Topology of Combinatorial Differential Manifolds

by

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Abstract

The central problem we attack is to show that all combinatorial differential manifolds (CD manifolds) are piecewise linear manifolds. We succeed in this aim for all CD manifolds involving only Euclidean oriented matroids, and we give evidence for the general case.

The major steps in our approach are as follows:

- We define a new notion of a **triangulation of an oriented matroid**, so that the boundary of the star of any simplex in a CD manifold is a triangulation of some oriented matroid. Our central problem is then reduced to showing that any oriented matroid triangulation is a PL sphere.

- We show that every uniform totally cyclic oriented matroid has a triangulation which is a PL sphere. The central problem is then reduced further to showing that any two triangulations of a fixed oriented matroid have a PL common refinement.

- We give a candidate for such a common refinement. This candidate can be defined for triangulations of any oriented matroid, and we show that it is in general a regular cell complex.

- Finally, we show that this candidate really is a common refinement for any two triangulations of a Euclidean oriented matroid.

In Chapter 5 we give applications of this work to the theory of matroid polytopes.

Thesis Supervisor: Dr. Robert MacPherson, Professor of Mathematics

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Contents

Chapter 1

Introduction

Combinatorial differential manifolds (CD manifolds) were introduced by Gelfand and MacPherson in [GM] as a combinatorial analog to differential manifolds. Their application in [GM] led to a combinatorial formula for the Pontrjagin classes, and they show promise for a number of applications in geometry and topology. This thesis explores some of the many open questions on the properties of CD manifolds. Our main aim is to show that all CD manifolds are piecewise linear manifolds. We succeed in this aim for all CD manifolds involving only Euclidean oriented matroids and give progress on the general case.

The theory of CD manifolds relies heavily on the theory of oriented matroids. Essentially, an oriented matroid is a combinatorial model for a real vector space, and a CD manifold is a simplicial complex together with a collection of oriented matroids which play the role of a tangent bundle. The well-developed theory of oriented matroids lends some powerful machinery to the study of CD manifolds. In return, the results described here on CD manifolds have application in oriented matroid theory, particularly in the theory of matroid polytopes.

1.1 The General Idea

A CD manifold is a combinatorial analog to a differential manifold, encoding not only the topological structure but also some combinatorial remnant of the differential structure. We will make this precise in the next chapter. In the meantime, we offer here a brief translation dictionary between the language of smooth manifolds and the CD language.

The basic idea of CD manifolds is given by the translation:

A real vector space	$\leftrightarrow$	An oriented matroid
Differential manifold = an n-dimensional topological manifold and an n-dimensional vector space at each point	$\leftrightarrow$	CD manifold = a simplicial complex X of pure dimension n, a cellular refinement $\hat{X}$ of X, and a rank n oriented matroid at each cell of $\hat{X}$

In particular, there is a natural way to convert a triangulation of a differential manifold into a CD manifold. To see the usefulness of this, we note the following natural translations.

Grassmannian $G(k,n)$ = all k-dimensional subspaces of a real rank n vector space	$\leftrightarrow$	Combinatorial Grassmannian $\mathsf{M}(k, M^n)$ = all rank k strong images of a rank n oriented matroid M^n
Sphere bundle=bundle derived from a real vector bundle by replacing each fiber with the unit vectors in that fiber	$\leftrightarrow$	CD sphere bundle=bundle derived from a matroid bundle by replacing each oriented matroid with the poset of all its rank 1 strong images

Translations like these allow us to find combinatorial analogs to some of our favorite topological methods, as described in the next section.

1.2 Earlier Work on CD Manifolds

This section is by necessity brief. The only published papers on CD manifolds are [GM] and [M], from 1992 and 1993, respectively. We describe here relevant work on combinatorial Grassmannians as well.

In [GM] Gelfand and MacPherson used CD manifolds to find a combinatorial formula for the Pontrjagin classes of a differential manifold. Their method converts a differential manifold to a CD manifold, then combinatorially mimics Chern-Weil theory to calculate Pontrjagin classes. They utilized the natural notions of Grassmanians and sphere bundles that exist for CD manifolds.

The CD notion of sphere bundles comes from the Topological Representation Theorem (cf. [FL], [BLSWZ]), which states that any rank r oriented matroid can be represented by an arrangement of pseudospheres on the unit sphere S^{r-1}. This tells us that the fibers of a CD sphere bundle really are topological spheres. The combinatorial Grassmannians are topologically more mysterious. The combinatorial Grassmannian of the previous section is a finite poset. There is a canonical map $\xi : G(k,n) \to \mathcal{O}(\mathsf{M}(k, M^n))$ from the real Grassmannian to the order complex of this finite poset. This map has no hope of being a homeomorphism, but we can hope that the two spaces are topologically similar. The Topological Representation Theorem tells us that $\mathcal{O}(\mathsf{M}(1, M^n))$ and $\mathcal{O}(\mathsf{M}(n-1, M^n))$ are homeomorphic to the corresponding Grassmannians. Babson in [Ba] showed that $\mathcal{O}(\mathsf{M}(2, M^n))$ and $G(k, \mathbf{R}^n)$ are homotopic, though they need not be homeomorphic. Little else is known about the topology of $\mathcal{O}(\mathsf{M}(k, M^n))$.

1.3 Results In This Thesis

The central problem we address is:

Conjecture 1.3.1 *All CD manifolds are piecewise linear manifolds.*

That is, we wish to show that the boundary of the star of any simplex in a CD manifold is a sphere.

In Chapter 3 we find a set of axioms for the boundary of the star of a simplex in an oriented matroid. This suggests a definition of a **triangulation of an oriented matroid**. The boundary of the star of a simplex is a triangulation of the oriented matroid at any cell of that simplex. We can then generalize Conjecture 1.3.1 to:

The PL Conjecture: If M is an oriented matroid, then any triangulation of M is a PL sphere.

In Section 3.1 we prove this conjecture for realizable oriented matroids. In Section 3.2 we prove that only totally cyclic oriented matroids have triangulations, and then that any totally cyclic uniform oriented matroid has a triangulation which is a PL sphere.

This might lead one to expect that the PL Conjecture holds. It is hard to imagine that an oriented matroid could have topologically different triangulations. We make this more concrete in Chapter 4 by the notion of a **common refinement** of two oriented matroid triangulations. If an oriented matroid M is Euclidean then we show that any two triangulations of M have a common refinement. Together with our results of Section 3.2 this implies:

Theorem 1.3.1 *The PL Conjecture holds for all Euclidean oriented matroids.*

Corollary 1.3.1 *Any CD manifold involving only Euclidean oriented matroids is a PL manifold.*

For triangulations of more general oriented matroids we define a candidate for such a refinement and show that it is a regular cell complex.

Triangulations of oriented matroids are closely related to triangulations of convex polytopes. In Chapter 5 we make this more explicit by defining **triangulations of matroid polytopes**. We show that every uniform matroid polytope has a triangulation which is a PL ball and give evidence that every triangulation of a matroid polytope is a PL ball.

Chapter 2

Preliminaries

Note: In the Appendix we summarize all the definitions and results we need from oriented matroid theory.

2.1 Definitions

The following definitions are from [M].

Note: Throughout the following, a "simplex" will be considered as simply a finite set (its set of vertices). For instance, a simplex is "independent" in an oriented matroid M if that set of vertices is independent in M. This introduces a dilemma of notation, which we resolve as follows: if Δ is a simplex, we use $|\Delta|$ to denote the order of the set Δ, and we use $||\Delta||$ to denote the geometric realization of the simplex Δ. For any simplicial complex X, we denote the set of 0-cells of X by X^0.

Definition 2.1.1 *(From [M]) An n-dimensional* **combinatorial differential manifold** *is a triple $(X, \hat{X}, M)$ such that:*

1. *X is a pseudomanifold of dimension n (i.e., a simplicial complex of pure dimension n such that every $(n-1)$-simplex is contained in exactly two n-simplices)*

2. *$\hat{X}$ is a cell complex refining $||X||$: every cell σ of $\hat{X}$ is contained in a simplex $||\Delta(\sigma)||$ of $||X||$*

3. *M is a function assigning to each cell σ of $\hat{X}$ a rank n oriented matroid $M(\sigma)$ with elements $\mathrm{star}(\Delta(\sigma))^0$.*

We have the following axioms:

- *The rank of $(\Delta(\sigma))^0$ in $M(\sigma)$ is equal to the dimension of $||\Delta(\sigma)||$.*

- *(Linear independence) If Δ' is a simplex in the boundary of $\mathrm{star}\,\Delta(\sigma)$, then Δ' is independent in $M(\sigma)$.*

- *(Convexity) If Δ' is in the boundary of* $\operatorname{star}\Delta(\sigma)$*, then no other simplex is in the convex hull of Δ' in $M(\sigma)$.*

- *(Continuity) If σ' is in the boundary of σ, then $M(\sigma')((\operatorname{star}\sigma)^0)$ is a specialization of $M(\sigma)$.*

For instance, given a differential manifold N and a smooth triangulation $\eta : ||X|| \to N$ we can associate an oriented matroid to any point $p \in ||X||$ as follows: Let $||\Delta||$ be the minimal simplex of $||X||$ containing p. Then there is a unique piecewise linear map $f_p : ||\operatorname{star}(\Delta)|| \to U \subset T_{\eta(p)}(N)$ (the "flattening" at p) such that $f_p(p) = 0$ and for every simplex Δ' of $\operatorname{star}(\Delta)$, $d_p f_p|_{||\Delta'||} = d_p\eta|_{||\Delta'||}$. Then $f_p((\operatorname{star}\Delta)^0)$ is a configuration of vectors in $T_{\eta(p)}(N)$ defining an oriented matroid $M(p)$. (See Figure 2-1.)

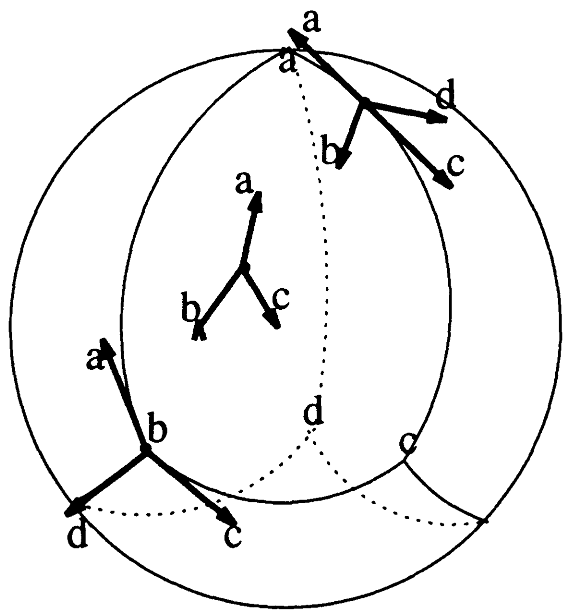

Figure 2-1: The minimal triangulation of S^2 and the resulting oriented matroids at three points

In this way we get an oriented matroid $M(p)$ at every point p of $||X||$. We say the triangulation η is **tame** if there is a regular cell complex $\hat{X}$ refining $||X||$ such that M is constant on every cell of $\hat{X}$. For instance, all piecewise analytic triangulations are tame. Any tame triangulation of N gives a CD manifold in this way.

2.2 An Example

This example follows from [M]. We include it here to help the reader get a feel for CD manifolds. Since all oriented matroids involved are rank 2 and all rank 2 oriented matroids are realizable, the reader may visualize with confidence.

