Aspects of Theories with Dynamical, Topological or Gauge Symmetries

by

Jean-Guy Demers

B. Sc. Physics, Université de Sherbrooke, 1986
M. Sc. Condensed Matter Physics, Université de Sherbrooke, 1988

Submitted to the Department of Physics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 1994

Author ....
Department of Physics
May 1994

Certified by ....
Roman Jackiw
Professor
Thesis Supervisor

Accepted by ....
George Koster
Chairman, Departmental Committee on Graduate Students

Aspects of Theories with Dynamical, Topological or Gauge Symmetries

by

Jean-Guy Demers

B. Sc. Physics, Université de Sherbrooke, 1986

M. Sc. Condensed Matter Physics, Université de Sherbrooke, 1988

Submitted to the Department of Physics
on May 1994, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Abstract

Three topics are considered. Firstly, the $so(2,1)$ dynamical symmetry of a charged particle in the field of a vortex in $2+1$ dimensions is used to solve the Schroedinger equation when an harmonic potential is present. Endowing the particle with a spin 1/2, we solve albraically the Pauli Hamiltonian in presence of a harmonic potential or a uniform magnetic field by identifying the representations of the $spl^*(2,1)$ symmetry present in that case. Secondly, problems of topological field theories are discussed. Constructing explicitly the twisted N=2 supersymmetry generators for the $3+1$ dimensional topological Yang-Mills theory, we provide an understanding for the lack of local excitations of this theory. Working in 2+1 dimensions and defining a twist that also invert the Grassmann parity, abelian gauged fixed BF and Chern-Simons theories are obtained by twisting $N = 4$ supersymmetric matter Lagrangians. Analogous results are given in $1+1$ dimensions. Thirdly, non-relativistic particles in thermal equilibrium are discussed in first quantization. The real time matrix propagator is recovered by making use of a parametrized form for the action.

Thesis Supervisor: Roman Jackiw
Title: Professor

À mes parents

love the Lord

your God

with all

your heart

and

with all

your soul

and

with all

your strength

Deut. 6:5

Acknowledgments

Considering those who instilled in me a desire to work toward this degree, I am grateful to my professors at the Université de Sherbrooke, in particular to Alain Caillé, André Lemieux, Marcel Banville and André-Marie Tremblay.

During my stay in the Center for Theoretical Physics, I had the opportunity to work with three professors. From each of them, I learned not only a great deal of physics, but also valuable insights on better ways of conducting research. I first thank my advisor, Roman Jackiw, for introducing me, among other things, to problems of dynamical symmetries. In this man of talent and experience, I especially appreciated a keen sense of organization. I thank Roger Brooks for guidance on the road of topological field theories, as well as for his constant composure amid our debates over various topics. I also express my gratitude to Samir Mathur for his availability and for motivating discussions on a number of physics problems, in which I enjoyed his amazing capacity to come up with new associations.

Of the numerous teachers at MIT from whom I have learned much physics, I thank especially Nihat Berker, John Joannopoulos, Mehran Kardar, Edward Farhi, Roman Jackiw and Barton Zwiebach.

The collaboration of Claudio Lucchesi in the production of chapter four was much appreciated.

My studies at MIT were made possible by the financial assistance of the Fonds pour la Formation de Chercheurs et l'Aide À la Recherche, the Physics Department and the Center for Theoretical Physics.

Finally, but not least, I am grateful to my wife Sylvie for support and enthousiasm over the years.

January 1994

Contents

Chapter 1

OVERVIEW

We provide here an introductory overview to the problems considered in the following chapters. Some extra references are also added for the reader interested in background and/or related works.

The material presented in chapter 2 orbits around the dynamical (super)symmetries of a charged particle in the field of an magnetic vortex in 2+1 dimensions. There is at least two context in which the notion of magnetic vortex arise. In a model of non-relativistic particles minimally coupled to a Chern-Simons gauge field, the gauss law is of the form $B(x) \; \alpha \; \rho(x)$ where B is the (scalar) magnetic field and ρ is the charge density while x is the coordinate on the plane[1]. Thus, pointlike charges also carry pointlike magnetic fluxes. Another motivation is the problem of a particle evolving in the field of a long and thin tube of flux, as in the Aharonov-Bohm effect or the idealized cosmic strings, and where the (trivial) motion along the tube is ignored. Dynamical symmetries on the other hand are especially useful when the hamiltonian can be expressed as a polynomial in the Casimirs of the symmetry group of the system. A well known application of this concept is on the hydrogen atom[2]. Combining rotations with the Runge-Lenz vector, the algebra $o(4)$ is obtained and both spectrum *and* degeneracies are accounted for by the representations of that algebra. Another well known application is the use of $su(3)$ in connection with the masses of light hadrons[3]. For a spinless particle, our dynamical symmetry will be $so(2,1)$ and in the presence of an harmonic potential, the hamiltonian will be the Casimir of the $so(2)$ compact subgroup, leading to an algebraic derivation of the (discrete) spectrum. The $so(2,1)$ symmetry has also been put to use in other problems, as the generalized harmonic oscillator[4] (for an hamiltonian $\frac{1}{2}(ap^2 + b(qp+pq) + cq^2)$ where p and q are the canonical variables and a,b,c are real, time dependent functions) as well as on more general potentials in parabolic coordinates[5]. It is also helpful in constructing Green functions for particles in various combinations of harmonic oscillator, Coulomb and Aharonov-Bohm potentials in three dimensions[6]. Moreover, it also exists in the two dimensional system of spinless particles mentioned above, with[7] and without[8] uniform external magnetic field. When our particle is endowed with a spin 1/2, the symmetry algebra becomes larger and graded[9] and is termed $spl^*(2,1)$. We show how the use of its representations will provide an algebraic solution of the Pauli equation when either an external magnetic field or a harmonic potential is present.

Chapters 3 and 4 deal with issues of topological field theories (TFT). The basic ingredients entering the definition of a TFT are[10]: 1. a collection of Grassmann graded fields Φ defined on a Riemannian manifold 2. a odd, nilpotent operator Q 3. an energy momentum tensor that is Q-exact ($T_{\alpha\beta} = \{Q, V_{\alpha\beta}\}$ for some $V_{\alpha\beta}$) and 4. the condition that physical states are in the cohomology of Q (that is, physical states are annihilated by Q, and are defined up to addition of states of the form $Q|\phi\rangle$ for some state $|\phi\rangle$; for BRST gauged fixed theories, these conditions reflect the requirement of gauge invariance of the physical states[11]). By defining, as usual, the energy momentum tensor through a variation of the action with respect to the metric ($\delta_g S_q = \frac{1}{2}\int_M d^n x\, \sqrt{g}\, \delta g^{\alpha\beta}\, T_{\alpha\beta}$ where S_q here is the quantum action and include the classical part, as well as the ghosts and multipliers required to gauge fixed it; M is the n dimensional Riemannian target space), one can display the essential characteristic that makes TFT's interesting from a mathematical point of view: the possibility to generate quantities that depend only on the global features of the target space (topological invariants). The most straightforward example of this is the partition function $Z = \int [d\Phi] e^{-S_q}$, as $\delta_g Z = \int [d\Phi] e^{-S_q}(-\delta_g S_q) = \int [d\Phi] e^{-S_q}\{Q, \chi\}$ where $\chi = -\frac{1}{2}\int_M d^n x\, \sqrt{g}\, \delta g^{\alpha\beta}\, V_{\alpha\beta}$. But since the average of $\{Q, \chi\}$ is nothing but a vacuum expectation value, $\delta_g Z$ vanishes. In the same way, one shows easily that if an operator

P is in the cohomology of Q, its vacuum expectation value will be a topological invariant. Interestingly, independence with respect to the metric is thus achieved without summing over metrics, a fact that generated much interest among physicists, as this is thought to be a pointer to the construction of a theory of quantum gravity.

One of the important developments in the young history of TFT's was made in 1988 with the construction of the so-called topological Yang-Mills (TYM) theory[12]. This relativistic TFT came as an important tool in the study of Donaldson invariants in four dimensions. When it was constructed, it was noticed that on a flat manifold, it can be obtained by twisting the N=2 super Yang-Mills (SYM) theory. In four euclidean dimensions, the Lorentz group $SO(4)$ is isomorphic to $SU_L(2) \times SU_R(2)$. This twisting consist in taking the diagonal sum of (say) $SU_L(2)$ with the automorphism $SU_I(2)$. Of the initial supersymmetries, one becomes a Lorentz scalar and plays the role of the nilpotent operator Q and the modified theory is just TYM. But shortly later, it was also shown[13] that TYM on an arbitrary manifold M can be recovered by BRST gauge fixing (with appropriate gauge parameters) the topological symmetry ($\delta A_\mu^a = \theta_\mu^a$ where A_μ^a is the gauge field with a the gauge group index and θ_μ^a is arbitrary) of either zero or the topological action $\int_M d^n x \, F \wedge F$. In that context, the nilpotent operator is just the usual BRST charge, and the requirement that the physical states be in the cohomology of Q is readily seen to imply that only ground states are physical. But when viewed through the twisting construction, the same requirement is somewhat unexpected, as the parent theory (SYM) does in fact possess physical degrees of freedom. In chapter 3, we explore this issue. The details of the twisting are provided, and some ambiguities on what constitutes the true Lorentz group after twisting are resolved. The disappearance of excited states is found to be tied to the lost of hermiticity of some of the Lorentz generators, and the consequences on the representations of the twisted algebra are discussed.

Now there is more to TFT's than TYM. In fact, TYM is a typical representative of a class of TFT, often referred to as Donaldson-Witten (or TQFT) type. A few characteristics define this type. The classical action is trivial (either zero or a total derivative as in the example above) whereas the total action is Q exact. Moreover, as for TYM, they can usually be obtained by a twisting from a supersymmetric theory. The other class of TFT is the so-called BF theories, of which the Chern-Simons is a special case. In the abelian version, their classical action S_c is of the form $\int_M B_{(k)} \wedge F_{(D-k)}$ (where the subscript is the form degree, F is the field strength and D is the dimension of spacetime), and is thus not simply a total derivative. It is also metric independent. When adding the contributions of the ghosts and gauge fixing terms, the total action appear as $S_c + \{Q, V\}$, which ensures that the energy momentum tensor is Q exact, and making apparent the topological nature. In chapter 4, we present results on how (at least some) TFT's of BF type can be obtained through twisting. We give examples in two dimensions, but work mostly in three dimensions, where the twisting make use of the unique $SU(2)$ of spacetime. This symmetry is mixed with the $SU(2)$ of the automorphism of the $N = 4$ (free) parent theory. The original feature of this twisting is that it also involves a change of Grassmann parity. The close relation with the usual twisting defined to reach Donaldson-Witten theories suggest an intimate connection between the two types of TFT and a general conjecture is made on this connection.

In the last chapter, we address an issue of finite temperature physics. The (non-relativistic) picture of path integrals is now a part of most books on quantum mechanics. It is interesting to ask how this construction requires modifications to account for the presence of a bath of identical particles. This is not just an academic exercise, even though most problems of many particles at finite temperature are usually treated using field theories. In analogy with particle creation in curved spacetime (and thus near black holes), one would expect string creation to occur, but so far they have not been (theoretically) found[15]. As string theory is only available in first quantized form, the question arise of how to describe correctly the propagation of strings in a bath of strings. As a toy model of this problem, we study the case of non-relativistic particles. We will take the point of view that the presence of the bath allows for the propagating particle to be exchanged with others from the bath as it propagates. This will involve the notion of "hole" (or absence of particle) effectively propagating backward in time. Dealing with this situation will be best done by making use of a parametrized version of the action, in which the time slicing will be made with respect with a parameter time τ rather than the physical time t. Of the various finite temperature formalisms, we will choose the so-called real time[14] mostly because it has been shown to be causal, a property clearly desirable when particles are created. It is also adapted to describe systems harboring out of equilibrium distributions of particles, as one finds in expanding cosmologies and black holes problems. In that formalism, the propagator is made of four components gathered in a matrix. We will give a rather literal interpretation to these components, and explain how they can be recovered when careful attention is given to the gauge fixing of the reparametrization invariance.

Bibliography

[1] R. Jackiw, Topics in Planar Physics, MIT-CTP preprint #1824 (1989).

[2] B.G. Wybourne, *Classical Groups for Physicists* (Wiley, New York, 1974); F. Iachello, *Rev. Mod. Phys.* **65** 1993 569.

[3] M. Gell-Mann *Phys. Rev.* **125**, (1962) 1067; Y. Ne'eman *Nucl. Phys.* **26** (1961) 222.

[4] M. Hirayama, *Prog. Theor. Phys.* **86** (1991) 343; S. Baskoutas, A. Jannussis and R. Mignani, Rome University preprint ROME-815-1991.

[5] H. Boschi-Filho, M. de Souza and A.N. Vaidya *J. Phys.* **A24** (1991) 4981.

[6] H. Boschi-Filho and A.N. Vaidya *Ann. Phys.* **212** (1991) 1.

[7] T. Awaji and M. Hotta *Intl. J. Mod. Phys.* **B6** (1992) 1229.

[8] R. Jackiw and S.-Y. Pi, *Phys. Rev.* **D42** (1990) 3500.

[9] More on supersymmetry in quantum mechanics can be found in P. Salomonson and J.W. Van Holten, *Nucl. Phys.* **B196** (1982) 509; M. De Crombrugghe and V. Rittenberg *Ann. Phys.* **151** (1983) 99 and references therein.

[10] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Phys. Rep.* **209** (1991) 129.

[11] M. Henneaux and C. Teitelboim *Quantization of Gauge Systems* (Princeton University Press, 1992), chap. 14.

[12] E. Witten, *Comm. Math. Phys,* **117** (1988) 91.

[13] L. Baulieu and I.M. Singer *Nucl. Phys. (Proc. Suppl.)* **5B** (1988) 12; R. Brooks, D. Montano and J. Sonnenschein *Phys. Lett.* **214B** (1988) 91.

[14] N.P. Landsman and C.G. van Weert, *Phys. Rep.* **145** (1987) 141.

[15] H.J. de Vega, preprint PAR-LPTHE-92-50, hep-th #9302052 and references therein.

Chapter 2

DYNAMICAL SUPERSYMMETRY AND SOLUTIONS FOR PAULI HAMILTONIANS

Abstract

A charged point particle interacting with a vortex in $2+1$ dimensions (or the relative coordinate of two anyons) possess a dynamical so(2,1) symmetry that can be exploited to solve for the case when an external harmonic oscillator potential is added. Moreover, if the particle carries spin $1/2$, its interaction with the vortex exhibits a dynamical $spl^*(2,1)$ supersymmetry, which can be imported to the cases where a harmonic oscillator or an external magnetic field is present. Using this supersymmetry, the corresponding Pauli Hamiltonians can be solved algebraically.

Appeared in *Modern Physics Letters* **A89** (1993) 827.

2.1 Introduction

The physics of a charged point particle interacting with an external magnetic vortex in 2 + 1 dimensions occurs in many physically interesting contexts. One case is when the particle is in the field of a long and thin solenoid and motion along the z-axis is ignored. More generally, because of Chern-Simons electrodynamics, this Aharonov-Bohm type of interaction also appears between anyons, so its study is of interest to various types of two-dimensional systems, including models for high-T_c superconductivity.

If the particle of charge e is taken to be spinless, its dynamics is governed by the Lagrangian ($c = \hbar = mass = 1$):

$$L = \frac{1}{2}v^2 + e\mathbf{v} \cdot \mathbf{A}(\mathbf{r}) \tag{2.1}$$

where the vector potential

$$A^i(\mathbf{r}) = -\frac{\Phi}{2\pi}\frac{\varepsilon^{ij} r^j}{r^2} \tag{2.2}$$

produces a vortex of strength Φ centered at the origin:

$$B = \nabla \times \mathbf{A} = \Phi\delta^2(\mathbf{r}) \quad . \tag{2.3}$$

Here, $\mathbf{a} \times \mathbf{b} \equiv \varepsilon^{ij} a^i b^j$ and $\mathbf{v} \equiv \frac{d\mathbf{r}}{dt}$. Eq.(2.1) also describes the relative coordinate of two anyons [1].

The Hamiltonian for (2.1) can be written as:

$$H = \frac{1}{2}v^2 \tag{2.4}$$

with $\mathbf{v} = \mathbf{p} - e\mathbf{A}$ and quantization is then achieved by postulating the commutators

$$\begin{aligned}
\left[r^i, r^j\right] &= 0 \\
\left[r^i, v^j\right] &= i\delta^{ij} \\
\left[v^i, v^j\right] &= i\varepsilon^{ij} eB(\mathbf{r}) = i\varepsilon^{ij} e\Phi\delta^2(\mathbf{r})
\end{aligned} \tag{2.5}$$

It was recently shown [2] that (4) possess a dynamical conformal symmetry generated by the following three constants of motion:

$$
\begin{aligned}
H &= \frac{1}{2}v^2 && (\textit{time translation}) \\
D &= tH - \frac{1}{4}(\mathbf{r}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{r}) && (\textit{time dilation}) \\
K &= -t^2H + 2tD + \frac{1}{2}r^2 && (\textit{time special conformal transformation}) \quad (2.6)
\end{aligned}
$$

When commuted among themselves, these charges reproduce the $so(2,1)$ algebra:

$$
\begin{aligned}
[D,H] &= -iH \\
[D,K] &= iK \\
[H,K] &= 2iD \quad .
\end{aligned}
\tag{2.7}
$$

The group $SO(2,1)$ is not a compact group, however, the combination

$$
R = \frac{1}{2}\left(\frac{1}{a}K + aH\right) \tag{2.8}
$$

generates the compact $SO(2)$ subgroup where the parameter a has dimensions of time. Because of its discrete spectrum, R is an interesting object to analyze and in this paper, we discuss how it can be use to solve eigenvalue problems. For the present case, we choose to consider its form at $t = 0$, when it reads:

$$
R = \frac{a}{2}\left(\frac{v^2}{2} + \frac{1}{2}\frac{r^2}{a^2}\right) \quad . \tag{2.9}
$$

But this is just the Hamiltonian obtained by adding to (4) an external harmonic oscillator potential of frequency $\omega = a^{-1}$:

$$
\begin{aligned}
H^h &\equiv \frac{1}{2}v^2 + \frac{1}{2}\omega^2 r^2 \\
&= \left.\frac{2R}{a}\right|_{a=\omega^{-1}}
\end{aligned}
\tag{2.10}
$$

This fact can be used immediately to provide group theoretically the spectrum and wavefunctions of H^h: the possible (infinite dimensional) representations of R are known and by considering its coordinate realization , one finds [2]:

$$H^h \psi_n^j(r,\theta) = 2\omega(d+n)\psi_n^j(r,\theta) \tag{2.11}$$

with

$$\psi_n^j(r,\theta) = \frac{e^{ij\theta}}{\sqrt{2\pi}} \frac{(-1)^n}{r} \left(\omega r^2\right)^d \left(\frac{2n!}{\Gamma(n+2d)}\right)^{1/2} e^{-\frac{1}{2}\omega r^2} L_n^{2d-1}\left(\omega r^2\right) \tag{2.12}$$

where the L_n^m's are the generalized Laguerre polynomials. Here,

$$d = \frac{1}{2} + \frac{1}{2}|j-\nu| \quad , \tag{2.13}$$

where j is the integer angular momentum , $\nu \equiv \frac{e\Phi}{2\pi}$ and n is a positive integer. This is indeed the well known result [1].

