising from the computations of this chapter.

We first observe that the scheme structures of spaces of connections in this section have arisen a posteriori, with no general reason to think that the obvious functor of transport-equivalence classes of connections on a vector bundle will be representable. We thus ask:

Question III.8.1. When is the functor of transport-equivalence classes of connections of vector bundles representable by a scheme?

We remark that we had hoped to approach statements beyond finiteness for the locus of connections with vanishing p-curvature via consideration of the locus of connections with nilpotent p-curvature. However, although we saw in Section III.7 that it is easy to show explicitly that the latter is finite flat, it turns out that the described approach is untenable, and indeed that the locus of vanishing p-curvature has strictly better behavior than the locus of connections with nilpotent p-curvature. In fact, a direct calculation via computer algebra software in characteristic 3 shows that although we have seen that the locus of connections with vanishing p-curvature is not merely finite flat but also reduced over all smooth curves, hence etale, there are non-reduced components of the locus of connections with nilpotent p-curvature, and in fact the locus of connections with vanishing p-curvature is imbedded inside such a non-reduced component. In contrast, Mochizuki has shown [42, II, Thm. 2.8, p. 153] that the locus of connections with vanishing p-curvature allowing the base curve to vary is finite flat as a stack over the stack of genus-2 curves, and is in addition smooth over the base field.

Finally, as remarked in the introduction, one advantage of the explicit approach of this chapter over the more general degeneration arguments of Mochizuki and of Chapter VI is that it allows study of arbitrary curves rather than solely general curves. In particular, we saw that in characteristic 3, all curves had reduced (equivalently, unramified) spaces of Frobenius-unstable bundles, and in characteristic 5, the locus of curves with ramified spaces of Frobenius-unstable bundles was cut out by a discriminant divisor in the space of genus 2 curves. One could ask what happens in characteristic 7, or more generally:

Question III.8.2. Is there a good description of the space of genus 2 curves in any given characteristic for which the space of Frobenius-unstable bundles is non-reduced?

Chapter IV

On the Degree of the Verschiebung

In this chapter we address the degree of the Verschiebung rational map V_2 induced by pullback under Frobenius on the moduli space M_2 of rank 2 vector bundles with trivial determinant on a smooth proper curve, in the case of genus 2. We bgein with some fairly straightforward and general results on degrees of rational maps of projective spaces and on V_2 itself, followed by a more involved use of deformation theory and hypercohomology spectral sequences to make some theoretical, characteristic-independent examinations of the relationship between the locus of p-curvature 0 connections on the unstable bundles $\mathscr{E}$ of Proposition III.3.4, the exceptional locus of the Verschiebung, and the image of the exceptional divisor if one blows up the exceptional locus.

Our main result is:

Theorem IV.0.1. *Let C be a smooth, proper genus 2 curve over an algebraically closed field k of characteristic $p > 2$, and suppose that the Frobenius-unstable locus for vector bundles of rank 2 and trivial determinant is composed of δ reduced points, where for our purposes "reduced" means that the corresponding bundle $\mathscr{F}$ satisfies $\mathrm{Def}^0(\mathscr{F}) \hookrightarrow \mathrm{Def}^0(F^*\mathscr{F})$, with Def^0 denoting first-order infinitesmal deformations preserving triviality of the determinant. Then:*

(i) Each Frobenius-unstable bundle corresponds to an undefined point of V_2, and V_2 has degree $p^3 - \delta$;

(ii) Each undefined point may be resolved by a single blowup;

(iii) The image of the exceptional divisor associated to such an undefined point is precisely

$\mathbb{P}\operatorname{Ext}(\mathscr{L}, \mathscr{L}^{-1}) \subset M_2(C)$, where $\mathscr{L}$ is a theta characteristic on C, and specifically is the destabilizing line bundle for $F^*\mathscr{F}$, where $\mathscr{F}$ is the Frobenius-unstable vector bundle associated to the undefined point.

We may apply this theorem to the results of Chapter III to draw conclusions in characteristics $3, 5, 7$, as in Corollary IV.6.1. Later, we will apply this theorem in more generality to conclude part (ii) of Theorem VI.0.1 from part (i); part (i) of that theorem already follows from Mochizuki's work (see the introduction to Chapter VI), but we will in Chapter VI give a self-contained argument for the same result, via a degeneration argument involving also the ideas of Chapters I, V, and III.

Section IV.1 consists of an analysis of degrees of rational maps from a projective space to itself which are defined at all but a finite set of points, in terms of the polynomial degree of the map, and the length of the undefined locus. Section IV.2 then computes the polynomial degree in the case of V_2. The remaining sections are devoted to exploring the relationship between abstract knowledge of the locus of Frobenius-unstable vector bundles, and more concrete (but harder to approach directly) questions such as the scheme structure on the undefined locus of the Verschiebung, and the image of exceptional divisors after blowing up to obtain a morphism. In Section IV.3, we describe deformations of connections together with their underlying bundles in terms of hypercohomology, and in Section IV.4 we make use of certain spectral sequences to obtain more information for the specific vector bundle we wish to study. Section IV.5 explores the geometric significance of the preceding calculations, and finally Section IV.6 concludes with the proof of Theorem IV.0.1, conclusions in low characteristics, and further questions. Appendix IV.A is a compilation of necessary technical background results on V_n, given in more generality and drawn primarily from unpublished work of A. J. de Jong. Finally, Appendix IV.B develops some slightly non-standard commutative algebra which arises in our vector bundle manipulations.

The existing literature on such geometric questions on the Verschiebung is considerably scarcer than on Frobenius-unstable vector bundles. The only other such results in this area were developed recently by Laszlo and Pauly, who gave explicit polynomials defining the Verschiebung in the particular cases of genus 2, rank 2, and characteristics 2 and 3, in [37] and [36]. Lange and Pauly also obtain our formula for the degree of V_2 via a quite different approach in [34], although their techniques thus far gives only that it is an upper bound in the case of ordinary curves, and not that it is an equality.

IV.1 On Degrees of Rational Self-Maps of $\mathbb{P}^n$

In this section we make some basic observations about degrees of rational maps from projective space to itself. We remark that unless otherwise specified, in this section the term 'point' shall always refer to a closed point. We suppose we have the following:

Situation IV.1.1. We are given a rational map $f : \mathbb{P}^n \to \mathbb{P}^n$ which is dominant and defined at all but a finite set of points. We suppose that we are given coordinate 'functions' X_i on $\mathbb{P}^n$, and f is represented by $n+1$ homogeneous polynomials F_i of degree d in the X_i.

Then we show:

Proposition IV.1.2. *In the above situation, we have the inequality:*

$$\deg f \leq d^n - \delta,$$

where δ is the total length of the 'undefined locus' subscheme E_f of $\mathbb{P}^n$ cut out by the F_i. Moreover, the following are equivalent:

a) The above inequality is an equality;

b) E_f is a locally complete intersection;

c) E_f is Gorenstein.

In particular, we get equality when the length of the points of E_f are all 1 *or* 2.

Proof. Choose P in $\mathbb{P}^n$, and write $H_1, \dots H_n$ for n hyperplanes cutting out P. Denote by E_P the (scheme-theoretic) intersection of the $f^*(H_i)$. Now, for a given X_{i_0} on the image space, we observe that on $\mathbb{P}^n_{f^*X_{i_0}} = \mathbb{P}^n_{F_{i_0}}$, we actually have $f^{-1}(P) \cong E_P$ (scheme-theoretically): Indeed, $\mathbb{P}^n_{F_{i_0}}$ is affine, say $\operatorname{Spec} R$, and following through the tensor product definition of $f^{-1}P$, we find it is given by $\operatorname{Spec} R/(\{f^*H_i\})$, which is precisely the scheme intersection of the f^*H_i on $\mathbb{P}^n_{F_{i_0}}$. On the other hand, as i_0 varies, the $\mathbb{P}^n_{F_{i_0}}$ will cover everything except the undefined locus E_f of f, since E_f is given precisely as the locus where all F_i vanish. But the $f^*(H_i)$ are given by polynomials homogeneous and linear in the F_i; this means that there is a closed immersion of E_f into E_P. Hence, we can write E_P as a set as $E_f \cup f^{-1}(P)$, with a closed immersion of the latter into the former, so assuming that everything is 0-dimensional, which we now do, the total length of the latter is less than or equal to the total length of the former. For P general, the total length of $f^{-1}(P)$ is $\deg f$ by Lemma A.26, and we are writing δ for the total length of E_f. But E_P is cut out by n hypersurfaces of degree d with no excess intersection, so by Bezout's theorem [26, Thm I.7.7] it has total length d^n, yielding the desired inequality.

Now, E_f being a locally complete intersection implies that it is Gorenstein (see [13, Cor 21.19]). For the converse, we invoke the theorem that if R is a regular local ring and I an ideal of codimension n, then if R/I is Gorenstein, either I is a complete intersection, or it is generated by at least $n+2$ generators (see [13, p. 542]). Since we know E_f is codimension n and cut out by $n+1$ hypersurfaces (namely, the F_i), it follows that if it is Gorenstein it must be a local complete intersection, and we have the equivalence of b) and c). Next, to see that a) implies b), we note that if E_f is not a locally complete intersection, we cannot have equality, as E_P is cut out by n hypersurfaces, the $f^*(H_i)$.

Finally, we need to show that b) implies a), which requires that if E_f is a locally complete intersection, the ideal generated by the $f^*(H_i)$ corresponding to a general point P on the image $\mathbb{P}^n$ is locally equal to the ideal of the F_i (that is, the ideal cutting out E_f), at every point Q of E_f. Since we only care about a general choice of P, it will be enough to check each of the finitely many points of E_f separately. We therefore fix a Q in E_f, and assume we have fixed a choice of dehomogenization, so that the F_i are actually functions on a neighborhood of Q. We also note that it will be enough to prove the statement for any particular choice of H_i cutting out the given P. We first note that the defining ideal I_{E_f} of E_f, generated by all the F_i, may actually be generated by all but one on the local ring: indeed, $I_{E_f}/\mathfrak{m}_Q I_{E_f}$ must be at most n-dimensional, since we assumed that E_f is a local complete intersection, and hence cut out by some n elements. But since the F_i generate I_{E_f}, they must span $I_{E_f}/\mathfrak{m}_Q I_{E_f}$, and in particular some n of the $n+1$ of them must span; then Nakayama's lemma implies that those n actually generate I_{E_f} locally at Q, as desired. Hence, we have $F_j = \sum_{i \neq j} a_i F_i$ for some integer j and $a_i \in \mathscr{O}_{\mathbb{P}^n,Q}$, and (allowing slight sloppiness with the indexing of the H_i) a general point P in the image $\mathbb{P}^n$ may be cut out by H_i of the form $X_i - \lambda_i X_j$. Since we will have $f^*(H_i) = F_i - \lambda_i F_j$, it is clear that it suffices to show that F_j is in the ideal generated by the $f^*(H_i)$ for a general choice of the λ_i, since then we can solve for the other F_i in terms of the $f^*(H_i)$ as well, and we will have $E_f \cong E_P$ at Q, as desired. Using our formula for F_j in terms of the F_i, we have then $\sum_{i \neq j} a_i f^*(H_i) = \sum_{i \neq j} a_i (F_i - \lambda_i F_j) = (1 - \sum_{i \neq j} a_i \lambda_i) F_j$; the coefficient in front of

F_j is always a regular function in R_Q by hypothesis, and for a general choice of λ_i, it will be nonvanishing at Q, so we can invert to solve for F_j, completing the proof of the proposition. $\square$

For posterity, we observe that the inequality of the proceeding proposition really isn't, in general, an equality (that is, E_f need not be a locally complete intersection):

Example IV.1.3. Consider the map from $\mathbb{P}^2$ to itself given by $(X, Y, Z) \mapsto (X^3, Y^3, XYZ)$. This is undefined only at $(0, 0, 1)$, where the subscheme E_f has length 5 (it is given on the affine open $Z = 1$ by $k[x, y]/(x^3, y^3, xy)$, which is a 5-dimensional k-vector space with basis $(1, x, x^2, y, y^2)$). We see however that it is dominant of degree 3: if we take a point (X_0, Y_0, Z_0) with all coordinates nonzero, we have 3 choices for X and 3 for Y, which then determine Z; but we can scale to fix a particular choice of X, which leaves us with 3 points in the preimage. Since $3^2 - 5 = 4 > 3$, we see that in this example the inequality is in fact strict.

IV.2 The Case of V_2

We now apply the results of the preceding section to examine what we can about the degree of the Verschiebung map induced by pullback under Frobenius on the moduli space of vector bundles of rank 2 with trivial determinant on a curve C of genus 2. We also address the related question of when the map can be made into a morphism by blowing up the undefined points only a single time. We begin by reviewing the basic facts about the moduli space and Verschiebung, without any hypotheses on the genus.

We restrict to rank 2 for ease of notation: given any fixed line bundle $\mathscr{L}$, there is a proper coarse moduli variety parametrizing vector bundles on C with rank 2 and determinant $\mathscr{L}$; however, away from the dense open subset of stable bundles, the moduli variety cannot distinguish between different isomorphism classes of semistable bundles which are extensions of the same pair of line bundles (two such vector bundles are said to be **S-equivalent**). We will denote by M_2 the moduli space of semistable vector bundles of rank 2 and trivial determinant on C, which will be the only space we will consider here. We will refer to the map induced by pullback under Frobenius on moduli spaces of vector bundles as the **Verschiebung**, since in the case of line bundles this is precisely what it gives (that is to say, on the Jacobian, the map induced by pulling back line bundles under Frobenius is actually the dual isogeny to Frobenius itself). We denote by V_2 the particular Verschiebung map on M_2. V_2 is a dominant rational map, with its undefined points contained among Frobenius-unstable vector bundles (in fact, the undefined points are precisely the Frobenius-unstable bundles, but we will only prove this for a general curve here). See Section IV.A for the precise technical definitions and proofs of these statements.

Now, in our situation of rank 2 bundles on a curve of genus 2, it is a theorem of Narasimhan and Ramanan (see Theorem IV.A.9) that $M_2 \cong \mathbb{P}^3$. A. J. de Jong provided the following argument to show that for this case, we have:

Proposition IV.2.1. *The Verschiebung map is given by polynomials of degree p.*

Proof. We have a diagram:

$$\begin{tikzcd} M_2(C^{(p)}) \cong \mathbb{P}^3 \arrow[r, dashed, "V_2"] & M_2(C) \cong \mathbb{P}^3 \\ J(C^{(p)}) \arrow[u] \arrow[r, "V"] & J(C) \arrow[u] \end{tikzcd}$$

where the Jacobians map into their respective M_2's as the Kummer surfaces given as $\{\mathscr{L} \oplus \mathscr{L}^{-1} : \mathscr{L} \in J\}$, which is precisely the semistable boundary locus inside M_2 (see [46, Prop. 6.3]), and V is the Verschiebung on the Jacobians. Showing that our original rational map is given by polynomials of degree p is equivalent to showing that $V_2^* H = pH$ on the open set $U_2 \subset M_2(C^{(p)})$ on which V_2 is defined. Since H generates $\operatorname{Pic}(\mathbb{P}^3)$, we have $V_2^* = mH$ for some m, and it suffices to invoke the following two facts: first, the pullbacks of H to the Jacobians are $2\Theta^{(p)}$ and 2Θ respectively, where Θ denotes the theta divisor on $J(C)$; and second, $V_2^*\Theta = p\Theta^{(p)}$.

For the first assertion, it is not difficult to check on functors that the map $J(C) \to M_2(C) \cong \mathbb{P}H^0(\operatorname{Pic}^1(C), 2\Theta)$ is unramified away from the two-torsion points, simply by looking at the associated Cartier divisors $j \mapsto j\Theta + j^{-1}\Theta$ under first-order deformations. Hence, if we take a general hyperplane in $M_2(C)$, its pullback will be reduced in $J(C)$, and we can compute it set-theoretically. Now, choose any $\mathscr{L} \in \operatorname{Pic}^1(C)$ not a theta characteristic; by Lemma IV.A.11, $\mathbb{P}\operatorname{Ext}(\mathscr{L}, \mathscr{L}^{-1})$ defines a hyperplane in $M_2(C)$, and by much the same argument, it is easy to check that the restriction to the image of $J(C)$ is precisely the set $\Theta_{\mathscr{L}} \cup [-1]^*\Theta_{\mathscr{L}}$; these two are distinct precisely when $\mathscr{L}$ is not a theta characteristic, but they always define the same line bundle, so we get $\mathscr{O}(2\Theta)$, as desired.

The second assertion follows from the formula $[m]^*\mathscr{L} \cong \mathscr{L}^{\otimes\frac{m^2+m}{2}} \otimes [-1]^*\mathscr{L}^{\otimes\frac{m^2-m}{2}}$ for pullback under the multiplication by m map of a line bundle on an abelian variety, and $\deg[m] = m^{2g}$ (see [45, p. 59, p. 61]). For any line bundle $\mathscr{L}$ with $[-1]^*\mathscr{L} \cong \mathscr{L}$, we get $[m]^*\mathscr{L} \cong \mathscr{L}^{\otimes m^2}$. We then have the factorization $F_J \circ V = [p]$ on $J(C^{(p)})$, and it easy to see that $F_J^*\mathscr{L} \cong (F_J^{-1}\mathscr{L})^p$, whereupon it follows that if $[-1]^*\mathscr{L} \cong \mathscr{L}$, $V^*F_J^{-1}\mathscr{L} \cong \mathscr{L}^p$ modulo p-torsion. Now, it is easy to check that $F^{-1}\Theta^{(p)} = \Theta$, and that $[-1]^*\mathscr{O}(\Theta^{(p)}) \cong \mathscr{O}(\Theta^{(p)})$ (indeed, if one chooses the particular theta divisor via a theta characteristic, this symmetry will hold on the divisor level). We can finally conclude that we get actual equality without p-torsion, because we already had that the hyperplane classes pull back to twice the theta divisors on the Jacobians, and that one hyperplane class pulls back to a multiple of the other under V_2. □

The above result was also proved independently by Laszlo and Pauly, [36, Prop. 7.2].

Corollary IV.2.2. *The degree of V_2 is bounded above by p^3; or more sharply, $p^3 - \delta$, where δ is the number of points at which V_2 it is undefined. If the undefined points are reduced, this upper bound is an equality.*

Proof. Immediate from the preceding proposition and Proposition IV.1.2. □

On the other hand, we also have:

Lemma IV.2.3. *The degree of V_2 is bounded below by p^2.*

Proof. Consider any point on the Kummer surface inside M_2; that is, a bundle of the form $\mathscr{L} \oplus \mathscr{L}^{-1}$ for $\mathscr{L}$ a line bundle of degree 0. There are p^2 line bundles of degree 0 mapping to $\mathscr{L}$ under V on the Jacobian, differing from each other by a p-torsion line bundle, and these will give p^2 different bundles in the Kummer surface as long as none of them is related to its inverse by a p-torsion bundle, which can only happen if $\mathscr{L}$ was originally a 2-torsion bundle. Thus, on an open part of the Kummer surface, each point has at least p^2 preimages, and because the Kummer surface has codimension 1 in the entire space, we can conclude that on a (non-empty) open subset of it, V has finite preimage, so this gives the desired lower bound on the degree by part (ii) of Proposition A.26. □

IV.3 Some Cohomology and Hypercohomology Groups

We begin by reviewing some fundamental facts about representing groups of deformations with cohomology and hypercohomology. Throughout the remainder of this chapter, we fix the notation:

Notation IV.3.1. A 'deformation' refers to a first order infinitesmal deformation, and ϵ is a square-zero element.

Let C be any curve, U_1, U_2 an open cover, and ω_i one-forms trivializing Ω^1_C on the U_i. We set the convention now that all our 1-cocycles will be written with coordinates on U_2. We will follow the trivialization notation of Section III.1, except that we will write all our connection matrices in terms of the trivializations provided by the ω_i, and will therefore write them as T_i rather than $\bar{T}_i$.

Let $\mathscr{E}$ be a vector bundle on C, given by a transition matrix E on $U_1 \cap U_2$; the only restriction on E is that its determinant be invertible. Next, suppose ∇ is a connection on $\mathscr{E}$, given by T_i on U_i. Here, the T_i must satisfy the relationship $T_1 = \frac{\omega_2}{\omega_1} E T_2 E^{-1} + E \frac{dE^{-1}}{\omega_1}$, where ω_i are a trivialization of Ω^1_C on the U_i. Then, it is a standard fact that a deformation of $\mathscr{E}$ is given by some invertible matrix $E(I + \epsilon E')$; indeed, this follows from the assertion that any deformation of a free module is free, since then the same U_i trivialize the deformation, and the freeness of the deformation follows from the argument for Proposition A.16, the injectivity of the map π of that proof being part of the definition of a deformation. We then see that invertibility of the transition matrix actually follows from E' being regular on $U_1 \cap U_2$, since $E(I - \epsilon E')$ will then provide an inverse. We also see that if we wish to take a deformation preserving the determinant of E, we simply restrict to E' having trace 0, since the determinant of $I + \epsilon E'$ is $1 + \epsilon \operatorname{Tr} E'$. Indeed, by considering E' as a 1-cocyle, we get:

Proposition IV.3.2. *The space of deformations of $\mathscr{E}$ is isomorphic to $H^1(C, \mathcal{E}nd(\mathscr{E}))$, and the space of deformations of $\mathscr{E}$ preserving the determinant of $\mathscr{E}$ is isomorphic to $H^1(C, \mathcal{E}nd^0(\mathscr{E}))$.*

Proof. We have seen the equivalences of deformations and cocycles, so we need only show that both objects have the same equivalence relation on them. A cocycle E', given with coordinates on U_2, is equivalent precisely to cocycles of the form $E' + S_2 - E^{-1} S_1 E$ for some 0-cochain S_i. On the other hand, two deformations of $\mathscr{E}$ are equivalent if there is an isomorphism between them, fixing $\mathscr{E}$. If we write the isomorphism as $I + \epsilon S_i$ on U_i, and the second deformation as $E(I + \epsilon E'')$, we have the condition that $E(I + \epsilon E'') = (I - \epsilon S_1)E(I + \epsilon E')(I + \epsilon S_2) = E + \epsilon(EE' - S_1 E + ES_2) = E(I + \epsilon(E' - E^{-1}S_1 E + S_2))$, giving precisely the desired formula for E''. □

Remark IV.3.3. The more naive identification of 1-cocycles with deformations might be to take $E + \epsilon E'$ rather than $E(I + \epsilon E')$, but this would have made the determinant-preserving choices for E' harder to classify, and more importantly, would have given the wrong equivalence under coboundaries.

Next, deformations of ∇ over a given deformation $\mathscr{E}'$ of $\mathscr{E}$ are given by $T_i + \epsilon T_i'$ regular on U_i, satisfying

$$(T_2 + \epsilon T_2') = \frac{\omega_1}{\omega_2}(I - \epsilon E')E^{-1}(T_1 + \epsilon T_1')E(I + \epsilon E') - \frac{d((I - \epsilon E')E^{-1})}{\omega_2}E(I + \epsilon E').$$

Note that rather than our usual writing of T_1 in terms of T_2, we have done the opposite, as this will be more convenient when we work with 1-cocycles having coordinate on U_2. Taking the ϵ term, expanding the differential, and then substituting back in with $T_1 = \frac{\omega_2}{\omega_1}ET_2E^{-1} + E\frac{dE^{-1}}{\omega_1}$ we get that the T_i' satisfy

$$T_2' = \frac{\omega_1}{\omega_2}(E^{-1}T_1'E) - E'T_2 + T_2E' + \frac{dE'}{\omega_2}. \tag{IV.3.4}$$

If we set $E' = 0$, we find the T_i' are subject to $T_2' = \frac{\omega_1}{\omega_2}E^{-1}T_1'E$, which is to say, they are a 0-cocyle of $\mathcal{E}nd(\mathscr{E}) \otimes \Omega_1^C$. Since the determinant of a connection is given by its trace, we once again find that fixing the determinant is equivalent to choosing trace 0 matrices. As before, we get:

Proposition IV.3.5. *The space of deformations of ∇ over $\mathscr{E}$ (respectively, fixing the determinant of ∇) is isomorphic to $H^0(C, \mathcal{E}nd(\mathscr{E}) \otimes \Omega_C^1)$ (respectively, $H^0(C, \mathcal{E}nd^0(\mathscr{E}) \otimes \Omega_C^1)$).*

However, we will actually be interested in transport equivalence classes of connections. Specifically, for a deformation of ∇ over a deformation of $\mathscr{E}$, we will mod out by transport under automorphisms of the form $I + \epsilon S_i$, which sends a connection given by $T_i + \epsilon T_i'$ to $T_i + \epsilon(T_i' + T_iS_i - S_iT_i + \frac{dS_i}{\omega_i})$. We therefore introduce the sheaf map which is simply the connection induced on $\mathcal{E}nd(\mathscr{E})$ by ∇; we denote it $d_\nabla : \mathcal{E}nd(\mathscr{E}) \to \mathcal{E}nd(\mathscr{E}) \otimes \Omega_C^1$, and it is given by $\phi \mapsto \nabla \circ \phi - \phi \circ \nabla$. We note that for an open U with a trivializing one-form ω, if ∇ is given on an open U (with respect to ω) by T, and a section of $\mathcal{E}nd(\mathscr{E})$ is given on U by S, then $d_\nabla(S) = TS - ST + \frac{dS}{\omega}$. Comparing this to our infinitesmal transport formula, we immediately see:

Proposition IV.3.6. *The space of transport equivalence classes of deformations of ∇ over $\mathscr{E}$ is isomorphic to $H^0(C, \mathcal{E}nd(\mathscr{E}) \otimes \Omega^1_C)/d_\nabla(H^0(C, \mathcal{E}nd(\mathscr{E})))$.*

We also have:

Proposition IV.3.7. *$d_\nabla \mathcal{E}nd^0(\mathscr{E})$ takes values in $\mathcal{E}nd^0(\mathscr{E})$. Further, if* char k *is prime to the rank of $\mathscr{E}$, the space of transport equivalence classes of deformations of ∇ over $\mathscr{E}$ with fixed determinant is isomorphic to $H^0(C, \mathcal{E}nd^0(\mathscr{E}) \otimes \Omega^1_C)/d_\nabla(H^0(C, \mathcal{E}nd^0(\mathscr{E})))$.*

Proof. The first part is given for the induced connection on $\mathcal{E}nd^0(\mathscr{E})$ in Section III.1. The only potentially tricky part of the second assertion is to show that $d_\nabla(H^0(C, \mathcal{E}nd^0(\mathscr{E}))) = d_\nabla(H^0(C, \mathcal{E}nd(\mathscr{E})))$, since there is no reason to assume *a priori* that our transport endomorphisms have trivial determinant. However, because char k doesn't divide the rank of $\mathscr{E}$, we can write any endomorphism of $\mathscr{E}$ as as scalar map plus a map of trace 0, and we see that the transport will be additive on these infinitesmal endomorphisms, and will not be affected by the scalar term, yielding the desired result. □

However, parametrizing pairs of deformations of $\mathscr{E}$ together with deformations of ∇ over $\mathscr{E}$ cannot be described naturally with sheaf cohomology, but rather requires sheaf hypercohomology. Given a complex $\mathscr{F}^\bullet$ of sheaves, the hypercohomology $\mathbb{H}^*(C, \mathscr{F}^\bullet)$ can be described as the comology of the double-complex whose p, q term is given by Cech p-chains with coefficients in $\mathscr{F}^q$. For notational convenience, we specify:

Notation IV.3.8. The complex $\mathcal{E}nd^0(\mathscr{E}) \stackrel{d_\nabla}{\to} \mathcal{E}nd^0(\mathscr{E}) \otimes \Omega^1_C$ will also be denoted by $\mathscr{K}^\bullet$.

We have:

Proposition IV.3.9. *The space of transport equivalence classes of deformations of the pair $\mathscr{E}$ together with ∇ (respectively, deformations fixing both determinants) is isomorphic to $\mathbb{H}^1(C, \mathcal{E}nd(\mathscr{E}) \stackrel{d_\nabla}{\to} \mathcal{E}nd(\mathscr{E}) \otimes \Omega^1_C)$ (respectively, $\mathbb{H}^1(C, \mathcal{E}nd^0(\mathscr{E}) \stackrel{d_\nabla}{\to} \mathcal{E}nd^0(\mathscr{E}) \otimes \Omega^1_C)$).*

Proof. An element of $\mathbb{H}^1(C, \mathcal{E}nd(\mathscr{E}) \stackrel{d_\nabla}{\to} \mathcal{E}nd(\mathscr{E}) \otimes \Omega^1_C)$ is a Cech 1-cocycle E' with coefficients in $\mathcal{E}nd(\mathscr{E})$ together with a Cech 0-cochain T'_i with coefficients in $\mathcal{E}nd(\mathscr{E}) \otimes \Omega^C_1$ which agree in their image in the group of Cech 1-cocyles with coefficients in $\mathcal{E}nd(\mathscr{E}) \otimes \Omega_1^C$. Since one has to change coordinates of one of the T_i' in order to substract it from the other, we find that, working on U_2, the image of the T_i' is $T_2' - \frac{\omega_1}{\omega_2} E^{-1} T_1' E$. This accounts for the term on the left and the first term on the right in Equation IV.3.4, so we need to check that the image of E' under the complex map accounts for the rest. Now, ∇ is given on U_i by $s_i \mapsto T_i s_i + \frac{ds_i}{\omega_i}$; we are working on U_2, and E' was given in terms of coordinates on U_2, so $\nabla \circ \phi - \phi \circ \nabla$ will be given by $T_2 E' + \frac{dE'}{\omega_2} - E' T_2$, and this precisely accounts for the remaining terms in Equation IV.3.4.

Lastly, equivalence of connections is given by transport under automorphisms, and equivalence of hypercohomology classes is given by changing our 1-cocyle and 0-cochain by the image of a 0-cochain of $\mathcal{E}nd(\mathscr{E})$. We had already shown that this gives exactly the right equivalence relations on the deformations of $\mathscr{E}$, and that for a fixed deformation of $\mathscr{E}$ and a 0-cocycle of $\mathcal{E}nd(\mathscr{E})$, it gives the right equivalence relation for deformations of ∇. But the latter really checked that 0-cochains give the right formula transport along an isomorphism of a connection from one deformation of $\mathscr{E}$ to an isomorphic one; we only used the cocycle hypothesis at the end to keep the (trivial) deformation of $\mathscr{E}$ fixed. Thus, we get the right equivalence relation in general, as desired.

This proves the proposition for $\mathcal{E}nd(\mathscr{E})$, but the $\mathcal{E}nd^0(\mathscr{E})$ case is precisely the same, as everything above was purely formal. □

IV.4 Spectral Sequences

We can use the standard spectral sequence for the cohomology of a double complex to place $\mathbb{H}^1(C, \mathcal{E}nd^0(\mathscr{E}) \to \mathcal{E}nd^0(\mathscr{E}) \otimes \Omega_C^1)$ into a short exact sequence. Starting with the Cech double complex associated to our complex, and taking differentials in the vertical direction, we see that the E_1 term is:

$$\begin{array}{cccc}
\vdots & \vdots & & \\
0 & 0 & & \\
H^1(C, \mathcal{E}nd^0(\mathscr{E})) \xrightarrow{d_2^1} & H^1(C, \mathcal{E}nd^0(\mathscr{E}) \otimes \Omega^1_C) & 0 & \cdots \\
H^0(C, \mathcal{E}nd^0(\mathscr{E})) \xrightarrow{d_2^0} & H^0(C, \mathcal{E}nd^0(\mathscr{E}) \otimes \Omega^1_C) & 0 & \cdots
\end{array}$$

where d_2^i are the maps induced on Cech i-cocycles by d_∇. We see immediately that the spectral sequence stabilizes at E_2, and yields a short exact sequence:

$$0 \to \operatorname{coker} d_2^0 \to \mathbb{H}^1(C, \mathcal{E}nd^0(\mathscr{E}) \to \mathcal{E}nd^0(\mathscr{E}) \otimes \Omega^1_C) \to \ker d_2^1 \to 0.$$

We will show:

Proposition IV.4.1. *Suppose ∇ has p-curvature 0, and the corresponding $\mathscr{F}$ which pulls back under Frobenius to $\mathscr{E}$ is stable. Then d_2^0 is injective, so we can consider $H^0(C, \mathcal{E}nd^0(\mathscr{E}))$ to be a subgroup of $H^0(C, \mathcal{E}nd^0(\mathscr{E}) \otimes \Omega^1_C)$, and we left get a left exact sequence*

$$0 \to H^0(C, \mathcal{E}nd^0(\mathscr{E}) \otimes \Omega^1_C)/H^0(C, \mathcal{E}nd^0(\mathscr{E})) \to \mathbb{H}^1(C, \mathscr{K}^\bullet) \to H^1(C, \mathcal{E}nd^0(\mathscr{E}))$$

where the image on the right is the image under Frobenius pullback of $H^1(C^{(p)}, \mathcal{E}nd^0(\mathscr{F}))$.

Proof. An element in the kernel of d_2^0 is a trace zero endomorphism of $\mathscr{E}$ which commutes with ∇. But this is precisely the condition for it to come from a trace zero endomorpism of $\mathscr{F}$ (see, for instance, [30, Thm 5.1]), which must be 0, since $\mathscr{F}$ is stable (this follows almost immediately from the definition of stability and the fact that there are no non-trivial division algebras over an algebraically closed field; see [27, Cor. 1.2.8]).

Similarly, an element in the kernel of d_2^1 comes from $H^1(C^{(p)}, \mathcal{E}nd^0(\mathscr{F}))$, as asserted. □

Reinterpreted in terms of deformations, we are saying that, as should be the case, every deformation ∇' of ∇ on $\mathscr{E}$ corresponds to a unique pair of deformations of $(\mathscr{E}, \nabla)$: namely, $(\mathscr{E}, \nabla')$. Of course, a pair $(\mathscr{E}', \nabla')$ also gives rise to a deformation $\mathscr{E}'$ of $\mathscr{E}$. What is interesting here is that not every deformation $\mathscr{E}'$ of $\mathscr{E}$ arises in this way; indeed, $\mathscr{E}'$ admits a ∇' if and only if it came from some deformation $\mathscr{F}'$ of $\mathscr{F}$.

We also get some information by computing the spectral sequence for the double complex in the other direction. Specifically, we have:

Proposition IV.4.2. *With the same notation and hypotheses as in Proposition IV.4.1, we have*

$$H^1(C^{(p)}, \mathcal{E}nd^0(\mathscr{F})) \hookrightarrow \mathbb{H}^1(C, \mathscr{K}^\bullet)$$

The image is described by 1-cocycles of $\mathcal{E}nd^0(\mathscr{E})$ in the kernel of d_∇ (together with the zero 0-cochain of $\mathcal{E}nd^0(\mathscr{E}) \otimes \Omega^1_C$).

Proof. Taking differentials in the opposite direction as before, we find our E_2 term looks like

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
H^2(C, \ker d_\nabla) & H^2(C, \operatorname{coker} d_\nabla) & 0 & \cdots \\
H^1(C, \ker d_\nabla) & H^1(C, \operatorname{coker} d_\nabla) & 0 & \cdots \\
H^0(C, \ker d_\nabla) & H^0(C, \operatorname{coker} d_\nabla) & 0 & \cdots
\end{array}$$

Now, $H^1(C, \ker d_\nabla)$ cannot have any further nonzero differentials mapping into or out of it, so it injects into $\mathbb{H}^1(C, \mathscr{K}^\bullet)$. We saw in the proof of the previous proposition that $\ker d_\nabla$ is precisely $F^{-1}\mathcal{E}nd^0(\mathscr{F})$. Although it isn't even an $\mathscr{O}_C$ module, because F is a homeomorphism (and because we are dealing with F^{-1}, and not F^*), we can compute its cohomology as being the same as the cohomology on $C^{(p)}$ of $\mathcal{E}nd^0(\mathscr{F})$. Thus, $H^1(C, \ker d_\nabla) = H^1(C^{(p)}, \mathcal{E}nd^0(\mathscr{F}))$, and we get the desired injection, and description of its image. □

All this is saying is that any nontrivial deformation of $\mathscr{F}$ gives a nontrivial deformation of $(\mathscr{E}, \nabla)$ when pulled back to C, which shouldn't be too surprising given the categorical equivalence between $\mathscr{F}$ on $C^{(p)}$ and pairs $(\mathscr{E}, \nabla)$ on C, but still needed to be proved. Note that it does not imply that every non-trivial deformation of $\mathscr{F}$ maps to a non-trivial deformation of $\mathscr{E}$.