Consider the triangulation of S^2 shown in Figure 2-1. We will derive the corresponding CD manifold $(X, \hat{X}, M)$. Here X is just the hollow tetrahedron with 0-simplices a, b, c, and d.

The following lemma is easy, and a useful thing to keep in mind throughout the next chapter or two of this thesis.

Lemma 2.2.1 *$\Delta(\sigma)^+$ is always a circuit of $M(\sigma)$.*

Proof: Our first axiom for CD manifolds tells us that $\Delta(\sigma)$ has rank $|\Delta(\sigma)|-1$ in M, so some subset of $\Delta(\sigma)$ is dependent in $M(\sigma)$. But our linear independence axiom tells us that any proper subset of $\Delta(\sigma)$ is independent in M. Thus $\Delta(\sigma)$ is the support of a circuit $\mathcal{C}$ in M.

We proceed by induction on $|\Delta(\sigma)|$. If $|\Delta(\sigma)| = 1$, 2, or 3, then the convexity axiom tells us that $\mathrm{C} = \Delta(\sigma)^+$, since every proper subset of $\Delta(\sigma)$ is a simplex in the boundary of star $\Delta(\sigma)$.

Say we know the result for $|\Delta(\sigma)| - 1$. Note that $\mathcal{C}$ will be a circuit of $M(\sigma')$ for every cell σ' in the interior of $||\Delta(\sigma)||$, since our weak maps between cells in the interior of $||\Delta(\sigma)||$ must preserve this circuit. Thus for any one $s \in \Delta(\sigma)$ we may assume that σ has in its boundary a cell τ such that $\Delta(\tau) = \Delta(\sigma) - \{s\}$. We know $M(\sigma) \rightsquigarrow M(\tau)$, and thus some circuit of $M(\tau)$ is contained in $\mathcal{C}$. Any proper subset of $\Delta(\sigma)$ except $\Delta(\tau)$ is a simplex in the boundary of star$(\Delta(\tau))$, and thus is independent in $M(\tau)$. By our induction hypothesis and the vector elimination axiom for oriented matroids, the only two circuits of $M(\tau)$ supported by $\Delta(\tau)$ are $\Delta(\tau)^+$ and $\Delta(\tau)^-$ So $\mathcal{C}$ has the same sign on all elements of $\Delta(\tau)$, i.e., on all elements of $\Delta(\sigma) - \{s\}$. But this is true for any $s \in \Delta(\sigma)$. Thus $\mathrm{C} = \Delta(\sigma)^+$.

QED

Thus the interior of each maximal simplex Σ of X is a single cell of $\hat{X}$, and the oriented matroid on that cell has the single circuit Σ^+.

At each vertex x of X, the oriented matroid $M(x)$ is uniquely defined by the convexity axiom. For example:

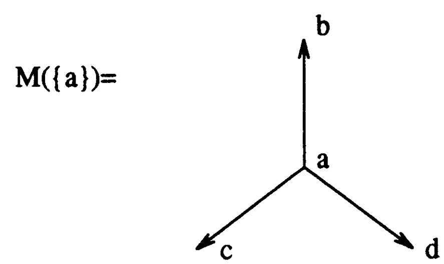

Figure 2-2:

It remains to find the cell decomposition of the 1-simplices of X and the oriented matroids at these cells.

Consider the cell ω_1 of $\{a, b\}$ which has $\{a\}$ in its boundary. Then $M(\omega_1) \rightsquigarrow M(\{a\})$. This, together with the independence axiom and Lemma 2.2.1, uniquely determines $M(\omega_1)$.

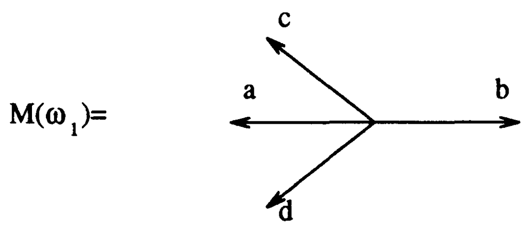

Figure 2-3:

Similarly, if ω_2 is the cell of $\{a, b\}$ which has $\{b\}$ in its boundary, then we have:

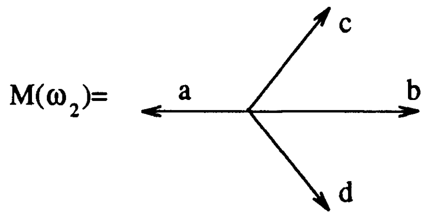

Figure 2-4:

So we see that $\omega_1 \neq \omega_2$. Figure 2-5 then gives the cell decomposition and oriented matroids for the 1-simplex $\{a, b\}$:

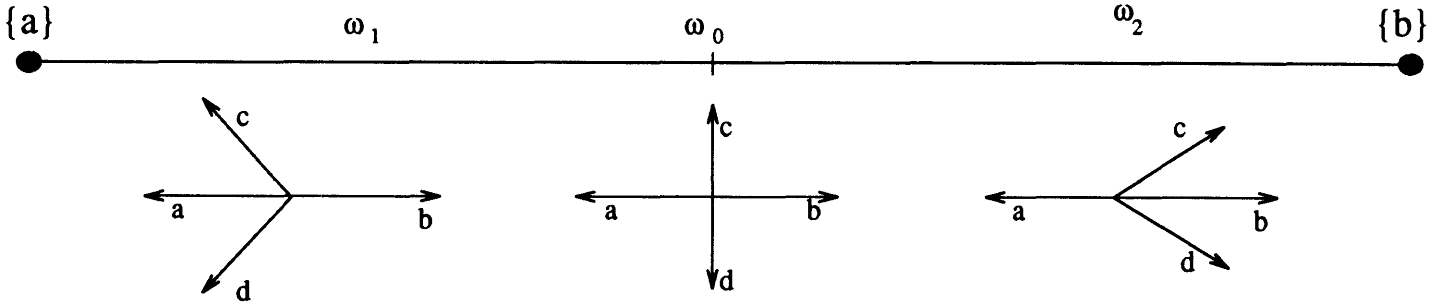

Figure 2-5: Cell decomposition and oriented matroids for a 1-simplex of Ex. 1

The final cell decomposition of X is shown in Figure 2-6.

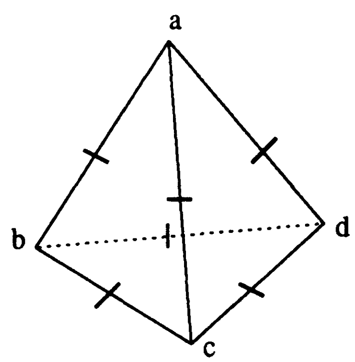

Figure 2-6: The cell decomposition of S^2 derived in Example 1

Chapter 3

Triangulations of Oriented Matroids

3.1 Definition, Connection to CD Manifolds

The axioms for a CD manifold restrict to give axioms for the boundary of the star of any simplex in a CD manifold. Looking back at the axioms for CD manifolds, we see obvious restrictions for all but the continuity axiom. The following lemma uses the continuity axiom to give one more condition.

Lemma 3.1.1 *Let σ be a cell of a CD manifold, M be the oriented matroid at σ, and L be the boundary of* $\operatorname{star}(\Delta(\sigma))$. *Let ω be a simplex of L, $L_\omega = \operatorname{link}_L(\omega)$, and $M_\omega = (M(\sigma)/\omega)(L^0_\omega)$. Then:*

- *L_ω is a pseudomanifold.*
- *If ν is a simplex of L_ω then ν is independent in M_ω and* $\operatorname{conv}(\nu) = \nu$ *in M_ω.*

Proof: That L_ω is pseudomanifold with independent simplices follows immediately from the same facts for L.

Let ν be a simplex of L_ω, and assume by way of contradiction M_ω has a circuit ν^+x^-, for some $x \in L^0_\omega$. If N is some basis of M_ω containing ν, then $\pm(\nu^+x^-)$ are the only two circuits supported on $N \cup \{x\}$: if there were another such circuit, circuit elimination would give us a circuit supported on N.

Consider a cell σ' of our CD manifold such that $\Delta(\sigma') = \Delta(\sigma) \cup \omega$ and $\sigma \subset \delta\sigma'$. (See Figure 3-1.) Then we know $M(\sigma') \rightsquigarrow M(\sigma)$, so every vector in $M(\sigma')$ contains a vector of $M(\sigma)$.

Let N be a basis of M/ω containing ν. Then in $M(\sigma')/\omega$ some vector is supported on a subset of $N \cup \{x\}$. This vector must contain a vector of $M(\sigma)/\omega$, by Lemma A.2. By our observations above, the only two vectors this could be are $\pm(\nu^+x^-)$. Thus $M(\sigma')$ contains a vector $\mathcal{V} = \omega_1^+\omega_2^-\nu^+x^-$, where $\omega_1, \omega_2 \subseteq \omega$.

By Lemma 2.2.1, we know also that $\mathcal{W} = \Delta(\sigma')^+$ is a vector of $M(\sigma')$. Composing these two vectors, we get the vector $\mathcal{W} \circ \mathcal{V} = \Delta(\sigma)^+\omega^+\nu^+x^-$ in $M(\sigma')$. By Lemma A.1 every vector is a composition of conformal circuits, so $M(\sigma)$ contains

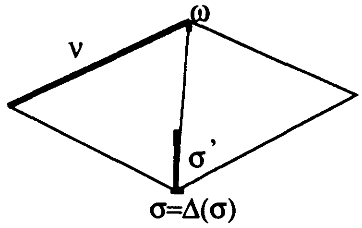

Figure 3-1: Figure for Lemma 3.1.1

a circuit B^+x^-, where $B \subseteq \Delta(\sigma') \cup \nu$. Certainly $\Delta(\sigma') \not\subseteq B$ since $\Delta(\sigma')^+$ is itself a circuit. But any subset of $\Delta(\sigma') \cup \nu$ not containing all of $\Delta(\sigma')$ is a simplex of the boundary of star($\Delta(\sigma')$). So x is in the convex hull of a simplex in the boundary of star($\Delta(\sigma')$), contradicting the convexity axiom for CD manifolds. Thus we have $\text{conv}(\nu) = \nu$ in M_ω.

QED

This result suggests the following definition:

Definition 3.1.1 *If M is a rank n oriented matroid with ground set E, a* **triangulation** *of M is an $(n-1)$-dimensional simplicial complex L such that:*

- $L^0 = E$.
- *L is a pseudomanifold.*
- *If ω is a simplex of L then ω is independent in M and* $\text{conv}(\omega) = \omega$ *in M.*
- *Either*
  - $n = 1$ *and* $L = S^0$, *or*
  - $n > 1$, *and if ω is a simplex of L, then* $\text{link}_L(\omega)$ *is a triangulation of* $(M/\omega)(\text{link}_L(\omega)^0)$.

The intuitive idea of an oriented matroid triangulation is:

Proposition 3.1.1 *If an oriented matroid M can be realized as a configuration C of unit vectors in R^n with 0 in their convex hull, then the triangulations of M are exactly the triangulations of the unit sphere by sectors from great spheres with vertices the set C.*

Proof: We show first, by induction on n, that every such triangulation L of the unit sphere gives a triangulation of M. Note that if $n = 1$ then the unit sphere consists of two points, and L is a triangulation of M.

For any n it is clear that L will satisfy the first three axioms for a matroid triangulation. To see the final axiom, let σ be a simplex of L. Then the vertices of $\text{link}_L(\sigma)$ give a configuration of unit vectors $\mathbf{R}^n/\langle\sigma\rangle$, and $\text{link}_L(\sigma)$ projects radially to a triangulation of the unit sphere in $\mathbf{R}^n/\langle\sigma\rangle$. Thus our induction hypothesis tells us that L satisfies the final axiom.

