

**On Geometric Constructions of the Universal
Enveloping Algebra $U(sl_n)$**

by

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M.S., Leningrad State University, Leningrad (1982)

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

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Abstract

In the thesis, the universal enveloping algebra $U(\mathfrak{sl}_n)$ and its modified version \dot{U} are constructed as subalgebras of the algebra \bar{U} . The algebra \bar{U} is constructed as a projective limit of finite dimensional subalgebras of convolution algebras of constructible functions on cotangent bundles of flag manifolds.

The construction provides a canonical basis of \dot{U} , which gives rise to distinguished bases of all irreducible finite dimensional representations of \mathfrak{sl}_n .

The basic steps follow those of Ginzburg's Lagrangian construction. We show how the latter is related to Lusztig's construction of the $-$ part U^- of $U(\mathfrak{sl}_n)$, which is done in terms of constructible functions on Lagrangian subvarieties of spaces of representations of quivers.

Using the geometric setting, we compute the canonical basis of \dot{U} for \mathfrak{sl}_2 and 12 series of monomials in the canonical basis for \mathfrak{sl}_3 .

Thesis Supervisor: George Lusztig
Title: Professor of Mathematics

To my mother and the memory of my father

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Introduction.

A Drinfeld-Jimbo algebra U over $\mathbf{Q}(v)$ is defined in terms of generators and relations, associated with root data (see e.g. [L3, n. 3.1]). G. Lusztig in [L3, Part IV] defined the algebra \dot{U} , a modified version of U . This is an algebra without unit, with Cartan part of U replaced by a direct sum of infinitely many one-dimensional algebras.

Every U -module with a weight decomposition can be naturally regarded as a \dot{U} -module, and the algebra \dot{U} turns out to be more suitable for studying these modules. In particular, \dot{U} has a canonical basis which is compatible with irreducible finite dimensional modules and their tensor products.

A. Beilinson, G. Lusztig, and R. MacPherson [BLM] constructed geometrically the algebra \dot{U} in type A . The construction used the geometry of relative positions of pairs of flags.

V. Ginzburg [G] gave a “micro-local” version of Beilinson-Lusztig-MacPherson construction. He constructed a projective system of finite dimensional algebras generated by certain Lagrangian cycles in the cotangent bundles of flag manifolds. An algebra multiplication was given by convolution in Borel-Moore homology. Each of these algebras was a surjective image of the classical $U(\mathfrak{sl}_n)$, and the homomorphisms from $U(\mathfrak{sl}_n)$ commuted with the homomorphisms of the projective system.

The algebra $U = U(\mathfrak{sl}_n)$ was shown to be embedded in the inverse limit of this system. In fact, the same was true for the algebra \hat{U} .

Ginzburg's approach also gave a geometric realization of all irreducible finite dimensional representations of \mathfrak{sl}_n , each equipped with a distinguished basis.

We construct the algebra \hat{U} for $U = U(\mathfrak{sl}_n)$ in terms of constructible functions on cotangent bundles of flag manifolds, using convolution of functions. The geometric setting and basic steps follow those of Ginzburg.

Working with constructible functions instead of homology makes proofs and computations rather elementary (though, sometimes lengthy). It also allows one to see the relation between the construction of the entire algebra $U(\mathfrak{sl}_n)$ and Lusztig's Lagrangian construction of its $-$ part U^- . The latter is given in terms of constructible functions on certain Lagrangian subvarieties of spaces of representations of quivers. It works for type A as well as for root data of other types (see [L1], [L2]). Understanding this relation might help to find a geometric realization of entire algebras of types other than A_n .

The thesis is structured as follows.

Section 1 describes the geometric setting which follows [BLM], [G]. We define the variety $Z_d = \cup_{A \in \Theta_d} Z[A]$, where $Z[A]$ are cotangent bundles to GL_d -orbits \mathcal{O}_A on the variety of pairs of flags in the space \mathbb{C}^d .

In section 2, we define the algebra U_d as a subalgebra of the convolution algebra of constructible functions on Z_d . We show that there is a surjective algebra homomorphism from $U(\mathfrak{sl}_n)$ onto U_d .

In section 3, we prove the existence of linearly independent functions $\{\varphi_A\}_{A \in \Theta_d}$, such that each φ_A is identically 1 on $Z[A]$, and vanishes on some open dense subset of $Z[A']$ for any $A' \neq A$.

In section 4, we show that U_d is finite dimensional, and that $\dim U_d = |\Theta_d|$. It follows that the functions $\{\varphi_A\}_{A \in \Theta_d}$ form a basis of U_d . Then we prove that the basis with such properties is unique.

In section 5, we construct all irreducible finite dimensional \mathfrak{sl}_n -modules. Every such module arises from an irreducible U_d -module for some d .

Following [G], we define a closed subvariety M^x of Z_d . We consider the space L_x of constructible functions on M^x , which are the restrictions on M^x of functions of U_d . It is finite dimensional. We show that L_x is an irreducible U_d -module. We indicate a highest weight vector s_x such that the functions $\varphi_A \cdot s_x$ behave with respect to irreducible components of M^x in the same way as the functions φ_A behave with respect to irreducible components of Z_d . Then we prove that $\{\varphi_A \cdot s_x \mid \varphi_A \cdot s_x \neq 0, A \in \Theta_d\}$ is a basis of L_x .

In section 6, we show that for any d there is a surjective algebra homomorphism from U_{d+n} onto U_d which commutes with the homomorphisms from $U(\mathfrak{sl}_n)$ described in section 2. We show that each basis element $\varphi_B \in U_{d+n}$ is mapped either to 0, or to a basis element $\varphi_{(B-I)}$ of U_d .

In section 7, we consider the inverse limit \bar{U} of the projective system of $\{U_d\}$. We show that the algebra $U(sl_n)$ can be imbedded into \bar{U} . Then, we define a subalgebra \hat{U} (without unit) of \bar{U} , which is spanned by the elements

$$\cdots \leftarrow \varphi_{(A-I)} \leftarrow \varphi_A \leftarrow \varphi_{(A+I)} \leftarrow \cdots,$$

corresponding to all the basis elements of the algebras U_d .

In section 8, we show that the algebra \hat{U} is isomorphic to the algebra \hat{U} of type A . Then we give a geometric interpretation of some purely algebraic results on \hat{U} obtained by Lusztig [L3].

In section 9, we describe the relation between the above construction and Lusztig's Lagrangian construction of the $-$ part U^- of $U(sl_n)$.

Section 10 contains examples of computations of canonical bases for $n = 2, 3$.

1 Preliminaries.

The setup closely follows [BLM], [G].

1.1. Let us fix $n \geq 2, d \geq 0$. Consider the variety \mathcal{F}_d of n -step partial flags

$$F = (0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1} \subseteq F_n = \mathbf{C}^d)$$

in the complex d -dimensional vector space. Let

$$\underline{\dim} F = (\dim F_1, \dim F_2, \dots, \dim F_n).$$

Connected components of \mathcal{F}_d are parametrized by sequences of non-negative integers $\underline{l} = (l_1, l_2, \dots, l_n)$, such that $0 \leq l_1 \leq l_2 \leq \cdots \leq l_n = d$. The connected component corresponding to \underline{l} consists of all the flags F such that $\underline{\dim} F = \underline{l}$. Each component is a single orbit under the natural action of the group $GL_d = GL(d, \mathbf{C})$ on \mathcal{F}_d .

1.2. Let us consider the variety $\mathcal{F}_d \times \mathcal{F}_d$ of pairs of flags. As in [BLM], we assign to each pair $(F, F') \in \mathcal{F}_d \times \mathcal{F}_d$ an $n \times n$ matrix $\Phi(F, F') = A = (a_{ij})$ such that

$$a_{ij} = \dim \left(\frac{F_i \cap F'_j}{F_{i-1} \cap F'_j + F_i \cap F'_{j-1}} \right).$$

Let $co(A)$ and $ro(A)$ be the vectors of column sums of A and row sums respectively, thus

$$co(A) = \left(\sum_{i=1}^n a_{i1}, \sum_{i=1}^n a_{i2}, \dots, \sum_{i=1}^n a_{in} \right),$$

$$ro(A) = \left(\sum_{j=1}^n a_{1j}, \sum_{j=1}^n a_{2j}, \dots, \sum_{i=1}^n a_{nj} \right).$$

We have

- (i) all a_{ij} are non-negative integers;
- (ii) $\sum_{i,j=1}^n a_{ij} = d$;
- (iii) $\dim(F_i/F_{i-1}) = ro(A)_i$, $\dim(F'_j/F'_{j-1}) = co(A)_j$ for all $1 \leq i, j \leq n$;
- (iv) $\dim(F_i \cap F'_j) = \sum_{l \leq i, m \leq j} a_{l,m}$, for all $1 \leq i, j \leq n$.

Let Θ_d be the set of $n \times n$ matrices satisfying (i) and (ii). The assignment $(F, F') \mapsto \Phi(F, F')$ defines a bijection from the set of orbits of GL_d under the diagonal action on $\mathcal{F}_d \times \mathcal{F}_d$ to the set Θ_d .

We denote by \mathcal{O}_A the orbit corresponding to $A \in \Theta_d$. We have

$$\mathcal{F}_d \times \mathcal{F}_d = \bigcup_{A \in \Theta_d} \mathcal{O}_A.$$

1.2.1. The number of elements in Θ_d is equal to the number of decompositions of d into a sum of n^2 non-negative integers, thus $|\Theta_d| = \binom{d+n^2-1}{d}$.

1.2.2. We denote by Δ_d the set of diagonal matrices in Θ_d . Note that $\Phi(F, F') = A \in \Delta_d$ if and only if $F = F'$, and then $\dim F_i = \sum_{k=1}^i a_{kk}$ for all $i = 1, \dots, n$.

1.3. The set Θ_d is partially ordered. For any $A, B \in \Theta_d$ we say that $B \leq A$ if \mathcal{O}_B is contained in the Zariski closure $\overline{\mathcal{O}_A}$ of \mathcal{O}_A . We write $B < A$ if $B \leq A$ and $B \neq A$.

If $A = \Phi(F, F')$, $B = \Phi(G, G')$, then $B \leq A$ if and only if $\underline{\dim} F = \underline{\dim} G$, $\underline{\dim} F' = \underline{\dim} G'$, and $\dim(F_i \cap F'_j) \leq \dim(G_i \cap G'_j)$ for all $i, j = 1, \dots, n$. Using 1.2 (iii), (iv), we have that $B \leq A$ if and only if for all $i, j = 1, \dots, n$

$$ro(A) = ro(B), \quad co(A) = co(B), \quad \sum_{p \leq i; q \leq j} a_{pq} \leq \sum_{p \leq i; q \leq j} b_{pq}.$$

1.4. Let N_d be the variety of all nilpotent linear maps $x : \mathbb{C}^d \rightarrow \mathbb{C}^d$ such that $x^n = 0$. For a flag $F \in \mathcal{F}_d$ and a map $x \in N_d$ we write $x \triangleright F$ if $x(F_i) \subseteq F_{i-1}$ for all $i = 1, \dots, n$. Clearly, $x \triangleright F$ implies $Im x^{n-i} \subseteq F_i \subseteq Ker x^i$ for all $i = 1, \dots, n$.

1.5. The cotangent bundle $T^*\mathcal{F}_d$ can be naturally identified with the set of pairs

$$M_d = \{(F, x) \in \mathcal{F}_d \times N_d \mid x \triangleright F\}.$$

Let $\pi : M_d \rightarrow N_d$ be defined by $\pi(F, x) = x$.

Following [G], we consider the subvariety $Z_d = M_d \times_{N_d} M_d$ of $M_d \times M_d \cong T^*(\mathcal{F}_d \times \mathcal{F}_d)$. We identify the variety Z_d with the set of triples

$$\{(F, F', x) \in \mathcal{F}_d \times \mathcal{F}_d \times N_d \mid x \triangleright F, x \triangleright F'\}.$$

Let $Z[A]$ be the conormal bundle of the orbit \mathcal{O}_A . This is a locally closed Lagrangian subvariety of $T^*(\mathcal{F}_d \times \mathcal{F}_d)$. We have

$$Z[A] = \{(F, F', x) \in Z_d \mid (F, F') \in \mathcal{O}_A\},$$

and

$$Z_d = \bigcup_{A \in \Theta_d} Z[A].$$

All the irreducible components of Z_d are of the form $\overline{Z[A]}$ for some $A \in \Theta_d$. Here \overline{X} denotes the Zariski closure of X .

1.6. The group GL_d acts on the variety Z_d by $g \cdot (F, F', x) = (gF, gF', gxg^{-1})$. The action leaves each $Z[A]$ stable.

Unlike the previously considered cases of GL_d -action on \mathcal{F}_d and $\mathcal{F}_d \times \mathcal{F}_d$, here the number of orbits is in general infinite.

1.7. We define a map $\tau_0 : Z_d \rightarrow Z_d$ by $\tau_0(F, F', x) = (F', F, x)$. This is clearly an involutive algebraic automorphism of the variety Z_d .

For any $F, F' \in \mathcal{F}_d$, if $\Phi(F, F') = A$, then $\Phi(F', F) = {}^tA$, where tA is the transpose of A (see [BLM, n. 1.1]). Therefore, $\tau_0(Z[A]) = Z[{}^tA]$ for any $A \in \Theta_d$.

2 The algebra U_d .

2.1. As defined by R. MacPherson [M], a function on a variety is called constructible if it takes a finite number of values, and the preimage of each value is a constructible set.

Let \mathcal{A}_d be the vector space over the field \mathbf{Q} of rational numbers of all constructible functions $\varphi : Z_d \rightarrow \mathbf{Q}$. Similarly to Lusztig's definition of the multiplication of functions on the space of representations of quivers [L1], we define an operation $*$ on \mathcal{A}_d by

$$\varphi_1 * \varphi_2(F, F', x) = \sum_{a \in \mathbf{Q}} a \cdot \chi\{\tilde{F} \in \mathcal{F}_d \mid x \triangleright \tilde{F}, \varphi_1(F, \tilde{F}, x) \cdot \varphi_2(\tilde{F}, F', x) = a\},$$

where χ denotes the Euler characteristic in cohomology over \mathbf{Q} with compact support. We set $\chi(\emptyset) = 0$.

The operation $*$ makes \mathcal{A}_d into an associative \mathbf{Q} -algebra. This follows from finite additivity of χ for constructible sets, multiplicativity of χ for fiber bundles, and the fact that for any regular map $f : X \rightarrow Y$ of algebraic varieties there exists a stratification of Y such that for each stratum S the restriction $f|_{f^{-1}(S)}$ is a fiber bundle.

The unit in \mathcal{A}_d is a function $\mathbf{1}$ such that $\mathbf{1}(F, F', x) = 0$ if $F \neq F'$, and $\mathbf{1}(F, F, x) = 1$ for all $F \in \mathcal{F}_d, x \triangleright F$.

2.2. We define now the algebra U_d .

For any $i, j \in [1, n]$ let $E_{i,j}$ be the $n \times n$ matrix such that its l, m entry is $\delta_{i,l}\delta_{j,m}$, where $\delta_{i,j}$ is the Kronecker δ -function. Following [BLM] and [G], we define functions $e_i, f_i \in \mathcal{A}_d, i = 1, \dots, n-1$ as follows.

$$e_i(F, F', x) = \begin{cases} 1, & \text{if } (F, F', x) \in \bigcup_{A \in \Delta_{d-1}} Z[A + E_{i,i+1}] \\ 0, & \text{otherwise;} \end{cases}$$

$$f_i(F, F', x) = \begin{cases} 1, & \text{if } (F, F', x) \in \bigcup_{A \in \Delta_{d-1}} Z[A + E_{i+1,i}] \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\Phi(F, F') = A + E_{i,i+1}$ with a diagonal $A \in \Delta_{d-1}$ means that $F'_i \subset F_i$, $\dim(F_i/F'_i) = 1$, and $F_j = F'_j$ for all $j \neq i$. Similarly, if $\Phi(F, F') = A + E_{i+1,i}$, we have that $F_i \subset F'_i$, $\dim(F'_i/F_i) = 1$, and $F_j = F'_j$ for all $j \neq i$. It follows (see 1.3) that all the conormal bundles in the above definition are closed.

Let U_d be a subalgebra of \mathcal{A}_d with $\mathbf{1}$, generated by the functions $\{e_i, f_i\}_{i=1}^{n-1}$.

Let $h_i = e_i * f_i - f_i * e_i \in U_d$.

2.3. The pullback $\tau = \tau_0^* : \mathcal{A}_d \rightarrow \mathcal{A}_d$ of the involution τ_0 (see 1.7) is an involutive vector space isomorphism defined by $\tau(\psi)(F, F', x) = \psi(F', F, x)$.

Let \mathcal{A}_d^{opp} be the algebra with the same underlying vector space as \mathcal{A}_d , but with the reversed multiplication $\hat{*}$ defined by:

$$\varphi \hat{*} \psi = \psi * \varphi.$$

By definitions of $*$ and τ

$$\begin{aligned}
\tau(\varphi * \psi)(F, F', x) &= \varphi * \psi(F', F, x) \\
&= \sum_{a \in \mathbf{Q}} a \cdot \chi\{\tilde{F} \in \mathcal{F}_d \mid x \triangleright \tilde{F}, \varphi(F', \tilde{F}, x) \cdot \psi(\tilde{F}, F, x) = a\} \\
&= \sum_{a \in \mathbf{Q}} a \cdot \chi\{\tilde{F} \in \mathcal{F}_d \mid x \triangleright \tilde{F}, \tau(\varphi)(\tilde{F}, F', x) \cdot \tau(\psi)(F, \tilde{F}, x) = a\} \\
&= \tau(\psi) * \tau(\varphi)(F, F', x) \\
&= \tau(\varphi) \hat{*} \tau(\psi)(F, F', x).
\end{aligned}$$

This shows that τ is an algebra isomorphism $\tau : \mathcal{A}_d \rightarrow \mathcal{A}_d^{opp}$.

By definition of the functions e_i, f_i , we have $\tau(e_i) = f_i, \tau(f_i) = e_i, 1 \leq i \leq n-1$. Therefore, U_d is stable under τ , and in fact the restriction $\tau : U_d \rightarrow U_d^{opp}$ is an algebra isomorphism.

Notice that $\tau(h_i) = h_i$ for all $i = 1, \dots, n-1$.

Proposition 2.4. *The functions $e_i, f_i, h_i \in U_d$ satisfy the following relations:*

$$(1) \quad h_i * h_j = h_j * h_i, \quad 1 \leq i, j \leq n;$$

$$(2) \quad e_i * f_j = f_j * e_i, \quad \text{for all } i \neq j;$$

$$(3a) \quad h_i * e_j - e_j * h_i = \begin{cases} 2e_j, & \text{if } i = j, \\ -e_j, & \text{if } |i - j| = 1, \\ 0, & \text{if } |i - j| > 1. \end{cases}$$

$$(3b) \quad h_i * f_j - f_j * h_i = \begin{cases} -2f_j, & \text{if } i = j, \\ f_j, & \text{if } |i - j| = 1, \\ 0, & \text{if } |i - j| > 1. \end{cases}$$

$$(4a) \quad \begin{aligned} e_i * e_i * e_j - e_i * e_j * e_i - e_j * e_i * e_i &= 0, \quad \text{if } |i - j| = 1; \\ e_i * e_j &= e_j * e_i, \quad \text{if } |i - j| > 1; \end{aligned}$$

$$(4b) \quad \begin{aligned} f_i * f_i * f_j - f_i * f_j * f_i - f_j * f_i * f_i &= 0, \quad \text{if } |i - j| = 1; \\ f_i * f_j &= f_j * f_i, \quad \text{if } |i - j| > 1. \end{aligned}$$

Let sl_n be a Lie algebra over \mathbf{Q} of all $n \times n$ matrices of trace 0 with rational entries. We denote by $\underline{e}_i, \underline{f}_i, \underline{h}_i$ the standard generators of the universal enveloping algebra $U(sl_n)$, so that $\underline{e}_i = E_{i,i+1}, \underline{f}_i = E_{i+1,i} \in sl_n, \underline{h}_i = [\underline{e}_i, \underline{f}_i], i = 1, \dots, n-1$.

Corollary 2.5. *There is a unique surjective algebra homomorphism $\gamma_d : U(sl_n) \rightarrow U_d$*

such that $\gamma_d(\underline{e}_i) = e_i$, $\gamma_d(\underline{f}_i) = f_i$ for all $i = 1, \dots, n-1$.

Proof. The relations of proposition 2.4 are precisely the relations for the standard generators of $U(\mathfrak{sl}_n)$.

We now prove proposition 2.4.

First, we compute the values of the functions h_i .