We now suppose that $\mathscr{E}$ is an extension of $\mathscr{L}^{-1}$ by $\mathscr{L}$, with $\mathscr{L}$ any line bundle on C. The short exact sequence

$$0 \longrightarrow \mathscr{L} \xrightarrow{i} \mathscr{E} \xrightarrow{j} \mathscr{L}^{-1} \longrightarrow 0$$

induces maps

$$d_1 : \mathrm{Hom}(\mathscr{L}^{-1}, \mathscr{L}) \to \mathcal{E}nd^0(\mathscr{E})$$

and

$$d_2 : \mathcal{E}nd^0(\mathscr{E}) \to \mathrm{Hom}(\mathscr{L}, \mathscr{L}^{-1}) \tag{IV.4.3}$$

by composition. Specifically, $d_1(\phi) = i \circ \phi \circ j$, and $d_2(\phi) = j \circ \phi \circ i$. Explicitly in terms of transition matrices, we see that:

$$d_1(\begin{bmatrix} f \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} f \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix} \tag{IV.4.4}$$

$$d_2(\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} f_{21} \end{bmatrix} \tag{IV.4.5}$$

We then see that the image of d_1 does indeed lie inside $\mathcal{E}nd^0(\mathscr{E})$, and we also see that it is in the kernel of d_2. Thus, we get a filtration of $\mathcal{E}nd^0(\mathscr{E})$ as $0 \subset \operatorname{im} d_1 \subset \ker d_2 \subset \mathcal{E}nd^0(\mathscr{E})$. We claim this induces the following filtration of our original complex:

$$\begin{tikzcd}
\mathcal{E}nd^0(\mathscr{E}) \arrow[r, "d_\nabla"] & \mathcal{E}nd^0(\mathscr{E}) \otimes \Omega^1_C \\
\ker d_2 \arrow[r, "d_\nabla"] \arrow[u, hook] & \mathcal{E}nd^0(\mathscr{E}) \otimes \Omega^1_C \arrow[u, hook] \\
\operatorname{im} d_1 \arrow[r, "d_\nabla"] \arrow[u, hook] & (\ker d_2) \otimes \Omega^1_C \arrow[u, hook] \\
0 \arrow[r, "d_\nabla"] \arrow[u, hook] & (\operatorname{im} d_1) \otimes \Omega^1_C \arrow[u, hook]
\end{tikzcd} \tag{IV.4.6}$$

The only part which requires any verification is that $d_\nabla(\operatorname{im} d_1) \subset (\ker d_2) \otimes \Omega^1_C$.

Something in the image of d_1 is of the form $i \circ \phi \circ j$ for some $\phi \in \mathrm{Hom}(\mathscr{L}^{-1}, \mathscr{L})$, by definition. We can evaluate d_∇ on this as $\nabla \circ i \circ \phi \circ j - i \circ \phi \circ j \circ \nabla$. Tensoring with Ω^1_C

doesn't change i or j at all (it simply allows coefficients to have certain poles they couldn't otherwise have), so d_2 makes sense equally well on $\mathcal{E}nd^0(\mathscr{E}) \otimes \Omega^1_C$ as on $\mathscr{E}$, and as sheaves, $(\ker d_2) \otimes \Omega^1_C$ is the same as the kernel of d_2 acting on $\mathcal{E}nd(\mathscr{E}) \otimes \Omega^1_C$. We now see that we get what we want, since $d_2(\nabla \circ i \circ \phi \circ j - i \circ \phi \circ j \circ \nabla) = j \circ \nabla \circ i \circ \phi \circ j \circ i - j \circ i \circ \phi \circ j \circ \nabla \circ i = 0$ because each term has a $j \circ i$ in it. Of course, this is a long-winded way of saying something that could easily be checked explicitly on the matrix level as well. Regardless, we get the desired filtration, and we can apply a resulting spectral sequence to get a different calculation of our hypercohomology group. We will see:

Proposition IV.4.7. *In our specific situation, where C has genus 2 and $\mathscr{L}$ is a theta characteristic, there is a short exact sequence*

$$0 \to \Gamma(\mathcal{H}om(\mathscr{L}^{-1}, \mathscr{L}) \otimes \Omega^1_C) \to \mathbb{H}^1(C, \mathscr{K}^\bullet) \to \operatorname{Ext}(\mathscr{L}, \mathscr{L}^{-1}) \to 0$$

Proof. When we have a filtration of our complex, we get (see [12, 1.4.5]) a spectral sequence converging to its hypercohomology, whose E_2 term is given in terms of the hypercohomology of the quotient complexes. Specifically, if $\mathscr{K}^\bullet$ is our complex, and $\mathscr{A}^\bullet_i$ the filtration, we get $E_2^{p,q} = \mathbb{H}^{p+q}(C, \operatorname{Gr}^q_{\mathscr{A}}(\mathscr{K})) \Rightarrow \mathbb{H}^{p+q}(C, \mathscr{K})$. We must therefore start by calculating the associated graded complexes of the filtration. The associated graded sheaves on the lefthand side are $0, \operatorname{im} d_1, \ker d_2 / \operatorname{im} d_1$, and $\mathcal{E}nd^0(\mathscr{E}) / \ker d_2$. Now, d_1 being a non-zero map from a line bundle is injective, so $\operatorname{im} d_1 \cong \mathcal{H}om(\mathscr{L}^{-1}, \mathscr{L})$. We see from our explicit descriptions of d_1 and d_2 that $\ker d_2 / \operatorname{im} d_1$ is isomorphic to the diagonal elements of $\mathcal{E}nd^0(\mathscr{E})$, which is just $\mathscr{O}_C$. Lastly, since d_2 is surjective onto $\mathcal{H}om(\mathscr{L}, \mathscr{L}^{-1})$, $\mathcal{E}nd^0(E)/\ker d_2 \cong \mathcal{H}om(\mathscr{L}, \mathscr{L}^{-1})$. Of course, the associated graded sheaves on the right are gotten by taking these, shifted them down by one, and tensoring with Ω^1_C. Noting that $\mathcal{H}om(\mathscr{L}^{-1}, \mathscr{L}) \cong \mathscr{L}^{\otimes 2} \cong \Omega^1_C$, we actually get isomorphic line bundles for the middle two associated graded complexes, and we have to check that the maps between them are isomorphisms. At first blush, they may look like they should actually be zero, but recall that d_∇ is not the obvious inclusion, and indeed not even linear (so we will have to check that the induced map on quotients is linear). We simply verify this directly:

We first verify that $\operatorname{im} d_1$ maps isomorphically to $(\ker d_2) \otimes \Omega^1_C / (\operatorname{im} d_1) \otimes \Omega^1_C$. Recall that $d_\nabla(S) = TS - ST - \frac{dS}{\omega}$. If S is 0 except in the upper right coordinate, then the deviation from linearity is also confined to the upper right coordinate, and therefore vanishes mod

$(\operatorname{im} d_1) \otimes \Omega^1_C$. In order to check that the map is nonzero, we will look on U_2. Here, we can make use of the fact that T can be written with 0's on the diagonal to see that TS has a nonzero entry only in the lower right, while ST has a nonzero entry only in the upper left, and conclude that $d_\nabla(S)$ is not concentrated in the upper right coordinate, and therefore has nonzero image mod $(\operatorname{im} d_1) \otimes \Omega^1_C$. Since we have a nonzero, linear map of isomorphic line bundles, it must be an isomorphism.

The next map is also linear by precisely the same argument. We show it is nonzero by showing that for some S, the lower left coordinate of $TS - ST$ is nonzero, again on U_2. We know that we can normalize T so that its diagonal is 0, and its lower left entry is 1. An $S \in \ker d_2$ is simply upper triangular, and since it has trace zero, must have diagonal coefficients of the form $f, -f$. But then the lower left coordinate of $d_\nabla(S)$ will be $2f$, which for any odd characteristic (where here 0 would receive honorary designation as odd, if we cared) is nonzero exactly when f is nonzero. So we see that $d_\nabla(\ker d_2)$ is not contained in $(\ker d_2) \otimes \Omega^1_C$, giving us that this induced map also is nonzero, and hence an isomorphism.

We see immediately from the spectral sequence for a double complex (or from the definition in terms of an injective resolution, if one takes that approach), that the hypercohomology of a 2-term complex for which the complex map is an isomorphism is simply 0. Additionally, the hypercohomology of a 1-term complex concentrated in the ith place is the cohomology of the nonzero term, shifted by i. This then gives us the E_2 term of our spectral sequence, since the middle two quotient complexes are isomorphisms, and the other two have only one nonzero term. We get as the E_2 term:

$$
\begin{array}{ccccccc}
H^0(C, \mathscr{H}) & H^1(C, \mathscr{H}) & 0 & \cdots & & & \\
 & 0 & 0 & 0 & \cdots & & \\
 & & 0 & 0 & 0 & \cdots & \\
 & & & 0 & H^0(C, \mathscr{H} \otimes \Omega^1_C) & H^1(C, \mathscr{H} \otimes \Omega^1_C) &
\end{array}
$$

where $\mathscr{H} := \mathcal{H}om(\mathscr{L}, \mathscr{L}^{-1})$.

Now, there are potentially non-zero differentials between the remaining terms, but because $H^0(C, \mathcal{H}om(\mathscr{L}, \mathscr{L}^{-1})) = 0$, the $H^0(C, \mathscr{H} \otimes \Omega^1_C)$ cannot have any further nonzero differentials, and we get the desired short exact sequence for $\mathbb{H}^1(C, \mathscr{K}^\bullet)$, making use of the fact that $H^1(C, \mathcal{H}om(\mathscr{L}, \mathscr{L}^{-1})) = \operatorname{Ext}(\mathscr{L}, \mathscr{L}^{-1})$. $\square$

We also note from the construction of the spectral sequence that the map from $\mathbb{H}^1(C, \mathscr{K}^\bullet)$ to $\operatorname{Ext}(\mathscr{L}, \mathscr{L}^{-1})$ is in fact precisely the map induced by first mapping to $H^1(C, \mathcal{E}nd^0(\mathscr{E}))$, and then taking the map induced by d_2 on H^1. Because this factors through our prior map to $H^1(C, \mathcal{E}nd^0(\mathscr{E}))$, we get a diagram:

$$\begin{tikzcd}
 & H^0(C, \mathcal{E}nd^0(\mathscr{E}) \otimes \Omega^1_C)/H^0(C, \mathcal{E}nd^0(\mathscr{E})) \arrow[dl, hook] \arrow[d, hook] & \\
H^0(C, (\Omega^1_C)^2) \arrow[r, hook] & \mathbb{H}^1(C, \mathscr{K}^\bullet) \arrow[r, two heads] \arrow[d] & \operatorname{Ext}(\mathscr{L}, \mathscr{L}^{-1}) \\
 & H^1(C, \mathcal{E}nd^0(\mathscr{E})) \arrow[ur] &
\end{tikzcd}$$

where the inclusion on the upper left follows formally from exactness in the middle. But now we compare the dimensions to conclude:

Proposition IV.4.8. *The kernel of $\mathbb{H}^1(C, \mathscr{K}^\bullet) \to H^1(C, \mathcal{E}nd^0(\mathscr{E}))$ is equal to the kernel of $\mathbb{H}^1(C, \mathscr{K}^\bullet) \to \operatorname{Ext}(\mathscr{L}, \mathscr{L}^{-1})$. Equivalently, a deformation of the pair $(\mathscr{E}, \nabla)$ induces the trivial extension in $\operatorname{Ext}(\mathscr{L}, \mathscr{L}^{-1})$ if and only if it was the trivial deformation of $\mathscr{E}$ (together with any deformation of ∇).*

Proof. We need only show that the dimensions of the two kernels are equal. Since $(\Omega^1_C)^{\otimes 2}$ has degree 4, Riemann-Roch for line bundles gives us that $H^0(C, (\Omega^1_C)^{\otimes 2})$ has dimension 3. On the other hand, the dimension of $H^0(C, \mathcal{E}nd^0(\mathscr{E}) \otimes \Omega^1_C)/H^0(C, \mathcal{E}nd^0(\mathscr{E}))$, invoking Serre duality and the fact that $\mathcal{E}nd^0(\mathscr{E})$ is self-dual (easily verified from the definition if one considers $\mathcal{E}nd(\mathscr{E}) \cong \mathscr{E}^\vee \otimes \mathscr{E}$), is exactly the Euler characteristic of $\mathcal{E}nd^0(\mathscr{E}) \otimes \Omega^1_C$. By Riemann-Roch for vector bundles (see Theorem A.4, this is $d + r(1-g)$. $\mathcal{E}nd^0(\mathscr{E})$ has rank 3 and degree 0, and Ω^1_C has rank 1 and degree 2, so $\mathcal{E}nd^0(\mathscr{E}) \otimes \Omega^1_C$ has rank 3 and degree 6, giving $6 + 3(1-2) = 3$, as desired. $\square$

IV.5 Geometric Significance

What we have done so far has been very formal, so we will now lend it some geometric substance, largely by characterizing the map $H^1(C, \mathcal{E}nd^0(\mathscr{E})) \to \mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1})$ in geometric terms. Suppose we have a family of vector bundles $\tilde{\mathscr{E}}$ on C with base T, and trivial determinant, such that for some k-valued point $0 \in T$, $\tilde{\mathscr{E}}|_0 \cong \mathscr{E}$. Since we have a map $\mathscr{E} \to \mathscr{L}^{-1}$, by adjointness we get a map $\tilde{\mathscr{E}} \to i_{0*}\mathscr{L}^{-1}$, and we can take the kernel to get a new family $\tilde{\mathscr{E}}'$ over T which is isomorphic to $\tilde{\mathscr{E}}$ away from 0 (this will be another vector bundle if T is a curve, but not quite, for instance, if T is $\operatorname{Spec} k[\epsilon]$; see Theorem IV.B.11 of the appendix).

We can then restrict back to the fiber at 0, where we will get a new $\mathscr{E}'$ on C which will also be a vector bundle (even if $T = \operatorname{Spec} k[\epsilon]$), of rank 2 and trivial determinant. Everything but the trivial determinant assertion actually follows immediately from Theorem *IV.B.*11, with T either a curve or $\operatorname{Spec} k[\epsilon]$. For the triviality of the determinant, and consequent remarks, we will for the moment assume that T is a curve. In this case, we get the desired result from [45, Cor 5.6], since we have triviality of the determinant away from 0 on T.

Moreover, we see that $\mathscr{E}'$ is an extension of $\mathscr{L}$ by $\mathscr{L}^{-1}$: Since i is a closed immersion, $(i_{0*}\mathscr{L}^{-1})|_0 = \mathscr{L}^{-1}$, so we have an exact sequence

$$\mathscr{E}' \to \mathscr{E} \to \mathscr{L}^{-1} \to 0 \tag{IV.5.1}$$

but since the kernel of $\mathscr{E} \to \mathscr{L}^{-1}$ is $\mathscr{L}$, $\mathscr{E}' \to \mathscr{E}$ factors through $\mathscr{L} \to \mathscr{E}$, and we see that we get a map from $\mathscr{E}'$ to $\mathscr{L}$, which is surjective, by exactness of equation IV.5.1. Since $\mathscr{E}'$ has trivial determinant, it follows that $\mathscr{E}'$ is an extension of $\mathscr{L}$ by $\mathscr{L}^{-1}$. Thus, in the case that T is a curve, we get a map

$$\phi_{\mathscr{E}} : \{\tilde{\mathscr{E}} \text{ over } T\} \to \mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1})$$

This will have the following significance for us in trying to understand the Verschiebung:

Lemma IV.5.2. *Suppose T is a curve, 0 a point of T, and $\tilde{\mathscr{F}}$ a nontrivial family of semistable vector bundles over T with trivial determinant, such that $F^*\mathscr{F} = \mathscr{E}$, where $\mathscr{F} := \tilde{\mathscr{F}}|_0$. That is to say, $\tilde{\mathscr{F}}$ gives a nonconstant map of T into the moduli space M_2, passing through a point where V_2 is undefined. Then writing $\tilde{\mathscr{E}} = F^*\tilde{\mathscr{F}}$, if $\mathscr{E}' = \phi_{\mathscr{E}}(\tilde{\mathscr{E}}) \neq 0$,* *the limit point of the image of the curve under V_2 at $\mathscr{F}$ is given by $\mathscr{E}'$.*

Proof. This is largely a formality. The limit point of the curve under V_2 is gotten, by definition, by looking at the limit of the family $\tilde{\mathscr{E}}$. But $\tilde{\mathscr{E}}'$ is isomorphic to $\tilde{\mathscr{E}}$ away from the limit point, so we can look at $\tilde{\mathscr{E}}'$ instead, and if $\mathscr{E}'$ is semistable, we see it must give the limit point of the curve in M_2, just by continuity of the corresponding map of T into M_2. So we just need to know that $\mathscr{E}'$ is semistable as long as it is nonzero in $\mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1})$. But suppose $\mathscr{M}$ of positive degree maps into $\mathscr{E}'$. It cannot be a subbundle of $\mathscr{L}^{-1}$ by degree considerations, so it must compose to give a nonzero map to $\mathscr{L}$. But then it must have degree 1, and the map must be an isomorphism, meaning there is a map from $\mathscr{L}$ back to $\mathscr{E}'$, and $\mathscr{E}'$ is the trivial element of $\mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1})$, as desired. □

The implication is that if we are lucky, we will be able to describe the image of the exceptional divisor of V_2 in terms of $\mathbb{P}\,\mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1})$, which defines a hyperplane inside of M_2 (see Lemma IV.A.11). To show that this is actually what happens, we first note that if $T = \operatorname{Spec} k[\epsilon]$, we actually still get a map $\phi_{\mathscr{E}} : \mathrm{Def}^0(\mathscr{E}) \to \mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1})$. In fact, more specifically, we have:

Lemma IV.5.3. *$\phi_{\mathscr{E}}$ still exists in the case $T = \operatorname{Spec} k[\epsilon]$, inducing a map from $\mathrm{Def}^0(\mathscr{E})$ to $\mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1})$ which arises as the negative of the map on 1-cocycles induced by d_2 : $\mathcal{E}nd^0(\mathscr{E}) \to \mathrm{Hom}(\mathscr{L}, \mathscr{L}^{-1})$ (IV.4.3), from the previous section.*

Proof. To prove this, we have to start by pinning down the identification of $\mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1})$ with $H^1(C, \mathrm{Hom}(\mathscr{L}, \mathscr{L}^{-1}))$. We will think of $\mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1})$ as being described by transition matrices of the form $F = \begin{bmatrix} \varphi_{12} & 0 \\ f & \varphi_{12}^{-1} \end{bmatrix}$, where f is any regular section on $U_1 \cap U_2$. We first show that rather than taking f itself as our 1-cocyle of $\mathrm{Hom}(\mathscr{L}, \mathscr{L}^{-1})$, we will have to take $-\varphi_{12}^{-1} f$. The 1-cocycle of $\mathrm{Hom}(\mathscr{L}, \mathscr{L}^{-1})$ starting with an extension $\mathscr{F} \in \mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1})$ is gotten by starting with the exact sequence

$$0 \to \mathscr{L}^{-1} \to \mathscr{F} \to \mathscr{L} \to 0$$

taking the induced sequence

$$0 \to \mathrm{Hom}(\mathscr{L}, \mathscr{L}^{-1}) \to \mathrm{Hom}(\mathscr{L}, \mathscr{F}) \to \mathrm{Hom}(\mathscr{L}, \mathscr{L}) \to 0$$

and finally, looking at the image in $H^1(C, \mathrm{Hom}(\mathscr{L}, \mathscr{L}^{-1}))$ of the identity in $\mathrm{Hom}(\mathscr{L}, \mathscr{L})$, under the boundary map (see [26, Exer. III.6.1]). Let s_i, t_i be the trivializing sections on U_i for $\mathscr{F}$ in terms of which our transition matrix is written, and u_i and u_i^* the corresponding trivializing sections of $\mathscr{L}$ and $\mathscr{L}^{-1}$, so that the exact sequence is given as the map $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ followed by $\begin{bmatrix} 1 & 0 \end{bmatrix}$ on both U_i. Tensoring all these by u_i^* to get induced trivializations for the $\mathrm{Hom}(\mathscr{L}, \cdot)$ sheaves, the maps of the induced exact sequence are given by the same matrices. So, we start with the identity 0-cocycle in $\mathrm{Hom}(\mathscr{L}, \mathscr{L})$, given on both U_i simply by $\begin{bmatrix} 1 \end{bmatrix}$. We choose any choice of lift to $\mathrm{Hom}(\mathscr{L}, \mathscr{F})$; the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ on both U_i will do fine. We take the induced 1-cochain (as always, with coefficients on U_2), getting $\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \varphi_{12}^{-1} F \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\varphi_{12}^{-1} f \end{bmatrix}$. Lastly, we lift this back to the 1-cocycle of $\mathrm{Hom}(\mathscr{L}, \mathscr{L}^{-1})$ given by $-\varphi_{12}^{-1} f$, which is exactly what we wanted to get.

Next, recalling that d_2, by definition, took the lower left coordinate of a matrix, we just need to show that if $EE' = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$ gives a deformation of $\mathscr{E}$, then $\phi_{\mathscr{E}}(E')$ is described by the transition matrix $\begin{bmatrix} \varphi_{12} & 0 \\ e_{21} & \varphi_{12}^{-1} \end{bmatrix}$. This suffices because the lower left coordinate of E' will be $\varphi_{12} e_{21}$, and we will have to multiply by $-\varphi_{12}^{-1}$ to get our 1-cocycle of $\mathrm{Hom}(\mathscr{L}, \mathscr{L}^{-1})$.

But we can calculate $\phi_{\mathscr{E}}(E')$ directly, in this case.

$$E(I + \epsilon E') = \begin{bmatrix} \varphi_{12} + \epsilon e_{11} & \varphi_{12}^{-2} + \epsilon e_{12} \\ \epsilon e_{21} & \varphi_{12}^{-1} + \epsilon e_{22} \end{bmatrix}$$

Denote by s_i, t_i our trivializing basis on U_i, in terms of which E, and $E + E'$, are written. Also write u_i for the trivialization of $\mathscr{L}^{-1}$ on U_i. This means we have $s_2 = (\varphi_{12} + \epsilon e_{11}) s_1 + \epsilon e_{21} t_1$, $t_2 = (\varphi_{12}^{-2} + \epsilon e_{12}) s_1 + (\varphi_{12}^{-1} + \epsilon e_{22}) t_1$, and $u_2 = \varphi_{12}^{-1} u_1$. Now, the induced map to $\mathscr{L}^{-1}$ sends $a s_i + b \epsilon s_i + c t_i + d \epsilon t_i$ simply to $c u_i$. This means that its kernel is generated by $s_i, \epsilon t_i$ on U_i. Using the above formulas for s_2, t_2 in terms of s_1, t_1, we find that this kernel has transition matrix

$$\begin{bmatrix} \varphi_{12} + \epsilon e_{11} & \epsilon \varphi_{12}^{-2} \\ e_{21} & \varphi_{12}^{-1} \end{bmatrix}$$

When restricted to $\operatorname{Spec} k$ (that is, upon modding out by ϵ), this gives precisely the desired form for $\phi_{\mathscr{E}}(E')$. The theory developed in the appendix, and particulary Corollary IV.B.3 and the subsequent discussion, justifies this calculation, even though the transition matrix is not unique, and the kernel itself (prior to restriction) is not characterized by it. $\square$

We next prove some statements which will ultimately relate the limit points of images of curves under the Verschiebung to $\phi_{\mathscr{E}}$ acting on $\operatorname{Def}^0(\mathscr{E})$.

Lemma IV.5.4. *In the same situation as Lemma IV.5.2, if we write $\bar{\mathscr{F}}$ for the induced first-order deformation of $\mathscr{F}$ gotten via some closed immersion $t_\epsilon : \operatorname{Spec} k[\epsilon] \hookrightarrow T$ deforming the point $0 \in T$, and $\bar{\mathscr{E}} := F^*\bar{\mathscr{F}}$, we have $\bar{\mathscr{E}} \cong t_\epsilon^*\tilde{\mathscr{E}}$. Further, $\phi_{\mathscr{E}}\bar{\mathscr{E}} = \phi_{\mathscr{E}}\tilde{\mathscr{E}}$.*

Proof. The first statement follows immediately from the fact that $\tilde{\mathscr{F}}$ is a sheaf on $C^{(p)} \times T$, and all we are saying is that pullback under t_ϵ commutes with pullback under F. Since t_ϵ only acts on T, and F only acts on $C^{(p)}$, they commute.

The second half of the lemma is just an application of Theorem IV.B.13 in the appendix to our specific situation. $\square$

Putting together this lemma with Lemma IV.5.2, we see:

Theorem IV.5.5. *In the same situation as the previous lemma, if we suppose $\phi_{\mathscr{E}} \circ F^*$ is injective on first-order deformations of $\mathscr{F}$, then the limit point at 0 of the image of $T \to M_2$ under the Verschiebung is given as $\phi_{\mathscr{E}} \circ F^*(\bar{\mathscr{F}})$. In particular, all such limit points are contained in $\mathbb{P}\operatorname{Ext}(\mathscr{L}, \mathscr{L}^{-1}) \subset M_2$. Further, the Verschiebung only needs to be blown up once at $\mathscr{F}$.*

Proof. Everything but the last assertion follows directly from the two lemmas. To show that the Verschiebung only needs to be blown up once at $\mathscr{F}$, we need to know that every curve through $\mathscr{F}$ has the limit point of its image under V_2 determined by its tangent at $\mathscr{F}$. If M_2 were a fine moduli space, we'd be done, as every curve in it would correspond to a family of vector bundles, and we could then apply our result that it suffices to look at the first order deformation induced by the family, which is exactly the same as the tangent vector to the curve. However, because all of our $\mathscr{F}$'s are in the stable locus of our moduli space, where it isn't too far from being fine, it turns out, thanks to Proposition A.12, that this property in fact still holds, completing the proof. □

Finally, we can draw some conclusions which have immediate consequences for our understanding of the Verschiebung:

Theorem IV.5.6. *Given $\mathscr{F}$ such that $F^*\mathscr{F} \cong \mathscr{E}$, let $E_{\mathscr{F}}$ be the exceptional divisor above $\mathscr{F}$ after blowing up M_2 to make the Verschiebung a morphism. Then of the following, a) and b) are equivalent, and either implies c) and d):*

a) The scheme of connections with vanishing p-curvature on $\mathscr{E}$ is reduced at the point corresponding to $\mathscr{F}$.

b) The map $\mathrm{Def}(\mathscr{F}) \to \mathrm{Def}(\mathscr{E})$ induced by F^* is injective.

c) The image of $E_{\mathscr{F}}$ under V_2 in M_2 is precisely $\mathbb{P}\,\mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1})$.

d) V_2 only needs to be blown up once at $\mathscr{F}$

Proof. a) is equivalent to there not being any non-trivial deformations of ∇ which hold $\mathscr{E}$ fixed, and have p-curvature 0. We claim that this is the same as there not being any nontrivial deformations of $\mathscr{F}$ which pull back to the trivial deformation of $\mathscr{E}$; that is, that a) is equivalent to b). Noting that $C \times k[\epsilon]$ is smooth over $k[\epsilon]$, [30, Thm 5.1] gives us that $k[\epsilon]$-vector bundles on $C^{(p)} \times k[\epsilon]$ are equivalent as a category to $k[\epsilon]$-vector bundles together with connections of p-curvature 0 on $C \times k[\epsilon]$, with the functor given by pulling back under F and taking the canonical connection. It is clear that this commutes with pulling back under closed immersions of the base, so in particular with restricting to $\mathrm{Spec}\, k$, and since F^* is a categorical equivalence, its inverse also commutes with restriction to $\mathrm{Spec}\, k$. This means that our categorical equivalence restricts to another categorical equivalence when the restriction to $\mathrm{Spec}\, k$ is specified (compatibly) for both categories, and lastly, noting that every vector bundle is flat over its base, so the flatness condition in the definition of a deformation is vacuous, we see that the deformations of $\mathscr{F}$ really are equivalent to deformations of $\mathscr{E}$ together with ∇, and this completes the proof of the claim.

On the other hand, b) is equivalent to the map $\mathrm{Def}(\mathscr{F}) \to \mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1})$ being injective, by Proposition IV.4.8, which implies it is an isomorphism, since both spaces have dimension 3 over k. Noting that Lemma IV.5.3 tells us our geometric and cocycles versions of this map are really the same up to sign, by the preceding theorem this implies c), as desired.

Lastly, the fact that d) follows from these conditions also follows from the preceding theorem, since as we just noted, b) gives us that $\mathrm{Def}(\mathscr{F}) \hookrightarrow \mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1})$. $\square$

IV.6 Conclusions and Further Questions

We may now easily put everything together to prove our main theorem:

Proof of Theorem IV.0.1. The assertion of (i) that each Frobenius-unstable bundle corresponds to an undefined point will follow from (iii), since the image of the exceptional divisor of a blow-up centered at such a point is not just a single point. The degree statement then follows from (ii) by Corollary IV.2.2. But now (ii) and (iii) follow from the implications b) implies c) and d) of Theorem IV.5.6. $\square$

If we apply the low-characteristic results of Chapter III, we can conclude:

Corollary IV.6.1. *Let C be a general smooth, proper genus 2 curve over an algebraically closed field k of characteristic $p > 2$, and suppose that $p \leq 7$. Then:*

(i) Each Frobenius-unstable bundle corresponds to an undefined point of V_2, and V_2 has degree $\frac{1}{3}(p^3 + 2p)$;

(ii) Each undefined point may be resolved by a single blowup;

(iii) The image of the exceptional divisor associated to such an undefined point is given by $\mathbb{P}\mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1}) \subset M_2(C)$, where $\mathscr{L}$ is a theta characteristic on C, and specifically is the destabilizing line bundle for $F^*\mathscr{F}$, where $\mathscr{F}$ is the Frobenius-unstable vector bundle associated to the undefined point.

Proof. Thanks to Theorem IV.0.1, all we need know is that the Frobenius-unstable locus consists of $\frac{2}{3}(p^3 - p)$ reduced points. That this is the correct number for $p \leq 7$ is simply Theorem III.0.1. That none of the Frobenius-unstable bundles have non-trivial deformations follows, in light of Remark III.5.8 and the equivalence of a) and b) in Theorem IV.5.6, directly from the calculations for the $p = 3$ and $p = 5$ cases in Section III.6, and for $p = 7$ from the conclusion of the argument for Theorem III.0.1 presented in Section III.7. $\square$

We conclude with some further questions:

Question IV.6.2. Are statements c) and d) of Theorem IV.5.6 in fact equivalent to a) and b)?

Question IV.6.3. Is the scheme of Frobenius-unstable bundles of rank two and trivial determinant (equivalently, the scheme of transport-equivalence classes of connections with trivial determinant and vanishing p-curvature on the appropriate unstable bundles) isomorphic to the scheme-theoretic undefined locus of V_2?

Question IV.6.4. Is the degree of V_2 constant over all smooth curves of genus 2?

We remark that an affirmative answer to the second question would give an affirmative answer to the third, thanks to a result of Mochizuki [42, II, Thm. 2.8, p. 153]: this gives that our scheme of Frobenius-unstable bundles is finite flat over our space of curves, and that it is smooth over the base field, from which it follows that its fiber over any fixed curve is a local complete intersection.

More generally, one could ask:

Question IV.6.5. How might one attempt to compute the degree of the Verschiebung for curves of higher genus, or vector bundles of higher rank?

We remark finally that to attempt to address this last question via similar techniques to those of this chapter, it would be necessary not only to generalize our understanding of the undefined locus of the Verschiebung, but also to appropriately generalize the degree formula of Proposition IV.1.2 to a substantially more general class of projective varieties; indeed, for higher genus and rank, the moduli spaces in question become singular along the strictly semi-stable locus.

IV.A Appendix: Some General Results on the Verschiebung

This appendix consists of the formal construction of, and some general results on, the generalized Verschiebung map on coarse moduli spaces of vector bundles. Most of the key arguments were developed by A. J. de Jong prior to the initial conceptualization of the present work; they are reproduced here because they are a necessary background component, and were never published elsewhere. We will work in the situation:

Situation IV.A.1. C is a smooth, proper curve over a field k of characteristic p. $C^{(p)}$ is the p-twist of C over k, and F is the relative Frobenius morphism from C to $C^{(p)}$.

Remark IV.A.2. The algebraically closed hypothesis is not actually necessary; the moduli space construction can be made to work over a non-algebraically closed field, and the arguments here will go through in this setting. However, it is harder to find references for the general case, and we will only apply the results here in the case of an algebraically closed base field.

Remark IV.A.3. There are a few points to be careful of with respect to characteristic and the general theory of moduli of vector bundles. The main obstruction is boundedness, which is not a problem in our situation of the base being a curve (see [27, Cor. 1.7.7]), and is now known in any dimension via the more involved argument of [35]. However, in characteristic p the statement that the moduli space universally corepresents the relevant functor is in fact no longer true. It is however true that it **uniformly** corepresents the functor, which is to say that it is universal for flat base change, and this is all we will need; see [44, Thm. 1.10, p. 38].

We recall the following definition and theorem:

Definition IV.A.4. Two vector bundles $\mathscr{E}, \mathscr{E}'$ of degree 0 are S**-equivalent** if and only if there are filtrations $F_{\mathscr{E}}$ and $F_{\mathscr{E}'}$ with $\mathrm{Gr}_{F_{\mathscr{E}}} \cong \mathrm{Gr}_{F_{\mathscr{E}'}}$ (this isomorphism is not required to preserve the grading), and the quotients of $F_{\mathscr{E}}$ and $F_{\mathscr{E}'}$ all stable sheaves of degree 0.

Theorem IV.A.5. *There is a coarse moduli scheme $M_n(C)$ which uniformly corepresents the functor of semistable vector bundles on C of rank n and trivial determinant. The closed points of $M_n(C)$ correspond to S-equivalence classes of vector bundles; in particular, there is an open subscheme $M_n^s(C)$ whose closed points parametrize stable vector bundles.*

Proof. See [27, Thm. 4.3.4]. □

We will want to know:

Lemma IV.A.6. *Let $\mathcal{U}$ be an open subfunctor of $\mathcal{M}_n(C)$ whose points are well-defined modulo S-equivalence. Then there is an open subscheme $U \subset M_n(C)$ which corepresents $\mathcal{U}$.*

Proof. One need only check that under the hypotheses on $\mathcal{U}$, in the construction of $M_n(C)$ used in [27, Thm. 4.3.3 and Thm. 4.3.4], we will have that the pullback of $\mathcal{U}$ to the R of that theorem, which will be representable as an open subscheme U_R of R, has the property that it is the preimage of its image U in $M_n(C)$, with U an open subscheme. Given this assertion, we can simply apply change of base to U and uniform corepresentability to get the desired result. But the desired property of U_R follows from the hypothesis that $\mathcal{U}$ is well-defined modulo S-equivalence, together with the description of the closed points of $M_n(C)$ of the previous theorem. Indeed, the set-theoretic image of U_R must be open because U_R is open in R, and $R \to M_n(C)$ is surjective, so we let U be the induced open subscheme, with a surjection $U_R \to U$. Since everything is of finite type over a field, constructible sets are determined on their closed points, and it is easily verified that every closed point of the preimage of U is in fact in U_R, as desired. $\square$

It will follow that:

Theorem IV.A.7. *In the above situation, and given $n > 0$, the operation of pulling back vector bundles under F induces a* **generalized Verschiebung** *rational map V_n : $M_n(C^{(p)}) \dashrightarrow M_n(C)$. If we denote by U_n the open subset of $M_n(C^{(p)})$ corresponding to bundles $\mathscr{E}$ such that $F^*(\mathscr{E})$ is semi-stable, we have further:*

(i) V_n is dominant

(ii) The domain of definition of V_n contains U_n.