The converse, that every triangulation of M gives a PL triangulation of the boundary of P, is proven by the following proposition.

QED

Proposition 3.1.2 *If L is a triangulation of a* rank n *oriented matroid M and M has a realization $V \subset \mathbf{R}^n$ as an arrangement of unit vectors in $\mathbf{R}^n$, let $[L]$ denote the corresponding PL immersion of $||L||$ into $\mathbf{R}^n$. Then every ray $\vec{0A}$ in $\mathbf{R}^n$ intersects the interior of exactly one simplex of $[L]$.*

Proof: For any such ray $\vec{0A}$ we will assume A is a unit vector, and use point-set topology on the unit sphere.

We induct on n. If $n = 1$ then $L = S^0$ and all is clear.

Let Q be the set of unit vectors A in $\mathbf{R}^n$ such that $\vec{0A}$ intersects $[L]$, and assume by way of contradiction $Q \neq S^{n-1}$. Certainly Q is a non-empty closed set of dimension $n-1$, and so it has a boundary. Let q be a vertex on ∂Q. Then q is in the interior of conv$([\sigma])$ for some non-maximal simplex σ of L.

Now consider the oriented matroid $M_\sigma = M(\text{link}(\sigma)^0)/\sigma$. The projection $\mathbf{R}^n \to \mathbf{R}^n/\langle[\sigma]\rangle$ gives a realization of M_σ. Our induction hypothesis then tells us that [link(σ)] is a sphere, and every half-space $\vec{0A} + \langle[\sigma]\rangle$ intersects [link(σ)] in exactly one point in $\mathbf{R}^n$. Thus [link(σ)] is linked to $\langle[\sigma]\rangle$ in $\mathbf{R}^n$, and so [star(σ)] is an n-ball with q in its interior, contradicting $q \in \partial Q$. Thus every ray intersects at least one simplex of $[L]$.

Now, let R be the set of unit vectors A in $\mathbf{R}^n$ such that $\vec{0A}$ intersects more than one simplex of $[L]$, and assume by way of contradiction $R \neq \emptyset$. Certainly R is a closed set of dimension $n-1$ not containing V, and so $\partial R \neq \emptyset$. Let r be a vertex on ∂R. Then r is in the interior of conv$([\sigma]) \cap$ conv$([\tau])$ for some non-maximal simplices σ, τ of L. Again, looking in $\mathbf{R}^n/\langle[\sigma]\rangle$ and $\mathbf{R}^n/\langle[\tau]\rangle$, we see that [star$(\sigma)$] and [star$(\tau)$] are n-balls with r in their interior, contradicting $r \in \partial R$. Thus every ray intersects exactly one simplex of $[L]$.

QED

In Chapter 5 we will return to the use of matroid triangulations as a tool for studying polytopes. Our more immediate interest in matroid triangulations comes from the following proposition.

Proposition 3.1.3 *If σ is a cell of a CD manifold, $M(\sigma)$ is the oriented matroid at σ, and $L(\sigma)$ is the boundary of the star of σ, then $L(\sigma)$ is a triangulation of $M(\sigma)$.*

Proof: The first three link axioms follow immediately from the definition of CD manifolds. Let ω be a simplex of $L(\sigma)$. Then Lemma 3.1.1 tells us that $\text{link}_L(\omega)$ satisfies the first three requirements to be a triangulation of $(M/\omega)(\text{link}_L(\omega)^0)$. To see the last, let ν be a simplex of $\text{link}_{L(\sigma)}(\omega)$. Then $\nu \cup \omega$ is a simplex of $L(\omega)$, and so we see again by Lemma 3.1.1 that $\text{link}_L(\omega)$ satisfies the final requirement to be a triangulation of $(M/\omega)(\text{link}_L(\omega)^0)$.

QED

Recall from the introduction:

The PL Conjecture: If M is an oriented matroid, then any triangulation of M is a PL sphere.

As a corollary to Proposition 3.1.3 we have:

Corollary 3.1.1 *If the PL conjecture holds then*

$$(X, \hat{X}, M) \text{ a CD manifold} \Rightarrow X \text{ a PL manifold.}$$

Thus the notion of triangulations allows us to study questions about CD manifolds by considering only individual oriented matroids and their associated links. This will be our line of attack in the following chapter, where we show that all CD manifolds involving only Euclidean oriented matroids are PL manifolds and that all CD manifolds are normal.

This approach also puts these questions about CD manifolds into a broader context in oriented matroid theory. As Lemma 3.1.1 suggests, results on triangulations of oriented matroids will have applications in the theory of convex polytopes.

As a corollary to Proposition 3.1.1 we have:

Corollary 3.1.2 *The PL conjecture holds for realizable oriented matroids.*

In Chapter 4 we will prove the PL Conjecture for the much broader class of Euclidean oriented matroids.

3.2 Existence of Triangulations

We first make life easier by restricting to uniform oriented matroids. Lemma A.3 says that any oriented matroid is the weak map image of a uniform oriented matroid of the same rank, and the following lemma is easily verified.

Lemma 3.2.1 *If M_1 and M_2 are two simple rank n oriented matroids, M_2 is a weak map image of M_1, and L is a triangulation of M_2, then L is a triangulation of M_1.*

Proof: Recall that M_2 is a weak map image of M_1 iff every circuit of M_1 contains a circuit of M_2. Thus any simplex of L which is independent in M_2 is independent in M_1, and $\mathrm{conv}_{M_2}(\omega) = \omega$ implies $\mathrm{conv}_{M_1}(\omega) = \omega$.

Assume by way of contradiction that for some simplex ω in L, we have $\mathrm{link}_L(\omega)$ is not a triangulation of M_2/ω. Consider such an ω of maximal order. Certainly $\mathrm{link}_L(\omega)$ is a pseudomanifold with simplices independent in M_1. If ν is a simplex of $\mathrm{link}(\omega)$ and $x \in \mathrm{link}(\omega)^0$ such that ν^+x^- is a circuit of M_1/ω, then by Lemma A.2 ν^+x^- contains a circuit of M_2/ω, contradicting our knowledge that L is a triangulation of M_1.

So there must be some simplex ν of $\mathrm{link}(\omega)$ such that $\mathrm{link}_{\mathrm{link}(\omega)}(\nu) = \mathrm{link}(\omega \cup \nu)$ is not a triangulation of $M_2/(\omega \cup \nu)$. But this contradicts the maximality of ω.

QED

Corollary 3.2.1 *If the PL Conjecture holds for all uniform oriented matroids, then it holds for general oriented matroids as well.*

With this assumption of uniformity, we get the following existence theorem.

Theorem 3.2.1 *For any simple uniform oriented matroid M, the following are equivalent:*

1. *M totally cyclic.*

2. *M has a triangulation which is a PL sphere.*

3. *M has a triangulation.*

Proof: We prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): We construct a sequence $S_1, S_2, \ldots, S_K$ of simplicial spheres. Each S_i will have vertex set $E_i \subseteq E$ and will be a triangulation of $M(E_i)$, and each S_i will be obtained from S_{i-1} by a stellar subdivision. The final sphere will be a triangulation of M. The key oriented matroid fact to keep in mind at each step is that any oriented matroid of rank r with $r+2$ or fewer elements is realizable.

Let E_1 be the set of elements of a positive circuit. Then S_1 is just the boundary of a rank(M)-simplex, with vertex set E_1. Once we've constructed S_{i-1}, consider some element x of M which has not been added to the picture yet, and let $E_i = E_{i-1} \cup \{x\}$. Since $M(E_1 \cup \{x\})$ is realizable, x is in the convex hull of exactly one face of S_1. If that face F_1 is not subdivided in S_{i-1}, then we construct S_i by the stellar subdivision of putting x in the middle of F_1. If F_1 was first subdivided in the sphere S_j by the point y, then by realizability of $M(F_1 \cup \{x, y\})$, we know x is in the convex hull of exactly one of these smaller faces in star(y). If that smaller face F_2 isn't subdivided in S_{i-1}, then we construct S_i by putting x in the middle of F_2. Otherwise, we keep looking in the same way until we find the unique face of S_{i-1} with x in its convex hull, and construct S_i with a stellar subdivision. It is easy to check that each S_j is an oriented matroid triangulation.

(3) $\Rightarrow$ (1): We prove this by induction on rank(M). Certainly it's true for rank 2 oriented matroids. Now, for a rank r oriented matroid, suppose M were acyclic, and let p be an extreme point of M. By the final condition of the definition, the link of p in our given triangulation is itself a triangulation of $(M/\{p\})(\text{star}(p)^0)$. But $M/\{p\}$ is acyclic, contradicting our induction hypothesis. Thus M must be totally cyclic.

QED

Chapter 4

Refinements of Triangulations

We now know that every uniform oriented matroid has a triangulation which is a sphere. Thus to prove the PL Conjecture, it suffices to prove:

Conjecture 4.0.1 (PL Conjecture, Reduced) *Given any two triangulations L_1, L_2 of an oriented matroid, there exists a regular cell complex $R_M(L_1, L_2)$ which is a common refinement of L_1 and L_2.*

Thus any two triangulations of an oriented matroid are PL equivalent.

Below we will describe such a common refinement for two triangulations of a realizable oriented matroid. We will then give a description of this refinement purely in terms of the oriented matroid (without referpence to the particular realization). Thus we can define this "refinement" even for two triangulations of a non-realizable oriented matroid.

The quotation marks are there because for a non-realizable oriented matroid it's not obvious that the "refinement" really is a common refinement of our two triangulations. Indeed, $R_M(L_1, L_2)$ is in general only defined as a poset, and it's not immediately obvious that this poset is even the face lattice of a regular cell complex. In Section 4.2 we give a way of looking at $R_M(L_1, L_2)$ that makes it clear that $R_M(L_1, L_2)$ really is a regular cell complex.

In Section 4.3 we show $R_M(L_1, L_2)$ is a common refinement of L_1 and L_2 if M is Euclidean. The property of being Euclidean is a purely combinatorial one. Thus from a purely combinatorial setup (a triangulation of a Euclidean oriented matroid) we derive a topological result (a PL sphere).

We do not yet have a proof that $R_M(L_1, L_2)$ is a common refinement of L_1 and L_2 in general. However, it is hard to imagine what else it might be. That a reasonable $R_M(L_1, L_2)$ can be defined at all is a heavy plausibility argument for Conjecture 4.0.1.

4.1 The Realizable Case

For a realizable oriented matroid we can see this refinement as follows: Imagine our oriented matroid is realized as a configuration of unit vectors in $\mathbf{R}^n$, and each triangulation L_i, $i = 1, 2$ is drawn on the unit sphere by sectors from great spheres. All these pieces of great spheres divide S^{n-1} into cells. These cells are regular (that is, the boundary of each cell is a sphere) since each cell is the intersection of two convex sets in S^{n-1}. Denote the resulting cell complex by $R_M(L_1, L_2)$. As a corollary to Proposition 3.1.2 we have:

Corollary 4.1.1 *For M a realizable oriented matroid and L_1 and L_2 as above, $R_M(L_1, L_2)$ is a common refinement of L_1 and L_2.*

A simplex σ of L_1 and a simplex τ of L_2 intersect to give a cell of $R_M(L_1, L_2)$ if and only if the convex hull of σ intersects the convex hull of τ on the unit sphere. That is, σ and τ intersect to give a cell if and only if there is some point p on the unit sphere that can be expressed as a linear combination in two ways:

$$p = \sum_{\vec{s} \in \sigma} \alpha_s \vec{s} = \sum_{\vec{t} \in \tau} \beta_t \vec{t} \quad \alpha_s, \beta_t \geq 0 \; \forall \vec{s}, \vec{t}.$$

Thus

$$\sum_{\vec{s} \in \sigma} \alpha_s \vec{s} - \sum_{\vec{t} \in \tau} \beta_t \vec{t} = 0 \quad \alpha_s, \beta_t \geq 0 \; \forall \vec{s}, \vec{t}.$$

Thus σ and τ give a cell of our refinement if and only if 0 is in the convex hull of $\sigma \cup (-\tau)$.