Lemma 2.6. *For any $i = 1, \dots, n-1$*

$$h_i|_{Z[A]} \equiv \begin{cases} a_{ii} - a_{i+1,i+1}, & \text{if } A \in \Delta_d, \\ 0, & \text{if } A \in \Theta_d - \Delta_d. \end{cases}$$

Proof. Since the functions e_i and f_i take only values 0 and 1, we have

$$\begin{aligned} e_i * f_i(F, F', x) &= \sum_{a \in \mathbf{Q}} a \cdot \chi\{\tilde{F} \in \mathcal{F}_d \mid x \triangleright \tilde{F}, e_i(F, \tilde{F}, x) \cdot f_i(\tilde{F}, F', x) = a\} \\ &= \chi\{\tilde{F} \in \mathcal{F}_d \mid x \triangleright \tilde{F}, e_i(F, \tilde{F}, x) = f_i(\tilde{F}, F', x) = 1\} \end{aligned}$$

As follows from 2.2, $e_i(F, \tilde{F}, x) = f_i(\tilde{F}, F', x) = 1$ if and only if \tilde{F} is such that

- (i) $\tilde{F}_i \subset (F_i \cap F'_i)$, $\dim(F_i/\tilde{F}_i) = \dim(F'_i/\tilde{F}_i) = 1$,
- (ii) $F_j = \tilde{F}_j = F'_j$ for all $j \neq i$,
- (iii) $x(F_{i+1} + F'_{i+1}) \subseteq \tilde{F}_i$.

Necessarily, $\dim F = \dim F'$.

If $F_i \neq F'_i$, then (i) implies that $F_i \cap F'_i$ has codimension 1 in both F_i and F'_i . The only \tilde{F} satisfying the conditions (i)-(iii) is

$$\tilde{F} = (F_0 \subseteq F_1 \subseteq \dots \subseteq F_{i-1} \subseteq (F_i \cap F'_i) \subset F_{i+1} \dots \subseteq F_n = \mathbf{C}^d).$$

If $F_i = F'_i$, then \tilde{F} has to be of the form

$$\tilde{F} = (F_0 \subseteq F_1 \subseteq \dots \subseteq F_{i-1} \subseteq V \subset F_{i+1} \dots \subseteq F_n = \mathbf{C}^d),$$

where V is such that $\dim(F_{i+1}/V) = 1$, and $x(F_{i+1}) \subseteq V$.

Therefore,

$$\begin{aligned} e_i * f_i(F, F, x) &= \chi\{V \mid (x(F_{i+1}) + F_{i-1}) \subseteq V \subset F_i, \dim(F_i/V) = 1\} \\ &= \dim(F_i/(x(F_{i+1}) + F_{i-1})), \end{aligned}$$

since the set of such spaces V is isomorphic to a projective space of dimension $m = \dim(F_i/(x(F_{i+1}) + F_{i-1})) - 1$, whose Euler characteristic is $m + 1$.

Thus, we have

$$e_i * f_i (F, F', x) = \begin{cases} \dim(F_i/(x(F_{i+1}) + F_{i-1})), & \text{if } F = F', \\ 1, & \text{if } \underline{\dim} F = \underline{\dim} F', F_j = F'_j \text{ for } j \neq i, \\ & \text{and } \dim(F_i/(F_i \cap F'_i)) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly,

$$f_i * e_i (F, F', x) = \begin{cases} \dim((x^{-1}(F_{i-1}) \cap F_{i+1})/F_i), & \text{if } F = F', \\ 1, & \text{if } \underline{\dim} F = \underline{\dim} F', F_j = F'_j \text{ for } j \neq i, \\ & \text{and } \dim(F_i/(F_i \cap F'_i)) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

This gives us that $h_i(F, F', x) = 0$ if $F \neq F'$, and that for any $F \in \mathcal{F}_d$ and any $x \triangleright F$

$$\begin{aligned} h_i(F, F, x) &= \dim(F_i/(x(F_{i+1}) + F_{i-1})) - \dim((x^{-1}(F_{i-1}) \cap F_{i+1})/F_i) \\ &= \dim F_i - \dim(x(F_{i+1}) + F_{i-1}) - \dim(x(F_{i+1}) \cap F_{i-1}) \\ &\quad - \dim(\text{Ker}(x) \cap F_{i+1}) + \dim F_i \\ &= 2 \dim F_i - \dim x(F_{i+1}) - \dim F_{i-1} - \dim(\text{Ker}(x) \cap F_{i+1}) \\ &= 2 \dim F_i - \dim F_{i-1} - \dim F_{i+1}. \end{aligned}$$

Together with 1.2.2, this shows that for all $i = 1, \dots, n-1$

$$h_i|_{Z[A]} \equiv \begin{cases} a_{ii} - a_{i+1, i+1}, & \text{if } A \in \Delta_d, \\ 0, & \text{if } A \in \Theta_d - \Delta_d. \end{cases}$$

2.7. Since $h_i(F, F', x) = 0$ unless $F = F'$, for any $\varphi \in U_d$ we have

$$\begin{aligned} h_i * \varphi (F, F', x) &= \sum_{a \in \mathbf{Q}} a \cdot \chi\{\tilde{F} \in \mathcal{F}_d \mid x \triangleright \tilde{F}, h_i(F, \tilde{F}, x) \cdot \varphi(\tilde{F}, F', x) = a\} \\ &= \sum_{a \in \mathbf{Q}} a \cdot \chi\{\tilde{F} \in \mathcal{F}_d \mid \tilde{F} = F, h_i(F, F, x) \cdot \varphi(F, F', x) = a\} \\ &= h_i(F, F, x) \varphi(F, F', x) \end{aligned}$$

Similarly,

$$\varphi * h_i (F, F', x) = \varphi(F, F', x) h_i(F', F', x).$$

In particular, $h_i * h_j(F, F', x) = 0$ unless $F = F'$, and

$$h_i * h_j (F, F, x) = h_i(F, F, x) h_j(F, F, x).$$

Therefore, $h_i * h_j = h_i h_j$, where the right hand side denotes the pointwise multiplication

of functions h_i, h_j .

2.8. By 2.7, $h_i * h_j = h_i h_j = h_j h_i = h_j * h_i$. This proves (1).

Also, 2.7 gives

$$(h_i * e_j - e_j * h_i)(F, F', x) = \{(2 \dim F_i - \dim F_{i-1} - \dim F_{i+1}) - (2 \dim F'_i - \dim F'_{i-1} - \dim F'_{i+1})\} e_j(F, F', x).$$

Let us denote the factor in braces by C . By 2.2, $e_j(F, F', x) = 0$ unless $F'_j \subset F_j$, $\dim(F_j/F'_j) = 1$, and $F_k = F'_k$ for all $k \neq j$. It follows that for any F, F' such that $e_j(F, F', x) \neq 0$

$$C = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } |i - j| = 1, \\ 0, & \text{if } |i - j| > 1. \end{cases}$$

This yields (3a).

We deduce (3b) from (3a) using the involution τ defined in 2.3. We have

$$(h_i * f_j - f_j * h_i) + C \cdot f_j = \tau((e_j * h_i - h_i * e_j) + C \cdot e_j) = \tau(0) = 0$$

for any $i, j = 1, \dots, n - 1$. This gives (3b).

2.9. Let $i \neq j$. Then

$$\begin{aligned} e_i * f_j(F, F', x) &= \sum_{a \in \mathbf{Q}} a \cdot \chi\{\tilde{F} \in \mathcal{F}_d \mid x \triangleright \tilde{F}, e_i(F, \tilde{F}, x) \cdot f_j(\tilde{F}, F', x) = a\} \\ &= \begin{cases} 1, & \text{if } F'_i \subset F_i, \dim(F_i/F'_i) = 1; F_j \subset F'_j, \dim(F'_j/F_j) = 1, \\ & F_k = F'_k \text{ for all } k \neq i, j; \\ 0, & \text{otherwise.} \end{cases} \\ &= f_j * e_i(F, F', x). \end{aligned}$$

This gives (2).

2.10. To prove the Serre relations we calculate the following functions.

$$e_i * e_i(F, F', x) = \begin{cases} 2, & \text{if } F'_i \subset F_i, \dim(F_i/F'_i) = 2; F_k = F'_k \text{ for all } k \neq i; \\ 0, & \text{otherwise.} \end{cases}$$

The functions $e_{i+1} * e_i * e_i$, $e_i * e_i * e_{i+1}$, and $e_i * e_{i+1} * e_i$ are all equal to 0 at (F, F', x) unless

- (i) $x \triangleright F, F'$;
- (ii) $F'_i \subset F_i, \dim(F_i/F'_i) = 2; F'_{i+1} \subset F_{i+1}, \dim(F_{i+1}/F'_{i+1}) = 1$;
- (iii) $F_k = F'_k$ for all $k \neq i, i+1$;

For F, F', x satisfying the conditions (i)-(iii) we have

$$e_i * e_i * e_{i+1} (F, F', x) = \begin{cases} 2, & \text{if } F_i \subseteq F'_{i+1}, \text{ and } x(F_{i+1}) \subseteq F'_i, \text{ or} \\ & \text{if } \dim(F_i/(F_i \cap F'_{i+1})) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$e_{i+1} * e_i * e_i (F, F', x) = \begin{cases} 2, & \text{if } F_i \subseteq F'_{i+1}; \\ 0, & \text{otherwise.} \end{cases}$$

$$e_i * e_{i+1} * e_i (F, F', x) = \begin{cases} 2, & \text{if } F_i \subseteq F'_{i+1}, \text{ and } x(F_{i+1}) \subseteq F'_i; \\ 1, & \text{if } F_i \subseteq F'_{i+1}, \text{ and } x(F_{i+1}) \not\subseteq F'_i; \\ 1, & \text{if } \dim(F_i/(F_i \cap F'_{i+1})) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$e_i * e_i * e_{i+1} - 2e_i * e_{i+1} * e_i + e_{i+1} * e_i * e_i = 0.$$

Using again the isomorphism τ , we have

$$f_i * f_i * f_{i+1} - 2f_i * f_{i+1} * f_i + f_{i+1} * f_i * f_i = \tau(e_{i+1} * e_i * e_i - 2e_i * e_{i+1} * e_i + e_i * e_i * e_{i+1}) = 0.$$

The rest of the relations (4) are proved by completely analogous computations.

Proposition 2.4 is proved.

Lemma 2.11. *All the functions in U_d are constant on GL_d -orbits.*

Proof. First we notice that the generators of U_d are constant on GL_d -orbits. Next, we claim that $*$ preserves this property. Indeed, let $\varphi_1, \varphi_2 \in U_d$ be such that $\varphi_i(g \cdot (F, F', x)) = \varphi_i(F, F', x)$ for any $(F, F', x) \in Z_d, g \in GL_d, i = 1, 2$. Then

$$\begin{aligned} & \varphi_1 * \varphi_2 (gF, gF', gxg^{-1}) \\ &= \sum_{a \in \mathbf{Q}} a \cdot \chi\{\tilde{F} \mid gxg^{-1} \triangleright \tilde{F}, \varphi_1(gF, \tilde{F}, gxg^{-1}) \cdot \varphi_2(\tilde{F}, gF', gxg^{-1}) = a\} \\ &= \sum_{a \in \mathbf{Q}} a \cdot \chi\{\tilde{F} \mid x \triangleright g^{-1}\tilde{F}, \varphi_1(F, g^{-1}\tilde{F}, x) \cdot \varphi_2(g^{-1}\tilde{F}, F', x) = a\} \\ &= \sum_{a \in \mathbf{Q}} a \cdot \chi(g \cdot \{\tilde{F} \mid x \triangleright \tilde{F}, \varphi_1(F, \tilde{F}, x) \cdot \varphi_2(\tilde{F}, F', x) = a\}) \\ &= \varphi_1 * \varphi_2 (F, F', x). \end{aligned}$$

The lemma follows.

3 The basis of U_d .

Theorem 3.1. *For any matrix $A \in \Theta_d$ there exists a function $\varphi \in U_d$ satisfying the following conditions:*

- (I) $\varphi|_{Z[A]} \equiv 1$;
- (II) φ vanishes on some open dense subset of $Z[A']$ for any $A' \neq A$,
- (III) the support $\text{supp } \varphi$ is contained in the union $\bigcup_{A' \leq A} Z[A']$.

*The function satisfying (I)-(II) is unique. We denote it by φ_A .
The functions $\{\varphi_A\}_{A \in \Theta_d}$ form a basis of the algebra U_d .*

In this section, we prove the existence of functions satisfying the conditions (I)-(II) for every $A \in \Theta_d$. We start with diagonal matrices. Then we construct φ_A for general A by an inductive procedure in the spirit of Lusztig's Lagrangian construction [L2] (see also section 9).

Let $\{\varphi_A\}_{A \in \Theta_d}$ be a collection of functions satisfying 3.1, (I)-(II). Suppose that for some $c_A \in \mathbf{Q}$

$$\sum_{A \in \Theta_d} c_A \cdot \varphi_A = 0.$$

Then by properties (I) and (II), for any $A \in \Theta_d$ the restriction of this sum on some open dense subset of the irreducible component $\overline{Z[A]}$ of Z_d is equal to c_A . Hence, all $c_A = 0$. Therefore, the functions $\{\varphi_A\}_{A \in \Theta_d}$ are linearly independent. This implies that $\dim U_d \geq |\Theta_d|$.

In section 4 we prove that $\{\varphi_A\}_{A \in \Theta_d}$ is a basis of U_d by showing that $\dim U_d \leq |\Theta_d|$. Then we prove the uniqueness. This will complete the proof of theorem 3.1.

Let D be a subalgebra of U_d containing $\mathbf{1}$, generated by $\{h_i\}_{i=1}^{n-1}$.

Proposition 3.2. *For any diagonal matrix $A \in \Delta_d$ there exists a function $\varphi_A \in D \subseteq U_d$ such that*

- (a) φ_A is equal to 1 on $Z[A] = \overline{Z[A]}$,
- (b) φ_A vanishes on $Z_d - Z[A]$.

Let \mathcal{C} be the algebra of all \mathbf{Q} -valued functions on the (finite) set Δ_d , with the usual

pointwise multiplication. Let $\{\delta_A\}_{A \in \Delta_d}$ be the standard basis of \mathcal{C} , where $\delta_A(B) = 0$ if $A \neq B$, and $\delta_A(A) = 1$. To prove the proposition we show that there is an isomorphism α from the algebra D onto \mathcal{C} , such that the functions $\varphi_A = \alpha^{-1}(\delta_A)$ have the required properties.

Lemma 3.3. *For any function $\varphi \in D$*

- (i) φ is constant on $Z[A]$ for any $A \in \Theta_d$;
- (ii) $\varphi|_{Z[A]} \equiv 0$ for all $A \in \Theta_d - \Delta_d$.

Proof. First, notice that by 2.6, all the generators of D satisfy (i), (ii).

Second, as we know from 2.7, $h_i * h_j = h_i h_j$, so the multiplication $*$ in D becomes just a pointwise multiplication of functions. This proves the lemma.

Lemma 3.3 allows us to define an algebra homomorphism $\alpha : D \rightarrow \mathcal{C}$ so that $\alpha(\varphi)(A)$ is the constant equal to $\varphi|_{Z[A]}$. It follows from lemma 3.3 that α is injective.

Lemma 3.4. *The homomorphism α is an isomorphism.*

Proof. Let $\hat{D} = \alpha(D)$. We want to show that $\hat{D} = \mathcal{C}$.

Let $A \in \Delta_d$. By lemma 2.6, $h_i|_{Z[A]} = a_{ii} - a_{i+1,i+1}$. Since $\sum_{i=1}^n a_{ii} = d$, this gives $h_{n-1}|_{Z[A]} = a_{n-1,n-1} - d + \sum_{i=1}^{n-1} a_{ii}$, so that

$$\begin{pmatrix} \alpha(h_1)(A) \\ \alpha(h_2)(A) \\ \vdots \\ \alpha(h_{n-2})(A) \\ \alpha(h_{n-1})(A) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{22} \\ \vdots \\ a_{n-2,n-2} \\ a_{n-1,n-1} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ d \end{pmatrix}$$

Adding all the columns of the matrix above to the last column, we compute its determinant:

$$\det \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & n-1 \end{pmatrix} = n-1 \neq 0.$$

Therefore, the vector of values $(\alpha(h_1)(A), \dots, \alpha(h_{n-1})(A))$ uniquely determines A ,

i.e. for any $A, B \in \Delta_d$

$$(\alpha(h_1)(A), \dots, \alpha(h_{n-1})(A)) = (\alpha(h_1)(B), \dots, \alpha(h_{n-1})(B))$$

if and only if $A = B$.

This means that the algebra \hat{D} separates points. Since also $1 \in \hat{D}$, by the Stone-Weierstrass theorem \hat{D} is dense in \mathcal{C} . But the algebra \mathcal{C} is finite dimensional, which implies $\hat{D} = \mathcal{C}$.

Lemma 3.4 is proved.

3.5. It follows directly from lemma 3.3 and the definition of α that $\varphi_A = \alpha^{-1}(\delta_A)$ with $A \in \Delta_d$ satisfy the conditions (a) and (b) of proposition 3.2.

Corollary 3.6. $\sum_{A \in \Delta_d} \varphi_A = 1$.

Corollary 3.7. For any $\psi \in U_d$ and any $A \in \Delta_d$

$$\varphi_A * \psi(F, F', x) = \begin{cases} \psi(F, F', x), & \text{if } \Phi(F, F') = A, \\ 0, & \text{otherwise;} \end{cases}$$

$$\psi * \varphi_A(F, F', x) = \begin{cases} \psi(F, F', x), & \text{if } \Phi(F', F') = A, \\ 0, & \text{otherwise.} \end{cases}$$

3.8. To proceed with the construction of the basis functions for non-diagonal matrices we introduce the following relation. For two matrices $A, B \in \Theta_d$ we say that $B \ll A$ if for all $i, j = 1, \dots, n$

$$\sum_{p \leq i} ro(B)_p \leq \sum_{p \leq i} ro(A)_p, \quad \sum_{q \leq j} co(B)_q \leq \sum_{q \leq j} co(A)_q,$$

and at least one of the inequalities is strict. By 1.2 (iii), this is equivalent to the following condition: for any $(F, F') \in \mathcal{O}_A, (G, G') \in \mathcal{O}_B$, and all $i = 1, \dots, n$

$$\dim G_i \leq \dim F_i, \quad \dim G'_i \leq \dim F'_i,$$

and at least one of the inequalities is strict.

3.9. For the matrix $O \in \Theta_d$ with the only non-zero entry $o_{nn} = d$ we have $O \ll A$ for all $A \in \Theta_d$. Since $O \in \Delta_d$, the existence of the function φ_O is given by proposition 3.2.

Let us now fix $A \in \Theta_d - \Delta_d$. Assume that we already constructed the functions $\varphi_{A'}$ satisfying 3.1, (I)-(III) for all $A' \ll A$, and all $A' < A$.

Let i be the minimal index for which $a_{pq} = 0$ for all p, q such that either $p < q$ and

$q \geq i + 2$, or $q < p$ and $p \geq i + 2$, i.e.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1,i+1} & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ a_{i+1,1} & \cdots & a_{i+1,i+1} & 0 & \vdots \\ 0 & \cdots & 0 & a_{i+2,i+2} & \vdots \\ \vdots & & \vdots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & a_{nn} \end{pmatrix}.$$

This means that for any $(F, F') \in \mathcal{O}_A$, $F_i \neq F'_i$, and $F_p = F'_p$ for all $p \geq i + 1$. Such $i \in [1, n - 1]$ exists since $A \notin \Delta_d$.

3.10. Because of the choice of i , there is at least one non-zero off-diagonal entry in the $(i + 1)$ -th column, or in the $(i + 1)$ -th row. Say, $a_{j,i+1} \neq 0$ for some $j \leq i$. (The case when $a_{j,i+1} = 0$ for all $j \leq i$ is discussed in 3.18.)

Let $B = A - E_{j,i+1} + E_{j+1,i+1}$ (see 2.2 for the definition of E_{pq}). Clearly, $B \ll A$. By induction assumption, there exists a function φ_B satisfying the conditions (I)-(III) of theorem 3.1. We define

$$\check{\varphi}_A = e_j * \varphi_B.$$

Lemma 3.11. (a) $\check{\varphi}_A$ is constant on $Z[A]$;

(b) $\text{supp } \check{\varphi}_A \subseteq \bigcup_{A' \leq A} Z[A']$.

Proof. Fix $(F, F') \in \mathcal{O}_A$. We want to compute

$$\check{\varphi}_A(F, F', x) = e_j * \varphi_B(F, F', x) = \sum_{t \in \mathbf{Q}} t \cdot \chi\{\tilde{F} \in \mathcal{F}_d \mid x \triangleright \tilde{F}, e_j(F, \tilde{F}, x) \cdot \varphi_B(\tilde{F}, F', x) = t\}$$

for any $x \triangleright F, F'$. Let

$$\begin{aligned} \mathcal{V} &= \{\tilde{F} \in \mathcal{F}_d \mid (F, \tilde{F}) \in \mathcal{O}_{\Phi(F,F) - E_{j,j} + E_{j,j+1}}\} \\ &= \{\tilde{F} \in \mathcal{F}_d \mid \tilde{F}_j \subset F_j, \dim(F_j/\tilde{F}_j) = 1, \tilde{F}_p = F_p \text{ for all } p \neq j\}. \end{aligned}$$

Then

$$e_j(\tilde{F}, F', x) = \begin{cases} 1, & \text{if } \tilde{F} \in \mathcal{V}, x \triangleright \tilde{F}, F'; \\ 0, & \text{otherwise,} \end{cases}$$

so we have

$$(*) \quad \check{\varphi}_A(F, F', x) = \sum_{t \in \mathbf{Q}} t \cdot \chi\{\tilde{F} \mid \tilde{F} \in \mathcal{V}, \varphi_B(\tilde{F}, F', x) = t\}.$$

3.12. We show that the only \tilde{F} which contribute to (*) are those for which $(\tilde{F}, F') \in$

\mathcal{O}_B .

