Examples of Solvable Quantum Groups and Their Representations

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Abstract

In this paper we first define a q-deformation of the universal enveloping algebra of the Heisenberg Lie algebra. We study this algebra and its finite-dimensional irreducible representations when $q = \varepsilon$, where ε is a primitive ℓth root of 1 with ℓ odd.

For each element of the Weyl group of a finite-dimensional simple Lie algebra, there is a corresponding solvable quantum group. We find generators and relations for each of these algebras in the case of the Lie algebra $sl_4(\mathbb{C})$, and we also find the central elements. Setting $q = \varepsilon$, where ε is a primitive ℓth root of 1 with ℓ odd, we then study the finite-dimensional irreducible representations of these algebras. It is shown that each representation has dimension either 1, ℓ, or ℓ^2, and that the dimension depends only on the central character.

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Contents

Chapter 1

The Quantum Heisenberg Algebra

In this chapter we define the quantum Heisenberg algebra $\mathcal{H}$, which is a q-deformation of the the universal enveloping algebra of the Heisenberg Lie algebra. Setting $q = \varepsilon$, we obtain the algebra $\mathcal{H}_\varepsilon$. We examine the finite-dimensional irreducible representations of this algebra when ε is a primitive ℓth root of 1, with $\ell > 2$.

1.1 Definition and Basic Properties

Definition 1.1 *The quantum Heisenberg algebra $\mathcal{H}$ is the associative algebra over the ring $\mathcal{A} = \mathbb{C}[q, q^{-1}, (q - q^{-1})^{-1}]$ with generators a, b, c and relations*

$$ab - qba = c \tag{1.1}$$

$$ac - q^{-1}ca = 0 \tag{1.2}$$

$$bc - qcb = 0 \tag{1.3}$$

We further define $\mathcal{H}_\varepsilon$, $\varepsilon \in \mathbb{C}$, $\varepsilon \neq 0, 1$, or -1, as the algebra $\mathcal{H}/(q - \varepsilon)\mathcal{H}$. We observe that $\mathcal{H}_1$ is the universal enveloping algebra of the Heisenberg Lie algebra.

Proposition 1.2 *(a) The elements $a^i b^j c^k$,$(i, j, k) \in \mathbb{Z}_+^3$, form a basis of $\mathcal{H}$ over $\mathcal{A}$ and of $\mathcal{H}_\varepsilon$ over $\mathbb{C}$.*
(b) The algebras $\mathcal{H}$ and $\mathcal{H}_\varepsilon$ have no zero divisors.

Proof: (a) The elements $a^ib^jc^k$ clearly span. To prove they are a basis, it suffices to show that the element cba reduces to the same element whether we begin by reducing cb or ba in the product. Checking, we have $(cb)a = (q^{-1}bc)a = q^{-1}b(ca) = b(ac) = (ba)c = q^{-1}(ab-c)c = q^{-1}abc - q^{-1}c^2$ and $c(ba) = q^{-1}c(ab-c) = q^{-1}(ca)b - q^{-1}c^2 = (ac)b - q^{-1}c^2 = a(cb) - q^{-1}c^2 = q^{-1}abc - q^{-1}c^2$.
(b) To see that there are no zero divisors, we note that ($a^ib^jc^k$ + lower-degree terms)($a^rb^sc^t$ + lower-degree terms)= $q^{kr-jr-ks}a^{i+r}b^{j+s}c^{k+t}$ + lower degree terms).

Proposition 1.3 *The element $(q-q^{-1})abc - q^{-1}c^2$ generates the center $\mathcal{Z}$ of $\mathcal{H}$.*

Proof: It is easily checked that this element commutes with each of the generators a, b, and c. Let z =($a^ib^jc^k$ + lower-degree terms) be central. Then $a^ib^jc^k$ must commute, modulo lower-degree terms, with each of the generators a, b, and c. This gives the condition that $i = j = k$. Then z =($a^mb^mc^m$ + lower-degree terms) - $q^{(1/2)m(m-1)}[abc - q^{-1}(q-q^{-1})^{-1}c^2]^m$ is a central element of degree less than that of z. By induction on degree, the proof is complete.

Lemma 1.4 *(a) In $\mathcal{H}$, for $m = 1,2,3,\ldots$*

$$ab^m = q^mb^ma + (q^{-(m-1)} + q^{-(m-3)} + \ldots + q^{m-3} + q^{m-1})b^{m-1}c \tag{1.4}$$

$$a^mb = q^mba^m + (1 + q^2 + \ldots + q^{2(m-2)} + q^{2(m-1)})a^{m-1}c \tag{1.5}$$

(b) in $\mathcal{H}_\varepsilon$, for $m = 1,2,3,\ldots$

$$ab^m = \varepsilon^mb^ma + \varepsilon^{1-m}\left(\frac{1-\varepsilon^{2m}}{1-\varepsilon^2}\right)b^{m-1}c \tag{1.6}$$

$$a^mb = \varepsilon^mba^m + \left(\frac{1-\varepsilon^{2m}}{1-\varepsilon^2}\right)a^{m-1}c \tag{1.7}$$

Proof: (a) By induction on m. Part (b) follows from part (a), with $\varepsilon \neq 0, 1$, or -1.

Proposition 1.5 *The center $\mathcal{Z}_\varepsilon$ of $\mathcal{H}_\varepsilon$, where ε is a primitive ℓth root of 1, is generated by a^ℓ, b^ℓ, c^ℓ, and $(\varepsilon - \varepsilon^{-1})abc + \varepsilon^{-1}c^2$.*

Proof: c^ℓ clearly commutes with a and b. a^ℓ commutes with c, and with b by the preceeding lemma. Likewise b^ℓ commutes with a and c. The element $(q - q^{-1})abc - q^{-1}c^2$ lies in the center $\mathcal{Z}$ of $\mathcal{H}$, so $(\varepsilon - \varepsilon^{-1})abc + \varepsilon^{-1}c^2$ lies in $\mathcal{Z}_\varepsilon$. Let z =($a^ib^jc^k$ + lower-degree terms) be central. Then $a^ib^jc^k$ must commute, modulo lower-degree terms, with each of the generators a, b, and c. This gives the condition that $i = j = k \pmod{\ell}$. Then z =($a^{m+\ell r}b^{m+\ell s}c^{m+\ell t}$ + lower-degree terms) - $\varepsilon^{(1/2)m(m-1)}a^{\ell r}b^{\ell s}c^{\ell t}[abc - \varepsilon^{-1}(\varepsilon - \varepsilon^{-1})^{-1}c^2]^m$ is a central element of degree less than that of z. By induction on degree, the proof is complete.

1.2 Irreducible Representations of $\mathcal{H}_\varepsilon$

We now consider the finite-dimensional irreducible representations of $\mathcal{H}_\varepsilon$, where ε is a primitive ℓth root of 1, with $\ell > 2$. Since a^ℓ, b^ℓ, c^ℓ, and $(\varepsilon - \varepsilon^{-1})abc + \varepsilon^{-1}c^2$ are central elements of $\mathcal{H}_\varepsilon$, by Schur's Lemma they act as scalars $a^\ell = x$, $b^\ell = y$, $c^\ell = z$, and $(\varepsilon - \varepsilon^{-1})abc + \varepsilon^{-1}c^2 = w$ in any finite-dimensional irreducible representation.

Proposition 1.6 *The finite-dimensional irreducible representations of $\mathcal{H}_\varepsilon$, where ε is a primitive ℓth root of 1, have the following dimensions:*

1	*if $z = 0$, and x or y is zero*
$\ell/2$	*if $z \neq 0$, $x = 0$, $y = 0$, and ℓ is even*
ℓ	*if $z = 0$, $x \neq 0$, and $y \neq 0$*
	if $z \neq 0$, and $x \neq 0$ or $y \neq 0$
	if $z \neq 0$, $x = 0$, $y = 0$, and ℓ is odd

Proof: Let V be an irreducible $\mathcal{H}_\varepsilon$-module.

Case 1: Suppose that $z = c^\ell = 0$ on V. Then, since c q-commutes (see definition 2.1) with a and b, it follows that $c = 0$ on V (see the proof of lemma 2.2). Then V is an irreducible module over the generators a and b, which satisfy the relation $ab = \varepsilon ba$ on V.

1a) If $x = a^\ell = 0$, then since a and b q-commute on V, it follows that $a = 0$ on V. Then V is one dimensional, spanned by an eigenvector of b. Likewise, if $y = b^\ell = 0$, then $\dim V = 1$.

1b) If $x = a^\ell \neq 0$ and $y = b^\ell \neq 0$, let v be an eigenvector of a; $av = \lambda v$, $\lambda \neq 0$. Then the vectors $v, bv, ..., b^{\ell-1}v$ are eigenvectors (these vectors are nonzero since $b^\ell \neq 0$) of a with distinct eigenvalues $\lambda, \varepsilon\lambda, ..., \varepsilon^{\ell-1}\lambda$, respectively. The space $\text{span}(v, bv, ..., b^{\ell-1}v)$ is invariant under a and b (and c, since $c = 0$ on V), so by irreduciblility is equal to V. Thus $\dim V = \ell$.

Case 2: $z = c^\ell \neq 0$, and $x = a^\ell \neq 0$ or $y = b^\ell \neq 0$. Suppose first that $x = a^\ell \neq 0$. V is also a module over the algebra with generators a and c and relation $ca = \varepsilon ac$. Let U be an irreducible submodule of V over this algebra. By the same reasoning as in case 1b), we see that $\dim U = \ell$. Then from $(\varepsilon - \varepsilon^{-1})abc + \varepsilon^{-1}c^2 = w$, we can solve for b, obtaining $b = [xz(\varepsilon - \varepsilon^{-1})]^{-1}a^{\ell-1}[w - \varepsilon^{-1}c^2]c^{\ell-1}$. Thus U is invariant under b, so $V = U$ by irreducibility and $\dim V = \ell$. Similarly, if $y = b^\ell \neq 0$ we have $\dim V = \ell$.

Case 3: $z = c^\ell \neq 0$, $x = a^\ell = 0$, and $y = b^\ell = 0$. Let U be an irreducible submodule of V over the algebra with generators a and c and relation $ca = \varepsilon ac$. Since a q-commutes with c and $a^\ell = 0$, it follows that $a = 0$ on U (see lemma 2.2). Thus U is one dimensional, spanned by an eigenvector u of c: $cu = \lambda u$, with $\lambda \neq 0$ since $c^\ell \neq 0$. The space $\text{span}(u, bu, ..., b^{\ell-1}u)$ is seen to be invariant under a, b, and c, so $V = \text{span}(u, bu, ..., b^{\ell-1}u)$. We note further that $cb^m u = \varepsilon^{-m}\lambda u$, so the spaces U, bU, ... ,$b^{\ell-1}U$ are eigenspaces of c with distinct eigenvalues. Thus $V = U \oplus bU \oplus ... \oplus b^{\ell-1}U$ (direct sum as vector spaces).

3a) Suppose ℓ is odd. Let m be the least positive integer such that $b^m u = 0$. Applying equation 1.6 to u, we obtain

$$0 = \lambda\varepsilon^{1-m}\left(\frac{1-\varepsilon^{2m}}{1-\varepsilon^2}\right)b^{m-1}u \tag{1.8}$$

Thus ℓ divides $2m$. Since ℓ is odd, it follows that $\ell = m$. Thus each of the spaces in the sum $V = U \oplus bU \oplus ... \oplus b^{\ell-1}U$ is one dimensional, and $\dim V = \ell$.

3b) Suppose ℓ is even. Then, applying equation 1.6 to u with $m = \ell/2$, we obtain

$$ab^{\ell/2}u = 0 \tag{1.9}$$

For $m = \ell/2+1, ..., \ell-1$ we have

$$ab^m u = \lambda\varepsilon^{1-m}\left(\frac{1-\varepsilon^{2m}}{1-\varepsilon^2}\right) b^{m-1}u \tag{1.10}$$

It follows that the space span$(b^{\ell/2}u, b^{\ell/2+1}u, ..., b^{\ell-1}u)$ is invariant under a. It is also invariant under c, and invariant under b since $b^\ell = 0$. Thus this is an invariant subspace of V. But it does not contain the vector u, because $V = U \oplus bU \oplus ... \oplus b^{\ell-1}U$ and $u \in U$. Thus by irreducibility, since this space is not equal to V, it must be 0. Thus $V = U \oplus bU \oplus ... \oplus b^{\ell/2-1}U$. Now let m be the least positive integer such that $b^m u = 0$. By the same reasoning as in case 3a, we see that ℓ divides $2m$. Since $m \leq \ell/2$, it follows that $m = \ell/2$. Thus each of the spaces in the sum $V = U \oplus bU \oplus ... \oplus b^{\ell/2-1}U$ has dimension one, so $\dim V = \ell/2$.

We now consider only the case where ε is a primitive ℓth root of 1 with ℓ *odd.*

Proposition 1.7 *In any finite-dimensional irreducible representation of $\mathcal{H}_\varepsilon$, with ε a primitive ℓth root of 1 with ℓ odd, we have the relation*

$$w^\ell = (\varepsilon - \varepsilon^{-1})^\ell xyz + z^2 \tag{1.11}$$

where $a^\ell = x$, $b^\ell = y$, $c^\ell = z$, and $(\varepsilon - \varepsilon^{-1})abc + \varepsilon^{-1}c^2 = w$.