4.2 General Definition

Let's restate this last conclusion in pure oriented matroid language:

Definition 4.2.1 *If L_1 and L_2 are two triangulations of an oriented matroid M, then we define*

$$R_M(L_1, L_2) = \left\{ (\sigma, \tau) \in L_1 \times L_2 : \textit{either} \begin{array}{c} \sigma = \tau \textit{ or} \\ \hat{\sigma}^+ \hat{\tau}^- \textit{ is a vector of } M \textit{ for some} \\ \hat{\sigma} \subseteq \sigma, \hat{\tau} \subseteq \tau \end{array} \right\}.$$

This set is ordered by inclusion: $(\hat{\sigma}, \hat{\tau}) < (\sigma, \tau)$ if $\hat{\sigma} \subseteq \sigma$ and $\hat{\tau} \subseteq \tau$.

This definition makes sense even for non-realizable M. If M is not realizable then $R_M(L_1, L_2)$ is only defined as a poset, and it is not obvious even that $R_M(L_1, L_2)$ is a regular cell complex.

We can make a slicker definition using the **Lawrence construction**. (cf. [BLSWZ]) Given an oriented matroid M with elements E, we let M_D "M doubled") be the oriented matroid obtained from M by replacing each element x of E with two antiparallel elements x_1, x_2. Thus M_D has elements $E_1 \dot{\cup} E_2$, with $M_D(E_1) \cong M_D(E_2) \cong M$, and $x_1^+ x_2^+$ is a circuit of M_D for every $x \in M$. Then M_D^* is an acyclic oriented matroid, called the **Lawrence polytope** $\Lambda(M^*)$ associated to M^*.

Definition 4.2.2 *Let L_1, L_2 be two triangulations of M and let M_D be as described above. Consider L_1 as a triangulation of $M_D(E_1)$ and L_2 as a triangulation of $M_D(E_2)$. Then we define $R_M(L_1, L_2)$ to be the poset of covectors of $\Lambda(M^*)$ of the form $\sigma^+\tau^+$, where σ is a simplex of L_1 and τ is a simplex of L_2, ordered by inclusion.*

It is not hard to see that this definition is equivalent to the preceding definition.

It also makes sense to talk about refinements of more general simplicial complexes on the elements of M:

Definition 4.2.3 *Let L_1, L_2 be two simplicial complexes on subsets of the elements of M with all simplices independent in M, and let M_D be as described above. Consider L_1 as a simplicial complex on a subset of E_1, L_2 as a simplicial complex on a subset of E_2. Then we define $R_M(L_1, L_2)$ to be the poset of covectors of $\Lambda(M^*)$ of the form $\sigma^+\tau^+$, where σ is a simplex of L_1 and τ is a simplex of L_2, ordered by inclusion.*

To reinforce the intuition that $\sigma^+\tau^+$ represents a geometric intersection of σ and τ, we often denote a cell of the form $\sigma^+\tau^+$ by $\boldsymbol{\sigma}|\cap|\boldsymbol{\tau}$.

Note that if the covector $\sigma^+\tau^+$ is an element of $R_M(L_1, L_2)$, then any face of that covector is also in $R_M(L_1, L_2)$. This tells us:

Proposition 4.2.1 *$R_M(L_1, L_2)$ is a subcomplex of the covector complex of M_D^*.*

We summarize reassuring properties of $R_M(L_1, L_2)$ in the following proposition.

Proposition 4.2.2

1. *$R_M(L_1, L_2)$ is the face lattice of a regular cell complex.*
2. *If X and Y are two simplicial complexes (not necessarily disjoint) then $R_M(X \cup Y, L) = R_M(X, L) \cup R_M(Y, L)$.*
3. *The dimension of a cell $\sigma|\cap|\tau$ is $|\sigma| + |\tau| - \mathrm{rank}(M(\sigma \cup \tau)) - 1$.*
4. *If L_1 and L_2 are triangulations of M, then*
   - *If M is uniform then $R_M(L_1, L_2)$ is a pseudomanifold of pure dimension* $\mathrm{rank}(M) - 1$.
   - *If M is any oriented matroid then*
     - $\dim(R_M(L_1, L_2)) = \mathrm{rank}(M) - 1$.
     - *Every cell of dimension* $\mathrm{rank}(M) - 2$ *is contained in exactly two cells of dimension* $\mathrm{rank}(M) - 1$.
     - *Every cell of the form $\sigma|\cap|\sigma$ is in the boundary of some cell of dimension* $\mathrm{rank}(M) - 1$.

This last statement is somewhat unsatisfying – we certainly hope $R_M(L_1, L_2)$ is a pseudomanifold of pure dimension for any M. In the next section we shall see that it is a pseudomanifold of pure dimension if M is Euclidean.

Proof: The first and third statements are corollaries of the preceding proposition, and the second follows directly from the definitions.

To prove the fourth, we first prove that every cell $\sigma|\cap|\tau$ of dimension rank$(M)-2$ is contained in exactly two cells of dimension rank$(M)-1$.

There are three possibilities here: either $\sigma = \tau$ is a codimension one simplex, or $|\sigma| = \text{rank}(M)-1, |\tau| = \text{rank}(M)-2$, or vice-versa. In the latter two cases the result follows immediately from L_2 being a pseudomanifold. In the first case, we know σ is contained in exactly two maximal simplices $\sigma \cup \{s_1\}$ and $\sigma \cup \{s_2\}$ of L_1 and τ is contained in exactly two maximal simplices $\tau \cup \{t_1\}$ and $\tau \cup \{t_2\}$ of L_2. The recursive axiom for oriented matroid triangulations implies that in the rank 1 oriented matroid M_D/σ we have circuits $s_1^+s_2^+$ and $t_1^+t_2^+$. Thus we may assume $s_1^+t_1^+$ is a circuit in M_D/σ. Composing the corresponding vector in M_D with the vector $\sigma^+\tau^+$, we get the vector $(\sigma\cup\{s_1\})^+(\tau\cup\{t_1\})^+$ in M_D, and hence the cell $(\sigma\cup\{s_1\})|\cap|(\tau\cup\{t_1\})$ in $R_M(L_1, L_2)$. Similarly, $(\sigma\cup\{s_2\})|\cap|(\tau\cup\{t_2\})$ is a cell. Because $s_1^+t_2^+$ is not a vector of M_D/σ, we see $(\sigma\cup\{s_1\})|\cap|(\tau\cup\{t_2\})$ is not a cell of $R_M(L_1, L_2)$, and similarly $(\sigma\cup\{s_2\})|\cap|(\tau\cup\{t_1\})$ is not a cell. So $\sigma|\cap|\tau$ is contained in exactly two top-dimensional cells.

The third statement of the proposition tells us no cell has dimension greater than rank$(M)-1$. Now we show any cell of $R_M(L_1, L_2)$ of the form $\hat\sigma|\cap|\hat\sigma$ (for instance, a cell corresponding to a 0-simplex) is contained in a rank$(M)-1$-dimensional cell. Thus $R_M(L_1, L_2)$ has dimension exactly rank$(M)-1$.

Let $\hat\sigma|\cap|\hat\sigma$ be a cell of $R_M(L_1, L_2)$. Order the elements of $\hat\sigma$ somehow: $\hat\sigma = \{s_1, s_2, \ldots, s_k\}$. Let $\sigma = \hat\sigma \cup \{s_{k+1}, \ldots, s_n\}$ be a maximal simplex of L_1. Let $\omega = \hat\sigma \cup \{w_{k+1}, \ldots, w_n\}$ be a maximal simplex of L_2. We now extend M with two lexicographic extensions: let $M' = M \cup \{s_\sigma, s_\omega\}$, where s_σ is the lexicographic extension by $[s_1^+s_2^+\ldots s_n^+]$ (so $s_\sigma \in \text{conv}(\sigma)$) and s_ω is the lexicographic extension by $[s_1^+s_2^+\ldots s_k^+w_{k+1}^+\ldots w_n^+]$ (so $s_\omega \in \text{conv}(\omega)$).

Now, $R_{M'}(\{s_\sigma, s_\omega\}, L_2)$ has a 1-cell $\{s_\sigma, s_\omega\}|\cap|\omega$. By Lemma A.6 and our observations above on codimension one cells, we see that the connected component of $R_{M'}(\{s_\sigma, s_\omega\}, L_2)$ containing $\{s_\sigma, s_\omega\}|\cap|\omega$ is a one-dimensional pseudomanifold with boundary. One boundary point is $s_\omega|\cap|\omega$. The only possible other boundary point is $s_\sigma|\cap|\tau$, where τ is some maximal simplex of L_2. Well, τ is a really nice simplex: since M'_D has vectors $\tau^+s_\sigma^+$ and $s_\sigma^-\sigma^+$, vector elimination gives a vector $\tau^+\sigma^+$, and thus a cell $\sigma|\cap|\tau$ in $R_M(L_1, L_2)$.

We claim $\hat\sigma|\cap|\hat\sigma$ is in the boundary of $\sigma|\cap|\tau$. We know $\hat\sigma \subseteq \sigma$, so we need only show $\hat\sigma \subseteq \tau$. We'll show by induction on i that $s_i \in \tau$ for every $i \le k$.

If $i = 1$, then consider $R_M(\{s_\sigma, s_1\}, L_2)$. This has a 1-cell $\{s_\sigma, s_1\}|\cap|\tau$, and so either $s_1 \in \tau$ or there's a 0-cell $\{s_\sigma, s_1\}|\cap|\hat\tau$, where $\hat\tau \subset \tau$. Since by Lemma A.7 s_σ and s_1 are contravariant, the latter idea is impossible, and so $s_1 \in \tau$.

Now assume the result for $i-1$, and consider the oriented matroid

$$M_i = (M'/\{s_1,\ldots,s_{i-1}\})(\mathrm{link}_{L_2}(\{s_1,\ldots,s_{i-1}\} \cup s_\sigma)$$

and the simplicial complex

$$\mathrm{link}_{L_2}(\{s_1,\ldots,s_{i-1}\}),$$

which is a triangulation of $M_i \backslash s_\sigma$. Both $\{s_i\}$ and $\tau\backslash\{s_1,\ldots,s_{i-1}\}$ are simplices of this triangulation, by the induction hypothesis. The elements s_i and s_σ are contravariant in M_i, and so $s_\sigma \in \mathrm{conv}(\tau)$ in M' implies $s_i \in \mathrm{conv}(\tau\backslash\{s_1,\ldots,s_{i-1}\})$ in M_i. By the convexity axiom for triangulations, this implies $s_i \in \tau$.

So, now we know that every cell of $R_M(L_1, L_2)$ of the form $\sigma|\cap|\sigma$ is contained in a cell of dimension $\mathrm{rank}(M)-1$. In particular, the existence of cells of the form $\{x\}|\cap|\{x\}$ promises that the dimension of $R_M(L_1, L_2)$ really is $\mathrm{rank}(M)-1$.