Thus, although H^h breaks the conformal symmetry, its occurrence in the $so(2,1)$ algebra is sufficient to solve for its spectrum and eigenfunctions. Interestingly, it is possible to show that H^h also possess a $so(2,1)$ dynamical symmetry. It is different from (6), and related by a coordinate transformation. This importation of symmetry has also been carried out [4] for the case where a uniform magnetic field is added to (4). In both cases, using group theory to diagonalize the generator of the compact subgroup leads to a solution of the time independent Schrödinger equation. Because it establishes the existence of a symmetry in addition to solve the eigenvalue problem, this method of importing symmetry is now adopted.

For the two situations just mentioned (with harmonic oscillator or magnetic field) we now consider the charged particle to be a spin 1/2 object. In the presence of the vortex alone, its Lagrangian acquires a fermionic contribution:

$$L = \frac{1}{2}v^2 + \frac{i}{2}\boldsymbol{\psi} \cdot \dot{\boldsymbol{\psi}} + e\mathbf{v} \cdot \mathbf{A} + eBS \quad . \tag{2.14}$$

with $\mathbf{A}$ given in (2.2) . Here , the $\psi^i (i = 1, 2)$ are anticommuting, time dependent Grassmann variables and describe the spin degree of freedom of a classical particle [5]; the spin itself being given by $S = -\frac{i}{2}\boldsymbol{\psi} \times \boldsymbol{\psi}$. Legendre transforming (2.14) gives the Pauli Hamiltonian:

$$H = \frac{1}{2}v^2 - eBS \tag{2.15}$$

with $\mathbf{v} = \mathbf{p} - e\mathbf{A}(\mathbf{r})$.

Recently [6], (2.15) was shown to possess a dynamical supersymmetry $spl^*(2,1)$, with $so(2,1)$ for its bosonic subalgebra. In section II, we show that when (2.15) is augmented to include an external harmonic oscillator or a uniform magnetic field, the system is still supersymmetric. In section III, we solve the eigenvalue problem for these augmented systems by making use, in each case, of the close connection between the generator of the compact subgroup of $SO(2,1)$ and the appropriate Pauli Hamiltonian.

2.2 Supersymmetry Imported

The Hamiltonian (2.15) can be quantized by supplementing (5) with the anticommutator:

$$\{\psi^i, \psi^j\} = \delta^{ij} \tag{2.16}$$

By studying the symmetries of (2.14) at the classical level and using Noether's theorem, one finds [6] that for the quantized version of (2.15), the following constants of motion can be obtained:

$$\begin{aligned}
H &= \frac{1}{2}v^2 - eBS \\
D &= tH - \frac{1}{4}(\mathbf{r}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{r}) \\
K &= -t^2 H + 2tD + \frac{1}{2}r^2 \\
Q_1 &= \mathbf{v}\cdot\boldsymbol{\psi} \\
Q_2 &= \mathbf{v}\times\boldsymbol{\psi}
\end{aligned}$$

$$
\begin{aligned}
S_1 &= -tQ_1 + \mathbf{r} \cdot \boldsymbol{\psi} \\
S_2 &= -tQ_2 + \mathbf{r} \times \boldsymbol{\psi} \\
Y &= S + \frac{1}{2}\mathbf{r} \times \mathbf{v} \quad .
\end{aligned} \tag{2.17}
$$

Being time dependent, these charges do not commute with (2.15) but their total time derivatives vanish:

$$
\frac{dC}{dt} = i\,[H, C] + \frac{\partial C}{\partial t} = 0 \quad . \tag{2.18}
$$

From (2.16) and (2.17), the following $spl^*(2,1)$ graded algebra is then verified:

$$
\begin{gathered}
[H, D] = iH \ , \qquad [H, K] = 2iD \ , \qquad [D, K] = iK \ , \\
\{Q_1, Q_1\} = 2H \ , \qquad \{Q_2, Q_2\} = 2H \ , \qquad \{Q_1, S_1\} = \{Q_2, S_2\} = -2D \\
[H, S_1] = -iQ_1 \ , \qquad [K, Q_1] = iS_1 \ , \qquad [H, S_2] = -iQ_2 \ , \qquad [K, Q_2] = iS_2 \\
[D, Q_1] = -\frac{i}{2}Q_1 \ , \qquad [D, S_1] = \frac{i}{2}S_1 \ , \qquad [D, Q_2] = -\frac{i}{2}Q_2 \ , \qquad [D, S_2] = \frac{i}{2}S_2 \\
[Y, Q_1] = \frac{i}{2}Q_2 \ , \qquad [Y, S_1] = \frac{i}{2}S_2 \ , \qquad [Y, Q_2] = -\frac{i}{2}Q_1 \ , \qquad [Y, S_2] = -\frac{i}{2}S_1 \\
\{S_1, S_1\} = \{S_2, S_2\} = 2K \ , \qquad \{Q_1, S_2\} = \{Q_2, S_1\} = 2Y \\
[H, Q_1] = [H, Q_2] = 0 \ , \qquad [K, S_1] = [K, S_2] = 0 \\
[Y, H] = [Y, K] = [Y, D] = 0 \quad .
\end{gathered} \tag{2.19}
$$

This set of symmetry is larger that the one found for the system of a spin 1/2 and a magnetic monopole[7], which is $OSp(1,1)$. It also differs in that in our case, the symmetries and the algebra are formally exact on the whole plane, including the origin.

Now the addition to (2.15) of a spectrum-discretizing harmonic oscillator term can be accomplished in the following way [8]. The action for Lagrangian (2.14)

$$
S = \int dt \left(\frac{1}{2}v^2 + \frac{i}{2}\boldsymbol{\psi} \cdot \dot{\boldsymbol{\psi}} + e\mathbf{v} \cdot \mathbf{A} + eBS \right) \tag{2.20}
$$

is transformed, under the change of coordinate

$$\begin{aligned} t &= \frac{1}{\omega}\tan\omega t' \\ \mathbf{r} &= \frac{\mathbf{r}'}{\cos\omega t'} \\ \psi &= \psi' \end{aligned} \tag{2.21}$$

into $S^h = \int dt'\, \mathcal{L}^h$ with the new Lagrangian

$$\mathcal{L}^h = \left(\frac{1}{2}v'^2 - \frac{1}{2}\omega^2 r'^2 + \frac{i}{2}\boldsymbol{\psi}'\cdot\dot{\boldsymbol{\psi}}' + e\mathbf{v}'\cdot\mathbf{A}' + eB'S'\right) \quad . \tag{2.22}$$

Here, $\dot{\boldsymbol{\psi}} = \frac{d\boldsymbol{\psi}}{dt'}$, $\mathbf{v}' = \frac{d\mathbf{r}'}{dt'}$ whereas $\mathbf{A}'$ and B' are as in (2.2) and (2.3) but in terms of $\mathbf{r}'$. The corresponding Hamiltonian (we now suppress the primes)

$$\mathcal{H}^h = \frac{1}{2}v^2 + \frac{1}{2}\omega^2 r^2 - eBS \tag{2.23}$$

with

$$\mathbf{v} = \mathbf{p} - e\mathbf{A}(\mathbf{r}) \tag{2.24}$$

still describe a spin 1/2 particle interacting with a vortex, but with the desired external harmonic oscillator potential now added. Using (2.21), one can transform (2.17) into

$$\begin{aligned} H^h &= \frac{1}{2}\cos^2\omega t v^2 - \frac{\omega}{2}\sin 2\omega t\,(\mathbf{r}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{r}) + \frac{1}{2}\omega^2\sin^2\omega t r^2 - e\Phi\cos^2\omega t\delta^2(\mathbf{r}) \\ D^h &= \frac{1}{\omega}\tan\omega t H^h - \frac{1}{4}(\mathbf{r}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{r}) + \frac{1}{2}\sin\omega t r^2 \\ K^h &= -\frac{1}{\omega^2}\tan^2\omega t H^h + 2\frac{1}{\omega}\tan\omega t D^h + \frac{1}{2}\,\frac{r^2}{\cos^2\omega t} \\ Q_1^h &= \cos\omega t\mathbf{v}\cdot\boldsymbol{\psi} - \omega\sin\omega t\mathbf{r}\cdot\boldsymbol{\psi} \\ S_1^h &= \cos\omega t\mathbf{v}\times\boldsymbol{\psi} - \omega\sin\omega t\mathbf{r}\times\boldsymbol{\psi} \\ Q_2^h &= -\frac{1}{\omega}\tan\omega t Q_1^h + \frac{1}{\cos\omega t}\mathbf{r}\cdot\boldsymbol{\psi} \\ S_2^h &= -\frac{1}{\omega}\tan\omega t Q_2^h + \frac{1}{\cos\omega t}\mathbf{r}\times\boldsymbol{\psi} \end{aligned}$$

$$Y^h = S + \frac{1}{2}\mathbf{r}\times\mathbf{v} \quad . \tag{2.25}$$

These charges are now conserved with respect to the new Hamiltonian $\mathcal{H}^h$:

$$\frac{dC}{dt} = i\left[\mathcal{H}^h, C\right] + \frac{\partial C}{\partial t} = 0 \tag{2.26}$$

where t is the transformed time. When $\omega \to 0, \mathcal{H}^h \to H$ and they become identical to (2.17). For finite ω, they coincide with (2.17) at $t = 0$; at finite t, they follow their own evolution according to $\mathcal{H}^h$. They are a consequence of symmetries obtained by using Eq. (2.21) to go from $\mathcal{L}^h$ to L, applying the known symmetry to the latter case, and then returning to $\mathcal{L}^h$ by inverting Eqs. (2.21) [4]. One can also check that this transformation is supercanonical: If $\mathcal{H}^h$ is quantized by imposition of the commutation relations (2.5) and (2.16), then the same relations follow for non-prime variables; the quantities in (2.25) are therefore supercharges obeying the superalgebra (2.19).

In a similar fashion, one can modify the action (2.20) through the transformation

$$\begin{aligned} t &= \frac{2}{\Omega}\tan\frac{\Omega t'}{2} \quad , \\ r^i &= r'^i - \tan\frac{\Omega t'}{2}\varepsilon^{ij}r'^j \quad , \\ \psi^i &= \cos\Omega t'\psi'^i - \sin\Omega t'\varepsilon^{ij}\psi'^j \quad . \end{aligned} \tag{2.27}$$

It becomes $S^B = \int dt'\, \mathcal{L}^B$ with the new Lagrangian

$$\mathcal{L}^B = \frac{1}{2}v'^2 + \frac{i}{2}\boldsymbol{\psi}'\cdot\dot{\boldsymbol{\psi}}' + e\mathbf{v}'\cdot\mathbf{A}' + e\mathbf{v}'\cdot\mathcal{A}' + e\left(\hat{\Omega} + B'\right)S' \tag{2.28}$$

and the corresponding Hamiltonian (again suppressing the primes)

$$\mathcal{H}^B = \frac{1}{2}v^2 - e\left(\hat{\Omega} + B\right)S \tag{2.29}$$

where $\mathbf{v} = \mathbf{p} - e\mathbf{A} - e\mathcal{A}$ and $\mathcal{A}^i = -\frac{\hat{\Omega}}{2}\varepsilon^{ij}r^j$ with $\hat{\Omega} \equiv \frac{\Omega}{e}$. This describes the experimentally relevant case when the spin 1/2 particle sees not only the vortex, but also a uniform magnetic field of strength $\hat{\Omega}$ and constitutes a generalization of the spinless case [4]. As for the previous case, assuming that (2.5) and (2.16) hold with $B(\mathbf{r}) = \hat{\Omega} + \Phi\delta^2(\mathbf{r})$, one shows that the transformation is supercanonical. The supercharges (2.19) can also be imported and they transform into:

$$
\begin{aligned}
H^B &= \frac{1}{2}\cos^2\frac{\Omega}{2}tv^2 + \frac{\Omega^2}{8}r^2 - \frac{\Omega}{2}\cos^2\frac{\Omega}{2}t\mathbf{v}\times\mathbf{r} \\
&\quad + \frac{\Omega}{8}\sin\Omega t\,(\mathbf{v}\cdot\mathbf{r}+\mathbf{r}\cdot\mathbf{v}) - e\Phi\cos^2\frac{\Omega}{2}t\delta^2(\mathbf{r})S \\
D^B &= \frac{2}{\Omega}\tan\frac{\Omega}{2}t\; H^B - \frac{1}{4}(\mathbf{v}\cdot\mathbf{r}+\mathbf{r}\cdot\mathbf{v}) - \frac{\Omega}{4}\tan\frac{\Omega}{2}t\; r^2 \\
K^B &= -\frac{4}{\Omega^2}\tan^2\frac{\Omega}{2}t\; H^B + \frac{1}{\Omega}\tan\frac{\Omega}{2}t\; D^B + \frac{r^2}{2\cos^2\frac{\Omega t}{2}} \\
Q_1^B &= \cos^2\frac{\Omega}{2}t\;\mathbf{v}\cdot\boldsymbol{\psi} - \sin\frac{\Omega}{2}t\;\mathbf{v}\times\boldsymbol{\psi} + \frac{\Omega}{2}\cos\Omega t\;\mathbf{r}\times\boldsymbol{\psi} + \frac{\Omega}{2}\sin\Omega t\;\mathbf{r}\cdot\boldsymbol{\psi} \\
Q_2^B &= \sin\frac{\Omega}{2}t\;\mathbf{v}\cdot\boldsymbol{\psi} + \cos^2\frac{\Omega}{2}t\;\mathbf{v}\times\boldsymbol{\psi} + \frac{\Omega}{2}\sin\Omega t\;\mathbf{r}\times\boldsymbol{\psi} - \frac{\Omega}{2}\cos\Omega t\;\mathbf{r}\cdot\boldsymbol{\psi} \\
S_1^B &= -\frac{2}{\Omega}\tan\frac{\Omega}{2}t\; Q_1^B + \mathbf{r}\cdot\boldsymbol{\psi} - \tan\frac{\Omega}{2}t\;\mathbf{r}\times\boldsymbol{\psi} \\
S_2^B &= -\frac{2}{\Omega}\tan\frac{\Omega}{2}t\; Q_2^B + \tan\frac{\Omega}{2}t\;\mathbf{r}\cdot\boldsymbol{\psi} + \mathbf{r}\times\boldsymbol{\psi} \\
Y^B &= S + \frac{1}{2}\mathbf{r}\times\mathbf{v} + \frac{\Omega}{4}r^2 \quad .
\end{aligned} \tag{2.30}
$$

They are time-independent when (2.29) is used to translate the time. They also form a dynamical supersymmetry by satisfying the graded superalgebra (2.19).

2.3 Spectrum and Wavefunctions

It was shown in the last section that the Hamiltonians (2.23) and (2.29) possess a dynamical supersymmetry. We discuss here how this can be used to construct their eigenfunctions and eigenvalues. Because our modified systems exhibit the same symmetry as (2.15), much of our group theoretical reasoning overlaps with that of Ref. [6], to which we refer the reader for details.

By considering the subgroup structure of $spl^*(2,1)$ one finds that six Casimirs are needed to specify a state in the representation space. Here, as in Ref. [6], the two

Casimirs of the full algebra $spl^*(2,1)$ vanish identically in our coordinate realizations (2.25) and (2.30). The four remaining and their eigenvalues are

$$J|j\alpha sn\rangle = j|j\alpha sn\rangle \tag{2.31}$$

$$S|j\alpha sn\rangle = S|j\alpha sn\rangle \tag{2.32}$$

$$A|j\alpha sn\rangle = \alpha|j-\nu|\,|j\alpha sn\rangle \tag{2.33}$$

$$R|j\alpha sn\rangle = \left(\frac{1}{2}|j-s-\nu|+\frac{1}{2}+n\right)|j\alpha sn\rangle \tag{2.34}$$

with the constraint

$$\alpha s(j-\nu) > 0 \ . \tag{2.35}$$

Here, S is the spin operator with eigenvalues $\pm\frac{1}{2}$. The total angular momentum operator

$$J = \mathbf{r}\times\mathbf{p} + S \tag{2.36}$$

has for spectrum

$$j = m + s \ , \tag{2.37}$$

where the integer m accounts for the orbital contribution. A is a fermion number operator given by

$$A \equiv i[Q_1, S_1] - \frac{1}{2} \tag{2.38}$$

and α is either $+1$ or -1. R is the generator of the $SO(2)$ subgroup of the bosonic sector $SO(2,1)$ of the full algebra, defined in (2.8). For our realization, it is positive definite and n is a positive integer.

Now consider our imported supercharges for the harmonic oscillating case (2.25). Because these charges are independent of time, we can always select $t=0$ to get:

$$\begin{aligned}\mathcal{H}^h &= \omega^2 K^h + H^h \\ &= 2\omega R^h\,|_{a=\omega^{-1}} \ .\end{aligned} \tag{2.39}$$

Thus, (2.34) gives immediately the spectrum of $\mathcal{H}^h$:

$$E^h = (|j - s - \nu| + 2n + 1)\,\omega \quad . \tag{2.40}$$

By use of (2.37), this is independent of s and our algebraically obtained result is in agreement with the spectrum for two flux-carrying spin 1/2 particles [9]. The corresponding eigenfunctions are found by writing R^h as:

$$R^h = \frac{1}{2a}K^h + \frac{a}{2K^h}\left((D^h)^2 - iD^h + C_0\right) \quad . \tag{2.41}$$

Here, C_0 is the Casimir of the $so(2,1)$ algebra [6] and its eigenvalue is given by:

$$\Delta_{j\alpha}(\Delta_{j\alpha} - 1) \quad , \tag{2.42}$$

with

$$\Delta_{j\alpha} = \frac{1}{2} \mid j - s - \nu \mid + \frac{1}{2} \quad . \tag{2.43}$$

Because $\mathbf{A}(\mathbf{r})$ has no radial component, D^h can be realized as:

$$D^h = \frac{i}{2}\left(r\frac{\partial}{\partial r} + 1\right) \quad . \tag{2.44}$$

Hence, (2.34) leads to the eigenvalue equation

$$\left(-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{(2\Delta_{j\alpha} - 1)^2}{r^2} + \omega^2 r^2\right)\psi^{h\pm}_{j\alpha n}(r,\theta) = 4\omega\,(\Delta_{j\alpha} + n)\,\psi^{h\pm}_{j\alpha n}(r,\theta) \quad . \tag{2.45}$$

The $\pm$ are for spin up/down. The angular part $e^{im\theta}$ of the wavefunction can be separated and the normalized solutions are given in terms of the generalized Laguerre polynomials

$$\psi^{h\pm}_{j\alpha n}(r,\theta) = e^{im\theta}(-1)^n\omega^{\Delta_{j\alpha}}\left(\frac{n!}{\pi\Gamma(2\Delta_{j\alpha} + n)}\right)^{1/2} r^{2\Delta_{j\alpha}-1}e^{-\frac{1}{2}\omega r^2}L_n^{2\Delta_{j\alpha}-1}(\omega r^2) \quad . \tag{2.46}$$

In the limit $\nu = 0$, (2.40) becomes

$$E^h(\nu = 0) = (|m| + 2n + 1)\,\omega \tag{2.47}$$

and we recover the spectrum and degeneracy of a spin 1/2 particle in a two-dimensional harmonic oscillator. Or setting $s = 0$, one gets

$$E^h(s = 0) = (|j - \nu| + 2n + 1)\,\omega \tag{2.48}$$

which is the spectrum for the spinless case obtained in (2.13).