Proof. The first task is to prove the existence of V_n; we show at the same time that U_n can be defined formally and makes good sense, and it will follow from the construction that U_n is in the domain of definition of V_n. The key statement is that in any family of vector bundles, the locus on the base over which the fibers are semistable is open; see [27, Prop. 2.3.1]. We thus get an open subfunctor $\mathcal{U}_n$ of the moduli space functor $\mathcal{M}_n(C^{(p)})$ corresponding to semi-stable vector bundles on $C^{(p)}$ which pull back under F to semi-stable vector bundles on C. We claim that it is enough to show that this subfunctor is stable under S-equivalence. Indeed, given this, by the preceding lemma, $\mathcal{U}_n$ corresponds naturally to an open subscheme U_n of $M_n(C^{(p)})$ which corepresents $\mathcal{U}_n$ and whose closed points are precisely S-equivalence classes of vector bundles on $C^{(p)}$ whose pullbacks under F are semi-stable. Now, Frobenius pullback induces a map from $\mathcal{U}_n$ to $\mathcal{M}_n(C)$; if we compose with the map $\mathcal{M}_n(C) \to M_n(C)$, since U_n corepresents $\mathcal{U}_n$, we obtain the desired morphism $V_n : U_n \to M_n(C)$.

We therefore show that our subfunctor is in fact stable under S-equivalence. Let $\mathscr{E}, \mathscr{E}'$ be as in Definition IV.A.4. Since F is flat (Proposition A.25), F^* behaves well with respect to the operation of Gr on filtrations, and we claim this implies that $F^*\mathscr{E}$ is semi-stable if and only if $F^* \operatorname{Gr}_{F_{\mathscr{E}}}$ is semi-stable: certainly, if $\mathscr{F} \subset F^*\mathscr{E}$ is a destabilizing subsheaf, then by considering the smallest subsheaf in the filtration $F^*F_{\mathscr{E}}$ into which $\mathscr{F}$ maps, there will be a nonzero map of $\mathscr{F}$ into the corresponding quotient in $F^* \operatorname{Gr}_{F_{\mathscr{E}}}$. Conversely, suppose that $\mathscr{F}$ is a destabilizing subsheaf of some quotient in $F^* \operatorname{Gr}_{F_{\mathscr{E}}}$; that is, $\mathscr{F}$ has positive degree and there is an injection $F^*F_{\mathscr{E}}^{(i)}/F^*F_{\mathscr{E}}^{(i+1)}$ for some i, from which it follows that the cokernel has negative degree. If $\mathscr{F}'$ is the subsheaf of $F^*F_{\mathscr{E}}^{(i)}$ generated by $\mathscr{F}$ and $F^*F_{\mathscr{E}}^{(i+1)}$, the cokernel of the inclusion is the same, hence has negative degree, and it follows that $\mathscr{F}'$ has positive degree and maps into $F^*\mathscr{E}$, giving the desired instability. We conclude that $F^*\mathscr{E}$ is semi-stable if and only if $F^*\mathscr{E}'$ is semi-stable, as desired.

We now prove part (ii). We simply exhibit a point of $M_n(C^{(p)})$ at which V_n induces a finite flat map on versal deformation spaces. This point will correspond to a vector bundle $\mathscr{E}_0$ of the form $\mathscr{L}_1 \oplus \mathscr{L}_2 \oplus \cdots \oplus \mathscr{L}_n$, where the $\mathscr{L}_i$ are distinct line bundles of degree 0, with $\bigotimes_i \mathscr{L}_i \cong \mathscr{O}_{C^{(p)}}$, so that $\mathscr{E}_0$ has trivial determinant. By the asserted vanishing of $h^0(B \otimes L_1)$ for general L_1 and sequence (3) of [50, p. 119], we may also require that the natural maps

$$H^1(C^{(p)}, \mathscr{L}_i \otimes \mathscr{L}_j^{-1}) \stackrel{F^*}{\to} H^1(C, F^*\mathscr{L}_i \otimes F^*\mathscr{L}_j)$$

are isomorphisms for all $i \neq j$.

We will consider deformations of $\mathscr{E}_0$ preserving the triviality of the determinant. One can check that this satisfies the criteria (H_1), (H_2), (H_3) of [53], so we let $R^{(p)}$ be a hull for the deformation problem and $\mathscr{E}$ on $C^{(p)} \times \operatorname{Spec} R^{(p)}$ be the corresponding versal deformation of $\mathscr{E}_0$. The first-order deformations are parametrized by $H^1(C^{(p)}, \mathcal{E}nd^0(\mathscr{E}_0))$, and deformations are visibly unobstructed: we can trivialize $\mathscr{E}_0$ on a pair of open sets U_1, U_2, and represent it and its deformations by a single transition matrix with coefficients in the desired Artin ring. We conclude that $R^{(p)} \cong k[[t_1, \dots t_N]]$, with $N = h^1(C^{(p)}, \mathcal{E}nd^0(\mathscr{E}_0))$. Since

$$\mathcal{E}nd^0(\mathscr{E}_0) \cong \mathscr{O}_{C^{(p)}}^{\oplus n-1} \oplus \bigoplus_{i \neq j} \mathscr{L}_i \otimes \mathscr{L}_j^{-1},$$

Riemann-Roch for vector bundles (see Theorem A.4) gives

$$\begin{aligned} h^1(C^{(p)}, \mathcal{E}nd^0(\mathscr{E}_0)) &= h^0(C^{(p)}, \mathcal{E}nd^0(\mathscr{E}_0)) - \deg \mathcal{E}nd^0(\mathscr{E}_0) + (\operatorname{rk} \mathcal{E}nd^0(\mathscr{E}_0)(g-1) \\ &= (n-1) - 0 + (n^2-1)(g-1) = gn^2 + n - g \end{aligned}$$

Now, let $I^{(p)} \subset R^{(p)}$ be the ideal defining the maximal closed subscheme of $\operatorname{Spec} R^{(p)}$ over which $\mathscr{E}$ remains isomorphic to a direct sum of n distinct line bundles. If $J^{(p)}$ denotes the Jacobian of $C^{(p)}$, and $(J^{(p)})^n \to J^{(p)}$ is the addition morphism, let $T^{(p)}$ be the fiber of this morphism over 0. Since the kernel of the addition map corresponds precisely to n-tuples of line bundles whose direct sum has trivial determinant, then by Lemma A.13 (ii) we may then describe $R^{(p)}/I^{(p)}$ as the completion of the local ring of $T^{(p)}$ at the point corresponding to $\mathscr{L}_1, \dots, \mathscr{L}_n$.

Now, if we set

$$\mathscr{F}_0 := F^*\mathscr{E}_0 \cong \bigoplus_i F^*\mathscr{L}_i,$$

we can as before let R be a hull for the deformations of $\mathscr{F}_0$, and I the ideal cutting out the locus preserving the direct sum decomposition. We then obtain the corresponding description of R/I as above. Now, by Lemma A.13 (i), the pullback under Frobenius induces a morphism $v : R \to R^{(p)}$. Next, since direct sum decompositions are preserved under pullback, $v(I) \subset I^{(p)}$, so we get an induced homomorphism $R/I \to R^{(p)}/I^{(p)}$. This clearly corresponds to the morphism $T^{(p)} \to T$ induced by the Verschiebung morphism $V : J^{(p)} \to J$; this last is finite flat, so we find that $R/I \to R^{(p)}/I^{(p)}$ is also finite flat.

Now, we also assert that we have

$$\operatorname{Hom}_k(I^{(p)}/\mathfrak{m}_{R^{(p)}}I^{(p)}, k) \cong \bigoplus_{i \neq j} H^1(C^{(p)}, \mathscr{L}_i \otimes \mathscr{L}_j^{-1})$$

and similarly for $\operatorname{Hom}_k(I/\mathfrak{m}_R I, k)$. Indeed, by the definition of a hull, $\mathfrak{m}_R/\mathfrak{m}_R^2$, the cotangent space to $\operatorname{Spec} R$, is dual to the space of first-order infinitesmal deformations of $\mathscr{F}_0$, with the subspace obtained by modding out by I and then dualizing corresponding to deformations preserving the direct sum decomposition. It is also easy to see that this subspace in turn corresponds to the summand of $H^1(C, \mathcal{E}nd^0(\mathscr{F}_0))$ given by $H^1(C, \mathscr{O}_C^{\oplus n-1})$, so we find that the quotient of the deformation space by this subspace, corresponding to deformations transverse to those preserving the direct sum decomposition, and obtained as the dual of the subspace $I/\mathfrak{m}_R I \subset \mathfrak{m}_R/\mathfrak{m}_R^2$, is then given by $H^1(C, \bigoplus_{i \neq j} F^*\mathscr{L}_i \otimes F^*\mathscr{L}_j^{-1}) = \bigoplus_{i \neq j} H^1(C, F^*\mathscr{L}_i \otimes \mathscr{L}_j^{-1})$, which is the desired statement for R. The argument for $R^{(p)}$ proceeds identically.

As a result, we find by our additional hypotheses on the choice of the $\mathscr{L}_i$ that v induces an isomorphism $I/\mathfrak{m}_R I \xrightarrow{\sim} I^{(p)}/\mathfrak{m}_{R^{(p)}} I^{(p)}$; we claim that this together with the finite-flatness of $R/I \to R^{(p)}/I^{(p)}$ is enough to imply that $v : R \to R^{(p)}$ is itself finite flat. Indeed, the isomorphism $I/\mathfrak{m}_R I \xrightarrow{\sim} I^{(p)}/\mathfrak{m}_{R^{(p)}} I^{(p)}$ implies that a set of generators of $I/\mathfrak{m}_R I$ over R will map to a set of generators of $I^{(p)}/\mathfrak{m}_{R^{(p)}} I^{(p)}$ over R and in particular over $R^{(p)}$, which by Nakayama's lemma implies that any lifts generate $I^{(p)}$ over $R^{(p)}$; we conclude that $I^{(p)} = v(I)R^{(p)}$. Now, we note that because $I_r \subset \mathfrak{m}_R$, we may write $R^{(p)}/v(\mathfrak{m}_R)R^{(p)} = (R^{(p)}/v(I)R^{(p)})/v(\mathfrak{m}_R)(R^{(p)}/v(I)R^{(p)}) = (R^{(p)}/I^{(p)})/v(\mathfrak{m}_R)(R^{(p)}/I^{(p)})$, which must be finite over R/I and hence over R because $R^{(p)}/I^{(p)}$ is. We thus have that the closed fiber of $\operatorname{Spec} R^{(p)} \to \operatorname{Spec} R$ is finite, and since both rings are regular of the same dimension, by [13, Thm 18.16 b] the map is flat, hence dominant. It remains to check that the map on hulls being dominant implies that the map on coarse moduli spaces is dominant. Because the coarse moduli spaces are irreducible (see [47, Rem. 5.5]), it suffices to show that the map from the hull to the coarse moduli space is dominant, which is Lemma A.13 (iii). □

Remark IV.A.8. In fact, U_n is the precise domain of definition of V_n, but the proof of this fact would take us too far afield. It will however follow in our case of primary interest, when the Frobenius-unstable locus is reduced (in the functorial sense of consisting of points which have no non-trivial deformations) from the main results of Chapter IV.

We recall the following theorem of Narasimhan and Ramanan:

Theorem IV.A.9. *If C has genus 2, $M_2(C) \cong \mathbb{P}^3$.*

Proof. See [46, Thm. 2, §7]; they use the language of Riemann surfaces, but the argument goes through unmodified in arbitrary odd characteristic. □

Lemma IV.A.10. *If C has genus 2, there are only finitely many semistable vector bundles of rank 2 and trivial determinant pulling back to unstable bundles under Frobenius; in particular, there are only finitely many undefined points of V_2.*

Proof. The argument for part (i) of Theorem IV.A.7 showed in particular that the locus of Frobenius-unstable bundles is closed in M_2, which in this case by the previous theorem is $\mathbb{P}^3$. In particular, if it were positive dimensional, it would have to intersect every surface inside the moduli space. But the Kummer surface made up of non-stable bundles (which is expressible as the bundles of the form $\mathscr{L} \oplus \mathscr{L}^{-1}$ for $\mathscr{L}$ a line bundle of degree 0) cannot contain any Frobenius-unstable bundles, as $F^*(\mathscr{L}\oplus\mathscr{L}^{-1}) = F^*\mathscr{L}\oplus F^*\mathscr{L}^{-1}$ cannot be unstable by Lemma III.3.3. Thus, the locus of Frobenius-unstable bundles is zero-dimensional, as desired. □

We conclude with a well-known observation not directly related to the Verschiebung.

Lemma IV.A.11. *If C has genus 2, then for any $\mathscr{L} \in \mathrm{Pic}^1(C)$, the subspace $\mathbb{P}\mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1})$ defines a hyperplane in $M_2 \cong \mathbb{P}^3$.*

Proof. In brief, the isomorphism between M_2 and $\mathbb{P}^3$ is more specifically obtained as an isomorphism $M_2 \cong \mathbb{P}H^0(\mathrm{Pic}^1(C), \mathscr{O}(2\Theta))$, under the map sending $\mathscr{E}$ to the divisor $\Theta_{\mathscr{E}}$ defined as the set of $\mathscr{L}' \in \mathrm{Pic}^1(C)$ such that $\mathscr{E}\otimes\mathscr{L}'$ has a section; $\mathscr{E} \in \mathbb{P}\mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1})$ if and only if $\mathscr{E}\otimes\mathscr{L}$ has a section if and only if $\mathscr{L} \in \Theta_{\mathscr{E}}$, so $\mathbb{P}\mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1}) \subset \mathbb{P}H^0(\mathrm{Pic}^1(C), \mathscr{O}(2\Theta))$ is defined by requiring vanishing of the section at $\mathscr{L} \in \mathrm{Pic}^1(C)$, which is a linear condition on sections of $\mathscr{O}(2\Theta)$. This proves that $\mathbb{P}\mathrm{Ext}(\mathscr{L}, \mathscr{L}^{-1})$ maps into a hyperplane in M_2, with surjectivity on the stable locus of the hyperplane. This argument breaks down *a priori* along the non-stable locus, where one could have a non-zero map $\mathscr{L}^{-1} \to \mathscr{E}$ which is not saturated; however, this is not in fact a problem, thanks to [46, Lem. 5.8]. □

IV.B Appendix: A Commutative Algebra Digression

We digress momentarily from the main thrust of our argument to develop some simple but non-standard commutative algebra over non-reduced rings which will be helpful in studying deformations of vector bundles. Throughout this section, R will denote a (typically non-reduced) Noetherian ring, $\mathfrak{n}$ the ideal of nilpotents of R, and M a finitely generated R-module. Note that everything we define in this section will be equivalent to their standard versions whenever R is integral.

Definition IV.B.1. M is NR-free of rank r if M is generated by some $m_1, \dots m_r$, such that given any relation $\sum_i a_i m_i = 0$, all the a_i must be nilpotent.

Lemma IV.B.2. *If M is NR-free of rank r, then M_{red} is free of rank r. In particular, rank is well-defined for NR-free modules. Further, the converse holds if R is local, or if $\mathfrak{n}^m = 0$ for some m.*

Proof. Suppose M is NR-free of rank r. Clearly, the m_i still generate M_{red}, but mod the nilpotents of R, all relations between them are 0, so they freely generate. Conversely, suppose M_{red} is free of rank r, with generators m_i. It is enough to show that some lifts of the m_i generate M, since they will clearly satisfy the desired relations restrictions. If R is local with maximal ideal $\mathfrak{m}$, this follows immediately from Nakayama's lemma, since $\mathfrak{n} \subset \mathfrak{m}$. If, on the other hand, $\mathfrak{n}^m = 0$, we will get the desired result by showing via induction that $M' := M/(\{m_i\}_i)$ is contained in (the image of) $\mathfrak{n}^j M$ for all j. The base case is $j = 0$, which is a triviality. Now suppose it holds for $j-1$, and we want to show it for j. Take $m \in M$; by the induction hypothesis, we can write $m = m' + \sum_i a_i m_i$ for some $m' \in \mathfrak{n}^{j-1}M$. Writing $m' = e_1 \cdots e_{j-1}\tilde{m}$, because the m_i generate M_{red}, we can find some $\tilde{a}_i$ such that $\tilde{m} - \sum_i \tilde{a}_i m_i$ is in the kernel of $M \to M_{\text{red}}$. But this kernel is precisely $\mathfrak{n}M$, so substituting back in, we find that $m - \sum_i (a_i + e_1 \cdots e_{j-1}\tilde{a}_i) m_i \in \mathfrak{n}^j M$, as desired. □

This argument also immediately gives us:

Corollary IV.B.3. *Given two generating sets, m_i and m_i' for an NR-free R-module M, there is a (non-unique) invertible matrix T relating the m_i to the m_i', such that if $\bar{m}_i$, $\bar{m}_i'$, and $\bar{T}$ are the images of m_i, m_i' and T in M_{red}, $\bar{T}$ is the matrix relating the $\bar{m}_i$ to the $\bar{m}_i'$.*

We can extend the standard definitions:

Definition IV.B.4. M is locally NR-free of rank r if M becomes NR-free of rank r over every local ring of R. Given a separated Noetherian scheme X, a coherent sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules is locally NR-free of rank r if $\mathscr{F}(U)$ is locally NR-free of rank r over $\mathscr{O}_X(U)$ for every U. Because our modules are finitely generated, this is equivalent to there being an open cover of X on which $\mathscr{F}$ becomes NR-free of rank r.

We see that while we can attach transition matrices to locally NR-free sheaves, they are not unique, nor do they uniquely determine $\mathscr{F}$. However, they do determine $\mathscr{F}_{\text{red}}$ over X_{red}, which is all we will need for our purposes.

To develop the results we want, we will make the hypothesis for the rest of this section that $\operatorname{Spec} R$ is irreducible. With this, we can prove the following proposition, which properly reformulated will kill several birds (pigeons, or perhaps vultures; nothing anyone would want to keep alive) with one stone:

Proposition IV.B.5. *Suppose that R_{red} is integral, and let M be a free R-module of rank r, and N an R-module generated by $n_1, \dots n_s$, with l non-zero relations of the form $fn_i = 0$ for some $f \in R$. Then if ϕ is a surjective map from M to N, $\ker \phi$ is an NR-free R-module of rank $r - s + l$. If further $l = 0$, then $\ker \phi$ is actually free.*

Proof. First, we can suppose that f is not a unit, as if it were N would just be a free module of rank $s - l$, and we could simply use $n_{l+1}, \dots n_s$ as generators. Our first claim is that if we lift ϕ from N to $\tilde{N} := R[n_1, \dots n_s]$, the map will remain surjective. Let $\tilde{\phi}$ be some such lift, we show that $\tilde{N}/\operatorname{im}\tilde{\phi} \subset f^j\tilde{N}/\operatorname{im}\tilde{\phi}$ for all j, once again by induction on j, and once again with the $j = 0$ case being trivial. Suppose it is true for $j - 1$; given $n \in N$, by hypothesis, there is some $m \in M$ such that $n - \tilde{\phi}m = f^{j-1}n'$ for some $n' \in \tilde{N}$. But since ϕ is surjective onto N, and the kernel of $\tilde{N} \to N$ is contained in (f), there is some $m' \in M$ with $n' - \tilde{\phi}m' \in (f)\tilde{N}$. Substituting back in, we find $n - \tilde{\phi}(m + f^{j-1}m') \in f^j\tilde{N}$, as desired. Next, the Krull Intersection Theorem (see [13, Cor. 5.4]) implies that for some $f' \in (f)$, $(1 - f')(\cap_j(f^j)) = 0$. Now, by the hypothesis that R_{red} is integral, either $1 - f'$ is nilpotent, or $\cap_j(f^j) = 0$. However, if $1 - f'$ were nilpotent, f' would have to be a unit, which is not possible, since f was assumed not to be a unit. It follows that $\cap_j(f^j) = 0$, and $\tilde{\phi}$ is surjective.

Next, since it is a surjective map between free modules, the kernel of $\tilde{\phi}$ must be free, of rank $r - s$ (for instance, because free modules are projective, and hence always admit a splitting map back, expressing the kernel as a summand of a free module; see, for instance, [13, pp. 621-622]). Note that this observation immediately proves the last assertion of the theorem. Now, let $\ker\tilde{\phi}$ be generated by $\tilde{m}_1, \dots \tilde{m}_{r-s}$. Next, let $\hat{m}_1, \dots \hat{m}_l$ be any elements mapping to $fy_l \in \tilde{N}$ under $\tilde{\phi}$. We assert that $\{\tilde{m}_i, \hat{m}_j\}$ is a generating set making $\ker\phi$ into an NR-free module. First, we show that it generates $\ker\phi$: clearly, $\ker\phi$ is precisely the subset of M mapping under $\tilde{\phi}$ to elements of the form $\sum_{i=1}^{s} fa_iy_i \in \tilde{N}$. But subtracting off $\sum_i a_i\hat{m}_i$ from such an element will place it in the kernel of $\tilde{N}$, and the latter is generated by the $\tilde{m}_i$, by construction. Finally, we need to show that any relation $\sum_i a_i\tilde{m}_i + \sum_i a_i'\hat{m}_i = 0$ must have all a_i, a_i' nilpotent. But looking at the image of this identity under $\tilde{\phi}$, we see that $\sum_i a_i'fy_i = 0$, which, since $\tilde{N}$ is free, implies $a_i'f = 0$ for all i, giving in particular (again, using that R_{red} is integral), that all the a_i' are nilpotent. But multiplying through the original identity by f now gives $\sum fa_i\tilde{m}_i = 0$, which, since the $\tilde{m}_i$ were free, gives $fa_i = 0$ for all i, and all a_i nilpotent, as desired. Thus, $\ker\phi$ is NR-free of rank $r - s + l$, as desired. $\square$

Remark IV.B.6. This is where we see the importance of working with the R-modules themselves, rather than the induced R_{red}-modules. If f is a nonzero nilpotent, we will actually obtain different rank modules by first taking the kernel and then restricting to R_{red} as we would by restricting to R_{red} and then taking the kernel. Even though (under very weak hypotheses) an R module is NR-free of a given rank if and only if its reduced version is free of the same rank over R_{red}, the R-module can carry a lot more information than its reduction does.

To apply this proposition, we define:

Definition IV.B.7. An effective NR-Cartier divisor on a Noetherian, separated, irreducible scheme X is a global section of the monoid sheaf $(\mathscr{O}_X \smallsetminus \{0\})/\mathscr{O}_X^*$.

Remark IV.B.8. This terminology is slightly misleading, in that it suggests that there should be a notion of a non-effective NR-Cartier divisor, which I have no intention of introducing. However, NR-effective Cartier divisor sound like it is in particular a Cartier divisor, which is more misleading. In any case, as with the rest of the section, if R is integral, this definition corresponds precisely to the usual notion, even though in the integral case one can avoid talking about monoids via use of $\mathscr{K}^*$.

Lemma IV.B.9. *Associated to any non-trivial effective NR-Cartier divisor f on X is a canonical closed immersion $X_f \hookrightarrow X$. Given any closed subscheme of X, there is at most one effective NR-Cartier divisor which induces it.*

Proof. It is clear that f is precisely equivalent to a nonzero, (and by the nontriviality hypothesis) non-unit principal ideal sheaf on X, so by the definition of a closed subscheme, it is exactly equivalent to a closed subscheme cut out locally by a single element of $\mathscr{O}_X$. $\square$

Remark IV.B.10. Unlike the case with standard effective Cartier divisors, X_f need not have codimension 1 in X for an effective NR-Cartier divisor. However, even when it has codimension 0, it behaves in certain ways as if it had codimension 1, as the following theorem demonstrates:

Theorem IV.B.11. *Let $X_f \hookrightarrow X$ be the closed immersion associated to a nontrivial effective NR-Cartier divisor f on X, $\mathscr{F}$ a locally free O_X-module of rank r on X, and $\mathscr{G}$ a coherent $\mathscr{O}_X$-module on X. Let $f : \mathscr{F} \to \mathscr{G}$ be surjective. Then:*

(i) if $\mathscr{G}$ is locally free of rank s on X, $\ker f$ *is locally free of rank $r - s$, and*

(ii) if $\mathscr{G}$ is the pushforward of a locally free sheaf of rank s on X_f, $\ker f$ *is locally NR-free on X of rank r.*

Proof. (i) follows immediately from Proposition IV.B.5, by restricting to a cover on which $\mathscr{F}$ and $\mathscr{G}$ are free, and noting that we are in the case $l = 0$. (ii) also follows from the same proposition, as if our cover is also fine enough give trivializing elements for f, we find that f matches up with the f of the theorem, and because $\mathscr{G}$ is a pushforward from X_f, it satisfies the hypotheses of the proposition with $l = s$, giving us the desired conclusion. $\square$

We will immediately apply this to prove a theorem to the effect that in certain cases, when one wants to replace an unstable vector bundle in a family with a semistable one, it suffices to look simply at the first order deformation induced by the family. We begin with:

Lemma IV.B.12. *Let $f : S \hookrightarrow T$ be a closed immersion of schemes, $\mathscr{F}$ a locally free sheaf on T, and $\mathscr{G}$ a locally free sheaf on S. Given $\psi : \mathscr{F} \to f_*\mathscr{G}$, then ψ arises via adjointness from $f^*\psi : f^*\mathscr{F} \to f^*f_*\mathscr{G} = \mathscr{G}$, and if $f^*\psi$ is surjective, then so is ψ. In this case, there is a natural surjection $f^* \ker \psi \twoheadrightarrow \ker f^*\psi$.*

Proof. First, $f^*f_*\mathscr{G}$ is $\mathscr{G}$ because f is a closed immersion. Next, the facts that ψ is surjective, and arises from $f^*\phi$ via adjointness are easily seen from the adjointness construction, using the canonical map $\mathscr{F} \to f_*f^*\mathscr{F}$. We thus get a short exact sequence

$$0 \to \ker \psi \to \mathscr{F} \to f_*\mathscr{G} \to 0$$

which, since pullbacks are right exact, gives us the diagram

$$\begin{tikzcd}
f^* \ker \psi \arrow[rr] \arrow[dr, dashed] & & f^*\mathscr{F} \arrow[r, "f^*\psi"] & \mathscr{G} \arrow[r] & 0 \\
& \ker f^*\psi \arrow[ur] & & & \\
0 \arrow[ur] & & & &
\end{tikzcd}$$

Here, the dotted arrow exists because, by exactness, $f^* \ker \psi$ maps into $\ker f^*\psi$. In fact, exactness implies that this is surjective, yielding the desired result. $\square$

Theorem IV.B.13. *Let $f : S \hookrightarrow T$ be a closed immersion, with T reduced, and such that if $i : S_{\text{red}} \hookrightarrow S$ is the standard inclusion, then both i and $f \circ i$ are effective NR-Cartier divisors (for instance, let T be C times some other curve, and $S = C \times \operatorname{Spec} k[\epsilon]/(\epsilon^2)$). Let $\mathscr{F}$ be a locally free sheaf of rank r on T, and $\mathscr{G}$ a locally free sheaf of rank s on S_{red}, and $\psi : \mathscr{F} \to f_* i_* \mathscr{G}$ arising via adjointness from a surjective map $f^*\mathscr{F} \to i_*\mathscr{G}$, which must then be $f^*\psi$. Then $i^* f^* \ker \psi = i^* \ker f^*\psi$, and both are locally free of rank r on S_{red}.*

Proof. The previous lemma gives us that ψ is surjective, and a surjection $f^* \ker \psi \twoheadrightarrow \ker f^*\psi$. Since pullback is right exact, this gives us $i^* f^* \ker \psi \twoheadrightarrow i^* \ker f^*\psi$, and by Theorem IV.B.11 (i), it suffices to show these are both locally free of rank r, since then we would have the kernel of this surjection also being locally free, of rank 0, and hence just the 0 sheaf. On the other hand, by (ii) of the same theorem, $\ker \psi$ is is locally NR-free on T of rank r, hence locally free of rank r, because T is reduced. Thus, $i^* f^* \ker \psi$ must also be locally free of rank r. But likewise, since $f^*\mathscr{F}$ must also be locally free of rank r, using the same theorem, $\ker f^*\psi$ is locally NR-free of rank r on S, which as we saw initially is equivalent to $i^* \ker f^*\psi$ being locally free of rank r on S_{red}, as desired. □

Chapter V

Logarithmic Connections With Vanishing p-Curvature

We develop in this chapter a basic theory of connections with simple poles and vanishing p-curvature on smooth curves, and apply it to the case of rank 2 vector bundles on $\mathbb{P}^1$ to classify such connections completely in terms of rational functions on $\mathbb{P}^1$ with prescribed ramification. We state as our main theorem the case which will be of most use to us, and which is simplest to state. For the most general versions, see Proposition V.4.5 together with Theorem V.5.7. We also obtain a similar classification for projective connections in Section V.6.

Theorem V.0.1. *Fix an integer $n > 0$, let $\epsilon = 0$ or 1 according to the parity of n, $d = \frac{n+\epsilon p}{2} - 1$, choose $P_1, \dots P_n$ general points on $\mathbb{P}^1_k$, with k an algebraically closed field of characteristic $p > 2$. Also fix $\mathscr{E}$ to be the vector bundle $\mathscr{O}(\epsilon p - d) \oplus \mathscr{O}(d)$. Then transport-equivalence classes of connections on $\mathscr{E}$ with trivial determinant, vanishing p-curvature, simple poles at the P_i, and not inducing a connection on $\mathscr{O}(d) \subset \mathscr{E}$ are in natural one to one correspondence with objects $(\{\alpha_i\}_i, \bar{f})$, where the α_i are integers between 1 and $\frac{p-1}{2}$, and $\bar{f}$ is a separable rational function on $\mathbb{P}^1$ of degree $n\frac{p-1}{2} + 1 - \sum \alpha_i$, ramified to order at least $p - 2\alpha_i$ at each P_i, and considered modulo automorphism of the image space. Furthermore, this classification holds even for first-order infinitesmal deformations.*

We conclude that the classes of such connections have no non-trivial deformations, and are counted by the recursive formula of Theorem I.0.4.

We use throughout the terminological conventions for vector bundles and connections given in Section III.1. Our methodology will be to work primarily over an algebraically closed field, with periodic examinations of the generalization to first-order infinitesmal deformations. As such, throughout the chapter we will follow the conventions of Notation IV.3.1. We also say that ∇ is a **rational connection** on a smooth scheme if it is a connection on a dense open subset U_∇, but may have poles away from U_∇; we say that ∇ is **logarithmic** if all such poles are simple.

Warning V.0.2. We will refer to connections with **trivial determinant** on vector bundles $\mathscr{E}$ on $\mathbb{P}^1$ in the case that $p|\deg\mathscr{E}$, even if $\deg\mathscr{E} \neq 0$, since in this case we have a unique canonical connection on $\det\mathscr{E}$, and can require that the determinant connection agree with it.

We begin in Section V.1 with some calculations holding on any smooth curve, the primary purpose of which is to show that a connection is logarithmic with vanishing p-curvature if and only if everywhere formally locally it decomposes as a direct sum of connections on line bundles. The purpose of Section V.2 is to re-establish the results of the previous section for certain first-order infinetesmal deformations. Section V.3 develops simpler criteria in the special case of vector bundles of rank 2, Section V.4 specializes further to the case of vector bundles on $\mathbb{P}^1$, and Section V.5 completes the classification in this situation in terms of self-maps of $\mathbb{P}^1$ with prescribed ramification. Section V.6 demonstrates how the same theory may be applied to classify projective connections, and finally, Section V.7 discusses the convoluted chronology intertwining the results of this chapter, of Chapter I, and of Mochizuki's closely-related work.

The only similar work in the literature appears, unsurprisingly, to be that of Mochizuki, who proves a special case of the main results of this chapter, in the situation of three poles on $\mathbb{P}^1$; in fact, he proves this result in the more general context of n-connections over an arbitrary base, so our result (in the case of three points) is simply the $n = 0$ case of [42, Thm. IV.2.3, p. 211].

V.1 Formal Local Calculations

In this section, we make some basic observations about kernels of connections with vanishing p-curvature and simple poles on smooth curves, and apply formal local analysis to show that, formally locally, they may be split as a direct sum of connections on line bundles;

equivalently, they may be diagonalized under transport.

We begin with:

Proposition V.1.1. *Let C be a smooth curve over an algebraically closed field k, and $\mathscr{E}$ a vector bundle of rank r on C. Then if we consider the operations of taking kernels of connections and of extending canonical connections of Frobenius pullbacks, we deduce:*

(i) There is a one-to-one correspondence between rational connections on $\mathscr{E}$ with vanishing p-curvature on one side, and pairs $(\mathscr{F}, \varphi)$ of vector bundles $\mathscr{F}$ of rank r on $C^{(p)}$ together with injective maps $\varphi : F^*\mathscr{F} \hookrightarrow \mathscr{E}$, subject to the restriction that $\varphi(\mathscr{F})$ must generate the full kernel of the connection on $\mathscr{E}$ obtained by extending ∇^{can} from $F^*\mathscr{F}$ to $\mathscr{E}$ via φ, and taken modulo automorphisms of $\mathscr{F}$.

(ii) Under this equivalence, the poles of a connection are precisely the points where φ fails to be surjective.

(iii) Under this equivalence, transport of connections on $\mathscr{E}$ corresponds to changing φ by the corresponding automorphism of $\mathscr{E}$.

Proof. Let ∇ be a rational connection on C. Then we find that $\mathscr{E}^\nabla$ is naturally a $\mathscr{O}_{C^{(p)}}$-submodule of $F_*\mathscr{E}$, and hence must be coherent and torsion-free, and therefore a vector bundle, from which we conclude that $F^*\mathscr{E}^\nabla$ is also a vector bundle. Indeed, $F^*\mathscr{E}^\nabla$ is naturally a subsheaf of $\mathscr{E}$, and can be understood concretely as the subsheaf spanned by the kernel of ∇ inside $\mathscr{E}$. We thus have a sequence $0 \to F^*\mathscr{E}^\nabla \to \mathscr{E} \to \mathscr{G} \to 0$ for some $\mathscr{G}$ on C, and the inclusion map giving us the φ from statement (i). We observe that $F^*\mathscr{E}^\nabla$ has rank r if and only if $\mathscr{G}$ is torsion if and only if ∇ has vanishing p-curvature, and that in this case $\mathscr{G}$ is actually supported at the poles of ∇: if ∇ has vanishing p-curvature, the inclusion $F^*\mathscr{E}^\nabla \hookrightarrow \mathscr{E}$ is an isomorphism wherever ∇ is regular (Theorem III.1.4), so $\mathscr{G}$ is supported at the poles of ∇, and conversely, if we restrict to the open set away from the support of $\mathscr{G}$, we find that the inclusion $F^*\mathscr{E}^\nabla \hookrightarrow \mathscr{E}$ is an isomorphism and that ∇ is therefore the canonical connection associated to a pullback under Frobenius, and thus has vanishing p-curvature on this open set, which gives vanishing p-curvature everywhere. This yields one direction of (i), as well as (ii).

On the other hand, given $\mathscr{E}$ on C and a vector bundle $\mathscr{F}$ of rank r on $C^{(p)}$, together with an inclusion $\varphi : F^*\mathscr{F} \hookrightarrow \mathscr{E}$, the quotient is torsion, so ∇^{can} on $F^*\mathscr{F}$ uniquely extends to a rational connection ∇ on $\mathscr{E}$, such that $\mathscr{E}^\nabla \supseteq \mathscr{F}$. Then by the above, ∇ has vanishing p-curvature, and if we add the hypothesis that $\mathscr{E}^\nabla = \mathscr{F}$, we find that ∇ is determined uniquely by $\mathscr{F}$ (as an abstract bundle on $C^{(p)}$, not as a subsheaf of $\mathscr{E}$) and φ. Furthermore, it is clear from above that such a ∇ determines $\mathscr{F}$ uniquely, and φ up to automorphisms of $\mathscr{F}$ (note: not up to automorphisms of $F^*\mathscr{F}$, which will change ∇), completing the proof of (i). Statement (iii) is now clear, completing the proof. □

We fix, for the remainder of the chapter, the following notation:

Notation V.1.2. $\mathscr{E}$ denotes a vector bundle of rank r on C, and $\mathscr{F}$ a vector bundle of the same rank on $C^{(p)}$. φ is an injection $F^*\mathscr{F} \hookrightarrow \mathscr{E}$, and ∇ is a connection on $\mathscr{E}$.

Definition V.1.3. We define a **pre-kernel map** to be a pair $(\mathscr{F}, \varphi)$ with $\mathscr{F}$ locally free of rank r on $C^{(p)}$, and $\varphi : F^*\mathscr{F} \to \mathscr{E}$ an injection. If $(\mathscr{F}, \varphi)$ satisfies the further condition of Proposition V.1.1 (i) to correspond to a connection, we call it a **kernel map**. By abuse of terminology, we will refer to modification of φ by $F^* \operatorname{Aut}(\mathscr{F})$ and $\operatorname{Aut}(\mathscr{E})$ as **transport**.