If M is uniform, we know that any cell $\sigma|\cap|\tau$ with $\sigma \neq \tau$ is contained in a cell $\bar{\sigma}|\cap|\bar{\tau}$, where $\bar{\sigma}$ and $\bar{\tau}$ are maximal simplices. So if M is uniform we can conclude that $R_M(L_1, L_2)$ is a pseudomanifold of pure dimension $\mathrm{rank}(M)-1$.

QED

To prove the PL Conjecture it remains to show that $R_M(L_1, L_2)$ is a common refinement of L_1 and L_2 for any two triangulations L_1, L_2 of M. Note that by the previous proposition it would suffice to show that $R_M(\sigma, L)$ is a refinement of σ for any single simplex σ and any triangulation L. As our first piece of evidence for this we offer the following propositions.

Lemma 4.2.1 *Let σ and τ be two independent sets in M. If $x \in \mathrm{conv}(\sigma)$ and Σ is the stellar subdivision of σ by x, then $R_M(\Sigma, \tau)$ is a PL refinement of $R_M(\sigma, \tau)$.*

Proof: We consider two cases:

1. If $R_M(\sigma, \tau) = \emptyset$, then we want to show $R_M(\Sigma, \tau) = \emptyset$. Assume by way of contradiction that $R_M(\Sigma, \tau)$ has a cell $\hat{\sigma}|\cap|\hat{\tau}$. Then certainly $x \in \hat{\sigma}$, and $\Lambda(M^*)$ has a covector $\hat{\sigma}^+\hat{\tau}^+$. But $x \in \mathrm{conv}(\sigma)$ implies that $\Lambda(M^*)$ has a covector σ^+x^-. Thus, by the vector elimination axiom for oriented matroids, $\Lambda(M^*)$ has a covector $\sigma^+\hat{\tau}^+$, a contradiction.

2. If $R_M(\sigma, \tau) \neq \emptyset$, then we want to show $R_M(\Sigma, \tau)$ is a PL ball. We know that $\Lambda(M^*)$ has a covector $x^+\sigma^+\tau^+$. In the pseudosphere picture of $\mathcal{V}^*(\Lambda(M^*))$, this covector is a PL ball with boundary $R_M(\sigma, \tau) \cup R_M(\Sigma, \tau) \cup \{x^+\hat{\sigma}^+\hat{\tau}^+ | \hat{\sigma} \subseteq \sigma, \hat{\tau} \subseteq \tau\}$. Since $R_M(\sigma, \tau)$ is a ball (the closure of a single cell), we induct on $|\tau|$ to see that $R_M(\Sigma, \tau)$ is the complement of a PL ball in the boundary of the ball $x^+\sigma^+\tau^+$, and thus is a PL ball.

QED

Proposition 4.2.3 *If L is any triangulation of a uniform oriented matroid M and S is a triangulation of M constructed by a sequence of stellar subdivisions as in Theorem 3.2.1, then $R_M(L, S)$ is a refinement of L.*

Proof: Consider the sequence $S_1, S_2, \dots S_k = S$ of spheres constructed in Theorem 3.2.1. We show that $R_M(\tau, S_1)$ is a refinement of τ for any $\tau \in L$. Lemma 4.2.1 then gives the inductive step to prove $R_M(\tau, S_i)$ is a refinement of τ for every i.

We know $(S_1^0)^+$ is an element of $\mathcal{V}^*(\Lambda(M^*))$. Also, since every element of τ is in the convex hull of some face of S_1^0, we know that for every $t \in \tau$ there is a $\sigma \subset S_1^0$ such that $\sigma^- t^+$ is an element of $\mathcal{V}^*(\Lambda(M^*))$. Composing these covectors, we see that $(S_1^0)^+\tau^+$ is an element of $\mathcal{V}^*(\Lambda(M^*))$. The boundary of this cell in the pseudosphere picture is a sphere. The only faces of this sphere not in $R_M(\tau, S_1)$ are those which are non-zero on every element of S_1^0. Thus the complement of $R_M(\tau, S_1)$ in this sphere is the star of the 0-cell $(S_1^0)^+$ in $\partial(S_1^0)^+\tau^+$ and thus is a ball. So $R_M(\tau, S_1)$ is a ball as well.

QED

4.3 For a Euclidean Oriented Matroid

In this section we prove the PL Conjecture for Euclidean oriented matroids, i.e., oriented matroids in which all programs are Euclidean. (Note that this includes all realizable oriented matroids and all rank 3 oriented matroids.)

Lemma 4.3.1 *If M is an oriented matroid, L is a triangulation of M, $\sigma = \{f, g\} \subset E$ is independent in M, and (M, f, g) is a Euclidean oriented matroid program, then $R_M(\sigma, L)$ is a refinement of σ.*

Proof: First we want to deal with the case that some 0-cell in $R_M(\sigma, L)$ is not of the form $\sigma|\cap|\tau$, with τ a codimension 1 simplex of L. In this case we'll perturb L to get rid of this cell.

Let ω be a simplex of L not of codimension 1 such that $\sigma|\cap|\omega$ is a 0-cell in $R_M(\sigma, L)$. Let t_0 be an element of ω and let $\tau = \{t_0, t_1, \dots t_k\}$ be a codimension 1 simplex of L containing ω. We perturb M by replacing t_0 with the lexicographic extension t_0' by $[t_0^+ t_1^- t_2^- \dots t_k^-]$. Let M' be the resulting oriented matroid. Let L' be the simplicial complex obtained from L by replacing t_0 with t_0'. Then Lemma 3.2.1 tells us L' is a triangulation of M'.

Let $\tau' = (\tau \cup t_0')\backslash t_0$. We then check that $\sigma|\cap|\tau'$ is a cell of $R_{M'}(\sigma, L')$. This is true because $t_0 \in \mathrm{conv}(\tau')$, and so composing the vectors $\sigma^+\omega^+$ and $t_0^-(\tau')^+$ in $(M \cup t_0')_D$, we get a vector $\sigma^+(\tau')^+$.

Now note $R_M(\sigma, L)$ is obtained from $R_{M'}(\sigma, L')$ by replacing $R_{M'}(\sigma, \mathrm{star}_{L'}(t_0'))$ with $R_M(\sigma, \mathrm{star}_L(t_0))$. It's then easy to check that the complex $\{\sigma^+\tau^+t_0^+ : \sigma^+\tau^+ \in R_{M'}(\sigma, \mathrm{star}_{L'}(t_0'))$ collapses to $R_M(\sigma, \mathrm{star}_L(t_0))$ by a sequence of elementary colapses through the cells $R_{M'}(\sigma, \mathrm{star}_{L'}(t_0))$. Thus $R_M(\sigma, L)$ is a retract of $R_{M'}(\sigma, L')$. By replacing t_0 with t_0' we've created no new "bad" 0-cells. Thus by a sequence of perturbations we can remove all bad 0-cells from $R_M(\sigma, L)$ to get a refinement $R_{M''}(\sigma.L'')$ which retracts to $R_M(\sigma, L)$. Below we will show that $R_{M''}(\sigma, L'')$ is a

PL 1-ball. So $R_M(\sigma, L)$ is a 1-dimensional retract of a PL 1-ball, and hence is also a 1-ball.

So, assuming all our 0-cells come from codimension 1 simplices:

We induct on the number of cells of $R_M(\sigma, L)$. Proposition 4.2.2 tells us $R_M(\sigma, L)$ has at least one 1-cell. If $R_M(\sigma, L)$ has more than one 1-cell, then it has a 0-cell of the form $\sigma|\cap|\tau$.

Since (M, f, g) is Euclidean, we can take a single-element extension $M' = M \cup x$ such that $x \in \mathrm{conv}(\sigma) \cap \mathrm{conv}(\tau)$. (See Figure 4-1.) By Lemma 4.2.1, it then suffices

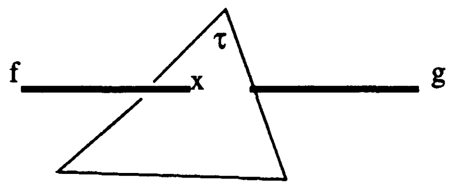

Figure 4-1:

to show that $R_{M'}(\{f,x\}, L) \cup R_{M'}(\{x,g\}, L)$ is a PL 1-ball.

Let $\tau_1 = \tau \cup \{t_1\}$ be one of the two maximal simplices of L containing τ. Then $\sigma|\cap|\tau_1$ is a cell of $R_M(\sigma, L)$, and so by Lemma 4.2.1 either $\{f,x\}|\cap|\tau_1$ is a cell or $\{x,g\}|\cap|\tau_1$ is a cell. Assume $\{f,x\}|\cap|\tau_1$ is a cell, so that $f^+x^+\tau_1^+$ is a vector of M'_D. Then in the rank 1 oriented matroid M'_D/τ, we have circuits $f^+t_1^+$ and f^+g^+, and thus we have the third vector $t_1^-g^+$. This is in fact a circuit since $g \cup \tau$ and $t_1 \cup \tau$ are independent sets in M. Thus in M'_D we have no vector $g^+t_1^+\tau^+$, and so $\{x,g\}|\cap|\tau_1$ is not a cell. Thus $R_M(\{x,g\}, L)$ has fewer cells than $R_M(\sigma, L)$, and so by induction it's a 1-ball.

Now consider the other simplex $\tau_2 = \tau \cup \{t_2\}$ of L containing τ. Certainly $\sigma|\cap|\tau_2$ is a cell of $R_M(\sigma, L)$. Assume by way of contradiction that both $\{f,x\}|\cap|\tau_1$ and $\{f,x\}|\cap|\tau_2$ are cells. Then in M'_D/τ we get vectors $f^+t_1^+$ and $f^+t_2^+$, and thus a vector $t_1^+t_2^-$. But this last vector is also a vector of M'/τ, contradicting the recursive axiom for oriented matroid triangulations. Thus $R_M(\{f,x\}, L)$ also has fewer cells than $R_M(\sigma, L)$, and so by induction it's a 1-ball.

Finally, we note that these two 1-balls have exactly one endpoint in common and no interior points in common (since their union is a pseudomanifold). Thus $R_M(\sigma, L)$ is a 1-ball.

QED

While we've been focusing on the PL Conjecture as the foremost property triangulations should satisfy, it would also be nice to know that Proposition 3.1.2 generalizes to triangulations of arbitrary oriented matroids. In the next proposition we use the notion of refinements to generalize Proposition 3.1.2 to oriented matroids with Euclidean extensions. We will use this fact in the proof of Theorem 4.3.1.

Proposition 4.3.1 *If L is a triangulation of M and $M' = M \cup x$ is a single-element extension of M such that for some element g of M the oriented matroid program (M, g, x) is Euclidean, then x is in the interior of the convex hull of exactly one simplex of L.*

Proof: Let $\sigma = \{g, x\}$. Then the same argument as in the proof of Lemma 4.3.1 says that $R_{M'}(\sigma, L)$ is a path from g to x. One end of this path is a cell $x|\cap|\omega$. This ω is the unique simplex of L with x in the interior of its convex hull.

QED

Theorem 4.3.1 *If M is an oriented matroid, L is a triangulation of a subset of M of rank* rank(M), *$\sigma \subset E$ is independent in M, and (M, f, g) is a Euclidean oriented matroid program for every $f, g \in \sigma$, then $R_M(\sigma, L)$ is a refinement of σ.*

Proof: We induct on rank(M). If rank$(M) = 1$ then $L = S^0$ and all is obvious.

So assume we know our result for rank$(M) - 1$. We first use this hypothesis to show that $R_M(\sigma, L)$ is a PL manifold. That is, we will show that the star of a cell $\hat{\sigma}|\cap|\tau$ in $R_M(\sigma, L)$ is a PL ball.