Proof: Case 1: If $z = 0$, then $c = 0$ so $w = 0$, and the relation is satisfied trivially.

Case 2: $z \neq 0$, and $x = 0$ or $y = 0$. Suppose first that $a^\ell = x = 0$. As in case 3 of Proposition 1.6, there is a vector u such that $au = 0$ and $cu = \lambda u$, where $\lambda^\ell = z$. We can rewrite the element $(\varepsilon - \varepsilon^{-1})abc + \varepsilon^{-1}c^2$ as $(\varepsilon^2 - 1)bac + c^2$. Applying this element to u, we get $wu = \lambda^2 u$. Thus $w = \lambda^2$, and raising this to the ℓth power gives $w^\ell = z^2 = (\varepsilon - \varepsilon^{-1})^\ell xyz + z^2$. If $b^\ell = 0$ the proof is similar.

Case 3: $z \neq 0$, $x \neq 0$, and $y \neq 0$. We have seen that in this case the representation is ℓ dimensional. Also in this case, a, b, and c are diagonalizable. For example, letting v be an eigenvector of a with eigenvalue λ, the vectors $v, cv, ..., c^{\ell-1}v$ are a basis for the irreducible $\mathcal{H}_\varepsilon$-module V, and these are eigenvectors of a with eigenvalues

$\lambda, \lambda\varepsilon^{-1}, \ldots, \lambda\varepsilon^{-(\ell-1)}$. Thus the determinant of a is the product of these eigenvalues, which is x. Similarly, the determinants of b and c are y and z, respectively. We now take the determinant of the equation

$$w - \varepsilon^{-1}c^2 = (\varepsilon - \varepsilon^{-1})abc \tag{1.12}$$

The determinant of the right-hand side is $(\varepsilon - \varepsilon^{-1})^{\ell} xyz$. The determinant of the left-hand side is

$$\prod_{j=0}^{\ell-1} (w - \varepsilon^{-1}\mu^2\varepsilon^{2j}), \tag{1.13}$$

where μ is an eigenvalue of c, so $\mu^\ell = z$. To compute this product we use the Gauss Binomial Formula

$$\prod_{j=0}^{m-1} (\alpha + q^{2j}\beta) = \alpha^m + q^{m(m-1)}\beta^m + \sum_{j=1}^{m-1} \left(\frac{[m]\ldots[m-j+1]}{[j]\ldots[1])}\right) q^{j(m-1)}\alpha^{m-j}\beta^j \tag{1.14}$$

where $[n] = (q^n - q^{-n})/(q - q^{-1})$, with $m = \ell$, $\alpha = w$, $\beta = -\varepsilon^{-1}\mu^2$, and $q = \varepsilon$. Noting that $[\ell] = 0$, this gives $w^\ell + (-\mu^2)^\ell = w^\ell - z^2$.

Proposition 1.8 *In $\mathcal{H}_\varepsilon$,*

a)

$$[(\varepsilon - \varepsilon^{-1})abc + \varepsilon^{-1}c^2]^\ell = (\varepsilon - \varepsilon^{-1})^\ell a^\ell b^\ell c^\ell + (c^\ell)^2 \tag{1.15}$$

b)

$$[(\varepsilon - \varepsilon^{-1})ab + \varepsilon^{-1}c]^\ell = (\varepsilon - \varepsilon^{-1})^\ell a^\ell b^\ell + c^\ell \tag{1.16}$$

Proof:

a) $\mathcal{Z}_\varepsilon$ is a finitely generated commutative algebra. Thus, given any nonzero element z of $\mathcal{Z}_\varepsilon$, there is a finite-dimensional irreducible representation which maps z to a nonzero scalar. Also, since $\mathcal{H}_\varepsilon$ is a finitely-generated module over $\mathcal{Z}_\varepsilon$ (as a $\mathcal{Z}_\varepsilon$-module, $\mathcal{H}_\varepsilon$ is generated by the monomials $a^ib^jc^k$ with $i, j, k < \ell$), the canonical map Spec $\mathcal{H}_\varepsilon \rightarrow$ Spec $\mathcal{Z}_\varepsilon$ is surjective. [4] Thus there is a finite-dimensional irreducible representation of $\mathcal{H}_\varepsilon$ which maps z to a nonzero scalar. Since we have shown that the element $[(\varepsilon-\varepsilon^{-1})abc+\varepsilon^{-1}c^2]^\ell-(\varepsilon-\varepsilon^{-1})^\ell a^\ell b^\ell c^\ell-(c^\ell)^2$ is mapped to zero in any finite-dimensional irreducible representation, it follows that this element must be zero in $\mathcal{H}_\varepsilon$.

b) Since c commutes with $[(\varepsilon-\varepsilon^{-1})ab+\varepsilon^{-1}c]$, we have $[(\varepsilon-\varepsilon^{-1})ab+\varepsilon^{-1}c]^\ell c^\ell = [(\varepsilon-\varepsilon^{-1})abc+\varepsilon^{-1}c^2]^\ell = (\varepsilon-\varepsilon^{-1})^\ell a^\ell b^\ell c^\ell+(c^\ell)^2$. Now use the fact that $\mathcal{H}_\varepsilon$ has no zero divisors.

1.3 Another Quantum Heisenberg Algebra

Consider the algebra over the ring $\mathcal{A}=\mathbb{C}[q,q^{-1},(q-q^{-1})^{-1}]$, with generators a_i, b_i, $(i=1,2)$ and c with relations

$$b_1b_2=b_2b_1 \tag{1.17}$$

$$a_1a_2=a_2a_1 \tag{1.18}$$

$$ca_i=qa_ic \tag{1.19}$$

$$b_ic=qcb_i \tag{1.20}$$

$$b_ia_j=qa_jb_i \qquad \text{for } i\neq j \tag{1.21}$$

$$a_ib_i-qb_ia_i=c \tag{1.22}$$

This is the algebra $\mathcal{U}^{s_1s_3s_2s_1s_3}$, which is examined in Chapter 2, with the relabeling $E_1\to b_1$, $E_3\to b_2$, $E_{23}\to a_1$, $E_{12}\to a_2$, and $E_{123}\to q^{-1}c$. We find in Chapter 2 for this algebra that the element $[(q-q^{-1})b_1a_1+c][(q-q^{-1})b_2a_2+c]$ generates the center. When $q=\varepsilon$ where ε is a primitive ℓth root of 1 with ℓ odd, we find that the finite-dimensional irreducible representations have the following dimensions:

1 if $c^\ell=0$, $a_1^\ell=0$, and $a_2^\ell=0$

1 if $c^\ell=0$, $b_1^\ell=0$, and $b_2^\ell=0$

ℓ^2 if $c^\ell\neq 0$ and $(\varepsilon-\varepsilon^{-1})^\ell a_1^\ell b_1^\ell+c^\ell\neq 0$

ℓ^2 if $c^\ell\neq 0$ and $(\varepsilon-\varepsilon^{-1})^\ell a_2^\ell b_2^\ell+c^\ell\neq 0$

ℓ in all other cases.

Chapter 2

Quantum Groups Associated With $\mathcal{U}_q^+(sl_4(C))$

Let W be the Weyl group of the finite-dimensional simple Lie algebra $sl_n(\mathbb{C})$. For each $w \in W$, there is a corresponding solvable quantum group $\mathcal{U}^w$. Each of these quantum groups is a subalgebra of $\mathcal{U}_q^+(sl_n(\mathbb{C}))$; when w is the longest element of W, we obtain $\mathcal{U}_q^+(sl_n(\mathbb{C}))$. In this chapter, we consider $sl_4(\mathbb{C})$ and give defining relations for $\mathcal{U}^w$ for each $w \in W$. Then, letting q $= \varepsilon$, a primitive ℓth root of 1 with ℓ odd, we obtain the algebras $\mathcal{U}_\varepsilon^w$. We study the finite-dimensional irreducible representations of these algebras, showing that they all have dimensions which are powers of ℓ. We also show that the dimensions depend only on the central character of the representation.

2.1 $\mathcal{U}_q(sl_n(C))$ and $\mathcal{U}^w$

Let a_{ij} be the Cartan matrix of $sl_n(\mathbb{C})$. Quantum $sl_n(\mathbb{C})$, which we shall designate from this point on as $\mathcal{U}$, is the algebra over the ring $\mathcal{A} = \mathbb{C}[q, q^{-1}, (q - q^{-1})^{-1}]$ with generators E_i, F_i, K_i, K_{-i} ($i = 1, ..., n-1$) and relations

$$K_iK_j = K_jK_i\,, \quad K_iK_{-i} = K_{-i}K_i = 1 \tag{2.1}$$

$$K_iE_j = q^{a_{ij}}E_jK_i\,, \quad K_iF_j = q^{-a_{ij}}F_jK_i \tag{2.2}$$

$$E_iF_j - F_jE_i = \delta_{ij}(K_i - K_i^{-1})/(q - q^{-1}) \tag{2.3}$$

$$E_i^2E_j - (q + q^{-1})E_iE_jE_i + E_jE_i^2 = 0 \qquad \text{if } a_{ij} = -1 \tag{2.4}$$

$$E_iE_j - E_jE_i = 0 \qquad \text{if } a_{ij} = 0 \tag{2.5}$$

$$F_i^2F_j - (q + q^{-1})F_iF_jF_i + F_jF_i^2 = 0 \qquad \text{if } a_{ij} = -1 \tag{2.6}$$

$$F_iF_j - F_jF_i = 0 \qquad \text{if } a_{ij} = 0. \tag{2.7}$$

We have the following automorphisms T_i $(i = 1, ..., n-1)$ of the algebra $\mathcal{U}$ [6]:

$$T_iE_i = -F_iK_i \quad ; \quad T_iE_j = E_j \quad \text{if } a_{ij} = 0 \tag{2.8}$$

$$T_iE_j = -E_iE_j + q^{-1}E_jE_i \quad \text{if } a_{ij} = -1 \tag{2.9}$$

$$T_iF_i = -K_i^{-1}E_i \quad ; \quad T_iF_j = F_j \quad \text{if } a_{ij} = 0 \tag{2.10}$$

$$T_iF_j = -F_jF_i + qF_iF_j \quad \text{if } a_{ij} = -1 \tag{2.11}$$

$$T_iK_j = K_jK_i^{-a_{ij}} \tag{2.12}$$

These automorphisms T_i satisfy the braid relations.

Let $w \in W$, and let $s_{i_1}...s_{i_m}$ be a reduced expression for w in terms of simple reflections. Let $\beta_1 = \alpha_{i_1}$, ..., $\beta_m = s_{i_1}...s_{i_{m-1}}(\alpha_{i_m})$. For $l = 1, ..., m$, let $E_{\beta_l} = T_{i_1}...T_{i_{l-1}}E_{i_l}$ (these depend on the choice of reduced expression for w). For $k = (k_1, ..., k_m) \in \mathbf{Z}_+^m$, let $E^k = E_{\beta_1}^{k_1}...E_{\beta_m}^{k_m}$. These elements form a basis of $\mathcal{U}^w$ over $\mathcal{A}$ [3]. And for $i < j$ we have:

$$E_{\beta_i}E_{\beta_j} - q^{(\beta_i|\beta_j)}E_{\beta_j}E_{\beta_i} = \sum_{k \in \mathbf{Z}_+^m} c_kE^k, \tag{2.13}$$

where $c_k \in \mathbf{C}[q, q^{-1}]$ and $c_k \neq 0$ only when $k = (k_1, ..., k_m)$ is such that $k_s = 0$ for $s \leq i$ and $s \geq j$ [5]. The algebra $\mathcal{U}^w$ is generated by the elements E_{β_1}, ...,E_{β_m} with defining relations 2.13. $\mathcal{U}^w$ is independent of the choice of reduced expression for w [3].

Setting $q = \varepsilon$, we obtain the algebra $\mathcal{U}_\varepsilon^w$. The elements $E_{\beta_i}^\ell$ $(i = 1, ..., m)$ are central in $\mathcal{U}_\varepsilon^w$ [3].

2.2 Preliminary Results on Irreducible $\mathcal{U}_\varepsilon^w$ Modules

Definition 2.1 *Let x and y be elements of $\mathcal{U}^w$ (respectively $\mathcal{U}_\varepsilon^w$). We shall say that x and y $q-commute$ if they satisfy $xy - q^s yx = 0$ (repectively $xy - \varepsilon^s yx = 0$) for some $s \in \mathbf{Z}$.*

The elements $E_{\beta_i}^\ell$ are central in the algebra $\mathcal{U}_\varepsilon^w$. Thus, by Schur's lemma, they act as scalars in any finite-dimensional irreducible representation of $\mathcal{U}_\varepsilon^w$.