Let $\tilde{F} \in \mathcal{V}$ be such that $(\tilde{F}, F', x) \in \text{supp } \varphi_B$ for some $x \in N_d$. We claim that then $\Phi(\tilde{F}, F') = B$.

Let $\Phi(\tilde{F}, F') = B'$. Then $B' \leq B$ by property 3.1, (III).

On the other hand, since $\tilde{F} \in \mathcal{V}$, for any $1 \leq l, m \leq n-1, l \neq j$

$$\sum_{p \leq l; q \leq m} b'_{pq} = \dim(\tilde{F}_l \cap F'_m) = \dim(F_l \cap F'_m) = \sum_{p \leq l; q \leq m} a_{pq} = \sum_{p \leq l; q \leq m} b_{pq},$$

and for any $m = 1, \dots, n-1$

$$\sum_{p \leq j; q \leq m} b'_{pq} = \dim(\tilde{F}_j \cap F'_m) \leq \dim(F_j \cap F'_m) - 1 \leq \sum_{p \leq j; q \leq m} a_{pq} - 1 = \sum_{p \leq j; q \leq m} b_{pq}.$$

This means that $B \leq B'$ (see 1.3). Therefore, $B' = B$.

By induction assumption $\varphi_B|_{Z[B]} \equiv 1$. Thus, we can continue (*):

$$\check{\varphi}_A(F, F', x) = \chi\{\tilde{F} \mid x \triangleright \tilde{F}, \tilde{F} \in \mathcal{V}, \Phi(\tilde{F}, F') = B\}.$$

3.13. Next, we show that the condition $x \triangleright \tilde{F}$ above is redundant. Namely, we claim that for any $\tilde{F} \in \mathcal{V}$, such that $\Phi(\tilde{F}, F') = B$

$$x \triangleright F, F' \Rightarrow x \triangleright \tilde{F}.$$

Indeed, for any such \tilde{F} ,

$$x(\tilde{F}_{k+1}) \subseteq x(F_{k+1}) \subseteq F_k = \tilde{F}_k, \text{ if } k \neq j.$$

Further, since $a_{pq} = b_{pq}$ for all $q \leq i$, we have $F_j \cap F'_i = \tilde{F}_j \cap F'_i$. Also, because of the choice of i and j , $F_{j+1} \subseteq F_{i+1} = F'_{i+1}$. It follows that

$$x(\tilde{F}_{j+1}) = x(F_{j+1}) = x(F_{j+1} \cap F'_{i+1}) \subseteq (F_j \cap F'_i) = (\tilde{F}_j \cap F'_i) \subseteq \tilde{F}_j.$$

By definition, this means that $x \triangleright \tilde{F}$.

3.14. Finally, for any $x \triangleright F, F'$ using 3.12 and 3.13 we can compute

$$\begin{aligned} \check{\varphi}_A(F, F', x) &= \chi\{\tilde{F} \mid \tilde{F} \in \mathcal{V}, \Phi(\tilde{F}, F') = B\} \\ &= \chi\{V \subseteq \mathbf{C}^d \mid ((F_j \cap F'_i) + F_{j-1}) \subseteq V \subset F_j, \dim(F_j/V) = 1\} \\ &= \dim(F_j / ((F_j \cap F'_i) + F_{j-1})) \\ &= a_{j,i+1}. \end{aligned}$$

This shows that $\check{\varphi}_A|_{Z[A]} \equiv a_{j,i+1}$, which proves 3.11 (a). To prove (b) we need the following

Lemma 3.15. (cf. [BLM, lemma 3.2]). *For any matrix $C \in \Theta_d$, and any constructible function $\psi \in \mathcal{A}_d$ such that $\text{supp } \psi \subseteq Z[C]$*

$$\text{supp } (e_j * \psi) \subseteq \bigcup_{\substack{p \in [1, n]: \\ c_{j+1, p} \geq 1}} Z[C + E_{jp} - E_{j+1, p}].$$

Proof. If $(G, G', y) \in \text{supp } (e_j * \psi)$, then there exists $\tilde{G} \in \mathcal{F}_d$ such that $(G, \tilde{G}, y) \in \text{supp } e_j$, and $(\tilde{G}, G', y) \in \text{supp } \psi \subseteq Z[C]$. Therefore,

$$(G, \tilde{G}) \in \mathcal{O}_{\Phi(G, G) - E_{jj} + E_{j, j+1}}, \quad (\tilde{G}, G') \in \mathcal{O}_C.$$

This means that

- (i) $\tilde{G}_p = G_p$ for all $p \neq j$,
- (ii) $\tilde{G}_j \subset G_j$, $\dim(G_j/\tilde{G}_j) = 1$,
- (iii) there exists $p \in [1, n]$ such that $G_j \cap G'_q = \tilde{G}_j \cap G'_q$ for all $q \neq p$, and
- (iv) $\tilde{G}_j \cap G'_p \subset G_j \cap G'_p$, $\dim(G_j \cap G'_p/\tilde{G}_j \cap G'_p) = 1$.

Since

$$\tilde{G}_j \cap G'_p + \tilde{G}_{j+1} \cap G'_{p-1} \subsetneq G_j \cap G'_p + G_{j+1} \cap G'_{p-1} \subseteq \tilde{G}_{j+1} \cap G'_p,$$

we have $c_{j+1, p} = \dim(\tilde{G}_{j+1} \cap G'_p) / (\tilde{G}_j \cap G'_p + \tilde{G}_{j+1} \cap G'_{p-1}) \neq 0$. Conditions (i)-(iv) imply that $(G, G') \in \mathcal{O}_{C + E_{jp} - E_{j+1, p}}$. Lemma 3.15 is proved.

3.16. We now prove 3.11, (b). By 3.1, (III) we know that $\text{supp } \varphi_B \subseteq \bigcup_{B' \leq B} Z[B']$. By lemma 3.15,

$$\text{supp } \check{\varphi}_A = \text{supp } (e_j * \varphi_B) \subseteq \bigcup_{B' \leq B} \bigcup_{\substack{p \in [1, n]: \\ b'_{j+1, p} \geq 1}} Z[B' + E_{jp} - E_{j+1, p}].$$

Fix $B' \leq B$ and p such that $b'_{j+1, p} \geq 1$. Let $A' = B' + E_{jp} - E_{j+1, p}$. We want to show that $A' \leq A$.

We claim that $p \leq i + 1$. Indeed, assume that $p \geq i + 2$.

We know that $b_{qr} = a_{qr} = 0$ for all q, r such that $q < r$ and $r \geq i + 2$ (see 3.9). Since $j \leq i$, then in particular $b_{qr} = 0$ for all q, r such that $q \leq j + 1 < i + 2$, and $r \geq p \geq i + 2$.

Therefore, since $b'_{j+1,p} \geq 1$ we have

$$\sum_{\substack{q \leq j+1 \\ r \leq p-1}} b_{qr} = \sum_{\substack{q \leq j+1 \\ r \leq n}} b_{qr} = \sum_{\substack{q \leq j+1 \\ r \leq n}} b'_{qr} \geq b'_{j+1,p} + \sum_{\substack{q \leq j+1 \\ r \leq p-1}} b'_{qr} > \sum_{\substack{q \leq j+1 \\ r \leq p-1}} b'_{qr}.$$

This contradicts the assumption $B' \leq B$. Hence $p \leq i + 1$.

By the definitions of B and A' , for any $l, m = 1, \dots, n$

$$\sum_{q \leq l; r \leq m} a_{qr} = \varepsilon + \sum_{q \leq l; r \leq m} b_{qr}; \quad \sum_{q \leq l; r \leq m} a'_{qr} = \varepsilon' + \sum_{q \leq l; r \leq m} b'_{qr},$$

where

$$\varepsilon = \begin{cases} 1, & \text{if } l = j; m \geq i + 1, \\ 0, & \text{otherwise,} \end{cases} \quad \varepsilon' = \begin{cases} 1, & \text{if } l = j; m \geq p, \\ 0, & \text{otherwise.} \end{cases}$$

Since $p \leq i + 1$, the inequality $\sum_{q \leq l; r \leq m} b'_{qr} \geq \sum_{q \leq l; r \leq m} b_{qr}$ implies that

$$\sum_{q \leq l; r \leq m} a'_{qr} \geq \sum_{q \leq l; r \leq m} a_{qr}$$

for all $l, m = 1, \dots, n$. This means that $A' \leq A$. Therefore, $\text{supp } \check{\varphi}_A \subseteq \cup_{A' \leq A} Z[A']$. Lemma 3.11 is proved.

3.17. We now finish the construction of φ_A .

For any $A' \in \Theta_d$ the variety $Z[A']$ is irreducible. Since the function $\check{\varphi}_A$ is constructible, there exists an open dense subset of $Z[A']$ such that the restriction of $\check{\varphi}_A$ on this subset is constant. Let us denote this constant by $c(A')$. We define

$$\varphi_A = \frac{1}{a_{j,i+1}} \cdot (\check{\varphi}_A - \sum_{A' < A} c(A') \cdot \varphi_{A'}).$$

We check that φ_A satisfies the conditions of theorem 3.1, using that by induction assumption $\varphi_{A'}$ has the properties (I)-(III) for any $A' < A$.

First, since $\text{supp } \varphi_{A'} \cap Z[A] = \emptyset$, by 3.14

$$\varphi_A|_{Z[A]} = \frac{1}{a_{j,i+1}} \cdot \check{\varphi}_A|_{Z[A]} \equiv 1.$$

Second, φ_A is a linear combination of functions with supports contained in the union $\cup_{A' \leq A} Z[A']$. For $\check{\varphi}_A$ this is given by lemma 3.11, (b) and for all $\varphi_{A'}$ - by property 3.1, (III).

Therefore, $\text{supp } \varphi_A \subseteq \cup_{A' \leq A} Z[A']$.

Finally, φ_A is defined so that for any $A' < A$ it is 0 on an open dense subset of $Z[A']$. This completes the proof of 3.1, (I)-(III) for φ_A .

3.18. We recall that the function φ_A was constructed under the assumption that $a_{j,i+1} \neq 0$ for some $j \leq i$ (see 3.10). If this is not the case, then as was explained in 3.10, there must exist $k \leq i$ such that $a_{i+1,k} \neq 0$. Then the matrix tA has a non-zero entry in the $(i+1)$ -th column. We prove the existence of φ_A using the involution τ (see 2.3), and the above procedure for the matrix tA .

We need the following facts.

(i) Transposition of matrices in Θ_d preserves the relations \leq and \ll , i.e.

$$\begin{aligned} A' \leq A &\Leftrightarrow {}^tA' \leq {}^tA, \\ A' \ll A &\Leftrightarrow {}^tA' \ll {}^tA. \end{aligned}$$

This is obvious from the definitions of \leq and \ll (see 1.3, 3.8.)

(ii) A function $\varphi \in U_d$ satisfies the conditions 3.1, (I)-(III) for a matrix $A \in \Theta_d$ if and only if the function $\tau(\varphi)$ satisfies the same conditions for tA .

This follows from (i) and the fact that τ_0 maps $Z[A]$ isomorphically onto $Z[{}^tA]$ for any $A \in \Theta_d$ (see 1.7).

By induction assumption, we have the functions $\varphi_{A'}$ for all $A' < A$ and all $A' \ll A$. For all B such that $B < {}^tA$, or $B \ll {}^tA$, but $B \not< A$, and $B \not\ll A$, we set $\varphi_B = \tau(\varphi({}^tB))$. By (ii), this function satisfies (I)-(III). By (i), we now have functions $\varphi_{B'}$ for all B' such that $B' < {}^tA$, or $B' \ll {}^tA$. We can apply the procedure 3.10-3.17 to construct the function $\varphi_{{}^tA}$. Then by (ii), the function

$$\varphi_A = \tau(\varphi_{{}^tA})$$

satisfies 3.1, (I)-(III).

Note that if we set $B = A - E_{i+1,k} + E_{i+1,k+1}$, then $B \ll A$, and

$$\varphi_A = \frac{1}{a_{j,i+1}} \cdot \varphi_B * f_k - \sum_{A' < A} \text{const} \cdot \varphi_{A'}.$$

4 Dimension of U_d .

In this section we show that $\dim U_d = \binom{d+n^2-1}{d}$, and complete the proof of theorem 3.1.

4.1. We recall that by corollary 2.5 we have the surjective homomorphism

$$\gamma_d : U(\mathfrak{sl}_n) \rightarrow U_d$$

such that $\gamma_d(\underline{e}_i) = e_i$, $\gamma_d(\underline{f}_i) = f_i$ for all $i = 1, \dots, n-1$. This turns U_d into a $U(\mathfrak{sl}_n)$ -module, where $X \in U(\mathfrak{sl}_n)$ acts on $\varphi \in U_d$ by

$$X \cdot \varphi = \gamma_d(X) * \varphi.$$

Lemma 4.2. (1) U_d can be decomposed into a direct sum of a finite number of weight spaces

$$U_d = \bigoplus_{\mu \in \mathcal{M}_d} (U_d)^\mu,$$

where $\mu = (\mu_1, \mu_2, \dots, \mu_{n-1})$, and $\mu(\underline{h}_i) = \mu_i - \mu_{i+1}$ for all $i = 1, \dots, n-1$ (we set $\mu_n = 0$);

(2) For any $\mu \in \mathcal{M}_d$ we have $\sum_{i=1}^{n-1} \mu_i = d - kn$ for some non-negative integer k .

Proof. By corollary 3.6,

$$(*) \quad U_d = \mathbf{1} * U_d = \left(\sum_{A \in \Delta_d} \varphi_A \right) * U_d = \sum_{A \in \Delta_d} (\varphi_A * U_d).$$

We claim that the sum in (*) is direct, and that each subspace $\varphi_A * U_d$ with $A \in \Delta_d$ is a weight space.

For any $A, B \in \Delta_d$, and any $\psi, \psi' \in U_d$, corollary 3.7 implies that

$$\text{supp}(\varphi_A * \psi) \cap \text{supp}(\varphi_B * \psi') = \emptyset,$$

if $A \neq B$. Hence, $\varphi_A * U_d \cap \varphi_B * U_d = 0$. This shows that the sum in (*) is direct.

Further, for any $A \in \Delta_d$ and $i \leq n-1$ we have $h_i|_{Z[A]} \equiv a_{ii} - a_{i+1, i+1}$ (see 2.6). Using corollary 3.7, for any $\psi \in U_d$, we can compute

$$\begin{aligned} h_i * \varphi_A * \psi(F, F', x) &= h_i(F, F, x)(\varphi_A * \psi)(F, F', x) \\ &= \begin{cases} (a_{ii} - a_{i+1, i+1}) \cdot \psi(F, F', x), & \text{if } (F, F) \in \mathcal{O}_A, \\ 0 & \text{otherwise.} \end{cases} \\ &= (a_{ii} - a_{i+1, i+1}) \cdot (\varphi_A * \psi)(F, F', x). \end{aligned}$$

Therefore, for any $\varphi \in (\varphi_A * U_d)$,

$$\underline{h}_i \cdot \varphi = \gamma_d(\underline{h}_i) * \varphi = h_i * \varphi = (a_{ii} - a_{i+1, i+1}) \cdot \varphi.$$

This implies that $(\varphi_A * U_d)$ is an eigenspace of $\gamma_d(\underline{h}_i)$ with the eigenvalue $a_{ii} - a_{i+1,i+1}$. Therefore, $\varphi_A * U_d = (U_d)^\mu$ with $\mu(\underline{h}_i) = a_{ii} - a_{i+1,i+1}$, for any $A \in \Delta_d$. The claim (1) is proved.

Further,

$$\mu_i = \sum_{p=i}^{n-1} \mu(\underline{h}_p) = \sum_{p=i}^{n-1} (a_{pp} - a_{p+1,p+1}) = a_{ii} - a_{nn},$$

and

$$\sum_{i=1}^{n-1} \mu_i = \sum_{i=1}^{n-1} (a_{ii} - a_{nn}) = \sum_{i=1}^n a_{ii} - n \cdot a_{nn} = d - a_{nn} \cdot n.$$

This proves (2).

Lemma 4.3. *The algebra U_d is finite dimensional.*

Proof. Let U^-, U^0 , and U^+ be the subalgebras of $U(\mathfrak{sl}_n)$ generated respectively by $\{\underline{f}_i\}_{i=1}^{n-1}$, $\{\underline{h}_i\}_{i=1}^{n-1}$, and $\{\underline{e}_i\}_{i=1}^{n-1}$. By the Poincaré-Birkhoff-Witt theorem,

$$U(\mathfrak{sl}_n) = U^- \otimes U^0 \otimes U^+.$$

The image $\gamma_d(U^-)$ is generated by monomials $\{f_{i_1} * \cdots * f_{i_k} \in U_d\}_{1 \leq i_1, \dots, i_k \leq n-1}$. For any such monomial we have $(f_{i_1} * \cdots * f_{i_k})(F, F', x) = 0$ unless $\dim F'_i - \dim F_i = \#\{p \mid i_p = i\}$ for all $i = 1, \dots, n-1$. Since $F_i, F'_i \subseteq \mathbf{C}^d$, all such monomials are 0 if $k > d(n-1)$.

This implies that $\dim \gamma_d(U^-) < \infty$. Similarly, $\dim \gamma_d(U^+) < \infty$. By proposition 3.2, $\dim \gamma_d(U^0) = \dim D = |\Delta_d| < \infty$.

Since $U_d = \gamma_d(U(\mathfrak{sl}_n)) = \gamma_d(U^-) \otimes \gamma_d(U^0) \otimes \gamma_d(U^+)$, we have $\dim U_d < \infty$.

4.4. As a finite dimensional \mathfrak{sl}_n -module, U_d can be decomposed into a direct sum of irreducible modules, $U_d \cong \bigoplus L_\lambda$. Each L_λ is a highest weight module with an integral dominant weight $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1})$, $\lambda_{n-1} \geq 0$.

Let $L_{\lambda^{(1)}}, \dots, L_{\lambda^{(l)}}$ be a complete set of pairwise non-isomorphic modules in the decomposition of U_d .

U_d acts on itself by left multiplication. By definition, this action is compatible with the action of $U(\mathfrak{sl}_n)$. Therefore, each of $L_{\lambda^{(p)}}$ is also an irreducible U_d -module. This gives an algebra homomorphism $r : U_d \rightarrow \bigoplus_{p=1}^l \text{End}(L_{\lambda^{(p)}})$.

Lemma 4.4.1. *The homomorphism r is injective.*

Proof. We want to show that for any non-zero $\psi \in U_d$ there is $\lambda^{(p)}$ such that ψ does not annihilate $L_{\lambda^{(p)}}$. But $\psi * \mathbf{1} = \psi \neq 0$, which means that ψ does not annihilate $U_d \cong \bigoplus L_\lambda$. Then for some λ , ψ does not annihilate L_λ . Since $L_\lambda \cong L_{\lambda^{(p)}}$ for some $p \in [1, l]$,

the lemma follows.

4.5. Let $\Lambda(d) = \{\lambda - \text{dominant integral} \mid \sum_{i=1}^{n-1} \lambda_i = d - kn \text{ for some integer } k \geq 0\}$. Note that by lemma 4.2, (2) we have $\lambda^{(1)}, \dots, \lambda^{(l)} \in \Lambda(d)$.

Corollary 4.6. $\dim U_d \leq \sum_{\lambda \in \Lambda(d)} (\dim L_\lambda)^2$.

Remark. In section 5 we construct the U_d -module L_λ for any $\lambda \in \Lambda(d)$. This shows that in fact $\{\lambda^{(1)}, \dots, \lambda^{(l)}\} = \Lambda(d)$.

4.7. We finish the computation of $\dim U_d$ using Weyl's decomposition of a tensor power of the standard representation of $GL_n = GL(n, \mathbf{C})$ into irreducible representations.

Let V be the n -dimensional complex vector space on which GL_n acts in the natural way. Consider the d -th tensor power $V^{\otimes d}$ of the module V . Let $\rho : GL_n \rightarrow \text{Aut}(V^{\otimes d})$ be the corresponding representation.

There is a natural action of the symmetric group S_d on $V^{\otimes d}$ given by

$$\sigma \cdot (v_1 \otimes v_2 \otimes \dots \otimes v_d) = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(d)}$$

for all $\sigma \in S_d$, $v_1, \dots, v_d \in V$.

4.8. Let \mathbf{A}_d be the subset of the set of integral dominant weights of GL_n such that

$$\mathbf{A}_d = \{\alpha = (\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n) \mid \alpha_n \geq 0, \alpha_i \in \mathbf{Z}, \sum_{i=1}^n \alpha_i = d\}.$$

We denote by W_α the irreducible module of GL_n with the highest weight α .