The previous proposition sets up a correspondence between kernel map classes and certain connections; the aim of the rest of the chapter will be to explore this correspondence more thoroughly, and ultimately use it to classify the relevant connections ∇ in the special case of $C = \mathbb{P}^1$, and ∇ having only simple poles.

We now carry out a straightforward calculation:

Proposition V.1.4. *Given $\mathscr{F}$ on $C^{(p)}$, and $\mathscr{E}$ on C, with an inclusion φ of $F^*\mathscr{F}$ into $\mathscr{E}$, let s_i and t_i be bases for $\mathscr{F}$ and $\mathscr{E}$ on some open set U of C, and suppose the inclusion φ is given on U by a matrix $S = (a_{ij})$. Let (b_{ij}) be the matrix (regular on U) $\det(S)S^{-1}$. Then if ∇ is the corresponding rational connection with vanishing p-curvature on $\mathscr{E}$, it has matrix $T = (c_{ij})$, with*

$$c_{ij} = \frac{1}{\det(S)} \left(\sum_k a_{ik}(db_{kj}) - \delta_{ij} d \det(S) \right)$$

where δ_{ij} is the Kronecker delta function. Equivalently, $T = S(dS^{-1})$. Further, $\operatorname{Tr}(T) = -\frac{d\det(S)}{\det(S)}$.

Proof. We have, by definition of S, that $s_j = \sum_i a_{ij} t_i$, and similarly that $t_j = \sum_i \frac{b_{ij}}{\det(S)} s_i$. Now, the s_i are in the kernel of ∇ by construction, and to find the connection matrix T

we simply write $\nabla(t_j)$ in terms of the t_i for all j. We have: $\nabla(t_j) = \sum_k d\left(\frac{b_{kj}}{\det(S)}\right) s_k = \sum_k d\left(\frac{b_{kj}}{\det(S)}\right) \sum_i a_{ik} t_i = \sum_{k,i} a_{ik} d\left(\frac{b_{kj}}{\det(S)}\right) t_i$, giving us that $T = S(dS^{-1})$. Continuing by expanding the differential, we get $\nabla(t_j) = \sum_{k,i} \frac{a_{ik}(db_{kj})}{\det(S)} t_i - \sum_{k,i} \frac{(d\det(S))a_{ik}b_{kj}}{\det(S^2)} t_i = \sum_{k,i} \frac{a_{ik}(db_{kj})}{\det(S)} t_i - \frac{d\det(S)}{\det(S)} t_j$, giving the desired formula for the c_{ij}.

For the trace formula, recall that the b_{ij} are constructed as the determinant of the matrix obtained from (a_{ij}) by removing the ith column and jth row, with sign alternating on i and j, from which one can obtain the identities $\sum_k a_{ik}b_{ki} = \det(S)$ and $\sum_{i,k}(da_{ik})b_{ki} = d\det(S)$. Now, we have from our formula for the c_{ij} that $\operatorname{Tr} T = \frac{1}{\det(S)}(\sum_i \sum_k a_{ik}(db_{ki})) - r\,d\det(S)$, where r is the rank of $\mathscr{E}$. But if for each i we write $d\det(S) = d(\sum_k a_{ik}b_{ki}) = \sum_k a_{ik}(db_{ki}) + \sum_k (da_{ik})b_{ki}$ and substitute in, we get $\operatorname{Tr} T = -\frac{1}{\det(S)}(\sum_i \sum_k (da_{ik})b_{ki}) = -\frac{d\det(S)}{\det(S)}$, as desired. $\square$

We next move on to formal local analysis of the situation at points where the determinant is not invertible (equivalently, points where the connection has poles).

Proposition V.1.5. *Formally locally (that is, over $k[[t]]$), any $r \times r$ matrix of nonzero determinant:*

(i) may be put via left change of basis into the following form, which is unique, and we will call **canonical row-reduced form***:*

$$\begin{bmatrix} t^{e_1} & f_{12} & \cdots & & f_{1r} \\ 0 & t^{e_2} & f_{23} & & \vdots \\ \vdots & & \ddots & \ddots & \\ & & & & f_{(r-1)r} \\ 0 & \cdots & & & t^{e_r} \end{bmatrix}$$

where each f_{ij} is a polynomial in t of degree less than e_j;

(ii) may, if one further allows right pth power change of basis, be put into canonical row-reduced form, with the further requirement that the f_{ij} do not have any terms with exponent congruent to e_i modulo p.

Proof. For (i), left multiplication by invertible matrices allows us to perform standard row reduction, applied over $k[[t]]$. Namely, we go from left to right, choosing the element with the smallest order in the first column, putting it in the first row if necessary by adding its row to the first, and subtracting off multiples of the first row from the rest to set the rest of the entries in the first column to 0, and finally multiplying by a diagonal matrix of all 1's except the first row to make the first entry some power of t. The same process is then repeated in the next column, except leaving the first row in place, and so forth, leaving an upper triangular matrix with powers of t along the diagonal, and arbitrary entries above the diagonal. We then go back once more from left to right, subtracting off appropriate multiples of each diagonal entry from each coefficient above it, making those coefficients into polynomials in t of degree strictly less than the power of t on the diagonal in that column.

We now show that this form is unique. That is, given M and M' in canonical row-reduced form, and an invertible N with $M' = NM$, we will show that $N = (n_{ij})$ is the identity matrix. This is probably standard, but the proof is easy enough, so we include it. Since M and M' are generically invertible (this hypothesis is probably unnecessary, but will always hold for us, and simplifies the argument), looking at $M' = NM$ over $k((t))$ immediately implies that N is upper triangular, since $N = M'M^{-1}$. Then, the fact that N is invertible implies that all its diagonal elements are units in $k[[t]]$, and the fact that both M and M' have powers of t on the diagonal implies that N in fact has 1's on the diagonal. We get immediately that all $e_i = e'_i$. Now, fixing i, we show that the i,jth coefficient of N for $j > i$ is 0 by induction on j. We start with $j = i+1$, and consider the i,jth coefficient of M', which is f'_{ij}, a polynomial in t of degree less than $e'_j = e_j$. It is given as $f'_{ij} = f_{ij} + n_{ij}t^{e_j}$, since $n_{ii} = 1$, and all other terms are 0 because both N and M are upper triangular. But f'_{ij} cannot have any t^{e_j} term, so we get n_{ij}. The induction step now proceeds via exactly the same argument, since we will have $n_{i(i+1)}, \dots n_{i(j-1)}$ all 0 by hypothesis. Thus, N is the identity, and M and M' were in fact the same, as desired.

For (ii), we remove the terms congruent to e_i mod p from each f_{ij} by going back from right to left, using pth-power column reduction via righthand multiplication by pth power matrices. *A priori*, it may seem that the degrees of the f_{ij} might be raised under this process, but note that any f_{ij} in the jth column will only require powers of e_i less than its degree, which is in turn bounded by e_j, to be removed. Further, the f_{ki} above the t^{e_i} all have degree less than e_i, so they will only be modifying the f_{kj} in degrees less than f_{ij} is being modified. $\square$

Remark V.1.6. Note that the hypothesis that the determinant be nonzero was not actually necessary to carry out this process, but merely insures that the powers of t actually end up on the diagonal, and not somewhere above them.

Remark V.1.7. Note the form of (ii) of the preceding proposition, tempting though it might be to assume otherwise, is not in fact unique. In particular, conjugation by permutation matrices is always allowed, and could be used to, for instance, rearrange the coefficients of a diagonal matrix (something which could not be accomplished using row reduction alone).

Proposition V.1.8. *For a pre-kernel map φ given on some open subset by $S = (a_{ij})$, the following are equivalent:*

a) φ corresponds to a logarithmic connection with vanishing p-curvature;

b) formally locally everywhere (equivalently, everywhere where the map fails to be invertible), S is transport-diagonalizable, with all diagonal coefficients having order of vanishing strictly less than p;

c) formally locally everywhere (equivalently, everywhere where the map fails to be invertible), when S is placed in the form of the preceding proposition, all $f_{ij} = 0$ and all e_i are strictly less than p.

Proof. First note that the condition that φ correspond to a ∇ with vanishing p-curvature and at most simple poles is clearly transport-invariant. We do the difficult direction first; namely, showing that a) implies c). For notational convenience, we prove this inductively on the rank r. The base case is $r = 1$, where the connection corresponding to a_{11} is simply $-\frac{da_{11}}{a_{11}}$, which always has at most simple poles. The condition that $e_1 < p$ comes from the fact that if $e_1 \geq p$, and we have $S = \left[t^{e_1}\right], T = \left[-e_1 t^{-1} dt\right]$, then $t^{e_1 - p}$ will also be a horizontal section formally locally, but is not in the image of φ (here we are using that in characteristic p, formation of the kernel of a connection commutes with completion, see Proposition A.29).

For the induction step, we first transport S formally locally into the form described in part (ii) of the previous proposition; this is in particular upper triangular, and noting that once S is upper triangular T is also upper triangular, we can (formally locally) restrict to the first $r-1$ rows and columns of T to get a connection with vanishing p-curvature and simple poles in rank one less, which is clearly already in the form of the previous proposition. Thus, by the induction hypothesis our entire $r \times r$ matrix will look like:

$$\begin{bmatrix} t^{e_1} & 0 & \dots & 0 & f_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & t^{e_{r-2}} & 0 & f_{r-2} \\ \vdots & & \ddots & t^{e_{r-1}} & f_{r-1} \\ 0 & \dots & \dots & 0 & t^{e_r} \end{bmatrix}$$

We wish to show that the f_i are all 0, and $e_r < p$. Now, the associated connection matrix will be:

$$\begin{bmatrix} -e_1 t^{-1}dt & 0 & \dots & 0 & (e_1 f_1 - f_1' t)t^{-e_r-1}dt \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & -e_{r-2}t^{-1}dt & 0 & (e_{r-2}f_{r-2} - f_{r-2}'t)t^{-e_r-1}dt \\ \vdots & & \ddots & -e_{r-1}t^{-1}dt & (e_{r-1}f_{r-1} - f_{r-1}'t)t^{-e_r-1}dt \\ 0 & \dots & \dots & 0 & -e_r t^{-1}dt \end{bmatrix}$$

This will have simple poles only if $e_i f_i - f_i' t$ vanishes to order at least e_r; since f_i has degree less than e_r by hypothesis, this difference must be 0. But it is clear that terms will cancel in a given degree if and only if the degree is congruent to e_i mod p, and also by hypothesis each f_i has no terms in degree congruent to e_i mod p. We conclude that each $f_i = 0$, as desired. Lastly, the condition that $e_r < p$ follows from the necessity of the image of φ to contain the kernel of ∇ just as it did in the rank 1 base case.

Now, c) implies b) is trivial, so we just need to show that b) implies a). If S is formally locally diagonalizable, as long as the e_i are less than p the diagonalized map corresponds to a connection with simple poles and vanishing p-curvature, and since this is a transport-invariant property, S must have as well. $\square$

Because under these equivalent conditions, all $e_i < p$, we note that it is actually only necessary to use constant column operations in our formal local transport-diagonalization procedure, so we conclude:

Corollary V.1.9. *A pre-kernel map φ given on some open subset by $S = (a_{ij})$ corresponds to a logarithmic connection with vanishing p-curvature if and only if at each point where φ*

fails to be surjective, for t a local coordinate at that point, there exist constants c_{ij} for all $0 < i < j \leq r$ and a formal local invertible M such that $MSU(c_{ij})$ is diagonal with t^{e_i} as its diagonal coefficients, and all $e_i < p$, where $U(c_{ij})$ is the upper triangular matrix having 1's on the diagonal and given by the c_{ij} above the diagonal.

We may also phrase this last result purely in terms of connections:

Corollary V.1.10. *A rational connection (having at least one pole) is logarithmic with vanishing p-curvature if and only if, formally locally at every pole, the connection may be transported so as to have diagonal matrix with each diagonal entry of the form $e_i t^{-1} dt$, with $e_i \in \mathbb{F}_p$.*

Proof. The only if direction follows immediately from our prior work: by Corollary V.1.9, the kernel map is formally locally diagonalizable with diagonal entries t^{e_i}, and by Proposition V.1.4 we see that this gives a connection of the desired form. Conversely, one computes directly that given a diagonal connection as described, the kernel mapping may be given explicitly by a diagonal matrix with t^{e_i} on the diagonals (where $0 \leq e_i < p$), and is in particular of full rank, implying that the p-curvature of the connection vanishes. □

From here on we assume that we are in:

Situation V.1.11. Our connection ∇ is logarithmic, with vanishing p-curvature. At every pole of ∇, we suppose that the e_i of Corollary V.1.10 are all non-zero.

The non-vanishing conditions on the e_i will come into play only when we attempt to study deformations of connections.

As another corollary, we can put together the preceding propositions to get the following relationship between $\det(S)$, $\mathrm{Tr}(T)$, the e_i, and the eigenvalues of the residue matrix $\mathrm{res}_t T$:

Corollary V.1.12. *With the notation of Proposition V.1.4, and in Situation V.1.11, $\mathrm{res}_t T$ is diagonalizable (in the usual sense), with eigenvalues given as the e_i mod p. The determinant satisfies $\mathrm{ord}_t \det(S) = \sum_i e_i \equiv \mathrm{Tr}(\mathrm{res}_t T) \pmod{p}$, but moreover, if we have the e_i only in terms of their reductions $\bar{e}_i$ mod p, we also have the formula $e_i = < \bar{e}_i >$, and hence $\mathrm{ord}_t \det(S) = \sum_i < \bar{e}_i >$, where $< a >$ for any $a \in \mathbb{Z}/p\mathbb{Z}$ denotes the unique integer representative for a between 0 and $p-1$. Finally, transport of T along an automorphism conjugates $\mathrm{res}_t T$ by the same automorphism.*

Remark V.1.13. The determinant of a connection (equivalently, the trace of the corresponding connection matrix), is well-determined under transport equivalence only globally on a proper curve, because transporting the connection under an automorphism corresponds to transporting the determinant connection on the determinant line bundle under the induced automorphism; globally, the only automorphisms of a line bundle are the scalars, which leave the determinant connection fixed, but locally this need not be the case. However, an automorphism given locally by a matrix S will act on a connection matrix T by $T \to S^{-1}TS + S^{-1}dS$; the trace of T therefore changed by the trace of $S^{-1}dS$. In particular, while the trace of T may change, the trace of the residue of T is preserved, since S is presumed to be invertible, and hence $S^{-1}dS$ is regular and has no residue.

Remark V.1.14. We cannot expect such nice behavior when we weaken the hypothesis that ∇ have only simple poles. First of all, there really are cases where this fails to hold, as soon as the rank is higher than one. In this situation, the relationship between the order of the determinant and the order of the poles is much less clear-cut. Moreover, it is easy to check that for rank higher than one, at a point with poles of order greater than one, the residue itself is no longer well-defined under transport.

V.2 Generalization to $k[\epsilon]$

The aim of this section is to generalize the results of the previous section to the case where we have changed base to $\operatorname{Spec} k[\epsilon]$. It turns out that the most difficult part of this is to show that the kernel of an appropriate deformation of a connection as in Situation V.1.11 will give a deformation of the kernel of the original connection. We proceed in several steps. We first pin down our situation and notation:

Situation V.2.1. We suppose that C is obtained from a smooth proper curve C_0 over k via change of base to $\operatorname{Spec} k[\epsilon]$, and similarly for a vector bundle $\mathscr{E}$ on C from $\mathscr{E}_0$ on C_0. ∇ is a connection on $\mathscr{E}$, and ∇_0 is the induced connection on $\mathscr{E}_0$.

Notation V.2.2. If D is the divisor of poles of ∇_0, so that ∇_0 takes values in $\mathscr{E}_0 \otimes \Omega^1_{C_0}(D)$, then we denote by $\mathscr{I}_{\nabla_0}$ the $\mathscr{O}_{C_0}$-submodule of $\mathscr{E}_0 \otimes \Omega^1_{C_0}(D)$ generated by the image of ∇_0.

Our first goal will be to show that in our situation, with very minor additional hypotheses, $\mathscr{E}^\nabla$ is in fact a deformation of $\mathscr{E}_0^{\nabla_0}$. Specifically:

Proposition V.2.3. *Suppose ∇ is a logarithmic connection with vanishing p-curvature, with ∇_0 having poles wherever ∇ does, and such that all the e_i of Corollary V.1.10 applied to ∇_0 are non-zero. Then:*

(i) $\mathscr{E}^{\nabla}$ is locally free on $C^{(p)}$ of rank equal to $\operatorname{rk}\mathscr{E}$*;*

(ii) the natural map $\mathscr{E}^{\nabla}/\epsilon\mathscr{E}^{\nabla} \to (\mathscr{E}_0)^{\nabla_0}$ is an isomorphism.

Proof. We first note that to prove (i), it will suffice to show that $\mathscr{E}^{\nabla}/\epsilon\mathscr{E}^{\nabla}$ is torsion-free over C_0: by Proposition A.16, it is enough see that $\mathscr{E}^{\nabla}/\epsilon\mathscr{E}^{\nabla}$ and $\epsilon\mathscr{E}^{\nabla}$ are both locally free over C_0, of rank equal to the rank of $\mathscr{E}$. Now, on the open subset of C on which ∇ is regular, by Theorem III.1.4 we have that $\mathscr{E}^{\nabla}$ is in fact locally free of the correct rank, and so then are $\mathscr{E}^{\nabla}/\epsilon\mathscr{E}^{\nabla}$ and $\epsilon\mathscr{E}^{\nabla}$. The required rank condition will thus follow automatically if we can show that both these sheaves are locally free on all of C_0, which is a smooth curve; this reduces the problem to showing that both these sheaves are torsion-free. Finally, $\epsilon\mathscr{E}^{\nabla}$ is a subsheaf of the locally free sheaf $\mathscr{E}$ and hence torsion-free, so we obtain the desired reduction of (i) to showing that $\mathscr{E}^{\nabla}/\epsilon\mathscr{E}^{\nabla}$ is torsion-free over C_0.

We now reduce both (i) and (ii) down to a certain divisibility lemma. Both statements are local on C, so we make our analysis entirely on stalks, letting P be an arbitrary point of C. Locally, $\mathscr{E}$ is free, so we can pick a splitting map $\mathscr{E}/\epsilon\mathscr{E} \to \mathscr{E}$, and we can then write $\nabla = \nabla_0 + \epsilon\nabla_1$, and it makes sense to view both ∇_0 and ∇_1 as taking values in $\mathscr{E}/\epsilon\mathscr{E}$ (since this is naturally isomorphic to $\epsilon\mathscr{E}$). The basic observation is that ∇_1 must take values in $\mathscr{I}_{\nabla_0}$: indeed, it may have simple poles only where ∇_0 does, so it takes values in $\mathscr{E}_0 \otimes \Omega^1_{C_0}(D)$, and by Corollary V.1.10, we see by the hypothesis that all the e_i are non-zero that in fact $\mathscr{I}_{\nabla_0}$ is all of $\mathscr{E}_0 \otimes \Omega^1_{C_0}(D)|_U$.

We first consider (i): since we are checking that $\mathscr{E}^{\nabla}/\epsilon\mathscr{E}^{\nabla}$ has no torsion as a module over $\mathscr{O}_{C^{(p)}}$, we need only consider multiplication by $f \in \mathscr{O}_{C,P}$ such that $df = 0$. We must show that given $s \in \mathscr{E}_P^{\nabla}$ with $fs \in \epsilon\mathscr{E}_P^{\nabla}$, s must itself be in $\mathscr{E}_P^{\nabla}$. If we write $fs = \epsilon s'$, with $s' \in \mathscr{E}_P^{\nabla}$, and $s' = s_1' + \epsilon s_2'$, then we see that $f|s_1' \in \mathscr{E}_{0,P}$, and it will suffice to show we can choose s_2' so that $f|s_2'$ as well, since then we can divide through by f to write s as ϵ times an element of $\mathscr{E}_P^{\nabla}$. Since the value of s' is only relevant modulo ϵ, we may replace s_2' by any element which keeps s' in the kernel of ∇. Now, we have $0 = \nabla(s') = \nabla_0(s_1') + \epsilon(\nabla_1(s_1') + \nabla_0(s_2'))$, and since $df = 0$, $f|\nabla_1(s_1')$, so since both ∇_1 and ∇_0 take values in $\mathscr{I}_{\nabla_0}$, f must likewise divide $\nabla_0(s_2')$ in $\mathscr{I}_{\nabla_0}$, and the divisibility lemma which follows completes the proof, taking

s_2' as our s in the lemma, and obtaining our new s_2' as the lemma's fs'.

Next, we wish to reduce (ii) down to the same lemma. Having already completed (i), we may assume that $\mathscr{E}^\nabla$ is locally free, with rank equal to $\operatorname{rk}\mathscr{E}$. It follows that $\mathscr{E}^\nabla/\epsilon\mathscr{E}^\nabla$ is locally free of the same rank on C_0, as is $\mathscr{E}_0^{\nabla_0}$ by Proposition V.1.1. It therefore suffices to show that the natural map is a surjection in order to conclude that it is an isomorphism. Let s_0 be a section of $\mathscr{E}_{0,P}^{\nabla}$; we need only lift it to a section $s \in \mathscr{E}_P^\nabla$. Moreover, we know that we can do so generically, since we have the Cartier isomorphism away from the poles of ∇ by Theorem III.1.4. Therefore, there exists some f such that fs_0 lifts to a section s of $\mathscr{E}_P^\nabla$; as before, we are working over $\mathscr{O}_{C^{(p)}}$, so as an element of $\mathscr{O}_{C,P}$, $df = 0$. But now we find ourselves in the same situation as before: if $s = s_1 + \epsilon s_2$, we have that $f|s_1$, we want f to divide s_2, and we may modify s_2 arbitrarily as long as s remains in the kernel of ∇. Thus by the same argument as for (i), we reduce to our divisibility lemma. □

Lemma V.2.4. *Given $f \in \mathscr{O}_{C,P}$ for some $P \in C$, with $df = 0$, and s in the stalk $\mathscr{E}_{0,P}$ with $f|\nabla_0(s)$ in the stalk $\mathscr{I}_{\nabla_0,P}$, then there exists $s' \in \mathscr{E}_{0,P}$ with $\nabla_0(fs') = \nabla_0(s)$.*

Proof. Under our hypotheses on ∇_0, which allow us to invoke Corollary V.1.10, the statement is essentially trivial. We first prove the result formally locally. In this setting, we claim it is enough to handle the case $f = t^p$: in general, write $f = (t^p)^i u$ for some $i \geq 0$ and some u; certainly, if we have handled the case of t^p, we can inductively "divide out" by t^p i times, and then since u is a unit, we can simply set $s' = u^{-1}s$. But for $f = t^p$, we simply carry out a direct computation; the diagonalizability obtained from Corollary V.1.10 expresses the connection formally locally as a direct sum of connections on line bundles, so it suffices to work with rank one, and a connection of the form $\nabla_0(s) = ds + et^{-1}dt$, for some $e \in \mathbb{F}_p$; our $\mathscr{I}_{\nabla_0}$ in this context is simply everything of the form $\sum_{i\geq -1} a_i t^i dt$. If we write $s = \sum_{i\geq 0} a_i t^i$, we get $\nabla_0(s) = \sum_{i\geq 0}(i+e)a_i t^{i-1}dt$; this is divisible by t^p in $\mathscr{I}_{\nabla_0}$ if and only if $(i+e)a_i = 0$ for all $i < p$. Now, for any $i < p$ with $i + e = 0$, we can replace a_i with 0 without changing $\nabla_0(s)$, and for all other i, we must have $a_i = 0$ to start with. Hence, we see that we can modify s in degree $p - e$, if necessary, so that all $a_i = 0$ for $i < p$, and we can then obtain our s' as t^{-p} times our modified s.

This gives the formal local result, but it is now easy enough to conclude the desired Zariski-local statement. We have $s - fs'$ in the kernel of ∇, and because in characteristic p formation of the kernel of a connection commutes with completion (see Proposition A.29), we can write $s - fs' = \sum_i f_i s_i$ where $s_i \in \mathscr{E}^{\nabla}_{0,P}$ and $f_i \in k[[t]]$. But by definition, we can approximate the f_i to arbitrary powers of t by elements of $\mathscr{O}_{C,P}$; if we let f_i' approximate the f_i to order at least $\operatorname{ord}_t f$, we find that f must divide $s - \sum_i f_i' s_i$, so we can set our desired Zariski-local section to be $\frac{1}{f}(s - \sum_i f_i' s_i)$. □

We now know the correct conditions for connections over $k[\epsilon]$. Specifically, after this section, whenever we are over $k[\epsilon]$, we assume we have:

Situation V.2.5. Our connection ∇ is logarithmic, with vanishing p-curvature. If ∇_0 is the connection obtained modulo ϵ, then every pole of ∇ must also be a pole of ∇_0, and we suppose that the e_i of Corollary V.1.10 as applied to ∇_0 are all non-zero.

Finally, we are ready to conclude:

Corollary V.2.6. *Corollary V.1.9 holds even over $k[\epsilon]$; more precisely, a pre-kernel map φ as in Proposition V.1.1, given by $S = (a_{ij})$ on some open subset which contains every point where φ fails to be surjective, corresponds to a connection satisfying the conditions of Situation V.2.5 if and only if at each point where φ fails to be surjective, for t a local coordinate at that point, there exist constants c_{ij} for all $0 < i < j \leq r$ and a formal local invertible M such that $MSU(c_{ij})$ is diagonal with t^{e_i} as its diagonal coefficients, and all $e_i < p$, where $U(c_{ij})$ is the upper triangular matrix having 1's on the diagonal and given by the c_{ij} above the diagonal.*

Proof. We first note that given an S, the calculation of Proposition V.1.4 is still valid because S and hence $\det S$ is still generically invertible. Hence, as before it is clear that if the desired $M, U(c_{ij})$ exist, then S corresponds to a connection of the desired type. Conversely, given such a connection, since S describes the kernel of our connection, by the previous proposition, we find that we have an S which agrees modulo ϵ with the S_0 obtained from taking the connection modulo ϵ; we can then apply Corollary V.1.9 to conclude that formally locally on C there is an invertible M_0 and a $U_0(c_{ij})$, both over k, such that $S' = M_0 SU(c_{ij})$ is of the desired form modulo ϵ. Thus, we can write

$$S' = \begin{bmatrix} t^{e_1} + \epsilon f_{11} & \dots & \epsilon f_{1r} \\ \vdots & \ddots & \vdots \\ \epsilon f_{r1} & \dots & t^{e_r} + \epsilon f_{rr} \end{bmatrix}$$

The following calculations are then easy:

$$\det(S') = t^{\sum_i e_i} + \epsilon t^{\sum_i e_i} \sum_i t^{-e_i} f_{ii}$$

$$S'^{-1} = \begin{bmatrix} t^{-e_1} - \epsilon t^{-2e_1} f_{11} & \dots & -\epsilon t^{-e_r - e_1} f_{1r} \\ \vdots & \ddots & \vdots \\ -\epsilon t^{-e_1 - e_r} f_{r1} & \dots & t^{-e_r} - \epsilon t^{-2e_r} f_{rr} \end{bmatrix}$$

$$dS'^{-1} = (D(-e_i t^{-e_i - 1}) + \epsilon (t^{-e_j - e_i - 1}((e_j + e_i) f_{ij} - t f'_{ij}))_{ij}) dt$$

(with $D(a_i)$ denoting the diagonal matrix with coefficients a_i, and $(a_{ij})_{ij}$ denoting the matrix with i, jth coefficient given by a_{ij}) and finally, $T' = S'(dS'^{-1})$ is given by

$$\begin{bmatrix} -e_1 t^{-1} + \epsilon t^{-e_1 - 1}(e_1 f_{11} - t f'_{11}) & \dots & \epsilon t^{-e_r - 1}(e_1 f_{1r} - t f'_{1r}) \\ \vdots & \ddots & \vdots \\ \epsilon t^{-e_1 - 1}(e_r f_{r1} - t f'_{r1}) & \dots & -e_r t^{-1} + \epsilon t^{-e_r - 1}(e_r f_{rr} - t f'_{rr}) \end{bmatrix} dt$$

Thus, in order to have simple poles, it is necessary and sufficient that $\operatorname{ord}_t(e_i f_{ij} - t f'_{ij}) \geq e_j$ for all i, j. But this is precisely the condition required to be able to remove all the f_{ij} via row and (constant) column operations, since the inequality above implies that all terms of f_{ij} in degree ℓ must vanish for $\ell < e_j$, unless $\ell = e_i$. Constant column operation can remove the terms of degree e_i from each f_{ij}, and then we have that $\operatorname{ord}_t f_{ij} \geq e_j$, so row operations can remove the f_{ij}, as desired. $\square$

V.3 Applications to Rank 2

As our case of primary interest, we will develop the theory further in the case of vector bundles of rank 2 and connections ∇ whose residue at all poles has trace zero. Note that in this case, at any pole the e_i of Corollary V.1.10 satisfy $e_1 = -e_2$, and in particular are automatically both non-zero as required in Situation V.1.11. We will work simultaneously over $k[\epsilon]$, assuming in this case the conditions of Situation V.2.5. In this scenario, we define:

Definition V.3.1. Given $f \in A[[t]]$, we say that $\operatorname{ord}_t f = e$ if and only if the first non-zero coefficient of f is t^e. If further this first non-zero coefficient is a unit in A, we say that f **vanishes uniformly to order** e at $t = 0$.

Now, the kernel map φ associated to any connection ∇ is given locally by the matrix $S = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$. The corresponding connection ∇ is then given locally by the matrix T, which Proposition V.1.4 allows us to write explicitly as

$$T = \frac{1}{\det S} \begin{bmatrix} (dg_{12})g_{21} - (dg_{11})g_{22} & (dg_{11})g_{12} - (dg_{12})g_{11} \\ (dg_{22})g_{21} - (dg_{21})g_{22} & (dg_{21})g_{12} - (dg_{22})g_{11} \end{bmatrix} \tag{V.3.2}$$

In this case, Corollary V.1.12 tells us that the simple poles of the connection will occur at precisely the places where $\det S$ vanishes, and that this will always occur to order precisely p. Over $k[\epsilon]$, Corollary V.2.6 implies that the determinant will vanish uniformly to order p. As before, choose a point where this is the case, and let t be a local coordinate at that point. Denote by e_{ij} the order at t of g_{ij}. We will develop more precisely the criterion for S to correspond to T (that is, for the image of S to contain the kernel of T), and for T to have simple poles. We find:

Proposition V.3.3. *Over k (respectively, $k[\epsilon]$), assuming that $\det S$ vanishes (uniformly) to order p at $t = 0$, for S to correspond to a connection T with a simple pole at $t = 0$ and vanishing p-curvature, it is necessary and sufficient that there exists a c_t such that after S is replaced by $S' = S \begin{bmatrix} 1 & -c_t \\ 0 & 1 \end{bmatrix}$, we have:*

$$\min\{\operatorname{ord}_t g_{11}, \operatorname{ord}_t g_{21}\} + \min\{\operatorname{ord}_t g_{12}, \operatorname{ord}_t g_{22}\} \geq p.$$

Over k, this may be stated equivalently as (after S is replaced by S'), the order of vanishing at $t = 0$ is greater than or equal to p for all of $g_{11}g_{22}, g_{21}g_{12}, g_{11}g_{12}, g_{21}g_{22}$.

Proof. First, if S corresponds to a connection with a simple pole at $t = 0$, by Corollary V.1.9 (respectively, Corollary V.2.6) we have a c_{12} such that $MS \begin{bmatrix} 1 & c_{12} \\ 0 & 1 \end{bmatrix}$ is diagonal with powers of t on the diagonal, and M is a formal local invertible matrix. Letting $c_t = -c_{12}$, we replace S by S', and are simply saying that MS is diagonal with powers of t on the diagonal, say t^e and t^{p-e}. Then if $M^{-1} = (m_{ij})$, we have that S is given by $\begin{bmatrix} m_{11}t^e & m_{12}t^{p-e} \\ m_{21}t^e & m_{22}t^{p-e} \end{bmatrix}$, which trivially gives the desired conditions on the g_{ij}.

Conversely, suppose that the required c_t exists, and we have replaced S by S'. Let $e = \min\{\operatorname{ord}_t g_{11}, \operatorname{ord}_t g_{21}\}$. By hypothesis, $\min\{\operatorname{ord}_t g_{12}, \operatorname{ord}_t g_{22}\} \geq p - e$. Thus, we can write S as $(m_{ij})D(t^e, t^{p-e})$ for some m_{ij} regular at $t = 0$, and once again the condition that S has determinant vanishing uniformly to order p at $t = 0$ implies that $\det(m_{ij})$ is a unit, and hence that (m_{ij}) is invertible and may be moved to the other side, letting us apply Corollary V.1.9 (respectively, Corollary V.2.6) to conclude that S corresponds to a connection with a simple pole at $t = 0$ and vanishing p-curvature, as desired. $\square$

Remark V.3.4. This criterion looks rather asymmetric on the face of it, but note that locally one may always conjugate by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to switch the rows and columns, after which application of the above criterion gives equivalent conditions in terms of subtracting the right column from the left rather than vice versa. We will refer to this as the **mirror criterion.**

Remark V.3.5. Our initial description of c_t was that there exist an invertible M such that $MS \begin{bmatrix} 1 & -c_t \\ 0 & 1 \end{bmatrix}$ is diagonal, from which it immediately follows that c_t is independent of transport of S via left multiplication. However, if we multiply by some invertible column-operation matrix N on the right, we will need to determine how to "move" this action over to the left, which is in general not a simple matter, and can result in substantial changes to the behavior of the c_t. This is a rather ironic situation, since it is the right multiplication which leaves the corresponding connection unchanged, and the left multiplication which applies automorphism transport to it. In any case, we will at least be able to characterize exactly how the c_t can change under global right multiplication in most cases on $\mathbb{P}^1$.

V.4 Global Computations on $\mathbb{P}^1$

Throughout this section, let $\mathscr{E}$ be $\mathscr{O}(\epsilon p - d) \oplus \mathscr{O}(d)$ on $\mathbb{P}^1$, where $\epsilon = 0$ or 1, and $\epsilon p \leq 2d$. We set up the basic situation to be used in this section and the next, and then classify in Proposition V.4.5 an "easy case" for the connections we wish to study, which will not be used in the degeneration arguments of Chapter VI, but we include nonetheless for the sake of completeness. Let $t_1, \dots t_n$ be local coordinates at n distinct points on $\mathbb{P}^1$; without loss of generality, write $t_i = x - \lambda_i$, if x is a coordinate for some $\mathbb{A}^1 \subset \mathbb{P}^1$ containing the relevant points, and let c_i be the c_t of Proposition V.3.3 for each t_i. If ∇ is a connection on $\mathscr{E}$ with vanishing p-curvature and simple poles at the λ_i, $F^*\mathscr{E}^\nabla$ must have degree $-p(n - \epsilon)$, so it must be of the form $\mathscr{O}(-mp) \oplus \mathscr{O}((m-n+\epsilon)p)$ for some integer m (without loss of generality, say $m \leq n - \epsilon - m$), and because it must map with full rank to $\mathscr{O}(\epsilon p - d) \oplus \mathscr{O}(d)$, we find that we also must have $(m - n + \epsilon)p \leq \epsilon p - d$, $-mp \leq d$, which gives us $-d \leq mp \leq np - d$. We now fix some choice of m, and consider possibilities for φ (together with the induced ∇, which should have simple poles at the λ_i, and corresponding matrices S and T on a particular choice of open subset of $\mathbb{P}^1$ which we identify with $\mathbb{A}^1$).