The case of $\mathcal{H}^B$ given in (2.29) offers itself to a similar analysis: using (2.8) with supercharges (2.30), we find that

$$\mathcal{H}^B = \Omega R^B \,|_{a=\frac{2}{\Omega}} \; - \frac{\Omega}{2}\,(J + S - \nu) \tag{2.49}$$

so the eigenvalues are by (2.34)

$$E^B = \left(\frac{1}{2}|j - s - \nu| + n + \frac{1}{2} - \frac{1}{2}(j + s - \nu)\right)\Omega \ . \tag{2.50}$$

Since $\mathcal{H}^B$ differs from R^B by only a constant, they have the same eigenfunctions, with the appropriate frequency:

$$\psi^{B\pm}_{j\alpha n}(r,\theta) = e^{im\theta}(-1)^n \left(\frac{\Omega}{2}\right)^{\Delta_{j\alpha}} \left[\frac{n!}{\pi\Gamma(2\Delta_{j\alpha} + n)}\right]^{1/2} r^{2\Delta_{j\alpha}-1} e^{-\frac{\Omega}{4}r^2} L_n^{2\Delta_{j\alpha}-1}\left(\frac{\Omega r^2}{2}\right) \ . \tag{2.51}$$

The limit $\nu = 0$ produce the spectrum

$$\begin{aligned} E^B(\nu = 0) &= \left(\frac{1}{2}|m| - \frac{m}{2} + n + \frac{1}{2} - s\right)\Omega \\ &\equiv \left(p + \frac{1}{2} - s\right)\Omega \end{aligned} \tag{2.52}$$

where p is a positive integer. We recognize the levels of a well-known supersymmetric system: an electron of gyromagnetic ratio 2 in a uniform magnetic field [10]. Here, the infinite degeneracy of the Landau levels shows up in the various possible values of $m > 0$. Also, in the limit $s = 0$, we have :

$$E^B(s=0) = \left(\frac{1}{2}|j-\nu| + n + \frac{1}{2} - \frac{1}{2}(j-\nu)\right)\Omega \ , \tag{2.53}$$

in agreement with the spinless result of Ref. [4].

2.4 Conclusion

A charged particle of spin 1/2 in the field of a vortex with an external harmonic oscillator or a uniform magnetic field exhibits, on the plane, a dynamical supersymmetry whose superalgebra, $spl^*(2,1)$, has $so(2,1)$ for its bosonic sector. By constructing a basis for the representation space of this superalgebra, it was showed that in both situations, the Casimir of the $so(2)$ subgroup of $so(2,1)$ is closely related to the Pauli Hamiltonian describing the system. In each case, this was used to determine all regular eigenstates and their eigenvalues.

Now even though the symmetries appear to be formally exact, the presence of the delta function in the hamiltonian calls for a careful analysis [12]. Further work could seek to identify the possible self-adjoint extensions required for the various generators involved [7]. Also, it is expected that supercoherent states can be constructed for these systems [11].

Bibliography

[1] For a recent review, see G. S. Canright and S. M. Girvin, *Science* **247** (1990) 1197.

[2] R. Jackiw, *Ann. Phys.* (NY) **201** (1990) 83.

[3] J. M. Leinaas, J. Myrheim, *Nuovo Cimento* **37B** (1977) 1.

[4] T. Awaji, M. Hotta, *Int. J. Mod. Phys.* **B6** (1992) 1229.

[5] F. A. Berezin and M. S. Marinov, *JETP Lett.* **21** (1975) 30; *Ann. Phys.* (NY) **104** (1977) 336; R. Casalbuoni, *Nuovo Cimento* **33A** (1976) 389.

[6] C. J. Park, *Nucl. Phys.* **B376** (1992) 99.

[7] E. D'Hoker, L. Vinet, *Phys. Lett.* **137** (1984) 72; *Comm. Math. Phys.* **97** (1985) 391.

[8] S. Takagi, *Prog. Theor. Phys.* **85** (1991) 463; **85** (1991) 723.

[9] T. Blum, C. R. Hagen, S. R. Ramaswamy *Phys. Rev. Lett.* **64** (1990) 709.

[10] A. Barducci, R. Casalbuoni, L. Lusanna, *Nuovo Cimento* **35A** (1976) 377; R. W. Haymaker, A. R. P. Rau, *Am. J. Phys.* **54** (1986) 928.

[11] B. W. Fatyga, V. A. Kostelecky, M. M. Nieto, D. R. Truax, *Phys. Rev. D* **43**, (1991) 1403.

[12] R. Jackiw, Delta-Function Potentials in Two and Three dimensional Quantum Mechanics, MIT-CTP preprint #1937 (1991).

Chapter 3

HERMITICITY AND THE COHOMOLOGY CONDITION IN TOPOLOGICAL YANG-MILLS THEORY

Abstract

The symmetries of the topological Yang-Mills theory are studied in the Hamiltonian formalism and the generators of the twisted N=2 super-Poincaré algebra are explicitly constructed. Noting that the twisted Lorentz generators do not generate the Lorentz symmetry of the theory, we relate the two by extracting from the latter the twisted version of the internal SU(2) generator. The hermiticity properties of the various generators are also considered throughout, and the boost generators are found to be non-hermitian. We then recover the BRST cohomology condition on physical states from representation theory arguments.

To appear in *Annals of Physics*

3.1 Introduction

In recent years, much attention has been devoted to the study of topological field theories [1]. Because these theories have no local dynamics, their correlation functions depend only on the global features of the target space. An important example is given by the topological Yang-Mills (TYM) theory, which was used to obtain the Donaldson invariants for smooth 4-manifolds[2]. Shortly after TYM was introduced, it was shown [3, 4] that it can also be obtained by BRST gauge fixing the topological symmetry ($\delta A_\alpha^a = \theta_\alpha^a$, with θ_α^a arbitrary) of either zero or the topological action $S = \int d^4x \; F^{\mu\nu} \tilde{F}_{\mu\nu}$. Under an appropriate choice of gauge parameters, the resulting action is identical to the one introduced in [2] and is given by:

$$\begin{aligned} S &= \int_M Tr\Big\{\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2}D_\alpha\phi D^\alpha\lambda - i\eta D_\alpha\psi^\alpha + iD_\alpha\psi_\beta\chi^{\alpha\beta} \\ &\qquad -\frac{i}{8}\phi\left[\chi_{\alpha\beta},\chi^{\alpha\beta}\right] - \frac{i}{2}\lambda\left[\psi_\alpha,\psi^\alpha\right] - \frac{i}{2}\phi\left[\eta,\eta\right] - \frac{1}{8}\left(\left[\phi,\lambda\right]\right)^2\Big\} \\ &\equiv \int_M \mathcal{L}. \end{aligned} \tag{3.1}$$

Here, all fields are Lie algebra valued and transform according to the adjoint representation of the gauge group, which is taken to be compact and semi-simple. The covariant derivative is $D_\alpha = \nabla_\alpha + [A_\alpha, \;]$, where ∇_α is the covariant derivative with respect to the diffeomorphisms on the curved manifold M of metric $g_{\mu\nu}$. The gauge field A_α and the scalars λ and ϕ are bosonic while η, ψ_α and $\chi_{\alpha\beta}$ are all anticommuting and respectively scalar, vector and self-dual tensor fields $\left(\chi_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta}{}^{\mu\nu}\chi_{\mu\nu}\right)$. Note that in this version (3.1) still possesses the usual (non-topological) Yang-Mills symmetry.

When (3.1) was introduced, its intimate relation with $N = 2$ super Yang-Mills (SYM) was already noticed [2]. In fact, formal representations of the Donaldson polynomials have also been obtained in SYM[11]. In Euclidean space-time, the latter theory enjoys the Lorentz symmetry $SO(4)$ (isomorphic to $SU_L(2) \otimes SU_R(2)$) as well as the internal global $SU_I(2)$ symmetry. If one "twists" this symmetry by replacing $SU_L(2)$ by the diagonal sum of $SU_L(2)$ and $SU_I(2)$, $SU_{L'}(2)$, the rotation group then becomes $SU_{L'}(2) \otimes SU_R(2)$ and the resulting theory is just (3.1). Through this procedure the original supersymmetry generators are also transformed and the Lorentz scalar supercharge thus obtained is identified as the BRST charge. The twist procedure has also been used to obtain extended ($N = 2$) TYM theories[5]. Furthermore, TYM has also been obtained via the use of Killing spinors in $N = 2$ conformal supergravity[17]; in this case, a "local" version of the twisting procedure is implemented by embedding the $SU(2)$ connection in the Lorentz spin connection.

In this paper, we will detail the twisting of the $N = 2$ supersymmetry (Section 3.2) and explicitly construct the various generators while studying their hermiticity properties (Section 3.3). We will argue that after twisting, the internal symmetry generators are transformed into a useful and hitherto unappreciated symmetry of (3.1). It will also be shown that the boost generators are not hermitian. This will be used in Section 3.4 to discuss the following issue. Despite their connection through twisting, TYM and SYM theories differ in that the former does not support any local excitations. When TYM is considered through the BRST construction, it is found that the only states in the cohomology of the BRST charge are those with vanishing energy [2]. Among other things, this absence of local excitations complicates any attempt to use topological field theories in a description of quantum gravity, a possibility suggested by the natural general covariance of these theories. It is hence usually thought that one must first establish a mechanism to break the topological symmetry. As a consequence, we find it compelling to study more closely the relation between SYM and TYM. Within the context of twisting and without appealing to the BRST derivation of (3.1), we will propose an explanation, based on representation theory arguments, of why TYM is indeed free of local excitations. As we are only interested in the details of the canonical quantization of the theory, such as its hermiticity properties and spectrum, we will work on flat manifolds. Our concluding remarks are contained in Section 3.5.

3.2 Twisted N=2 Supersymmetry Algebra

Our starting point is the $N = 2$ superPoincaré algebra (without central charge) [7, 8]:

$$
\begin{aligned}
[P_\alpha, P_\beta] &= 0, \qquad [P_\mu, J_{\alpha\beta}] = i\, g_{\mu[\alpha}\, P_{\beta]}, && (3.2a)\\
[J_{\alpha\beta}, J_{\mu\nu}] &= i g_{\alpha[\mu}\, J_{\nu]\beta} + i g_{\beta[\mu}\, J_{\alpha|\nu]}, && (3.2b)\\
[P_\mu, Q_{Ai}] &= 0 = [P_\mu, \bar{Q}_{\dot{A}j}], && (3.2c)\\
[Q_{Ai}, J_{\alpha\beta}] &= (\sigma_{\alpha\beta})_A{}^B Q_{Bi}, \qquad [\bar{Q}^{\dot{A}}{}_j, J_{\alpha\beta}] = (\bar{\sigma}_{\alpha\beta})^{\dot{A}}{}_{\dot{B}} \bar{Q}^{\dot{B}}{}_j, && (3.2d)\\
\{Q_{Ai}, \bar{Q}_{\dot{B}}{}^j\} &= 2\delta_i{}^j P_\alpha (\sigma^\alpha)_{A\dot{B}}, && (3.2e)\\
\{Q_{Ai}, Q_{Bj}\} &= 0 = \{\bar{Q}_{\dot{A}}{}^i, \bar{Q}_{\dot{B}}{}^j\}, && (3.2f)\\
[T^i{}_j, J_{\alpha\beta}] &= 0 = [T^i{}_j, P_\alpha], && (3.2g)\\
[T^i{}_j, T^k{}_l] &= \frac{1}{2}\left(\delta^k_j\, T^i{}_l - \delta^i{}_l\, T^k{}_j\right), && (3.2h)\\
[T^i{}_j, Q_{Ak}] &= -\frac{1}{2}\left(\delta^i_k\, Q_{Aj} - \frac{1}{2}\delta^i_j\, Q_{Ak}\right), && (3.2i)\\
[T^i{}_j, \bar{Q}_{\dot{A}k}] &= -\frac{1}{2}\left(\delta^i_k\, \bar{Q}_{\dot{A}j} - \frac{1}{2}\delta^i_j\, \bar{Q}_{\dot{A}k}\right). && (3.2j)
\end{aligned}
$$

Our convention closely follows Ref. [9]. Bracketed indices are to be antisymmetrized, ignoring the ones just preceeding a vertical bar (thus $g_{\beta[\mu}J_{\alpha|\nu]} \equiv g_{\beta\mu}J_{\alpha\nu} - g_{\beta\nu}J_{\alpha\mu}$). Greek letters denote Lorentz indices, with P_α and $J_{\alpha\beta}$ standing for translation and Lorentz generators respectively. Capital latin letters are two-spinor indices with undotted ones referring to $SU_R(2)$ and dotted ones to $SU_L(2)$. Raising and lowering these indices is done with the help of the antisymmetric matrices $\varepsilon_{AB}, \varepsilon^{AB}, \varepsilon_{\dot{A}\dot{B}}$ and $\varepsilon^{\dot{A}\dot{B}}$. They are given by: $\varepsilon_{12} = -\varepsilon^{12} = \varepsilon_{\dot{1}\dot{2}} = -\varepsilon^{\dot{1}\dot{2}} = -1$ and act on Weyl spinors as: $\bar{\psi}_{\dot{A}} = \varepsilon_{\dot{A}\dot{B}}\bar{\psi}^{\dot{B}}, \chi^A = \varepsilon^{AB}\chi_B$. The internal indices are $i, j...$ (in subsequent sections, these symbols will be used as spatial components of Lorentz indices); $T^i{}_j$ is traceless and generate $SU_I(2)$. The metric $g_{\alpha\beta}$ is euclidean ($= \delta_{\alpha\beta}$) whereas $\sigma_\alpha = (-i, \sigma_j), \bar{\sigma}_\alpha = (i, \sigma_j)$ where σ_i are the usual Pauli matrices. Similarly to the Minkowski case, we define $\sigma_{\alpha\beta} = \frac{i}{4}(\sigma_\alpha\bar{\sigma}_\beta - \sigma_\beta\bar{\sigma}_\alpha)$ and $\bar{\sigma}_{\alpha\beta} = \frac{i}{4}(\bar{\sigma}_\alpha\sigma_\beta - \bar{\sigma}_\beta\sigma_\alpha)$.

We now perform the twisting of this algebra. Replacing $SU_L(2)$ by the diagonal sum of $SU_L(2)$ and $SU_I(2)$ translates into the identification of the internal indices with left handed Weyl spinor indices, leading to:

$$Q_{Ai} \;\rightarrow\; Q_{A\dot{C}} = a_1 (\sigma^{\alpha})_{A\dot{C}} \bar{Q}_{\alpha}, \tag{3.3a}$$

$$\bar{Q}^{\dot{B}}{}_{j} \;\rightarrow\; \bar{Q}^{\dot{B}}{}_{\dot{D}} = a_2\, \delta^{\dot{B}}_{\dot{D}}\, Q_{+}\, a_3 (\bar{\sigma}^{\mu\nu})^{\dot{B}}{}_{\dot{D}}\, S_{\mu\nu}, \tag{3.3b}$$

$$T^{i}{}_{j} \;\rightarrow\; T^{\dot{A}}{}_{\dot{B}} = a_4\, (\bar{\sigma}^{\mu\nu})^{\dot{A}}{}_{\dot{B}}\, R_{\mu\nu}, \tag{3.3c}$$

where on the RHS, the twisted quantities are expressed in terms of their Lorentz components: $\bar{Q}_{\alpha}$ is a vector, Q a scalar, whereas $S_{\mu\nu}$ and $R_{\mu\nu}$ are self-dual tensors; a_i's are arbitrary constants. At this stage, these constants could be absorbed in the definition of the generators, but they will be useful in the next section, as we will use already known expressions for Q and $\bar{Q}_{\alpha}$. $Q, \bar{Q}_{\alpha}$ and $S_{\mu\nu}$ are Grassman odd, whereas $R_{\mu\nu}$ is Grassman even. Note that vectorial Grassman charges are also known to exist in non-critical string theory[10]. The relations (3.3) can be inverted:

$$\bar{Q}_{\alpha} = \frac{1}{2a_1} (\sigma_{\alpha})^{\dot{C}A} Q_{A\dot{C}}, \tag{3.4a}$$

$$Q = \frac{1}{2a_2}\, \delta^{\dot{D}}_{\dot{B}}\, \bar{Q}^{\dot{B}}{}_{\dot{D}}, \tag{3.4b}$$

$$S_{\mu\nu} = \frac{1}{2a_3} (\bar{\sigma}_{\mu\nu})^{\dot{B}}{}_{\dot{D}}\, \bar{Q}^{\dot{D}}{}_{\dot{B}}, \tag{3.4c}$$

$$R_{\mu\nu} = \frac{1}{2a_4} (\bar{\sigma}_{\mu\nu})^{\dot{A}}{}_{\dot{B}}\, T^{\dot{B}}{}_{\dot{A}}, \tag{3.4d}$$

where we have made use of the identity:

$$\mathrm{Tr}\ (\bar{\sigma}_{\alpha\beta}\, \bar{\sigma}_{\mu\nu}) = \frac{1}{2} \delta_{\alpha[\mu}\, \delta_{\nu]\beta} + \frac{1}{2} \epsilon_{\alpha\beta\mu\nu}. \tag{3.5}$$

Under the twisting (3.3), the superPoincaré algebra (3.2) is transformed into:

$$[P_{\alpha}, P_{\beta}] = 0, \qquad [P_{\mu}, J_{\alpha\beta}] = i\delta_{\mu[\alpha}\, P_{\beta]}, \tag{3.6a}$$

$$[J_{\alpha\beta}, J_{\mu\nu}] = i\delta_{\alpha[\mu}\, J_{\nu]\beta} + i\delta_{\beta[\mu}\, J_{|\alpha\nu]}, \tag{3.6b}$$

$$[P_{\beta}, Q] = [P_{\beta}, \bar{Q}_{\alpha}] = [P_{\beta}, S_{\mu\nu}] = 0, \tag{3.6c}$$

$$[Q, J_{\alpha\beta}] = \frac{a_3}{a_2} S_{\alpha\beta}, \tag{3.6d}$$

$$
\begin{aligned}
[\bar{Q}_\mu, J_{\alpha\beta}] &= \frac{i}{2}\left(\delta_{\mu[\alpha}\, S_{\beta]\nu} - \delta_{\nu[\alpha}\, S_{\beta]\mu}\right), && (3.6e)\\
[S_{\mu\nu}, J_{\alpha\beta}] &= \frac{Q}{4}\frac{a_2}{a_3}\left(\delta_{\alpha[\mu}\, \delta_{\nu]\beta} + \epsilon_{\alpha\beta\mu\nu}\right) + \frac{i}{2}\left(\delta_{\mu[\alpha}\, S_{\beta]\nu} - \delta_{\nu[\alpha}\, S_{\beta]\mu}\right), && (3.6f)\\
\{Q, \bar{Q}_\alpha\} &= -\frac{1}{a_1 a_2} P_\alpha, && (3.6g)\\
\{\bar{Q}_\alpha, S_{\mu\nu}\} &= -\frac{i}{2a_1 a_3}\left(\delta_{\alpha[\mu}\, P_{\nu]} + \epsilon_{\alpha\mu\nu\beta} P^\beta\right), && (3.6h)\\
\{\bar{Q}_\alpha, \bar{Q}_\beta\} &= \{Q, Q\} = \{S_{\mu\nu}, S_{\alpha\beta}\} = 0, && (3.6i)\\
[Q, R_{\mu\nu}] &= \frac{a_3}{4a_2 a_4} S_{\mu\nu}, && (3.6j)\\
[R_{\mu\nu}, S_{\alpha\beta}] &= -\frac{a_2}{16a_3 a_4} Q\left(\delta_{\alpha[\mu}\, \delta_{\nu]\beta} + \epsilon_{\alpha\beta\mu\nu}\right) - \frac{i}{8a_4}\left(\delta_{\mu[\alpha}\, S_{\beta]\nu} - \delta_{\nu[\alpha}\, S_{\beta]\mu}\right) && (3.6k)\\
[\bar{Q}_\alpha, R_{\mu\nu}] &= -\frac{i}{8a_4}\left(\delta_{\alpha[\mu}\, \bar{Q}_{\nu]} + \varepsilon_{\mu\nu\alpha\beta}\, \bar{Q}^\beta\right), && (3.6l)\\
[R_{\mu\nu}, R_{\alpha\beta}] &= -\frac{i}{4a_4}\left(\delta_{\mu[\alpha}\, R_{\beta]\nu} - \delta_{\nu[\alpha}\, R_{\beta]\mu}\right), && (3.6m)\\
[R_{\mu\nu}, J_{\alpha\beta}] &= 0 = [R_{\mu\nu}, P_\beta]. && (3.6n)
\end{aligned}
$$

The existence of (3.6) was conjectured in [11, 12]. The Poincaré sector of the algebra, Eqs. (3.6a — 3.6b), is of course left unchanged by the twisting. This would suggest that for the twisted theory, $J_{\alpha\beta}$ also generate Lorentz rotations. However, a look at (3.6) reveals that the fermionic charges, as well as $R_{\mu\nu}$, do not transform in the expected way (e.g. Q does not transform as a scalar). In the following section, we will examine how the algebra (3.6) is realized in TYM, and identify the correct Lorentz generators.