Proposition 4.9. (H. Weyl, [W])

(1) $\{W_\alpha \mid \alpha \in \mathbf{A}_d\}$ is the complete list of irreducible GL_n -modules such that $\text{Hom}_{GL_n}(W_\alpha, V^{\otimes d}) \neq 0$.

(2) The image $\rho(GL_n)$ coincides with the set $\text{Aut}_{S_d}(V^{\otimes d})$ of all automorphisms of $V^{\otimes d}$ commuting with the action of S_d .

4.10. As is known, the restriction of GL_n -action to SL_n turns W_α into an irreducible SL_n -module. The corresponding $sl_n(\mathbf{C})$ -module is isomorphic to $L_{\lambda(\alpha)} \otimes \mathbf{C}$ with $\lambda_i(\alpha) = \alpha_i - \alpha_n$ for all $i = 1, \dots, n-1$; $\dim_{\mathbf{C}} W_\alpha = \dim_{\mathbf{Q}} L_{\lambda(\alpha)}$. Note that

$$\lambda(\alpha) \in \Lambda(d) \Leftrightarrow \alpha \in \mathbf{A}_d.$$

Proposition 4.9, (1) and corollary 4.6 imply that

$$\dim U_d \leq \sum_{\lambda \in \Lambda(d)} (\dim L_\lambda)^2 = \sum_{\alpha \in \mathbf{A}_d} (\dim W_\alpha)^2 = \dim \rho(GL_n).$$

4.11. By 4.9, (2), $\dim \rho(GL_n) = \dim \text{Aut}_{S_d}(V^{\otimes d}) = \dim \text{End}_{S_d}(V^{\otimes d})$. The last space is a linear space of all $T = (t_{i_1 i_2 \dots i_d; j_1 j_2 \dots j_d})_{1 \leq i_p, j_p \leq n}$, such that $t_{i_1 i_2 \dots i_d; j_1 j_2 \dots j_d} \in \mathbf{C}$, and for any $\sigma \in S_d$

$$t_{i_{\sigma(1)} i_{\sigma(2)} \dots i_{\sigma(d)}; j_{\sigma(1)} j_{\sigma(2)} \dots j_{\sigma(d)}} = t_{i_1 i_2 \dots i_d; j_1 j_2 \dots j_d}.$$

The dimension of this space equals the number of orbits of S_d acting on the set of d -tuples of pairs of indices

$$\{((i_1, j_1), (i_2, j_2), \dots, (i_d, j_d)) \mid 1 \leq i_1, i_2, \dots, i_d, j_1, j_2, \dots, j_d \leq n\}$$

by $\sigma \cdot ((i_1, j_1), (i_2, j_2), \dots, (i_d, j_d)) = ((i_{\sigma(1)}, j_{\sigma(1)}), (i_{\sigma(2)}, j_{\sigma(2)}), \dots, (i_{\sigma(d)}, j_{\sigma(d)}))$. This in turn is equal to the number of combinations (possibly, with repetitions) of d elements out of n^2 elements of the set of all pairs $\{(i, j) \mid 1 \leq i, j \leq n\}$. The latter is equal to $\binom{d+n^2-1}{d}$.

Combining this with 4.10, we conclude that $\dim U_d \leq \binom{d+n^2-1}{d}$.

4.12. As was shown in section 3, functions $\{\varphi_A\}_{A \in \Theta_d} \in U_d$ are linearly independent. Therefore, $\dim U_d \geq |\Theta_d| = \binom{d+n^2-1}{d}$ (see 1.2.1).

Together with 4.11 this shows that $\dim U_d = \binom{d+n^2-1}{d}$, and that $\{\varphi_A\}_{A \in \Theta_d}$ form a basis of U_d .

Corollary 4.13. *If $\psi \in U_d$ vanishes on some non-empty open subset of $Z[A]$ for any $A \in \Theta_d$, then $\psi = 0$.*

Proof. Since $\{\varphi_A\}_{A \in \Theta_d}$ is a basis of U_d , we have

$$\psi = \sum_{A \in \Theta_d} c(A) \cdot \varphi_A$$

for some constants $c(A) \in \mathbf{Q}$. Then for each $A \in \Theta_d$ the right hand side equals $c(A)$ on some open dense subset of $Z[A]$. But ψ vanishes on an open subset. Since $Z[A]$ is irreducible, any two of its open subsets have a non-empty intersection. Hence all $c(A) = 0$, which implies $\psi = 0$.

4.14. We now prove the uniqueness part of theorem 3.1. Assume that for some $A \in \Theta_d$ functions $\varphi, \varphi' \in U_d$ satisfy the conditions (I) and (II) of theorem 3.1 for the

matrix A . Then $\varphi - \varphi'$ vanishes on some open subsets of all $Z[B]$, $B \in \Theta_d$. By corollary 4.13, $\varphi - \varphi' = 0$.

Corollary 4.15. *The set $\{\varphi_A\}_{A \in \Theta_d}$ is invariant under the involution τ .*

Indeed, as was shown in 3.18, the function $\tau(\varphi_A)$ satisfies the conditions of theorem 3.1 for the matrix tA . By uniqueness, $\tau(\varphi_A) = \varphi({}^tA)$.

5 U_d -modules.

5.1. Let $x \in N_d$. We recall that $x^n = 0$ (see 1.4).

For any $i \in [1, n-1]$, let $\lambda_i(x)$ be the number of Jordan blocks of sizes $i, i+1, \dots, n-1$ in the Jordan decomposition of x . We set $\lambda_n(x) = 0$. Let $k(x)$ be the number of blocks of size n . Then

$$\lambda_1(x) \geq \lambda_2(x) \geq \dots \geq \lambda_{n-1}(x) \geq \lambda_n(x) = 0; \quad \sum_{i=1}^{n-1} \lambda_i(x) = d - k(x) \cdot n.$$

Note that for any $x \in N_d$ the weight λ such that $\lambda_i = \lambda_i(x), i = 1, \dots, n$ lies in Λ_d (see 4.4). Conversely, for any $\lambda \in \Lambda_d$ there obviously exists $x \in N_d$ such that $\lambda(x) = \lambda$.

Let $C(x) = \{g x g^{-1} \mid g \in GL_d\}$ be the conjugacy class of x . As is well known, the number of conjugacy classes in N_d is finite, and $C(x) = C(y)$ if and only if $\lambda(x) = \lambda(y)$.

5.2. For flags $F, F' \in \mathcal{F}_d$ we say that $F \subseteq F'$ if $F_i \subseteq F'_i$ for all $i = 1, \dots, n-1$. We define the flag

$$K^x = (0 \subseteq \text{Ker } x \subseteq \text{Ker } x^2 \subseteq \dots \subseteq \text{Ker } x^{n-1} \subseteq \text{Ker } x^n = \mathbb{C}^d).$$

Obviously, $x \triangleright K^x$. We have $\dim K_i^x = \dim \text{Ker } x^i = \sum_{p=1}^i \lambda_p(x) + i \cdot k(x)$.

Therefore,

$$\underline{\dim} K^x = \underline{\dim} K^y \Leftrightarrow \lambda(x) = \lambda(y).$$

If $x \triangleright F$, then necessarily $F_i \subseteq K_i^x$ for all i , therefore $F \subseteq K^x$.

5.3. Following [G], for any $x \in N_d$ we consider

$$M^x = \{(F, F', y) \in Z_d \mid F' = K^x, y = x\}.$$

This is a closed subvariety of Z_d isomorphic to the subvariety $\pi^{-1}(x)$ of M_d (see 1.5).

The subspace

$$\{\psi \in \mathcal{A}_d \mid \text{supp } \psi \subseteq M^x\} \subset \mathcal{A}_d$$

is clearly a left ideal of \mathcal{A}_d . In particular, it is stable under left multiplication by U_d , which makes it a U_d -module.

Let $s_x \in \mathcal{A}_d$ be such that $s_x(F, F', y) = 0$ unless $F = F' = K^x$, $y = x$, and $s_x(K^x, K^x, x) = 1$. Let L_x be a submodule generated by s_x ,

$$L_x = \{\psi * s_x \mid \psi \in U_d\}.$$

Since U_d is finite dimensional, so is L_x .

We have

$$\psi * s_x(F, F', y) = \begin{cases} \psi(F, K^x, x), & \text{if } F' = K^x, y = x, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, multiplication of a function ψ by s_x from the right amounts to restriction of ψ on M^x followed by extension by 0.

Via the homomorphism γ_d defined in 2.5, L_x is also a $U(\mathfrak{sl}_n)$ -module, and therefore an \mathfrak{sl}_n -module.

Theorem 5.4. *For any $\lambda \in \Lambda_d$ and any $x \in N_d$ such that $\lambda(x) = \lambda$*

- (1) *L_x is an irreducible \mathfrak{sl}_n -module isomorphic to the highest weight module L_λ ; s_x is a highest weight vector.*
- (2) *Functions $\{\varphi_A * s_x \mid A \in \Theta_d, \varphi_A * s_x \neq 0\}$ form a basis of L_x .*

The rest of this section is devoted to the proof of theorem 5.4.

5.5. First we show that s_x is a highest weight vector.

By definition, $e_i(F, K^x, x)$ could be not 0 only if $\dim F_i = \dim K_i^x + 1$. But this would contradict $x \triangleright F$, since we must have $F \subseteq K^x$ (see 5.2). Hence $e_i(F, K^x, x) = 0$ for all $(F, K^x, x) \in Z_d$. This gives

$$\underline{e}_i \cdot s_x = e_i * s_x = 0$$

for all $i = 1, \dots, n - 1$.

Next, we compute the weight of s_x :

$$\underline{h}_i \cdot s_x = h_i * s_x = h_i(K^x, K^x, x) \cdot s_x.$$

By 5.2 and lemma 2.6,

$$\begin{aligned}
h_i(K^x, K^x, x) &= 2 \dim \text{Ker } x^i - \dim \text{Ker } x^{i-1} - \dim \text{Ker } x^{i+1} \\
&= 2 \cdot \left(\sum_{p=1}^i \lambda_p(x) + i \cdot k(x) \right) - \left(\sum_{p=1}^{i-1} \lambda_p(x) + (i-1) \cdot k(x) \right) - \left(\sum_{p=1}^{i+1} \lambda_p(x) + (i+1) \cdot k(x) \right) \\
&= \lambda_i(x) - \lambda_{i+1}(x).
\end{aligned}$$

Therefore, $\underline{h}_i \cdot s_x = (\lambda_i(x) - \lambda_{i+1}(x)) \cdot s_x$ for all $i = 1, \dots, n-1$.

Further, since $L_x = U_d * s_x = \gamma_d(U(\mathfrak{sl}_n)) * s_x = U(\mathfrak{sl}_n) \cdot s_x$, and L_x is finite dimensional, it is irreducible. Therefore, $L_x \cong L_\lambda$ with $\lambda(\underline{h}_i) = \lambda_i(x) - \lambda_{i+1}(x)$. This completes the proof of (1).

5.6. To prove (2) we have to understand the set of irreducible components of M^x .

This variety is closely related to the variety of all parabolic subgroups of GL_d which contain a fixed unipotent element. This is a well understood object for GL_d as well as for the other classical groups, see R. Steinberg [St], N. Spaltenstein [Sp]. The varieties M^x also occur in studying resolutions of nilpotent varieties. W. Borho and R. MacPherson in [BM] call them Spaltenstein-varieties.

In the following proposition we reformulate some of the results, and prove them in our setting.

$$\text{Let } \Theta_d(x) = \{A \in \Theta_d \mid Z[A] \cap M^x \neq \emptyset\}.$$

$$\text{Let } M^x(A) = Z[A] \cap M^x. \text{ This is a locally closed subvariety of } M^x.$$

Proposition 5.7. (cf. [St].) (i) *The set*

$$GL_d \cdot M^x(A) = \{(gF, gK^x, gxg^{-1}) \mid (F, K^x, x) \in M^x(A), g \in GL_d\}$$

is dense in $Z[A]$, *i.e.* $\overline{GL_d \cdot M^x(A)} = \overline{Z[A]}$ *for any* $A \in \Theta_d(x)$;

(ii) $M^x(A)$ *is irreducible*; $\dim M^x(A) = \dim Z[A] - \dim C(x)$;

(iii) $\{\overline{M^x(A)}\}_{A \in \Theta_d(x)}$ *is the complete set of irreducible components of* M^x .

Proof. Let $N_{(F, F')} = \{y \in N_d \mid y \triangleright F, F'\}$ be the fiber of the conormal bundle

$Z[\Phi(F, F')]$ at the point (F, F') . We have

$$N_{(F, F')} = \bigcup_{y \in N_d} (N_{(F, F')} \cap C(y)).$$

This is a disjoint union of a finite number of sets, since the number of conjugacy classes in N_d is finite. But $N_{(F, F')}$ is irreducible, therefore there exists $z \in N_d$ such that $N_{(F, F')} \cap C(z)$ is open dense in $N_{(F, F')}$.

Lemma 5.8. *For any $F \in \mathcal{F}_d$ such that $x \triangleright F$, the set $N_{(F, K^x)} \cap C(x)$ is open dense in $N_{(F, K^x)}$.*

Proof. Let $z \in N_d$ be such that $N_{(F, K^z)} \cap C(z)$ is dense in $N_{(F, K^z)}$. Then $x \in \overline{C(z)}$. It follows that $\text{rank } x^i \leq \text{rank } z^i$ for any $i \in [1, n-1]$. Therefore, $\dim \text{Ker } x^i \geq \dim \text{Ker } z^i$.

But $z \triangleright K^x$, and hence $K^x \subseteq K^z$ (see 5.2). This shows that $\underline{\dim} K^x = \underline{\dim} K^z$, and therefore by 5.2, $\lambda(x) = \lambda(z)$. This implies that $C(x) = C(z)$. The lemma is proved.

5.9. Let us fix a flag G such that $x \triangleright G$, $(G, K^x) \in \mathcal{O}_A$. By definition,

$$\begin{aligned} M^x(A) &= Z[A] \cap M^x = \{(F, K^x, x) \in M^x \mid (F, K^x) \in \mathcal{O}_A\} \\ &= \{(hG, hK^x, x) \in M^x \mid h \in GL_d\}. \end{aligned}$$

Then

$$\begin{aligned} GL_d \cdot M^x(A) &= \{g \cdot (F, K^x, x) \mid (F, K^x, x) \in M^x(A), g \in GL_d\} \\ &= \{g \cdot (hG, hK^x, x) \mid g, h \in GL_d, (hG, hK^x, x) \in M^x\} \\ &= \{g \cdot (G, K^x, hxh^{-1}) \mid g, h \in GL_d, hxh^{-1} \triangleright G, K^x\} \\ &= \{g \cdot (G, K^x, y) \mid g \in GL_d, y \in N_{(G, K^x)} \cap C(x)\} \end{aligned}$$

By lemma 5.8, $N_{(G, K^x)} \cap C(x)$ is dense in the fiber $N_{(G, K^x)}$. Since

$$Z[A] = \{g \cdot (G, K^x, y) \mid g \in GL_d, y \in N_{(G, K^x)}\},$$

this shows that $GL_d \cdot M^x(A)$ is dense in $Z[A]$. This proves 5.7, (i).

5.10. We now prove 5.7, (ii).

Let X_1 be an irreducible component of $M^x(A)$. Assume that $M^x(A) = X_1 \cup X_2$, where $X_2 = \overline{M^x(A)} - X_1 \neq \emptyset$. Then $GL_d \cdot X_1$ is closed in $GL_d \cdot M^x(A)$, and $GL_d \cdot X_1 \cup GL_d \cdot X_2 = GL_d \cdot M^x(A)$.

The latter set is dense in $\overline{Z[A]}$, which is irreducible. Therefore, $GL_d \cdot X_1 = GL_d \cdot X_2$. Then $X_2 = (GL_d)_x \cdot X_1$, where $(GL_d)_x$ denotes the centralizer of x . But all the centralizers of nilpotent elements in GL_d are known to be connected, and therefore $(GL_d)_x$ stabilizes

every irreducible component of $M^x(A)$. This implies that $X_2 = (GL_d)_x \cdot X_1 \subseteq X_1$, which contradicts the choice of X_1, X_2 . Therefore, we must have $X_2 = \emptyset$, so $M^x(A) = X_1$ is irreducible.

Let us consider the map $pr : Z_d \rightarrow N_d$ given by $pr(F, F', y) = y$. Then $pr(GL_d \cdot M^x(A)) \subseteq C(x)$, and the restriction $pr : GL_d \cdot M^x(A) \rightarrow C(x)$ is a GL_d -equivariant locally trivial fibration with the fiber $M^x(A)$. Hence

$$\dim Z[A] = \dim GL_d \cdot M^x(A) = \dim C(x) + \dim M^x(A),$$

which yields (ii).

5.11. Being conormal bundles, all $Z[A]$ contained in the same connected component of Z_d have the same dimension. Therefore 5.7, (ii) implies that all $M^x(A)$ lying in the same connected component of M^x have the same dimension. It follows that $M^x(A') \subseteq \overline{M^x(A)}$ if and only if $A' = A$, and hence each $\overline{M^x(A)}$ is an irreducible component of M^x . Further,

$$M^x \subset Z_d = \bigcup_{A \in \Theta_d} Z[A] \Rightarrow M^x = \bigcup_{A \in \Theta_d} (M^x \cap Z[A]) = \bigcup_{A \in \Theta_d(x)} M^x(A).$$

Therefore, each irreducible component must be of the form $\overline{M^x(A)}$ for some $A \in \Theta_d(x)$. Proposition 5.7 is proved.

Corollary 5.12. *For any $A \in \Theta_d(x)$*

- (i) $(\varphi_A * s_x)|_{M^x(A)} \equiv 1$;
- (ii) $\varphi_A * s_x$ vanishes on some open dense subset of $M^x(A')$ for all $A' \neq A$.

Proof. This corollary is a counterpart of theorem 3.1 for the functions $\varphi_A * s_x \in L_x$.

We recall that $\varphi|_{M^x} = (\varphi * s_x)|_{M^x}$ for all $\varphi \in U_d$.

Since $M^x(A) \subset Z[A]$, by 3.1, (I)

$$(\varphi_A * s_x)|_{M^x(A)} = \varphi_A|_{M^x(A)} \equiv 1.$$

Now suppose that $\text{supp}(\varphi_A * s_x)$ contains an open subset of $M^x(A')$ for some $A' \neq A$. Then this set is contained in $\text{supp} \varphi_A$. But by lemma 2.11, φ_A is constant on GL_d -orbits, and by 5.7, $GL_d \cdot M^x(A')$ is dense in $Z[A']$. This implies that $\text{supp} \varphi_A$ contains an open subset of $Z[A']$, which contradicts 3.1, (II).

Corollary 5.13. *The functions $\{\varphi_A * s_x\}_{A \in \Theta_d(x)}$ are linearly independent.*

Proof. Let

$$\sum_{A \in \Theta_d(x)} c_A \cdot (\varphi_A * s_x) = 0$$

for some $c_A \in \mathbf{Q}$. Then by corollary 5.12, for any $A \in \Theta_d(x)$ the restriction of this sum on some open subset of the irreducible component $\overline{M^x(A)}$ of M^x is equal to c_A . Hence, all $c_A = 0$.

5.14. We are now ready to finish the proof of theorem 5.4. The key fact which we need is that $\dim L_x = |\Theta_d(x)|$.

Let $\alpha_i = \lambda_i(x) + k(x)$, $i = 1, \dots, n$. Then $\alpha = (\alpha_1 \geq \dots \geq \alpha_n)$ is a partition of d (see 5.1).

A semi-standard α -tableau is the Young diagram of type α with the nodes replaced by integers $1, 2, \dots, n$ so that the numbers are nondecreasing along each row, and strictly increasing down each column.

The number of irreducible components of M^x equals the number of semi-standard α -tableaux. This follows from N. Shimomura's theorem on the fixed point subvarieties of unipotent transformations on flag varieties (see [Sh]).

On the other hand, the dimension of the irreducible GL_n -module W_α of the highest weight α also equals the number of semi-standard α -tableau (see e.g. [CL]). By 5.4, (1) we know that $L_x \cong L_\lambda$, and by 4.10, $\dim L_\lambda = \dim W_\alpha$. Therefore, $\dim L_x$ equals the number of irreducible components of M^x .

By proposition 5.7, (iii) the latter is equal to $|\Theta_d(x)|$. Since the functions $\{\varphi_A * s_x\}_{A \in \Theta_d(x)}$ are linearly independent (see 5.13), this shows that they form a basis of L_x .

Remark. In Ginzburg's construction (see [G, theorem 4.4]), the basis of L_x is given by fundamental classes of irreducible components of M^x . This also shows that $\dim L_x$ equals the number of irreducible components of M^x .

5.15. Finally, we have to show that $\varphi_B * s_x = 0$ if $B \in \Theta_d - \Theta_d(x)$. We use the same argument as in corollary 4.13. Since $\varphi_B * s_x \in L_x$, it has to be a linear combination of the basis elements:

$$\varphi_B * s_x = \sum_{A \in \Theta_d(x)} c_A \cdot (\varphi_A * s_x).$$

For any $A \in \Theta_d(x)$ the restriction of the right hand side on some open dense subset of $M^x(A)$ is equal to c_A . But by corollary 5.12, (ii) we know that $\varphi_B * s_x$ vanishes on some open dense subset of $M^x(A)$. Since $M^x(A)$ is irreducible, the intersection of these two open subsets is not empty. Hence all $c_A = 0$, so $\varphi_B * s_x = 0$.