We may write $\mathrm{Hom}(F^*\mathscr{E}^\nabla, \mathscr{E})$ as

$$\begin{bmatrix} \mathscr{O}((m+\epsilon)p - d) & \mathscr{O}((n-m)p - d) \\ \mathscr{O}(mp + d) & \mathscr{O}((n - \epsilon - m)p + d) \end{bmatrix}$$

The matrix S can therefore be written with coefficients g_{ij} being polynomials in x of the appropriate degrees, with products along both the diagonal and antidiagonal having degree bounded by np. Moreover, there are n points where the determinant must vanish to order p, so up to scalar multiplication, the determinant must be $\prod_i (x - \lambda_i)^p$. Global transport of our kernel map corresponds to left multiplication by matrices of the form

$$\begin{bmatrix} \mathscr{O}(0) & \mathscr{O}(2d) \\ \mathscr{O}(-2d) & \mathscr{O}(0) \end{bmatrix}$$

and right multiplication by

$$F^* \begin{bmatrix} \mathscr{O}(0) & \mathscr{O}(n - \epsilon - 2m) \\ \mathscr{O}(2m - n + \epsilon) & \mathscr{O}(0) \end{bmatrix}$$

Then we have:

Proposition V.4.1. *Although the c_i are not invariants of a connection, for the most part they change predictably under transport of their kernel maps. It is always possible to scale them simultaneously. It is also possible to translate them simultaneously by (any constant times) λ_i^{pj} for any j between 0 and $n - \epsilon - 2m$. If $m < n - \epsilon - m$, the i for which the c_i are uniquely defined do not change under transport, and the above modifications are the only possible ones for these c_i.*

Proof. We make use only of the criterion of Corollary V.1.9 (recalling that the c_i were by definition the negatives of the constants arising there). We first show that the asserted modifications are possible. If we begin with S, and at each λ_i an M_i and upper triangular

$U(-c_i)$ with $M_iSU(-c_i)$ diagonal, we can transport S to simultaneously scale the c_i by any μ simply by replacing S by $SD(1,\mu) = S\begin{bmatrix}1 & 0\\ 0 & \mu\end{bmatrix}$, $U(-c_i)$ by $D(1,\mu^{-1})U(-c_i)D(1,\mu) = U(-\mu c_i)$, and M_i by $D(1,\mu^{-1})$, whereupon our original diagonal matrix is conjugated by $D(1,\mu)$, and remains unchanged (and in particular, diagonal).

Next, translation of all c_i by $\mu\lambda_i^{pj}$ is accomplished simply by right multiplication of S by $U(\mu x^{pj})$: at each λ_i, we can write $x^{pj} = \lambda_i^{pj} + (x^j - \lambda_i^j)^p$, and then if $M_iSU(-c_i) = D(d_1, d_2)$ was diagonal, it follows that $M_i(SU(\mu x^{pj}))U(-c_i - \mu\lambda_i^{pj}) = M_iSU(-c_i)U(\mu(x^j - \lambda_i^j)^p) = \begin{bmatrix}d_1 & \mu d_1(x^j - \lambda_i^j)^p\\ 0 & d_2\end{bmatrix}$. Now, since $\mathrm{ord}_{\lambda_i} d_2 < p$, we can multiply M on the left by $U(-\mu\frac{d_1(x^j-\lambda_i^j)^p}{d_2})$ to recover the initial diagonal matrix, so we see that $c_i + \mu\lambda_i^{pj}$ has taken the role of c_i, as desired.

Lastly, when $m < n - \epsilon - m$, we simply need to verify that the above cases are the only possible forms of transport that can affect the c_i: we have seen that only right multiplication can affect the c_i, and when $m < n - \epsilon - m$, the only matrices we can right multiply by are upper triangular with scalars on the diagonal and inseparable polynomials of degree $\leq (n - \epsilon - 2m)p$ in the upper right. These are generated by the two cases above together with $D(\mu, 1)$, but $D(\mu, 1) = D(\mu, \mu)D(1, \mu^{-1})$, and the $D(\mu, \mu)$ can be commuted to the left and absorbed into M, so we see that $D(\mu, 1)$ acts the same as $D(1, \mu^{-1})$ via right multiplication on S, so we have already dealt with it as well. In particular, all methods of acting on the c_i change them invertibly, so whether or not they are uniquely determined is tranport-invariant as long as $m < n - \epsilon - m$. $\square$

Example V.4.2. When $m = n - \epsilon - m$, it is not true that the c_i behave well under transport, and they may even go from uniquely determined to arbitrary and back. For instance, consider a diagonal matrix vanishing to order $e < p/2$, $p - e$ along the diagonal at a chosen point λ_i. In this case, c_i is well-determined as 0, since if we multiplied by any $U(c_i)$ with $c_i \neq 0$, we would have that the product of the entries on the top row of our matrix only vanished to order $2e < p$. But because $m = n - \epsilon - m$, we can right-multiply by $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ to switch the columns of our matrix, at which point c_i may be chosen arbitrarily, because $2p - 2e > p$.

Before moving on to the next results, we fix some combinatorial notation which will come up as soon as we actually start trying to count classes of connections.

Notation V.4.3. For a given p, n, and s, denote by $N_p(n,s)$ the number of mononomials of degree s in n variables subject to the restriction that each variable occur with positive exponent strictly less than p. Also denote by $N_p^D(n,s)$ the number of such monomials in which exactly D variables appear with degree less than $p/2$.

We give explicit formulas for these numbers:

Lemma V.4.4. *We have:*

$$N_p(n,s) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{s-i(p-1)-1}{n-1}.$$

For any fixed n, this is expressed by the jth of $n+1$ polynomials of combined degree $n-1$ in s and p, with j being the largest integer ($\leq n+1$) such that $j(p-1) \leq s-n$. We also have

$$N_p^D(n,s) = \binom{n}{D} \sum_{j=0}^{s} N_{(p+1)/2}(D,j) N_{(p+1)/2}(n-D, s-j-(n-D)\frac{p-1}{2}).$$

Proof. The number of monomials of degree s in n variables, with each variable appearing with positive exponent, is the same as the number of monomials of degree $s-n$ in n variables with no restriction, which is given by $\binom{s-1}{n-1}$. We then use the inclusion-exclusion principle, subtracting off for each individual variable the number of monomials where that variable occurs with exponent at least p, adding back for each pair of variables the monomials where both variables occur with exponent at least p, and so on. Each time we change the restriction on a given variable from occurring with positive exponent to occurring with exponent at least p, this corresponds to dropping the degree by $p-1$, and the $\binom{n}{i}$ comes from the number of choices of which variables to bound in this manner. This gives the desired formula, and the statement that it can be expressed (within the appropriate range) as a polynomial of combined degree $n-1$ in s and p follows trivially, since $\binom{s-i(p-1)-1}{n-1} = \frac{1}{(n-1)!}\prod_{j=0}^{n-2}(s-i(p-1)-1-j)$ if $s-i(p-1)-1 \geq n-1$, and 0 otherwise.

The asserted formula for $N_p^D(n,s)$ also follows easily from the definitions, since the count may be split up over the choice of which D variables have degree less than $p/2$, and then one has a product of a monomial with those variables, each with degree less than $p/2$, with a monomial of the remaining variables, each with degree greater than $p/2$. This last condition is equivalent to dropping the degree by $\frac{p-1}{2}$ for each variable, giving the desired formula. $\square$

Proposition V.4.5. *We can classify completely all kernel map classes with kernel isomorphic to $\mathscr{O}(-m) \oplus \mathscr{O}(m-n+\epsilon)$ which can be made via transport to have either g_{11} or g_{12} equal to 0. We may describe them as (note that despite the geometry in the description, we make no claim of any a priori scheme or variety structure):*

(1) there are $N_p(n, mp+d)$ transport-antidiagonalizable classes.

(2) For each D, there are $N_p^D(n, mp+d)$ $\mathbb{P}^{D-2-n+2m}$'s of classes of kernel maps for which g_{11} may be transported to 0 but which are not transport-antidiagonalizable.

(3) If $m \neq n-\epsilon-m$, there are an additional (distinct) $N_p(n,(n-\epsilon-m)p+d)$ transport-diagonalizable classes.

(4) Again if $m \neq n-\epsilon-m$, for each D there are an additional (distinct) $N_p^D(n,(n-\epsilon-m)p+d)$ $\mathbb{P}^{D-1}$'s of classes of non-transport-diagonalizable kernel maps for which g_{12} may be transported to 0.

In particular, in the case $mp < d$, all possible kernel maps are classified by (1) and (2).

Proof. We begin with the case that $g_{11} = 0$. We have $g_{21}g_{12} = \prod_{i=1}^n (x-\lambda_i)^p$, so fix the orders at each λ_i of g_{21} (equivalently, g_{12}). There are $N_p(n, mp+d)$ choices, by definition. Clearly, we get only one antidiagonalizable one given the choices of orders. The interesting part is to classify the non-antidiagonalizable ones. Let D be the number of i such that g_{21} has order less than $p/2$ at λ_i; this will be the number of c_i which are uniquely determined under our criterion of Proposition V.3.3. Indeed, for such c_i, this criterion tells us precisely that for each λ_i, there is a c_i such that $(x-\lambda_i)^{p-\operatorname{ord}_{\lambda_i} g_{21}} | (g_{22} - c_i g_{21})$. Note that this is equivalent to saying that $g_{22} \equiv c_i g_{21} \pmod{(x-\lambda_i)^{\operatorname{ord}_{\lambda_i} g_{12}}}$. From this perspective, in the cases that c_i can be arbitrary, we may as well always consider them to be 0, since that is what the righthand side will be. We then observe that if we choose values for the c_i, there is at most one transport-class with those values, since any choice of g_{22} is determined modulo $(x-\lambda_i)^{\operatorname{ord}_{\lambda_i} g_{12}}$ for all i, and hence modulo g_{12}. On the other hand, the Chinese Remainder Theorem says that for any choice of the c_i, we can find an appropriate g_{22}. Now, there are D of the c_i which must be specified, and they cannot all be the same, since in that case one could arrange by a single column operation for g_{12} to divide g_{22}, which then means we are in the transport-antidiagonalizable case. Moreover, by Proposition V.4.1 we can do a global column operation to set the first $n-\epsilon-2m+1$ of the c_i to 0 (since powers of distinct numbers are always linearly independent), reducing us to $D-n+\epsilon+2m-1$ choices, and we can also scale all the remaining c_i. So, we have a $\mathbb{P}^{D-2-n+\epsilon+2m}$ of distinct choices for the c_i, each corresponding to a unique class of kernel maps. When $m < n-\epsilon-m$, from Proposition V.4.1 we know these are the only possibilities, so we are done. On the other hand, when $m = n-\epsilon-m$, we note that the only possibilities for right transport which preserve $g_{11} = 0$ are the upper triangular ones, which correspond precisely to the translation and scaling we have already used, so this case works out exactly the same way. This finishes cases (1) and (2).

The proofs of cases (3) and (4) proceed in exactly the same way, except that for convenience we classify kernel map classes by the c_i for the mirror criterion, and we also have to note that globally in this case we cannot translate the c_i at all, since any non-trivial column operation would make $g_{12} \neq 0$, so we get a $\mathbb{P}^{D-1}$ rather than a $\mathbb{P}^{D-2-n+\epsilon+2m}$. Finally, when $m = n-\epsilon-m$, the kernel map classes in (3) and (4) are the same as the ones in (1) and (2), since one can globally switch columns, but whenever $m < n-\epsilon-m$, they are distinct, since if either of the g_{1j} is 0, it is clear no transport-equivalent matrix could have the other 0 instead. $\square$

Remark V.4.6. With this proposition, we already see polynomials in p arising in counting connections with a fixed set of poles on a fixed vector bundle. Ultimately, the numbers of this proposition will not come into the calculation of the number of connections we are interested in for Chapter VI, but that number will also be a polynomial in p, strongly suggesting the existence of a more general underlying phenomenon.

V.5 Maps from $\mathbb{P}^1$ to $\mathbb{P}^1$

We have fully analyzed classes of kernel maps in which one of g_{11} or g_{12} may be transported to 0. To analyze the remaining kernel maps, we shift focus considerably. For motivation, we initially restrict our attention to the case $m = n-\epsilon-m$. We then note that the pair g_{11}, g_{12} can be viewed as defining a map $\bar{\varphi}$ from $\mathbb{P}^1$ to itself, of degree equal to $mp-d-\deg\gcd(g_{11}, g_{12})$. For later convenience, we will reverse the standard order and consider this as corresponding to the rational function $\frac{g_{12}}{g_{11}}$. We see that this map is invariant under left transport, since that only affects the bottom row of S, and because of the hypothesis that $m = n - \epsilon - m$, right multiplication occurs only by invertible matrices of scalars, which act precisely as post-composition by fractional linear transformations of $\mathbb{P}^1$ (although note that this action is by the transpose of the usual matrix represention of $\mathrm{Aut}(\mathbb{P}^1)$). Thus, we get a well-defined map from transport-equivalence classes of our kernel maps to maps from $\mathbb{P}^1$ to $\mathbb{P}^1$ modulo post-composition by $\mathrm{Aut}(\mathbb{P}^1)$. We also note that if we consider the mirror criterion of Proposition V.3.3, the resulting c_i will actually be determined as the inverses of the values of this map at the λ_i, and the diagonal terms in the criterion describe ramification conditions for the map at the λ_i. With this as a guiding example, we drop the hypothesis that $m = n - \epsilon - m$, reinterpret this 'map' (which will no longer be well-defined on kernel map classes), and proceed to give a precise description of its image in the space of rational functions, to show that each rational function in this image corresponds to exactly one class of kernel maps, and to give a precise description of the space of rational functions arising from each kernel map class.

Warning V.5.1. In order to streamline the proofs in this section, whenever we refer to the c_i or criterion of Proposition V.3.3, we will mean the mirror criterion under which scalar multiples of the right column are added to the left. Of course, all proofs could be carried through via the same approach under the non-mirror criterion, but this will simply let us make certain simplifications to the argument, which would otherwise be more cumbersome.

We begin with some notation and observations: first, since $\det(S)$ is supported at the λ_i, the GCD of the coefficients of S must likewise be.

Notation V.5.2. Set α_i so that the GCD of the coefficients of S is $\prod(x-\lambda_i)^{\alpha_i}$. Factoring this out from S, write $\hat{S} = (\hat{g}_{ij})$ for the resulting matrix, whose coefficients have no nontrivial common divisor. Now, let g_1 be the GCD of $\hat{g}_{11}$ and $\hat{g}_{12}$, write $\beta_i = \mathrm{ord}_{\lambda_i} g_1$, and finally write $f_g := \frac{g_{12}}{g_{11}}$, considered as an endomorphism of $\mathbb{P}^1$.

We make the following observation: formally locally at each λ_i, we can transport-diagonalize S with powers of t_i on the diagonal, obtaining two positive integers summing to p as the exponents. Momentarily writing α_i' for the lesser of the two, we note that $t_i^{\alpha_i'}$ is the GCD of the coefficients of the diagonalized matrix, and since GCDs are unchanged by multiplication by invertible matrices, it must also have been the GCD of the coefficients of

S (over $k[[t_i]]$); hence, $\alpha_i' = \alpha_i$. We also see easily that one of g_{11}, g_{12} may be transported to 0 if and only if S is transport-equivalent to a kernel map with f_g having degree 0, hence constant. Thus:

Corollary V.5.3. *The kernel map classes classified in Proposition V.4.5 are precisely those for which the associated endomorphism $f_g : \mathbb{P}^1 \to \mathbb{P}^1$ can be made constant under transport.*

We note that in general we can use (constant) row operations to insure that at each λ_i, $\operatorname{ord}_{\lambda_i} g_{2j} \leq \operatorname{ord}_{\lambda_i} g_{1j}$ for both j, and without further comment we assume we have done this. Because $m \leq n - \epsilon - m$, we can similarly add constant multiples of the left column to the right, allowing us to set $\operatorname{ord}_{\lambda_i} g_{j2} \leq \operatorname{ord}_{\lambda_i} g_{j1}$, and in particular, $\operatorname{ord}_{\lambda_i} g_{22} = \alpha_i$ for all i. For later convenience, we can and will also use choose a (possibly nonconstant) column operation to make sure that g_{12} has maximal degree $(n-m)p-d$. To summarize, to analyze transport-classes of kernel maps, it is sufficient to consider the following situation, which we will assume we are in for the remainder of the section:

Situation V.5.4. We have normalized so that $\operatorname{ord}_{\lambda_i} g_{22} = \alpha_i$ for all i, $\operatorname{ord}_{\lambda_i} g_{12} = \alpha_i + \beta_i$, and $\deg g_{12} = (n-m)p - d$.

We now analyze the situation with f_g:

Proposition V.5.5. *f_g has degree $(n-m)p - d - \sum_i \alpha_i - \sum_i \beta_i$, and is ramified to order at least $p - 2\alpha_i - \beta_i$ at each λ_i, and $(n - \epsilon - 2m)p$ (when this is non-zero) at infinity.*

Proof. By definition, f_g has degree $(n-m)p - d - \sum \alpha_i - \deg g_1$. Noting that g_1 will divide the determinant of $\hat{S}$, it must be supported at the λ_i, so we also have $\deg g_1 = \sum \beta_i$. Next, examining the (mirror) criterion of Proposition V.3.3, we see that the requirement that $(g_{11} - c_i g_{12})(g_{22})$ vanish to order at least p at λ_i, since we had arranged for $\operatorname{ord}_{\lambda_i} \hat{g}_{22} = 0$, gives the desired ramification condition at λ_i. The ramification at infinity follows because we had set $\deg g_{12} = (n - \epsilon - m)p - d$, so it has degree at least $(n - \epsilon - 2m)p$ greater than g_{12}. ☐

In particular, we see that when f_g is nonconstant, the c_i are actually all uniquely determined as $\frac{1}{f_g(\lambda_i)}$. It follows that we need not worry about running into trouble with maps taking on values at ramification points which would force the c_i to be infinite, since this would mean $f_g(\lambda_i) = 0$ for some i, which would imply $\operatorname{ord}_{\lambda_i} \hat{g}_{12} > \operatorname{ord}_{\lambda_i} \hat{g}_{11}$, and this is precisely what we had arranged to avoid. In determining necessary and sufficient conditions to fill in the $\hat{g}_{2j}$ from the $\hat{g}_{1j}$ in such a way as to satisfy our criterion, we find:

Proposition V.5.6. *For any given choice of α_i and $\hat{g}_{1j}$ as prescribed by the previous proposition, there is a unique corresponding kernel map class if and only if for all i such that $\beta_i > 0$, f_g has precisely the minimum required ramification, i.e. f_g ramifies to order precisely $p - 2\alpha_i - \beta_i$ at λ_i. Otherwise, there will be no corresponding kernel map.*

Proof. We first show that if the conditions on $\hat{g}_{1j}$ are satisfied, we do in fact get a unique corresponding kernel map class: that is, given $\hat{g}_{11}$ and $\hat{g}_{12}$, there is a unique way (up to transport) to fill in $\hat{g}_{21}$ and $\hat{g}_{22}$ which satisfies the (mirror) criterion of Proposition V.3.3. This will follow from standard results on generators of ideals over PIDs: we need to choose the bottom row so that the determinant is $\Delta = \prod(x - \lambda_i)^{p-2\alpha_i}$; the solutions to $\hat{g}_{11}h_1 - \hat{g}_{12}h_2 = \Delta$ are expressible for some particular choice of h_1, h_2 as $h_1 + q\frac{\hat{g}_{12}}{g_1}, h_2 + q\frac{\hat{g}_{11}}{g_1}$ as q varies freely. In particular, two ways of filling in the bottom row are transport equivalent if and only if their corresponding q's differ by a multiple of g_1, so we will need to check that the criterion determines q precisely modulo g_1. We also observe that given q modulo g_1, we can always choose a representative polynomial for it so that the resulting g_{21}, g_{22} have the correct degrees: changing q by a multiple of g_{21} corresponds to subtracting a multiple of the first row from the second, which can always, for instance, force the degree of g_{22} to be strictly smaller than $(n-m)p - d \leq (n - \epsilon - m)p + d$, without changing the determinant, and this forces g_{21} to have degree exactly $mp + d$. Note also that some as above h_1, h_2 must exist because our hypotheses immediately imply that $\beta_i < p - 2\alpha_i$.

Now, note that any q yields a solution satisfying the order conditions along the diagonal, antidiagonal, and the top row: indeed, our ramification condition gives order at least p along the top row and diagonal (after column operation by c_i), and the determinant then forces the antidiagonal to also have order at least p at all λ_i. In particular, $\operatorname{ord}_{\lambda_i} h_1 - c_i h_2 \geq p - 2\alpha_i - \beta_i$, since we arranged for $\operatorname{ord}_{\lambda_i} g_{12} \leq \operatorname{ord}_{\lambda_i} g_{11}$ at all i, so we have $\operatorname{ord}_{\lambda_i} \hat{g}_{12} = \beta_i$. Next, we know that if we can fill in the bottom row so to satisfy our criterion, we can do it with $\hat{g}_{22}$ invertible at all λ_i, and conversely, if $\hat{g}_{22}$ is invertible at all λ_i, our criterion requires precisely (in addition to the determinant being correct) that $\operatorname{ord}_{\lambda_i}(g_{21} - c_i g_{22})g_{22} \geq p$, or equivalently, $\operatorname{ord}_{\lambda_i}(\hat{g}_{21} - c_i\hat{g}_{22}) \geq p - 2\alpha_i$. Plugging in our expressions for possibilities for

$\hat{g}_{21}$ and $\hat{g}_{22}$, we get

$$\operatorname{ord}_{\lambda_i}(h_1 - c_i h_2 + q(\frac{\hat{g}_{12}}{g_1} - c_i\frac{\hat{g}_{11}}{g_1})) \geq p - 2\alpha_i.$$

Now, we observed earlier that $h_1 - c_i h_2$ has order at least $p - 2\alpha_i - \beta_i$; by hypothesis, either $\beta_i = 0$, in which case we are done, or $\beta_i \neq 0$ and the latter term above has order precisely $p - 2\alpha_i - \beta_i$, in which case q will be determined uniquely modulo $(x - \lambda_i)^{\beta_i}$ by our order condition. Of course, by the Chinese remainder theorem, combining these for all i determines a unique q modulo g_1, giving us our unique kernel map class corresponding to f_g, as desired.

Conversely, if g_1 has support at λ_i, then f_g has to ramify to precisely the required order at λ_i, and no higher: If g_1 is supported at λ_i, we have $\operatorname{ord}_{\lambda_i} g_{12} > \operatorname{ord}_{\lambda_i} g_{22}$, so under our criterion, after column translation, since $\operatorname{ord}_{\lambda_i} g_{21}g_{22} \geq p$, we obtain $\operatorname{ord}_{\lambda_i} g_{12}g_{21} > p$, and the determinant condition then implies that $\operatorname{ord}_{\lambda_i} g_{11}g_{22} = p$, meaning that we cannot have any extra ramification at λ_i, as desired. $\square$

Now, we have already observed that in the case $m = n - \epsilon - m$, we get each kernel map class corresponding to a unique function, modulo automorphism of $\mathbb{P}^1$. In the case $m < n - \epsilon - m$, it is clear from the definition of f_g that transport of a kernel map can change f_g precisely by an inseparable polynomial of degree at most $(n - \epsilon - 2m)p$. Putting this together with the previous propositions, we conclude:

Theorem V.5.7. *If we fix α_i and β_i, the set of classes of kernel maps with vanishing p-curvature, the chosen α_i and β_i, and such that f_g cannot be made constant under transport is in one to one correspondence with the set of rational functions on $\mathbb{P}^1$ of degree $(n-m)p - d - \sum \alpha_i - \sum \beta_i$, ramified to order at least $p - 2\alpha_i - \beta_i$ at each λ_i (with equality whenever $\beta_i > 0$), and further mapping infinity to infinity to order at least $(n - \epsilon - 2m)p$, modulo automorphism of the image space, and modulo the relation that two such rational functions which differ by an inseparable polynomial of degree $\leq (n - \epsilon - 2m)p$ are equivalent.*

In particular, when $m = n - \epsilon - m$, we have no ramification condition at infinity, and no extra relation on our rational functions, so we recover our original assertion that our kernel map classes are in one to one correspondence simply with rational functions with the appropriate ramification at the λ_i, up to automorphism.

Additionally, in the case f_g is inseparable, the ramification away from infinity is automatic, and indeed strictly greater than required, which implies that all the β_i must be 0.

We further show:

Proposition V.5.8. *In the case that $m = n - \epsilon - m$ and $\beta_i = 0$ for all i, the classification of Theorem V.5.7 holds also over $k[\epsilon]$.*

Proof. Suppose we have a connection over $k[\epsilon]$ whose kernel map is described by a matrix $(g_{ij} + \epsilon h_{ij})$, where we continue with the notation of Notation V.5.2 for the kernel map over k given by (g_{ij}), and assume the g_{ij} have been normalized as in Situation V.5.4. We know from Corollary V.2.6 that our kernel matrix must still be formally locally diagonalizable with the same eigenvalues over $k[\epsilon]$, so our observation that α_i was alternatively described as the smaller eigenvalue of the formally locally diagonalized kernel map gives us that each of the h_{ij} must also vanish to order at least α_i at λ_i, and we set $\hat{h}_{ij} := \frac{h_{ij}}{\prod_i (x-\lambda_i)^{\alpha_i}}$. Because we have assumed $\beta_i = 0$, it follows that $\frac{\hat{g}_{12}+\epsilon\hat{h}_{12}}{\hat{g}_{11}+\epsilon\hat{h}_{11}}$ is a deformation of f_g maintaining the same degree. It is easy to check that the fact that Proposition V.3.3 holds over $k[\epsilon]$ allows the same analysis as before to show that our deformation preserves the required ramification, and it is clear that transport still corresponds to postcomposition by an automorphism of $\mathbb{P}^1$.

It therefore remains only to show that given an appropriate deformation of f_g, we can still uniquely produce a corresponding kernel map over $k[\epsilon]$. We therefore suppose we are given α_i for each i, $\hat{g}_{11} + \epsilon\hat{h}_{11}$, and $\hat{g}_{12} + \epsilon\hat{h}_{12}$. We may further suppose that we have $\hat{g}_{21}$ and $\hat{g}_{22}$ satisfying the required determinant, degree, and vanishing conditions modulo ϵ, so we are simply trying to uniquely produce $\hat{h}_{21}, \hat{h}_{22}$ to do likewise over $k[\epsilon]$. We first consider the determinant condition: with $\hat{h}_{21} = \hat{h}_{22} = 0$, the determinant will be off by $\epsilon(\hat{g}_{22}\hat{h}_{11} - \hat{g}_{21}\hat{h}_{12})$ from the desired $\prod_i (x - \lambda_i)^{p-2\alpha_i}$. We see that we want to choose $\hat{h}_{21}, \hat{h}_{22}$ so that we have $\hat{g}_{11}\hat{h}_{22} - \hat{g}_{12}\hat{h}_{21} = \hat{g}_{22}\hat{h}_{11} - \hat{g}_{21}\hat{h}_{12}$, and this will be possible if and only if $g_1 := \gcd(\hat{g}_{11}, \hat{g}_{12}) | (\hat{g}_{22}\hat{h}_{11} - \hat{g}_{21}\hat{h}_{12})$. However, since we have assumed that all $\beta_i = 0$, we have $g_1 = 1$, and may choose $\hat{h}_{21}, \hat{h}_{22}$ to give the desired determinant. Moreover, given any fixed way of filling in the bottom row to give the right determinant, all possible choices (with the same $\hat{g}_{21}, \hat{g}_{22}$) are given precisely as those obtained by adding ϵ-multiples of the top row to the bottom, which gives the desired uniqueness. We can then use the same argument as in the proof of Proposition V.5.6 to force the degrees of $\hat{h}_{21}, \hat{h}_{22}$ to be bounded by $mp + d - \sum_i \alpha_i, (n - \epsilon - m)p + d - \sum_i \alpha_i$ as required. Lastly, we must verify the vanishing condition imposed by Proposition V.3.3; since everything will be multiplied through by $\prod_i (x - \lambda_i)^{\alpha_i}$, it is enough to verify that at each λ_i, after column operation by c_i, we will have $\min\{\operatorname{ord}_{\lambda_i}(\hat{g}_{11} + \epsilon\hat{h}_{11}), \operatorname{ord}_{\lambda_i}(\hat{g}_{21} + \epsilon\hat{h}_{21})\} \geq p - 2\alpha_i$. By hypothesis, since $\operatorname{ord}_{\lambda_i} \hat{g}_{22} = 0$, this will already be satisfied modulo ϵ, and the ramification condition gives precisely $\operatorname{ord}_{\lambda_i}(\hat{h}_{11}) \geq p - 2\alpha_i$, so it remains only to check that $\operatorname{ord}_{\lambda_i}(\hat{h}_{21}) \geq p - \alpha_i$. However, we now see that $\operatorname{ord}_{\lambda_i} \hat{g}_{11}\hat{h}_{22} - \hat{g}_{12}\hat{h}_{21} = \operatorname{ord}_{\lambda_i} \hat{g}_{22}\hat{h}_{11} - \hat{g}_{21}\hat{h}_{12} \geq p - 2\alpha_i$, so since $\operatorname{ord}_{\lambda_i} \hat{g}_{11} \geq p - 2\alpha_i$ and $\operatorname{ord}_{\lambda_i} \hat{g}_{12} = 0$, we get the desired inequality. □

Remark V.5.9. The condition that $\beta_i = 0$ in the above proposition is unnecessary if one is willing to look at g^1_d's rather than maps, and do slightly more analysis of vanishing conditions. However, we will only need the case $\beta_i = 0$.

We are now in a position to give:

Proof of Theorem V.0.1. We begin by noting that the hypothesis that the connections in question do not induce a connection on $\mathscr{O}(d) \subset \mathscr{E}$ is equivalent to f_g being nonconstant and separable, since this is precisely when the upper right coefficient in Equation V.3.2 will be non-zero. To see that this is equivalent to restricting to separable f_g in the "non-constant" case described by Theorem V.5.7, it suffices to observe that if a kernel map is transport equivalent to one with f_g constant, then its f_g is necessarily inseparable. We next note that the degree and ramification conditions imposed in Theorem V.5.7, by the separability of f_g and the Riemann-Hurwitz formula, mean that no additional ramification can occur and the β_i must all be zero. It therefore suffices to show that the only case which can actually occur when the P_i are general is the case $n - \epsilon - m = m$.

Now, suppose that $n - \epsilon - m > m$; since $n = 2d + 2$ is even, we must have $n - \epsilon - 2m \geq 2$, and we see from Riemann-Hurwitz that there are two cases to consider: either the ramification at infinity is exactly $(n - \epsilon - 2m)p$, or it is $(n - \epsilon - 2m)p + 1$. The latter case is impossible for general P_i by Proposition I.4.5, while the former is handled by the remark following that proposition. We conclude that for P_i general, $n - \epsilon - m = m$ is the only case that occurs, as desired. Finally, given this, the previous proposition shows that the classification still holds for first-order infinitesmal deformations. □

V.6 Projective Connections

We review and develop the theory of projective connections, showing that at least on $\mathbb{P}^1$, they may be understood in terms of the theory of standard connections already developed. We assume throughout that the characteristic of our base field does not divide the rank of our vector bundles. We begin with a smooth proper curve C and an etale $\mathbb{P}^{r-1}$ bundle $\mathscr{P}$; this must always be the projectivization of some rank r (Zariski) vector bundle $\mathscr{E}$ on C (see Proposition A.31), which we refer to as a **deprojectivization** of $\mathscr{P}$, and a vector bundle $\mathscr{E}'$ has the same projectivization if and only if $\mathscr{E}' = \mathscr{E} \otimes \mathscr{L}$ for some line bundle $\mathscr{L}$. It follows that we can define a **degree class** of $\mathscr{P}$, well-defined modulo r. We recall the following definitions:

Definition V.6.1. A **projective connection** on $\mathscr{P}$ is a global section of the sheafification of the presheaf associating to an open subset U of C the set of connections on $\mathscr{E}|_U$, modulo the equivalence relation that two connections on U are equivalent if they differ by a scalar multiple of the identity in $\mathcal{E}nd(\mathscr{E}) \otimes \Omega^1_C$. A **rational projective connection** on $\mathscr{P}$ is a section of the same sheaf over some open subset U of C. A **pole** of a rational projective connection is a point P of C over which U cannot be extended, and its order is defined to be the minimum order of poles at P as the non-projective connections in U are allowed to vary within the chosen equivalence class.

It is easily checked that given $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{L}$ for some $\mathscr{L}$, there is a natural induced bijection of projective connections as defined above, so that the definition is in fact independent of $\mathscr{E}$: indeed, given connections on $\mathscr{E}$ on U_1 and U_2 (chosen small enough to trivialize $\mathscr{E}$ and $\mathscr{L}$) which differ (after application of the appropriate transition formula; see (III.1.8)) by a scalar, the transition matrix of $\mathscr{E} \otimes \mathscr{L}$ is simply multiplied by the transition function of $\mathscr{L}$, so the same connections on U_1 and U_2 will differ simply by a different scalar.

Lemma V.6.2. *Let $\mathscr{E}$ be a deprojectivization of $\mathscr{P}$, U any open subset of C trivializing $\mathscr{E}$, and ∇ a rational projective connection on $\mathscr{P}$. Then on U, ∇ can be represented uniquely as a standard rational connection ∇^0 on $\mathscr{E}$ with vanishing trace, and ∇^0 connection will have the correct pole orders everywhere.*

Proof. Choose any representative $\tilde{\nabla}$ for ∇ on U, without regard for extraneous poles. Since we have assumed that the characteristic does not divide the rank, we may define $\nabla^0 =$

$\tilde{\nabla} - \frac{1}{r} \operatorname{Tr} \tilde{\nabla}$. This visibly has vanishing trace, and we note that this must have the correct pole orders everywhere: otherwise there would be some equivalent $\nabla' = \nabla^0 + \omega$ which has a smaller order pole at some point P of C, but then ω would have to have the same order pole at P as ∇^0, and since $\operatorname{Tr} \nabla' = r\omega$, ∇' would have to have the same order pole at P as well, a contradiction. Finally, uniqueness of ∇^0 is obvious, since modifying it by some ω will change the trace by $r\omega$. □

We can also define transport along automorphisms of $\mathscr{E}$ in the obvious way; since $\mathcal{A}ut(\mathscr{E}) = \mathcal{A}ut(\mathscr{E} \otimes \mathscr{L})$ for any line bundle $\mathscr{L}$, this will make sense on the level of projective connections, independent of choice of $\mathscr{E}$. We will also want to make an independent definition of vanishing p-curvature, for which it will be necessary to know:

Lemma V.6.3. *Let ∇ be any connection on any open set U, and suppose ∇' is a connection on U with $\nabla - \nabla' = \omega \mathrm{I}$ a scalar endomorphism. Then if θ is any derivation on U, we have*

$$\psi_\nabla(\theta) - \psi_{\nabla'}(\theta) = ((\hat{\theta}(\omega))^p + \theta^{p-1}(\hat{\theta}(\omega)) - f_{\theta^p}\hat{\theta}(\omega))\,\mathrm{I}.$$

Proof. We make use of the explicit formulas of Propositions III.2.6 and III.2.7. Indeed, we can compare the p-curvatures of ∇ and ∇' term by term; if $\bar{T}$ is a matrix for ∇' on U with respect to a derivation θ, we have $\bar{T} + \hat{\theta}(\omega)\,\mathrm{I}$ as the matrix for ∇, and we see that if we expand each term of $\psi_\nabla(\theta)$, we get $\psi_{\nabla'}(\theta)$ from expanding out only terms involving $\bar{T}$, and $((\hat{\theta}(\omega))^p + \theta^{p-1}(\hat{\theta}(\omega)) - f_{\theta^p}\hat{\theta}(\omega))\,\mathrm{I}$ from expanding out only terms involving $\hat{\theta}(\omega)$, since these last all commute with one another, and we therefore see that the argument of Corollary III.2.8 still works, since the rank was used only to insure that the terms all commuted. We thus want to show that all of the coefficients of the cross terms are always zero mod p.