Lemma 2.2 *Suppose E_{β_i} q-commutes with each of the generators E_{β_j} of $\mathcal{U}_\varepsilon^w$. If $E_{\beta_i}^\ell = 0$ on a finite-dimensional irreducible $\mathcal{U}_\varepsilon^w$-module V, then $E_{\beta_i} = 0$ on V.*

Proof: Let $v \in V$ be an eigenvector of E_{β_i}. Then $E_{\beta_i} v = 0$. By irreducibility, v generates V as a $\mathcal{U}_\varepsilon^w$ module. Thus any element of V may be written as a linear combination of terms having the form $E_{\beta_1}^{k_1} ... E_{\beta_m}^{k_m} v$. Then $E_{\beta_i} E_{\beta_1}^{k_1} ... E_{\beta_m}^{k_m} v = \varepsilon^s E_{\beta_1}^{k_1} ... E_{\beta_m}^{k_m} E_{\beta_i} v = 0$ (for some $s \in \mathbf{Z}$). Thus $E_{\beta_i} = 0$ on V.

Lemma 2.3 *Let $s_{i_1} ... s_{i_{m-1}}$ be a reduced expression for w in terms of simple reflections, and let $s_{i_1} ... s_{i_m}$ be a reduced expression for $\tilde{w}$. Let V be a finite-dimensional irreducible module over the algebra $\mathcal{U}_\varepsilon^{\tilde{w}}$. V is also a module over the algebra $\mathcal{U}_\varepsilon^w$. Let U be an irreducible submodule of V over the algebra $\mathcal{U}_\varepsilon^w$. Then $V = U \oplus E_{\beta_m} U \oplus ... \oplus E_{\beta_m}^k U$ (direct sum as vector spaces) for some $0 \leq k \leq \ell - 1$, where $dim E_{\beta_m}^j U = dim U$ for $j = 0, ..., k$.*

Proof: Let r be the smallest postive integer such that there exists $u \in U$, $u \neq 0$, satisfying $E_{\beta_m}^{r+1} \in U + E_{\beta_m} U + ... + E_{\beta_m}^r U$. We know that $r \leq \ell - 1$, because $E_{\beta_m}^\ell$ acts as a scalar on V. The sum $U + E_{\beta_m} U + ... + E_{\beta_m}^r U$ is direct (by our choice of r).

From 2.13 it follows that for $i < m$ we have

$$E_{\beta_i} E_{\beta_m}^k = \varepsilon^{k(\beta_i|\beta_m)} E_{\beta_m}^k E_{\beta_i} + E_{\beta_m}^{k-1} f_{k-1}(E_{\beta_1}, ..., E_{\beta_{m-1}}) + ... + f_0(E_{\beta_1}, ..., E_{\beta_{m-1}}), \quad (2.14)$$

where $f_j(E_{\beta_1}, ..., E_{\beta_{m-1}})$ is a polynomial in $E_{\beta_1}, ..., E_{\beta_{m-1}}$. Applying 2.14 to a vector in U, we see that $U \oplus E_{\beta_m} U \oplus ... \oplus E_{\beta_m}^r U$ is invariant under $E_{\beta_1}, ..., E_{\beta_{m-1}}$. We also have by 2.13

$$E_{\beta_m}^{r+1}(E_{\beta_{m-1}}^{k_{m-1}} ... E_{\beta_1}^{k_1}) = \varepsilon^s (E_{\beta_{m-1}}^{k_{m-1}} ... E_{\beta_1}^{k_1}) E_{\beta_m}^{r+1} + E_{\beta_m}^r g_{k_m - 1}(E_{\beta_1}, ..., E_{\beta_{m-1}}) + ...$$
$$+ g_0(E_{\beta_1}, ..., E_{\beta_{m-1}}), \quad (2.15)$$

for some $s \in \mathbf{Z}$. By irreducibility of U over $\mathcal{U}_\varepsilon^w$, the element u generates U over $\mathcal{U}_\varepsilon^w$, and any element of U may be written as a linaear combination of terms each having the form $E_{\beta_{m-1}}^{k_{m-1}} ... E_{\beta_1}^{k_1} u$. Applying (2.15) to u, we see that the right-hand side of this equation lies in $U \oplus E_{\beta_m} U \oplus ... \oplus E_{\beta_m}^r U$. Thus $U \oplus E_{\beta_m} U \oplus ... \oplus E_{\beta_m}^r U$ is also invariant under E_{β_m}, and by irreducibility over the algebra $\mathcal{U}_\varepsilon^{\tilde{w}}$, we have $V = U \oplus E_{\beta_m} U \oplus ... \oplus E_{\beta_m}^r U$.

Finally, if $r = 0$, the proof is complete. If $r > 0$, consider the maps $E_{\beta_m} : E_{\beta_m}^{i-1} U \to E_{\beta_m}^i U$ $(i = 1, ..., r)$. Suppose $E_{\beta_m}^i \tilde{u} = 0$. Then by choice of r, $\tilde{u} = 0$ so $E_{\beta_m}^{i-1} \tilde{u} = 0$. Thus the nullspace of each of these maps is 0, which implies $dim E_{\beta_m}^{i-1} U \leq dim E_{\beta_m}^i U$, hence $dim E_{\beta_m}^{i-1} U = dim E_{\beta_m}^i U$.

2.3 $\mathcal{U}^w$ for $w = s_1, s_1 s_2, s_1 s_2 s_1, s_1 s_2 s_1 s_3, s_1 s_2 s_1 s_3 s_2,$ and $s_1 s_2 s_1 s_3 s_2 s_1$

For $w = s_1 s_2 s_1 s_3 s_2 s_1$, we find that $\beta_1 = \alpha_1$, $\beta_2 = \alpha_1 + \alpha_2$, $\beta_3 = \alpha_2$, $\beta_4 = \alpha_1 + \alpha_2 + \alpha_3$, $\beta_5 = \alpha_2 + \alpha_3$, and $\beta_6 = \alpha_3$. We then find, using (2.8) through (2.12), that $E_{\beta_1} = E_1$, $E_{\beta_2} = -E_1 E_2 + q^{-1} E_2 E_1$, $E_{\beta_3} = E_2$, $E_{\beta_4} = E_1 E_2 E_3 - q^{-1} E_2 E_1 E_3 - q^{-1} E_3 E_1 E_2 + q^{-2} E_3 E_2 E_1$, $E_{\beta_5} = -E_2 E_3 + q^{-1} E_3 E_2$, and $E_{\beta_6} = E_3$. We shall write $E_{\beta_2} = E_{\alpha_1 + \alpha_2}$

as E_{12}, etc. With some computation, we find the relations (2.13) are as follows:

$$E_1E_{12} = qE_{12}E_1 \tag{2.16}$$

$$E_1E_2 = q^{-1}E_2E_1 - E_{12} \tag{2.17}$$

$$E_{12}E_2 = qE_2E_{12} \tag{2.18}$$

$$E_1E_{123} = qE_{123}E_1 \tag{2.19}$$

$$E_{12}E_{123} = qE_{123}E_{12} \tag{2.20}$$

$$E_2E_{123} = E_{123}E_2 \tag{2.21}$$

$$E_1E_{23} = q^{-1}E_{23}E_1 - E_{123} \tag{2.22}$$

$$E_{12}E_{23} = E_{23}E_{12} + (q - q^{-1})E_2E_{123} \tag{2.23}$$

$$E_2E_{23} = qE_{23}E_2 \tag{2.24}$$

$$E_{123}E_{23} = qE_{23}E_{123} \tag{2.25}$$

$$E_1E_3 = E_3E_1 \tag{2.26}$$

$$E_{12}E_3 = q^{-1}E_3E_{12} - E_{123} \tag{2.27}$$

$$E_2E_3 = q^{-1}E_3E_2 - E_{23} \tag{2.28}$$

$$E_{123}E_3 = qE_3E_{123} \tag{2.29}$$

$$E_{23}E_3 = qE_3E_{23} \tag{2.30}$$

$\mathcal{U}^{s_1}$ has the generator E_1 (and no relations). $\mathcal{U}^{s_1s_2}$ has generators E_1, E_{12} with relation (2.16).

$\mathcal{U}^{s_1s_2s_1}$ has generators E_1, E_{12}, E_2 with relations (2.16) through (2.18). In this algebra, we find that the element $E_{12}[(q - q^{-1})E_1E_2 + qE_{12}]$ is central (it is easily checked that it commutes with each of the generators).

$\mathcal{U}^{s_1s_2s_1s_3}$ has generators E_1, E_{12}, E_2, E_{123} with relations (2.16) through (2.21).

$\mathcal{U}^{s_1s_2s_1s_3s_2}$ has generators E_1, E_{12}, E_2, E_{123}, E_{23} with relations (2.16) through (2.25). We find that the element $E_{12}E_{23} - qE_2E_{123}$ is central in this algebra.

$\mathcal{U}^{s_1s_2s_1s_3s_2s_1}$ (which is equal to $\mathcal{U}_q^+(sl_4(\mathbb{C}))$, because $s_1s_2s_1s_3s_2s_1$ is the longest element of W for $sl_4(\mathbb{C})$) has generators E_1, E_{12}, E_2, E_{123}, E_{23}, E_3 with relations (2.16) through (2.30). The elements $E_{12}E_{23} - qE_2E_{123}$ and $(q-q^{-1})E_3[(q-q^{-1})E_1E_2 + qE_{12}]E_{123} + E_{123}[(q-q^{-1})E_1E_{23} + qE_{123}]$ are central in this algebra.

2.4 Irreducible Representations of $\mathcal{U}_\varepsilon^w$ for $w = s_1$, s_1s_2, $s_1s_2s_1$, $s_1s_2s_1s_3$, $s_1s_2s_1s_3s_2$, and $s_1s_2s_1s_3s_2s_1$

We now let $q = \varepsilon$, where ε is a primitive ℓth root of 1 (with ℓ odd in most cases). All representations considered will be finite-dimensional. Recall that if $s_{i_1}...s_{i_m}$ is a reduced expression for w in terms of simple reflections, then the elements $E_{\beta_i}^\ell$ $(i = 1, ..., m)$ are central in $\mathcal{U}_\varepsilon^w$, so they act as scalars in any finite-dimensional representation.

Proposition 2.4 *The finite-dimensional irreducible representations of $\mathcal{U}_\varepsilon^{s_1}$, where ε is a primitive ℓth root of unity, are one dimensional.*

Proof: Let V be an irreducible $\mathcal{U}_\varepsilon^{s_1}$-module. Since $\mathcal{U}_\varepsilon^{s_1}$ is generated by E_1, V is spanned by an eigenvector of E_1. So dim$V$=1.

Proposition 2.5 *The finite-dimensional irreducible representations of $\mathcal{U}_\varepsilon^{s_1s_2}$, where ε is a primitive ℓth root of 1, have the following dimensions:*

ℓ *if $E_1^\ell \neq 0$ and $E_{12}^\ell \neq 0$*

1 in all other cases.

Proof: Let V be an irreducible $\mathcal{U}_\varepsilon^{s_1s_2}$-module. If $E_1^\ell = 0$ on V, then $E_1 = 0$ on V by lemma 2.2. Then V is one-dimensional, spanned by an eigenvector of E_{12}. Similarly if $E_{12}^\ell = 0$, then dim$V$=1. If $E_1^\ell \neq 0$ and $E_{12}^\ell \neq 0$, let v be an eigenvector of E_1, with eigenvalue λ $(\lambda \neq 0)$. Then span$(v, E_{12}v, ..., E_{12}^{\ell-1}v)$ is invariant under E_1 and E_{12}, so this space is equal to V. $v, E_{12}v, ..., E_{12}^{\ell-1}v$ are eigenvectors of E_1 with eigenvalues $\lambda, \varepsilon\lambda, ..., \varepsilon^{\ell-1}\lambda$, respectively (each of these vectors is nonzero, because $E_{12}^\ell \neq 0$). Therefore these vectors are linearly independent and $\dim V = \ell$.

Proposition 2.6 *The finite-dimensional irreducible representations of $\mathcal{U}_\varepsilon^{s_1 s_2 s_1}$, where ε is a primitive ℓth root of unity with ℓ odd, have dimensions*

1 *if $E_{12}^\ell = 0$, and E_1^ℓ or E_2^ℓ is zero*

ℓ *in all other cases*

Proof: The algebra $\mathcal{U}^{s_1 s_2 s_1}$ is isomorphic to the quantum Heisenberg algebra $\mathcal{H}$ discussed in Chapter 1, with the identification $E_1 \to b$, $E_{12} \to q^{-1}c$, and $E_2 \to a$. The element $E_{12}[(q-q^{-1})E_1E_2 + qE_{12}]$ is central in this algebra, corresponding to the element $q^{-2}c[(q-q^{-1})ab + q^{-1}c]$ in $\mathcal{H}$. We also note from applying Proposition 1.8 to this case that $[(\varepsilon-\varepsilon^{-1})E_1E_2 + \varepsilon E_{12}]^\ell = (\varepsilon-\varepsilon^{-1})^\ell E_1^\ell E_2^\ell + E_{12}^\ell$.