This completes the proof of theorem 5.4.

6 Stabilization.

In this section we show that for any d there is a surjective algebra homomorphism $U_{d+n} \rightarrow U_d$ which commutes with homomorphisms γ_d, γ_{d+n} from $U(\mathfrak{sl}_n)$, and which is compatible with the bases of U_d and U_{d+n} given by theorem 3.1.

6.1. Following [G], we fix a decomposition $\mathbf{C}^{d+n} = \mathbf{C}^d \oplus \mathbf{C}^n$, and a nilpotent operator $x^\sharp : \mathbf{C}^n \rightarrow \mathbf{C}^n$ whose Jordan form has a single $n \times n$ block. There is a unique complete flag $F^\sharp \in \mathcal{F}_n$ such that $x^\sharp \triangleright F^\sharp$. Clearly,

$$F_i^\sharp = \text{Im}(x^\sharp)^{n-i} = \text{Ker}(x^\sharp)^i, \quad i = 1, \dots, n.$$

If $F \in \mathcal{F}^d$, let $F \oplus F^\sharp$ denote a flag in \mathcal{F}_{d+n} whose i -th space is $F_i \oplus F_i^\sharp$ for all $i = 1, \dots, n$. For any $x \in N_d$ obviously

$$x \triangleright F \Leftrightarrow (x \oplus x^\sharp) \triangleright (F \oplus F^\sharp).$$

For $G \in \mathcal{F}_{d+n}$ by $\underline{G} \in \mathcal{F}_d$ we denote a flag such that $\underline{G}_i = G_i \cap \mathbf{C}^d$, $i = 1, \dots, n$.

Lemma 6.2. *If $x \in N_d$ and $G \in \mathcal{F}_{d+n}$ are such that $(x \oplus x^\sharp) \triangleright G$, then $x \triangleright \underline{G}$, and $G = \underline{G} \oplus F^\sharp$.*

Proof. Let us fix $i \in [1, n-1]$. Since $(x \oplus x^\sharp) \triangleright G$, we have

$$\text{Im}(x \oplus x^\sharp)^{n-i} \subseteq G_i \subseteq \text{Ker}(x \oplus x^\sharp)^i$$

(see 1.4). But $\text{Ker}(x \oplus x^\sharp)^i = \text{Ker } x^i \oplus \text{Ker}(x^\sharp)^i = \text{Ker } x^i \oplus F_i^\sharp$. Similarly, $\text{Im}(x \oplus x^\sharp)^{n-i} = \text{Im } x^{n-i} \oplus F_i^\sharp$ (see 6.1). It follows that

$$F_i^\sharp \subseteq G_i \subseteq \mathbf{C}^d \oplus F_i^\sharp.$$

Therefore, any vector in G_i can be written uniquely as a sum $u + v$, where $u \in \mathbf{C}^d$, $v \in F_i^\sharp \subseteq G_i$. Then also $u \in G_i$. Hence $G_i = (G_i \cap \mathbf{C}^d) \oplus F_i^\sharp$.

The statement $x \triangleright \underline{G}$ is obvious. The lemma is proved.

6.3. Following [G], we consider an embedding $\iota : Z_d \hookrightarrow Z_{d+n}$ defined by

$$\iota(F, F', x) = (F \oplus F^\sharp, F' \oplus F^\sharp, x \oplus x^\sharp).$$

Let I denote the $n \times n$ identity matrix. For any $A \in \Theta_d$ obviously $A + I \in \Theta_{d+n}$.

Lemma 6.4. (cf. [G, n. 5.1 (ii)]). *For any $A \in \Theta_d$ we have $\iota(Z[A]) \subset Z[A + I]$.*

Proof. We have to prove that for $F, F' \in \mathcal{F}_d$

$$(F, F') \in \mathcal{O}_A \Leftrightarrow (F \oplus F^\sharp, F' \oplus F^\sharp) \in \mathcal{O}_{A+I}.$$

For $i, j \in [1, n]$ let $V = (F_i \oplus F_i^\sharp) \cap (F'_j \oplus F_j^\sharp)$. It suffices to show that

$$\dim V = \dim(F_i \cap F'_j) + \min(i, j)$$

for all $i, j = 1, \dots, n$. Let $k = \min(i, j)$. Then $V = (F_i \oplus F_k^\sharp) \cap (F'_j \oplus F_k^\sharp)$. Clearly, $F_k^\sharp \subseteq V \subseteq \mathbf{C}^d \oplus F_k^\sharp$. Then, as in lemma 6.2,

$$V = (V \cap \mathbf{C}^d) \oplus F_k^\sharp = (F_i \cap F'_j) \oplus F_k^\sharp.$$

Hence $\dim V = \dim(F_i \cap F'_j) + k$. This proves the lemma.

6.5. The pullback $\iota^* : U_{d+n} \rightarrow \mathcal{A}_d$ is defined by

$$\iota^*(\psi)(F, F', x) = \psi(\iota(F, F', x)) = \psi(F \oplus F^\sharp, F' \oplus F^\sharp, x \oplus x^\sharp).$$

Proposition 6.6. (1) ι^* is an algebra homomorphism;

(2) $\iota^*(U_{d+n}) \subseteq U_d$, and the following diagram is commutative

$$\begin{array}{ccc} & & U(\mathfrak{sl}_n) \\ & \nearrow \gamma_{d+n} & \downarrow \gamma_d \\ U_{d+n} & \xrightarrow{\iota^*} & U_d \end{array}$$

Here γ_d is the homomorphism given by 2.5.

Proof. Let $\psi_1, \psi_2 \in U_{d+n}$. We want to show that $\iota^*(\psi_1 * \psi_2) = \iota^*(\psi_1) * \iota^*(\psi_2)$.

For any $(F, F', x) \in Z_d$ we have

$$\begin{aligned} \iota^*(\psi_1 * \psi_2)(F, F', x) &= \psi_1 * \psi_2(F \oplus F^\sharp, F' \oplus F^\sharp, x \oplus x^\sharp) \\ &= \sum_{a \in \mathbf{Q}} a \cdot \chi\{\tilde{F} \mid (x \oplus x^\sharp) \triangleright \tilde{F}, \psi_1(F \oplus F^\sharp, \tilde{F}, x \oplus x^\sharp) \cdot \psi_2(\tilde{F}, F' \oplus F^\sharp, x \oplus x^\sharp) = a\} \end{aligned}$$

For any \tilde{F} which contributes to the above expression, by lemma 6.2 we have $\tilde{F} = \underline{\tilde{F}} \oplus F^\sharp$,

and $x \triangleright \underline{\tilde{F}}$. Therefore we can write

$$\begin{aligned}\psi_1(F \oplus F^\sharp, \tilde{F}, x \oplus x^\sharp) &= \iota^*(\psi_1)(F, \underline{\tilde{F}}, x) \\ \psi_2(\tilde{F}, F' \oplus F^\sharp, x \oplus x^\sharp) &= \iota^*(\psi_2)(\underline{\tilde{F}}, F', x).\end{aligned}$$

We can continue

$$\begin{aligned}\iota^*(\psi_1 * \psi_2)(F, F', x) &= \sum_{a \in \mathbf{Q}} a \cdot \chi\{\underline{\tilde{F}} \in \mathcal{F}_d \mid x \triangleright \underline{\tilde{F}}, \iota^*(\psi_1)(F, \underline{\tilde{F}}, x) \cdot \iota^*(\psi_2)(\underline{\tilde{F}}, F', x) = a\} \\ &= \iota^*(\psi_1) * \iota^*(\psi_2)(F, F', x).\end{aligned}$$

This proves (1).

6.7. Let us temporarily indicate the generators $e_i, f_i \in U_d$ by writing $e_i^{(d)}, f_i^{(d)}$. Since we already know that ι^* is an algebra homomorphism, to prove (2) it suffices to show that $\iota^*(e_i^{(d+n)}) = e_i^{(d)}$, $\iota^*(f_i^{(d+n)}) = f_i^{(d)}$ for all $i = 1, \dots, n-1$.

By lemma 6.4, $\iota(Z[A]) \subset Z[A + I]$ for any $A \in \Theta_d$. Obviously,

$$A - E_{i,i+1} \in \Delta_{d-1} \Leftrightarrow A + I - E_{i,i+1} \in \Delta_{d+n-1}.$$

By definition of e_i (see 2.2) this implies that $\iota^*(e_i^{(d+n)}) = e_i^{(d)}$.

Next, we notice that the involution τ_0 defined in 1.7 commutes with ι . Therefore, $\tau = \tau_0^*$ (see 2.3) commutes with ι^* . Hence,

$$\iota^*(f_i^{(d+n)}) = \iota^* \tau(e_i^{(d+n)}) = \tau \iota^*(e_i^{(d+n)}) = \tau(e_i^{(d)}) = f_i^{(d)}.$$

The proposition is proved.

Corollary 6.8. *The homomorphism ι^* is surjective.*

This follows from 6.6, (2) and the fact that γ_d is surjective.

6.9. Next we show that the bases of U_d and U_{d+n} constructed in section 3 are compatible with ι^* . Let $\Theta'_{d+n} \subset \Theta_{d+n}$ be the set of matrices with strictly positive diagonal entries. There is a bijection $\Theta'_{d+n} \xrightarrow{\sim} \Theta_d$ given by $B \mapsto B - I$, where I is the identity matrix.

Theorem 6.10. *The image of a basis element of U_{d+n} under ι^* is either 0, or a basis element of U_d . Namely,*

$$(1) \quad \iota^*(\varphi_B) = \varphi_{(B-I)} \text{ for all } B \in \Theta'_{d+n};$$

(2) $\iota^*(\varphi_B) = 0$ for all $B \in \Theta_{d+n} - \Theta'_{d+n}$.

Let \mathcal{V} be any open dense subset of an irreducible component $\overline{Z[A]}$ of Z_d . The key fact in the proof of the theorem is that the set

$$GL_{d+n} \cdot \iota(\mathcal{V}) = \{g \cdot (F \oplus F^\sharp, F' \oplus F'^\sharp, x \oplus x^\sharp) \mid (F, F', x) \in \mathcal{V}, g \in GL_{n+d}\}$$

is open dense in $Z[A + I]$.

Let $\mathcal{U} = \{(G, G', y) \in Z_{d+n} \mid y^{n-1} \neq 0\}$. This is a GL_{d+n} -stable open subset of Z_{d+n} .

The condition $y^{n-1} \neq 0$ means that the canonical Jordan form of y has at least one $n \times n$ block. Note that the image $\iota(Z_d)$ is contained in \mathcal{U} .

Let $\mathcal{U}_B = \mathcal{U} \cap Z[B]$.

Lemma 6.11. (i) For any $B \in \Theta_{d+n}$ the set \mathcal{U}_B is not empty if and only if $B \in \Theta'_{d+n}$;
(ii) If $B \in \Theta'_{d+n}$, then $B - I \in \Theta_d$, and $GL_{d+n} \cdot \iota(Z[B - I]) = \mathcal{U}_B$.

Proof. Let $B \in \Theta'_{d+n}$. Then by lemma 6.4, $\iota(Z[B - I]) \subset Z[B] \cap \mathcal{U} = \mathcal{U}_B$. It follows that \mathcal{U}_B is not empty. Since \mathcal{U}_B is GL_{d+n} -stable, this also gives the inclusion $GL_{d+n} \cdot \iota(Z[B - I]) \subseteq \mathcal{U}_B$.

Now let $(G, G', y) \in \mathcal{U}_B \neq \emptyset$ for some $B \in \Theta_{d+n}$. Since the canonical Jordan form of y has at least one $n \times n$ block, there exists an element $g \in GL_{d+n}$ such that $gyg^{-1} = x \oplus x^\sharp$ for some $x \in N_d$. Then $(x \oplus x^\sharp) \triangleright gG, gG'$. By lemma 6.2, the flags gG, gG' can be decomposed into direct sums

$$gG = F \oplus F^\sharp, gG' = F' \oplus F'^\sharp, \text{ where } F = \underline{gG}, F' = \underline{gG'}.$$

This shows that $(gG, gG', x \oplus x^\sharp) = \iota(F, F', x)$. Then lemma 6.4 implies that $B = \Phi(F, F') + I$. Hence $B \in \Theta'_{d+n}$, and $(F, F') \in \mathcal{O}_{B-I}$.

Thus $(G, G', y) = g^{-1} \cdot \iota(F, F', x)$, $(F, F', x) \in Z[B - I]$. Since (G, G', y) was an arbitrary point of \mathcal{U}_B , this shows that $\mathcal{U}_B \subseteq GL_{d+n} \cdot \iota(Z[B - I])$. The lemma is proved.

Corollary 6.12. If $B \in \Theta_{d+n}$, then $\iota^*(\varphi_B)$ vanishes on an open dense subset of $Z[A]$ for any $A \in \Theta_d$ such that $A \neq B - I$.

Proof. Fix $A \in \Theta_d$ such that $B' = A + I \neq B$. Then $B' \in \Theta'_{d+n}$. By 3.1, (II) the function φ_B vanishes on some open dense subset \mathcal{V} of $Z[B']$. By lemma 2.11, φ_B is constant on GL_{d+n} -orbits, so we can assume that \mathcal{V} is GL_{d+n} -invariant.

By lemma 6.11, $\mathcal{U}_{B'} \neq \emptyset$. Thus, both \mathcal{V} and $\mathcal{U}_{B'}$ are non-empty open subsets of $Z[B']$,

which is irreducible. Hence, their intersection $\mathcal{V} \cap \mathcal{U}_{B'}$ is not empty. It is also open and GL_{d+n} -invariant. Lemma 6.11, (ii) implies that $\mathcal{V} \cap \mathcal{U}_{B'} \cap \iota(Z[A])$ is a non-empty open subset of $\iota(Z[A])$, on which φ_B vanishes. Therefore, $\iota^*(\varphi_B)$ vanishes on an open dense subset of $Z[A]$.

6.13. To prove theorem 6.10, (1) we show that $\iota^*(\varphi_B)$ satisfies the conditions (I)-(II) of theorem 3.1 for the matrix $B - I$, and then use the uniqueness part of theorem 3.1.

By 3.1, (I) we have $\varphi_B|_{Z[B]} \equiv 1$. But $\iota(Z[B - I]) \subset Z[B]$, therefore for any $(F, F', x) \in Z[B - I]$

$$\iota^*(\varphi_B)(F, F', x) = \varphi_B(\iota(F, F', x)) = 1.$$

This shows that $\iota^*(\varphi_B)$ satisfies the condition 3.1, (I).

Next, let $A \in \Theta_d$ be such that $A \neq B - I$. By corollary 6.12, $\iota^*(\varphi_B)$ vanishes on an open subset of $Z[A]$. This proves 3.1, (II).

Thus, $\iota^*(\varphi_B)$ satisfies the conditions (I) and (II) of theorem 3.1 for the matrix $B - I$. By uniqueness, $\iota^*(\varphi_B) = \varphi_{(B-I)}$.

6.14. Let $B \in \Theta_{d+n} - \Theta'_{d+n}$. To prove 6.10, (2) we show that the function $\iota^*(\varphi_B)$ vanishes on an open dense subset of Z_d , and therefore has to be 0.

For any $A \in \Theta_d$ we have $A + I \in \Theta'_{d+n}$, and therefore $A + I \neq B$. By corollary 6.12, $\iota^*(\varphi_B)$ vanishes on an open dense subset of $Z[A]$. Therefore $\iota^*(\varphi_B)$ vanishes on an open dense subset of $Z[A]$ for any $A \in \Theta_d$. By corollary 4.13, $\iota^*(\varphi_B) = 0$.

This completes the proof of theorem 6.10.

6.15. The constructions of finite dimensional representations of U_d and U_{d+n} also agree with ι^* . It is easy to see that $\Lambda(d) \subset \Lambda(d+n)$ (see 4.5), and for any $x \in N_d$ we have

$$\iota : M^x \xrightarrow{\cong} M^{x+x^\sharp}.$$

The modules L_{x+x^\sharp} and L_x are isomorphic via ι^* . Moreover, $\iota^*(s_{x+x^\sharp}) = s_x$, and ι^* takes the basis given by theorem 5.4 into the basis. Namely, for any basis vector $\varphi_A * s_{x+x^\sharp}$, $A \in \Theta_{d+n}(x + x^\sharp)$, we have $A - I \in \Theta_d(x)$, and

$$\iota^*(\varphi_A * s_{x+x^\sharp}) = \varphi_{(A-I)} * s_x.$$

7 The algebras \bar{U} and \hat{U} .

7.1. The homomorphisms ι^* give rise to a projective system of algebras

$$\bigoplus_{d=0}^{n-1} U_d \leftarrow \bigoplus_{d=0}^{n-1} U_{d+n} \leftarrow \bigoplus_{d=0}^{n-1} U_{d+2n} \leftarrow \dots$$

Following [G], we consider the inverse limit \bar{U} of this system.

7.2. Let $\Theta = \bigcup_{d=0}^{\infty} \Theta_d$, $\Delta = \bigcup_{d=0}^{\infty} \Delta_d$.

Let $\tilde{\Theta} \subset \Theta$ be the set of matrices with at least one zero diagonal entry. Let $\tilde{\Delta} = \Delta \cap \tilde{\Theta}$.

For a matrix $A \in \Theta$ let \tilde{A} denote the matrix

$$\tilde{A} = A - I \cdot \min \{a_{ii}, 1 \leq i \leq n\}.$$

Then $\tilde{A} \in \tilde{\Theta}$, and $A = \tilde{A}$ for any $A \in \tilde{\Theta}$.

Let $k(A)$ be the integral part of $(\sum_{i,j} \tilde{a}_{ij})/n$.

7.3. For $A \in \Theta$ let us consider the sequence $\hat{\varphi}_A = \{(\hat{\varphi}_A)_0, (\hat{\varphi}_A)_1, (\hat{\varphi}_A)_2, \dots\}$, such that

$$\begin{aligned} (\hat{\varphi}_A)_j &\in \bigoplus_{d=0}^{n-1} U_{d+jn}, \\ (\hat{\varphi}_A)_j &= 0 \text{ for } 0 \leq j \leq k(A) - 1, \text{ and} \\ (\hat{\varphi}_A)_{k(A)+j} &= \varphi_{(\tilde{A}+jI)} \text{ for } j = 0, 1, 2, \dots \end{aligned}$$

Thanks to theorem 6.10, $\hat{\varphi}_A \in \bar{U}$.

Note that $\hat{\varphi}_A = \hat{\varphi}_{\tilde{A}}$. For any $A \in \tilde{\Theta}$ we have

$$\hat{\varphi}_A = (\underbrace{0, \dots, 0}_{k(A)-1}, \varphi_A, \varphi_{(A+I)}, \varphi_{(A+2I)}, \dots).$$

Every element of \bar{U} can be uniquely written as a formal linear combination $\sum_{A \in \tilde{\Theta}} c_A \cdot \hat{\varphi}_A$, $c_A \in \mathbf{Q}$.

Since all the projections ι^* are algebra homomorphisms, there is a well defined algebra structure on \bar{U} . With $U_d * U_{d'}$ understood to be 0 if $d \neq d'$, we have

$$(\hat{\varphi}_A * \hat{\varphi}_B)_i = (\hat{\varphi}_A)_i * (\hat{\varphi}_B)_i,$$

for all $i \geq 0$.

The unit element $\bar{1} \in \bar{U}$ is the infinite sum $\bar{1} = \sum_{A \in \tilde{\Delta}} \hat{\varphi}_A$.

By proposition 6.2, the homomorphisms

$$\gamma_{d,k} = \bigoplus_{d=0}^{n-1} \gamma_{d+kn} : U(\mathfrak{sl}_n) \rightarrow \bigoplus_{d=0}^{n-1} U_{d+kn}$$

are compatible with the projections. Therefore, there is a well defined homomorphism $\gamma : U(\mathfrak{sl}_n) \rightarrow \bar{U}$.

Lemma 7.4. *The homomorphism γ is injective.*

Proof. The algebra $U(\mathfrak{sl}_n)$ is spanned by monomials of the form

$$\underline{e}_{s_1}^{t_1} \underline{f}_{p_1}^{q_1} \dots \underline{e}_{s_l}^{t_l} \underline{f}_{p_l}^{q_l},$$

where all $s_i, p_i \in [1, n-1]$, $t_i, q_i \geq 0$.

Let $t(j) = \sum_{i: s_i=j} t_i$, $q(j) = \sum_{i: p_i=j} q_i$ for each $j \in [1, n-1]$.

Let $d = \sum_{j=1}^{n-1} (t(j) + q(j))$. We define flags $F, F' \in \mathcal{F}_d$ so that $F_n = F'_n = \mathbb{C}^d$, and all F_j, F'_j are the canonical subspaces of the form $\mathbb{C}^k \subseteq \mathbb{C}^d$, of dimensions

$$\dim F_j = t(j) + \sum_{i \leq j-1} (t(i) + q(i)), \quad \dim F'_j = q(j) + \sum_{i \leq j-1} (t(i) + q(i)),$$

for all $j = 1, \dots, n-1$.