If we consider a particular term $(\theta_0^{i_1-1}(\bar{T} + \hat{\theta}(\omega)\,\mathrm{I})) \dots (\theta_0^{i_\ell-1}(\bar{T} + \hat{\theta}(\omega)\,\mathrm{I}))$ corresponding to a vector $\mathfrak{i}$, a cross term will arise by choosing a subset $\Lambda \subset \{1, \dots, \ell\}$ from which the $\bar{T}$ term will be chosen, with the $\hat{\theta}(\omega)\,\mathrm{I}$ term being chosen for all indices outside Λ. Since these last are scalar matrices and commute with all other matrices, to compute the relevant coefficient we can essentially sum over all permutations in S_ℓ^Λ as in Proposition III.2.7. The only caveat is that we will end up counting some terms more than once this way. Specifically, if $\sigma \in S_\ell^\Lambda$ fixes Λ and, when applied to the vector $\mathfrak{i}$, leaves the vector unchanged, then it will give the same term in the expansion as the identity. Such σ form a subgroup of S_ℓ, and we see that precomposition by this subgroup described precisely which elements of S_ℓ^Λ

give the same terms in the expansion, so if we denote the order of this subgroup by $P_{\mathfrak{i}}^{\Lambda}$, we find that the coefficient we want to compute is given by, still in the notation of Proposition III.2.7, the expression $n_{\mathfrak{i}}^{\Lambda}/P_{\mathfrak{i}}^{\Lambda}$. Now, since there is an $n!$ in the numerator of $n_{\mathfrak{i}}^{\Lambda}$, and since in our case $n = p$, the only way that our coefficient could be nonzero would be if either $P_{\mathfrak{i}}^{\Lambda}$ or the denominator of $n_{\mathfrak{i}}^{\Lambda}$ were also divisible by p. Now, the denominator of $n_{\mathfrak{i}}^{\Lambda}$ cannot be divisible by p, since the i_j add up to p, and the only way that p could appear in the denominator would therefore be when Λ is all of $\{1, \dots, \ell\}$, which corresponds to the terms which only involve $\bar{T}$, or when $\ell = 1$, which gives the $\theta^{p-1}(\hat{\theta}(\omega))$ term, and both of these have already been dealt with. Now, $P_{\mathfrak{i}}^{\Lambda}$ is the order of a subgroup of S_ℓ which fixes Λ, so may be considered as a subgroup of $S_{\ell-|\Lambda|}$, and can be a multiple of p only if $\ell = p$ and $|\Lambda| = 0$, which corresponds to the term $((\hat{\theta}(\omega))^p$ and has likewise already been taken into account. We can conclude finally that all the cross terms vanish, as desired. □

We now see:

Lemma V.6.4. *Continuing with the notation of Lemma V.6.2, if any representative of* ∇ *on* U *has vanishing* p*-curvature, then so must* ∇^0.

Proof. We know by Proposition A.30 that for any connection ∇', $\operatorname{Tr} \nabla' = 0$ implies that $\operatorname{Tr} \psi_{\nabla'} = 0$. So, suppose $\nabla' = \nabla^0 + \omega$ for some ω, and has vanishing p-curvature. Let θ be the derivation (possibly on a smaller open subset of U); by the previous lemma, we have $\psi_{\nabla^0}(\theta) = \psi_{\nabla^0}(\theta) - \psi_{\nabla'}(\theta)$ is a scalar matrix, with vanishing trace, and since r is prime to p, it must be zero, as desired. □

With this lemma, we can now make the following definition:

Definition V.6.5. We say that a projective connection has **vanishing** p**-curvature** if its induced connection with vanishing trace on some open subset U does, or equivalently, if the induced connection with vanishing trace on every open subset U does.

We now specialize to the case of $C = \mathbb{P}^1$, and $r = 2$. In this case, we will refer to $\mathscr{P}$ being **even** or **odd** as dictated by its degree class. We immediately see that by appropriate choice of deprojectivization, we get:

Lemma V.6.6. *In the case that* $C = \mathbb{P}^1$ *and* $r = 2$, *we have a uniquely determined* d *such that* $\mathscr{P} \cong \mathbb{P}(\mathscr{E}_{\mathscr{P}})$ *for* $\mathscr{E}_{\mathscr{P}} := \mathscr{O}(d + \epsilon p) \oplus \mathscr{O}(-d))$, *with* ϵ *equal to the degree class of* $\mathscr{P}$.

We can classify projective connections in terms of standard connections:

Proposition V.6.7. *Let $C = \mathbb{P}^1$ and $r = 2$. Then projective connections on $\mathscr{P}$ with vanishing p-curvature and simple poles at points $P_1, \dots, P_n$ are in one-to-one correspondence with logarithmic connections on the $\mathscr{E}_{\mathscr{P}}$ of the preceding lemma, with poles at the P_i, trivial determinant, and vanishing p-curvature. This correspondence descends naturally to transport-equivalence classes.*

Proof. It is clear that the described connections on $\mathscr{E}_{\mathscr{P}}$ give connections on $\mathscr{P}$. Conversely, given a connection ∇ on $\mathscr{P}$, choose U open, but small enough so that ∇ may be represented by a ∇^0 on U with vanishing trace, simple poles at the P_i inside U, and no other poles on U. Given any other sufficiently small open set U', since the determinant of $\mathscr{E}_{\mathscr{P}}$ is a Frobenius pullback, we can write the transition matrix from U to U' to have determinant with vanishing differential, so the same connection after transition to U' still has vanishing trace (this is in essence the trace formula of Proposition V.1.4), and by Lemma V.6.2 and the hypothesis that P_i are the only poles of ∇ as a projective connection, must be regular away from the P_i, with simple poles at the P_i; since U' was arbitrary, we conclude that ∇^0 is in fact a connection as required on $\mathscr{E}_{\mathscr{P}}$.

Finally, it is clear from the definition of transport of a projective connection that the constructed correspondence is compatible with transport, and hence descends to transport equivalence classes. $\square$

V.7 Mochizuki's Work and Backwards Solutions

Upon discovering the classification of logarithmic connections on $\mathbb{P}^1$ with vanishing p-curvature in terms of self-maps of $\mathbb{P}^1$ with prescribed ramification developed in this chapter, the original intent was to study such maps directly, and then apply the results to obtain conclusions on connections on $\mathbb{P}^1$, and ultimately on curves of higher genus. Ultimately, the results of this thesis did in fact fall into this form, but the actual path of research, and in particular the motivation for a key, otherwise extremely obscure, result along the way, was not nearly so straightforward. Indeed, after some study of the problem of self-maps of $\mathbb{P}^1$, addressed in Chapter I, the author determined that the problem was subtler than initially expected, and after some work, reduced it down to being able to control the degeneration of separable maps to inseparable maps, ultimately addressed in Section I.5. The author was, however, completely unable to approach this problem via any of the standard techniques.

Some time later, the author was informed of Mochizuki's work on connections on curves in [42]. In particular, upon translating from his language for projective line bundles to our situation with vector bundles of rank 2, the theorem [42, II, Thm. 2.8, p. 153], applied to logarithmic connections with vanishing p-curvature on $\mathbb{P}^1$ with marked points, asserts that the stack of those connections classified by Theorem V.5.7 is finite flat over $\bar{\mathcal{M}}_{0,n}$, and in particular, that the number of such connections (counted with multiplicity) shouldn't depend on the choice of marked points. The same theorem of Mochizuki includes etaleness results implying that the stack of such connections is reduced at a totally degenerate curve, and hence for a general choice of marked points for any fixed genus-0 configuration defining a stratum of the boundary of $\bar{\mathcal{M}}_{0,n}$. This suggested a roundabout way of studying self-maps of $\mathbb{P}^1$ with prescribed ramification: rather than degenerating the maps themselves as was ultimately carried out in Chapter I, one could translate to connections, use Mochizuki's generic etaleness results to degenerate to connections on reducible rational curves, and then translate back to maps to obtain the same result.

This approach avoided the issue of separable maps degenerating to inseparable maps, but it was rather unsatisfying in its roundabout nature. The breakthrough was ultimately provided by a simple observation: by Mochizuki's work, the stack of connections in question is finite flat; however, on the maps side, separable maps can certainly degenerate to inseparable maps, whereupon they no longer correspond to connections. On the other hand, there were also the "asymmetric" connections that only existed for special configurations of marked points, corresponding to infinite families of maps with additional ramification at infinity, as described in Theorem V.5.7. In this context, the answer became clear: a family of connections degenerating in such a way that it corresponds generically to a separable family of maps which become inseparable in the limit must, in the limit, give an "asymmetric" connection, which we had already seen can only exist for special configurations of marked points, precisely because it corresponds to a certain infinite family of maps which can only exist for special configurations. With this realization, it was not overly difficult to write out some explicit examples of connections degenerating in such a way, to write down the corresponding families of maps, and to ultimately determine the explicit translation process from a family of separable maps degenerating to an inseparable map, to the appropriate infinite family of maps in the limit, yielding finally the direct arguments of Section I.5.

Remark V.7.1. We see in particular that the relationship between connections and maps really is more complicated in the case of more than three poles/ramification points, and one cannot hope to treat it as generally as Mochizuki treated the three-point case; specifically, we see that connections with $m \neq n - \epsilon - m$, which is to say those corresponding to maps with additional ramification at infinity will deform, as the poles move, to connections with $m = n - \epsilon - m$, which are lower-degree maps. This is essentially tied to the fact that the classification of locally free sheaves on $\mathbb{P}^1$ used in an essential way in our argument only holds over a field.

Remark V.7.2. In fact, there is still one significant result on self-maps of $\mathbb{P}^1$ which as of yet is only accessible via translation to connections. Specifically, for any specified degree and ramification indices such that the expected number of maps is finite, if all ramification indices are odd and less than p, one can use Theorem V.5.7 and Proposition V.6.7 (although some additional translation of conditions is still necessary) to classify such maps in terms of connections which in Mochizuki's language are "dormant indigenous bundles", and application of his finiteness result then implies that there are finitely many such maps, without requiring the ramification points to be general.

Chapter VI

Gluing to Nodal Curves and Deforming to Smooth Curves

The purpose of this chapter is to use largely standard gluing and deformation techniques in order to apply Chapters I, III, and V to obtain a new proof of a result of Mochizuki describing the locus of Frobenius-unstable vector bundles of rank 2 and trivial determinant on a general curve of genus 2. We then invoke the main theorem of Chapter IV to compute the degree of the associated Verschiebung rational map, as well as certain additional properties. Specifically, we show:

Theorem VI.0.1. *Let C be a general smooth, proper curve of genus 2, and $M_2(C)$ the coarse moduli space of semistable vector bundles of rank 2 and trivial determinant on C, and $V_2 : M_2(C^{(p)}) \dashrightarrow M_2(C)$ the rational Verschiebung map induced via pullback under the relative Frobenius morphism $F : C \to C^{(p)}$. Then:*

(i) There are $\frac{2}{3}p(p^2-1)$ Frobenius-unstable bundles of rank 2 and trivial determinant on C, all without non-trivial deformations (in the sense of "reducedness" in Theorem IV.0.1);

(ii) The undefined points of V_2 are precisely the Frobenius-unstable bundles, and each may be resolved by a single blowup at the reduced point. The degree of V_2 is given by $\frac{1}{3}p(p^2+2)$, and the image of the exceptional divisor above a Frobenius-unstable bundle $\mathscr{F}$ is given by $\operatorname{Ext}(\mathscr{L}, \mathscr{L}^{-1})$ where $\mathscr{L}$ is the theta characteristic destabilizing $F^*\mathscr{F}$.

The basic approach is to first glue the results of Chapter V to obtain results on a general rational nodal curve of genus 2, and then use deformation theory to obtain the corresponding results for a general smooth curve of genus 2. More precisely, we begin in Section VI.1 with a brief discussion of connections on nodal curves, and examine the gluing in Section VI.2; we carry out an auxiliary computation for the deformation theory in Section VI.3, and the deformation theory itself in Section VI.4, finally putting all the results together in Section VI.5 to conclude our main theorem. Both the gluing and deformation steps are special cases of results of Mochizuki [42]; the gluing is a special case of the statement on "torally indigenous bundles" on [42, p. 118], while the deformation result follows, for instance, from the finite flatness statement for "dormant torally indigenous bundles" in the $n = 0$ case of [42, Thm. II.2.8, p. 153]. While the results of this chapter are thus technically superfluous, the proofs are not long, and it seems desireable to have a less abstract, completely self-contained proof of Theorem VI.0.1, particularly given that our overall approach is somewhat different in that it involves degeneration to irreducible nodal curves rather than totally degenerate curves. Furthermore, there are a few distinctions worth pointing out: for the gluing statement, we provide some additional details, and in fact have superficially different behavior because we are working with vector bundles rather than Mochizuki's projective bundles; for the deformation theory, Mochizuki makes use of De Rham cohomology, whereas our perspective is substantially more naive, making use rather of the perspective and results of Chapter V.

It is perhaps also worth a brief comparison of Mochizuki's overall strategy with that carried out here. In Mochizuki, the proof of Theorem VI.0.1 (i) follows from three facts: first, that on each "atom" (that is, $\mathbb{P}^1$ with three marked points) certain choices of "radii" at the three marked points give a unique "dormant torally indigeneous bundle", which yields that the number of "dormant atoms" is $\frac{1}{24}p(p^2 - 1)$ [42, §V.1, p. 232, and Cor. V.3.7, p. 267]; second, that gluing torally indigenous bundles simply requires that the radii agree at marked points [42, p. 118], so that if one takes the totally degenerate curve of genus 2 obtained by gluing together two atoms along the three marked points, the number of dormant torally indigenous bundles on this curve is equal to the number on either atom alone, which is simply the number of dormant atoms by definition; finally, the stack of dormant indigenous bundles is finite flat over the stack of curves, and etale at a totally degenerate curve [42, Thm. II.2.8, p. 153], so that the $\frac{1}{24}p(p^2-1)$ dormant torally indigenous bundles we obtain for the degenerate curve deform to a general smooth curve, and have no non-trivial deformations (over the fixed base curve) in either case. Here, the fundamental idea is similar, with the main difference being that we degenerate to an irreducible rational nodal curve rather than a reducible one. This has the disadvantage that the relevant connections are harder to analyze in this case, requiring the full results of Chapter I rather than the easier three-point case. However, if one were to generalize to higher genus, there is a natural notion of "level" which measures how unstable a bundle is; indigenous bundles are those of maximal level, and on totally degenerate curves, all bundles with positive level are necessarily indigenous. However, on irreducible nodal curves, one can have bundles of intermediate level, possibly providing a more direct tool for studying bundles of intermediate level than the indigenization construction of Mochizuki's theory [42, §II.1.4].

We should perhaps also remark that the approach to the gluing theory of Section VI.2 follows Mochizuki's ideas in order to give a cleaner presentation than the approach originally envisioned. Aside from the role of Mochizuki's results discussed in Section V.7, this is the only part of the present work not developed fully independently of Mochizuki.

VI.1 Connections on Nodal Curves

Let C be a proper nodal curve, and $\mathscr{E}$ a vector bundle on C. We define:

Definition VI.1.1. A **logarithmic connection** on $\mathscr{E}$ is a k-linear map $\nabla : \mathscr{E} \to \mathscr{E} \otimes \omega_C$, where ω_C is the dualizing sheaf on C, and ∇ satisfies the Liebnitz rule induced by the canonical map $\Omega^1_C \to \omega_C$.

Remark VI.1.2. The above terminology conflicts slightly with that of Chapter V, where a logarithmic connection was defined to be a connection allowed to have simple poles. They are indeed both special cases of a more general theory, and one should more properly specify that the former is logarithmic with respect to the log structure induced by the nodes, while the latter is logarithmic with respect to the log structure given by the divisor along which the connection has poles. In our specific context, this would be undesirably unwieldy, but to avoid confusion in this chapter, we will refer to the latter as a D-logarithmic connection, where D is a reduced divisor supported on the smooth locus of a curve, and the connection is allowed to have simple poles along D.

We note that all the background material on connections in Section III.1 up to (but not including) the Cartier isomorphism still holds if one replaces Ω^1_C by ω_C (and in particular, the sheaf of derivations by $\omega_C^\vee$) throughout. We summarize:

Proposition VI.1.3. *All statements on induced connections for operations of vector bundles, and statements (III.1.1) and (III.1.2) on the p-curvature map, hold in the case of logarithmic connections on nodal curves, with ω_C in place of Ω^1_C. One still has a canonical connection on a Frobenius pullback with vanishing p-curvature whose kernel recovers the original sheaf.*

Although it is true that taking the kernel of the canonical connection of a Frobenius pullback still recovers the original sheaf on $C^{(p)}$ when C is singular, the Cartier isomorphism fails because given a logarithmic connection with vanishing p-curvature on C, the Frobenius pullback of the kernel will not in general map surjectively onto the original sheaf at the singularities of C.

VI.2 Gluing Connections and Underlying Bundles

Let $\tilde{C}$ be the normalization of C, and $\tilde{\mathscr{E}}$ the pullback of $\mathscr{E}$ to $\tilde{C}$. Given a logarithmic connection ∇ on $\mathscr{E}$, we get a D_C-logarithmic connection on $\tilde{\mathscr{E}}$, where D_C is the divisor of points lying above the nodes of C. We want a complete description of connections on $\tilde{\mathscr{E}}$ arising this way, and a correspondence between these and connections on $\mathscr{E}$. We claim:

Proposition VI.2.1. *Logarithmic connections ∇ on $\mathscr{E}$ are equivalent to connections on $\tilde{\mathscr{E}}$ having simple poles at the points $P_1, Q_1, \dots, P_\delta, Q_\delta$ lying above the nodes of C, and such that under the gluing maps $G_i : \tilde{\mathscr{E}}|_{P_i} \to \tilde{\mathscr{E}}|_{Q_i}$ giving $\mathscr{E}$, for each i we have $\mathrm{Res}_{P_i}(\nabla) = -G_i^{-1} \circ \mathrm{Res}_{Q_i}(\nabla) \circ G_i$. The properties of having trivial determinant and vanishing p-curvature are preserved under this correspondence.*

Proof. The main assertion follows easily from [8, Thm. 5.2.3] together with the remark [8, p. 226] for nodal curves, which together state that sections of ω_C correspond to sections of $\Omega^1_{\tilde{C}}(D_C)$ with residues at the pair of points above any given node adding to zero.

Since vanishing p-curvature can be verified on open sets, and the normalization map is an isomorphism, it is clear that logarithmic connections with vanishing p-curvature on C will correspond to logarithmic connections with vanishing p-curvature on $\tilde{C}$. The same may be said of trivial determinant, with the additional trivial observation that on the open subset on which the normalization is an isomorphism, the differential map d is the same on C and $\tilde{C}$. □

We can in particular conclude:

Corollary VI.2.2. *Let $\mathscr{L}$ be a line bundle on C. Then $\mathscr{L}$ can have a logarithmic connection ∇ with vanishing p-curvature only if $p | \deg \tilde{\mathscr{L}}$.*

Proof. Applying the previous proposition, if we pull back to $\tilde{\nabla}$ on $\tilde{\mathscr{L}}$ we find that the residues of $\tilde{\nabla}$ come in additive inverse pairs mod p. We obviously have $p|\mathscr{F}^*(\tilde{\mathscr{L}}^{\tilde{\nabla}})$, and then by Corollary V.1.12 we have that the determinant of the inclusion map $\mathscr{F}^*(\tilde{\mathscr{L}}^{\tilde{\nabla}}) \hookrightarrow \tilde{\mathscr{L}}$ has total order equal to the sum of the residues mod p, which is zero, so we conclude that $\deg \tilde{\mathscr{L}}$ must also vanish mod p, as asserted. □

We now restrict to the situation:

Situation VI.2.3. Suppose that $\mathscr{E}$ has rank 2 and trivial determinant, and we have fixed an exact sequence

$$0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{L}^{-1} \to 0.$$

The same statements then hold for $\tilde{\mathscr{E}}$.

We introduce some terminology in this situation:

Definition VI.2.4. Given a logarithmic connection ∇ on $\mathscr{E}$ (respectively, a D-logarithmic connection ∇ on $\tilde{\mathscr{E}}$), the **Kodaira-Spencer map** associated to ∇ and a sub-line-bundle $\mathscr{L}$ (respectively, $\tilde{\mathscr{L}}$) is the natural map $\mathscr{L} \to \mathscr{L}^{-1} \otimes \omega_C$ (respectively, $\tilde{\mathscr{L}} \to \tilde{\mathscr{L}}^{-1} \otimes \Omega^1_{\tilde{C}}(D)$) obtained by composing the inclusion map, ∇, and the quotient map (tensored with the identity). One verifies directly that this is a linear map.

Note that with this terminology, our initial observation giving Proposition III.3.2 boils down to the statement that the Frobenius-pullback of a Frobenius-unstable bundle necessarily has a connection such that the Kodaira-Spencer map of the destabilizing line bundle is an isomorphism. It should perhaps therefore not be surprising that we will consider connections for which the Kodaira-Spencer is an isomorphism. We note:

Lemma VI.2.5. *Suppose that the arithmic genus of C (respectively, the genus of $\tilde{C}$ plus $\frac{\deg D}{2}$) is greater than or equal to $3/2$; that is to say, we are in the "hyperbolic" case. Then* *if the Kodaira-Spencer map associated to* $\nabla, \mathscr{L}$ *is an isomorphism for any* ∇, *then* $\mathscr{L}$ *is a destabilizing line bundle for* $\mathscr{E}$ *(respectively,* $\tilde{\mathscr{E}}$*), and is thus uniquely determined even independent of* ∇.

Proof. In the case of C, we have $\mathscr{L} \cong \mathscr{L}^{-1} \otimes \omega_C$, so $\mathscr{L}^{\otimes 2} \cong \omega_C$, which by the hypothesis on the arithmetic genus has positive degree (see for instance Theorem A.4). Thus, $\mathscr{L}$ is a destabilizing line bundle for $\mathscr{E}$, and we see that the proof of Lemma III.3.3 proceeds unmodified in this case, making use of the fact that we can define a degree on line bundles on a nodal curve simply by pulling back to the normalization, and noting that the properties necessary from the argument will follow formally.

In the case of $\tilde{C}$, we similarly have $\mathscr{L}^{\otimes 2} \cong \Omega^1_{\tilde{C}}(D)$, which is positive by the hypothesis on the genus of $\tilde{C}$ and degree of D. Again, $\mathscr{L}$ is a destabilizing line bundle, and we can apply Lemma III.3.3 directly in this case to obtain the desired result. □

We will see in the following arguments that it sometimes appears that we should be allowing the line bundle $\mathscr{L}$ to vary with the connection; however, as a result of the above lemma, since we will be restricting our attention in the subsequent results to connections whose Kodaira-Spencer maps are isomorphisms for $\mathscr{L}$, the fact that we have fixed $\mathscr{L}$ in our situation will not pose any problems, and the Kodaira-Spencer maps in question will depend only on the connections.

One can approach the issue of gluing connections from two perspectives: either fixing the glued bundle $\mathscr{E}$ on C, and exploring which connections on $\tilde{\mathscr{E}}$ will glue to yield connections on $\mathscr{E}$, or allowing the gluing of $\mathscr{E}$ itself to vary as well. The first approach may appear better-suited to our goals, since we ultimately wish to classify the connections on a particular unstable bundle on a nodal curve, and indeed one may carry through the calculation directly via this method in our situation. However, the second approach offers a more transparent view of the more general setting, particularly as far as issues of transport equivalence are concerned, and ultimately yields a cleaner argument even for our specific application, so following Mochizuki [42, p. 118], we will take the approach of allowing our gluings to vary as well. As such, we now fix $\tilde{\mathscr{E}}$ on $\tilde{C}$, but do not assume a fixed gluing $\mathscr{E}$ on C. That is to say:

Situation VI.2.6. Fix $\tilde{\mathscr{E}}$ of rank 2 and trivial determinant, together with an exact sequence

$$0 \to \tilde{\mathscr{L}} \to \tilde{\mathscr{E}} \to \tilde{\mathscr{L}}^{-1} \to 0.$$

The main statement on gluing is:

Proposition VI.2.7. *In Situation VI.2.6, let $\tilde{\nabla}$ be a D_C-logarithmic connection on $\tilde{C}$ with trivial determinant and vanishing p-curvature, such that the Kodaira-Spencer map associated to $\tilde{\mathscr{L}}$ is an isomorphism. Further suppose that the e_1, e_2 of Corollary V.1.10 match one another (up to permutation) for pairs of points lying above given nodes of C. Then if one fixes a gluing $\mathscr{L}$ of $\tilde{\mathscr{L}}$, there is a unique gluing of $(\tilde{\mathscr{E}}, \tilde{\nabla})$ to a pair $(\mathscr{E}, \nabla)$ on C, such that one obtains a sequence*

$$0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{L}^{-1} \to 0,$$

and the resulting $(\mathscr{E}, \nabla)$ will also have Kodaira-Spencer map an isomorphism. If C has arithmetic genus at least 2, transport equivalence is preserved under this correspondence.

Proof. We first claim that the condition that the Kodaira-Spencer map for $\tilde{\mathscr{L}}$ be an isomorphism implies that for any $P \in \{P_i, Q_i\}$, $\tilde{\mathscr{L}}|_P$ is not contained in an eigenspace of $\mathrm{Res}_P \tilde{\nabla}$, and that the eigenvalues are both non-zero. But because of the triviality of the determinant, the sum of the eigenvalues is zero, so because the residue matrices are diagonalizable (see Corollary V.1.12, noting that it does not use the e_i non-zero hypothesis of Situation V.1.11), the latter assertion is actually a consequence of the former. Now, considering the definition of the Kodaira-Spencer map $\tilde{\mathscr{L}} \to \tilde{\mathscr{L}}^{-1} \otimes \Omega^1_{\tilde{C}}(D_C)$, if we restrict to P we get a map which is clearly equal to zero if and only if $\nabla(\tilde{\mathscr{L}})|_P \subset \tilde{\mathscr{L}} \otimes \Omega^1_{\tilde{C}}|_P$, which is the case precisely when $\tilde{\mathscr{L}}|_P$ is contained in an eigenspace of $\mathrm{Res}_P \tilde{\nabla}$, as desired.

Given this, for each pair P_i, Q_i, Proposition VI.2.1 and our hypothesis on the matching eigenvalues of the residue matrices at P_i, Q_i imply that in order to glue the connection, it is necessary and sufficient to map eigenspaces of opposing sign to each other. To glue $\tilde{\mathscr{L}}$, we also map its image at P_i to its image at Q_i. We thus see that the two eigenspaces of $\mathrm{Res}_{P_i} \tilde{\nabla}$ and $\mathrm{Res}_{Q_i} \tilde{\nabla}$ and the images of $\tilde{\mathscr{L}}$ form a set of three one-dimensional subspaces which must be matched under G_i, and it is easy to see that this determines G_i up to scaling. But finally, scaling of G_i is equivalent to scaling the induced gluing map on $\tilde{\mathscr{L}}$, which is precisely what determines the isomorphism class of the glued $\mathscr{L}$; thus, $\mathscr{L}$ may be specified arbitrarily, and given a choice of $\mathscr{L}$, the G_i and hence the pair $(\mathscr{E}, \nabla)$ are uniquely determined, as desired.

Finally, it is trivial that if two connections on $\mathscr{E}$ are transport-equivalent, then their pullbacks to $\tilde{\mathscr{E}}$ are, and for the converse, the uniqueness of the gluing makes it clear that if two connections $\tilde{\nabla}$ and $\tilde{\nabla}'$ on $\tilde{\mathscr{E}}$ are transport-equivalent under an automorphism φ of $\tilde{\mathscr{E}}$, then φ naturally gives an isomorphism of the two gluings $\mathscr{E}$ and $\mathscr{E}'$ which takes ∇ to ∇'. Finally, the hypothesis that the arithmetic genus of C is at least 2 implies, as discussed in the previous lemma, that $\mathscr{L}$ and $\tilde{\mathscr{L}}$ are uniquely determined as the destabilizing sub-bundles of $\mathscr{E}$ and $\tilde{\mathscr{E}}$, so there is no concern that they might change under transport. $\square$

Putting together the previous propositions, we finally conclude:

Corollary VI.2.8. *Let $\tilde{\mathscr{E}}$ be a vector bundle on $\tilde{C}$ of rank 2, with the arithmetic genus of C being at least 2, and suppose there exists an exact sequence*

$$0 \to \tilde{\mathscr{L}} \to \tilde{\mathscr{E}} \to \tilde{\mathscr{L}}^{-1} \to 0.$$

Fix a gluing of $\tilde{\mathscr{L}}$ to a line bundle $\mathscr{L}$ on C; then there exists a bijective equivalence between transport-equivalence classes of D_C-logarithmic connections $\tilde{\nabla}$ on $\tilde{\mathscr{E}}$ with trivial determinant and vanishing p-curvature, the e_i of Corollary V.1.10 matching at the pairs of points above each node, and having the Kodaira-Spencer map an isomorphism on one side, and on the other side, pairs $(\mathscr{E}, \nabla)$ of gluings of $\mathscr{E}$ preserving an exact sequence

$$0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{L}^{-1} \to 0,$$

together with logarithmic connections ∇ on $\mathscr{E}$ with vanishing p-curvature and trivial determinant and having the Kodaira-Spencer map an isomorphism, up to isomorphism/transport equivalence.

Further, this correspondence holds for first-order infinitesmal deformations.

Proof. We can immediately conclude the statement over a field from our previous propositions. For first-order deformations, the same arguments will go through, with the aid of the following facts: first and most substantively, it follows from Corollary V.2.6 that the residue matrices on $\tilde{C}$ will still be diagonalizable, albeit over $k[\epsilon]/\epsilon^2$, with the eigenvalues e_i the same as for the connection being deformed. Next, since we are simply taking a base change of our original situation over k, the general gluing description given by Proposition VI.2.1

still holds for formal reasons. Finally, one can easily verify that even over an arbitrary ring, we still have the assertion we used that an automorphism of a rank two free module is determined uniquely by sending any three pairwise independent lines to any other three. We therefore conclude the desired statement for first-order deformations as well. □

Remark VI.2.9. The statement of Proposition VI.2.7 is slightly cleaner in Mochizuki's setting [42, p. 118], because in the context of projective bundles, the choice of $\mathscr{L}$ made in the statement is unnecessary, since the isomorphism class of $\mathbb{P}(\mathscr{E})$ will not depend on which $\mathscr{L}$ was chosen. He therefore obtains a simpler uniqueness of gluing statement, while we will have to rigidify our situation by specifying $\mathscr{L}$. Since our ultimate goal is to classify connections on a particular specified $\mathscr{E}$, however, this will not pose any obstacle.

VI.3 An Auxiliary Computation

In this section, we perform an auxiliary computation using the techniques of Chapter V and the previous section, which will be necessary to our final goal of deforming connections from a nodal curve to a smooth curve, carried out in the following section.

Situation VI.3.1. We suppose that C is a general irreducible, rational proper curve with two nodes, $\tilde{C} \cong \mathbb{P}^1$ its normalization, with P_1, Q_1, P_2, Q_2 being the points lying above the two nodes. We let $\mathscr{E}$ be the vector bundle described by Situation III.5.1, and ∇ a logarithmic connection on $\mathscr{E}$ with trivial determinant and vanishing p-curvature.

Remark VI.3.2. We do not expect that generality of C should be necessary for the result of this section to hold. However, we will only apply our result in the case that C is general, and the hypothesis simplifies our argument considerably.

We observe that in this situation the normalization $\tilde{\mathscr{E}}$ of $\mathscr{E}$ is isomorphic to $\mathscr{O}(1)\oplus\mathscr{O}(-1)$: we certainly have $\tilde{\mathscr{L}} \cong \mathscr{O}(1)$, so by Lemma III.3.3, $\mathscr{O}(1)$ is the maximal line bundle in $\tilde{\mathscr{E}}$, and then the desired splitting follows from [27, Proof of Thm. 1.3.1]. Also, by Proposition VI.2.1 $\tilde{\nabla}$ is a D_C-logarithmic connection on $\tilde{\mathscr{E}}$ with trivial determinant and vanishing p-curvature. We wish to show:

Proposition VI.3.3. *The space of sections of $\mathcal{E}nd^0(\mathscr{E}) \otimes F^*\omega_{C^{(p)}}$ horizontal with respect to the connection ∇^{ind} induced by ∇ on $\mathscr{E}$ and ∇^{can} on $F^*\omega_{C^{(p)}}$ has dimension equal to three.*

Proof. Our method will be to first carry out the computation on $\tilde{C}$, and then impose the necessary gluing conditions to obtain the desired result. Noting that $F^*\tilde{\omega}_{C^{(p)}} \cong \mathscr{O}(2p)$, and also that the proof of Theorem V.0.1 in Section V.5 gives by the generality of C that the kernel of $\tilde{\nabla}$ is isomorphic to $\mathscr{O}(-2) \oplus \mathscr{O}(-2)$, we claim that an endomorphism as in the statement is equivalent to a diagram:

$$\begin{tikzcd}
\mathscr{O}(1) \oplus \mathscr{O}(-1) \arrow[r, "B"] & \mathscr{O}(2p+1) \oplus \mathscr{O}(2p-1) \\
\mathscr{O}(-2p) \oplus \mathscr{O}(-2p) \arrow[u, "S"] \arrow[r, "A"] & \mathscr{O} \oplus \mathscr{O} \arrow[u, "S"']
\end{tikzcd}$$

where A is the Frobenius pullback of a map $\mathscr{O}(-2)\oplus\mathscr{O}(-2) \to \mathscr{O}\oplus\mathscr{O}$, B has vanishing trace, and S is the kernel inclusion map as in Chapter V. Indeed, it is easy to see that the induced kernel map $\mathscr{O} \oplus \mathscr{O} \to \mathscr{O}(2p+1) \oplus \mathscr{O}(2p-1)$ will still be S, and that for the map given by B to be in the kernel of the induced connection on $\mathcal{H}om(\mathscr{O}(1) \oplus \mathscr{O}(-1), \mathscr{O}(2p+1) \oplus \mathscr{O}(2p-1))$ is equivalent to mapping the kernel of ∇ on $\mathscr{O}(1) \oplus \mathscr{O}(-1)$ into the kernel of the induced connection on $\mathscr{O}(2p+1) \oplus \mathscr{O}(2p-1)$.

We will therefore work within the space of possibilities for A; this lies *a priori* in

$$F^* \begin{bmatrix} \mathscr{O}(2) & \mathscr{O}(2) \\ \mathscr{O}(2) & \mathscr{O}(2) \end{bmatrix},$$

and we note that since $B = SAS^{-1}$, the trace of B vanishes if and only if the trace of A vanishes. We are therefore simply looking for matrices A with vanishing trace such that SAS^{-1} is regular, at least prior to considering the gluing conditions to get from $\mathbb{P}^1$ to the nodal curve. The determinant of S vanishes only at the four points of D_C, vanishing to order p at those points, so it suffices to analyze the situation formally locally at those points. More precisely, for P any point of $\{P_i, Q_i\}$, if we argue as for the mirror criterion of Proposition V.3.3, which is to say by conjugating formally locally by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we can apply Corollary V.1.9 to obtain a constant c and an invertible matrix M such that $MSU(c) = D(t^\alpha, t^{p-\alpha})$, where M is invertible at P, $D(a_1, a_2) := \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$, and $U(c) := \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$. Further, c is the image of the ramification point P of the non-constant map associated to S in Theorem V.0.1, so in particular c is well-defined, and see as a result that $\alpha > p/2$. Further, by the generality of C, we have from part (ii) of Theorem I.2.3 that the c obtained for the different P are distinct. Now, since $U(c)U(-c) = \mathrm{I}$, we have

$$D(t^{\alpha}, t^{p-\alpha})U(-c)AU(c)D(t^{-\alpha}, t^{\alpha-p}) = MSAS^{-1}M^{-1}$$

and since M is invertible at P, this is regular at P if and only if SAS^{-1} is. Let $A' = U(-c)AU(c)$, with coefficients (a'_{ij}). Then we have

$$D(t^{\alpha}, t^{p-\alpha})A'D(t^{-\alpha}, t^{\alpha-p}) = \begin{bmatrix} a'_{11} & t^{2\alpha-p}a'_{12} \\ t^{p-2\alpha}a'_{21} & a'_{22} \end{bmatrix},$$

so since $\alpha > p/2$, it is necessary and sufficient that a'_{21} vanish to order at least $2\alpha - p$, which is less than p. Since A and hence A' is a Frobenius-pullback, vanishing of a'_{21} is equivalent to vanishing to order at least p, and we find that we need simply impose the condition that a'_{21} vanish at P. Now, we see that

$$a'_{21} = a_{21} - ca_{11} + ca_{22} - c^2a_{12} = a_{21} - 2ca_{11} - c^2a_{12}$$

since the trace of A is required to be zero. We have a nine-dimensional space of choices for a_{11}, a_{21}, a_{12}, and we will need to show that these vanishing conditions at the four P impose four independent conditions on this space. However, we will be able to approach the gluing condition similarly, so we postpone the calculation and do both at once.