Proposition 2.7 *The finite-dimensional irreducible representations of $\mathcal{U}_\varepsilon^{s_1 s_2 s_1 s_3}$, where ε is a primitive ℓth root of unity with ℓ odd, have dimensions:*

1 *if $E_{123}^\ell = 0$, $E_{12}^\ell = 0$, and $E_2^\ell = 0$*

1 *if $E_{12}^\ell = 0$ and $E_1^\ell = 0$*

ℓ^2 *if $E_{123}^\ell \neq 0$, $E_{12}^\ell \neq 0$, and $(\varepsilon-\varepsilon^{-1})^\ell E_1^\ell E_2^\ell + E_{12}^\ell \neq 0$*

ℓ *in all other cases*

Proof: Let V be an irreducible $\mathcal{U}_\varepsilon^{s_1 s_2 s_1 s_3}$-module.

Case 1: $E_{123}^\ell = 0$ on V. Since E_{123} q-commutes with E_1, E_{12}, and E_2, $E_{123}^\ell = 0$ on V implies that $E_{123} = 0$ on V. Thus, by lemma 2.3, $V = U$ where U is an irreducible $\mathcal{U}_\varepsilon^{s_1 s_2 s_1}$-module. So $\dim V$=1 or ℓ.

Case 2: $E_{123}^\ell \neq 0$, $E_{12}^\ell = 0$, and $E_1^\ell = 0$ on V. $E_{12}^\ell = 0$ implies $E_{12} = 0$. It then follows that E_1 q-commutes with E_2, so $E_1^\ell = 0$ implies $E_1 = 0$. We are thus left with the two generators E_2 and E_{123}, which commute. V is spanned by a common eigenvector of these two generators, so $\dim V$=1.

Case 3: $E_{123}^\ell \neq 0$, $E_{12}^\ell = 0$, and $E_1^\ell \neq 0$ on V. $E_{12}^\ell = 0$ implies $E_{12} = 0$. Let v be a common eigenvector of E_2 and E_{123}, which commute. Then $E_{123}v = \lambda v$, with $\lambda \neq 0$. The space $\mathrm{span}(v, E_1 v, ..., E_1^{\ell-1})$ is invariant under E_1, E_2, and E_{123}, so is equal to V. Furthermore, the vectors $v, E_1v, ..., E_1^{\ell-1}$ are all eigenvectors of E_{123} with distinct eigenvalues λ, $\varepsilon^{-1}\lambda$,..., $\varepsilon^{-(\ell-1)}\lambda$, respectively (the vectors are all nonzero because $E_1^\ell \neq 0$). Thus $\dim V = \ell$.

Case 4: $E_{123}^\ell \neq 0$ and $E_{12}^\ell \neq 0$ on V. V is a module over $\mathcal{U}_\varepsilon^{s_1s_2s_1}$; let U be an irreducible submodule of V over $\mathcal{U}_\varepsilon^{s_1s_2s_1}$. $E_{12}[(\varepsilon-\varepsilon^{-1})E_1E_2+\varepsilon E_{12}]$ is a central element of $\mathcal{U}_\varepsilon^{s_1s_2s_1}$, so it acts as a scalar on U. Let $x = E_{12}[(\varepsilon-\varepsilon^{-1})E_1E_2+\varepsilon E_{12}]$. We see by checking directly that $xE_{123} = \varepsilon^2 E_{123}x$. We consider the following two subcases.

Case 4a: $x = E_{12}[(\varepsilon-\varepsilon^{-1})E_1E_2+\varepsilon E_{12}]$ acts as a nonzero scalar α on U (thus $(\varepsilon-\varepsilon^{-1})^\ell E_1^\ell E_2^\ell + E_{12}^\ell \neq 0$). We know that $V = U + E_{123}U + ... + E_{123}^{\ell-1}U$. The spaces U, $E_{123}U$, ... , $E_{123}^{\ell-1}U$ are eigenspaces of x with corresponding to eigenvalues α, $\varepsilon^2\alpha$, ..., $\varepsilon^{\ell-1}\alpha$, $\varepsilon^{\ell+1}\alpha$, ..., $\varepsilon^{2\ell-2}\alpha$; these eigenvalues are all distinct (because ℓ is odd). Thus $V = U \oplus E_{123}U \oplus ... \oplus E_{123}^{\ell-1}U$. Each of the spaces in the direct sum is nonzero and each has dimension equal to the dimension of U, because $E_{123}^\ell \neq 0$. Thus $\dim V = \ell \dim U$. From our previous results we know that $E_{12}^\ell \neq 0$ on U implies that $\dim U = \ell$. Therefore $\dim V = \ell^2$.

Case 4b: $x = E_{12}[(\varepsilon-\varepsilon^{-1})E_1E_2+\varepsilon E_{12}] = 0$ on U (thus $(\varepsilon-\varepsilon^{-1})^\ell E_1^\ell E_2^\ell + E_{12}^\ell = 0$). Since $V = U + E_{123}U + ... + E_{123}^{\ell-1}U$ and $xE_{123} = \varepsilon^2 E_{123}x$, it follows that $x = 0$ on V. Since $E_{12}^\ell \neq 0$, E_{12} is invertible so we have $(\varepsilon-\varepsilon^{-1})E_1E_2 + \varepsilon E_{12} = 0$ on V. Solving for E_{12}, we find that $E_{12} = (\varepsilon^{-2}-1)E_1E_2$. Substituting this into the relation $E_1E_2 = \varepsilon^{-1}E_2E_1 - E_{12}$, we find that $E_1E_2 = \varepsilon E_2E_1$ on V. Thus we see that E_1 and E_2 q-commute with all generators. Thus $E_1^\ell = 0$ would imply that $E_1 = 0$, which would further imply that $E_{12} = 0$, contrary to assumption. Therefore $E_1^\ell \neq 0$, and likewise $E_2^\ell \neq 0$. Direct verification shows that the element $E_{123}E_2^{\ell-1}$ commutes with each of the generators E_1, E_{12}, E_2, and E_{123}. Thus this element is central and acts as a scalar α on V. E_2 acts as a scalar c ($c \neq 0$) on V. Multiplying both sides of the equation $E_{123}E_2^{\ell-1} = \alpha$ on the right by E_2, we obtain $E_{123} = (\alpha/c)E_2$. Since we can express E_{12} and E_{123} in terms of E_1 and E_2, V must be irreducible over the generators E_1 and E_2, which satisfy $E_1E_2 = \varepsilon E_2E_1$. Since $E_1^\ell \neq 0$ and $E_2^\ell \neq 0$, we have $\dim V = \ell$ (shown in the same way as when we let V be an irreducible $\mathcal{U}_\varepsilon^{s_1s_2}$-module).

Proposition 2.8 *The finite-dimensional irreducible representations of* $\mathcal{U}_\varepsilon^{s_1s_2s_1s_3s_2}$, *where ε is a primitive ℓth root of unity with ℓ odd, have dimensions:*

1 *if $E_{123}^{\ell} = 0$, $E_{12}^{\ell} = 0$, and any two or three of E_1^{ℓ}, E_2^{ℓ}, E_{23}^{ℓ} are zero*

ℓ^2 *if $E_{123}^{\ell} \neq 0$ and $(\varepsilon - \varepsilon^{-1})^{\ell} E_1^{\ell} E_2^{\ell} + E_{12}^{\ell} \neq 0$*

ℓ *in all other cases*

Proof: Let V be an irreducible $\mathcal{U}_{\varepsilon}^{s_1 s_2 s_1 s_3 s_2}$-module. The element $E_{12}E_{23} - qE_2E_{123}$ commutes with the generators E_1, E_{12}, E_2, E_{123}, and E_{23} of the algebra $\mathcal{U}^{s_1 s_2 s_1 s_3 s_2}$, so is central. Thus $E_{12}E_{23} - \varepsilon E_2 E_{123}$ acts as a scalar α on V.

Case 1: $E_{12}^{\ell} \neq 0$ on V. E_{12}^{ℓ} acts as a scalar b ($b \neq 0$) on V. From $E_{12}E_{23} - \varepsilon E_2 E_{123} = \alpha$, we may solve for E_{23}: $E_{23} = (1/b)[\varepsilon E_{12}^{\ell-1} E_2 E_{123} + \alpha E_{12}^{\ell-1}]$. Thus if U is an irreducible $\mathcal{U}_{\varepsilon}^{s_1 s_2 s_1 s_3}$-submodule of V, we see that U is E_{23}-invariant, so $U = V$. From previous results, we know that dim$U$=ℓ or ℓ^2 when $E_{12}^{\ell} \neq 0$ on U. Thus dim$V$=ℓ or ℓ^2.

Case 2: $E_{12}^{\ell} = 0$ and $E_{123}^{\ell} = 0$. E_{123} q-commutes with all the generators, so $E_{123}^{\ell} = 0$ implies $E_{123} = 0$ on V. It then follows that E_{12} now q-commutes with all other generators in the representation, so $E_{12}^{\ell} = 0$ implies $E_{12} = 0$ on V. We are left with the generators E_1, E_2, and E_{23}, which satisfy the relations $E_1E_2 = \varepsilon^{-1}E_2E_1$, $E_1E_{23} = \varepsilon^{-1}E_{23}E_1$, and $E_2E_{23} = \varepsilon E_{23}E_2$. We find that the elements $E_1E_2E_{23}^{\ell-1}$ and $E_1^{\ell-1}E_2^{\ell-1}E_{23}$ are central in the representation, so they act as scalars: $E_1E_2E_{23}^{\ell-1} = \beta$ and $E_1^{\ell-1}E_2^{\ell-1}E_{23} = \gamma$. Because E_1, E_2, and E_{23} all q-commute, if the ℓth power of any of these generators is 0, then the generator itself is zero. Thus if any two (or all three) of these generators have ℓth powers equal to zero, then V will be one-dimensional, spanned by an eigenvector of the third generator. Now suppose any two (or all three) of these generators have ℓth powers not equal to zero. Let U be an irreducible submodule of V over the algebra with those two generators and their relation. Then (as before) dim$U$ = ℓ. But U is invariant under the third generator, because we can solve for the third generator in terms of the first two from $E_1E_2E_{23}^{\ell-1} = \beta$ or $E_1^{\ell-1}E_2^{\ell-1}E_{23} = \gamma$. Thus in this situation dim$V$ = ℓ. So in this case we then have dim$V$=1 or ℓ.

Case 3: $E_{12}^{\ell} = 0$, $E_{123}^{\ell} \neq 0$, $E_1^{\ell} = 0$. Let U be an irreducible $\mathcal{U}_{\varepsilon}^{s_1 s_2 s_1 s_3}$-submodule of V. We have seen that in the case $E_{12}^{\ell} = 0$, $E_{123}^{\ell} \neq 0$, and $E_1^{\ell} = 0$ on U, that U is one-dimensional, spanned by a vector u which satisfies $E_1 u = 0$, $E_{12}u = 0$, $E_2 u = \lambda u$, and $E_{123}u = \mu u$, where $\mu \neq 0$ since $E_{123}^\ell \neq 0$. We have $V = U + E_{23}U + ... + E_{23}^{\ell-1}U$. $E_{123}(E_{23}^m u) = \mu\varepsilon^m(E_{23}^m u)$, and μ, $\mu\varepsilon$, ... , $\mu\varepsilon^{\ell-1}$ are distinct, so $V = U \oplus E_{23}U \oplus ... \oplus E_{23}^{\ell-1}U$. It remains to show that each of these summands is nonzero. If $E_{23}^\ell \neq 0$ this is clear. If $E_{23}^\ell = 0$, we use the following formula, which is proven by induction on m ($m = 1, 2, ...$):

$$E_1 E_{23}^m = \varepsilon^{-m} E_{23}^m E_1 - \varepsilon^{1-m}\left(\frac{1-\varepsilon^{2m}}{1-\varepsilon^2}\right) E_{23}^{m-1} E_{123}. \tag{2.31}$$

Let m be the least positive integer such that $E_{23}^m u = 0$. Then, applying (2.31) to u, we obtain

$$0 = -\varepsilon^{1-m}\left(\frac{1-\varepsilon^{2m}}{1-\varepsilon^2}\right)\mu E_{23}^{m-1}u. \tag{2.32}$$

It follows that ℓ divides $2m$, which implies that ℓ divides m, since ℓ is odd. Thus $\ell = m$, and $\dim V = \ell$ in this case.

Case 4: $E_{12}^\ell = 0$, $E_{123}^\ell \neq 0$, $E_1^\ell \neq 0$, $E_2^\ell = 0$ on V. Let U be an irreducible $\mathcal{U}_\varepsilon^{s_1 s_2 s_1 s_3}$-submodule of V. From previous results we know that $E_{12}^\ell = 0$ implies $E_{12} = 0$ on U. Now E_2 q-commutes with the other generators in the representation U, so $E_2^\ell = 0$ implies $E_2 = 0$ on U. Now a simple induction argument shows that $E_{12}(E_{23}^m U) = 0$, so $E_{12} = 0$ on V (since $V = U + E_{23}U + ... + E_{23}^{\ell-1}U$). From the relation $E_{12}E_{23} = E_{23}E_{12} + (\varepsilon - \varepsilon^{-1})E_2 E_{123}$, we now have $0 = (\varepsilon - \varepsilon^{-1})E_2 E_{123}$ on V. Since $E_{123}^\ell \neq 0$, this implies that $E_2 = 0$ on V. Then V irreducible over the generators E_1, E_{123}, E_{23}, with relations $E_1 E_{123} = \varepsilon E_{123} E_1$, $E_1 E_{23} = \varepsilon^{-1} E_{23} E_1 - E_{123}$, and $E_{123}E_{23} = \varepsilon E_{23} E_{123}$. Relabeling these generators $E_1 \to E_1$, $E_{123} \to E_{12}$, and $E_{23} \to E_2$, we see that we have the algebra $\mathcal{U}_\varepsilon^{s_1 s_2 s_1}$. Then from previous results (noting that $0 \neq E_{123}^\ell \to E_{12}^\ell$) we have $\dim V = \ell$.