Let $\hat{\sigma}|\cap|\tau$ be a cell of $R_M(\sigma, L)$. Then its star is the join of $\hat{\sigma}|\cap|\tau$ with $\text{link}_{R_M(\sigma,L)}(\hat{\sigma}|\cap|\tau)$. So it suffices to show this link is a PL ball or a PL sphere. We will find a PL ball or sphere in $R_{M/\tau}(\Sigma, \text{link}_L(\tau))$, for some Σ, which is a refinement of this link.

Let σ_1 be a basis for $(M/\tau)(\hat{\sigma})$. Extend this to a basis $\sigma_1 \cup \sigma_2$ for $(M/\tau)(\sigma)$. Extend M/τ by adding the negatives of all elements of σ_1. Let σ_1' be the set of all these negatives. Let Σ be the the simplicial complex consisting of all independent subsets of $\sigma_1 \cup \sigma_1' \cup \sigma_2$. This is the join of a generalized octahedron with a $(|\sigma_2|-1)$-simplex, and hence is a PL sphere (if $|\sigma_2| = \emptyset$) or a PL ball. So by our induction hypothesis $R_{M/\tau}(\Sigma, \text{link}_L(\tau))$ is a PL sphere or PL ball. It's not hard to check that $R_{M/\tau}(\Sigma, \text{link}_L(\tau))$ is a PL refinement of $\text{link}_{R_M(\sigma,L)}(\hat{\sigma}|\cap|\tau)$.

Now, to show $R_M(\sigma, L)$ is a ball, we induct on $|\sigma|$. If $|\sigma| = 1$ then Proposition 4.3.1 tells us $R_M(\sigma, L)$ is a point. The case $|\sigma| = 2$ is covered by the same argument as in Lemma 4.3.1.

For $|\sigma| > 2$ we rely on Lemma A.8 in Appendix. This lemma tells us that for any collection of cocircuits $\{T_1, T_2, \ldots, T_j\}$ there exists an extension $(M \cup \{x_1, x_2, \ldots, x_j\}, g, f)$ of (M, g, f) so that each x_i is parallel to f and goes through T_i.

The idea is: using this lemma, we'll choose two elements $f, g \in \sigma$ and slice $R_M(\sigma, L)$ into a sequence of thin wedges $R_{M'}(\sigma_i, L)$ by stellar subdivisions along the edge $|\{f, g\}|$. Then we'll show each wedge collapses to one of its faces, so by our induction hypothesis $R_{M'}(\sigma_i, L)$ is a refinement of σ_i. Finally, we note that the union of these wedges is a PL ball, so by Lemma 4.2.1 we know $R_M(\sigma, L) \cong \cup_i R_{M'}(\sigma_i, L)$ is a PL ball.

Now, for the details:

Assume we know our result for $|\sigma| - 1$. If $\hat{\sigma}|\cap|\tau$ is a 0-cell of $R_M(\sigma, L)$ for $f, g \in \hat{\sigma} \subseteq \sigma$ and $|\tau| \leq \text{rank}(M) - 2$, we define a rank $(\text{rank}(M) - 1)$ set $H(\tau)$ by:

- If $\text{rank}(M(\tau \cup (\hat{\sigma}\backslash\{f, g\}))) = \text{rank}(M) - 1$ then let $H(\tau) = \tau \cup (\hat{\sigma}\backslash\{f, g\})$. (This will be the case if M is uniform, for instance.)
- If not, choose some $T \subset E$ such that $\text{rank}(M(\tau \cup (\sigma\backslash\{f, g\}) \cup T)) = \text{rank}(M) - 1$, $\text{rank}(M(\tau \cup (\sigma\backslash\{f\}) \cup T)) = \text{rank}(M)$, and $\text{rank}(M(\tau \cup (\sigma\backslash\{g\}) \cup T)) = \text{rank}(M)$, and let $H(\tau) = \tau \cup (\sigma\backslash\{f, g\}) \cup T$.

We first check that every extension of (M, g, f) parallel to f through $H(\tau)$ is in $\text{conv}(\{f, g\})$:

If x is parallel to f in (M, g, f), then either $x^+f^-g^-$ or $x^+f^-g^+$ is a circuit of M. Assume by way of contradiction $x^+f^-g^+$ is a circuit of M. Then in $M/H(\tau)$, the signed set $x^+f^-g^+$ is a vector. But also x^- is a vector in $M/H(\tau)$, and so f^-g^+ is a vector of $M/H(\tau)$. By the independence conditions we put in our choice of $H(\tau)$, we know this is in fact a circuit. But we have a cell $\hat{\sigma}|\cap|\tau$, where $f, g \in \hat{\sigma}$ and $\tau, \hat{\sigma}\backslash\{f, g\} \subset H(\tau)$, and so the vector $\hat{\sigma}^+\tau^-$ in M gives a circuit f^+g^+ in $M/H(\tau)$, a contradiction. Thus the extension x is in $\text{conv}(\{f, g\})$.

We now extend (M, g, f) by extensions through all our $H(\tau)$. (See figure 4-2.) Let M' be the resulting oriented matroid.

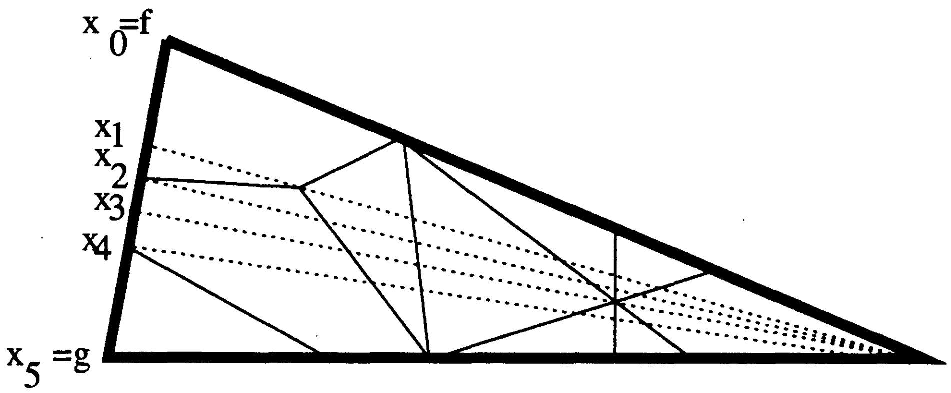

Figure 4-2: Slicing up $R_M(\sigma, L)$.

This gives an ordered sequence of extensions $f = x_{0,1}, \ldots, x_k = g$ as shown in Figure 4-2. (We know this sequence is ordered because it gives a refinement of the 1-simplex $\{f, g\}$.) Now consider the set of simplices $\sigma_i = (\sigma\backslash\{f, g\}) \cup \{x_i, x_{i+1}\}$, for $0 \leq i < k$.

Assume by way of contradiction that $R_{M'}(\sigma_i, L)$ has a 0-cell of the form $\hat{\sigma}_i|\cap|\tau$, where $x_i, x_{i+1} \in \hat{\sigma}_i \subseteq \sigma_i$. Then since x_i and x_{i+1} are in $\text{conv}(\{f, g\})$, we get a 0-cell $\hat{\sigma}|\cap|\tau$ in $R_M(\sigma, L)$, with $f, g \in \hat{\sigma}$. So consider the $H(\tau)$ we chose earlier. An argument like our earlier one shows that the extension x given by $H(\tau)$ comes between x_i and x_{i+1} in our order along the edge $||\{f, g\}||$, a contradiction. So, just as in Figure 4-2, each of our cell complexes $R_{M'}(\sigma_i, L)$ has no 0-cells in its interior.

To make things really easy from here, we'll take even thinner wedges. For each $x_i \neq g$, we extend M' lexicographically by $[x_i^+ g^+]$ to get a new element y_i. (See Figure 4-3.) Let M'' denote the resulting oriented matroid.

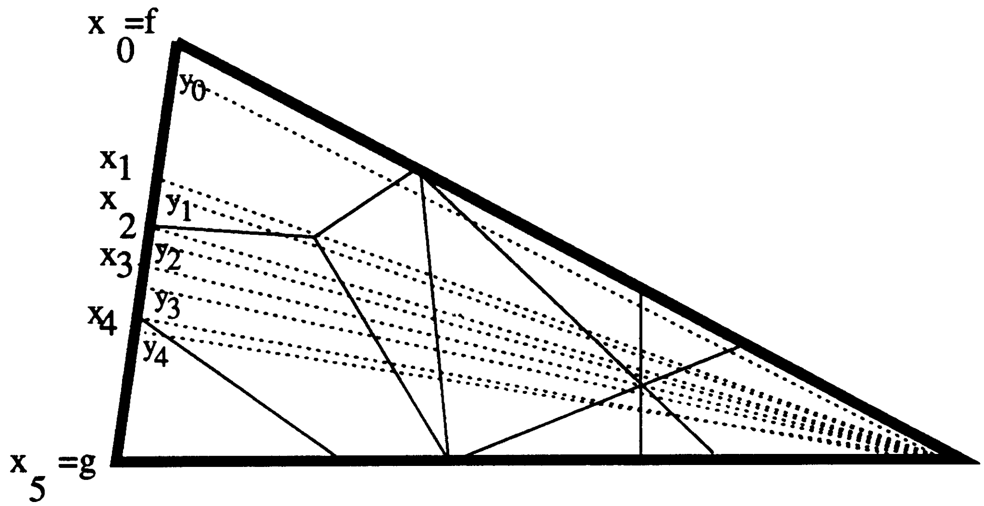

Figure 4-3: Slicing $R_M(\sigma, L)$ even finer

Each wedge σ_i has been divided into $\alpha_i = (\sigma_i \backslash x_i) \cup \{y_i\}$ and $\beta_i = (\sigma_i \backslash x_{i-1}) \cup \{y_i\}$. Our thinner wedges will again have no 0-simplices in their interior.

Now consider any maximal cell $\alpha_i |\cap| \tau$ of $R_{M''}(\alpha_i, L)$. This cell must have a codimension one face in $R_{M''}(\alpha_i \backslash x_{i-1}, L)$ and some face in $R_{M''}(\alpha_i \backslash y_i, L)$. Thus this cell collapses to its face in $R_{M''}(\alpha_i \backslash y_i, L)$. So $R_{M''}(\alpha_i, L)$ collapses to $R_{M''}(\alpha_i \backslash y_i, L)$. Similarly, $R_{M''}(\beta_i, L)$ collapses to $R_{M''}(\beta_i \backslash y_i, L)$. By our induction hypothesis, the cell complexes $R_{M''}(\alpha_i \backslash y_i, L)$ and $R_{M''}(\beta_i \backslash y_i, L)$ are contractible. Thus $R_{M''}(\alpha_i, L)$ and $R_{M''}(\beta_i, L)$ are contractible. It is a standard result in PL topology (cf. [RS]) that a contractible PL manifold with boundary is a ball. Thus $R_{M''}(\alpha_i, L)$ and $R_{M''}(\beta_i, L)$ are PL refinements of α_i and β_i, respectively.

Putting these wedges together, we see $\cup_{1 \leq i < k}(R_M(\alpha_i, L) \cup R_M(\beta_i, L)) = R_M(\cup_{1 \leq i < k} \alpha_i \cup \beta_i, L)$ is a refinement of $\cup(\alpha_i \cup \beta_i)$. Now note that $\cup(\alpha_i \cup \beta_i)$ is obtained from σ by a sequence of stellar subdivisions, so Lemma 4.2.1 tells us $R_M(\sigma, L)$ is a PL refinement of σ.