Then $\gamma_d(\underline{e}_{s_1}^{t_1} \underline{f}_{p_1}^{q_1} \dots \underline{e}_{s_l}^{t_l} \underline{f}_{p_l}^{q_l})(F, F', 0) = (e_{s_1}^{t_1} * f_{p_1}^{q_1} \dots * e_{s_l}^{t_l} * f_{p_l}^{q_l})(F, F', 0) \neq 0$.

This shows that the image of every monomial under γ is not 0. Since γ is linear, the lemma follows.

The image $\gamma(U(\mathfrak{sl}_n))$ is generated by the infinite sums

$$\bar{e}_i = \gamma(\underline{e}_i) = \sum_{A \in \bar{\Delta}} \hat{\varphi}_{(A+E_{i,i+1})}, \quad \bar{f}_i = \gamma(\underline{f}_i) = \sum_{A \in \bar{\Delta}} \hat{\varphi}_{(A+E_{i+1,i})}.$$

Let \hat{U} be the subspace of \bar{U} spanned by $\{\hat{\varphi}_A\}_{A \in \bar{\Theta}}$.

Lemma 7.5. *\hat{U} is a subalgebra of \bar{U} .*

Proof. It suffices to show that for any $A, A' \in \bar{\Theta}$ the product $\hat{\varphi}_A * \hat{\varphi}_{A'}$ is a linear combination of a finite number of $\hat{\varphi}_B$.

This statement is an analogue of the stabilization phenomenon discovered in [BLM]. From proposition 4.2, [BLM], proved by an explicit computation, it follows that there

exists $p_o \geq 0$, and $B_1, \dots, B_m \in \Theta_{p_o}$ such that

$$\text{supp} ((\hat{\varphi}_A)_p * (\hat{\varphi}_{A'})_p) \subseteq \bigcup_{j=1}^m Z[B_j + (p - p_o)I]$$

for any $p \geq p_o$. This implies that

$$\hat{\varphi}_A * \hat{\varphi}_{A'} = \sum_{B \in \Theta_{p_o}} c_B \cdot \hat{\varphi}_B.$$

This is a finite sum. The lemma is proved.

7.6. The involution τ defined in 2.3, commutes with all the projections ι^* . Hence, it gives rise to an involution $\bar{\tau}$ of \bar{U} . Corollary 4.15 implies that the set of all $\hat{\varphi}_A$ is invariant under $\bar{\tau}$. It follows that $\bar{\tau}$ leaves both $\gamma(U(\mathfrak{sl}_n))$ and \hat{U} stable.

Note that $\bar{\tau}(\bar{e}_i) = \bar{f}_i$.

7.7. The algebras $\bar{U}, \hat{U}, \gamma(U(\mathfrak{sl}_n))$ correspond respectively to the algebras $\hat{\mathbf{K}}, \mathbf{K}, \mathbf{U}$ constructed in [BLM].

8 The algebra \hat{U} .

8.1. Following Lusztig [L3, Part IV], we define the algebra \hat{U} for $U = U(\mathfrak{sl}_n)$.

Let $X = \{\lambda = (\lambda_1, \dots, \lambda_{n-1}) \mid \lambda_i \in \mathbf{Z}\}$ be the weight lattice, so that $\lambda(\underline{h}_i) = \lambda_i$ for all $i = 1, \dots, n-1$.

Let $\mathbf{I} = \{\mathbf{i}\}_{i=1}^{n-1}$. Let $\mathcal{L} \cong \mathbf{Z}[\mathbf{I}]$ be the root lattice, imbedded in X so that $\mathbf{i}(\underline{h}_j) = 2\delta_{i,j} - \delta_{i,j-1} - \delta_{i,j+1}$. For $\nu \in \mathcal{L}$ we have $\nu(\underline{h}_i) = 2\nu_i - \nu_{i-1} - \nu_{i+1}$ for all $i = 1, \dots, n-1$, with $\nu_0 = \nu_n = 0$ understood.

For $\lambda', \lambda'' \in X$ let

$${}_{\lambda'} U_{\lambda''} = \frac{U}{\sum_{i=1}^{n-1} (\underline{h}_i - \lambda'_i) U + \sum_{i=1}^{n-1} U (\underline{h}_i - \lambda''_i)}.$$

We have ${}_{\lambda'} U_{\lambda''} = 0$ unless $\lambda' - \lambda'' \in \mathcal{L}$.

Let

$$\hat{U} = \bigoplus_{\lambda', \lambda'' \in X} {}_{\lambda'} U_{\lambda''}.$$

Let $\pi_{\lambda', \lambda''} : U \rightarrow {}_{\lambda'} U_{\lambda''}$ be the canonical projection.

There is a natural associative algebra structure on \hat{U} defined as follows.
Let $U(\nu)$, $\nu \in \mathcal{L}$ be the subspace of U generated by monomials

$$e_{p_1} \cdots e_{p_k} h_{t_1} \cdots h_{t_l} f_{s_1} \cdots f_{s_m},$$

such that

$$\#\{q \mid p_q = i\} - \#\{r \mid s_r = i\} = \nu_i$$

for all $i = 1, \dots, n-1$. We have $U = \bigoplus_{\nu \in \mathcal{L}} U(\nu)$.

The product on \hat{U} is uniquely defined by the following conditions: for any $\lambda'_1, \lambda''_1, \lambda'_2, \lambda''_2 \in X$, and any $a \in U(\lambda'_1 - \lambda''_1), b \in U(\lambda'_2 - \lambda''_2)$ the product $\pi_{\lambda'_1, \lambda''_1}(a)\pi_{\lambda'_2, \lambda''_2}(b)$ is equal to $\pi_{\lambda'_1, \lambda''_2}(ab)$ if $\lambda'_1 = \lambda''_2$, and is zero otherwise.

Let $1_\lambda = \pi_{\lambda, \lambda}(1)$. Then we have

$$1_\lambda 1_{\lambda'} = \delta_{\lambda, \lambda'} 1_\lambda, \quad \lambda' U \lambda'' = 1_{\lambda'} \hat{U} 1_{\lambda''}$$

8.2. We now show that the algebra \hat{U} defined in section 6 is isomorphic to the algebra \hat{U} .

For $\lambda', \lambda'' \in X$, let

$$\Theta(\lambda', \lambda'') = \{A \in \Theta \mid ro(A)_i - ro(A)_{i+1} = \lambda'_i, \quad co(A)_i - co(A)_{i+1} = \lambda''_i; \quad i = 1, \dots, n-1\}.$$

Then $\Theta = \bigcup_{\lambda', \lambda'' \in X} \Theta(\lambda', \lambda'')$. The union is disjoint, and $\Theta(\lambda', \lambda'')$ is empty unless $\lambda' - \lambda'' \in \mathcal{L}$.

Note that for any integer k , and any $i = 1, \dots, n-1$ we have

$$\begin{aligned} ro(A)_i - ro(A)_{i+1} &= ro(A + kI)_i - ro(A + kI)_{i+1}, \\ co(A)_i - co(A)_{i+1} &= co(A + kI)_i - co(A + kI)_{i+1}. \end{aligned}$$

It follows that all $A + kI$ lie in the same $\Theta(\lambda', \lambda'')$.

For any $\lambda \in X$, and any diagonal $A \in \Delta \cap \Theta(\lambda, \lambda)$, we have $a_{ii} - a_{i+1, i+1} = \lambda_i$ for all $i = 1, \dots, n-1$. Hence all $a_{ii} - a_{nn}$ are uniquely defined by λ . Since all the entries of A are non-negative, it follows that there is a unique matrix $A(\lambda) = \tilde{\Delta} \cap \Theta(\lambda, \lambda)$. We define $\hat{1}_\lambda = \hat{\varphi}_{A(\lambda)}$. Then

$$\hat{1}_\lambda \hat{1}_{\lambda'} = \delta_{\lambda, \lambda'} \hat{1}_\lambda, \quad \bar{1} = \sum_{\lambda \in X} \hat{1}_\lambda.$$

Note that $\hat{1}_\lambda \in \hat{U}$ for all $\lambda \in X$.

Lemma 8.3. (cf. [L3, n. 23.1.3]). *For any $\lambda \in X$, and any $i = 1, \dots, n-1$, we have*

$$\begin{aligned}\hat{1}_\lambda * \bar{e}_i &= \bar{e}_i * \hat{1}_{\lambda-i}, \\ \hat{1}_\lambda * \bar{f}_i &= \bar{f}_i * \hat{1}_{\lambda+i}, \\ \hat{1}_\lambda * \bar{h}_i &= \bar{h}_i * \hat{1}_\lambda = \lambda_i \hat{1}_\lambda.\end{aligned}$$

Proof. Fix an integer $q > 1$. Let $A = A(\lambda) + qI$. By definition, $a_{jj} - a_{j+1,j+1} = \lambda_j$ for all $j = 1, \dots, n-1$.

Let $d = qn + \sum_{i,j} a_{ij}$. By the definitions of e_i, f_i in 2.2, and corollary 3.7, we have the following equalities in U_d

$$\begin{aligned}\varphi_A * e_i &= \sum_{B \in \Delta_{d-1}} \varphi_A * \varphi_{(B+E_{i,i+1})} \\ &= \varphi_A * \varphi_{(A-E_{ii}+E_{i,i+1})} \\ &= \varphi_{(A-E_{ii}+E_{i,i+1})} * \varphi_{(A-E_{ii}+E_{i+1,i+1})} \\ &= e_i * \varphi_{(A-E_{i+1,i+1}+E_{ii})}.\end{aligned}$$

Let $C = A - E_{i+1,i+1} + E_{ii}$. Then

$$c_{jj} - c_{j+1,j+1} = \lambda_j - 2\delta_{j,i} + \delta_{j,i-1} + \delta_{j,i+1}$$

for all $j = 1, \dots, n-1$. Hence $C \in \Theta(\lambda - i, \lambda - i)$. This shows that $\hat{1}_\lambda * \bar{e}_i = \bar{e}_i * \hat{1}_{\lambda-i}$.

Using $\bar{\tau}$, we have the equalities for \bar{f}_i .

By corollary 3.7 and lemma 2.6, we have the following equality in U_d

$$h_i * \varphi_A|_{Z[B]} = \delta_{B,A}(b_{ii} - b_{i+1,i+1}) = \lambda_i \cdot \varphi_A.$$

This implies that $\hat{1}_\lambda * \bar{h}_i = \lambda_i \hat{1}_\lambda$. The lemma is proved.

Lemma 8.4. *For any $\lambda \in X$, and any $i = 1, \dots, n-1$,*

$$\hat{1}_\lambda * \bar{e}_i, \quad \hat{1}_\lambda * \bar{f}_i \in \hat{U}.$$

Proof. Indeed,

$$\hat{1}_\lambda * \bar{e}_i = \hat{1}_\lambda * \left(\sum_{B \in \hat{\Delta}} \hat{\varphi}_{(B+E_{i,i+1})} \right).$$

Let $A = A(\lambda)$, so that $\hat{1} = \hat{\varphi}_A$. For any $q > 0$, there is a unique matrix $B \in \hat{\Delta}$ and an integer $p > 0$ such that $co(A + qI) = ro(B + E_{i,i+1} + pI)$. Necessarily, $p = q$ if $a_{ii} > 0$, $p = q - 1$ if $a_{ii} = 0$, and $B = A + (q - p)I - E_{ii}$.

This shows that

$$\hat{1}_\lambda * \bar{e}_i = \hat{\varphi}_A * \hat{\varphi}_{(B+E_{i,i+1})}.$$

By lemma 7.5, this product lies in \hat{U} .

Since the involution $\bar{\tau}$ leaves \hat{U} stable, we have

$$\hat{1}_\lambda * \bar{f}_i = \bar{\tau}(\bar{e}_i * \hat{1}_\lambda) = \hat{1}_{\lambda+i} * \bar{e}_i \in \hat{U}.$$

The lemma is proved.

8.5. Let ${}_{\lambda'}\bar{U}_{\lambda''}$ be the subspace of \bar{U} , consisting of all (possibly, infinite) sums $\sum c_A \cdot \hat{\varphi}_A$, such that $c_A = 0$ unless $A \in \Theta(\lambda', \lambda'')$.

By corollary 3.7 and lemma 2.6, we have

$${}_{\lambda'}\bar{U}_{\lambda''} = \hat{1}_{\lambda'} * \bar{U} * \hat{1}_{\lambda''}.$$

Let $\bar{\pi}_{\lambda', \lambda''} : \bar{U} \rightarrow {}_{\lambda'}\bar{U}_{\lambda''}$ be the projection defined by

$$\bar{\pi}_{\lambda', \lambda''} \left(\sum_{A \in \bar{\Theta}} c_A \cdot \hat{\varphi}_A \right) = \sum_{A \in \bar{\Theta} \cap \Theta(\lambda', \lambda'')} c_A \cdot \hat{\varphi}_A.$$

We have $\bar{\pi}_{\lambda, \lambda}(\bar{1}) = \hat{1}_\lambda$.

Let ${}_{\lambda'}\hat{U}_{\lambda''} = {}_{\lambda'}\bar{U}_{\lambda''} \cap \hat{U}$. This space is spanned by $\{\hat{\varphi}_A \mid A \in \Theta(\lambda', \lambda'')\}$. We have

$$\hat{U} = \bar{1} * \hat{U} * \bar{1} = \bigoplus_{\lambda', \lambda'' \in X} \hat{1}_{\lambda'} * \hat{U} * \hat{1}_{\lambda''} = \bigoplus_{\lambda', \lambda'' \in X} {}_{\lambda'}\hat{U}_{\lambda''}.$$

Proposition 8.6. (1) $\bar{\pi}_{\lambda', \lambda''}(\gamma(U)) = {}_{\lambda'}\hat{U}_{\lambda''}$ for any $\lambda', \lambda'' \in X$;

(2) There is a unique algebra isomorphism $\dot{\gamma} : \dot{U} \rightarrow \hat{U}$ such the following diagram is commutative:

$$\begin{array}{ccc} U & \xrightarrow[\cong]{\gamma} & \gamma(U) \\ \downarrow \bigoplus_{\lambda', \lambda'' \in X} \bar{\pi}_{\lambda', \lambda''} & & \downarrow \bigoplus_{\lambda', \lambda'' \in X} \bar{\pi}_{\lambda', \lambda''} \\ \dot{U} & \xrightarrow[\cong]{\dot{\gamma}} & \hat{U} \end{array}$$

Proof. We want to show that $\bar{\pi}_{\lambda', \lambda''}(\psi) \in \hat{U}$ for any $\psi \in \gamma(U(\mathfrak{sl}_n))$, and any $\lambda', \lambda'' \in X$.

First, notice that the statement is true for $\bar{1} \in \gamma(U)$.

Next, we show that if the statement holds for $\psi \in \gamma(U)$, then it is also true for $\bar{e}_i * \psi$, and $\psi * \bar{e}_i$, for any $i = 1, \dots, n-1$.

We have

$$\bar{\pi}_{\lambda', \lambda''}(\bar{e}_i * \psi) = \hat{1}_{\lambda'} * (\bar{e}_i * \psi) * \hat{1}_{\lambda''} = (\hat{1}_{\lambda'} * \bar{e}_i) * (\hat{1}_{\lambda'-i} * \psi * \hat{1}_{\lambda''}).$$

The first factor lies in \hat{U} by lemma 8.4, the second - by the assumption on ψ . Since \hat{U} is closed under multiplication, we have $\bar{\pi}_{\lambda', \lambda''}(\bar{e}_i * \psi) \in \hat{U}$.

Similarly,

$$\bar{\pi}_{\lambda', \lambda''}(\psi * \bar{e}_i) = \hat{1}_{\lambda'} * (\psi * \bar{e}_i) * \hat{1}_{\lambda''} = (\hat{1}_{\lambda'} * \psi * \hat{1}_{\lambda''+i}) * (\hat{1}_{\lambda''+i} * \bar{e}_i) \in \hat{U}.$$

Using the involution $\bar{\tau}$, we see that also $\bar{f}_i * \psi, \psi * \bar{f}_i \in \hat{U}$. This proves 8.6, (1).

8.7. Using 8.3, for any $\lambda', \lambda'' \in X$, and any $i = 1, \dots, n-1$ we have

$$\begin{aligned} & \bar{\pi}_{\lambda', \lambda''} \cdot \gamma((\underline{h}_i - \lambda'_i) U + U(\underline{h}_i - \lambda''_i)) \\ &= \hat{1}_{\lambda'} * ((\bar{h}_i - \lambda'_i) * \gamma(U)) + \gamma(U) * (\bar{h}_i - \lambda''_i \hat{1}_{\lambda'}) * \hat{1}_{\lambda''} \\ &= (\hat{1}_{\lambda'} * \bar{h}_i - \lambda'_i) * \gamma(U) + \gamma(U) * (\hat{1}_{\lambda''} * \bar{h}_i - \lambda''_i \hat{1}_{\lambda''}) \\ &= 0. \end{aligned}$$

This shows that the image of $(\sum_{i=1}^{n-1} (\underline{h}_i - \lambda'_i) U + \sum_{i=1}^{n-1} U(\underline{h}_i - \lambda''_i))$ under γ is contained in $\text{Ker } \bar{\pi}_{\lambda', \lambda''}$. Therefore, a map $\dot{\gamma} : \hat{U} \rightarrow \hat{U}$ is well defined by the requirement

$$\dot{\gamma}(\pi_{\lambda', \lambda''}(s)) = \bar{\pi}_{\lambda', \lambda''}(\gamma(s))$$

for all $s \in U(sl_n)$, $\lambda', \lambda'' \in X$. This map is obviously surjective.

Let $\dot{\gamma}(u) = 0$ for some $u \in \hat{U}$. Let $t \in U$ be such that $\bigoplus_{\lambda', \lambda''} \pi_{\lambda', \lambda''}(t) = u$. Then

$$\gamma(t) = \left(\sum_{\lambda' \in X} \hat{1}_{\lambda'} \right) * \gamma(t) * \left(\sum_{\lambda'' \in X} \hat{1}_{\lambda''} \right) = \sum_{\lambda', \lambda'' \in X} \hat{1}_{\lambda'} * \gamma(t) * \hat{1}_{\lambda''} = \sum_{\lambda', \lambda'' \in X} \bar{\pi}_{\lambda', \lambda''}(\gamma(t)) = 0.$$

Since γ is injective (see Lemma 7.4), we have $t = 0$, and hence $u = 0$. Therefore, $\dot{\gamma}$ is injective.

Finally, for any $\lambda'_1, \lambda''_1, \lambda'_2, \lambda''_2 \in X$, and any $s \in U(\lambda'_1 - \lambda''_1), t \in U(\lambda'_2 - \lambda''_2)$ we have

$$\begin{aligned} \dot{\gamma}(\pi_{\lambda'_1, \lambda''_1}(s) \pi_{\lambda'_2, \lambda''_2}(t)) &= \delta_{\lambda''_1, \lambda'_2} \dot{\gamma}(\pi_{\lambda'_1, \lambda''_2}(st)) = \delta_{\lambda''_1, \lambda'_2} \bar{\pi}_{\lambda', \lambda''}(\gamma(st)) \\ &= \hat{1}_{\lambda'_1} * \gamma(s) * \hat{1}_{\lambda''_1} * \hat{1}_{\lambda'_2} * \gamma(t) * \hat{1}_{\lambda''_2} = \dot{\gamma}(\pi_{\lambda'_1, \lambda''_1}(s)) * \dot{\gamma}(\pi_{\lambda'_2, \lambda''_2}(t)). \end{aligned}$$

This shows that $\dot{\gamma}$ is an algebra isomorphism.

Proposition 8.6 is proved.

Let us identify $U = U(sl_n)$ with its image $\gamma(U)$.

8.8. As shown in [L3, n. 23.1.3], \hat{U} has a natural U -bimodule structure. In our

situation, both \hat{U} and $\gamma(U)$ are subalgebras of the algebra \bar{U} , and this bimodule structure is realized as a multiplication in \bar{U} . It is easy to see that \hat{U} is stable under multiplication by U .

Indeed, for any $\lambda', \lambda'' \in X$, and any $i = 1, \dots, n-1$ we have

$$\bar{e}_i * {}_{\lambda'}\hat{U}_{\lambda''} = \bar{e}_i * \hat{1}_{\lambda'} * \hat{U} * \hat{1}_{\lambda''} = \hat{1}_{\lambda'-i} * \bar{e}_i * \hat{U} * \hat{1}_{\lambda''} \subset {}_{\lambda'-i}\bar{U}_{\lambda''}.$$

By lemma 8.4, $\hat{1}_{\lambda'-i} * \bar{e}_i \in \hat{U}$. This implies that

$$\bar{e}_i * {}_{\lambda'}\hat{U}_{\lambda''} \subseteq {}_{\lambda'-i}\hat{U}_{\lambda''}.$$

Similar for multiplication by \bar{f}_i . Then, using the involution $\bar{\tau}$, we see that \hat{U} is stable under multiplication by U from the right.

8.9. The structure arising in \hat{U} from the comultiplication on U (see [L3, n. 23.1.5]) can be interpreted geometrically as follows.