Case 5: $E_{12}^\ell = 0$, $E_{123}^\ell \neq 0$, $E_1^\ell \neq 0$, $E_2^\ell \neq 0$ on V. Let U be an irreducible $\mathcal{U}_\varepsilon^{s_1 s_2 s_1 s_3}$-submodule of V. From previous results we know that U has a basis given by $(u, E_1 u, ..., E_1^{\ell-1})$, where u is a common eigenvector of E_{123} and E_2; $E_{123}u = \lambda u$ ($\lambda \neq 0$, since $E_{123}^\ell \neq 0$), $E_2 u = \mu u$ ($\mu \neq 0$, since $E_2^\ell \neq 0$). We also know that $E_{12} = 0$ on U. $E_2 E_{123}$ commutes with E_1, so we see that $E_2 E_{123} = \mu\lambda$ on U.

$V = U + E_{23}U + \ldots + E_{23}^{\ell-1}U$, and $E_2E_{123}(E_{23}^m u) = \mu\lambda\varepsilon^{2m}(E_{23}^m u)$ for any $u \in U$. Thus U, $E_{23}U$, ... , $E_{23}^{\ell-1}U$ are eigenspaces of E_2E_{123} with respective eigenvalues $\mu\lambda$, $\mu\lambda\varepsilon^2$, ... , $\mu\lambda\varepsilon^{2\ell-2}$ (distinct, since ℓ is odd). Thus $V = U \oplus E_{23}U \oplus \ldots \oplus E_{23}^{\ell-1}U$. If $E_{23}^\ell \neq 0$, then $\dim V = \ell \dim U = \ell^2$. If $E_{23}^\ell = 0$, we use the following formula, which is proven by induction on m ($m = 1, 2, \ldots$):

$$E_{12}E_{23}^m = E_{23}^m E_{12} + \left(\frac{\varepsilon^{2m}-1}{\varepsilon}\right) E_{23}^{m-1}E_2E_{123}. \tag{2.33}$$

Let $u \in U$, $u \neq 0$, and let m be the least positive integer such that $E_{23}^m u = 0$. Applying (2.33) to u, we have

$$0 = \mu\lambda\left(\frac{\varepsilon^{2m}-1}{\varepsilon}\right) E_{23}^{m-1}u. \tag{2.34}$$

It follows that ℓ divides $2m$, which implies that ℓ divides m, since ℓ is odd. Thus $\ell = m$. It then follows that each space in the sum $V = U \oplus E_{23}U \oplus \ldots \oplus E_{23}^{\ell-1}U$ has dimension equal to the dimension of U, and $\dim V = \ell \dim U = \ell^2$.

Proposition 2.9 *The finite-dimensional irreducible representations of $\mathcal{U}_\varepsilon^{s_1s_2s_1s_3s_2s_1}$, where ε is a primitive ℓth root of unity with ℓ odd, have dimensions:*

1 *if* $E_{123}^\ell = 0$, $E_{12}^\ell = 0$, $E_{23}^\ell = 0$, *and* $E_2^\ell = 0$

1 *if* $E_{123}^\ell = 0$, $E_{12}^\ell = 0$, $E_{23}^\ell = 0$, $E_1^\ell = 0$, *and* $E_3^\ell = 0$

ℓ^2 *if* $E_{123}^\ell = 0$, $E_{12}^\ell = 0$, $E_1^\ell \neq 0$, $E_{23}^\ell \neq 0$, *and* $(\varepsilon - \varepsilon^{-1})^\ell E_2^\ell E_3^\ell + E_{23}^\ell \neq 0$

ℓ^2 *if* $E_{123}^\ell = 0$, $E_{23}^\ell = 0$, $E_3^\ell \neq 0$, $E_{12}^\ell \neq 0$, *and* $(\varepsilon - \varepsilon^{-1})^\ell E_1^\ell E_2^\ell + E_{12}^\ell \neq 0$

ℓ^2 *if* $E_{123}^\ell \neq 0$, $(\varepsilon - \varepsilon^{-1})^\ell E_1^\ell E_2^\ell + E_{12}^\ell \neq 0$

ℓ^2 *if* $E_{123}^\ell \neq 0$, $(\varepsilon - \varepsilon^{-1})^\ell E_1^\ell E_2^\ell + E_{12}^\ell = 0$, *and* $(\varepsilon - \varepsilon^{-1})E_3[(\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}]E_{123} + E_{123}[(\varepsilon - \varepsilon^{-1})E_1E_{23} + \varepsilon E_{123}] \neq 0$

ℓ^2 *if* $E_{123}^\ell \neq 0$, $(\varepsilon - \varepsilon^{-1})^\ell E_1^\ell E_2^\ell + E_{12}^\ell = 0$, $(\varepsilon - \varepsilon^{-1})E_3[(\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}]E_{123} + E_{123}[(\varepsilon - \varepsilon^{-1})E_1E_{23} + \varepsilon E_{123}] = 0$. $E_1^\ell \neq 0$. $E_{23}^\ell \neq 0$, *and* $(\varepsilon - \varepsilon^{-1})^\ell E_2^\ell E_3^\ell + E_{23}^\ell \neq 0$

ℓ *in all other cases*

Proof: Let V be an irreducible $\mathcal{U}_\varepsilon^{s_1s_2s_1s_3s_2s_1}$-module.

Case 1: $E_{123}^{\ell} = 0$, $E_{23}^{\ell} = 0$, $E_3^{\ell} = 0$. It then follows that $E_{123} = 0$, $E_{23} = 0$, and $E_3^{\ell} = 0$ in the representation. We conclude then that V is an irreducible $\mathcal{U}_{\varepsilon}^{s_1 s_2 s_1}$-module, so $\dim V = 1$ or ℓ.

Case 2: $E_{123}^{\ell} = 0$, $E_{23}^{\ell} = 0$, $E_3^{\ell} \neq 0$, $E_{12}^{\ell} = 0$. Then $E_{123} = 0$, $E_{23} = 0$, and $E_{12} = 0$ on V. Thus V is an irreducible module over the algebra with generators E_1, E_2, and E_3 with relations $E_1E_2 = \varepsilon^{-1}E_2E_1$, $E_1E_3 = E_3E_1$, and $E_2E_3 = \varepsilon^{-1}E_3E_2$. Let v be a common eigenvector of E_1 and E_3. If $E_2^{\ell} = 0$, then $E_2 = 0$ and $\dim V = 1$. If $E_2^{\ell} \neq 0$, then the vectors $(v, E_2v, ..., E_2^{\ell-1}v)$ form a basis for V (they are eigenvectors of E_1 and E_3, with distinct eigenvalues as eigenvectors of E_3), so $\dim V = \ell$.

Case 3: $E_{123}^{\ell} = 0$, $E_{23}^{\ell} = 0$, $E_3^{\ell} \neq 0$, $E_{12}^{\ell} \neq 0$. Then $E_{123} = 0$ and $E_{23} = 0$ on V. Let U be an irreducible submodule of V over the algebra with generators E_1, E_{12}, E_2 and their relations, i.e. $\mathcal{U}_{\varepsilon}^{s_1 s_2 s_1}$. Since $E_{12}^{\ell} \neq 0$, we have $\dim U = \ell$. We know that the element $E_{12}[(\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}]$ is central and acts as a scalar α on U. Let $x = E_{12}[(\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}]$. We find that $xE_3 = \varepsilon^{-2}E_3x$, using $E_{123} = 0$ and $E_{23} = 0$. We consider the following two subcases.

Case 3a: $x = E_{12}[(\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}]$ acts as a nonzero scalar α on U (Thus $(\varepsilon - \varepsilon^{-1})^{\ell}E_1^{\ell}E_2^{\ell} + E_{12}^{\ell} \neq 0$). Then $V = U + E_3U + ... + E_3^{\ell-1}U$, and since $xE_3 = \varepsilon^{-2}E_3x$, we see that U, E_3U, ... , $E_3^{\ell-1}U$ are eigenspaces of x with eigenvalues α, $\alpha\varepsilon^{-2}$, ... , $\alpha\varepsilon^{2\ell-2}$ (distinct, since ℓ is odd). Furthermore, each of these spaces has dimension equal to the dimension of U, since $E_3^{\ell} \neq 0$. So $\dim V = \ell \dim U = \ell^2$.

Case 3b: $x = E_{12}[(\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}] = 0$ on U (Thus $(\varepsilon - \varepsilon^{-1})^{\ell}E_1^{\ell}E_2^{\ell} + E_{12}^{\ell} = 0$). Then since $V = U + E_3U + ... + E_3^{\ell-1}U$ and $xE_3 = \varepsilon^{-2}E_{123}x$, it follows that $x = 0$ on V. Since $E_{12}^{\ell} \neq 0$, this implies that $[(\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}] = 0$ on V. As in case 4b of Proposition 2.7, we find that $E_{12} = (\varepsilon^{-2} - 1)E_1E_2$ and $E_1E_2 = \varepsilon E_2E_1$ on V. Thus V is an irreducible module over the algebra with generators E_1, E_2, and E_3 with relations $E_1E_2 = \varepsilon E_2E_1$, $E_1E_3 = E_3E_1$, and $E_2E_3 = \varepsilon^{-1}E_3E_2$. We then find in the same manner as for case 2 of this proposition that $\dim V = \ell$.

Case 4: $E_{123}^{\ell} = 0$, $E_{12}^{\ell} = 0$ on V. It follows that $E_{123} = 0$, $E_{12} = 0$ on V. Now relabel the generators as follows: $E_1 \to E_3$, $E_2 \to E_2$, $E_3 \to E_1$, and $E_{23} \to -\varepsilon^{-1}E_{12}$. Now we find that we have the same generators and relations as we have for $\mathcal{U}_{\varepsilon^{-1}}^{s_1 s_2 s_1 s_3 s_2 s_1}$

when $E_{123}^{\ell} = 0$, $E_{23}^{\ell} = 0$ on V. ε^{-1} is also a primitive ℓth root of 1, so this is covered in cases 1 through 3 of this proposition. We conclude as in those cases that $\dim V = 1$, ℓ, or ℓ^2. Note that the condition (i.e. whether or not it is zero) on the element $(\varepsilon^{-1} - \varepsilon)^{\ell} E_1^{\ell} E_2^{\ell} + E_{12}^{\ell}$ of $\mathcal{U}_{\varepsilon^{-1}}^{s_1 s_2 s_1 s_3 s_2 s_1}$ becomes here the corresponding condition on the element $(\varepsilon - \varepsilon^{-1})^{\ell} E_2^{\ell} E_3^{\ell} + E_{23}^{\ell}$.

Case 5: $E_{123}^{\ell} = 0$, $E_{12}^{\ell} \neq 0$, $E_{23}^{\ell} \neq 0$, and $E_1^{\ell} \neq 0$ on V. $E_{123}^{\ell} = 0$ implies $E_{123} = 0$ on V. The element $E_{12}E_{23} - \varepsilon E_2 E_{123} = E_{12}E_{23}$ acts as a scalar α on V, so $E_{23} = (\alpha/b)E_{12}^{\ell-1}$ on V, where $b = E_{12}^{\ell}$. Let U be an irreducible submodule of V over the algebra with generators E_1, E_{12}, E_{23}, and E_3 with their relations. Let u be a common eigenvector of E_1 (with eigenvalue $\lambda \neq 0$ for E_1) and E_3 (with eigenvalue μ), which commute. Then the space $\mathrm{span}(u, E_{12}u, ..., E_{12}^{\ell-1}u)$ is seen to be invariant under E_1, E_{12}, E_{23}, and E_3. Also, the vectors $u, E_{12}u, ..., E_{12}^{\ell-1}u$ are eigenvalues of E_1 with distinct eigenvalues, since $E_1(E_{12}^m u) = \varepsilon^m \lambda (E_{12}^m u)$. Thus $U = \mathrm{span}(u, E_{12}u, ..., E_{12}^{\ell-1}u)$ and $\dim U = \ell$. Note also that $E_3(E_{12}^m u) = \varepsilon^m \mu (E_{12}^m u)$, so $E_3 = (\mu/\lambda)E_1$ on U. $E_{12}[(\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}]$ commutes with E_1, E_{12}, and E_2, and thus must also commute with E_{23} and E_3, since $E_{23} = (\alpha/b)E_{12}^{\ell-1}$ and $E_3 = (\mu/\lambda)E_1$. Thus $E_{12}[(\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}] = \beta$ (β=constant) on U. Since $E_{12}^{\ell} \neq 0$ and $E_1^{\ell} \neq 0$, E_{12} and E_1 are invertible and we can solve for E_2 in terms of β, E_{12} and E_1. Thus U is E_2-invariant, so $U = V$ and $\dim V = \ell$.