QED

Corollary 4.3.1 *The PL Conjecture holds for all Euclidean oriented matroids.*

Chapter 5

Connection with Matroid Polytopes

Las Vergnas in [LV1] defined a **matroid polytope** to be an acyclic oriented matroid in which all elements are extreme. In particular, the vertices of a convex polytope in affine space define a matroid polytope. Matroid polytopes are the most natural context to study certain aspects of the theory of convex polytopes, such as Gale transforms. Our definition of a triangulation of an oriented matroid suggests a similar definition for a triangulation of a matroid polytope. In fact, we will make a definition for a triangulation of any totally cyclic oriented matroid.

Definition 5.0.1 *If M is a rank n acyclic oriented matroid with elements E, a* **polytope triangulation** *of M is an $(n-1)$-dimensional simplicial complex L such that*

- $L^0 = E$.
- *L is a pseudomanifold with boundary.*
- *If ω is a simplex of L then ω is independent in M and* $\operatorname{conv}(\omega) = \omega$ *in M.*
- *Either*
  - $n = 1$, *or*
  - $n > 1$, *and*
    - *if ω is a simplex of ∂L, then* $\operatorname{link}_L(\omega)$ *is a polytope triangulation of* $(M/\omega)(\operatorname{link}_L(\omega)^0)$.
    - *if ω is a simplex in $L - \partial L$, then* $\operatorname{link}_L(\omega)$ *is a matroid triangulation of* $(M/\omega)(\operatorname{link}_L(\omega)^0)$.

In [BM] Billera and Munson gave another definition of a triangulation of a matroid polytope, in terms of single-element extensions of the oriented matroid. From Proposition 4.3.1 we can see that the two definitions are equivalent for oriented matroids in which all extensions are Euclidean. By stating our definition only in terms of the vectors of the oriented matroid, we avoid the usual anxieties associated with single-element extensions of oriented matroids (c.f. [MR]).

Proposition 5.0.1 *Every uniform acyclic oriented matroid M has a polytope triangulation which is a PL ball.*

Proof: Take a single-element extension $M \cup p$ of M such that $M \cup p$ is totally cyclic. Order the elements of $M \cup p$ as $\{p, e_1, \ldots, e_a, f_1, \ldots, f_b\}$, where $\{e_1, \ldots, e_a\}$ is the set of extreme elements of M and $\{f_1, \ldots, f_b\}$ are all other elements of M. Then the proof of Theorem 3.2.1 gives an algorithm for producing a sequence of simplicial complexes $S_0, S_2, \ldots, S_K$ such that each S_i is a PL sphere, S_i is a matroid triangulation of the first $\text{rank}(M) + i$ elements of $M \cup p$, and S_K is a matroid triangulation of $M \cup p$. It's then not hard to check that $S_i - \text{star}_{S_i}(p)$ is a PL ball which is a polytope triangulation of the first $\text{rank}(M) + i - 1$ elements of M.

QED

Note that this proof shows that to every ordering of the elements of a matroid polytope there is an associated triangulation. It turns out that this association is the same as the one described by Billera and Munson in [BM] for their notion of triangulation.

Similarly, we can describe a "common refinement" of any two polytope triangulations and show that it is a regular cell complex. One would expect that any proof to our conjecture that any triangulation of a totally cyclic oriented matroid is a sphere would also prove that any polytope triangulation is a ball.

As a corollary to the results in the previous chapter we have:

Corollary 5.0.1 *If M is an acyclic Euclidean oriented matroid, then any polytope triangulation of M is a PL ball.*

Chapter 6

Conjectures

Conjecture 6.0.1 *Let M be an oriented matroid. A simplicial complex L is a triangulation of M iff:*

- $L^0 = E$.
- *L is a pseudomanifold.*
- *Each simplex of L is independent in M.*
- *If σ and τ are two simplices of L, then $\sigma^+\tau^-$ is not a vector of M.*

Indeed, I'd be tempted to use this as the definition of a triangulation, if I could show that it described boundaries of stars in CD manifolds.

Note this conjecture implies that our definition of triangulations is at least as strong as the definition of Billera and Munson's mentioned on page 31.

Recalling Proposition 4.3.1, we also conjecture its generalization to arbitrary oriented matroids:

Conjecture 6.0.2 *If L is a triangulation of M then every single-element extension of M is in interior of the convex hull of a unique simplex of L.*

This is kind of a bizarre conjecture in light of results that the topological space of single-element extensions of an oriented matroid need not be a sphere (cf. [MR], in which examples are given of oriented matroids with disconnected extension spaces). An argument similar to the proof of Proposition 4.3.1 shows that the previous conjecture implies this one.

Recall our result from Section 3.2 that every uniform totally cyclic oriented matroid has a triangulation. This suggests the conjecture:

Conjecture 6.0.3 *Every totally cyclic oriented matroid appears in some CD manifold*

That is, we would like to know that every oriented matroid extends to a differential structure on some CD manifold. To see the difficulty of this conjecture, we note the result of Richter-Gebert([RG]) that there exist oriented matroids with "isolated" elements – elements which cannot be perturbed at all. Thus it's not clear that we can construct the "smoothly varying" oriented matroids we need for a CD manifold containing such an obstreperous oriented matroid.

It is known (cf. [K]) that not every PL manifold can be given a differential structure. However, the CD analog is open:

Conjecture 6.0.4 *Every PL manifold can be given the structure of a CD manifold.*

If this were true, then the theory of CD manifolds would give new tools for studying PL topology.

Appendix A

Oriented Matroids

For a more complete exposition on oriented matroids, see [BLSWZ]. This appendix summarizes those results from [BLSWZ] which we use in this thesis.

Let E be a finite set. A **signed subset** of E is a function $X : E \to \{-, 0, +\}$. If X and Y are two signed subsets of E, define their **composition** $X \circ Y$ to be

$$X \circ Y(e) = \begin{cases} X(e) & \text{if } X(e) \neq 0 \\ Y(e) & \text{otherwise} \end{cases}$$

Write $\underline{X}$ for the support of X, X^- for $X^{-1}(-)$, and X^+ for $X^{-1}(+)$. If X is a signed set, $\sigma = X^-$, and $\tau = X^+$, then we often denote X by $\sigma^-\tau^+$.

For any two sets S and T write $S \backslash T$ for $\{e | e \in S, e \notin T\}$.

Definition A.1 *(From [BLSWZ]) An* **oriented matroid** *is a finite set E together with a collection $\mathcal{V}$ of signed subsets of E such that*

1. $0 \in \mathcal{V}$,

2. *(symmetry)* $\mathcal{V} = -\mathcal{V}$,

3. *(composition)If* $X, Y \in \mathcal{V}$ *then* $X \circ Y \in \mathcal{V}$,

4. *(vector elimination) For all* $X, Y \in \mathcal{V}$ *and* $e \in X^+ \cap Y^-$ *there is a* $Z \in \mathcal{V}$ *such that*
$$Z^+ \subseteq (X^+ \cup Y^+) \backslash e,$$
$$Z^- \subseteq (X^- \cup Y^-) \backslash e,$$
and $(\underline{X} \backslash \underline{Y}) \cup (\underline{Y} \backslash \underline{X}) \cup (X^+ \cap Y^+) \cup (X^- \cap Y^-) \subseteq \underline{Z}$.

The elements of E are called the **elements** *of the oriented matroid. The elements of $\mathcal{V}^*$ are called the* **vectors** *of the oriented matroid.*

The motivating example: Consider a finite set $E = \{x_1, x_2, \ldots x_k\}$ of vectors in $\mathbf{R}^n$. For any non-trivial equality $\sum a_i \vec{x_i} = 0$ we get a signed set $X : [k] \to \{-, 0, +\}$ given by $X(i) = \text{sign}(a_i)$. For a given E, the collection of all such signed sets is the set of **vectors** of an oriented matroid.

Definition A.2 *Any oriented matroid which arises in this way is called* **realizable**. *Any $E \in \mathbf{R}^n$ which gives the oriented matroid M is called a* **realization** *of M.*

A.1 Lots of Terminology, A Few Lemmas

Let $M = (E, \mathcal{V})$ be an oriented matroid.

Definition A.3 *A* **circuit** *of M is a minimal non-0 vector of M.*

Note: It is more common to use "circuits" to denote the minimal dependent sets of an ordinary matroid and to use "signed circuits" for the minimal vectors of an oriented matroid. Since ordinary matroids never appear in this thesis, we're letting it slide.

We say a composition $X \circ Y$ is **conformal** if $X(e)Y(e) \geq 0$ for every $e \in E$. We will use the following lemma from [BLSWZ].

Lemma A.1 *Any vector X of an oriented matroid is a composition of circuits conforming to X.*

Definition A.4 *A subset I of E is* **independent** *if no subset of E is the support of a vector of M. I is a* **basis** *for M if I is an independent set of maximal order. The* **rank** *of M is the order of a basis for M.*

Definition A.5 *Let $A \subseteq E$ and $x \in E$. The* **convex hull** *of A is the set* $\mathrm{conv}_M(A) = A \cup \{x \in E :$ *there is a subset B of A such that B^+x^- is a circuit of M*$\}$

Definition A.6 *Two elements x, y of M are* **parallel** *if x^+y^- is a circuit of M. They are* **antiparallel** *if x^+y^+ is a circuit of M.*

Definition A.7 *M is* **simple** *if there is no $e \in E$ such that $e^+ \in \mathcal{V}$.*

Definition A.8 *M is* **uniform** *if every maximal independent subset of E has the same order.*

Let $A \subset E$.
If $X : E \to \{-, 0, +\}$ is a signed set and $A \subset E$, we denote $X|_{E \setminus A}$ by $X \setminus A$.

Definition A.9 *The* **deletion** *$M \setminus A$ of A from M is the oriented matroid $(E \setminus A, \mathcal{V} \setminus A)$, where*

$$\mathcal{V} \setminus A = \{X \setminus A : X \in \mathcal{V} \text{ and } X(A) = 0\}.$$

We will also denote $M \setminus A$ by $M(E \setminus A)$.

Definition A.10 *Let $M = (E, \mathcal{V})$ be an oriented matroid, and let $A \subseteq E$. The* **contraction of** *M* **by** *A is the oriented matroid with elements $E \setminus A$ and vectors $\{X \setminus A : X \in \mathcal{V}\}$.*

Definition A.11 *M is* **acyclic** *if $\mathcal{V}$ contains no positive vector. M is* **totally cyclic** *if $\mathcal{V}$ contains a positive vector of rank* $\text{rank}(M)$.

If $E \in \mathbf{R}^n$ is a realization of M, and $A \subseteq E$, then:

- $E \backslash A$ is a realization of $M \backslash A$, and
- If $\pi : \mathbf{R}^n \to \mathbf{R}^n / \langle A \rangle$ is the orthogonal projection, then $\pi(E \backslash A)$ is a realization of M/A.
- M is acyclic iff E is contained in some open half-space of $\mathbf{R}^n$. M is totally cyclic iff E is not contained in any closed half-space of $\mathbf{R}^n$.

Definition A.12 *If M is acyclic, then $x \in E$ is* **extreme** *if $x \notin \text{conv}(E \backslash x)$.*

A.2 Strong Maps and Weak Maps

Definition A.13 *Let M_1 and M_2 be two oriented matroids on the same ground set E. Then we say there is a* **strong map** *from M_1 to M_2, denoted $M_1 \to M_2$, if every vector of M_1 is a vector of M_2.*

Strong maps are the oriented matroid analog to linear maps. If $M = (E, \mathcal{V})$ is an oriented matroid, and $A \subset E$, then $M(E \backslash A) \to M/A$.