3.3 The Algebra Realized

In order to study the twisted $N = 2$ superPoincaré symmetries of (3.1), we make use of Noether's theorem in its Lagrangian form. Under a symmetry transformation, the variation of the Lagrangian density is a total derivative $\delta\mathcal{L} = \partial_\mu \Lambda^\mu$ and using the equations of motion, the current $J^\mu = \sum_{fields\Phi} \delta\Phi \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} - \Lambda^\mu$ is conserved [13]. The simplest of the symmetries is the invariance under translation, for which $\delta\mathcal{L} = a^\mu \partial_\mu \mathcal{L}$ with a^μ a constant infinitesimal parameter. The corresponding form of the energy-momentum tensor is given by:

$$\begin{aligned}\theta^{\alpha\gamma} &= F^{\gamma}{}_{\mu}F^{\alpha\mu} - \frac{1}{2}D^{\alpha}\phi\, D^{\gamma}\lambda - \frac{1}{2}D^{\alpha}\lambda\, D^{\gamma}\phi + iD^{\gamma}\psi^{\alpha}\ \eta + iD^{\gamma}\psi_{\mu}\ \chi^{\alpha\mu} \\ &\quad + D_{\mu}A^{\gamma}\ F^{\alpha\mu} + A^{\gamma}J^{\alpha} - g^{\alpha\gamma}\mathcal{L} \end{aligned} \tag{3.7}$$

with

$$J_{\alpha} = \frac{1}{2}\left[\phi, D_{\alpha}\lambda\right] + \frac{1}{2}\left[\lambda, D_{\alpha}\phi\right] - i\left[\psi_{\alpha}, \eta\right] - i\left[\psi^{\mu}, \chi_{\alpha\mu}\right], \tag{3.8}$$

and where $\mathcal{L}$ is the Lagrangian density given in (3.1). The conservation of this tensor, $\partial_{\alpha}\theta^{\alpha\gamma} = 0$, gives rise to the energy and momentum generators:

$$\begin{aligned} P_0 &= \int d^3x \left\{ \frac{1}{2}\left(F_{0i}F_{0i} - \tilde{F}_{0i}\tilde{F}_{0i}\right) - \frac{1}{2}D_0\phi\, D_0\lambda + \frac{1}{2}D_i\phi\, D_i\lambda - i\varepsilon_{ijk}D_j\psi_k\, \chi_i \right. \\ &\qquad - i\psi_0 D_i\chi_i + i\eta D_i\psi_i + \frac{i}{2}\phi\left[\chi_i, \chi_i\right] + \frac{i}{2}\lambda\left[\psi_i, \psi_i\right] \\ &\qquad \left. + \frac{i}{2}\lambda\left[\psi_0, \psi_0\right] + \frac{i}{2}\phi\left[\eta, \eta\right] + \frac{1}{8}\left(\left[\phi, \lambda\right]\right)^2 - A_0 G \right\}, \end{aligned} \tag{3.9}$$

$$P_i = \int d^3x \left\{ F_{0k}F_{ik} - \frac{1}{2}D_0\lambda\, D_i\phi - \frac{1}{2}D_0\phi\, D_i\lambda + iD_i\psi_j\, \chi_j + iD_i\eta\, \psi_0 - A_i\, G \right\}, \tag{3.10}$$

where $\chi_i \equiv \chi_{0i}$. Here and in the rest, integrations are over "spatial" coordinates, with traces understood. We also ignore ordering ambiguities. In Eq. (3.9), A_0 should be viewed as the Lagrange multiplier which imposes the generalized Gauss law constraint $G \equiv D_iF_{0i} - J_0 \approx 0$. The Hamiltonian (3.9) can equally be obtained by Legendre transforming (3.1). In order to compute the algebra of these charges, we first identify the various momenta of (3.1), and impose on them the appropriate equal-time canonical commutators:

$$\begin{aligned} P_{\psi_i^a} &= i\chi_i^a, & \{\chi_i^a(x), \psi_j^b(y)\} &= -\delta^{ab}\delta_{ij}\delta(x-y), \\ P_{\psi_0^a} &= i\eta^a, & \{\psi_0^a(x), \eta^b(y)\} &= -\delta^{ab}\delta(x-y), \\ P_{A_i^a} &= F_{0i}^a, & [A_i^a(x), F_{0j}^b(y)] &= i\delta^{ab}\delta_{ij}\delta(x-y), \\ P_{\phi^a} &= -\frac{1}{2}(D_0\lambda)^a, & [\phi^a(x), -\frac{1}{2}(D_0\lambda)^b(y)] &= i\delta^{ab}\delta(x-y), \end{aligned}$$

$$P_{\lambda^a} \;=\; -\frac{1}{2}(D_0\phi)^a, \qquad\qquad [\lambda^a(x), -\frac{1}{2}(D_0\phi)^b(y)] = i\delta^{ab}\delta(x-y), \quad (3.11)$$

where x and y denote here space coordinates. Making use of these commutators, P_0 and P_i are found to correctly translate the fields, and when commuted among themselves yield:

$$[P_i, P_j] = 0, \qquad\qquad [P_0, P_j] = i\int d^3x\; \partial_j A_0(x) G(x). \qquad (3.12)$$

As with various forthcoming commutators, we find that because of the remnant Yang-Mills symmetry in the action (3.1), the algebra (3.6) is only realized on physical states, annihilated by the constraint G.

We now wish to study the hermiticity properties of our generators. We take for adjoint assignments:

$$\begin{aligned} A_\alpha^\dagger &= A_\alpha, \\ \psi_i^\dagger &= -\chi_i, \\ \psi_0^\dagger &= \eta, \\ \phi^\dagger &= \lambda. \end{aligned} \qquad (3.13)$$

Despite its non-covariance, this choice is natural for various reasons. In order for the field theory to be well defined, P_0 should be hermitian and it is under (3.13). Moreover, as is shown below, this choice also leads to a semi-positive definite spectrum for P_0, in analogy with SYM theory. Also, P_i and the Lagrangian (3.1) are equally hermitian with this prescription. Note that because of the peculiarity of the self-duality operation in euclidean metric[14] ($\chi_{\alpha\beta} = \frac{1}{2\lambda}\varepsilon_{\alpha\beta\mu\nu}\chi^{\mu\nu}$ with $\lambda = 1, i$ for euclidean and Minkowskian metrics respectively), we require $\varepsilon_{\alpha\beta\mu\nu}$ to change sign when taking the adjoint. Given that the presence of $\varepsilon_{\alpha\beta\mu\nu}$ in the various generators has its origin in the self-dualtiy of $\chi_{\alpha\beta}$, this prescription in effect reproduce the study of hermiticity in Minkowskian metric. An alternative road would be to study the Lagrangian (3.1) in Minkowski spacetime. The symmetry generators would then, up to signs, be the same as the ones presented here for euclidean metric. The algebra of the generators in that case would be a Wick rotated version of (3.6), obtained by the change: $\delta_{\mu\nu} \to \eta_{\mu\nu}(\equiv diag(-1,1,1,1))$, $\sigma^\alpha \to (1,\sigma_i)$, $\bar{\sigma}^\alpha \to (-1,\sigma_i)$, $\varepsilon_{\alpha\beta\mu\nu} \to \frac{1}{i}\varepsilon_{\alpha\beta\mu\nu}$.

Under Lorentz transformations, the variation of the fields is $\delta\Phi = \omega^{\mu\nu}x_\mu\partial_\nu\Phi$ where $\omega^{\mu\nu}$ is an infinitesimal antisymmetric parameter. The corresponding currents:

$$\begin{aligned} M^{\alpha\beta\gamma} &= \Big[x^\beta\Big\{-\frac{1}{2}D^\alpha\lambda\, D^\gamma\phi - \frac{1}{2}D^\alpha\phi\, D^\gamma\lambda + F^\gamma{}_\mu F^{\alpha\mu} + iD^\gamma\eta\,\psi^\alpha + iD^\gamma\psi_\mu\,\chi^{\alpha\mu} \\ &\quad +\frac{A^\gamma}{2}[\lambda, D^\alpha\phi] + \frac{A^\gamma}{2}[\phi, D^\alpha\lambda] - iA^\gamma[\eta,\psi^\alpha] - iA^\gamma[\psi_\mu,\chi^{\alpha\mu}] - g^{\alpha\gamma}\mathcal{L}\Big\} \\ &\quad +D_\mu(x^\beta A^\gamma)\,F^{\alpha\mu} + i\psi^\gamma\chi^{\alpha\beta}\Big] - \Big[\beta \leftrightarrow \gamma\Big], \end{aligned} \tag{3.14}$$

are conserved ($\partial_\alpha M^{\alpha\beta\gamma} = 0$) and lead to the constants of motion associated with boosts and rotations:

$$M_{0i} = x_0 P_i - \int d^3x\; x_i\mathcal{P}_0 + i\int d^3x\; \psi_i\eta, \tag{3.15}$$

$$M_{kj} = \int d^3x\; \{(x_k\mathcal{P}_j - i\psi_k\chi_j) - (k \leftrightarrow j)\}, \tag{3.16}$$

where $\mathcal{P}_0$ is the energy density, as integrated in Eq.(3.9) and similarly for $\mathcal{P}_j$ from Eq.(3.10). One can readily check, using (3.13), that M_{kj} is hermitian but that M_{0i} is *not*. We will return to this point in Section 3.4, in connection with the possible excited states of the theory. Using Eq.(3.11), the following commutators are obtained:

$$[P_0, M_{0i}] = ix_0\int d^3x\,\partial_i A_0(x)G(x) + i\int d^3x\,A_i(x)G(x) + iP_i, \qquad [P_0, M_{kj}] = 0, \tag{3.17}$$

$$[P_j, M_{0i}] = -i\delta_{ij}P_0, \qquad\qquad [P_i, M_{kj}] = i\delta_{i[k}P_{j]}, \tag{3.18}$$

$$[M_{0i}, M_{0j}] = -iM_{ij}, \qquad\qquad [M_{0i}, M_{kj}] = i\delta_{i[k}M_{0]j} \tag{3.19}$$

$$[M_{kj}, M_{lm}] = i\delta_{k[l}M_{m]j} + i\delta_{j[l}M_{k|m]}. \tag{3.20}$$

Together with (3.12), we thus recover the Poincaré sector of (3.6).

Turning to the twisted supersymmetries, we have the scalar charge Q, identified in [2]. It is preserved on an arbitrary manifold and the energy momentum tensor can by expressed as a Q variation. In the context of the BRST construction of (3.1), it is precisely the BRST charge. Its expression is:

$$Q = \int d^3x \left\{ \left(F_{0i} + \tilde{F}_{0i} \right) \psi_i - \eta D_0 \phi - D_i \phi \, \chi_i - \frac{\psi_0}{2} [\lambda, \phi] \right\}. \tag{3.21}$$

Under translation and rotation, it transforms as:

$$[P_0, Q] = 0 = [P_j, Q], \tag{3.22}$$

$$[M_{0i}, Q] = i \int d^3x \, x_i \psi_0(x) G(x), \qquad [M_{kj}, Q] = 0, \tag{3.23}$$

$$\frac{1}{2}\{Q, Q\} = - \int d^3x \, \phi(x) G(x). \tag{3.24}$$

We thus recover the nilpotency of Q (up to gauge transformations), but (3.23) shows that $M_{\alpha\beta}$ does not correspond to the generator $J_{\alpha\beta}$ appearing in (3.6). This is confirmed by the study of $\bar{Q}_\alpha$, also identified in [2]. Its time and space components are given by:

$$\bar{Q} \equiv \bar{Q}_0 = \int d^3x \left\{ \left(F_{0i} - \tilde{F}_{0i} \right) \chi_i + \psi_0 D_0 \lambda - \psi_i D_i \lambda + \frac{\eta}{2} [\phi, \lambda] \right\}, \tag{3.25}$$

$$\begin{aligned} \bar{Q}_i \; = \; \int d^3x \Big\{ & \varepsilon_{ijk} \left(F_{0j} - \tilde{F}_{0j} \right) \chi_k + \psi_0 D_i \lambda + \psi_i D_0 \lambda - \left(F_{0i} - \tilde{F}_{0i} \right) \eta \\ & + \varepsilon_{ijk} \psi_j D_k \lambda + \frac{1}{2} [\phi, \lambda] \chi_i \Big\}. \end{aligned} \tag{3.26}$$

The spacetime symmetry transformations of $\bar{Q}$ and $\bar{Q}_i$ are made clear by:

$$[P_0, \bar{Q}] = -i \int d^3x \, \eta(x) G(x), \qquad [P_i, \bar{Q}] = 0, \tag{3.27}$$

$$[M_{0i}, \bar{Q}] = i\bar{Q}_i - i \int d^3x \, x_i \eta(x) G(x), \qquad [M_{kj}, \bar{Q}] = 0, \tag{3.28}$$

$$\frac{1}{2}\{\bar{Q}, \bar{Q}\} = - \int d^3x \, \lambda(x) G(x), \tag{3.29}$$

as well as:

$$[P_0, \bar{Q}_i] = -i \int d^3x\, \chi_i(x) G(x), \qquad\qquad [P_j, \bar{Q}_i] = 0, \tag{3.30}$$

$$[M_{0i}, \bar{Q}_j] = i\delta_{ij}\bar{Q}, \qquad\qquad [M_{kj}, \bar{Q}_i] = i\delta_{i[j}\bar{Q}_{k]}, \tag{3.31}$$

$$\frac{1}{2}\{\bar{Q}_i, \bar{Q}_j\} = -\delta_{ij} \int d^3x\, \lambda(x) G(x), \qquad\qquad \frac{1}{2}\{\bar{Q}, \bar{Q}_i\} = 0 \tag{3.32}$$

When $\bar{Q}_\alpha$ is anticommuted with the BRST generator, it gives:

$$\frac{1}{2}\{Q, \bar{Q}_\alpha\} = -P_\alpha - \int d^3x\, A_\alpha(x)\, G(x). \tag{3.33}$$

Thus, given our choice of generators, (3.21) (3.25) and (3.26), the relation (3.6g) is obtained, provided $a_1 a_2 = \frac{1}{2}$. Observe how the adjoint assignments (3.13) produce:

$$Q^\dagger = -\bar{Q}, \tag{3.34}$$

and as announced they render the Hamiltonian (3.9) semi-positive definite (as is the case in SYM).

To identify $S_{\mu\nu}$, we compute the adjoint of $\bar{Q}_i$, obtaining:

$$\begin{aligned} S_{0i} \;=\; \int d^3x \Big[& \varepsilon_{ijk} \left(F_{0j} + \tilde{F}_{0j}\right) \psi_k + \eta D_i\phi - \chi_i D_0\phi - \left(F_{0i} + \tilde{F}_{0i}\right) \psi_0 \\ & + \varepsilon_{ijk}\chi_j D_k\phi - \frac{1}{2}[\lambda, \phi]\, \psi_i \Big]. \end{aligned} \tag{3.35}$$

Its spacetime symmetry transformations and nilpotency are revealed by the following set of commutators:

$$[P_0, S_{0i}] = -i \int d^3x\, \psi_i(x) G(x), \qquad\qquad [P_j, S_{0i}] = 0, \tag{3.36}$$

$$[M_{0i}, S_{0j}] = -i\varepsilon_{ijk} S_{0k} - i \int d^3x\, x_j \psi_i(x) G(x), \qquad\qquad [M_{kj}, S_{0i}] = i\delta_{i[j} S_{0|k]}, \tag{3.37}$$

$$\frac{1}{2}\{S_{0i}, S_{0j}\} = -\delta_{ij} \int d^3x\, \phi(x) G(x). \tag{3.38}$$

Eqs.(3.36) and (3.37) show that S_{0i} generates a symmetry if Gauss's law is imposed and that it is a self-dual object. Relating to previous fermionic symmetries, we compute:

$$\{S_{0i}, Q\} = 0, \tag{3.39}$$

$$\frac{1}{2}\{S_{0i}, \bar{Q}\} = P_i + \int d^3x \, A_i(x)G(x), \tag{3.40}$$

$$\frac{1}{2}\{S_{0i}, \bar{Q}_j\} = -\delta_{ij}\left(P_0 + \int d^3x \, A_0(x)G(x)\right) + \varepsilon_{ijk}\left(P_k + \int d^3x \, A_k(x)G(x)\right), \tag{3.41}$$

which reproduces (3.6h), if $a_1 a_3 = \frac{-i}{4}$.