Case 6: $E_{123}^{\ell} = 0$, $E_{12}^{\ell} \neq 0$, $E_{23}^{\ell} \neq 0$, and $E_3^{\ell} \neq 0$ on V. Relabeling $E_1 \to E_3$, $E_2 \to E_2$, $E_3 \to E_1$, $E_{12} \to -\varepsilon^{-1}E_{23}$ and $E_{23} \to -\varepsilon^{-1}E_{12}$, we obtain the same algebra we had in case 5, with ε^{-1} in place of ε. Since ε^{-1} is also a primitive ℓth root of 1, by case 5 we have $\dim V = \ell$.

Case 7: $E_{123}^{\ell} = 0$, $E_{12}^{\ell} \neq 0$, $E_{23}^{\ell} \neq 0$, $E_1^{\ell} = 0$, and $E_3^{\ell} = 0$ on V. Then $E_{123} = 0$ on V. Let U be an irreducible submodule of V over the algebra with generators E_1, E_{12}, E_{23}, and E_3. These generators all q-commute (since $E_{123} = 0$ on V), so $E_1^{\ell} = 0$ and $E_3^{\ell} = 0$ on U imply $E_1 = 0$ and $E_3 = 0$ on U. Let u be an eigenvector of E_{12}; $E_{12}u = \lambda u$, $\lambda \neq 0$. As in case 5, we have $E_{23} = (\alpha/b)E_{12}^{\ell-1}$ on V, where $b = E_{12}^{\ell}$. Thus u is also an eigenvector of E_{23}, so $\dim U = 1$. Now $V = U + E_2U + ... + E_2^{\ell-1}U$, so $\dim V \leq \ell$. But V is also a module over the algebra with generators E_1, E_{12}, and

E_2, and we have seen that an irreducible module over this algebra with $E_{12}^\ell \neq 0$ has dimension equal to ℓ, thus $\dim V \geq \ell$. So $\dim V = \ell$.

We now consider the cases where $E_{123}^\ell \neq 0$. Direct verification shows that the element $(\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}$ q-commutes with each of the generators E_1, E_{12}, E_2, E_{123}, and E_{23} (but not with E_3). Let $x = (\varepsilon - \varepsilon^{-1})E_1E_2 + \varepsilon E_{12}$. Also recall that the element $(q-q^{-1})E_3[(q-q^{-1})E_1E_2+qE_{12}]E_{123}+E_{123}[(q-q^{-1})E_1E_{23}+qE_{123}]$ is central in $\mathcal{U}^{s_1s_2s_1s_3s_2s_1}$, so $(\varepsilon-\varepsilon^{-1})E_3[(\varepsilon-\varepsilon^{-1})E_1E_2+\varepsilon E_{12}]E_{123}+E_{123}[(\varepsilon-\varepsilon^{-1})E_1E_{23}+\varepsilon E_{123}]$ acts as a scalar in a finite-dimensional irreducible representation of $\mathcal{U}_\varepsilon^{s_1s_2s_1s_3s_2s_1}$. Let $y = [(\varepsilon-\varepsilon^{-1})E_1E_{23}+\varepsilon E_{123}]$ and $z = (\varepsilon-\varepsilon^{-1})E_3[(\varepsilon-\varepsilon^{-1})E_1E_2+\varepsilon E_{12}]E_{123}+E_{123}[(\varepsilon-\varepsilon^{-1})E_1E_{23}+\varepsilon E_{123}]$, so $z = (\varepsilon - \varepsilon^{-1})E_3xE_{123} + E_{123}y$. $z = \alpha$ for some scalar α on V. For the following cases, let U be an irreducible submodule of V over the algebra with generators E_1, E_{12}, E_2, E_{123}, and E_{23} with their relations. Since x q-commutes with each of the generators, x^ℓ is central in this algebra. By the same proof as in lemma 2.2, either $x = 0$ on U or $x^\ell \neq 0$ on U (i.e. x is invertible on U). Also we have $x^\ell = (\varepsilon - \varepsilon^{-1})^\ell E_1^\ell E_2\ell + E_{12}^\ell$, and likewise $y^\ell = (\varepsilon - \varepsilon^{-1})^\ell E_1^\ell E_{23}\ell + E_{123}^\ell$.

Case 8: $E_{123}^\ell \neq 0$ on V, $x^\ell \neq 0$ on U. In this case we may solve for E_3 in terms of E_1, E_{12}, E_2, E_{123}, and E_{23} from the equation $z = (\varepsilon - \varepsilon^{-1})E_3xE_{123} + E_{123}y = \alpha$. Thus U is E_3-invariant, so $V = U$ and $\dim U = \ell^2$ in this case.

Case 9: $E_{123}^\ell \neq 0$ on V, $x = 0$ on U (thus $(\varepsilon - \varepsilon^{-1})^\ell E_1^\ell E_2\ell + E_{12}^\ell = 0$), and $z = (\varepsilon - \varepsilon^{-1})E_3xE_{123} + E_{123}y = \alpha \neq 0$ on V. It follows that $E_{123}y = \alpha$ on U. $V = U + E_3U + ... + E_3^{\ell-1}U$, and we find that $(E_{123}y)E_3^m = \varepsilon^{2m}E_3^m(E_{123}y)$, so the spaces in the sum are eigenspaces of $E_{123}y$ with distinct (since ℓ is odd) eigenvalues α, $\alpha\varepsilon^2$, ... , $\alpha\varepsilon^{2\ell-2}$. Thus $V = U \oplus E_3U \oplus ... \oplus E_3^{\ell-1}U$. If $E_3^\ell \neq 0$, we conclude that $\dim V = \ell \dim U$. If $E_3^\ell = 0$, We use the following formula, which is proved by induction:

$$(E_{123}x)E_3^m = E_3^m(E_{123}x) - \left(\frac{1-\varepsilon^m}{1-\varepsilon}\right) E_3^{m-1}(E_{123}y) \tag{2.35}$$

Let $u \in U$, $u \neq 0$. Let m be the least positive integer such that $E_3^m u = 0$. Applying equation 2.35 to u, we obtain

$$0 = \alpha \left(\frac{1 - \varepsilon^m}{1 - \varepsilon} \right) E_3^{m-1} u, \tag{2.36}$$

from which we conclude that $m = \ell$. Thus we again have $\dim V = \ell \dim U$. In this case we previously found that $\dim U = \ell$, so $\dim V = \ell^2$.

Case 10: $E_{123}^\ell \neq 0$ on V, $x = 0$ on U (thus $(\varepsilon - \varepsilon^{-1})^\ell E_1^\ell E_2 \ell + E_{12}^\ell = 0$), and $z = (\varepsilon - \varepsilon^{-1}) E_3 x E_{123} + E_{123} y = 0$ on V. It follows that $E_{123} y = 0$ on U, so $y = 0$ on U. We find that $y E_3 = \varepsilon E_3 y$, and since $V = U + E_3 U + \ldots + E_3^{\ell-1} U$ we conclude that $y = 0$ on V. Equation 2.35 now becomes $(E_{123} x) E_3^m = E_3^m (E_{123} x)$, so we also see that $E_{123} x = 0$ on V, hence $x = 0$ on V. From $x = 0$ we find that $E_{12} = (\varepsilon^{-2} - 1) E_1 E_2$ and $E_1 E_2 = \varepsilon E_2 E_1$ on V. From $y = 0$ we find that $E_{123} = (\varepsilon^{-2} - 1) E_1 E_{23}$ and $E_1 E_{23} = \varepsilon E_{23} E_1$ on V. Also, $y^\ell = (\varepsilon - \varepsilon^{-1})^\ell E_1^\ell E_{23} \ell + E_{123}^\ell = 0$; since we have assumed that $E_{123}^\ell \neq 0$, it follows that we must also have $E_1^\ell \neq 0$ and $E_{23}^\ell \neq 0$. V is thus an irreducible module over the generators E_1, E_2, E_{23}, and E_3, which satisfy the relations:

$$E_1 E_2 = \varepsilon E_2 E_1 \tag{2.37}$$

$$E_1 E_{23} = \varepsilon E_{23} E_1 \tag{2.38}$$

$$E_1 E_3 = E_3 E_1 \tag{2.39}$$

$$E_2 E_{23} = \varepsilon E_{23} E_2 \tag{2.40}$$

$$E_2 E_3 = \varepsilon^{-1} E_3 E_2 - E_{23} \tag{2.41}$$

$$E_{23} E_3 = \varepsilon E_3 E_{23} \tag{2.42}$$

Let W be an irreducible submodule of V over the algebra with generators E_2, E_{23}, and E_3. This algebra is obviously isomorphic to $\mathcal{U}_\varepsilon^{s_1 s_2 s_1}$, which was considered earlier. The element $E_{23}[(\varepsilon - \varepsilon^{-1}) E_2 E_3 + \varepsilon E_{23}]$ is central in this algebra, and so acts as a scalar β on W. Let $\tilde{x} = E_{23}[(\varepsilon - \varepsilon^{-1}) E_2 E_3 + \varepsilon E_{23}]$. We have $V = W + E_1 W + \ldots + E_1^{\ell-1} W$. We now consider the following subcases:

Case 10a: $E_1^\ell \neq 0$, $E_{23}^\ell \neq 0$, and $\tilde{x} = E_{23}[(\varepsilon - \varepsilon^{-1})E_2E_3 + \varepsilon E_{23}] \neq 0$ on W (thus $\varepsilon - \varepsilon^{-1})^\ell E_2^\ell E_3^\ell + E_{23}^\ell \neq 0$). We find that $\tilde{x}E_1 = \varepsilon^{-2}E_1\tilde{x}$. Thus the spaces W, E_1W,...,$E_1^{\ell-1}W$ are eigenspaces of $\tilde{x}$ with distinct (since ℓ is odd) eigenvalues β, $\beta\varepsilon^{-2}$, ... , $\beta\varepsilon^{2\ell-2}$. Since $E_1^\ell \neq 0$, we have $\dim V = \ell \dim W$. $\dim W = \ell$, since$E_{23}^\ell \neq 0$, so $\dim V = \ell^2$.

Case 10b: $E_1^\ell \neq 0$, $E_{23}^\ell \neq 0$, and $\tilde{x} = E_{23}[(\varepsilon - \varepsilon^{-1})E_2E_3 + \varepsilon E_{23}] = 0$ on W. Then $[(\varepsilon - \varepsilon^{-1})E_2E_3 + \varepsilon E_{23}] = 0$ on W. We find that $[(\varepsilon - \varepsilon^{-1})E_2E_3 + \varepsilon E_{23}]E_1 = \varepsilon^{-1}E_1[(\varepsilon - \varepsilon^{-1})E_2E_3 + \varepsilon E_{23}]$. It then follows from $V = W + E_1W + ... + E_1^{\ell-1}W$ that $[(\varepsilon - \varepsilon^{-1})E_2E_3 + \varepsilon E_{23}] = 0$ on V. We then find that $E_{23} = (\varepsilon^{-2} - 1)E_2E_3$ on V, and that $E_2E_3 = \varepsilon E_3E_2$ on V. V is then irreducible over the generators E_1, E_2, and E_3, which satisfy the relations $E_1E_2 = \varepsilon E_2E_1$, $E_1E_3 = E_3E_1$, and $E_2E_3 = \varepsilon E_3E_2$. Since $(\varepsilon - \varepsilon^{-1})^\ell E_2^\ell E_3^\ell + E_{23}^\ell = 0$ and $E_{23}^\ell \neq 0$, we have $E_2^\ell \neq 0$. V then has dimension ℓ.

2.5 $\mathcal{U}^w$ for $w = s_1s_3$, $s_1s_3s_2$, $s_1s_3s_2s_1$, and $s_1s_3s_2s_1s_3$

For $w = s_1s_3s_2s_1s_3$, we find that $\beta_1 = \alpha_1$, $\beta_2 = \alpha_3$, $\beta_3 = \alpha_1 + \alpha_2 + \alpha_3$, $\beta_4 = \alpha_2 + \alpha_3$, and $\beta_5 = \alpha_1 + \alpha_2$. We then find, using (2.8) through (2.12), that $E_{\beta_1} = E_1$, $E_{\beta_2} = E_3$, $E_{\beta_3} = E_{123} = E_3E_1E_2 - q^{-1}E_3E_2E_1 - q^{-1}E_1E_2E_3 + q^{-2}E_2E_1E_3$. $E_{\beta_4} = E_{23} = -E_3E_2 + q^{-1}E_2E_3$, $E_{\beta_5} = E_{12} = -E_1E_2 + q^{-1}E_2E_1$. With some computation, we find the relations (2.13) are as follows:

$$E_1E_3 = E_3E_1 \tag{2.43}$$

$$E_1E_{123} = qE_{123}E_1 \tag{2.44}$$

$$E_3E_{123} = qE_{123}E_3 \tag{2.45}$$

$$E_1E_{23} = q^{-1}E_{23}E_1 - E_{123} \tag{2.46}$$

$$E_3E_{23} = qE_{23}E_3 \tag{2.47}$$

$$E_{123}E_{23} = qE_{23}E_{123} \tag{2.48}$$

$$E_1E_{12} = qE_{12}E_1 \tag{2.49}$$

$$E_3E_{12} = q^{-1}E_{12}E_3 - E_{123} \tag{2.50}$$

$$E_{123}E_{12} = qE_{12}E_{123} \tag{2.51}$$

$$E_{23}E_{12} = E_{12}E_{23} \tag{2.52}$$

$\mathcal{U}^{s_1s_3}$ has generators E_1, E_3 with relation (2.43). $\mathcal{U}^{s_1s_3s_2}$ has generators E_1, E_3, E_{123} with relations (2.43) through (2.45). $\mathcal{U}^{s_1s_3s_2s_1}$ has generators E_1, E_3, E_{123}, and E_{23} with relations (2.43) through (2.48).