We now examine the gluing, wishing to show that gluing P_1 to Q_1 and P_2 to Q_2 impose two independent conditions on our space of global sections, so that we end up with desired final dimension of three. We observe from our formula for $D(t^{\alpha}, t^{p-\alpha})A'D(t^{-\alpha}, t^{p-\alpha})$ that we have vanishing at P away from the diagonal, so that evaluated at P, we actually get a diagonal matrix of vanishing trace. Since we know from Section VI.2 that our gluing maps are required to glue eigenspaces for ∇, it is easy to check that the relevant diagonals will be identified under gluing, so we impose at most one condition for each gluing. One could attempt to examine the gluing maps in more detail, but we will take a simpler approach: we will show that imposing vanishing of B at each of the four P imposes four further independent conditions, which implies that gluing conditions would be independent regardless of the chosen gluing maps. Vanishing of B at P is of course equivalent to vanishing of $D(t^{\alpha}, t^{p-\alpha})A'D(t^{-\alpha}, t^{p-\alpha})$ which is in turn equivalent to vanishing of the diagonal terms, which is simply vanishing of a'_{11} at P. We calculate

$$a'_{11} = a_{11} + ca_{12},$$

so this condition together with our earlier vanishing condition is equivalent to the two conditions

$$a_{11}(P) + ca_{12}(P) = a_{21}(P) - ca_{11}(P) = 0.$$

We will finish the proof by showing that these impose eight independent conditions.

For convenience of calculation, we will assume (visibly without any loss of generality) that $P_1 = 0, P_2 = \infty, Q_1 = 1, Q_2 = \lambda$, and we denote the corresponding c's by $c_1, \ldots, c_4$, which we recall could be assumed to be distinct by the generality of C. Our conditions can then be encoded in the 8×9 matrix imposing conditions on the coefficients of a_{11}, a_{12}, a_{21}:

$$\begin{bmatrix} 1 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & c_2 & 0 & 0 & 0 \\ 1 & 1 & 1 & c_3 & c_3 & c_3 & 0 & 0 & 0 \\ 1 & \lambda & \lambda^2 & c_4 & c_4\lambda & c_4\lambda^2 & 0 & 0 & 0 \\ -c_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -c_2 & 0 & 0 & 0 & 0 & 0 & 1 \\ -c_3 & -c_3 & -c_3 & 0 & 0 & 0 & 1 & 1 & 1 \\ -c_4 & -c_4\lambda & -c_4\lambda^2 & 0 & 0 & 0 & 1 & \lambda & \lambda^2 \end{bmatrix}.$$

If we drop the fourth column and take the determinant of the resulting 8×8 matrix, we find that we get:

$$(c_4 - c_3)(c_2 - c_3)(c_2 - c_4)\lambda^2(\lambda - 1).$$

λ is distinct from 0 or 1 and the c_i are distinct from one another by hypothesis, so this is non-zero, as desired. □

Remark VI.3.4. One could also approach the problem by considering the induced connection on $\mathcal{E}nd^0(\mathscr{E})$ abstractly and applying the ideas of Chapter V to it; via this approach, it is easy to see that if the eigenvalues of $\operatorname{Res}\nabla$ were $e_1, p - e_1$ at a point, the eigenvalues of $\operatorname{Res}\nabla^{\text{ind}}$ are $|2e_1 - p|, p - |2e_1 - p|, 0$ at the same point, and it is fairly straightforward to show that there must be a five-dimensional space of global sections prior to gluing. Unfortunately, imposing the gluing conditions seems substantially harder in this setting.

VI.4 Deforming to a Smooth Curve

Suppose we have a proper, nodal curve C_0, a vector bundle $\mathscr{E}_0$ on C_0 of trivial determinant, and a logarithmic connection ∇_0 of trivial determinant on $\mathscr{E}$. By Proposition VI.1.3, p-curvature gives an algebraic morphism $\psi_p : H^0(\mathcal{E}nd^0(\mathscr{E}_0)\otimes\omega_{C_0}) \to H^0(\mathcal{E}nd^0(\mathscr{E}_0)\otimes F^*\omega_{C_0^{(p)}})$ such that for $\varphi \in H^0(\mathcal{E}nd^0(\mathscr{E}_0)\otimes\omega_{C_0})$, $\psi_p(\nabla_0+\varphi)$ in fact lies in $H^0(\mathcal{E}nd^0(\mathscr{E}_0)\otimes F^*\omega_{C_0^{(p)}})^{(\nabla_0+\varphi)^{\mathrm{ind}}}$. Now, we first claim:

Lemma VI.4.1. *If ∇_0 has vanishing p-curvature, the differential of ψ_p at 0 gives a linear map*

$$d\psi_p : H^0(\mathcal{E}nd^0(\mathscr{E}_0)\otimes\omega_{C_0}) \to H^0(\mathcal{E}nd^0(\mathscr{E}_0)\otimes F^*\omega_{C_0^{(p)}})^{\nabla_0^{\mathrm{ind}}}.$$

Proof. We simply consider the induced map on first-order deformations of ∇_0; that is, we denote for the moment by C, $\mathscr{E}$ the base change of C_0, $\mathscr{E}_0$ to $k[\epsilon]/(\epsilon^2)$, and we suppose that $\varphi \in \epsilon H^0(\mathcal{E}nd^0(\mathscr{E})\otimes\omega_C) \cong H^0(\mathcal{E}nd^0(\mathscr{E}_0)\otimes\omega_{C_0})$, and consider $\nabla_0+\varphi$; since ∇_0 has vanishing p-curvature, the image under ψ_p is in $\epsilon H^0(\mathcal{E}nd^0(\mathscr{E})\otimes F^*\omega_{C^{(p)}})^{(\nabla_0+\epsilon\varphi)^{\mathrm{ind}}}$, which is naturally isomorphic to $H^0(\mathcal{E}nd^0(\mathscr{E}_0)\otimes F^*\omega_{C_0^{(p)}})^{\nabla_0^{\mathrm{ind}}}$, giving the desired result. $\square$

Our main assertion is:

Proposition VI.4.2. *If the map $d\psi_p$ of the previous lemma is surjective, then given a deformation C of C_0 and $\mathscr{E}$ of $\mathscr{E}_0$ on C, such that the functor of connections with trivial determinant is formally smooth at ∇_0, then the functor of connections with trivial determinant and vanishing p-curvature on $\mathscr{E}$ is formally smooth at ∇_0.*

Proof. By hypothesis, there is no obstruction to deforming ∇_0 as a connection with trivial determinant. Following [53, Def. 1.2, Rem. 2.3], we say that a map $B \twoheadrightarrow A$ of local Artin rings over the base ring of our deformation and having residue field k is a **small extension** if the kernel is a principal ideal (ϵ) with $(\epsilon)\mathfrak{m}_B = 0$; it follows then that $\epsilon B \subset B$ is isomorphic to k. To verify (formal) smoothness, by virtue of [64, Prop. 17.14.2] it is easily checked inductively that it is enough to check on small extensions. We show therefore that for such a small extension, when $d\psi_p$ is surjective there is no obstruction to lifting a deformation of ∇_0 over A to a deformation over B, even with the addition of the vanishing p-curvature hypothesis. Let $C_B, \mathscr{E}_B$ be the given deformations over B of $C_0, \mathscr{E}_0$ respectively, with $C_A, \mathscr{E}_A$ the induced deformations over A, and suppose that ∇_B is a connection on $\mathscr{E}_B$ such that ∇_A has vanishing p-curvature. The main point is that it is straightforward to check that the hypothesis that $\epsilon B \cong k$ implies that $\epsilon H^0(\mathcal{E}nd^0(\mathscr{E}_B) \otimes \omega_{C_B}) \cong H^0(\mathcal{E}nd^0(\mathscr{E}_0) \otimes \omega_{C_0})$, and for any $\varphi \in \epsilon H^0(\mathcal{E}nd^0(\mathscr{E}_B) \otimes \omega_{C_B})$, we have $\epsilon H^0(\mathcal{E}nd^0(\mathscr{E}_B) \otimes F^*\omega_{C_B^{(p)}})^{(\nabla_B+\varphi)^{\text{ind}}} \cong H^0(\mathcal{E}nd^0(\mathscr{E}_0) \otimes F^*\omega_{C_0^{(p)}})^{\nabla_0^{\text{ind}}}$. We want to show that for some choice of $\varphi \in \epsilon H^0(\mathcal{E}nd^0(\mathscr{E}_B) \otimes \omega_{C_B})$, $\nabla_B + \varphi$ has vanishing p-curvature. But as before, since ∇_A has vanishing p-curvature, the image under ψ_p of $\nabla_B + \varphi$ is in $\epsilon H^0(\mathcal{E}nd^0(\mathscr{E}_B) \otimes F^*\omega_{C_B^{(p)}})^{(\nabla_B+\varphi)^{\text{ind}}}$, and under the above isomorphisms, the induced map is equal to $d\psi_p + \frac{1}{\epsilon}\psi_p(\nabla_B)$, where $\frac{1}{\epsilon}$ is simply shorthand for the isomorphism $\epsilon H^0(\mathcal{E}nd^0(\mathscr{E}_B) \otimes F^*\omega_{C_B^{(p)}})^{(\nabla_B+\varphi)^{\text{ind}}} \xrightarrow{\sim} H^0(\mathcal{E}nd^0(\mathscr{E}_0) \otimes F^*\omega_{C_0^{(p)}})^{\nabla_0^{\text{ind}}}$. Hence if $d\psi_p$ is surjective, we can choose φ so that $\nabla_B + \varphi$ has vanishing p-curvature, as desired. □

Finally, we apply this result in our specific situation:

Theorem VI.4.3. *Let C_0 be a nodal rational curve of genus 2, and $\mathscr{E}_0$ as in Situation III.5.1. Let ∇_0 have vanishing p-curvature and trivial determinant, and suppose that ∇_0 has no deformations preserving the p-curvature and not arising from transport. Then the map $d\psi_p$ of Lemma VI.4.1 is surjective; in particular, given any deformation C of C_0, if $\mathscr{E}$ is the corresponding deformation of $\mathscr{E}_0$, then the space of connections with trivial determinant and vanishing p-curvature on $\mathscr{E}$ is formally smooth at ∇_0.*

Proof. The main point is that by Remark III.5.8, the space of transport-equivalence classes of connections with trivial determinant on $\mathscr{E}_0$ or $\mathscr{E}$ is explicitly parametrized by $\mathbb{A}^3$ over the appropriate base. In particular, deformations of ∇_0 as a connection with trivial determinant are unobstructed, and it also follows that the space of first-order deformations of ∇_0 with trivial determinant, modulo those arising from transport, is three-dimensional. By Proposition VI.3.3, the image space of $d\psi_p$ is three-dimensional. We therefore get surjectivity precisely when transport accounts for the entire kernel, which is to say, when there are no deformations of ∇_0 having vanishing p-curvature and trivial determinant other than those obtained by transport. We can thus apply the previous proposition to conclude smoothness. □

VI.5 Implications for the Verschiebung

In this section we put together the results of the preceding sections and chapters to prove Theorem VI.0.1:

Proof of Theorem VI.0.1. We note that (ii) follows immediately from (i) by virtue of Chapter IV, and specifically Theorem IV.0.1. It thus suffices to prove (i), which by the results of Section III.3 is equivalent to showing that, for the particular $\mathscr{E}$ of Situation III.5.1, there are precisely $\frac{1}{24}p(p^2-1)$ transport-equivalence classes of connections with trivial determinant and vanishing p-curvature on $\mathscr{E}$, and that none of these have any non-trivial deformations. We will show that this statement holds in the situation that C is a general rational nodal curve, and then conclude the same result must hold for a general smooth curve.

We observe that even in the situation of a nodal curve, there is a unique extension $\mathscr{E}$ of $\mathscr{L}^{-1}$ by $\mathscr{L}$; indeed, the proof of Proposition III.3.4 goes through with ω_C in place of Ω^1_C and using more general forms of Riemann-Roch and Serre duality as given in Theorem A.4. We also note that by Corollary VI.2.2, the argument of Proposition III.3.2 still shows that any connection must have its Kodaira-Spencer map be an isomorphism. It then follows from Corollary VI.2.8 that it suffices to prove the same result for D-logarithmic connections on $\mathscr{O}(1)\oplus\mathscr{O}(-1)$ on $\mathbb{P}^1$ satisfying the hypotheses of Situation V.1.11 and having the Kodaira-Spencer map an isomorphism, where D is made up of four general points on $\mathbb{P}^1$, and the eigenvalues of the residues at the points match in the appropriate pairs. We note that by degree considerations, the Kodaira-Spencer map in this case is always either zero or an isomorphism, so if we fix α_i for each pair (P_i, Q_i), by Theorem V.0.1 we find that we are looking for separable rational functions on $\mathbb{P}^1$ of degree $2p-1-2\sum\alpha_i$, and ramified to order at least $p-2\alpha_i$ at P_i and Q_i (note that the coefficient doubling for the degree is due to our use of a single, matching α_i for both P_i and Q_i). We could use the second formula of Corollary I.6.3 to compute the answer directly, but the first formula yields a more elegant solution. In either case, we are already given the lack of non-trivial deformations, so it suffices to show that the number of maps is correct. The formula gives that for each (α_1, α_2) there are

$$\min\{\{p-2\alpha_i\}_i, \{p-2\alpha_{3-i}\}_i, \{2\alpha_i\}_i, \{2\alpha_{3-i}\}_i\}$$

such maps, which reduces to

$$\min\{\{p-2\alpha_i\}_i,\{2\alpha_i\}_i\}.$$

Rather than summing up over all α_i, as we would with the second formula, we note that the number of maps will also be given by:

$$\sum_{1\leq j\leq (p-1)/2} \#\{(\alpha_1,\alpha_2) : j\leq 2\alpha_i, j\leq p-2\alpha_i\}$$

which then reduces to

$$\sum_{1\leq j\leq (p-1)/2} (\frac{p+1}{2}-j)^2 = \sum_{1\leq j\leq (p-1)/2} j^2 = \sum_{1\leq j\leq (p-1)/2} (2\binom{j}{2}+j)$$

$$=2\binom{(p+1)/2}{3}+\frac{p+1}{2}\frac{p-1}{4}=\frac{1}{24}(p+1)((p-1)(p-3)+3(p-1))=\frac{(p+1)(p-1)p}{24},$$

giving the desired result for a general nodal curve.

We can now apply Theorem VI.4.3 to conclude that since none of our connections on the general nodal curve have non-trivial deformations, the space of connections with trivial determinant and vanishing p-curvature on our chosen bundle over our parameter space of genus 2 curves is formally smooth at each connection on the general nodal curve. Furthermore, by Corollary III.7.4 (in light of Remark III.5.8), this space of connections is finite, so we conclude that it is finite etale at the general nodal curve, and finite everywhere, which then implies (i) for a general smooth curve, as desired. □

Appendix A

Auxiliary Lemmas and Well-Known Results

This appendix is a compendium of individual results, containing a few auxiliary lemmas of a general nature, and a variety of results which are well-known to experts. Some of the latter are difficult to impossible to locate in the literature, while others are rephrased from the references given to make them more appropriate for their use here. Certain results of this type were left in the main text when they fit well into the flow of the argument; those which have instead been included below were deemed to be likely to distract from the main ideas of the sections in which they arose. The goal of this appendix is solely completeness; references were used over including arguments as much as was possible.

Finite generation of fundamental groups of curves

Used in Theorem I.2.3:

Theorem A.1. *Let C be a smooth proper curve over an algebraically closed field k, $P_1, \ldots, P_n$ points on C, and $C' = C \smallsetminus \{P_1, \ldots, P_n\}$. Then the tame fundamental group $\pi_1^t(C')$ is topologically finitely generated as a profinite group.*

Proof. If $k = \mathbb{C}$, we have the explicit description given in [4, 9, Cor. 3.2]. Now suppose k has characteristic 0. Then C and the P_i may be defined over some $K \subset k$, a finitely generated extension of $\mathbb{Q}$, and we can imbed K (and thereby $\bar{K}$) inside of $\mathbb{C}$. Then the (automatically tame) fundamental groups of C' over $\bar{K}$, k, and $\mathbb{C}$ are all isomorphic, thanks to [4, 11, Thm. 6.1]. Finally, suppose k has characteristic p. Let A be the Witt vectors of

k [54, Thm. II.5.3], then by [4, 11, Thm. 1.1] we can find a $\tilde{C}$ over $\operatorname{Spec} A$ whose special fiber is C, and since A is complete and the P_i smooth points we can lift them to sections $\tilde{P}_i$ of $\tilde{C}$ (for instance, by definition of formal smoothness [3, Prop. 2.2.6]). Now, the geometric generic fiber has characteristic 0, so we know its tame fundamental group is topologically finitely generated, and we can conclude the same for C itself in light of [4, 11, Thm. 4.4]. $\square$

Families of nodal curves

Used in Lemma II.2.3, Theorem II.2.4, and Situation II.3.1:

Theorem A.2. *Let X/B be a family of proper nodal curves. Then:*

(i) The singular locus of X over B is finite and unramified over B

(ii) Suppose that either B is regular, or each connected component of the singular locus of X over B maps isomorphically to its scheme-theoretic image. Then each connected component of the singular locus of X over B has scheme-theoretic image which is locally principally generated inside B

Proof. The singular locus of X/B is naturally defined by the first Fitting ideal of the sheaf $\Omega_{X/B}$ of relative differentials. This is a closed subscheme, compatible with base change by [13, Cor. 20.5], and by hypothesis contains no fibers, so it is immediately finite. It is enough to check that it is unramified on fibers, and indeed for a curve over a field to have unramified singular locus may be taken as the definition of it being nodal, or may be checked easily from the formal local definition of a node. This gives (i).

For (ii), we primarily make use of the deformation theory of nodal curves described in [11]. Specifically, the space of deformations of a nodal curve is prorepresentable (see [11, Lem. 1.4]), and the universal family has smooth base, with each node mapping isomorphically to the vanishing locus of a coordinate on the base (see [11, p. 82]). Now, the singular locus commutes with base change, and taking scheme-theoretic image commutes with flat base change in our case by Proposition A.14, so if we choose a point $b \in B$ and pass to the complete local ring, the image of the singular locus of the completed family will be the base change of the image of the original singular locus, and the image of the base change of a connected component Δ will be the base change of the image. It thus suffices to check that the image of the base change to the completion is principal; indeed, since completion is flat, the ideal of each Δ in the completion is obtained by tensoring the ideal of Δ in the local ring by the completion of the local ring, and since the completion map is surjective (and in particular has non-empty closed fiber), it follows from Nakayama's lemma that if the tensor product is principally generated over the completion, Δ was principally generated to start with.

After passing to the completion, by the definition of prorepresentation, our family is obtained by pullback from a map from the complete local scheme at b to the scheme prorepresenting the deformation functor. Since each connected component of the singular locus of the universal deformation maps as a closed immersion with principally generated image into the base, its pullback will likewise be a closed immersion with locally principal ideal sheaf. It follows that the image of each of the connected components of the completion of Δ are locally principally generated, so the only potential issue is that their union might not be. Our additional hypotheses give us two cases to consider. If B is regular, a principally generated ideal sheaf is either a Cartier divisor or all of B, and the union of Cartier divisors is again Cartier, so there is no problem. Similarly, if Δ mapped isomorphically onto its image, then it remains connected after completion, and there is no union to be concerned about. Thus, in either case the scheme-theoretic image is locally principal, as desired. $\square$

Remark A.3. One might note that in the universal deformation of the above proof, the image of the union of any collection of connected components of the singular locus is still principally generated. From this perspective, the problem is that the map from the base to the deformation space need not be flat, so scheme-theoretic image need not commute with base change. This may seem like a technicality, but in fact (ii) is false in general; one can, for instance, construct a family over a quadric cone in $\mathbb{P}^3$ having a node whose image is three lines through the cone point.

Serre duality and Riemann-Roch for LCI curves

Used in Proposition A.5, Lemma II.4.2, Remark III.5.8, Proposition IV.4.8, Theorem IV.A.7, Lemma VI.2.5, and Theorem VI.0.1:

Theorem A.4. *Let X be a reduced, proper, geometrically connected, local complete intersection curve over a perfect field k. Then there exists a coherent sheaf ω_X on X satisfying:*

(i) ω_X *is invertible, and there is a natural sheaf map* $\Omega^1_X \to \omega_X$ *which is an isomorphism away from the singularities of* X.

(ii) For all coherent $\mathscr{O}_X$*-modules* $\mathscr{F}$*, there is a canonical isomorphism* $H^1(X, \mathscr{F}) \xleftarrow{\sim} \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \omega_X)^\vee$*; in particular, if* $\mathscr{F}$ *is locally free, we have* $H^1(X, \mathscr{F}) \xleftarrow{\sim} H^0(X, \mathscr{F}^\vee \otimes \omega_X)$.

(iii) For any divisor D *on* X, $h^0(X, \mathscr{O}(D)) - h^0(X, \mathscr{O}(-D) \otimes \omega_X) = \deg D + 1 - p_a(X)$.

(iv) For any $\mathscr{F}$ *locally free of rank* r *and degree* d, $h^0(X, \mathscr{F}) - h^0(X, \mathscr{F}^\vee \otimes \omega_X) = d + r(1 - p_a(X))$

Here $p_a(X) := h^0(X, \omega_X)$*, but may be explicitly computed using, for instance, [8, Thm. 5.2.3].*

Proof. For (i), ω_X is invertible by [1, Rem VIII.1.17, (ii), p. 170]. Since k is perfect, the normalization $\nu : X' \to X$ is smooth over k, so $\omega_{X'} = \Omega^1_{X'}$ by [1, Thm. I.4.6, p. 14]; we have the natural map $\Omega^1_X \to \nu_* \Omega^1_{X'}$, and we have an injective map $\nu_* \omega_{X'} \to \omega_X$ by [1, Prop. VIII.1.16, (i)], with cokernel supported at the singularities of X. Since the first map is visibly an isomorphism away from the singularities of X, we get the desired result.

(ii) is Roch's half of Riemann-Roch, and the original version of Serre/Grothendieck duality, see [1, Thm VIII.1.15, p. 167].

(iii) is Riemann's half of Riemann-Roch (see [1, Thm. VIII.1.4, p. 164]) together with (ii) and the fact that $h^0(X, \mathscr{O}_X) = 1$ for X proper and geometrically reduced and connected (see Lemma A.24).

Finally, to derive (iv) from (ii), all one needs is to check that $\deg \mathscr{F}$ (which we define to be the degree of the determinant line bundle of $\mathscr{F}$), is also given by $\chi(\mathscr{F}) - r\chi(\mathscr{O}_X)$. This may be checked inductively on the rank, with Riemann's formula used for (iii) as the base case, and using the formula that for a short exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$, χ is additive, and $\det \mathscr{F} \cong \det \mathscr{F}' \otimes \det \mathscr{F}''$. ◻

Ampleness of sections of relative curves

Used in Lemma II.4.2:

Proposition A.5. *Let* $\pi : X \to Y$ *be a flat proper morphism, all of whose geometric fibers are connected, reduced, local complete intersection curves, and let* D_i *be a collection* *of sections into the smooth locus of π, such that every component of every geometric fiber of π meets some D_i. Then the divisor $D = \sum_i D_i$ is π-ample.*

Proof. By [59, Prop. 4.7.1] one can check that D is π-ample fiber by fiber. Let X_y be a fiber, then if $\mathscr{L}$ is any line bundle on X_y, we can check ampleness after passing to the algebraic closure of $\kappa(y)$; this is clear from, for instance, [59, Prop. 2.6.1], which says that ampleness is equivalent to the vanishing of $H^1(X_y, \mathscr{F} \otimes \mathscr{L}^{\otimes n})$, for all coherent sheaves $\mathscr{F}$ and n sufficiently large (and allowed to depend on $\mathscr{F}$). Denote our geometric fiber by $\bar{X}_y$; we check this criterion directly, using Theorem A.4; we have $H^1(\bar{X}_y, \mathscr{F} \otimes \mathscr{O}(D)^n) \cong \operatorname{Hom}(\mathscr{F} \otimes \mathscr{O}(nD), \omega_{\bar{X}_y}) \cong \Gamma(\mathcal{H}om(\mathscr{F}, \omega_{\bar{X}_y}) \otimes \mathscr{O}(-nD))$. Now, since $\omega_{\bar{X}_y}$ is invertible and in particular torsion-free, $\mathcal{H}om(\mathscr{F}, \omega_{\bar{X}_y})$ is torsion-free, and no section not vanishing on a component of $\bar{X}_y$ can be in every power of the maximal ideal at any point of that component, no non-zero sections persists after twisting by arbitrarily high powers of $\mathscr{O}(-D)$. Moreover, by coherence the global sections are finitely generated, so for sufficiently high powers of $\mathscr{O}(-D)$, none of the global sections persist, giving the desired vanishing of H^1. $\square$

Remark A.6. The previous proposition is true for considerably more general schemes of relative dimension one, but the argument becomes more difficult in full generality, and we will only apply it to the nodal case.

The relative Picard scheme of curves of compact type

Used in Theorem II.4.3:

Theorem A.7. *Let $\pi : X \to B$ be a flat, proper family of genus-g curves of compact type over a Noetherian scheme B, and suppose that the singular locus of X/B has at most a single component Δ', which maps isomorphically onto its image Δ, with $X|_\Delta = Y \cup Z$ breaking into distinct components with $Y \cap Z = \Delta'$. Assume further that there are sections D_i of the smooth locus of X/B such that every components of every geometric fiber meets at least one D_i. Then if we consider the functor $\mathcal{P}ic(X/B)$ associating to each T/B the group $\operatorname{Pic}(X_T)/\operatorname{Pic}(T)$, we have the following subfunctors, and each is representable by a smooth, projective scheme over B:*

(1) If Δ' is empty, the subfunctor $\mathcal{P}ic^d(X/B)$ for any d, corresponding to line bundles of degree d on each fiber.

(2) *If Δ' is non-empty, the subfunctor $\mathcal{P}ic^{d,i}(X/B)$ for any d and i, corresponding to line bundles of degree d on each fiber, with degree i on each fiber of Y and $d-i$ on each fiber of Z.*

Proof. By [3, Thm. 9.4.1], in either case $\mathcal{P}ic^0(X/B)$ is representable by a smooth, separated B-scheme with B-ample line bundle, and parametrizes line bundles with degree 0 on each component of each fiber. Note that the functor is of the claimed form by [3, Prop. 8.1.4], also using A.24. We claim that this is the scheme we want. In the first case, if D_1 is a section of π, we can simply twist by $\mathscr{O}_X(dD_1)$ to construct a correspondence between line bundles of degree 0 and of degree d. In the second case, if D_1 is on Y and D_2 is on Z when restricted to $X|_\Delta$, we can twist by $\mathscr{O}_X(iD_1 + (d-i)D_2)$ to construct the desired correspondence.

Because we have a B-ample line bundle, projectivity will follow from properness. Since our base is Noetherian, by the strong form of the valuative criterion for properness we can check properness after base change to $\operatorname{Spec} A$ for A an arbitrary DVR, in which case we have B regular, and by the smoothness of the Picard scheme, it is also regular and in particular integral. We can then check properness in this particular case on fibers: the hypotheses of Proposition A.23 are satisfied (see, for instance, [25, p. 250], and/or Lemma A.9 (ii) below). Alternatively, we could apply [63, Cor. 15.7.11] with $Y' = Y$ and the identity section of the Picard scheme as the required section. □

The behavior of the relative Picard scheme for X/B when X breaks into components

Used in Section II.3, Theorem II.4.3, and Theorem A.7:

Definition A.8. For the purpose of the next lemma, we say that a scheme X is a **union** of two closed subschemes Y and Z if there is an exact sequence

$$0 \to \mathscr{O}_X \to \mathscr{O}_Y \oplus \mathscr{O}_Z \to \mathscr{O}_{Y\cap Z} \to 0;$$

if X is reduced, this is equivalent to the topological notion, otherwise it is stronger in its requirements on the choice of Y and Z.

Lemma A.9. *Suppose we have $\pi : X \to B$, and X is a union of two closed subschemes Y and Z, with $Y \cap Z$ flat over B. Then:*

(i) For any $B' \to B$, if we denote by X', Y', Z' the schemes obtained by base change, there is an exact sequence:

$$1 \to \mathscr{O}^*_{X'} \to \mathscr{O}^*_{Y'} \oplus \mathscr{O}^*_{Z'} \to \mathscr{O}^*_{Y' \cap Z'} \to 1.$$

(ii) Suppose further that π is quasi-compact and separated, that $Y \cap Z$ has a section over B, and that $Y \cap Z$ has geometrically reduced and connected fibers. Then $\mathcal{P}ic(X/B) \cong \mathcal{P}ic(Y/B) \times_{\mathcal{P}ic((Y \cap Z)/B)} \mathcal{P}ic(Z/B)$, where $\mathcal{P}ic(T/B)$ denotes the functor associating to each B-scheme B' the group $\operatorname{Pic}(T \times_B B')/\operatorname{Pic}(B')$.

(iii) In particular, if $\pi|_{Y \cap Z}$ is an isomorphism onto B, then we have the product decomposition $\mathcal{P}ic(X/B) \cong \mathcal{P}ic(Y/B) \times_B \mathcal{P}ic(Z/B)$

Proof. By hypothesis, we have an exact sequence

$$0 \to \mathscr{O}_X \to \mathscr{O}_Y \oplus \mathscr{O}_Z \to \mathscr{O}_{Y \cap Z} \to 0.$$

Because we have supposed that $Y \cap Z$ is flat over B (and because intersection always commutes with base change), exactness of this sequence is preserved under base change, and passing to units (and correspondingly changing the second map from subtraction to division) clearly preserves injectivity, which reduces (i) down to the observation that on sheaves, restriction of units to a closed subscheme is surjective.

To prove (ii), we obtain from (i) an exact sequence

$$H^0(Y', \mathscr{O}^*_{Y'}) \oplus H^0(Z', \mathscr{O}^*_{Z'}) \to H^0(Y' \cap Z', \mathscr{O}^*_{Y' \cap Z'}) \to H^1(X', \mathscr{O}^*_{X'})$$

$$\to H^1(Y', \mathscr{O}^*_{Y'}) \oplus H^1(Z', \mathscr{O}^*_{Z'}) \to H^1(Y' \cap Z', \mathscr{O}^*_{Y' \cap Z'}).$$

By Lemma A.24, $\pi_* \mathscr{O}_{Y' \cap Z'} = \mathscr{O}_{B'}$, in particular every global unit on $Y' \cap Z'$ is the pullback of a global unit on B', which gives surjectivity of the first map in the above sequence, and consequently implies that we have

$$1 \to H^1(X', \mathscr{O}^*_{X'}) \to H^1(Y', \mathscr{O}^*_{Y'}) \oplus H^1(Z', \mathscr{O}^*_{Z'}) \to H^1(Y' \cap Z', \mathscr{O}^*_{Y' \cap Z'}).$$

Now, the existence of a section from B' to T for $T = X', Y', Z', Y' \cap Z'$ (with the first three following from the last) implies that the natural map $\pi^* : \mathrm{Pic}(B') \to \mathrm{Pic}(T)$ is injective, and it is easily verified that modding out the exact sequence above by $H^1(B', \mathscr{O}^*_{B'}) = \mathrm{Pic}(B')$ preserves exactness: for instance, injectivity is preserved because any $\eta = (\pi^*\eta_1, \pi^*\eta_2) \in \mathrm{Pic}(B') \oplus \mathrm{Pic}(B') \subset \mathrm{Pic}(Y') \oplus \mathrm{Pic}(Z')$ which comes from $\mathrm{Pic}(X')$ must map to the trivial element of $\mathrm{Pic}(B') \subset \mathrm{Pic}(Y' \cap Z')$, and hence must have $\eta_1 = \eta_2$, so that η is the image of $\pi^*\eta_1 \in \mathrm{Pic}(X')$. This completes the proof of (ii), and (iii) then follows trivially. $\square$

Example A.10. If we weaken the flatness hypotheses of the lemma, we no longer have that exactness of our initial short exact sequence of sheaves is preserved under base change. For instance, if we start with $A = k[x,y]/(xy)$ and ideals $I = (x)$ and $J = (y)$, and set $X = \mathrm{Spec}(A), Y = \mathrm{Spec}\, A/I, Z = \mathrm{Spec}\, A/J$, we have $A \hookrightarrow A/I \times A/J$, but if we tensor with $B = k[x,y,u,v,w]/(xy, w-xu, w-yv)$, we find that w is in the kernel of $B \to B/IB \times B/JB$.

Grassmannians and Schubert cycles

Used in Theorem II.4.3, Lemma II.A.3, and Theorem II.A.14:

Theorem A.11. *Let S be a scheme, $\mathscr{E}$ a vector bundle of rank d on S, and $r < d$ a positive integer. Then:*

(i) The Grassmannian functor associating to any T/S the sub-bundles $\mathscr{F}$ of $\mathscr{E}_T$ of rank r is representable by a projective, smooth scheme $G = G(\mathscr{E}, r)$ over S, of relative dimension $r(d-r)$. Indeed, G is locally isomorphic to $\mathbb{A}_S^{r(d-r)}$.

(ii) If we are also given a sequence of locally free quotients

$$\mathscr{E} = \mathscr{Q}_d \twoheadrightarrow \mathscr{Q}_{d-1} \twoheadrightarrow \cdots \twoheadrightarrow \mathscr{Q}_1 \twoheadrightarrow \mathscr{Q}_0 = 0$$

with $\mathrm{rk}\, \mathscr{Q}_i = i$, and integers $0 \leq a_0 < a_1 < \cdots < a_{r-1} \leq d$ then we have the associated Schubert cycle $\Sigma = \Sigma(\mathscr{E}, r, \mathscr{Q}_i, a_i)$ representing the functor of sub-bundles $\mathscr{F}$ of $\mathscr{E}_T$ of rank r and such that $\mathrm{rk}(\mathscr{F} \to \mathscr{Q}_{a_i}) \leq i$ for all i. It is integral when S is integral, and when S is also Cohen-Macaulay the Schubert cycle has pure codimension $\sum_i (a_i - i)$ inside the Grassmannian scheme.

Proof. For (i), see [32]: Proposition 1.2 gives representability, with smoothness and the dimension following from the argument by virtue of the local cover by copies of $\mathbb{A}_S^{r(d-r)}$; projectivity is a consequence of Proposition 1.5.

For (ii), we work on the local rings of S so that everything becomes free, and refer to [5]. Note that there they write $G(X;\gamma)$ for (the associated graded ring) of a Schubert cycle, with a switch in their own index notation in the statement of Theorem 5.4 (a) (note especially the final sentence on p. 52). Due to slight indexing differences, the a_i of their γ are equal to our $a_{i-1}+1$. For the statement that the Schubert cycle is integral when S is, see their Theorem 6.3. For the codimension statement, we only use that S is Cohen-Macaulay to conclude that the ambient Grassmannian scheme is catenary; it then suffices to choose any point $x \in \Sigma \in G$ and show that $\dim \mathscr{O}_{G,x} - \dim \mathscr{O}_{\Sigma,x} = \sum_i (a_i - i)$. One approach would be to produce an open subset of Σ which is in fact locally isomorphic to affine space of the right dimension (see [18, Proof of Prop. 14.6.5]); for our choice of reference, it is more convenient to apply Corollary 5.12 (b), which says that $\dim G - \dim \Sigma = \sum_i (a_i - i)$, since $G = \Sigma$ for the a_i set to i. We note that the argument for dimension (in Proposition 5.10) was simply a flat fibers argument, and because the fibers are integral of finite type over a field, they have the same dimension at every closed point x of G inside Σ, so we have $\dim G = \dim \mathscr{O}_{G,x}$ and $\dim \Sigma = \dim \mathscr{O}_{\Sigma,x}$ and conclude the desired codimension statement. □

Curves in the moduli space of vector bundles

Used in Theorem IV.5.5 and Proposition A.31:

Proposition A.12. *Let C be a smooth, proper curve over an algebraically closed field k, and M a coarse moduli space of stable vector bundles on C of fixed rank and determinant. Then if S is a smooth curve over k, any map $S \to M$ is induced by a family $\mathscr{E}$ of vector bundles on $S \times C$.*

Proof. The key point is that the obstruction to the existence of $\mathscr{E}$ is in the Brauer group of S. Given this, the fact that S is a smooth curve over an algebraically closed field implies that the obstruction vanishes, giving the desired result; see [17, Lem. 5.2], or for a non-cohomological formulation, apply Tsen's theorem [17, Rem. 1.14] and [22, Cor. 1.10]. The statement on the obstruction being an element of the Brauer group is proven in the analytic setting in [6, Prop. 3.3.2]; the argument requires some slight modifications in characteristic p, but the crux of the matter is the argument that the GIT quotient is an etale principal PGL_n-bundle: [27, Cor. 4.3.5] does not go through because of the use of Luna's etale slice theorem, but the argument does still show that the stabilizer in PGL_n of any stable point is trivial, and [44, Cor. 2.5, p. 55] then asserts that the action is proper; putting these together we conclude that the action is free, and then [44, Prop. 0.9, p. 16] gives that it is a flat principal PGL_n-bundle. Finally, since PGL_n is smooth, one gets from [24, Thm. 11.7] that it is an etale principal PGL_n-bundle, as desired. $\square$

Properties of Hulls (following the terminology of [53])

Used in Theorem IV.A.7:

Lemma A.13. *Let F_1, F_2 be moduli functors over an algebraically closed field k, and $\hat{F}_1$ and $\hat{F}_2$ the induced deformation functors at chosen k-valued points of F_1 and F_2. Assume that $\hat{F}_1$ and $\hat{F}_2$ have hulls R_1 and R_2. Then:*

(i) If $f : \hat{F}_1 \to \hat{F}_2$ is a morphism of functors, there is an induced morphism (not necessarily unique) of hulls $\operatorname{Spec} R_1 \to \operatorname{Spec} R_2$.