Noting now that the boost generators are not hermitian, we extract from them the twisted internal generators by taking the anti-hermitian part $R_{0i} \equiv M_{0i}^\dagger - M_{0i}$. In terms of the fields, it is simply:

$$R_{0i} = \int d^3x \, \left(-i\psi_i\eta + i\psi_0\chi_i + i\varepsilon_{ilm}\psi_l\chi_m\right). \tag{3.42}$$

Commuting with the Poincaré generators produces:

$$[P_0, R_{0i}] = 0, \qquad [P_j, R_{0i}] = 0, \tag{3.43}$$

$$[M_{0i}, R_{0j}] = -i\varepsilon_{ijk}R_{0k}, \qquad [M_{kj}, R_{0i}] = i\delta_{i[j}R_{0|k]}, \tag{3.44}$$

which shows that R_{0i} is also a self-dual object. When commuted with the fermionic symmetries and with itself, we get:

$$[R_{0i}, Q] = iS_{0i}, \qquad [R_{0i}, \bar{Q}] = i\bar{Q}_i, \tag{3.45}$$

$$[R_{0i}, \bar{Q}_j] = i\varepsilon_{ijk}\bar{Q}_k - i\delta_{ij}\bar{Q}, \qquad [R_{0i}, S_{0j}] = i\varepsilon_{ijk}S_{0k} - i\delta_{ij}Q, \tag{3.46}$$

$$[R_{0i}, R_{0j}] = 2i\varepsilon_{ijk}R_{0k}. \tag{3.47}$$

We thus find that (3.6) is realized in TYM with the following values of parameters: $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{-i}{4}$ and $a_4 = \frac{1}{8}$. As an infinitesimal transformation, $R_{\alpha\beta}$ only acts on fermionic fields (as is obvious from (3.42) and in parallel with SYM) :

$$\begin{aligned}
\delta_R \eta &= \frac{1}{2}\zeta_{\rho\sigma}\chi^{\rho\sigma}, \\
\delta_R \psi_\alpha &= -2\zeta_{\alpha\lambda}\psi^\lambda, \\
\delta_R \chi_{\alpha\beta} &= 2\zeta_{\alpha\beta}\eta - \zeta_{\lambda[\alpha}\chi_{\beta]}{}^\lambda, \qquad (3.48)
\end{aligned}$$

where $\zeta_{\alpha\beta}$ is an infinitesimal, commuting and self-dual parameter. Although relatively simple, this symmetry appears to have escaped notice. It would be interesting to investigate its use, for instance, in the perturbative renormalization of TYM [3, 15] or determine the class of manifolds on which it is preserved [12].

So far, we have thus identified for TYM all the generators in the twisted $N = 2$ superalgebra (3.6), with the exception of $J_{\alpha\beta}$. This generator should be hermitian, since it is so before twisting. The more or less natural object to consider here is the hermitian part of M_{0i}. So we conjecture:

$$\begin{aligned}
J_{0i} &= M_{0i} + \frac{R_{0i}}{2}, \qquad (3.49a) \\
J_{kj} &= M_{kj} + \frac{1}{2}\varepsilon_{kjl}R_{0l}, \qquad (3.49b)
\end{aligned}$$

where in (3.49b), we have used the self-duality of $R_{\mu\nu}$. Using the relations previously obtained, we find that on physical states, (3.6a — 3.6b), (3.6d — 3.6f), and (3.6n) are verified, with the above mentioned values of a_i's. Thus (3.49) is indeed the correct identification. In fact, this relation should be expected. After twisting, the Lorentz algebra is isomorphic to $SU_{L'}(2) \otimes SU_R(2)$ and thus some hybridization of the internal symmetry with the old Lorentz generators $J_{\alpha\beta}$ is expected.

3.4 Hermiticity and Excited States

As shown in the last section, TYM theory in flat Euclidian spacetime realizes the $SO(4)$ "Lorentz" algebra in such a way that the boost generators M_{0i} are non-hermitian (neither are they antihermitian). In order to classify the possible states of the theory, we wish to identify the unitary representations of the symmetry algebra. Let us concentrate here on the compact sector $SO(4)$. As is well known, the irreducible and unitary representations are in that case finite dimensional (dimension $(2\ell_1+1)(2\ell_2+1)$ with $\ell_1, \ell_2 = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots$), the generators are represented by hermitian matrices and the group elements related to the identity can be written as $e^{i\,\alpha_{\mu\nu} M^{\mu\nu}}$ with real parameters $\alpha_{\mu\nu}$. Now if M_{0i} is not hermitian, it is clear that as far as $SO(4)$ is concerned, the only admissible unitary representation will be the trivial one, in which $M_{0i} = 0$. The $SO(3)$ subgroup of spatial rotations generated by M_{kj} does not suffer this problem, and the Hilbert space of the theory could carry the usual labels ℓm of the $SO(3)$ representation since this subgroup commutes with P_0. But because M_{0i} are not hermitian, only $\ell = m = 0$ will be present in that case. This can be seen in the algebra: acting with both sides of (3.19) on the representation space will give the same result provided M_{ij} is also vanishing.

If TYM is considered in Minkowski spacetime, with $g_{\mu\nu} = \eta_{\mu\nu}$, M_{0i} will also be non-hermitian, with equally dramatic consequences. Suppose we are interested in the unitary representations of the twisted algebra (3.6), assumed to be rotated to Minkowski metric, as specified in section 3.3. To investigate them, we make use, as in the case of the superPoincaré algebra [7], of Wigner's method of induced representations [19]. This method is also appropriate here since our symmetry group possesses the same abelian invariant subgroup, namely the translations. In this method, one first makes a choice of "standard vector", eigenstates of P_μ and a representative member of the possible classes of eigenvalues of the Casimir P_μ^2. One then identifies the little group, formed by the generators that leave the standard vector intact, and excluding the abelian subgroup. Once the irreducible unitary representations of the little group have been identified (restricting to finite dimensional ones), they are then used to induce an irreducible unitary representation of the whole group. This is done by acting on the standard vector with the generators that change its eigenvalue of P_μ. These infinite dimensional representations then form the plane-wave basis, to which particles are associated.

Consider the massless case. The little supergroup is formed by $C_1 \equiv M_{10} +$

$M_{13}, C_2 \equiv M_{20} + M_{23},\ M_{12}, Q, \bar{Q}_\alpha, S_{\mu\nu}, R_{\mu\nu}$. Acting with any of these will leave the vector $|\, p_0^\mu = (m, 0, 0, m)\,\rangle$ unrotated. Now since

$$
\begin{aligned}
[C_1,\ M_{12}] &= -C_2, \\
[C_2,\ M_{12}] &= C_1, \\
[C_1,\ C_2] &= 0,
\end{aligned}
\tag{3.50}
$$

is the Lie algebra E_2, and since we seek a finite dimensional representation, we are led to $C_1 = C_2 = 0$ when acting on the standard vector, just as in the superPoincaré algebra [7]. Thus, at this level, the non-hermiticity of M_{10} and M_{20} appears irrelevant. However, in order to induce a representation of the entire group, we need a unitary realization of the finite transformation generated by M_{30}, $M_{10} - M_{13}$ and $M_{20} - M_{23}$. But with M_{30} non-hermitian, this can only be implemented through a trivial realization: $M_{03} = 0$. This in turn implies that if we consider the first part of Eq. (3.17) and choose $i = 3$ when acting on $|\, p_0\,\rangle$, the LHS will vanish, and lead to $P_3|p_0\rangle = 0$. (We refer to euclidean commutators for convenience; at this point the results clearly do not depend on the signs appearing in them.) One thus conclude that massless excitations will not occur in TYM.

A similar situation occurs if one attempts to construct massive representations. Taking as the standard vector $|p_1^\mu = (m, 0, 0, 0)\rangle$, the little group is made of (M_{kj}, Q, $\bar{Q}_\alpha$, $S_{\mu\nu}$, $R_{\mu\nu}$). Inducing a representation of the whole group will require a unitary operator for finite boosts, again this is only possible if the action of M_{0i} is trivial: $M_{0i}|p_1\rangle = 0$. But using now the first part of (3.18), we find $P_0|p_1\rangle = 0$, again contradicting the assumption on $|p_1\rangle$. In this way, we recover, in a group theoretical context, the absence of dynamics in TYM.

We now focus on the last possibility : null representations with standard vector $|p_3^\mu = 0\rangle$. (We will not consider spacelike representations). This vector is left unchanged by any Lorentz transformation and the little group is made of all the generators: $M_{\alpha\beta}$, Q, $\bar{Q}_\alpha$, $S_{\alpha\beta}$ and $R_{\alpha\beta}$. Here, representations of the full group and the little group coincide. As before, because we seek unitary representations, we will require that $M_{0i}|p_3\rangle = 0$. When used in (3.19) we obtain $M_{kj}|p_3\rangle = 0$, showing the rotational invariance of $|p_3\rangle$, which has thus the characteristics of a vacuum state. Turning now to the action of Q, consider the time component of (3.33), it reads:

$$\{Q, Q^\dagger\}\,|p_3\,\rangle = 0 \tag{3.51}$$

since $|p_3\rangle$ is by construction a physical state. Projecting on $\langle p_3|$, we find

$$\langle Q\,p_3|Q\,p_3\rangle + \langle Q^\dagger\,p_3|Q^\dagger p_3\rangle = 0 \tag{3.52}$$

and conclude that $Q|p_3\rangle = Q^\dagger|p_3\rangle = 0$.

Similarly, we can easily determine that the other generators have eigenvalue 0. By (3.34), $\bar{Q}|p_3\rangle = 0$. Making use of (3.28), we then find $\bar{Q}_i|p_3\rangle = 0$. Applying the same reasoning with (3.41) and (3.47), we find $S_{0i}|p_3\rangle = 0$ and $R_{0i}|p_3\rangle = 0$. Thus, all generators act trivially in TYM.

Now as mentioned before, the Lagrangian in (3.1) can be obtained by gauge fixing of a topological symmetry . The BRST charge introduced in that construction is the scalar Q given in (3.21). In that context, the physical states are assumed to be annihilated by Q, and such that they are not of the form $Q|\alpha\rangle$. Having shown the former, we now argue for the latter, following Ref. [2]. Consider a state $|\psi\rangle = Q|\alpha\rangle$, with $P_0|\psi\rangle = 0$. Because $[P_0, Q] = 0$, $|\psi\rangle$ and $|\alpha\rangle$ can be chosen to have the same eigenvalue under P_0. But with $P_0|\alpha\rangle = 0$, applying the steps given in (3.51) and (3.52) will lead to $|\psi\rangle = 0$. We thus obtain, in the context of twisted $N = 2$ SYM, the BRST cohomology condition of Refs. [2, 3, 4] on physical states.

3.5 Conclusion

We have used the Hamiltonian formalism to study the symmetries of (3.1). This formalism offers the inconvenience of a non manifest covariance, but made explicit the generators, as well as the "propagation" of the Gauss law constraint through the algebra. In this context, it would be interesting to see how the algebra we have obtained is modified by the gauge fixing of the Yang-Mills symmetry [18]. We were also able to make precise the relation between the Lorentz generators of TYM ($M_{\alpha\beta}$) and the twisted version of Lorentz and $SU_I(2)$ generators of SYM ($J_{\alpha\beta}$ and $R_{\alpha\beta}$ respectively) as displayed in (3.49). It is usually not illuminating to add symmetries to obtain new ones, but the interest here lies in their physical significance. One could avoid introducing the non-hermitian M_{0i}. But in order to understand the Lorentz structure of the various objects (fields, charges, etc) of the theory, they are needed. It is thus more sensible to discard $J_{\alpha\beta}$, keeping $M_{\alpha\beta}$ and $R_{\alpha\beta}$. In this way, $R_{\alpha\beta}$ appears as a symmetry of (3.1) unappreciated in previous work. In fact, its existence may seem odd at first sight, in view of the Coleman-Mandula theorem[20]. But as we have shown in Section 3.4, no massive unitary representations are realised in TYM, and in this way, the conclusions of the theorem are inapplicable. Nevertheless, more could be learned about $R_{\alpha\beta}$. Extending to more general manifolds, is it preserved[12]? Can it be used, along the lines of [16], to draw conclusions on the quantum theory at all orders in perturbation theory by restricting the possible counterterms (provided anomalies are absent)? It would also be interesting to investigate the extent of that symmetry in other topological theories. For instance, the symmetry algebra of the Chern-Simons theory in the Laudau gauge has been found to coincide with a twisted $N = 4$ superalgebra [13, 21]. It is expected that a twisted internal symmetry will also exist in that case.

Bibliography

[1] For a review, see D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Phys. Rep.* **209** (1991) 129.

[2] E. Witten, *Comm. Math. Phys.* **117** (1988) 353.

[3] R. Brooks, D. Montano and J. Sonnenschein, *Phys. Lett.* **B214** (1988) 91.

[4] L. Baulieu and I.M. Singer, *Nucl. Phys. (Proc. Suppl.)* **5B** (1988) 12.

[5] J.P. Yamron, *Phys. Lett.* **B213** (1988) 325.

[6] A. Karlhede and M. Roček, *Phys. Lett.* **B212** (1988) 51.

[7] P. West, *Introduction to Supersymmetry and Supergravity* (2nd ed. World Scientific, 1990).

[8] S.J. Gates, M.T. Grisaru, M. Roček and W. Siegel, *Superspace* (Frontiers in Physics, Benjamin Cummings, 1983).

[9] M.F. Sohnius, *Phys. Rep.* **128** (1985) 39.

[10] R. Brooks, *Phys. Lett.* **B267** (1991) 481

[11] R. Brooks and D. Kastor, *Phys. Lett.* **B246** (1990) 99.

[12] A. Galperin and O. Ogievetsky, *Phys. Lett.* **B236** (1990) 33.

[13] S.B. Treiman, R. Jackiw, B. Zumino and E.Witten, *Current Algebra and Anomalies* (Princeton Series in Physics, 1985), see R. Jackiw in *Field Theoretic Investigations in Current Algebra.*

[14] C. Nash and S. Sen, *Topology and Geometry for Physicists* (Academic Press, 1983).

[15] H.D Dahmen, S. Marculescu and L. Szymanowski, *Nucl. Phys.* **B383**, (1992) 110; *Phys. Lett.* **B252** (1990) 591.

[16] M. Werneck de Oliveira, *Phys. Lett.* **B307** (1993) 347.

[17] F. Delduc, F. Gieres and S.P. Sorella, *Phys. Lett.* **B225** (1989) 367.

[18] S. Ouvry, R. Stora and P. Van Baal, *Phys. Lett.* **B220** (1989) 159.

[19] E.P. Wigner, Ann. of Math. **40** (1939) 149.

[20] S. Coleman and J. Mandula, *Phys. Rev.* **159** (1975) 1251.

[21] R. Brooks, J.-G. Demers and C. Lucchesi, MIT-CTP preprint 2237, hep-th 9308127.

Chapter 4

TWISTING TO ABELIAN BF/CHERN-SIMONS THEORIES

Abstract

Starting from a $D = 3$, $N = 4$ supersymmetric theory for matter fields, a twist with a Grassmann parity change is defined which maps the theory into a gauge fixed, abelian BF theory on curved 3-manifolds. After adding surface terms to this theory, the twist is seen to map the resulting supersymmetric action to two uncoupled copies of the gauge fixed Chern-Simons action. In addition, we give a map which takes the BF and Chern-Simons theories into Donaldson-Witten TQFT's. A similar construction, but with $N = 2$ supersymmetry, is given in two dimensions.

Produced in collaboration with Roger Brooks and Claudio Lucchesi, and will appear in *Nuclear Physics* **B**.

4.1 Introduction

This paper deals with the problem of mapping supersymmetric field theories into topological field theories (TFT's) [1-6] and of mapping different classes of TFT's among themselves. TFT's fall under two classes. The first of the TFT's are the Schwarz-type [2], commonly known as BF, theories. Chern-Simons theory in three dimensions is a special case of BF theory. The second are Donaldson-Witten or Topological Quantum Field Theories (TQFT's) [3]. A sub-class of the TQFT's, the topological Yang-Mills (TYM) theories are gauge invariant. Another sub-class of the TQFT's is given by the topological sigma models which do not possess gauge invariances.

To date, these two classes of theories have had vastly different origins. On the one hand, the BF theories have non-trivial classical actions and first order equations of motion. Their classical (abelian) actions on manifolds of dimension D are metric independent as they are of the form $\int_M B_{(k)} \wedge F_{(D-k)}$, where $F_{(D-k)} = dA_{(D-k-1)}$ and the subscript denotes the form's degree. These theories are invariant under Maxwell (or Yang-Mills) gauge symmetries. They are also symmetric under the k-form symmetry which shifts $B_{(k)}$ into the exterior derivative of a $(k-1)$-form. On the other hand, the TQFT's classical lagrangians are either 0 or a total derivative and are devoid of classical equations of motion. Apart from the possible surface term, the entire lagrangian of a TQFT is obtained [7, 8, 9] as a BRST gauge fixing of a symmetry (topological symmetry) which manifests itself as compactly supported shifts of some field in the theory (for example, the gauge field in TYM). A large class of the latter theories may also be obtained from $N = 2$ [3, 10] or even $N \geq 2$ [11] supersymmetric theories via a procedure known as twisting.

We will work in three and two dimensions restricting ourselves to abelian BF theories. Placed in this context, we will solve a problem which has existed since the birth of these theories; namely, how to obtain the BF theories via the twisting of some supersymmetric theory. Furthermore, we will make substantial progress towards solving an equally long-standing problem; namely, what (if any) is the relation between BF theories and TQFT's.

As the twisting process will play an important role in our work, it is appropriate to give a quick review [3] using the example of $\mathbf{R}^4$. Starting with a $N = 2$ supersymmetric field theory and writing the Lorentz group as $SO_L(4) \simeq SU_l(2) \times SU_r(2)$, we then take the diagonal sum of $SU_l(2)$ with the automorphism group of the $N = 2$ superalgebra, $SU_I(2)$. The result is a $SU_d(2)$, which we use to form a new Lorentz group, $SO_{L'}(4) \simeq SU_d(2) \times SU_r(2)$. As a result, spin-$\frac{1}{2}$ fields, which also transformed as doublets of $SU_I(2)$, now become integer spinned, Grassmann odd fields.

The twisted super-Poincaré algebra, along with its implications for physical states, has recently been investigated in ref. [12]. In three dimensions, the Lorentz group is $SO_L(3) \simeq SU_L(2)$. In order to define a twist, the supersymmetric theory will have to possess a $SU_I(2)$ automorphism group so that the new Lorentz group may be taken to be the diagonal sum of the two $SU(2)$'s. This means that the $D = 3$ theory should be $N = 4$ supersymmetric. In two dimensions, we will require a $U(1)$ automorphism group, hence an $N = 2$ supersymmetric theory.

Glancing at the BF lagrangian (see above), we see that the Grassmann even fields are first order in derivatives. Whereas, upon gauge fixing, the Grassmann odd fields are second order in derivatives. This is an inversion of the usual structure in supersymmetric theories. Scaling this hurdle will be achieved by a second stage of the twisting wherein we will change the Grassmann parity of the fields; (bosons) fermions will become (anti-) commuting. As the supersymmetric theory we will apply our twisting procedure to will not be gauge invariant, the BF/Chern-Simons theory obtained will be gauge fixed. In this way, we will obtain the abelian BF and Chern-Simons theories from $N = 4$ supersymmetric theories in three dimensions. Similarly, $N = 2$ theories will be twisted to the abelian $D = 2$ BF theory. As an artifact of the process, we will actually obtain two (uncoupled) copies of Chern-Simons theory.