We also have an oriented matroid analog to specializations of vector arrangements:

Definition A.14 *Let M_1 and M_2 be two oriented matroids on the same ground set E. Then we say there is a* **weak map** *from M_1 to M_2, denoted $M_1 \rightsquigarrow M_2$, if every vector of M_1 contains a vector of M_2.*

For an example of two oriented matroids with a common weak map image, see Figure 2-5.

The first lemma below is quite easy. The second is a corollary of Theorem 7.3.1 in [BLSWZ].

Lemma A.2 *If $M_1 \rightsquigarrow M_2$ then $M_1/A \rightsquigarrow M_2/A$.*

Lemma A.3 *Any oriented matroid is the weak map image of a uniform oriented matroid of the same rank.*

A.3 Duality

Let $E = \{x_1, x_2, \ldots x_k\} \subset \mathbf{R}^n$ be a realization of a simple oriented matroid M. Let h_i be the normal hyperplane to x_i, and let h_i^+ be the half-space bounded by h_i containing x_i.

This arrangement $\{h_1, h_2, \ldots, h_k\}$ of hyperplanes in $\mathbf{R}^n$ decompose the unit sphere S^{n-1} into regular cells. Each cell can be specified by its relationship to each hyperplane H – whether it lies on the positive side of H, the negative side, or is contained in H. Thus each cell gives a signed set $X : E \rightarrow \{-, 0, +\}$. These signed sets are exactly the nonzero vectors of an oriented matroid M^*, called the **dual** to M.

We can define the dual purely in terms of the vectors of M, so that *any oriented matroid has a unique dual.* (We won't go through this definition here.) The vectors of M^* are called the **covectors** of M. So an oriented matroid can be specified by either its vectors or its covectors.

A.4 Realizability and The Topological Representation Theorem

Lemma A.4 *Any rank r oriented matroid with less than $r+3$ elements is realizable.*

Not every oriented matroid is realizable. However, the Topological Representation Theorem ([FL]) tells us that in some geometric sense, every oriented matroid is "almost realizable".

Let $\mathcal{A} = (S_e)_{e \in E}$ be a signed arrangement of pseudospheres, i.e., an arrangement of pseudospheres on S^{n-1} with a choice of positive side for each maximal pseudosphere. (For example, see Figure A-1.) Then $\mathcal{A}$ decomposes S^{n-1} into regular cells, which we can specify just as in the preceding example, with signed sets $X : E \rightarrow \{-, 0, +\}$. Let $\mathcal{V}(A)$ be the family of all such signed sets.

Theorem A.1 Topological Representation Theorem: *(from [FL])*

1. If $\mathcal{A} = (S_e)_{e \in E}$ is a signed arrangement of pseudospheres on S^{n-1}, then $\mathcal{V}(A)$ is the family of covectors of a simple oriented matroid on E.

2. If $(E, \mathcal{V})$ is a rank n simple oriented matroid then there exists a signed arrangement of pseudospheres $\mathcal{A}$ in S^{n-1} such that $\mathcal{V} = \mathcal{V}(A)$.

3. $\mathcal{V}(A) = \mathcal{V}(A')$ for two signed arrangements $\mathcal{A}$ and $\mathcal{A}'$ iff there exists a homeomorphism $h : S^{n-1} \rightarrow S^{n-1}$ such that $h(\mathcal{A}) = \mathcal{A}'$.

A.5 Extensions

Definition A.15 *Let $M = (E, \mathcal{V})$ be an oriented matroid, and let $A \subset E$. M is an* **extension** *of M' by A if $M \backslash A = M'$. We write this $M = M' \cup A$.*

A particular type of extension we'll use in Chapter 4 is the **lexicographic extension**:

Proposition A.1 *(From [LV2].) Let M be an oriented matroid, $I = [e_1, \ldots, e_k]$ an ordered subset of E, and $\alpha = [\alpha_1, \ldots, \alpha_k] \in \{+, -\}^k$. Then we can describe a*

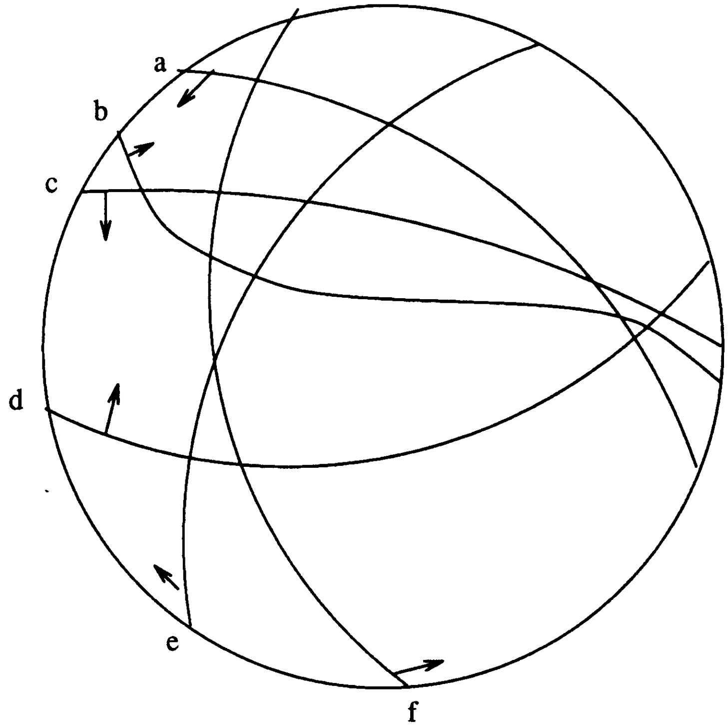

Figure A-1: A rank 3 arrangement of pseudospheres

single element extension $M \cup x$ by giving the sign of x on each cocircuit of M as follows:

$$\textit{if } Y \textit{ is a cocircuit of } M, \textit{ then} Y(x) = \begin{cases} \alpha_i Y(e_i) & \textit{if } i \textit{ is minimal such that } Y(e_i) \neq 0 \\ 0 & \textit{if } Y(e_i) = 0 \textit{ for all } i. \end{cases}$$

This extension is called the **lexicographic extension** *of M by $[e_1^{\alpha_1} \dots e_k^{\alpha_k}]$.*

If M is realizable then we can see the geometric sense of this inductively. The extension of M by $[e_1^+]$ is the oriented matroid $M \cup x$ with x parallel to e_1. The extension of M by $[e_1^-]$ has x antiparallel to e_1. The extension by $[e_1^{\alpha_1} \dots e_i^{\alpha_i} e_{i=1}^+]$ is obtained from the extension by $[e_1^{\alpha_1} \dots e_i^{\alpha_i}]$ by nudging x just a bit *towards* e_{i+1}. The extension by $[e_1^{\alpha_1} \dots e_i^{\alpha_i} e_{i=1}^-]$ is obtained from the extension by $[e_1^{\alpha_1} \dots e_i^{\alpha_i}]$ by nudging x just a bit *away* from e_{i+1}.

As a consequence of a theorem of Todd (c.f. [T],[BLSWZ]) we have the following two lemmas.

Lemma A.5 *Let $M = (E, \mathcal{V})$ be an oriented matroid. If $\sigma = \{s_1, s_2, \dots, s_k\} \subseteq E$ and $M \cup x$ is the lexicographic extension of M by $[s_1^+ s_2^+ \dots s_k^+]$ then $x \in$* conv(σ).

Lemma A.6 *If σ as above is a basis for M then x is not in any circuit of $M \cup x$ of rank less than* rank(M).

Definition A.16 *Two elements x, y of M are* **contravariant** *if they have opposite signs in all circuits containing them.*

Lemma A.7 *(From [BLSWZ]) If $M' = M \cup x$ is the lexicographic extension of M by $[s_1^+ s_2^+ \dots s_k^+]$ then:*

1. *s_1 and x are contravariant in $M \cup x$.*

2. *s_i and x are contravariant in $M'/\{s_1, s_2, \dots, s_{i-1}\}$.*

A.6 Euclidean Oriented Matroids

Definition A.17 *An* **oriented matroid program** *contained in M is a triple (M, g, f), where $f, g \in M$, $g^+ \notin \mathcal{V}$, and $f^+ \notin \mathcal{V}^*$.*

An element e is **parallel** *to f in (M, f, g) if $e^+ f^-$ is a circuit in M/g.*

If M is realizable then e and f are parallel in (M, f, g) if e lies on the ray $\overrightarrow{gf}$ in an affine realization.

Definition A.18 *An oriented matroid program (M, g, f) is* **Euclidean** *if for every cocircuit Y of M such that $Y(g) \neq 0$ there exists a single-element extension $\tilde{M} = M \cup p$ such that p is parallel to f in (M, g, f) and the extension $Y(p) = 0$ makes Y a cocircuit of $\tilde{M}$. An oriented matroid is* **Euclidean** *if all of its programs are Euclidean.*

For instance, any realizable oriented matroid is Euclidean. The element p extending a program (M, g, f) then is the intersection of the line $\overline{gf}$ with the hyperplane spanned by Y^0 in an affine realization putting g at infinity.

Lemma A.8 *(from [BLSWZ]) Let (M, g, f) be a Euclidean oriented matroid program, and let $(M \cup p, g, f)$ be an extension by an element parallel to f. Then $(M \cup p, g, f)$ is again Euclidean.*

Note that this does *not* imply that any extension of a Euclidean oriented matroid is Euclidean. This will be a cause for concern in Section 4.3.

Bibliography

[Ba] Babson, E., Thesis, MIT, 1993

[BLSWZ] A. Bjorner, M. Las Vergnas, B. Sturmfels, N. White, G. Ziegler, **Oriented Matroids**, Encyclopedia of Math. and its Appl., Cambridge Univ. Press (1992)

[BM] Billera, L.J, and Munson, B.S., Triangulations of oriented matroids and convex polytopes, *SIAM J. Algebraic and Discrete Methods*, **5** (1984)

[FL] Folkman, J., and Lawrence, J., Oriented matroids, *J. Combinatorial Theory,* Ser. B, **25** (1978)

[GM] Gelfand, I.M. and MacPherson, R., A combinatorial formula for the Pontrjagin classes, *Bull. Amer. Math. Soc.*, **26** (1992)

[K] Kervaire, M.A., A manifold which does not admit any differentiable structure, *Comm. Math. Helv.*, **34** (1960)

[LV1] Las Vergnas, M.,Matroïdes orientables, *C.R. Acad. Sci. Paris,* Ser. A, **280** (1975)

[LV2] Las Vergnas, M., Extensions ponctuelles d'une géométrie combinatoire orienté, *Problémes combinatoires et théorie des graphes* (Actes Coll. Orsay 1976), Colloques internationaux, C.N.R.S., No. 260 (1978)

[M] MacPherson, R., Combinatorial differential manifolds, *Topological Methods in Modern Mathematics: a Symposium in Honor of John Milnor's Sixtieth Birthday*, Publish or Perish (1993)

[MR] Mnëv, N, and Richter-Gebert, J., Two constructions of oriented matroids with disconnected extension space, *Discrete and Computational Geometry* **10** (1993)

[RG] Richter-Gebert, J., Oriented matroids with few mutations, *Discrete and Computational Geometry* **10** (1993)

[RS] Rourke, C.P., and Sanderson, B.J., **Introduction to Piecewise Linear Topology**, Springer Verlag (1972)

[T] Todd, M.J., Complementarity in oriented matroids, *SIAM J. Algebraic Discrete Methods*, bf 5 (1984)

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