(ii) If $\hat{F}_1$ is a closed subfunctor of $\hat{F}_2$, and is in fact prorepresentable by $\operatorname{Spec} R_1$, *then* $\operatorname{Spec} R_1$ *is naturally a closed subscheme of* $\operatorname{Spec} R_2$.

(iii) If F_1 has a coarse moduli scheme M_1 constructed via geometric invariant theory as a quotient of a rigidified moduli scheme (as in [44, Thm. 1.10, p. 38]), and M_1 is irreducible, then the natural map $\operatorname{Spec} R_1 \to \operatorname{Spec} M_1$ *is dominant, in the sense that the generic point of M_1 is in the image.*

Proof. The assertions follow from the definition of a hull [53, Def. 2.7]. For (i), because R_2 is formally smooth over $\hat{F}_2$, we may lift a map (not necessarily uniquely) inductively from $R_2/\mathfrak{m}_{R_2}^n$ to $R_2/\mathfrak{m}_{R_2}^{n+1}$ for each n, agreeing with the given map on the underlying functors at each stage, and finally since R_2 is complete we obtain the desired map.

For (ii), we must also use that a hull induces an isomorphism on tangent spaces with the underlying functor; since $\hat{F}_1$ is a closed subfunctor of $\hat{F}_2$, we certainly obtain a natural closed subscheme of $\operatorname{Spec} R_2$, which is smooth over $\operatorname{Spec} R_1$ and induces an isomorphism on tangent spaces. This implies that it is in fact isomorphic to $\operatorname{Spec} R_1$; see [53, Prop. 2.5 (i)].

Finally, for (iii), we show that the image of $\operatorname{Spec} R_1$ is not contained in any proper closed subset of M_1; given any such subset, we get a corresponding closed subset of the GIT rigidification M_1^r, which is strictly contained in M_1^r because the rigidification maps surjectively to M_1. We can then find a curve in M_1^r through our k-valued point and not contained in the closed subset; if we normalize, take the local ring, and complete, we obtain a map from $\operatorname{Spec} A$ to M_1^r for A a complete DVR with residue field k, with the closed point mapping to our chosen k-valued point, and image not contained in the chosen closed subset of M_1. Since the rigidification is representable, this gives a point of $F_1(A)$. Now, the quotients $A/\mathfrak{m}_A^n$ are all Artin k-algebras with residue field k, so we have induced points $\hat{F}_1(A/\mathfrak{m}_A^n)$ for all n, and by the smoothness of R_1 over $\hat{F}_1$, these may be lifted to a collection of compatible points $R_1(A/\mathfrak{m}_A^n)$. Since A is complete, this induces a map $\operatorname{Spec} A \to \operatorname{Spec} R_1$ factoring our original map $\operatorname{Spec} A \to M_1$, which implies that $\operatorname{Spec} R_1 \to M_1$ is not contained in the chosen closed subset of M_1. Since this subset was arbitrary, we obtain the desired dominance. □

Scheme-theoretic image and base change

Used in Theorem A.2:

Proposition A.14. *Let $f : X \to Y$ be a quasi-separated, quasi-compact morphism of schemes. Then taking the scheme-theoretic image under f commutes with flat base change.*

Proof. With the hypotheses on f, the scheme-theoretic image is defined by the kernel of the induced map $\mathscr{O}_Y \to f_*\mathscr{O}_X$ (see [57, Cor. 9.5.2], together with [65, Prop. 6.7.1]). Pushforward commutes with flat base change (see [26, Prop. III.9.3]), and it is easy to check directly from the definition that taking the kernel of a map of $\mathscr{O}_Y$-modules commutes with flat base change, giving the desired result. □

Remark A.15. The above statement is false if either the hypotheses on f or the base change is dropped. For instance, without the quasi-compactness hypothesis, one could take the natural inclusion $\coprod_n \operatorname{Spec} k[t]/t^n \to \operatorname{Spec} k[[t]]$, which is scheme-theoretically surjective, but becomes the empty inclusion after base change to the generic fiber. Without flatness, one could take an open immersion into an integral scheme, and base change to any point not in the set-theoretic image.

Deformations of locally free modules

Used in Section IV.3 and Proposition V.2.3:

Proposition A.16. *Let S be a scheme, and denote by S' the fiber product $S \times \operatorname{Spec} \mathbb{Z}[\epsilon]/(\epsilon^2)$. Suppose $\mathscr{F}$ is a coherent sheaf on S'. Then $\mathscr{F}/\epsilon\mathscr{F}$ and $\epsilon\mathscr{F}$ may both naturally be considered coherent sheaves on S, and $\mathscr{F}$ is locally free of rank r on S' if and only if $\mathscr{F}/\epsilon\mathscr{F}$ and $\epsilon\mathscr{F}$ are locally free of rank r on S.*

Proof. The statement being local, we may restrict to an affine open of S, so that $S = \operatorname{Spec} R$, $S' = \operatorname{Spec} R[\epsilon]$, and $\mathscr{F} = \tilde{M}$ for some finitely-generated $R[\epsilon]$-module M. One direction being obvious, we assume that $\mathscr{F}/\epsilon\mathscr{F}$ and $\epsilon\mathscr{F}$ are locally free, and further localize so that $M/\epsilon M$ and ϵM are free. We note that there is a natural surjective map $\pi : M/\epsilon M \twoheadrightarrow \epsilon M$. It is easy to check that if $M/\epsilon M$ is free, and π is an isomorphism, then any lift of any basis of $M/\epsilon M$ forms a basis of M. On the other hand, if ϵM is free, since π is surjective, $\ker \pi$ is also free, and if the ranks of $M/\epsilon M$ are equal, $\ker \pi$ must be zero, π is an isomorphism, and we find that M is free of the same rank, as desired. $\square$

Reducedness from fibers

Used in Corollary II.5.12:

Proposition A.17. *Let $f : X \to Y$ be a morphism, with all fibers of f reduced. Then if either*

(1) f is flat of finite type and Y is reduced and Noetherian, or

(2) $f' : X_{\mathrm{red}} \to Y$ is flat

then X is reduced.

Proof. For (1), reducedness is equivalent to having $\mathscr{O}_Y \hookrightarrow \prod_P \mathscr{O}_Y|_P$ for all points $P \in Y$, or equivalently, for P all generic points of components of Y; since Y is Noetherian, this is a finite product, so tensoring with $\mathscr{O}_X$ commutes, and by flatness we have $\mathscr{O}_X \hookrightarrow \prod_P \mathscr{O}_X|_{f^{-1}(P)}$. Since we have supposed that the fibers are reduced, we have $\mathscr{O}_X|_{f^{-1}(P)} \hookrightarrow \prod_{Q_P} \mathscr{O}_X|_{Q_P}$ where Q_P are the generic points of $f^{-1}(P)$, and composition gives $\mathscr{O}_X \hookrightarrow \prod_{P,Q_P} \mathscr{O}_X|_{Q_P}$, which implies that X is reduced.

For (2), let $\mathscr{F}$ be the sheaf of nilpotents inside $\mathscr{O}_X$; we have

$$0 \to \mathscr{F} \to \mathscr{O}_X \to \mathscr{O}_{X_{\mathrm{red}}} \to 0,$$

and the hypothesis that $\mathscr{O}_{X_{\text{red}}}$ is flat over $\mathscr{O}_Y$ means that for all $y \in Y$, $\mathscr{F} \otimes_{\mathscr{O}_Y} \kappa(y) \hookrightarrow \mathscr{O}_X \otimes_{\mathscr{O}_X} \kappa(y)$, so by the reducedness of fibers, $\mathscr{F} \otimes_{\mathscr{O}_Y} \kappa(y) = 0$ for all y. Because X is Noetherian, $\mathscr{F}$ is finitely generated, so we can apply Nakayama's lemma to conclude that $\mathscr{F} = 0$. $\square$

Remark A.18. In case (1), we only used that the generic fibers were reduced. However, in case (2), this is not enough, as one could consider a line with an imbedded point mapping to the line.

Question A.19. Given that either (1) or (2) is a sufficient hypothesis, it appears that there could be an underlying topological phenomenon. We ask: if f is open with reduced fibers, and Y reduced, is X reduced?

Example A.20. Note that f dominant (on every component of X) is not enough: the standard example of a morphism with constructible image is easily modified to provide the explicit counterexample: $\operatorname{Spec} k[x,y,z]/(xz, yz, z^2) \to \operatorname{Spec} k[s,t]$ by $s \mapsto x$, $t \mapsto xy + z$.

Remark A.21. The locus of reduced fibers is not very well-behaved. For instance, it need not be constructible, as demonstrated by $\operatorname{Spec} k[t,x]/(x^p - t)$ over $\operatorname{Spec} k[t]$, which has all its closed fibers non-reduced, but reduced generic fiber. On the other hand, [63, Lem. 9.7.2] implies that the opposite cannot occur.

Corollary A.22. *Let $f : X \to Y$ be a dominant morphism, with X irreducible and $Y =$* $\operatorname{Spec} A$, *with A a DVR. Then if either X is reduced, or if the fibers of f are reduced, one has that X is reduced and f is flat.*

Proof. The case that X is reduced is simply [26, Prop. III.9.7]. On the other hand, it follows that X_{red} is flat over Y, at which point case (2) of the preceding Proposition implies that if f has reduced fibers, X is reduced, so $X = X_{\text{red}}$ is flat over Y, as desired. $\square$

Properness from fibers

Used in Theorem A.7:

Proposition A.23. *Let $f : X \to Y$ be a separated morphism of finite type, with X, Y integral and Y Noetherian and normal, and suppose that all geometric fibers of f are proper, and consist of exactly one connected component. Then f is proper.*

Proof. First, because Y is Noetherian, and the hypotheses are all preserved under base change, it suffices by the valuative criterion of properness to consider that case that $Y = \operatorname{Spec} A$ for some DVR A, and in particular Y may be assumed to be integral and normal. By Nagata's compactification theorem [39], we can realize X as an open (scheme-theoretically) dense subscheme of some $\bar{X}$ which is proper over Y (note that we can argue instead using the considerably easier Chow's lemma, but Nagata's theorem simplifies the argument). We wish to show simply that under the hypotheses, $X = \bar{X}$. The crux of the argument is that $\bar{X}$ must also have connected fibers over Y: given this, and given $P \in \bar{X}$, the hypotheses imply that the fiber $X_{f(P)}$ of X inside the fiber $\bar{X}_{f(P)}$ of $\bar{X}$ containing P must be non-empty, open, and closed, so if $\bar{X}_{f(P)}$ is connected, it is equal to $X_{f(P)}$, and P is contained in X.

However, because X is open dense in $\bar{X}$, its geometric generic fiber must be dense in the geometric generic fiber of $\bar{X}$, which must then be connected. This implies that $K(Y)$ is separably closed in $K(X)$, and then by [59, Cor. 4.3.7], all the fibers of $\bar{X}$ must be connected, and we are done. □

Fibers and cohomological flatness

Used in Lemma II.3.2, Theorem II.4.3, Theorem A.4, Theorem A.7, and Lemma A.9:

Lemma A.24. *Any flat proper morphism $\pi : X \to B$ with geometrically reduced and connected fibers universally satisfies $\pi_* \mathscr{O}_X = \mathscr{O}_B$.*

Proof. Since π is proper with geometrically reduced and connected fibers, the global sections of $\mathscr{O}_X$ restricted to any geometric fiber, and hence any fiber, is simply the field itself: indeed, considering any global section as a morphism to $\mathbb{A}^1$, the image is affine, proper, and connected, and hence a single point; since the geometric fiber and $\mathbb{A}^1$ are both reduced and of finite type over an algebraically closed field, morphisms are determined on points, giving the desired isomorphism between global sections and the base field. Hence, by cohomology and base change (see Theorem A.32), $\pi_* \mathscr{O}_X$ is a line bundle on B. Finally, the identity section on $\mathscr{O}_X$ pushes forward to give a nowhere vanishing section of $\pi_* \mathscr{O}_X$, so we find that $\pi_* \mathscr{O}_X \cong \mathscr{O}_B$. Moreover, this property is preserved under base change, as we only used properness, flatness, and geometric properties of the fibers. □

Flatness of the relative Frobenius morphism

Used in Theorem III.1.4 and Theorem IV.A.7:

Proposition A.25. *Let X be a smooth S-scheme, with S of characteristic p. Then the relative Frobenius morphism $F : X \to X^{(p)}$ is flat.*

Proof. Since the relative Frobenius map commutes with base change, the criterion on flatness and fibers (see [63, Thm. 11.3.10]) reduces the question to the case that $S = \mathrm{Spec}(k)$. It is also clear that it suffices to prove flatness after faithfully flat base change, so we may further assume that k is algebraically closed. Now, X is regular, so its absolute Frobenius morphism F_X is flat [40, Thm. 107, p. 300]; on the other hand, since k is algebraically closed, $F_{\mathrm{Spec}(k)}$ is an isomorphism. If we denote by $\pi_{X/\mathrm{Spec}(k)}$ the base change of $F_{\mathrm{Spec}(k)}$ to $X^{(p)}$, the relative Frobenius morphism $F_{X/\mathrm{Spec}(k)}$ for X may therefore be composed with the isomorphism $\pi_{X/\mathrm{Spec}(k)}$ to obtain the flat map F_X, and is hence flat, as desired. □

Degrees of rational maps

Used in Proposition IV.1.2, Lemma IV.2.3, and Remark III.7.5:

Proposition A.26. *Let X and Y be integral schemes of finite type and the same dimension over a field k, and $f : X \to Y$ a dominant morphism (equivalently, since we do not assume properness, a dominant rational map). Then:*

(i) f induces a finite extension $K(X)/K(Y)$, and there exists a non-empty open subset $U \subset Y$ such that for each $y \in U$, the length of X_y is equal to $[K(X) : K(Y)]$.

(ii) If X and Y are regular, then for all $y \in Y$, either X_y has positive dimension, or the length of X_y is less than or equal to $[K(X) : K(Y)]$.

Proof. For (i), the dominance and dimension hypotheses immediately imply that the generic fiber is precisely the generic point of X, which is then of finite type over the generic point of Y, giving the first assertion of (i). Next, by Nagata's compactification theorem (see [39]), we may assume that X is proper over Y; indeed, the "boundary" of the compactification has strictly smaller dimension, so its image is contained in a closed subset of Y, and it follows that after restriction to an open subset of Y contained in the complement, f will be proper. Now, the loci of flatness and quasifiniteness are both open on X (see [63, Thm.

11.1.1] and [63, Thm. 13.1.3]), and dimensional considerations imply that if we restrict to an appropriate open set U of Y, we get a morphism which is finite flat. We thus have that restricted to U, the morphism is finite flat, and must have length of all fibers equal to the length of the generic fiber, as desired.

For (ii), we can first remove the closed subset of X on which f is not quasi-finite; if we then prove that for all $y \in Y$, the length of X_y is less than or equal to $[K(X) : K(Y)]$, this will give the desired result. Neither the lengths of fibers, nor the regularity hypotheses, will be affected by etale base change, so for any $y \in Y$, make an etale base change to decompose f into a disjoint union of a finite morphism with a morphism missing y (this is possible by repeatedly applying [3, Prop. 2.3.8 (a)] for the finitely many points of X_y). Now f is finite flat at all the points of X_y, so the generic fiber has length at least as large as the length of X_y, as desired. □

Corollary A.27. *Let X and Y be integral schemes of finite type and of the same dimension, and C a curve, all smooth over a field k, and $f : X \times C \dashrightarrow Y \times C$ a family of dominant rational maps from X to Y. Then the degree of f over any given point of C is less than or equal to the degree at a general fiber.*

Proof. We apply the previous proposition. It immediately follows from the hypotheses that f itself is a dominant rational map, and hence has some finite degree d. Then on some Zariski open of $Y \times C$, each fiber of f has length exactly d. In particular, over a general point P of C, there is a Zariski open on which each point has fiber under f of length d, and since f is a family of maps, which is to say a morphism over C, the entire fiber is over the same point P of C, so the degree of $f|_{X_C}$ is still d. Now, over the remaining (finitely many) points of C, the degree may only be lower than d, since given a degree d rational map, no point with a fiber of finite length can have length greater than d. □

Even in the simplest cases, the degree need not be constant over families, and can drop at particular fibers:

Example A.28. Consider the family of maps from $\mathbb{P}^2$ to itself given by $(X, Y, Z) \mapsto (X^2, Y^2, YZ + tZ^2)$. We immediately see that if $t \neq 0$, the map is actually a morphism. But for $t = 0$, while the map remains dominant, it is undefined at $(0, 0, 1)$. By Proposition IV.1.2, this means that the degree is 4 for a general fiber, but strictly less than that at $t = 0$

(in fact, it is easy to check that the undefined point has length 2, so the degree at $t = 0$ is 2).

Kernels of connections and completion in characteristic p

Used in Proposition V.1.8 and Lemma V.2.4:

Proposition A.29. *Let X be a scheme of finite type over a base scheme S of characteristic p. Suppose $\mathscr{E}$ is a vector bundle on X, and ∇ a connection on $\mathscr{E}$. Then the kernel of ∇ may be computed formally locally. More precisely, given any point x of X, if we denote the relative Frobenius morphism by F, the kernel of ∇ on the stalk of $\mathscr{E}$ at x is naturally an $\mathscr{O}_{X^{(p)},F(x)}$-module, and its completion maps naturally to the kernel of the formal local connection obtained from ∇ by completion at x; the assertion is that this map is an isomorphism.*

Proof. The main observation is that ∇ may actually be considered as an $\mathscr{O}_{X^{(p)}}$-linear map $F_*\mathscr{E} \to F_*(\mathscr{E} \otimes \Omega^1_{X/S})$; since F is a homeomorphism, no information is lost by pushing forward. Because $\mathscr{E}$ and $\mathscr{E} \otimes \Omega^1_{X/S}$ are coherent, and F_* is finite, the completion is obtained simply by tensoring with the completion of the local ring [13, Thm. 7.2 a.]; this is an exact process, since the completion is flat [13, Thm. 7.2 b.], so the kernel of this map of $\mathscr{O}_{X^{(p)}}$-modules may be computed formally locally. To complete the proof, it then suffices to observe that completion commutes with translation from the connection setting to the $\mathscr{O}_{X^{(p)}}$-linear map setting, which is clear from the definitions. $\square$

Vector bundles and connections with trivial determinants

Used in Section III.1, Theorem III.1.4, and Lemma V.6.4:

Proposition A.30. *Let $\mathscr{E}$ be a vector bundle with trivialized determinant on a smooth variety X over a field k of characteristic p, and ∇ an integrable connection on $\mathscr{E}$. Then:*

(i) Suppose ∇ has vanishing p-curvature. Then ∇ has trivial determinant if and only if the vector bundle $\mathscr{E}^{\nabla}$ has trivial determinant on $X^{(p)}$.

(ii) Suppose ∇ has trivial determinant. Then the p-curvature of ∇ has image in the traceless endomorphisms of $\mathscr{E}$.

Proof. For (i), we apply Proposition V.1.4, while (ii) will require Proposition III.2.7. The former made no use of the hypothesis that the base was a curve. The latter did use this hypothesis in order be able to locally trivialize Ω^1_X with a single one-form, but this was in fact entirely superficial, and used only so that it would suffice to consider a single connection matrix of functions in our calculations. Indeed, if T is a connection matrix of one-forms on U, and θ a derivation on U and $\hat{\theta}$ the corresponding linear map from Ω^1_X to $\mathscr{O}_X$, we have $\nabla_\theta(s) = \hat{\theta}(T)s + \theta s$, and if we write $\bar{T} = \hat{\theta}(T)$, it is easy to see that the calculations of Proposition III.2.6 and hence Proposition III.2.7 give the formula for the endomorphism $\psi_\nabla(\theta)$ for our particular θ.

Now, for (i), if $\mathscr{E}^\nabla$ has trivial determinant, it is easy to check that there is some collection of open sets $U_i \subset X^{(p)}$ on which $\mathscr{E}^\nabla$ is trivialized, with trivializations such that all transition matrices E^∇_{ij} have trivial determinant. Then $\mathscr{E} = F^*\mathscr{E}^\nabla$ will be trivialized on $F^{-1}(U_i)$, and will have transition matrices $F^*E^\nabla_{ij}$, which will again have trivial determinant. Moreover, under this trivialization, ∇ will simply be given by the zero matrix on each $F^{-1}(U_i)$, so it certainly has trivial determinant. Conversely, suppose that ∇ and $\mathscr{E}$ have trivial determinant; we write $\mathscr{E}$ as trivialized on $U_i \subset X$, with transition matrices E_{ij} having trivial determinant, and ∇ given by matrices T_i, which then have vanishing trace. By the formula of Proposition V.1.4, $\operatorname{Tr} T_i = 0$ implies $d \det S_i = 0$ where S_i is any inclusion of $\mathscr{E}^\nabla$ into $\mathscr{E}$ on U_i; such an inclusion is F-linear, so we obtain a trivialization of E^∇ on the $F(U_i)$, with transition matrices E^∇_{ij} satisfying $S_j^{-1}E_{ij}S_i = F^*E^\nabla_{ij}$. Change of basis of $E^\nabla|_{U_i}$ by an invertible matrix M_i will change S_i to $S_iF^*M_i$; we want to show that we can modify each S_i in this manner to make $F^*E^\nabla_{ij}$ have trivial determinant for all i, j. In fact, we note that since each $d \det S_i$ is zero, we can modify each S_i by some F^*M_i (specifically, a diagonal matrix with the desired scaling factor in the first coordinate) to make each S_i itself have trivial determinant, giving the desired result.

For (ii), we need only note that since the operation of taking traces is zero on matrix commutators, the same argument used to derive Corollary III.2.8 from Proposition III.2.7 still works, so we find we that for any derivation θ on some open U,

$$\psi_\nabla(\theta) = \bar{T}^p + (\theta^{p-1}\bar{T}) - f_{\theta^p}\bar{T}.$$

The second and third terms visibly have vanishing trace because $\bar{T}$ does, while it is easy to see (for instance, by passing to the algebraic closure of k and taking the Jordan normal form) that the trace of the pth power of a matrix is the pth power of the trace. Since θ was arbitrary, this gives the desired result. $\square$

Projective bundles on curves

Used in Section V.6:

Proposition A.31. *Let $\mathscr{P}$ be an etale projective bundle on a smooth curve C over an algebraically closed field k. Then $\mathscr{P}$ is the projectivization of a (Zariski) vector bundle $\mathscr{E}$ on C.*

Proof. $\mathscr{P}$ is an element of $H^1_{\text{et}}(C, \mathrm{PGL}_n)$, and will be the projectivization of an etale vector bundle if and only if it is in the image $H^1_{\text{et}}(C, \mathrm{GL}_n)$, which is to say, if and only if its image in the Brauer group $H^2_{\text{et}}(C, G_m)$ vanishes. But the Brauer group is zero in this case by Tsen's theorem (see proof of Proposition A.12), so $\mathscr{P}$ is the projectivization of an etale vector bundle $\mathscr{E}$. However, by faithfully flat descent for coherent sheaves [3, Thm. 6.4], the etale vector bundle comes from a quasicoherent Zariski sheaf, which must then also be locally free of the correct rank, so we get the desired result. $\square$

Cohomology and base change

Used in Lemma II.4.2, Remark III.5.8, and Lemma A.24:

Theorem A.32. *Let $f : X \to S$ be a proper morphism to a locally Noetherian scheme, and $\mathscr{F}$ a coherent $\mathscr{O}_X$-module which is flat over S. Then given an integer i, the following are equivalent:*

a) Base change commutes with $R^i f_*$ for $\mathscr{F}$.

b) Base change to any point $s \in S$ commutes with $R^i f_*$ for $\mathscr{F}$.

c) For all $s \in S$, the canonical map $R^i f_*(\mathscr{F}) \to H^i(X_s, \mathscr{F}_s)$ is surjective.

Additionally, of the following conditions, a) implies b) and c), and when S is reduced, we also have that b) implies a) (and hence c)).

a) Base change commutes with $R^i f_*$ and $R^{i+1} f_*$ for $\mathscr{F}$.

b) The function $s \mapsto \dim H^{i+1}(X_s, \mathscr{F}_s)$ *is constant with value* r.

c) $R^{i+1}f_*(\mathscr{F})$ *is locally free of rank* r.

Proof. This simply translates and pieces together several results from [60, §7], with S replacing Y, and $\mathscr{P}_\bullet$ the single-term complex consisting of $\mathscr{F}$ in degree zero. The main point of the translation is the notation for §7.7 defined in 7.7.1.1. For the equivalence of the first three conditions, $R^i f_*$ commuting with base change is simply condition d) of Theorem 7.7.5; it is clear that our a) implies our b), and our b) implies our c), and the last statement of Proposition 7.7.10, combined with Theorem 7.7.5, implies that our c) is equivalent to our a).

For the next three conditions, we combine the exactness conditions of Theorem 7.7.5, with the local-to-global statement of Proposition 7.7.10, at which point the desired assertions are obtained directly from parts b), d), and e) of Proposition 7.8.4. $\square$

Bibliography

[1] Allen Altman and Steven Kleiman. *Introduction to Grothendieck Duality Theory.* Number 146 in Lecture Notes in Mathematics. Springer-Verlag, 1970.

[2] Prakash Belkale. Geometric proofs of Horn and saturation conjectures. preprint.

[3] Siegfried Bosch, Werner Lutkebohmert, and Michel Raynaud. *Neron Models.* Springer-Verlag, 1991.

[4] Jean-Benoit Bost, Francois Loeser, and Michel Raynaud, editors. *Courbes Semi-stables et groupe fondamental en geometrie algebrique.* Birkhauser, 1998.

[5] Winfried Bruns and Udo Vetter. *Determinantal Rings.* Number 1327 in Lecture Notes in Mathematics. Springer-Verlag, 1988.

[6] Andrei Caldararu. Derived categories of twisted sheaves on Calabi-Yau manifolds. PhD thesis, Cornell University.

[7] Ciro Ciliberto. Geometric aspects of polynomial interpolation in more variables and of Waring's problem. In *European Congress of Mathematics, Vol. I*, pages 289–316. Birkhauser, 2001.

[8] Brian Conrad. *Grothendieck Duality and Base Change.* Number 1750 in Lecture Notes in Mathematics. Springer-Verlag, 2000.

[9] A. J. de Jong. A conjecture on arithmetic fundamental groups. *Israel J. Math.*, 121:61–84, 2001.

[10] A. J. de Jong and F. Oort. On extending families of curves. *J. Algebraic Geometry*, 6:545–562, 1997.

[11] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Institut Des Hautes Etudes Scientifiques Publications Mathematiques*, (36):75–109, 1969.

[12] Pierre Deligne. Théorie de Hodge. II. *Institut Des Hautes Etudes Scientifiques Publications Mathematiques*, (40):5–57, 1971.

[13] David Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry.* Springer-Verlag, 1995.

[14] David Eisenbud and Joe Harris. Divisors on general curves and cuspidal rational curves. *Inventiones Mathematicae*, 74:371–418, 1983.

[15] David Eisenbud and Joe Harris. Limit linear series: Basic theory. *Inventiones Mathematicae*, 85:337–371, 1986.

[16] Eduardo Esteves. Linear systems and ramification points on reducible nodal curves. *Matematica Contemporanea*, 14:21–35, 1998.

[17] Eberhard Freitag and Reinhardt Kiehl. *Etale Cohomology and the Weil Conjecture.* Springer-Verlag, 1980.

[18] William Fulton. *Intersection Theory.* Springer-Verlag, second edition, 1998.

[19] D. Gieseker. Stable vector bundles and the Frobenius morphism. *Ann. Sci. Ecole Norm. Sup. (4)*, 6:95–101, 1973.

[20] Alessandro Gimigliano. Our thin knowledge of fat points. In *The Curves Seminar at Queen's, Vol. VI.* Queen's University, 1989. Exp. No. B.

[21] Lisa R. Goldberg. Catalan numbers and branched coverings by the Riemann sphere. *Advances in Mathematics*, 85:129–144, 1991.

[22] Alexandre Grothendieck. Le groupe de Brauer, II: Théorie cohomologique.

[23] Alexandre Grothendieck. Techniques de construction et theoremes d'existence en geometrie algebrique, IV: Les schemas de Hilbert. *Seminaire Bourbaki*, (221), 1961.

[24] Alexandre Grothendieck. Le groupe de Brauer, III: Exemples et compléments. In *Dix exposès sur la cohomologie des schèmas*, pages 88–188. Masson & Cie, 1968.

[25] Joe Harris and Ian Morrison. *Moduli of Curves.* Springer-Verlag, 1998.

[26] Robin Hartshorne. *Algebraic Geometry.* Springer-Verlag, 1977.

[27] Daniel Huybrechts and Manfred Lehn. *The Geometry of Moduli Spaces of Sheaves.* Max-Planck-Institut fur Mathematik, 1997.

[28] Kirti Joshi, S. Ramanan, Eugene Z. Xia, and Jiu-Kang Yu. On vector bundles destabilized by Frobenius pull-back. preprint.

[29] Kirti Joshi and Eugene Z. Xia. Moduli of vector bundles on curves in positive characteristic. *Compositio Math.*, 122(3):315–321, 2000.

[30] Nicholas M. Katz. Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. *Inst. Hautes Etudes Sci. Publ. Math.*, 39:175–232, 1970.

[31] Nicholas M. Katz. Algebraic solutions of differential equations (p-curvature and the Hodge filtration). *Inventiones Mathematicae*, 18:1–118, 1972.

[32] Steven L. Kleiman. Geometry on Grassmannians and applications to splitting bundles and smoothing cycles. *Inst. Hautes Etudes Sci. Publ. Math.*, 36:281–297, 1969.

[33] Steven L. Kleiman. The transversality of a general translate. *Compositio Mathematica*, 28:287–297, 1974.

[34] Herbert Lange and Christian Pauly. On Frobenius-destabilized rank-2 vector bundles over curves. preprint.

[35] Adrian Langer. Semistable sheaves in positive characteristic. *Annals of Mathematics*, 159, 2004. to appear.

[36] Yves Laszlo and Christian Pauly. The Frobenius map, rank 2 vector bundles and Kummer's quartic surface in characteristic 2 and 3. *Advances in Mathematics.* to appear.

[37] Yves Laszlo and Christian Pauly. The action of the Frobenius maps on rank 2 vector bundles in characteristic 2. *Journal of Algebraic Geometry*, 11(2):129–143, 2002.

[38] A. Logan. The Kodaira dimension of moduli spaces of curves with marked points. *American Journal of Mathematics*, 125(1):105–138, 2003.

[39] W. Lutkebohmert. On compactification of schemes. *Manuscripta Mathematica*, 80(1):95–111, 1993.

[40] Hideyuki Matsumura. *Commutative Algebra.* The Benjamin/Cummings Publishing Company, second edition, 1980.

[41] Hideyuki Matsumura. *Commutative Ring Theory.* Cambridge University Press, 1986.

[42] Shinichi Mochizuki. *Foundations of p-adic Teichmüller Theory.* American Mathematical Society, 1999.

[43] E. Mukhin and A. Varchenko. Solutions to the XXX type Bethe ansatz equations and flag varieties. preprint.

[44] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric Invariant Theory.* Springer-Verlag, third enlarged edition, 1994.

[45] David Mumford. *Abelian Varieties.* Oxford University Press, second edition, 1974.

[46] M. S. Narasimhan and S. Ramanan. Moduli of vector bundles on a compact Riemann surface. *Annals of Mathematics*, 89(2):14–51, 1969.

[47] P. E. Newstead. *Introduction to Moduli Problems and Orbit Spaces.* Springer-Verlag, 1978.

[48] B. Osserman. Deformations of covers and wild ramification. In preparation.

[49] B. Osserman. The number of linear series on curves with given ramification. *International Mathematics Research Notices*, 2003(47):2513–2527.

[50] M. Raynaud. Sections des fibrés vectoriels sur une courbe. *Bull. Soc. Math. France*, 110(1):103–125, 1982.

[51] Michel Raynaud. *Anneaux Locaux Henseliens.* Number 169 in Lecture Notes in Mathematics. Springer-Verlag, 1970.

[52] I. Scherbak. Rational functions with prescribed critical points. *Geometric And Functional Analysis*, 12(6):1365–1380, 2002.

[53] Michael Schlessinger. Functors of Artin rings. *Transactions of the AMS*, 130:208–222, 1968.

[54] J. P. Serre. *Local Fields.* Springer-Verlag, 1979.

[55] Ravi Vakil. Schubert induction. preprint.

[56] Gayn B. Winters. On the existence of certain families of curves. *American Journal Of Mathematics*, 96(2):215–228, 1974.

[57] A. Grothendieck with J. Dieudonné. *Éléments De Géométrie Algébrique: I. Le Langage des Schémas*, volume 4 of *Publications mathématiques de l'I.H.É.S.* Institut des Hautes Études Scientifiques, 1960.

[58] A. Grothendieck with J. Dieudonné. *Éléments De Géométrie Algébrique: II. Étude Globale Élémentaire de Quelques Classes de Morphismes*, volume 8 of *Publications mathématiques de l'I.H.É.S.* Institut des Hautes Études Scientifiques, 1961.

[59] A. Grothendieck with J. Dieudonné. *Éléments De Géométrie Algébrique: III. Étude Cohomologique des Faisceaux Cohérents, Premiére Partie*, volume 11 of *Publications mathématiques de l'I.H.É.S.* Institut des Hautes Études Scientifiques, 1961.

[60] A. Grothendieck with J. Dieudonné. *Éléments De Géométrie Algébrique: III. Étude Cohomologique des Faisceaux Cohérents, Seconde Partie*, volume 17 of *Publications mathématiques de l'I.H.É.S.* Institut des Hautes Études Scientifiques, 1963.

[61] A. Grothendieck with J. Dieudonné. *Éléments De Géométrie Algébrique: IV. Étude Locale des Schémas et des Morphismes de Schémas, Premiére Partie*, volume 20 of *Publications mathématiques de l'I.H.É.S.* Institut des Hautes Études Scientifiques, 1964.

[62] A. Grothendieck with J. Dieudonné. *Éléments De Géométrie Algébrique: IV. Étude Locale des Schémas et des Morphismes de Schémas, Seconde Partie*, volume 24 of *Publications mathématiques de l'I.H.É.S.* Institut des Hautes Études Scientifiques, 1965.

[63] A. Grothendieck with J. Dieudonné. *Éléments De Géométrie Algébrique: IV. Étude Locale des Schémas et des Morphismes de Schémas, Troisiéme Partie*, volume 28 of *Publications mathématiques de l'I.H.É.S.* Institut des Hautes Études Scientifiques, 1966.

[64] A. Grothendieck with J. Dieudonné. *Éléments De Géométrie Algébrique: IV. Étude Locale des Schémas et des Morphismes de Schémas, Quatriéme Partie*, volume 32 of *Publications mathématiques de l'I.H.É.S.* Institut des Hautes Études Scientifiques, 1967.

[65] A. Grothendieck with J. Dieudonné. *Éléments De Géométrie Algébrique: I.* Springer-Verlag, 1971.