Previously, it had been shown that the gauge fixed Chern-Simons theories [13] (along with a related construction for the BF theories [14, 15, 16]) are invariant under a set of symmetries generated by a pair of scalar and a pair of vector charges, all Grassmann odd. The algebra of these charges allows a $SL_I(2, \mathbf{R}) \simeq SU_I(2)$ automorphism group. The number of components of these charges matches the number of components of four Majorana fermions and it was shown that this algebra is a twisted version of a $D = 3$, $N = 4$ supersymmetry algebra[1]. As part of our work , we will find the missing $N = 4$ supersymmetric theory which realizes the untwisted algebra. Since, as we will show, the supersymmetric theory may also be twisted to a TQFT, we will then formally relate a subset of TQFT's to the abelian BF theory.

Our paper is organized as follows. In the next section, the two $N = 4$ supersymmetric actions (which differ only by surface terms) we will use throughout our three dimensional discussion will be presented. Following this, in section 5.3, we will twist the first of these actions to the abelian BF theory in three dimensions. After writing down the action for a $D = 2$, $N = 2$ scalar supermultiplet, we will show how to twist this theory to the two dimensional BF theory, in sub-section 5.3.3. In section 5.4, we shall return to three dimensions and use the second action from section 5.2, which we will twist to the abelian Chern-Simons theory. The structure and transformations generated by the three dimensional twisted superalgebra will be given in section 5.5. In section 5.6, we will show how to connect TQFT's obtained from our supersymmetric theory with BF theories via a change in Grassmann parity. We conclude in section 5.7. The conventions used in this paper may be found in the appendix.

4.2 The $N = 4$ Supersymmetric Actions

Let us begin by introducing the two $N = 4$ supersymmetric actions we will be using in our discussion of the three dimensional topological theories. In order to establish the main features of the twisting process it is best to work on a flat manifold. Later, we will extend the procedure to curved manifolds (see sub-section (5.3.2). Although the actions constructed in this section exist in either Minkowski space-time or $\boldsymbol{R}^3$, in the rest of the paper we will restrict our discussion to manifolds with Euclidean signature.

Our supersymmetric matter multiplet contains the following complex fields:

[1]This algebra was termed $N = 2$ in ref. [13].

FIELD	SPIN	GRASSMANN PARITY
ϕ	0	even
λ	0	even
ψ_α	1/2	odd
χ_α	1/2	odd

There are a number of possible actions we could write down for these fields. Even within a given action, we can add surface terms. We will see the importance of this later. As our basic action we take[2]

$$S^{SUSY} = \int d^3x[\partial^a\bar{\phi}\partial_a\lambda + \partial^a\phi\partial_a\bar{\lambda} + i\frac{1}{2}\chi^\alpha(\gamma^a)_\alpha{}^\beta\partial_a\bar{\psi}_\beta - i\frac{1}{2}\bar{\chi}^\alpha(\gamma^a)_\alpha{}^\beta\partial_a\psi_\beta] \ , \quad (4.1)$$

where the bar denotes complex conjugation. This action is invariant under the following rigid supersymmetry transformations[3]

$$\begin{aligned} [\mathcal{Q}_\alpha, \phi] &= i\psi_\alpha \ , & [\mathcal{Q}_\alpha, \lambda] &= i\chi_\alpha \ , \\ \{\mathcal{Q}_\alpha, \bar{\psi}_\beta\} &= -2(\gamma^a)_{\alpha\beta}\partial_a\bar{\phi} \ , & \{\mathcal{Q}_\alpha, \bar{\chi}_\beta\} &= 2(\gamma^a)_{\alpha\beta}\partial_a\bar{\lambda} \ , \\ [\bar{\mathcal{Q}}_\alpha, \bar{\phi}] &= -i\bar{\psi}_\alpha \ , & [\bar{\mathcal{Q}}_\alpha, \bar{\lambda}] &= -i\bar{\chi}_\alpha \ , \\ \{\bar{\mathcal{Q}}_\alpha, \psi_\beta\} &= -2(\gamma^a)_{\alpha\beta}\partial_a\phi \ , & \{\bar{\mathcal{Q}}_\alpha, \chi_\beta\} &= 2(\gamma^a)_{\alpha\beta}\partial_a\lambda \ . \end{aligned} \quad (4.2)$$

The $\mathcal{Q}$-super–charges form the $N = 2$ supersymmetry algebra

$$\{\bar{\mathcal{Q}}_\alpha, \mathcal{Q}_\beta\} = -i2(\gamma^a)_{\alpha\beta}\partial_a \ , \quad \{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = 0 \ , \quad \{\bar{\mathcal{Q}}_\alpha, \bar{\mathcal{Q}}_\beta\} = 0 \ . \quad (4.3)$$

The action is invariant under the interchange $\lambda \leftrightarrow \phi$. From this, it follows that there is a second $N = 2$ supersymmetry of (5.1),

[2]The ordering of the fields in the various terms is important since our twisting procedure involves changing the Grassmann character of the fields. We will take the ordering as given in this action throughout.

[3]Throughout this paper we will discard surface terms while establishing the existence of supersymmetries.

$$
\begin{array}{rclrcl}
[\mathcal{S}_\alpha, \phi] & = & i\chi_\alpha \ , & [\mathcal{S}_\alpha, \lambda] & = & i\psi_\alpha \ , \\
\{\mathcal{S}_\alpha, \bar{\psi}_\beta\} & = & -2(\gamma^a)_{\alpha\beta}\partial_a\bar{\lambda} \ , & \{\mathcal{S}_\alpha, \bar{\chi}_\beta\} & = & 2(\gamma^a)_{\alpha\beta}\partial_a\bar{\phi} \ , \\
[\bar{\mathcal{S}}_\alpha, \bar{\phi}] & = & -i\bar{\chi}_\alpha \ , & [\bar{\mathcal{S}}_\alpha, \bar{\lambda}] & = & -i\bar{\psi}_\alpha \ , \\
\{\bar{\mathcal{S}}_\alpha, \psi_\beta\} & = & -2(\gamma^a)_{\alpha\beta}\partial_a\lambda \ , & \{\bar{\mathcal{S}}_\alpha, \chi_\beta\} & = & 2(\gamma^a)_{\alpha\beta}\partial_a\phi \ ,
\end{array} \tag{4.4}
$$

The $\mathcal{S}$-super–charges also form an $N = 2$ supersymmetry algebra. The automorphism group of each of the supersymmetry algebras is $U(1)$.

It will prove useful to re-write S^{SUSY} in terms of real/imaginary fermions rather than the complex ones. To do this, we define the real and imaginary parts of the fermions via: $\chi_\alpha \equiv \chi_{\alpha 1} + i\chi_{\alpha 2}$ and $\psi_\alpha \equiv \psi_{\alpha 1} + i\psi_{\alpha 2}$. Consequently, the action becomes

$$
S^{SUSY} = \int d^3x[\partial^a\bar{\phi}\partial_a\lambda + \partial^a\phi\partial_a\bar{\lambda} + \chi^{\alpha A}(\gamma^a)_\alpha{}^\beta\partial_a\psi_\beta{}^B\epsilon_{AB}] \ . \tag{4.5}
$$

The lagrangian in this action is equivalent to that in (5.1); *i.e.*, no surface terms were incurred in this re-writing. In twisting to the BF theory, we will use this form of the action.

From $\psi_{\alpha A}$ and $\chi_{\alpha A}$, we can construct another action whose lagrangian differs from (5.1) by a total derivative term. To do this we define $\Psi_{\alpha A} \equiv \psi_{\alpha A} + i\chi_{\alpha A}$ ($\bar{\Psi}_\alpha{}^A \equiv \psi_\alpha{}^A - i\chi_\alpha{}^A$). As $\Psi_{\alpha A}$ is a complex doublet, we take it to transform as a **2** of $SU_I(2)$ while $\bar{\Psi}_\alpha{}^A$ is in the conjugate representation. Using this in the action (5.1) we arrive at

$$
\begin{aligned}
S'^{SUSY} &= \int d^3x[\partial^a\bar{\phi}\partial_a\lambda + \partial^a\phi\partial_a\bar{\lambda} + i\frac{1}{2}\bar{\Psi}^{\alpha B}(\gamma^a)_\alpha{}^\beta\partial_a\Psi_{\beta B}] \\
&= S^{SUSY} + (\text{surface terms}) \ ,
\end{aligned} \tag{4.6}
$$

and discard the surface terms. The original two $N = 2$ supersymmetries now become invariances of the action under the following transformations

$$
\begin{aligned}
[Q_{\alpha A}, \phi] &= i(\Psi_{\alpha A} + \bar{\Psi}_{\alpha A}) \ , \\
[Q_{\alpha A}, \lambda] &= i(\Psi_{\alpha A} - \bar{\Psi}_{\alpha A}) \ ,
\end{aligned}
$$

$$
\begin{aligned}
\{Q_{\alpha A}, \Psi_{\beta B}\} &= -2\epsilon_{AB}(\gamma^a)_{\alpha\beta}\partial_a(\bar{\phi} - \bar{\lambda}) \ , \\
\{Q_{\alpha A}, \bar{\Psi}_{\beta B}\} &= -2\epsilon_{AB}(\gamma^a)_{\alpha\beta}\partial_a(\bar{\phi} + \bar{\lambda}) \ , \\
[\bar{Q}_{\alpha A}, \bar{\phi}] &= -i(\bar{\Psi}_{\alpha A} + \Psi_{\alpha A}) \ , \\
[\bar{Q}_{\alpha A}, \bar{\lambda}] &= -i(\bar{\Psi}_{\alpha A} - \Psi_{\alpha A}) \ , \\
\{\bar{Q}_{\alpha A}, \bar{\Psi}_{\beta B}\} &= -2\epsilon_{AB}(\gamma^a)_{\alpha\beta}\partial_a(\phi - \lambda) \ , \\
\{\bar{Q}_{\alpha A}, \Psi_{\beta B}\} &= -2\epsilon_{AB}(\gamma^a)_{\alpha\beta}\partial_a(\phi + \lambda) \ .
\end{aligned} \tag{4.7}
$$

This shows explicitly that the both actions, S^{SUSY} and S'^{SUSY} are invariant under an $N = 4$ supersymmetry. Indeed, the algebra of charges defined by (5.7) is

$$
\{\bar{Q}_\alpha{}^A, Q_{\beta B}\} = i4\delta_B{}^A(\gamma^a)_{\alpha\beta}\partial_a \ . \tag{4.8}
$$

This algebra has a $SU_I(2)$ automorphism invariance with the $Q_{\alpha A}$ transforming in the doublet representation.

4.3 Mapping to BF Theories

This section is divided into three parts. First, in sub-section (5.3.1), we present the twisting procedure while working with the action S^{SUSY}. As advertised, we will find the twisted action to be the gauge fixed, abelian BF theory on $\boldsymbol{R}^3$. Then, in sub-section (5.3.2), we will discuss how to obtain the BF theory on curved manifolds. Finally, in sub-section (5.3.3), as another example of the procedure, we will write down a $D = 2$, $N = 2$ supersymmetric action from which the two-dimensional abelian BF theory may be obtained via twisting.

4.3.1 g–Twisting S^{SUSY}

The Lorentz algebra in three dimensions is $SO_L(3) \simeq SU_L(2)$. As the first stage of our twisting we take all internal indices to be $SU_L(2)$ indices. This amounts [13] to re-defining the Lorentz group to be the diagonal subgroup of $SU_L(2) \times SU_I(2)$. With this, the original scalar fields remain Lorentz singlets while the real spin-$\frac{1}{2}$ fields become Lorentz bi-spinors: $\psi_{\alpha B} \rightarrow \psi_{\alpha\beta}$ and $\chi_{\alpha B} \rightarrow \chi_{\alpha\beta}$. This means that we can decompose $\psi_{\alpha\beta}$ as a real vector plus a scalar field; similarly for $\chi_{\alpha\beta}$.

As the second stage of our twist, we declare the fields to have opposite Grassmann parity to those of the parent supersymmetric theory. This second step does not exist in the known [3] twisting of supersymmetric theories to obtain Donaldson-like topological quantum field theories (TQFT's). We call this two stage mapping a "g–twist " and define it by the map

$$
\begin{aligned}
\mathcal{T}_g \ &: \ \psi_{\alpha B} \rightarrow \ \psi_{\alpha\beta} \ \equiv \ \frac{1}{\sqrt{2}}[i(\gamma^a)_{\alpha\beta}A_a \ - \ C_{\alpha\beta}\Sigma] \ , \\
\mathcal{T}_g \ &: \ \chi^{\alpha B} \ \rightarrow \ \chi^{\alpha\beta} \ \equiv \ \frac{1}{\sqrt{2}}[(\gamma^a)^{\alpha\beta}B_a \ + \ iC^{\alpha\beta}\Lambda] \ , \\
\mathcal{T}_g \ &: \ \phi \ \rightarrow \ \frac{1}{\sqrt{2}}(c \ - \ ib') \ , \\
\mathcal{T}_g \ &: \ \bar{\phi} \ \rightarrow \ \frac{1}{\sqrt{2}}(c \ + \ ib') \ , \\
\mathcal{T}_g \ &: \ \lambda \ \rightarrow \ \frac{1}{\sqrt{2}}(c' \ + \ ib) \ , \\
\mathcal{T}_g \ &: \ \bar{\lambda} \ \rightarrow \ \frac{1}{\sqrt{2}}(c' \ - \ ib) \ , \\
\mathcal{T}_g \ &: \ \epsilon_{AB} \ \rightarrow \ iC_{\alpha\beta} \ .
\end{aligned} \tag{4.9}
$$

The fields on the right hand side of the arrows are defined by this map to have Grassmann parity opposite to those on the left. The factors of "i" have been inserted so that the process of complex conjugation commutes with $\mathcal{T}_g$. Additionally, the other numerical factors are for later convenience. We summarize the new field content in the following table:

FIELD	SPIN	GRASSMANN PARITY
A_a	1	even
Σ	0	even
B_a	1	even
Λ	0	even
c	0	odd
b	0	odd
c'	0	odd
b'	0	odd

Performing the map, $\mathcal{T}_g$, on the action S^{SUSY} as given in eqn. (5.5) we find, up to surface terms,

$$S_{BF} = \int d^3x[\epsilon^{abc}B_a\partial_b A_c + (\partial^a A_a)\Lambda + (\partial^a B_a)\Sigma + c'\Box c + b'\Box b] \ . \quad (4.10)$$

This is the action of the fully gauge fixed abelian BF theory in three dimensions[4]. The first term is the classical BF action. In this term, the Levi-Cevita tensor arises from a trace on the product of three gamma matrices. The second and third terms represent the gauge fixings of the local $U(1)$ and 1-form symmetry on B_a (see section (5.5) for details). In these terms, the Lorentz dot product arises from the trace of products of two gamma matrices. The ghost actions for these gauge fixings are given by the last two terms in (5.10). Note that only the Landau gauge appears in this procedure. The surface terms mentioned above appear only from the gauge fixing and ghost terms. They are needed in order to write these terms in their conventional forms.

4.3.2 Curved 3-Manifolds

The classical BF action is topological. It is only after gauge fixing that a metric appears in the action. We would like to recover this peculiar metric dependence.

[4]The ordering of the fields in the gauge fixing terms is chosen so as not to introduce additional minus signs when we later map to the TQFT.

We could simply g–twist the action S^{SUSY} on $\mathbb{R}^3$ to obtain (5.10) and then covariantize it with respect to some background metric on a curved manifold, M. By definition, the subsequent action,

$$S_{BF}^{M} = \int d^3x \epsilon^{abc} B_a \partial_b A_c + \int d^3x \sqrt{g}[(\nabla^a A_a)\Lambda + (\nabla^a B_a)\Sigma + c'\Delta c + b'\Delta b] \ , \quad (4.11)$$

is the gauge fixed BF theory on M. The derivative ∇_a is covariant with respect to diffeomorphisms of M: $\nabla_a \equiv e_a{}^m \partial_m + \omega_a{}^b J_b$. Here $e_a{}^m$ is the driebein with determinant e. The object $\omega_a{}^b(e)$ is the dual of the Lorentz spin-connection for which the dual of the Lorentz generator is J_a.

Instead, suppose we started with the $N = 4$ gauged supergravity[5] version of S^{SUSY}. Among the new fields introduced would be four gravitini and a $SU_I(2)$ gauge field, V_a. As an example, the gravitini appear in the spin-connection in the covariant derivative. The latter is also covariant with respect to local $SU_I(2)$ gauge transformations due to the introduction of V_a. The action (5.11) does not contain either of these fields as it is neither $N = 4$ locally supersymmetric or $SU_I(2)$ gauge invariant. Thus, in the g–twisting, we must set the gravitini to zero. In order to maintain this ansatz, however, we must restrict the local supersymmetry of the action so that the gravitini may not be transformed away from zero. Since the local supersymmetry variations of the gravitini, $\zeta_{a\alpha}{}^A$, are given by the covariant derivative of the local supersymmetry parameter, we must find a covariantly constant anti-commuting parameter:

$$\delta\zeta_{a\alpha}{}^A \ = \ D_a\epsilon_\alpha^A \ = \ \partial_a\epsilon_\alpha^A \ - \ \omega_{a\alpha}{}^\beta\epsilon_\beta{}^A \ + \ V_{aB}{}^A\epsilon_\alpha^B \ = \ 0 \ . \quad (4.12)$$

To do this, we accentuate our procedure in analogy with the twisting in $D = 4$, $N = 2$ conformal supergravity backgrounds [17]. We introduce a scalar anti-commuting parameter, ϵ by $\epsilon_\alpha{}^A \equiv \epsilon\delta_\alpha{}^A$ having embedded the $SU_I(2)$ gauge field in the $SU(2)$ spin connection: $\omega_{a\alpha}{}^\beta\delta_\beta{}^A \equiv V_{aB}{}^A\delta_\alpha{}^B$. All supersymmetries are then lost with the exception of the one generated by the scalar charges. The corresponding transformations

[5]The construction of $D = 3$, $N = 4$ gauged supergravity along with its explicit couplings to matter is beyond the scope of this work.

will be given later. We then identify this curved background with the geometry of M.