Let us fix $d = d' + d'' \geq 0$, and the decomposition $\mathbf{C}^d = \mathbf{C}^{d'} \oplus \mathbf{C}^{d''}$.

We define the map $c_{d', d''} : Z_{d'} \times Z_{d''} \rightarrow Z_d$ by

$$c_{d', d''}((F, F', x), (G, G', y)) = (F \oplus G, F' \oplus G', x \oplus y).$$

Then the map $c_{d', d''}^* : U_d \rightarrow U_{d'} \otimes U_{d''}$ is well defined, and is an algebra homomorphism.

To see this, we notice first that $c_{d', d''}^*(e_i) = e_i \otimes 1 + 1 \otimes e_i$, $c_{d', d''}^*(f_i) = f_i \otimes 1 + 1 \otimes f_i$. (We use the same letters for generators of $U_d, U_{d'}, U_{d''}$; there should not be any confusion.)

For any $((F, F', x), (G, G', y)) \in Z_{d'} \times Z_{d''}$ we have

$$c_{d', d''}^*(e_i)((F, F', x), (G, G', y)) = e_i(F \oplus G, F' \oplus G', x \oplus y).$$

This function equals 1 if and only if

- (i) $x \triangleright F, F'$; $y \triangleright G, G'$;
- (ii) $(F \oplus G)_j = (F' \oplus G')_j \Leftrightarrow F_j = G_j, F'_j = G'_j$ for all $j \neq i$;
- (iii) $(F \oplus G)_i \supset (F' \oplus G')_i$, $\dim (F \oplus G)_i / (F' \oplus G')_i = 1$.

Otherwise, the function vanishes.

Condition (iii) is equivalent to the following condition

$$F_i = F'_i, G_i \supset G'_i, \dim G_i / G'_i = 1 \quad \text{or} \quad F_i \supset F'_i, \dim F_i / F'_i = 1, G_i = G'_i.$$

This shows that $c^*(e_i) = e_i \otimes 1 + 1 \otimes e_i$. Since $c_{d', d''}^*$ obviously commutes with τ , we have similar equality for f_i .

Then, it is not difficult to show that for any $\psi \in U_d$

$$c_{d',d''}^*(e_i * \psi) = (e_i \otimes 1 + 1 \otimes e_i) * c_{d',d''}^*(\psi).$$

Similar for f_i and multiplication from the right.

Therefore, for any d we have a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\Delta} & U \otimes U \\ \gamma_d \downarrow & & \downarrow \bigoplus_{d'+d''=d} \gamma_{d'} \otimes \gamma_{d''} \\ U_d & \xrightarrow{\bigoplus_{d'+d''=d} c_{d',d''}^*} & \bigoplus_{d'+d''=d} U_{d'} \otimes U_{d''} \end{array}$$

The maps c^* agree with the projections induced by ι^* .

8.10. There are three natural involutions of U , defined by

$$\begin{aligned} \omega(e_i) &= e_i, \quad \omega(f_i) = f_i, \quad \omega(h_i) = -h_i; \\ \sigma(e_i) &= f_i, \quad \sigma(f_i) = e_i, \quad \sigma(h_i) = -h_i; \\ \delta(e_i) &= f_{n-i}, \quad \delta(f_i) = e_{n-i}, \quad \delta(h_i) = -h_{n-i}, \end{aligned}$$

for all $i = 1, \dots, n-1$, such that σ and δ are algebra automorphisms, and ω is an anti-automorphism (see [L3, n. 3.1.3]).

The anti-automorphism $\bar{\tau}$ (see 7.6) corresponds to the composition $\sigma\omega = \omega\sigma$. A geometric realization of ω or σ is not seen, which seems to be a defect of the construction.

The automorphism δ is realized as follows.

For $F \in \mathcal{F}_d$, let F^* denote the flag in the dual space $(\mathcal{C}^d)^*$, such that F_i^* is the space of all linear operators vanishing on F_{n-i} . We have $\dim F_i^* = \text{codim } F_{n-i}$ for all $i = 0, \dots, n$.

If $\Phi(F, G) = A$, then $\Phi(F^*, G^*) = B$, such that $b_{ij} = a_{n-i+1, n-j+1}$, i.e. the matrix B is the result of transposition of A along both diagonals.

For $x \in N_d$ let x^* be defined by $x^*(v)(F) = v(x(F))$. It is easy to check that $x \triangleright F$ if and only if $x^* \triangleright F^*$.

Let us fix some isomorphism between $(\mathcal{C}^d)^*$ and \mathcal{C}^d . Then the map $(F, G, x) \mapsto (F^*, G^*, x^*)$ induces an algebra automorphism $U_d \rightarrow U_d$. By lemma 2.11, it is independent of the choice made. It clearly commutes with ι^* , and therefore gives rise to an algebra automorphism $\bar{U} \rightarrow \bar{U}$. This automorphism preserves the canonical basis, and its restriction to U coincides with δ .

8.11. Each U_d can be naturally imbedded into \hat{U} by the \mathbf{Q} -linear map

$$j_d : U_d \hookrightarrow \hat{U}$$

such that $j_d(\varphi_A) = \hat{\varphi}_A$ for all $A \in \Theta_d$. The projection onto the image $\hat{U} \rightarrow j_d(U_d)$ composed with j_d^{-1} is a surjective algebra homomorphism $p_d : \hat{U} \rightarrow U_d$.

By theorem 6.10, these homomorphisms commute with ι^* , i.e. we have

$$\iota^* \cdot p_{d+n} = p_d.$$

Via the homomorphisms p_d , all the finite dimensional sl_n -modules constructed in section 5, become naturally \hat{U} -modules. Thank to theorem 5.4 and n. 6.15, the basis $\hat{B} = \{\hat{\varphi}_A\}_{A \in \Theta^0}$ of \hat{U} has the property similar to the property of the distinguished basis of U_d with respect to U_d -modules. Namely, for any irreducible module L_x with the highest weight vector s_x , the elements

$$\{\hat{\varphi}_A \cdot s_x \mid \hat{\varphi}_A \cdot s_x \neq 0, A \in \Theta^0\}$$

form a basis of L_x .

8.12. Let $X^+ = \{\lambda \in X \mid \lambda_i \geq 0, i = 1, \dots, n-1\}$ be the set of dominant weights. The partition $\hat{B} = \sqcup_{\lambda \in X^+} \hat{B}[\lambda]$ of the canonical basis of \hat{U} into two-sided cells is defined in [L3, n. 29.1]. It can be seen geometrically as follows.

As was explained in the proof of proposition 5.7, for any matrix $A \in \Theta_d$ there exists a unique conjugacy class $C(x)$ in N_d such that the set $\{(F, F', y) \in Z[A] \mid y \in C(x)\}$ is open dense in $Z[A]$. Let $\lambda^A \in X^+$ be such that λ_i^A equals the number of $i \times i$ blocks in the Jordan decomposition of x for all $i = 1, \dots, n-1$. Clearly, $\lambda^A \in X^+$.

Note that the class corresponding to $A + I$ is $C(x \oplus x^\sharp)$ (see 6.1). Since the above definition does not take into account $n \times n$ Jordan blocks, we have $\lambda^A = \lambda^{A+I}$. Therefore, it makes sense to assign λ^A to $\hat{\varphi}_A$. We define

$$\hat{B}[\lambda] = \{\hat{\varphi}_A \mid \lambda^A = \lambda\}.$$

This gives a partition $\hat{B} = \sqcup_{\lambda \in X^+} \hat{B}[\lambda]$.

8.13. The partition of \hat{B} into cells is compatible with the homomorphisms p_d defined in n. 8.11.

We recall that for an integer $d \geq 0$, the set $\Lambda_d \subset X^+$ consists of all λ such that $\sum_j = 1^{n-1} j \lambda_j = d - kn$ for some integer $k \geq 0$. This is the same set as in 4.5, but now we have chosen a different basis for X .

By theorem 5.4, $\{L_\lambda\}_{\lambda \in \Lambda_d}$ is the complete set of irreducible U_d -modules. We know that $\dim U_d = \sum_{\lambda \in \Lambda_d} (\dim L_\lambda)^2$. Therefore, using n. 8.11, we have

$$\text{Ker } p_d = \text{span } \sqcup_{\lambda \in \Lambda_d} \hat{B}[\lambda].$$

The algebra $U_d \cong \hat{U} / \text{Ker } p_d$ is isomorphic to the finite dimensional factor of \hat{U}

$$\hat{U} / \hat{U}[X^+ - \Lambda_d],$$

defined in [L3, n. 29.2].

9 Lagrangian construction of U^- .

9.1. Let U^- be the subalgebra of $U = U(\mathfrak{sl}_n)$ generated by $\{\underline{f}_i\}_{i=1}^{n-1}$.

We describe the construction of U^- which is a special case of G. Lusztig's construction of the $-$ parts of universal enveloping algebras of Kac-Moody algebras (see [L1], [L2]). The construction was given in terms of constructible functions on certain Lagrangian subvarieties of spaces of representations of quivers.

Let $V = \bigoplus_{i=1}^l V_i$ be a graded finite dimensional vector space over \mathbf{C} .

The space of representations of the graph A_l is the vector space

$$E_V = \bigoplus_{i=1}^{l-1} (\text{Hom}_{\mathbf{C}}(V_i, V_{i+1}) \oplus \text{Hom}_{\mathbf{C}}(V_{i+1}, V_i))$$

A representation $t \in E_V$ is written $t = \bigoplus (t_{i,i+1} \oplus t_{i+1,i})$.

The group $G_V = \prod_{i=1}^l GL(V_i)$ acts naturally on E_V by

$$(g, t) \mapsto \bigoplus_{i=1}^{l-1} (g_{i+1} t_{i,i+1} g_i^{-1} \oplus g_i t_{i+1,i} g_{i+1}^{-1}).$$

Two representations are called isomorphic if they lie in the same orbit.

9.2. Consider the G_V -stable variety

$$\Lambda_V = \{t \in E_V \mid t_{21} t_{12} = 0 = t_{l-1,n} t_{l,l-1}, t_{i-1,i} t_{i,i-1} = t_{i+1,i} t_{i,i+1}, i = 2, \dots, l-1\}.$$

A point $t \in \Lambda_V$ corresponds to the commutative diagram

$$0 \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{0} \end{array} V_1 \begin{array}{c} \xrightarrow{t_{12}} \\ \xleftarrow{t_{21}} \end{array} V_2 \begin{array}{c} \xrightarrow{t_{23}} \\ \xleftarrow{t_{32}} \end{array} \cdots \begin{array}{c} \xrightarrow{t_{l-2,l-1}} \\ \xleftarrow{t_{l-1,l-2}} \end{array} V_{l-1} \begin{array}{c} \xrightarrow{t_{l-1,l}} \\ \xleftarrow{t_{l,l-1}} \end{array} V_l \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{0} \end{array} 0.$$

9.3. For $\nu \in (\mathbf{Z}_{\geq 0})^l$, let \mathcal{V}_ν be the space of all $V = \bigoplus_{i=1}^l V_i$ such that $\nu(V)_i = \dim V_i$ for all $i = 1, \dots, l$.

Let $M(V)$ be the \mathbf{Q} -vector space of all constructible functions on Λ_V which are constant on G_V -orbits. Because of this condition, for any $\nu \in (\mathbf{Z}_{\geq 0})^l$ and all $V \in \mathcal{V}_\nu$ we can identify $M(V)$ with a single vector space $M(\nu)$.

For two graded vector spaces V, \tilde{V} we write $\tilde{V} \subseteq V$ if $\tilde{V}_i \subseteq V_i$ for all $i = 1, \dots, l$. If $t \in \Lambda_V$, then for $\tilde{V} \subseteq V$ we write $t \sim \tilde{V}$ if $t_{ij}(\tilde{V}_i) \subseteq \tilde{V}_j$ for all $|i - j| = 1$; we denote by $t|_{\tilde{V}}$ the element of $\Lambda_{\tilde{V}}$ obtained by restriction of t on \tilde{V} .

Let $V' \in \mathcal{V}_{\nu'}$ be such that $R_i : V'_i \xrightarrow{\cong} V_i/\tilde{V}_i$ is an isomorphism for all i . There is a unique $s \in \Lambda_{V'}$ such that $R_j s_{ij} = t_{ji} R_i$ for all $|i - j| = 1$. For a function $f \in M(V')$ we define

$$f(\bar{t}_{\tilde{V}}) = f(s).$$

In fact, this defines the value $f(\bar{t}_{\tilde{V}})$ for any $f \in M(\nu')$. It is independent of choices of V' and R .

Let $M = \bigoplus_{\nu} M(\nu)$. There is an associative algebra structure on M defined as follows.

Let $V \in \mathcal{V}_\nu$, and let $\nu = \nu' + \nu''$. Let $f' \in M(\nu'), f'' \in M(\nu'')$. Then for any $t \in \Lambda_V$ we define

$$f' * f''(t) = \sum_{a \in \mathbf{Q}} a \cdot \chi\{\tilde{V} \in \mathcal{V}_{\nu''} \mid \tilde{V} \subseteq V, t \sim \tilde{V}, f'(\bar{t}_{\tilde{V}}) \cdot f''(t|_{\tilde{V}}) = a\}.$$

9.4. For $i \in [1, l]$, let μ be such that $\mu_j = \delta_{i,j}$. Then \mathcal{V}_μ consists of one point, $\mathbf{0}$. Let $F_i \in M(\mu)$ be such that $F_i(\mathbf{0}) = 1$. Let M_0 be the subalgebra of M generated by $F_i, i = 1, \dots, l$.

Let $n = l + 1$.

Theorem 9.5. (Lusztig, [L1, Theorem 12.13]) *There is an algebra isomorphism $\sigma : U^- \xrightarrow{\cong} M_0$ such that $\sigma(\underline{f}_i) = F_i$ for all $i = 1, \dots, n - 1$.*

9.6. We now explain how the above construction is related to the construction of \bar{U} .

Let a matrix $A \in \Theta_d$ be lower triangular. It means that for any $(G, G') \in \mathcal{O}_A$ we have $G \subseteq G'$ (see 5.2). Let us fix $(G, G', x) \in Z[A]$.

Let $\nu = (\dim G'_i - \dim G_i)_{i=1}^{n-1}$. Note that ν depends only on A .

We choose $V \in \mathcal{V}_\nu$ and isomorphisms

$$r_i : G'_i/G_i \rightarrow V_i$$

for all $i = 1, \dots, n-1$. We set $V_0 = G'_0/G_0 = 0$, $V_n = G'_n/G_n = 0$.

By definition, $x \triangleright G, G'$, i.e. for all $i \in [1, n-1]$ we have

$$x(G_i) \subseteq G_{i-1}, \quad x(G'_i) \subseteq G'_{i-1}.$$

Therefore, x induces linear maps $\bar{x}_i : G'_i/G_i \rightarrow G'_{i-1}/G_{i-1}$. For each i we define

$$t_{i,i-1} = r_{i-1}\bar{x}_i r_i^{-1} : V_i \rightarrow V_{i-1}.$$

Similarly, $G_i \subseteq G_{i+1}$, $G'_i \subseteq G'_{i+1}$. Hence the identity map $\epsilon : \mathbb{C}^d \rightarrow \mathbb{C}^d$ induces linear maps $\bar{\epsilon}_i : G'_i/G_i \rightarrow G'_{i+1}/G_{i+1}$. Therefore, for each i we can define a linear map

$$t_{i,i+1} = r_{i+1}\bar{\epsilon}_i r_i^{-1} : V_i \rightarrow V_{i+1}.$$

This gives us $t \in E_V$. Furthermore, for any $i = 1, \dots, n-1$ and any $v \in G'_i$ we have

$$\bar{\epsilon}_{i-1}\bar{x}_i(v + G_i) = \bar{\epsilon}_{i-1}(x(v) + x(G_i) + G_{i-1}) = x(v) + G_i,$$

and

$$\bar{x}_{i+1}\bar{\epsilon}_i(v + G_i) = \bar{x}_{i+1}(v + G_{i+1}) = x(v) + x(G_{i+1}) + G_i = x(v) + G_i.$$

Therefore, the following diagram is commutative.

$$\begin{array}{ccccccc} 0 & \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{0} \end{array} & G'_1/G_1 & \begin{array}{c} \xrightarrow{\bar{\epsilon}_1} \\ \xleftarrow{\bar{x}_2} \end{array} & G'_2/G_2 & \begin{array}{c} \xrightarrow{\bar{\epsilon}_2} \\ \xleftarrow{\bar{x}_3} \end{array} & \cdots & \begin{array}{c} \xrightarrow{\bar{\epsilon}_{n-2}} \\ \xleftarrow{\bar{x}_{n-1}} \end{array} & G'_{n-1}/G_{n-1} & \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{0} \end{array} & 0 \\ & & r_1 \downarrow & & r_2 \downarrow & & & & r_{n-1} \downarrow & & \\ 0 & \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{0} \end{array} & V_1 & \begin{array}{c} \xrightarrow{t_{12}} \\ \xleftarrow{t_{21}} \end{array} & V_2 & \begin{array}{c} \xrightarrow{t_{23}} \\ \xleftarrow{t_{32}} \end{array} & \cdots & \begin{array}{c} \xrightarrow{t_{n-2,n-1}} \\ \xleftarrow{t_{n-1,n-2}} \end{array} & V_{n-1} & \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{0} \end{array} & 0 \end{array}$$

This shows that $t = t(G, G', x) \in \Lambda_V$.

For any $f \in M$ we define a function $\varphi = R_d(f) \in \mathcal{A}_d$ by

$$\varphi(G, G', x) = \begin{cases} f(t(G, G', x)), & \text{if } G \subseteq G', \\ 0 & \text{otherwise.} \end{cases}$$

This map is well defined, and is independent of the choice of the assignment $(G, G', x) \mapsto t(G, G', x)$.

Let $U_d^- = \gamma_d(U^-)$.

Proposition 9.7. (1) R_d is an algebra homomorphism;

(2) $R_d(F_i) = f_i$ for all $i = 1, \dots, n-1$, so that $R_d(M_0) = U_d^-$, and we have the following commutative diagram.

$$\begin{array}{ccc}
 & U^- & \\
 & \sigma \downarrow \parallel & \\
 & M_0 & \\
 \gamma_d \swarrow & & \searrow \gamma_{d+n} \\
 U_d^- & \xleftarrow{R_d \quad R_{d+n}} & U_{d+n}^-
 \end{array}$$

Proof. The equality $R_d(F_i) = f_i$ obviously follows from the definitions of F_i and f_i (see 9.4 and 2.2).

Let us fix $\phi \in M_0$ and $j \in [1, n-1]$. For $(G, G', x) \in Z_d$, let $V \in \mathcal{V}_\nu, \{r_i\}, t(G, G', x)$ be as in 9.6.

If $G_j = G'_j$, then we have $R_d(F_j * \phi)(G, G', x) = 0 = f_j * R_d(\phi)(G, G', x)$. If $\dim G'_j/G_j \geq 1$, let ν' be such that $\nu'_i = \nu_i - \delta_{i,j}$. Then

$$\begin{aligned}
 R_d(F_j * \phi)(G, G', x) &= \sum_{a \in \mathbb{Q}} a \cdot \chi\{\tilde{V} \in \mathcal{V}_{\nu'} \mid \tilde{V} \subseteq V, t \sim \tilde{V}, \phi(t|_{\tilde{V}}) = a\} \\
 &= \sum_{a \in \mathbb{Q}} a \cdot \chi\{\tilde{V}_j \subset V_j \mid \dim V_j/\tilde{V}_j = 1, \text{Im } t_{j-1,j} + \text{Im } t_{j+1,j} \subseteq \tilde{V}_j, \phi(t|_{\tilde{V}}) = a\} \\
 &= \sum_{a \in \mathbb{Q}} a \cdot \chi\{\tilde{G} \in \mathcal{F}_d \mid (G, \tilde{G}, x) \in \bigcup_{A \in \Delta_{d-1}} Z[A + E_{j+1,j}], R_d(\phi)(\tilde{G}, G', x) = a\} \\
 &= f_j * R_d(\phi)(G, G', x).
 \end{aligned}$$

Therefore, R_d is an algebra homomorphism.

The proposition follows.

Proposition 9.7 shows that the homomorphisms $\{R_d\}$ give rise to a homomorphism $R : M_0 \rightarrow \bar{U}$.

9.8. Lusztig's construction provides the canonical basis for U^- . It is compatible with the decomposition $M_0 = \bigoplus_\nu M_0(\nu)$, where $M_0(\nu) = M_0 \cap M(\nu)$.

The basis of $M_0(\nu)$ is parametrized by irreducible components of $\Lambda_\nu, V \in \mathcal{V}_\nu$. For an irreducible component Y of Λ_ν , the basis function f_Y has the following properties (see [L2, Proposition 3.6]):

- (a) $f_Y|_O = 1$ for some open dense G_V -stable subset O of Y ;
(b) $f_Y = 0$ outside $Y \cup H$ for some closed G_V -stable subset $H \subset \Lambda_V$ of dimension $< \dim \Lambda_V$.