$\mathcal{U}^{s_1s_3s_2s_1s_3}$ has generators E_1, E_3, E_{123}, E_{23}, and E_{12} with relations (2.43) through (2.52). We find that the element $[(q-q^{-1})E_1E_{23}+qE_{123}][(q-q^{-1})E_3E_{12}+qE_{123}]$ is central in this algebra.

2.6 Irreducible Representations of $\mathcal{U}_\varepsilon^w$ for $w = s_1s_3$, $s_1s_3s_2$, $s_1s_3s_2s_1$, and $s_1s_3s_2s_1s_3$

Proposition 2.10 *The finite-dimensional irreducible representations of $\mathcal{U}_\varepsilon^{s_1s_3}$, where ε is a primitive ℓth root of unity, have dimension* 1.

Proof: Let V be an irreducible $\mathcal{U}_\varepsilon^{s_1s_3}$-module. E_1 and E_3 commute, so V is spanned by a common eigenvector of E_1 and E_3.

Proposition 2.11 *The finite-dimensional irreducible representations of $\mathcal{U}_\varepsilon^{s_1s_3s_2}$, where ε is a primitive ℓth root of unity, have dimensions:*

1   *if* $E_{123}^\ell = 0$

1   *if* $E_1^\ell = 0$ *and* $E_3^\ell = 0$

ℓ *in all other cases*

Proof: Let V be an irreducible $\mathcal{U}_\varepsilon^{s_1s_3s_2}$-module. If $E_{123}^\ell = 0$, then $E_{123} = 0$ and V is spanned by a common eigenvector of E_1 and E_3. If $E_1^\ell = 0$ and $E_3^\ell = 0$, then $E_1 = 0$ and $E_3 = 0$ and V is spanned by an eigenvector of E_{123}. If $E_{123}^\ell \neq 0$ and

$E_1^\ell \neq 0$, let v be a common eigenvector of E_1 and E_3. Then span$(v, E_{123}v, ..., E_{123}^{\ell-1}v)$ is invariant under each of the three generators, and the vectors $v, E_{123}v, ..., E_{123}^{\ell-1}v$ are eigenvectors of E_1 with distinct eigenvalues. Thus $\dim V = \ell$. Similarly, $\dim V = \ell$ if $E_{123}^\ell \neq 0$ and $E_3^\ell \neq 0$.

Proposition 2.12 *The finite-dimensional irreducible representations of* $\mathcal{U}_\varepsilon^{s_1 s_3 s_2 s_1}$, *where* ε *is a primitive* ℓ*th root of unity with* ℓ *odd, have dimensions:*

1   *if* $E_{123}^\ell = 0$ *and* $E_{23}^\ell = 0$

1   *if* $E_{123}^\ell = 0$, $E_1^\ell = 0$, *and* $E_3^\ell = 0$

ℓ^2 *if* $E_3^\ell \neq 0$. $E_{123}^\ell \neq 0$. *and* $(\varepsilon - \varepsilon^{-1})^\ell E_1^\ell E_{23}^\ell + E_{123}^\ell \neq 0$

ℓ *in all other cases*

Proof: If we relabel the generators $E_1 \to E_1$, $E_3 \to E_3$, $E_{123} \to E_{12}$, and $E_{23} \to E_2$, we find that we have the same generators and relations as we had for $\mathcal{U}_\varepsilon^{s_1 s_2 s_1 s_3 s_2 s_1}$ in the case where $E_{123}^\ell = 0$, $E_{23}^\ell = 0$. From these results we see that $\dim V = 1$, ℓ, or ℓ^2.

Proposition 2.13 *The finite-dimensional irreducible representations of* $\mathcal{U}_\varepsilon^{s_1 s_3 s_2 s_1 s_3}$, *where* ε *is a primitive* ℓ*th root of unity with* ℓ *odd, have dimensions:*

1   *if* $E_{123}^\ell = 0$, $E_1^\ell = 0$, *and* $E_3^\ell = 0$

1   *if* $E_{123}^\ell = 0$, $E_{12}^\ell = 0$, *and* $E_{23}^\ell = 0$

ℓ^2 *if* $E_{123}^\ell \neq 0$ *and* $(\varepsilon - \varepsilon^{-1})^\ell E_1^\ell E_{23}^\ell + E_{123}^\ell \neq 0$

ℓ^2 *if* $E_{123}^\ell \neq 0$ *and* $(\varepsilon - \varepsilon^{-1})^\ell E_3^\ell E_{12}^\ell + E_{123}^\ell \neq 0$

ℓ *in all other cases*

Proof: Let V be an irreducible $\mathcal{U}_\varepsilon^{s_1 s_3 s_2 s_1 s_3}$-module. The element $[(q - q^{-1})E_1 E_{23} + qE_{123}][(q - q^{-1})E_3 E_{12} + qE_{123}]$ is central in $\mathcal{U}^{s_1 s_3 s_2 s_1 s_3}$, so letting $[(\varepsilon - \varepsilon^{-1})E_1 E_{23} + \varepsilon E_{123}][(\varepsilon - \varepsilon^{-1})E_3 E_{12} + \varepsilon E_{123}]$ acts as a scalar α on V. Let $x = [(\varepsilon - \varepsilon^{-1})E_1 E_{23} + \varepsilon E_{123}]$, and let $y = [(\varepsilon - \varepsilon^{-1})E_3 E_{12} + \varepsilon E_{123}]$, so $xy = \alpha$ on V. We also note that x and y each q-commute with each of the generators, so each is either 0 or invertible on V.

Case 1: $E_{123}^\ell = 0$ on V. Then $E_{123} = 0$ on V, and we are left with the generators

E_1, E_3, E_{23}, and E_{12}, which satisfy

$$\begin{aligned} E_1E_3 &= E_3E_1 \\ E_1E_{23} &= \varepsilon^{-1}E_{23}E_1 \\ E_1E_{12} &= \varepsilon E_{12}E_1 \\ E_3E_{23} &= \varepsilon E_{23}E_3 \\ E_3E_{12} &= \varepsilon^{-1}E_{12}E_3 \\ E_{12}E_{23} &= E_{23}E_{12} \end{aligned}$$

We find that E_1E_3 and $E_{23}E_{12}$ are central in the representation. If $E_1^\ell = 0$ and $E_3^\ell = 0$, then $E_1 = 0$ and $E_3 = 0$ on V, and V is one dimensional, spanned by a common eigenvector of E_{23} and E_{12}. Similarly, if $E_{23}^\ell = 0$ and $E_{12}^\ell = 0$, $\dim V = 1$. In all other cases we find that $\dim V = \ell$. For example, if $E_1^\ell \neq 0$ and $E_{23}^\ell \neq 0$, let v be an common eigenvector of E_1 and E_3. $E_{23}E_{12} = \beta$ on V, and since $E_{23}^\ell \neq 0$ we can solve for E_{12} in terms of E_{23} from this equation. Thus span$(v, E_{23}v, ..., E_{23}^{\ell-1})$ is invariant under each of the generators. Also the vectors $v, E_{23}v, ..., E_{23}^{\ell-1}$ are eigenvectors of E_1 with distinct eigenvalues, so $\dim V = \ell$. The other cases are similar.

Case 2: $E_{123}^\ell \neq 0$, $x \neq 0$ on V, and $E_3^\ell \neq 0$. Let U be an irreducible submodule of V over the generators E_1, E_3, E_{123}, and E_{23}. From $x[(\varepsilon - \varepsilon^{-1})E_3E_{12} + \varepsilon E_{123}] = \alpha$, and using the fact that x and E_3 are invertible, we can solve for E_{12} in terms of the other generators. Thus U in E_{12}-invariant, and $V = U$. By previous considerations we know then that $\dim V = \ell$ or ℓ^2.

Case 3: $E_{123}^\ell \neq 0$, $x \neq 0$ on V, and $E_{12}^\ell \neq 0$. Relabeling $E_1 \to E_{23}$, $E_{23} \to E_1$, $E_{12} \to E_3$, and $E_{123} \to -\varepsilon^{-1}E_{123}$, we obtain the same algebra as in case 2, with ε^{-1} in place of ε. Thus by case 2 we have $\dim V = \ell$ or ℓ^2.

Case 4: $E_{123}^\ell \neq 0$, $x \neq 0$ on V, $E_3^\ell = 0$, and $E_{12}^\ell = 0$. Let U be an irreducible submodule of V over the algebra with generators E_1, E_3, E_{123}, and E_{23}. Since E_3 q-commutes with each of these generators and $E_3^\ell = 0$, $E_3 = 0$ on U. The remaining generators E_1, E_{123}, and E_{23} on U satisfy the same relations as the generators in $\mathcal{U}_\varepsilon^{s_1s_2s_1}$ (with $E_1 \to E_1$, $E_{123} \to E_{12}$, and $E_{23} \to E_2$). Thus $\dim U = \ell$, since $E_{123}^\ell \neq 0$.

Also, the element $E_{123}x$ acts as a nonzero scalar on U. We find that $(E_{123}x)E_{12} = \varepsilon^2 E_{12}(E_{123}x)$, so the spaces U, $E_{12}U$, ... , $E_{12}^{\ell-1}U$ are eigenspaces of $E_{123}x$ with distinct eigenvalues (since ℓ is odd). Thus $V = U \oplus E_{12}U \oplus ... \oplus E_{12}^{\ell-1}U$. We shall show that each of the spaces in the sum is nonzero. For that we need the following formula, which is proved by induction on m:

$$E_3 E_{12}^m = \varepsilon^{-m} E_{12}^m E_3 - \varepsilon^{1-m} \left(\frac{1-\varepsilon^{2m}}{1-\varepsilon^2} \right) E_{12}^{m-1} E_{123} \tag{2.53}$$

Let u be an eigenvector of E_{123} in U; $E_{123}u = \lambda u$ with $\lambda \neq 0$. Let m be the least positive integer such that $E_{12}^m u = 0$. Applying 2.53 to u, we obtain:

$$0 = -\lambda \varepsilon^{1-m} \left(\frac{1-\varepsilon^{2m}}{1-\varepsilon^2} \right) E_{12}^{m-1} u \tag{2.54}$$

It follows that $m = \ell$, since ℓ is odd. Thus each of the spaces in the sum $V = U \oplus E_{12}U \oplus ... \oplus E_{12}^{\ell-1}U$ is nonzero, and by lemma 2.3 this gives $\dim V = \ell \dim U$. Thus $\dim V = \ell^2$.

Case 5: $E_{123}^\ell \neq 0$, $y \neq 0$ on V. If we relabel the generators $E_1 \to E_3$, $E_3 \to E_1$, $E_{23} \to E_{12}$, $E_{12} \to E_{23}$, and $E_{123} \to E_{123}$, we find that we have not changed the relations. Thus this case is covered by cases 2, 3, and 4.

Case 6: $E_{123}^\ell \neq 0$, $x = 0$, and $y = 0$ on V. Then $E_{123} = (\varepsilon^{-2} - 1)E_1 E_{23}$ and $E_{123} = (\varepsilon^{-2} - 1)E_3 E_{12}$. We then find that $E_1 E_{23} = \varepsilon E_{23} E_1$ and $E_3 E_{12} = \varepsilon E_{12} E_3$. Thus V is irreducible over the generators E_1, E_3, E_{23}, and E_{12}, which satisfy the relations

$$\begin{aligned}
E_1 E_3 &= E_3 E_1 \\
E_1 E_{23} &= \varepsilon E_{23} E_1 \\
E_1 E_{12} &= \varepsilon E_{12} E_1 \\
E_3 E_{12} &= \varepsilon E_{12} E_3 \\
E_3 E_{23} &= \varepsilon E_{23} E_3 \\
E_{12} E_{23} &= E_{23} E_{12}
\end{aligned}$$

Note that the ℓth power of each of these generators must be nonzero. For example, if $E_1^\ell = 0$, then $E_1 = 0$ on V, which would imply that $E_{123} = 0$, a contradiction. We also find that $E_{23}E_{12}^{\ell-1}$ commutes with each of the generators, so is equal to a scalar β on V. Thus we can solve for E_{23} in terms of E_{12}. Let v be a common eigenvector of E_1 and E_3. The space span$(v, E_{12}v, ..., E_{12}^{\ell-1}v)$ is invariant under each of the generators, and the vectors $v, E_{12}v, ..., E_{12}^{\ell-1}v$ are eigenvectors of E_1 with distinct eigenvalues. Thus $\dim V = \ell$.