4.3.3 Two Dimensions

To illustrate the generality of our g–twisting procedure, we offer an example in two dimensions. As the Lorentz group in two dimensions is $U(1)$, our supersymmetric theory must have this abelian automorphism group. This means that the theory must be $N = 2$ supersymmetric. As our action we take

$$S^{SUSY}_{D=2} \;=\; \int d^2x[\partial^a\phi\partial_a\lambda \;+\; i\frac{1}{2}\bar{\psi}^\alpha(\gamma^a)_\alpha{}^\beta\partial_a\psi_\beta] \;\;, \tag{4.13}$$

where ϕ and λ are scalar fields and ψ is a complex spin-$\frac{1}{2}$ field. This action is invariant under the supersymmetry transformations,

$$\begin{aligned} [Q_\alpha,\phi] &= i\psi_\alpha \;, & [\bar{Q}_\alpha,\lambda] &= i\bar{\psi}_\alpha \;, \\ [Q_\alpha,\bar{\psi}_\beta] &= -2(\gamma^a)_{\alpha\beta}\partial_a\phi \;, & [\bar{Q}_\alpha,\psi_\beta] &= -2(\gamma^a)_{\alpha\beta}\partial_a\lambda \;. \end{aligned} \tag{4.14}$$

These form the $D = 2$, $N = 2$ supersymmetry algebra

$$\{\bar{Q}_\alpha, Q_\beta\} \;=\; -i2(\gamma^a)_{\alpha\beta}\partial_a \;\;. \tag{4.15}$$

Upon defining $\psi_\alpha = \psi_{\alpha 1} + i\psi_{\alpha 2}$ and denoting the new fermions as $\psi_{\alpha A}$, $A = 1, 2$, we define the g–twist to be

$$\begin{aligned} \mathcal{T}_g \;&:\; \psi_{\alpha B} \to \psi_{\alpha\beta} \equiv [i(\gamma^a)_{\alpha\beta}A_a \;-\; (\gamma^3)_{\alpha\beta}B \;-\; \frac{1}{2}C_{\alpha\beta}\Lambda] \;, \\ \mathcal{T}_g \;&:\; \phi \to c \;, \\ \mathcal{T}_g \;&:\; \lambda \to c' \;. \end{aligned} \tag{4.16}$$

The $D = 2$ analog of the procedure discussed in the previous sub-section but with $SU(2)$ replaced by $U(1)$, may now be applied to the action (5.13). It results in the

g–twisted action

$$S_{BF}^{D=2} = 2\int d^2x \epsilon^{ab} B\partial_a A_b + \int d^2x\sqrt{g}[(\nabla^a A_a)\Lambda + c'\triangle c] \ . \tag{4.17}$$

This is the gauged fixed, abelian BF action in two dimensions.

4.4 Mapping to Chern-Simons

As is well known, Chern-Simons theory is a special case of a BF theory in which the A_a and B_a fields are identified[6]. At the level of the fields this is a purely formal operation. However, when one considers that A_a is a $U(1)$-valued gauge field and B_a is a singlet under that gauge group, one realizes the absence of a representation theory prescription for the identification. At the level of symmetries, both fields transform as the exterior derivative of a scalar parameter. Thus, Chern-Simons theory is strictly a special case of BF theory only at the level of the structure of the fields in the action. Although this is a phenomenon in the gauge sector of the theory, we might expect similar behaviour with the space-time symmetries, if we try to obtain Chern-Simons via the g–twisting of a supersymmetric theory. Indeed, we will see that if we use the naive version of S^{SUSY}, there is no group theoretic prescription, in terms of $SU(2)$ representations, for the g–twist . Our map will be purely in terms of the fields. After seeing this, we will then turn to S'^{SUSY} (in the second sub-section), for which both the twist on the fields and the group theoretic interpretation are available.

4.4.1 g–Twisted S^{SUSY} with χ and ψ Identified

Since we already know that A_a and B_a must be identified, we start by identifying χ and ψ in eqn. (5.5) so that we take the action to be

$$S_0^{SUSY} = \int d^3x[\partial^a\phi\partial_a\lambda + \frac{1}{2}\psi^{\alpha B}(\gamma^a)_\alpha{}^\beta\partial_a\psi_{\beta B}] \ . \tag{4.18}$$

[6] In order to get the non-abelian Chern-Simons theory, a term which is cubic in the B_a field must be added to the BF lagrangian.

Here, λ and ϕ are now real bosons[7] and $\psi^{\alpha A}$ represents a pair of real spin$-\frac{1}{2}$ fields, $A = 1,2$. Naively, we might define the g–twist by the first line in eqn. (5.9) along with $\mathcal{T}_g : \lambda \to c'$ and $\mathcal{T}_g : \phi \to c$. Using this in S_{CS}^{SUSY} and applying the procedure outlined in sub-section (5.3.2), we arrive at the action

$$S_{CS} \;=\; \frac{1}{2}\int d^3x\epsilon^{abc}A_a\partial_bA_c \;+\; \int d^3x\sqrt{g}[(\nabla^aA_a)\Sigma \;+\; c'\Delta c] \;. \qquad (4.19)$$

Once again, we have switched the Grassmann parity of the fields. Of course, this is the gauge fixed abelian Chern-Simons action.

As there are only two real fermions in this action, there is only a global $SO(2)$ invariance, not $SU(2)$. Thus we are unable to associate the Lorentz symmetry of S_{CS} with the diagonal sum of two $SU(2)$'s and there is no group theoretic justification for taking the internal index on the fermions to be Lorentz spinor indices, in the definition of the twist. However, we simply point out that if this is done at the level of the fields, then the Chern-Simons action is obtained.

4.4.2 g–Twisting S'^{SUSY}

There is, however, a way to obtain the Chern-Simons action – actually two copies – while having a group theoretic justification. We start with the action S'^{SUSY} (5.6) which differs from S^{SUSY} by surface terms. Now we take the internal $SU_I(2)$ indices on $\Psi_{\alpha A}$ to be Lorentz spin-$\frac{1}{2}$ indices. Again this amounts to re-defining the Lorentz group to be the diagonal sum of the two $SU(2)$'s. Then the g–twist is defined by

$$\mathcal{T}_g : \Psi_{\alpha B} \;\to\; \Psi_{\alpha\beta} \equiv \frac{1}{\sqrt{2}}[(\gamma^a)_{\alpha\beta}(A_a + iB_a) \;+\; iC_{\alpha\beta}(\Sigma + i\Lambda)] \;\;, \qquad (4.20)$$

along with a change of Grassmann parity. $\mathcal{T}_g$ acts on the scalar fields as before (5.9). Performing these replacements in S'^{SUSY} and applying the procedure outlined

[7]The real parts of the corresponding fields from the previous sub-sections.

in sub-section (5.3.2), we obtain

$$\begin{aligned} S^2_{CS} \;=\; & -\tfrac{1}{2}\int d^3x \quad [\epsilon^{abc}A_a\partial_b A_c \;+\; \epsilon^{abc}B_a\partial_b B_c] \\ & -\int d^3x\sqrt{g} \quad [(\nabla^a A_a)\Sigma \;+\; (\nabla^a B_a)\Lambda \;-\; c'\triangle c \;-\; b'\triangle b] \;, \end{aligned} \tag{4.21}$$

This is the action for two uncoupled copies of the gauge fixed Chern-Simons theory. Curiously, the appearance of more than one gauge field is a phenomena in extended supersymmetric Chern-Simons theories [18]. Identifying the set of fields (B_a, Λ, b', b) with the set (A_a, Σ, c', c) reduces this to (twice) the action for one Chern-Simons gauge field (5.19).

4.5 The g–Twisted Super-Algebra

In the context of gauge fixed theories, "supersymmetry" is to be understood as a set of transformations generated by Grassmann odd charges which take fields of ghost number n into fields of ghost number $n \pm 1$. Vector super–charges of ghost number 1 were discovered for the three–dimensional Chern-Simons theory in the Landau gauge in ref. [19]. It was soon thereafter realized that the same theory is further invariant under the anti-BRST transformations and another vector generator both of ghost number -1 [13]. The BRST generator and the ghost number -1 vector generator were found to close on translations, thereby forming an $N = 2$ supersymmetry algebra. In addition, the anti-BRST generator and the ghost number 1 generator form another $N = 2$ superalgebra. The $N = 2$ algebra, including the $BRST$ generator, was then found to hold for the two– and four–dimensional non-abelian BF theories [14, 15], and was generalized to arbitrary dimensions in ref. [16]. It was used to prove the perturbative finiteness of the $D = 3$ Chern-Simons theory [20] and of the BF theory (see [16] and references therein). We will now extract these charges and algebras from our $N = 4$ supersymmetry algebra (5.8) via twisting.

The g–twist acts on the super–charges as

$$\begin{aligned} \mathcal{T}_g &: \quad Q_{\alpha B} \rightarrow Q_{\alpha\beta} \equiv (\gamma^a)_{\alpha\beta} Q_a + i C_{\alpha\beta} Q \ , \\ \mathcal{T}_g &: \quad \bar{Q}_{\alpha B} \rightarrow \bar{Q}_{\alpha\beta} \equiv (\gamma^a)_{\alpha\beta} \bar{Q}_a + i C_{\alpha\beta} \bar{Q} \ . \end{aligned} \tag{4.22}$$

In the absence of covariantly constant vectors, only the scalar super–charges are conserved on curved manifolds. On $\mathbb{R}^3$, the full set of super–charges is conserved. Note that since the supercurrents were originally a product of a Grassmann odd and the derivative of a Grassmann even field, the Grassmann parity of the super-charges remains the same, namely odd.

Performing the map on the $N = 4$ supersymmetry algebra (5.8) we find the **g**–twisted algebra whose only non-trivial anti-commutators are

$$\{\bar{Q}_a, Q_b\} = -i2\epsilon_{abc}\partial^c \ , \qquad \{\bar{Q}_a, Q\} = -i2\partial_a \ , \qquad \{\bar{Q}, Q_a\} = i2\partial_a \ . \tag{4.23}$$

Q and its complex conjugate are nilpotent.

The supersymmetry transformations (5.7) now take the forms:

$$
\begin{array}{rclrcl}
[Q, A_a] & = & \partial_a(c + ib') \ , & [Q, B_a] & = & i\partial_a(c' - ib) \ , \\
[Q, \Lambda] & = & 0 \ , & [Q, \Sigma] & = & 0 \ , \\
\{Q, c\} & = & i\Sigma \ , & \{Q, b\} & = & i\Lambda \ , \\
\{Q, c'\} & = & -\Lambda \ , & \{Q, b'\} & = & -\Sigma \ , \\
 & & & & & \\
[\bar{Q}, A_a] & = & \partial_a(c - ib') \ , & [\bar{Q}, B_a] & = & -i\partial_a(c' + ib) \ , \\
[\bar{Q}, \Lambda] & = & 0 \ , & [\bar{Q}, \Sigma] & = & 0 \ , \\
\{\bar{Q}, c\} & = & -i\Sigma \ , & \{\bar{Q}, b\} & = & -i\Lambda \ , \\
\{\bar{Q}, c'\} & = & -\Lambda \ , & \{\bar{Q}, b'\} & = & -\Sigma \ , \\
 & & & & & \\
[Q_a, A_b] & = & -\epsilon_{abc}\partial^c(c + ib') \ , & [Q_a, B_b] & = & -i\epsilon_{abc}\partial^c(c' - ib) \ , \\
[Q_a, \Lambda] & = & -i\partial_a(c' - ib) \ , & [Q_a, \Sigma] & = & -\partial_a(c + ib') \ , \\
\{Q_a, c\} & = & iA_a \ , & \{Q_a, b\} & = & iB_a \ , \\
\{Q_a, c'\} & = & -B_a \ , & \{Q_a, b'\} & = & -A_a \ , \\
 & & & & & \\
[\bar{Q}_a, A_b] & = & -\epsilon_{abc}\partial^c(c - ib') \ , & [\bar{Q}_a, B_b] & = & i\epsilon_{abc}\partial^c(c' + ib) \ , \\
[\bar{Q}_a, \Lambda] & = & i\partial_a(c' + ib) \ , & [\bar{Q}_a, \Sigma] & = & -\partial_a(c - ib') \ , \\
\{\bar{Q}_a, c\} & = & -iA_a \ , & \{\bar{Q}_a, b\} & = & -iB_a \ , \\
\{\bar{Q}_a, c'\} & = & -B_a \ , & \{\bar{Q}_a, b'\} & = & -A_a \ .
\end{array}
\tag{4.24}
$$

These are symmetry transformations for the three–dimensional, gauge fixed BF action. Upon defining $Q \equiv s + is'$ we find the BRST (s) and anti-BRST (s') transformations to be

$$
\begin{aligned}
[s, A_a] &= \partial_a c \ , & [s, B_a] &= \partial_a b \ , \\
[s, \Lambda] &= 0 \ , & [s, \Sigma] &= 0 \ , \\
\{s, c\} &= 0 \ , & \{s, b\} &= 0 \ , \\
\{s, c'\} &= -\Lambda \ , & \{s, b'\} &= -\Sigma \ , \\
\\
[s', A_a] &= \partial_a b' \ , & [s', B_a] &= \partial_a c' \ , \\
[s', \Lambda] &= 0 \ , & [s', \Sigma] &= 0 \ , \\
\{s', c\} &= \Sigma \ , & \{s', b\} &= \Lambda \ , \\
\{s', c'\} &= 0 \ , & \{s', b'\} &= 0 \ .
\end{aligned} \tag{4.25}
$$

Similarly, the transformations generated by the real, vector super-charges, s_a and s'_a defined by $Q_a \equiv s_a + is'_a$ are found from (5.24) to be

$$
\begin{aligned}
[s_a, A_b] &= -\epsilon_{abc}\partial^c c \ , & [s_a, B_b] &= -\epsilon_{abc}\partial^c b \ , \\
[s_a, \Lambda] &= -\partial_a b \ , & [s_a, \Sigma] &= -\partial_a c \ , \\
\{s_a, c\} &= 0 \ , & \{s_a, b\} &= 0 \ , \\
\{s_a, c'\} &= -B_a \ , & \{s_a, b'\} &= -A_a \ , \\
\\
[s'_a, A_b] &= -\epsilon_{abc}\partial^c b' \ , & [s'_a, B_b] &= -\epsilon_{abc}\partial^c c' \ , \\
[s'_a, \Lambda] &= -\partial_a c' \ , & [s'_a, \Sigma] &= -\partial_a b' \ , \\
\{s'_a, c\} &= A_a \ , & \{s'_a, b\} &= B_a \ , \\
\{s'_a, c'\} &= 0 \ , & \{s'_a, b'\} &= 0 \ .
\end{aligned} \tag{4.26}
$$

The vector super–charges along with the scalar BRST and anti-BRST super–charges satisfy the superalgebra

$$
\{s'_a, s_b\} = \epsilon_{abc}\partial^c \ , \qquad \{s_a, s'\} = \partial_a \ , \qquad \{s'_a, s\} = -\partial_a \ , \tag{4.27}
$$

with all other combinations vanishing. The BRST symmetry and the symmetry generated by the vector super–charge, s'_a, are in agreement with the results of [16]. The transformations of the anti-BRST and s_a charges were not previously given for the case of BF theories. Our results verify the general statement that s' and s'_a may be obtained from s and s_a, respectively, via interchanges of ghosts and anti-ghosts. Due to the first order nature of the classical BF action this takes the form $c \rightarrow b'$, $b' \rightarrow -c$, $b \rightarrow c'$ and $c' \rightarrow -b$.

Our superalgebras close on-shell only. Superfield formulations of the supersymmetric theories in section 5.2 are expected to yield, upon $\mathbf{g}$–twisting, off-shell closure of the algebras (5.23) and (5.27).

4.6 Relating BF to TQFT's

As mentioned before, twisting a supersymmetric action to a TQFT requires only the first step in our $\mathbf{g}$–twisting process in that the Grassmann parity of the fields is not changed. Performing the Grassmann parity change twice is equivalent to the identity. Thus if we perform a Grassmann parity change on the BF action, we expect to find a TQFT. Let us see this explicitly.

Upon making the replacements,

$$
\begin{aligned}
A_a &\rightarrow \rho_{a1} \ , \qquad B_a \rightarrow \rho_{a2} \ , \\
\Lambda &\rightarrow \xi_1 \ , \qquad \Sigma \rightarrow \xi_2 \ , \\
c &\rightarrow \varpi_1 \ , \qquad b \rightarrow \varpi_2 \ , \\
c' &\rightarrow \varphi_1 \ , \qquad b' \rightarrow \varphi_2 \ ,
\end{aligned}
\tag{4.28}
$$

with the Grassmann parity assignments,

FIELD	SPIN	GRASSMANN PARITY
ρ_{ai}	1	odd
ξ_i	0	odd
φ_i	0	even
ϖ_i	0	even

in the three dimensional BF action (5.11), we obtain

$$
S'_{TQFT} = S_{TQFT} - i\frac{1}{2}\int d^3x \epsilon^{abc}\rho_{ai}\partial_b\rho_{cj}\epsilon^{ij} \ , \tag{4.29}
$$

where

$$S_{TQFT} \;=\; -\int d^3x\sqrt{g}\sum_{i=1}^{2}\left[\nabla^a\varpi_i\nabla_a\varphi_i \;+\; \rho_i{}^a\nabla_a\xi_i\right] \;. \tag{4.30}$$

Making the same replacements in (5.25) yields the BRST transformations under which S'_{TQFT} is invariant. We record them for completeness:

$$\begin{aligned}
\{s,\rho_{ai}\} &= \partial_a\varpi_i \;, \\
[s,\varpi_i] &= 0 \;, \\
[s,\varphi_i] &= -\xi_i \;, \\
\{s,\xi_i\} &= 0 \;.
\end{aligned} \tag{4.31}$$

It is then easy to see that

$$S_{TQFT} \;=\; \{s, -\int d^3x\sqrt{g}\sum_{i=1}^{2}\rho_i^a\nabla_a\varphi_i\} \;. \tag{4.32}$$

Since the last term in S'_{TQFT} is metric independent, the energy-momentum tensor from the latter action is s–exact. Of course, starting with this TQFT action and inverting the replacements (5.28) leads us back to the BF theory.

Alternatively, we could start with our action (5.6) and perform the usual TQFT twist defined to be the map

$$\mathcal{T}_{TQFT} : \Psi_{\alpha B} \;\rightarrow\; \Psi_{\alpha\beta} \equiv \frac{1}{\sqrt{2}}[(\gamma^a)_{\alpha\beta}(\rho_{a1}+i\rho_{a2}) \;+\; iC_{\alpha\beta}(\xi_1+i\xi_2)] \;, \tag{4.33}$$

which leaves the spin-0 fields, $\phi \equiv \frac{1}{\sqrt{2}}(\varphi_1+i\varphi_2)$ and $\lambda \equiv \frac{1}{\sqrt{2}}(\varpi_1+i\varpi_2)$ along with the Grassmann parity of the fields unchanged. With this prescription, we find that the action, S'^{SUSY} becomes S'_{TQFT} up to surface terms. If we denote the operation of changing the Grassmann parity of the fields by g, then this information may be encoded in the following diagram:

$$\begin{tikzcd}
 & \boxed{S'^{SUSY}} \arrow[dl, "\mathcal{T}_g"'] \arrow[dr, "\mathcal{T}_{TQFT}"] & \\
\boxed{S_{BF}} \arrow[rr, leftrightarrow, "g"'] & & \boxed{S'_{TQFT}}
\end{tikzcd}$$

Figure 1: The TFT Triangle.

The last term in S'_{TQFT} does not normally appear in topological sigma models (even flat ones). Its presence is idiosyncratic to three dimensions. It is invariant under the BRST transformations of eqn. (5.31). Although this part of the action has ghost number -2, the full action remains invariant under the $U(1)$ transformation with weights $(-)^i$ for ρ_{ai} and $(-)^{i+1}$ for ξ_i.