9.9. Let $E_{V,+} \subseteq E_V$ (resp. $E_{V,-}$) be the subspace of representations s such that $s_{i+1,i} = 0$ (resp. $s_{i,i+1} = 0$) for all $i = 1, \dots, n-2$. Clearly, $E_V = E_{V,+} \oplus E_{V,-}$. For $t \in E_V$, let us write $t = t^+ \oplus t^-$, where $t^+ \in E_{V,+}$, $t^- \in E_{V,-}$.

The space E_V can be naturally regarded as a cotangent bundle of $E_{V,+}$. As was shown in [L1, n. 14], any irreducible component of Λ_V is the closure of the conormal bundle of some G_V -orbit on $E_{V,+}$.

Proposition 9.10. *There is a bijection between the canonical basis elements of M_0 and strictly lower triangular matrices in Θ . It is given by $f_Y \mapsto C$, such that*

$$R(f_Y) = \sum_{D \in \tilde{\Delta}} \hat{\varphi}_{(C+D)}.$$

9.11. First, we show that for any lower triangular $A \in \Theta_d$ there is $V \in \mathcal{V}_\nu$, and an irreducible component Y of Λ_V such that

$$(*) \quad R_d(f_Y) = \sum_{\substack{B \in \Theta_d \\ A-B \text{ diagonal}}} \varphi_B.$$

For any $(G, G', x) \in Z[A]$ let $\nu = (\dim G'_i/G_i)_{i=1}^{n-1}$. Let $V, \{r_i\}$ and $t(G, G', x)$ be as in n. 9.6. Note that t^+ is independent of x . Therefore, we have a morphism from the fiber $N_{(G,G')}$ of $Z[A]$ to the fiber N_{t^+} of the conormal bundle of the orbit $G_V \cdot t^+$. It is surjective.

Let Y be the closure of the conormal bundle of $G_V \cdot t^+$. By property 9.8 (a), the function $f_Y = 1$ on some open dense subset of N_{t^+} . Therefore, $R_d(f_Y) = 1$ on some open dense subset of $N_{(G,G')}$. Since $R_d(f_Y) \in U_d$, it is constant on all GL_d -orbits. Hence, $R_d(f_Y) = 1$ on some open dense subset of $Z[A]$.

It is easy to show that the $+$ -part of $t(F, F', y)$ is isomorphic to t^+ if and only if $(F, F') \in Z[B]$ such that $B \in \Theta_d$, and all the off-diagonal entries of A and B coincide. By the same argument as above, $R_d(f_Y)$ is identically 1 on an open dense subset of such $Z[B]$.

Similarly, we show that $R_d(f_Y)$ vanishes on an open dense subset of $Z[B]$ if $A - B$ is not diagonal. Therefore, the difference of $R_d(f_Y)$ and the right hand side of $(*)$ is a function in U_d , vanishing on an open dense subset of Z_d . By corollary 4.13, it has to be 0.

9.12. Next, we show that for any V and any irreducible component Y of Λ_V , there

exists d and a lower triangular matrix $A \in \Theta_d$ such that $(*)$ holds.

Let Y be the closure of the conormal bundle of the G_V orbit of $s \in E_{V,+}$. It is easy to show that there exists a pair of flags $G, G' \in \mathcal{F}_d$ for d large enough, such that for any $x \triangleright G, G'$ the $+$ -part of $t(G, G', x)$ is isomorphic to s . By 9.11, the matrix $A = \Phi(G, G')$ satisfies $(*)$.

By proposition 9.7 (2), the homomorphisms R_d commute with ι^* . Since by theorem 6.10, ι^* is compatible with the bases of U_{d+n} and U_d , the proposition follows.

Corollary 9.13. *For any $\lambda \in X^+$, the basis $\{\hat{\varphi}_A\}$ of \hat{U} and the basis $\{f_Y\}$ of M_0 give rise to the same canonical basis of the sl_n -module L_λ .*

Indeed, for any lower triangular matrix $A \in \Theta_d$ and any module L_x as in n. 5.3, there is at most one function $\varphi_B \in U_d$ such that $A - B$ is diagonal, and $\varphi_B * s_x \neq 0$.

10 Examples: $n = 2, 3$.

In this section we compute the canonical basis for $n = 2$, and some monomials in the canonical basis for $n = 3$.

We omit $*$, writing $\psi\phi$ instead of $\psi * \phi$.

For an element ψ of U_d or \bar{U} , and an integer $a \in \mathbf{Z}_{\geq 0}$, let

$$\psi^{(a)} = \frac{\psi^a}{a!}.$$

10.1. The canonical basis of \hat{U} for $U = U(sl_2)$ is computed in [L3, n. 25.3]. We describe the computation in the geometric setting.

The basis of \hat{U} in this case is parametrized by 2×2 matrices in $\tilde{\Theta}$ (see 7.2):

$$\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}, \quad \begin{pmatrix} c & a \\ b & 0 \end{pmatrix}, \quad a, b, c \in \mathbf{Z}_{\geq 0}.$$

The corresponding basis elements are given by

$$\hat{\varphi}_{\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}} = \bar{e}_1^{(a)} \hat{1}_{-a-b-c} \bar{f}_1^{(b)}, \quad \hat{\varphi}_{\begin{pmatrix} c & a \\ b & 0 \end{pmatrix}} = \bar{f}_1^{(b)} \hat{1}_{a+b+c} \bar{e}_1^{(a)}.$$

Let us prove it. Let $k \in \mathbf{Z}_{\geq 0}$, and let

$$A = \begin{pmatrix} k & a \\ b & c+k \end{pmatrix}, \quad D = \begin{pmatrix} k & 0 \\ 0 & a+b+c+k \end{pmatrix},$$

$A, D \in \Theta_d$, where $d = a + b + c + 2k$.

Let

$$\varphi = e_1^{(a)} \varphi_D f_1^{(b)}.$$

We have $\varphi(G, G', x) \neq 0$ if and only if

$$\dim G_1 = a + k; \dim G'_1 = b + k; \dim(G_1 \cap G'_1) \geq k; \text{Im } x \subseteq G_1 \cap G'_1; \text{rank } x \leq k.$$

As can be easily seen, $\varphi|_{Z[A]} \equiv 1$, and $\varphi|_{Z[B]} \equiv 0$ for all $B \not\subseteq A$.

Let now $(G, G', x) \in Z[A']$, $A' < A$. Then $\dim(G_1 \cap G'_1) > k$, and $\dim(G_1 + G'_1) < a + b + k$. Therefore, the maximal possible rank of x is

$$\min(\dim(G_1 \cap G'_1), \text{codim}(G_1 + G'_1)) > k.$$

But $\varphi(G, G', x) \neq 0$ only if $\text{rank } x \leq k$. Hence, φ vanishes on an open dense subset of $Z[A']$.

By theorem 3.1, $\varphi = \varphi_A$. Then theorem 6.10 implies

$$\hat{\varphi}_{\left(\begin{smallmatrix} a & \\ & b \end{smallmatrix}\right)} = \bar{e}_1^{(a)} \hat{1}_{-a-b-c} \bar{f}_1^{(b)}.$$

The computation for $\hat{\varphi}_{\left(\begin{smallmatrix} a & \\ & b \end{smallmatrix}\right)}$ is similar.

Note that $\bar{e}_1^{(a)} \hat{1}_{-a-b} \bar{f}_1^{(b)} = \bar{f}_1^{(b)} \hat{1}_{a+b} \bar{e}_1^{(a)}$ for all $a, b \geq 0$.

Let now $U = U(sl_3)$.

Proposition 10.2. *The following monomials are contained in the canonical basis of \hat{U} :*

$$(1) \quad \bar{e}_1^{(a)} \bar{f}_1^{(b)} \bar{e}_2^{(c)} \hat{1}_{(l,m)} \bar{f}_2^{(s)} \bar{e}_1^{(t)} \bar{f}_1^{(r)}, \quad \bar{f}_2^{(a)} \bar{e}_2^{(b)} \bar{f}_1^{(c)} \hat{1}_{(-m,-l)} \bar{e}_1^{(s)} \bar{f}_2^{(t)} \bar{e}_2^{(r)},$$

$$l \geq b + t, \quad -l \geq a + b - c, \quad -l \geq t + r - s, \quad -m \geq c + s;$$

$$(2) \quad \bar{e}_2^{(a)} \bar{f}_2^{(b)} \bar{e}_1^{(c)} \hat{1}_{(l,m)} \bar{f}_1^{(s)} \bar{e}_2^{(t)} \bar{f}_2^{(r)}, \quad \bar{f}_1^{(a)} \bar{e}_1^{(b)} \bar{f}_2^{(c)} \hat{1}_{(-m,-l)} \bar{e}_2^{(s)} \bar{f}_1^{(t)} \bar{e}_1^{(r)},$$

$$-l \geq c + s, \quad m \geq b + t, \quad -m \geq a + b - c, \quad -m \geq t + r - s;$$

$$(3) \quad \bar{e}_1^{(a)} \bar{e}_2^{(b)} \bar{e}_1^{(c)} \hat{1}_{(l,m)} \bar{f}_1^{(s)} \bar{f}_2^{(t)} \bar{f}_1^{(r)}, \quad \bar{f}_2^{(a)} \bar{f}_1^{(b)} \bar{f}_2^{(c)} \hat{1}_{(-m,-l)} \bar{e}_2^{(s)} \bar{e}_1^{(t)} \bar{e}_2^{(r)},$$

$$-l \geq a + c + s + r, \quad -m \geq b + t, \quad b \geq a + c, \quad t \geq s + r;$$

$$(4) \quad \bar{e}_1^{(a)} \bar{e}_2^{(b)} \bar{e}_1^{(c)} \hat{1}_{(l,m)} \bar{f}_2^{(s)} \bar{f}_1^{(t)} \bar{f}_2^{(r)}, \quad \bar{f}_2^{(a)} \bar{f}_1^{(b)} \bar{f}_2^{(c)} \hat{1}_{(-m,-l)} \bar{e}_1^{(s)} \bar{e}_2^{(t)} \bar{e}_1^{(r)},$$

$$\bar{e}_2^{(r)} \bar{e}_1^{(t)} \bar{e}_2^{(s)} \hat{1}_{(l,m)} \bar{f}_1^{(c)} \bar{f}_2^{(b)} \bar{f}_1^{(a)}, \quad \bar{f}_1^{(r)} \bar{f}_2^{(t)} \bar{f}_1^{(s)} \hat{1}_{(-m,-l)} \bar{e}_2^{(c)} \bar{e}_1^{(b)} \bar{e}_2^{(a)},$$

$$-l \geq a + c + t, \quad -m \geq b + s + r, \quad b \geq a + c, \quad t \geq s + r;$$

$$(5) \quad \bar{e}_2^{(a)} \bar{e}_1^{(b)} \bar{e}_2^{(c)} \hat{1}_{(l,m)} \bar{f}_2^{(s)} \bar{f}_1^{(t)} \bar{f}_2^{(r)}, \quad \bar{f}_1^{(a)} \bar{f}_2^{(b)} \bar{f}_1^{(c)} \hat{1}_{(-m,-l)} \bar{e}_1^{(s)} \bar{e}_2^{(t)} \bar{e}_1^{(r)},$$

$$-l \geq b + t, \quad -m \geq a + c + s + r, \quad b \geq a + c, \quad t \geq s + r.$$

Proof. Let $A = (a_{ij})$ be a 3×3 matrix, $A \in \Theta_d$. Let us consider the following functions in U_d .

$$\begin{aligned} \psi_1 &= e_1^{(a_{13})} f_1^{(a_{21})} e_2^{(a_{23}+a_{13})} 1_\lambda f_2^{(a_{32}+a_{31})} e_1^{(a_{12})} f_1^{(a_{31})}, \\ \psi_2 &= e_2^{(a_{23})} f_2^{(a_{31})} e_1^{(a_{12}+a_{13})} 1_\mu f_1^{(a_{21}+a_{31})} e_2^{(a_{13})} f_2^{(a_{32})}, \\ \psi_3 &= e_1^{(a_{13})} e_2^{(a_{23}+a_{13})} e_1^{(a_{12})} 1_\omega f_1^{(a_{21})} f_2^{(a_{32}+a_{31})} f_1^{(a_{31})}, \\ \psi_4 &= e_1^{(a_{13})} e_2^{(a_{23}+a_{13})} e_1^{(a_{12})} 1_\omega f_2^{(a_{31})} f_1^{(a_{21}+a_{31})} f_2^{(a_{32})}, \\ \psi_5 &= e_2^{(a_{13})} e_1^{(a_{12}+a_{13})} e_2^{(a_{23})} 1_\omega f_2^{(a_{32})} f_1^{(a_{21}+a_{31})} f_2^{(a_{31})}, \end{aligned}$$

where

$$\begin{aligned} \lambda &= (a_{11} - a_{22} + a_{12} + a_{21}, a_{22} - a_{33} - a_{32} - a_{23} - a_{31} - a_{13}), \\ \mu &= (a_{11} - a_{22} - a_{12} - a_{21} - a_{31} - a_{13}, \\ &\quad a_{22} - a_{33} + a_{12} + a_{21} - a_{32} - a_{23} + a_{31} + a_{13}), \\ \omega &= (a_{11} - a_{22} - a_{12} - a_{21}, a_{22} - a_{33} + a_{12} + a_{21} - a_{32} - a_{23} - a_{31} - a_{13}). \end{aligned}$$

For all $j = 1, \dots, 5$ we have $\psi_j|_{Z[A]} \equiv 1$, and $\psi_j|_{Z[B]} \equiv 0$ if $B \not\leq A$.

10.3. Any matrix $B \leq A$ has to be of the form

$$B = \begin{pmatrix} a_{11} + u & a_{12} + v & a_{13} - u - v \\ a_{21} + w & a_{22} + z & a_{23} - w - z \\ a_{31} - u - w & a_{32} - v - z & a_{33} + u + v + w + z \end{pmatrix}$$

for some integers u, v, w, z such that $u, u + v, u + w, u + v + w + z \geq 0$ (see 1.3).

10.4. For each $j \in [1, 5]$, we list the conditions on the coefficients of a matrix A . For A satisfying these conditions, and for $B \leq A$ as in 10.3, let $(G, G', y) \in \text{supp } \psi_j \cap Z[B]$.

For the fiber $N_{(G, G')}$ of the conormal bundle $Z[B]$ at (G, G') , we compare $\text{rank } y$, $\text{rank } y^2$, $\text{rank } y|_{G_2}$ etc. to those maximal possible for $x \in N_{(G, G')}$. We are looking for u, v, w, z such that the condition $\psi_j(G, G', y) \neq 0$ does not force any of these numbers for y to be less than maximal. In each case we show that we must have $u = v = w = z = 0$.

This means that ψ_j vanishes on an open dense subset of $Z[B]$ for $B \neq A$. Therefore, by theorem 3.1 it is equal to φ_A . Since all the conditions involved hold for A if and only if they hold for $A + I$, this will show that ψ_j gives rise to the canonical basis element $\hat{\varphi}_A$.

The proof repeatedly uses the fact that $x \triangleright G, G'$ if and only if $x(G_i \cap G'_j) \subseteq G_{i-1} \cap G'_{j-1}$ for all i, j (see 1.4). In particular, we must have

$$x(\mathbf{C}^d) \subseteq G_2 \cap G'_2, \quad x(G_2 \cap G'_2) \subseteq G_1 \cap G'_1, \quad x(G_2) \subseteq G_1 \cap G'_2, \quad x(G'_2) \subseteq G_2 \cap G'_1,$$

and also

$$x(G_1 + G'_1) = 0, \quad x(G_2 + G'_2) \subseteq G_1 + G'_1.$$

10.5. Let $j = 1$, and let $a_{11} \geq a_{22}, a_{33} \geq a_{22}, a_{23} \geq a_{12} + a_{11} - a_{22}, a_{32} \geq a_{21} + a_{11} - a_{22}$.

If $z > 0$, then the maximal rank of x^2 is strictly greater than a_{22} . But $\dim \text{Im } y \leq a_{11} + a_{12} + a_{21} + a_{22}$, and $\dim(\text{Ker } y \cap \text{Im } y) \geq a_{11} + a_{12} + a_{21}$, hence $\text{rank } y^2 \leq a_{22}$. This implies $z \leq 0$, hence $u + v + w \geq 0$.

If $v < 0$, and $u + w > 0$, then

$$\max \dim x(G'_2) > a_{11} + a_{21} \geq \dim y(G'_2).$$

If $v < 0$, and $u + w = 0$, then $v + z < 0$ contradicts $u + v + w + z \geq 0$. Hence, $v \geq 0$. Similarly, $w \geq 0$, and also $u + v + z, u + w + z \geq 0$.

If $u + v + w = 0$, then $z = 0, v + w = -u \leq 0$. It follows that $u = v = w = z = 0$.

If $u + v + w > 0$, then

$$\max \text{rank } (x|_{G_2 + G'_2}) = \min(a_{11} + a_{12} + a_{21} + u + v + w, a_{22} + a_{23} + a_{32} - v - w - z),$$

and $\dim y(G_2 + G'_2) \leq a_{11} + a_{12} + a_{21}$. Therefore, $a_{22} + a_{23} + a_{32} - v - w - z \leq a_{11} + a_{12} + a_{21}$. Then

$$\max \text{rank } x = \min(a_{11} + a_{12} + a_{21} + a_{22} + u + v + w + z, a_{22} + a_{23} + a_{32} + a_{33} + u),$$

and $\text{rank } y \leq a_{11} + a_{12} + a_{21} + a_{22}$.

If $u + v + w + z = 0$, then $u = v = w = z = 0$. If $u + v + w + z > 0$, then $u = 0$, and $a_{22} + a_{23} + a_{32} + a_{33} \leq a_{11} + a_{12} + a_{21} + a_{22}$, which also leads to $v = w = z = 0$.

10.6. Let $j = 2$, and let $a_{22} \geq a_{11}$, $a_{33} \geq a_{22}$, $a_{12} \geq a_{23} + a_{33} - a_{22}$, $a_{21} \geq a_{32} + a_{33} - a_{22}$.

Since the space $G_1 + G'_1 + (G_2 \cap G'_2)$ contains a subspace of dimension $a_{11} + a_{12} + a_{21} + a_{22} + a_{13} + a_{21}$, we have

$$\begin{aligned} \text{codim } (G_1 + G'_1 + (G_2 \cap G'_2)) &= a_{23} + a_{32} + a_{33} - z + u \\ &\leq a_{23} + a_{32} + a_{33}. \end{aligned}$$

Therefore, $z \geq u \geq 0$.

Also,

$$\max \dim x(G_2 \cap G'_2) = a_{11} + u \geq a_{11} \geq \dim y(G_2 \cap G'_2).$$

Hence, $u = 0$, and $v, w \geq 0$.

Finally,

$$\begin{aligned} \max \text{rank } x &= a_{11} + a_{12} + a_{21} + a_{22} + v + w + z \\ &\geq a_{11} + a_{12} + a_{21} + a_{22} \geq \text{rank } y. \end{aligned}$$

Hence, $v + w + z = 0$, so that $u = v = w = z = 0$.

10.7. Let $j = 3, 4$, or 5 , and let $a_{22} \geq a_{11} + a_{13} + a_{31}$, $a_{33} \geq a_{22} + a_{21} + a_{12}$.

Let also

$$\begin{aligned} a_{23} &\geq a_{12}, a_{32} \geq a_{21}, \text{ if } j = 3; \\ a_{23} &\geq a_{12}, a_{32} \leq a_{21}, \text{ if } j = 4; \\ a_{23} &\leq a_{12}, a_{32} \leq a_{21}, \text{ if } j = 5. \end{aligned}$$

Since all the entries of B are non-negative, we have $u + v \leq a_{13}$, $u + w \leq a_{31}$. Then $u + v + w + z \geq 0$ implies $-z \leq u + v + w \leq a_{13} + a_{31} - u$. Therefore, $a_{22} + z \geq a_{11} + u$.

For all $j = 3, 4, 5$ we have

$$\begin{aligned} \max \dim x(G_2 \cap G'_2) &= a_{11} + u \geq a_{11} \geq \dim y(G_2 \cap G'_2); \\ \max \text{rank } x &= \sum_{i,j \leq 2} a_{ij} + u + v + w + z \geq \sum_{i,j \leq 2} a_{ij} \geq \dim \text{Im } y. \end{aligned}$$

It follows that $u + v + w + z = u = 0$.

Now we have $-w - z = v \geq 0$, $-v - z = w \geq 0$. We can compute

$$\begin{aligned} \max \dim x(G_2) &= a_{11} + \min(a_{12}, a_{23}) + v \geq a_{11} + \min(a_{12}, a_{23}) \geq \dim y(G_2), \\ \max \dim x(G'_2) &= a_{11} + \min(a_{21}, a_{32}) + w \geq a_{11} + \min(a_{21}, a_{32}) \geq \dim y(G'_2). \end{aligned}$$

Hence $v = w = 0$, and therefore also $z = 0$.

10.8. We have shown that all the functions ψ_j , $j \in [1, 5]$ give rise to the basis elements $\hat{\varphi}_A$ for the corresponding A .

All the monomials listed in the proposition can be obtained from the corresponding functions ψ_j , using involutions τ and δ (see 2.3 and 8.10).

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