2.7 $\mathcal{U}^w$ for $w = s_2s_1s_3$ and $s_2s_1s_3s_2$

For $w = s_2s_1s_3s_2$, we find that $\beta_1 = \alpha_2$, $\beta_2 = \alpha_1+\alpha_2$, $\beta_3 = \alpha_2+\alpha_3$, $\beta_4 = \alpha_1+\alpha_2+\alpha_3$. We then find, using (2.8) through (2.12), that $E_{\beta_1} = E_2$, $E_{\beta_2} = E_{12} = -E_2E_1 + q^{-1}E_1E_2$, $E_{\beta_3} = E_{23} = -E_2E_3 + q^{-1}E_3E_2$, $E_{\beta_4} = E_{123} = E_2E_3E_1 - q^{-1}E_3E_2E_1 - q^{-1}E_1E_2E_3 + q^{-2}E_1E_3E_2$. With some computation, we find the relations (2.13) are as follows:

$$E_2E_{12} = qE_{12}E_2 \tag{2.55}$$

$$E_2E_{23} = qE_{23}E_2 \tag{2.56}$$

$$E_{12}E_{23} = E_{23}E_{12} \tag{2.57}$$

$$E_2E_{123} = E_{123}E_2 + (q-q^{-1})E_{12}E_{23} \tag{2.58}$$

$$E_{12}E_{123} = qE_{123}E_{12} \tag{2.59}$$

$$E_{23}E_{123} = qE_{123}E_{23} \tag{2.60}$$

$\mathcal{U}^{s_2s_1s_3}$ has generators E_2, E_{12}, and E_{23} with relations (2.55) through (2.57).

$\mathcal{U}^{s_2s_1s_3s_2}$ has generators E_2, E_{12}, E_{23}, and E_{123} with relations (2.55) through (2.60). We find that the element $E_2E_{123} - qE_{12}E_{23}$ is central in this algebra.

2.8 Irreducible Representations of $\mathcal{U}_\varepsilon^w$ for $w = s_2s_1s_3$ and $s_2s_1s_3s_2$

Proposition 2.14 *The finite-dimensional irreducible representations of $\mathcal{U}_\varepsilon^{s_2s_1s_3}$, where ε is a primitive ℓth root of unity, have dimensions:*

1   *if* $E_2^\ell = 0$

1   *if* $E_{12}^\ell = 0$ *and* $E_{23}^\ell = 0$

ℓ *in all other cases*

Proof: Let V be an irreducible $\mathcal{U}_\varepsilon^{s_2s_1s_3}$-module. If $E_2^\ell = 0$, then $E_2 = 0$ and V is spanned by a common eigenvector of E_{12} and E_{23}. If $E_{12}^\ell = 0$ and $E_{23}^\ell = 0$, then $E_{12} = 0$ and $E_{23} = 0$, and V is spanned by an eigenvector of E_2, so $\dim V = 1$. If $E_2^\ell \neq 0$ and $E_{12}^\ell \neq 0$, then letting v be a common eigenvector of E_{12} and E_{23}, we find that $\mathrm{span}(v, E_2v, ..., E_2^{\ell-1}v)$ is invariant under each of the generators, and the vectors $v, E_2v, ..., E_2^{\ell-1}v$ are eigenvectors of E_{12} with distinct eigenvalues. Thus $\dim V = \ell$. Similarly, if $E_2^\ell \neq 0$ and $E_{23}^\ell \neq 0$, then $\dim V = \ell$.

Proposition 2.15 *The finite-dimensional irreducible representations of $\mathcal{U}_\varepsilon^{s_2s_1s_3s_2}$, where ε is a primitive ℓth root of unity with ℓ odd, have dimensions:*

1   *if* $E_{12}^\ell = 0$ *and* $E_{23}^\ell = 0$

1   *if* $E_{123}^\ell = 0$, $E_2^\ell = 0$. *and* $E12^\ell$ *or* E_{23}^ℓ *is zero*

ℓ *in all other cases*

Proof: Let V be an irreducible $\mathcal{U}_\varepsilon^{s_2s_1s_3s_2}$-module. The element $E_2E_{123} - qE_{12}E_{23}$ is central in $\mathcal{U}^{s_2s_1s_3s_2}$, so $E_2E_{123} - \varepsilon E_{12}E_{23} = \alpha$ for some scalar α on V.

Case 1: $E_2^\ell \neq 0$. Let U be an irreducible submodule of V over the generators E_2, E_{12}, and E_{23}. From previous results we know that $\dim U = 1$ or ℓ. From $E_2E_{123} - \varepsilon E_{12}E_{23} = \alpha$ we can solve for E_{123} in terms of the other generators, so U is E_{123}-invariant and $U = V$.

Case 2: $E_2^\ell = 0$, $E_{12}^\ell = 0$, and $E_{23}^\ell = 0$. Since E_{12} and E_{23} q-commute with the other generators, we have $E_{12} = 0$, and $E_{23} = 0$ on V. It then follows that E_2

q-commutes with E_{123}, so $E_2 = 0$. Thus V is spanned by an eigenvector of E_{123}, and $\dim V = 1$.

Case 3: $E_2^\ell = 0$, $E_{12}^\ell \neq 0$, and $E_{23}^\ell = 0$. $E_{23}^\ell = 0$ implies $E_{23} = 0$ on V. This implies that E_2 q-commutes with the other generators, so $E_2 = 0$. We are left with the generators E_{12} and E_{123}, which satisfy $E_{123}E_{12} = \varepsilon^{-1}E_{12}E_{123}$. Since $E_{12}^\ell \neq 0$, $\dim V = \ell$ if $E_{123}^\ell \neq 0$ and $\dim V = 1$ if $E_{123}^\ell = 0$.

Case 4: $E_2^\ell = 0$, $E_{12}^\ell = 0$, and $E_{23}^\ell \neq 0$. By the same argument as in case 3, we have $\dim V = \ell$ if $E_{123}^\ell \neq 0$ and $\dim V = 1$ if $E_{123}^\ell = 0$.

Case 5: $E_2^\ell = 0$, $E_{12}^\ell \neq 0$, and $E_{23}^\ell \neq 0$. Let U be an irreducible submodule of V over the generators E_2, E_{12}, and E_{23}. Since E_2 q-commutes with E_{12} and E_{23}, $E_2 = 0$ on U. Then U is spanned by a common eigenvector u of E_{12} and E_{23}; $E_{12}u = \lambda u$ and $E_{23}u = \mu u$, where $\lambda \neq 0$ and $\mu \neq 0$. $V = U + E_{123}U + \ldots + E_{123}^{\ell-1}U$, and the spaces in the sum are eigenspaces of E_{12} with distinct eigenvalues, so $V = U \oplus E_{123}U \oplus \ldots \oplus E_{123}^{\ell-1}U$. If $E_{123}^\ell \neq 0$, it follows immediately that each of the spaces in the sum is nonzero, and $\dim V = \ell \dim U = \ell$. If $E_{123}^\ell = 0$, we use the following formula, which is proven by induction on m:

$$E_2 E_{123}^m = E_{123}^m E_2 + (\varepsilon - \varepsilon^{-1})\left(\frac{1-\varepsilon^{2m}}{1-\varepsilon^2}\right) E_{123}^{m-1} E_{12} E_{23} \tag{2.61}$$

Let m be the least positive integer such that $E_{123}^m u = 0$. Applying equation 2.61 to u, we obtain:

$$0 = \lambda\mu(\varepsilon - \varepsilon^{-1})\left(\frac{1-\varepsilon^{2m}}{1-\varepsilon^2}\right) E_{123}^{m-1} u \tag{2.62}$$

We conclude that $\ell = m$, since ℓ is odd. Thus we again have $\dim V = \ell \dim U = \ell$.

2.9 $\mathcal{U}^w$ and Irreducible Representations of $\mathcal{U}_\varepsilon^w$ for $w = s_1 s_2 s_3$

For $w = s_1 s_2 s_3$, we find that $\beta_1 = \alpha_1$, $\beta_2 = \alpha_1 + \alpha_2$, $\beta_3 = \alpha_1 + \alpha_2 + \alpha_3$. We then find, using (2.8) through (2.12), that $E_{\beta_1} = E_1$, $E_{\beta_2} = E_{12} = -E_1E_2 + q^{-1}E_2E_1$, $E_{\beta_3} =$

$E_{123} = E_1E_2E_3 - q^{-1}E_2E_1E_3 - q^{-1}E_3E_1E_2 + q^{-2}E_3E_2E_1$. With some computation, we find the relations (2.13) for $\mathcal{U}^{s_1s_2s_3}$ are as follows:

$$E_1E_{12} = qE_{12}E_1 \tag{2.63}$$

$$E_1E_{123} = qE_{123}E_1 \tag{2.64}$$

$$E_{12}E_{123} = qE_{123}E_{12} \tag{2.65}$$

Proposition 2.16 *The finite-dimensional irreducible representations of $\mathcal{U}_\varepsilon^{s_1s_2s_3}$, where ε is a primitive ℓth root of unity, have dimensions:*

1 *if any two or three of E_1^ℓ, E_{12}^ℓ, and E_{123}^ℓ are zero*

ℓ *in all other cases*

Proof: Let V be an irreducible $\mathcal{U}_\varepsilon^{s_1s_2s_3}$-module. We find that $E_1E_{12}^{\ell-1}E_{123}$ and $E_1^{\ell-1}E_{12}E_{123}^{\ell-1}$ are central in $\mathcal{U}_\varepsilon^{s_1s_2s_3}$, so we have $E_1E_{12}^{\ell-1}E_{123} = \alpha$ and $E_1^{\ell-1}E_{12}E_{123}^{\ell-1} = \beta$ on V for some scalars α and β.

Case 1: Any two (or all three) of the ℓth powers of the generators E_1, E_{12}, E_{123} are 0. Then those two generators are 0 on V, and V is spanned by an eigenvector of the third generator. So $\dim V = 1$.

Case 2: Any two (or all three) of the ℓth powers of the generators E_1, E_{12}, E_{123} are nonzero. Suppose, for example, that $E_1^\ell \neq 0$ and $E_{12}^\ell \neq 0$. Let U be an irreducible submodule of V over the generators E_1 and E_{12}. Then $\dim U = \ell$, and from $E_1E_{12}^{\ell-1}E_{123} = \alpha$ we can solve for E_{123} in terms of E_1 and E_{12}, so U is E_{123}-invariant and $V = U$, so $\dim V = \ell$. The other cases are similar.

2.10 $\mathcal{U}^w$ for the Remaining Elements of the Weyl Group

The remaining (nonidentity) elements of the Weyl Group for $sl_4(\mathbb{C})$ are s_2, s_3, s_2s_1, s_2s_3, s_3s_2, $s_3s_2s_3$, $s_3s_2s_1$, $s_3s_2s_3s_1$, $s_3s_1s_2s_3$, and $s_3s_2s_3s_1s_2$. Each of their algebras $\mathcal{U}^w$ have the same (with a change of indices) generators and relations as algebras already considered. For example, with the change of indices $1 \to 3$, $2 \to 2$, and $3 \to 1$, $\mathcal{U}^{s_3 s_2 s_3 s_1 s_2}$ has the same generators and relations as the algebra $\mathcal{U}^{s_1 s_2 s_1 s_3 s_2}$.

2.11 A Final Word

In the paper [3], a conjecture is made regarding the dimensions of the irreducible representations of solvable quantum groups. Namely, this conjecture states that the dimension should be $\ell^{(1/2) dim \mathcal{O}_\pi}$, where $\mathcal{O}_\pi$ is the symplectic leaf containing the restriction of the central character of π to $\mathcal{Z}_0$. This conjecture has been shown by Kac to hold for the quantum Heisenberg algebra considered in Chapter 1. For the algebras of Chapter 2, this conjecture has not been checked but it does predict the possible dimensions of these representations correctly.

Bibliography

[1] C. De Concini, V. G. Kac, and C. Procesi, "Quantum Coadjoint Action", *Journal of AMS,* **5** 1992, pp. 151-190.

[2] C. De Concini and V. G. Kac, "Representations of Quantum Groups at Roots of 1", *Progress in Math.*, **92**, Birkhauser, 1990, pp. 471-506.

[3] C. De Concini, V. G. Kac, and C. Procesi, "Some Quantum Analogues of Solvable Lie Groups", preprint.

[4] C. De Concini, V. G. Kac, and C. Procesi, "Some Remarkable Degenerations of Quantum Groups", preprint, 1991.

[5] S. Z. Levendorskii, Ya. S. Soibelman, "Algebras of Functions on Compact Quantum Groups, Schubert Cells and Quantum Tori", *Comm. Math. Physics,* **139**, 1991, pp. 141-170.

[6] G. Lusztig, "Quantum Groups at Roots of 1", *Geom. Ded.*, **35**, 1990, pp. 89-114.