A similar procedure may be performed using the Chern-Simons action, S^2_{CS}, (5.21). We find only S_{TQFT} instead of S'_{TQFT}; that is $g{:}S_{CS} \to S_{TQFT}$. The map is not invertible as we cannot obtain the Chern-Simons action from g: S_{TQFT}. In other words, only the gauge fixing and ghost actions of the Chern-Simons theory may be obtained from S_{TQFT} (or S'_{TQFT}).

4.7 Conclusion

We have defined supersymmetric actions for matter fields which when g–twisted (a twist plus Grassmann parity change) yield gauge fixed, abelian BF theories in three and two dimensions. In three dimensions, our theory is $N = 4$ supersymmetric while in two dimensions it is $N = 2$ supersymmetric. It has also been shown how to obtain the gauge fixed Chern-Simons theory via a g–twist . Furthermore, a Donaldson-Witten TQFT is obtained via the usual twisting applied to our supersymmetric action. This yields a scheme for mapping the BF theories into TQFT's. For the examples studied we can associate a topological field theory triangle explicitly illustrating the maps which relate the supersymmetric, BF and $TQFT$ actions.

The non-abelian case has not been addressed in this work. It would also be interesting to check for possible connections between the observables of the BF theories (linking numbers) and those of the TQFT's. Indeed, we expect that our procedure may be generalized to arbitrary dimensional manifolds (without torsion).

Appendix: Conventions

Our conventions are as follows. A Majorana spinor, ψ^α, in three dimensions is real and has two components. Our gamma matrix conventions in Minkowski space are $\gamma^a \equiv (\sigma^2, -i\sigma^1, i\sigma^3)$. We have the useful identity $(\gamma^a\gamma^b)_{\alpha\beta} = \eta^{ab}C_{\alpha\beta} - i\epsilon^{abc}(\gamma_c)_{\alpha\beta}$. The charge conjugation matrix, $C_{\alpha\beta} = \gamma^0 = \sigma^2$ acts as $\psi^\alpha = C^{\alpha\beta}\psi_\beta$ with $C_{\alpha\beta}C^{\gamma\delta} = \delta_\alpha{}^{[\gamma}\delta_\beta{}^{\delta]}$. Note that since C is imaginary, ψ_α is imaginary. The metric in Minkowski space is $\eta = \mathrm{diag}(1, -1, -1)$. For manifolds with Euclidean signature, the gamma matrices are $\gamma^a = (\sigma^2, \sigma^1, \sigma^3)$. With these conventions, $\psi^\alpha(\psi_\alpha)$ is still real (imaginary). The space-time Levi-Cevita tensor is defined by $\epsilon^{012} \equiv 1$ such that $\epsilon_{abc}\epsilon^{def} = \delta_a{}^{[d}\delta_b{}^{e}\delta_c{}^{f]}$. Internal or $SU(2)$ doublet indices are lowered with the real symplectic metric ϵ_{AB} as $\psi^A\epsilon_{AB} = \psi_B$ and raised as $\epsilon^{AB}\psi_B = \psi^A$. A bar is used to indicate complex conjugation.

In two dimensions, our gamma matrices are $\gamma^a = (\sigma^2, -i\sigma^1)$ and $\gamma^3 = \sigma^3$. These satisfy $\gamma^a\gamma^b = \eta^{ab} - \epsilon^{ab}\gamma^3$ and $\gamma^3\gamma^a = -\epsilon^{ab}\gamma_b$. Otherwise, our conventions are in analogy with three dimensions.

Bibliography

[1] J. Schonfeld, *Nucl. Phys.* **B185**(1981) 157 ; R. Jackiw and S. Templeton, *Phys. Rev.* **D23** (1981) 2291; C.R. Hagen, *Ann. Phys.* **157** (1984) 342; *Phys. Rev.* **D31** (1985) 331.

[2] A. S. Schwarz, *Lett. Math. Phys.* **2** (1978) 247; *Commun. Math. Phys.* **67** (1979) 1.

[3] E. Witten, *Commun. Math. Phys.* **117** (1988) 353; *Commun. Math. Phys.* **118** (1988) 411

[4] G. Horowitz, *Commun. Math. Phys.* **125** (1989) 417.

[5] M. Blau and G. Thompson, *Phys. Lett.* **228B** (1989) 64.

[6] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Phys. Rev.* **209** (1991) 129.

[7] L. Baulieu and I. M. Singer, *Nucl. Phys.* **B5** (Proc. Suppl.) (1988) 12.

[8] R. Brooks, D. Montano and J. Sonnenschein, *Phys. Lett.* **214B** (1988) 91.

[9] J.M.F. Labastida and M. Pernici, *Phys. Lett.* **212B** (1988) 56.

[10] R. Brooks and D. Kastor, *Phys. Lett.* **246B** (1990) 99; T. Eguchi and S.-K. Yang, *Mod. Phys. Lett.* **A5** (1990) 1693.

[11] J. P. Yamron, *Phys. Lett.* **213B** (1988) 325.

[12] J.-G. Demers, *Hermiticity and Cohomology Condition in Topological Yang-Mills Theory*, MIT-CTP preprint # 2233.

[13] F. Delduc, F. Gieres and S. P. Sorella, *Phys. Lett.* **225B** (1989) 367.

[14] A. Blasi and N. Maggiore, *Class. Quantum Grav.* **10** (1993) 37.

[15] E. Guadagnini, N. Maggiore and S. P. Sorella, *Phys. Lett.* **255B** (1991) 65.

[16] C. Lucchesi, O. Piguet and S. P. Sorella, *Nucl. Phys.* **B395** (1993) 325.

[17] A. Karlhede and M. Roček, *Phys. Lett.* **212B** (1988) 51.

[18] H. Nishino and S. J. Gates, Jr., *Intl. J. Mod. Phys.* **A8** (1993) 3371.

[19] D. Birmingham, M. Rakowski and G. Thompson, *Nucl. Phys.* **B329** (1990) 83; D. Birmingham and M. Rakowski, *Mod. Phys. Lett.* **A4** (1989) 1753.

[20] F. Delduc, C. Lucchesi, O. Piguet and S. P. Sorella, *Nucl. Phys.* **B346** (1990) 313.

Chapter 5

REAL TIME PROPAGATOR FROM FIRST QUANTIZATION

Abstract

We modify the usual path integral for a non-relativistic particle to include the effect of a bath of identical particles. For a thermal background, the matrix propagator of the real time formalism is recovered by making use of a parametrized form for the action.

To appear in the Proceedings of the *Third International Workshop on Thermal Field Theories*, Banff, Canada, Aug. 16-27 1993.

5.1 Introduction

The understanding of field theories has in many ways benefitted from studying their behavior in curved space-time. Among the interesting features revealed, the most well known is perhaps the possibility of particle creation, which occurs even for a free theory. Whether in cosmological context or in explaining Hawking's radiation, the creation of particle exist on general ground, and has its origin in the ambiguity in identifying a vacuum state for the theory[1]. Because of these created particles, field theories in curved spacetime raise issues similar to field theories at finite temperature (in and out of equilibrium). Also, it has been suggested that various cosmological and black hole models are best described by string theory (especially near singularities). But at present, the only available version of this theory is in a first quantized form. In this context the question arise of how to describe physics in a bath from a first quantized language. As a step in that direction, we investigate the same issue for particles.

But care must be exercised in the choice of quantities to be computed. For a relativistic scalar field in curved spacetime, it can be shown that the usual Feynman path integral in fact correspond to an expectation value between $|0\rangle_{in}$ and $|0\rangle_{out}$ vacua[2] (at least when these asymptotic regions exist). But this raises difficulties, as can be seen in a simplified model for cosmological expansion in (compactified) $1+1$ dimension. Suppose the background geometry of the scalar field has a metric given by

$$ds^2 = C(\eta)(d\eta^2 - dx^2) \tag{5.1}$$

where $-\infty < \eta < +\infty$ and $0 < x < 2\pi$. Consider the case where the conformal factor $C(\eta)$ undergoes a sudden jump from η_1 to η_2 at $\eta = 0$. With $T_{\mu\nu}(\eta, x)$ denoting the energy-momentum tensor, one finds the following for this model[2]. For $\eta < 0$, ${}_{in}\langle 0|T_{00}|0\rangle_{in}$ is simply the vacuum energy, while for $\eta > 0$, the same quantity is the sum of the vacuum energy and of the created particles, a sensible result. In contrast, ${}_{out}\langle 0|T_{00}|0\rangle_{in}$ is just the vacuum energy for both range of η. Moreover, ${}_{out}\langle 0|T_{11}|0\rangle_{in}$ is a complex quantity, and is thus unsuited as a source term in Einstein's equations.

To avoid such problems, we will make use of the real time formalism, which, owing to its causal structure, is appropriate for treating initial value problems[3]. As part of an effort to understand how this formalism can be used in a first quantized theory, we will construct the (bare) propagator for non-relativistic bosons in the presence of a (thermal) background.

5.2 An Interpretation of the Real Time Propagator

The time contour we adopt is the one where time goes from - ∞ to $+\infty$ just above the real axis, and returns to -∞ just below. With this path, the Green function of the system takes a matrix form[4]:

$$\begin{aligned} G(x,y) &= \langle T_p \psi(x)\psi^\dagger(y)\rangle \equiv Tr\{T_p(\psi(x)\psi^\dagger(y))\hat{\rho}\} \\ &= \begin{pmatrix} G_F(x,y) & G_+(x,y) \\ G_-(x,y) & \tilde{G}_F(x,y) \end{pmatrix} , \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} G_F(x,y) &= \langle T(\psi(x)\psi^\dagger(y))\rangle \\ G_+(x,y) &= \langle \psi^\dagger(y)\psi(x)\rangle \\ G_-(x,y) &= \langle \psi(x)\psi^\dagger(y)\rangle \\ \tilde{G}_F(x,y) &= \langle \tilde{T}(\psi(x)\psi^\dagger(y))\rangle \ . \end{aligned} \tag{5.3}$$

Here, T_p is the path ordering operator along the time contour, x, y hold for spacetime points and $\hat{\rho}$ is the density matrix. As the time component of each field in the bracketed pair can either be above of below the real axis, there are clearly four distinct objects, as given in Eq.(5.3). T and $\tilde{T}$ are respectively the time-ordering and anti time-ordering operators.

Suppose we now restrict ourselves to the case of free, non- relativistic bosons in thermal equilibrium in 1+1 dimensions. The evolution is governed by the Lagrangian:

$$L = \int dx \left(\psi^\dagger(x) \left(i\partial_t - \frac{\nabla^2}{2m} \right) \psi(x) \right) . \tag{5.4}$$

where the field $\psi(x)$ obeys the equal time canonical relation (x and y obviously denote here only the space variable):

$$[\psi(x,t), \psi^\dagger(y,t)] = \delta(x-y) . \tag{5.5}$$

By expanding $\psi(x)$ in modes, constructing the corresponding Fock space and taking note of the translation invariance, one then get for Eq.(5.3) in momentum space:

$$\begin{aligned} G_F(p_0, p_x) &= \frac{i}{p_0 - \frac{p_x^2}{2m} + i\varepsilon} - 2\pi n(p)\delta(p_0 - \frac{p_x^2}{2m}) , && (5.6) \\ G_+(p_0, p_x) &= -2\pi n(p)\delta(p_0 - \frac{p_x^2}{2m}) , && (5.7) \\ G_-(p_0, p_x) &= 2\pi(1-n(p))\delta(p_0 - \frac{p_x^2}{2m}) , && (5.8) \\ \tilde{G}_F(p_0, p_x) &= G_F^* , && (5.9) \end{aligned}$$

where $n(p) = \frac{1}{e^{\beta(p_0-\mu)}-1}$ is the Bose-Einstein distribution.

In first quantization, we identify G_F in Eq.(5.2) as representing the following series of processes. A particle evolving (in the path integral sense) from the spacetime point (x_i, t_i) to (x_f, t_f) will, in the presence of a bath, undergo various and distinct class of evolutions. The first term in Eq.(5.6) represents the usual Feynman propagator. But because of the bath, particle exchange can also occur during the evolution. As the initial particle evolves, it can be absorbed in the bath, and the final point (x_f, t_f) be effectively reached by another particle. Eq.(5.6) shows that this process requires the exchanging particles to be on shell. It is also clear that this exchange process could appear an arbitrary number of times between (x_i, t_i) and (x_f, t_f).

In a similar fashion, G_+ is taken to represent many processes. The simplest of them is when a particle evolves from (x_i, t_i) to some point (x', t') (with $t' > t_f$), where it is annihilates with a "hole" that departed at (x_f, t_f). Also, in analogy with the G_F

case, one must also consider all the possible number of exchanges with other particles during the evolution. $\tilde{G}_F$ and G_- can be thought as respectively representing the same processes as $\tilde{G}_F$ and G_+ but inverted (holes playing the role of particles and vice-versa).

We now proceed to make these ideas quantitative by using a formulation that allows for evolution forward and backward in time.

5.3 Parametrization and Gauge Fixing

In first quantization, a parametrized version of the free particle is given by the action[5]:

$$S = \int_0^1 d\tau \left(p_x \dot{x} - p_0 \dot{t} - \lambda (p_0 - \frac{p_x^2}{2m}) \right) , \tag{5.10}$$

where τ is the parameter time along the world line of the particle in space-time; without loss of generality, we take the length of the world line to be unity. In this formulation, the dynamical variables are t and x, along with the associated momentum p_0 and p_x respectively. The Lagrange multiplier λ enforces the constraint $\mathcal{H} \equiv p_0 - \frac{p_x^2}{2m} = 0$. The Green function is then computed as:

$$G(x_f t_f; x_i t_i) = \int \mathcal{D}p_0 \; \mathcal{D}t \; \mathcal{D}p_x \; \mathcal{D}x \; \mathcal{D}\lambda \; e^{iS} , \tag{5.11}$$

with boundary conditions $x(0) = x_i, t(0) = t_i, x(1) = x_f$, and $t(1) = t_f$. By discretizing Eq.(5.11), one obtains[6]:

$$G(x_f t_f; x_i t_i) = \int \frac{dp_x}{2\pi} \, e^{ip_x(x_f - x_i)} \int \frac{dp_0}{2\pi} \, e^{-ip_0(t_f - t_i)} \int \mathcal{D}\lambda \, e^{-i \int_0^1 d\tau \, \lambda (p_0 - \frac{p_x^2}{2m})} , \tag{5.12}$$

or in Fourier space:

$$G(p_0, p_x) = \int \mathcal{D}\lambda \; e^{-i(p_0 - \frac{p_x^2}{2m}) \; \int_0^1 d\tau \; \lambda} . \tag{5.13}$$

Using the constraint $\mathcal{H}$, we consider two versions of the gauge symmetry of the action in Eq.(5.10):

$$\begin{aligned} I. \qquad \delta x &= [x, \xi\mathcal{H}] = \xi\frac{p_x}{m} \\ \delta t &= [t, \xi\mathcal{H}] = \xi \\ \delta\lambda &= \dot{\xi} \\ \delta p_x &= \delta p_0 = 0 \ , \end{aligned} \tag{5.14}$$

or

$$\begin{aligned} II. \qquad \delta x &= [x, \varepsilon\lambda\mathcal{H}] = \varepsilon\lambda\frac{p_x}{m} \\ \delta t &= [t, \varepsilon\lambda\mathcal{H}] = \varepsilon\lambda \\ \delta\lambda &= (\dot{\varepsilon\lambda}) \\ \delta p_x &= \delta p_0 = 0 \ . \end{aligned} \tag{5.15}$$

In both cases, we assume that the symmetry parameters vanishes at the end points $\xi(0) = \xi(1) = \varepsilon(0) = \varepsilon(1) = 0$ and that the orientation of the world line is preserved.

Making use of Eq.(5.14), the finite transformation on λ is $\lambda'(\tau) = \frac{dh(\tau)}{d\tau} + \lambda(\tau)$ (with $h(\tau)$ an arbitrary function that vanishes at the end points of the world line) and one may gauge fix the system by imposing $\dot{\lambda} = 0$. Now since $\Lambda \equiv \int_0^1 d\tau\, \lambda$ is gauge invariant, Eq.(5.13) may easily be evaluated with $\int \mathcal{D}\lambda \rightarrow \int_{-\infty}^{\infty} d\Lambda$ with the result:

$$G(p_0, p_x) = 2\pi\delta(p_0 - \frac{p_x^2}{2m}) \ . \tag{5.16}$$

Contrary to expectations, one does not recover the Feynman propagator when using the symmetry of Eq.(5.14). A similar feature also occurs for the relativistic particle[2]. An interesting alternative is offered by Eq.(5.15), which in the case of a relativistic particle is in direct relation with the diffeomorphism of the einbein Lagrangian[7].

Using the gauge symmetry of Eq.(5.15), the finite transformation on λ is $\lambda'(\tau) = \frac{d\tau}{d\tau'}\lambda(\tau)$. As before, $\Lambda \equiv \int_0^1 d\tau\ \lambda$ is gauge invariant. But now, a function $\lambda_1(\tau)$ that goes to zero once (say), is not gauged related to another $\lambda_2(\tau)$ that does not go to zero, even though $\Lambda_1 = \Lambda_2$. Consider for instance all functions $\lambda(\tau)$ such that $\lambda(0) > 0$ and $\lambda(1) < 0$. We associate all the corresponding contributions to $G_+(p_0, p_x)$. There are first the class of functions that cross $\lambda = 0$ for only one value of τ. These contribute to Eq.(5.13),

$$\int_0^\infty d\Lambda_1 \int_{-\infty}^0 d\Lambda_2 \, e^{-i(p_0 - \frac{p_x^2}{2m})(\Lambda_1 + \Lambda_2)} \ . \tag{5.17}$$

To this process, we include a weight $\varepsilon A(p)$ that accounts for the presence of the bath. Here, ε is the infinitesimal quantity that regularize Eq.(5.17), and one gets:

$$\pi A(p) \delta(p_0 - \frac{p_x^2}{2m}) \ . \tag{5.18}$$

Another class is formed by the functions $\lambda(\tau)$ that are zero for three values of τ. If the weight to go from $\lambda < 0$ to $\lambda > 0$ is $\varepsilon B(p)$, we obtain for this class:

$$\frac{\pi}{2} A(p) B(p) A(p) \delta(p_0 - \frac{p_x^2}{2m}) \ . \tag{5.19}$$

Adding the contributions from classes made of functions with $5, 7, 9...$ crossing points, we recover G_+ given in Eq.(5.7) as long as $A(p) = 1 - 2n(p)$ and $B(p) = 2n(p)$.

In a similar fashion, G_F will be constructed of all $\lambda(\tau)$ such that $\lambda(0) > 0$ and $\lambda(1) > 0$. Clearly, the simplest class is the set of $\lambda(\tau)$ with no crossing points. Again, using Eq.(5.13), and $\int \mathcal{D}\lambda \to \int_0^\infty d\Lambda$, we get the zero temperature Feynman propagator, the first term in Eq.(5.6). But there are also contributions by functions with two crossing points, they are:

$$\frac{\pi}{2} A(p) B(p) \delta(p_0 - \frac{p_x^2}{2m}) \ , \tag{5.20}$$

where we appealed to the same weights as before. Summing functions with $4, 6, 8...$ crossing points, we recover G_F given in Eq.(5.6)

In just the opposite way, we may look at the functions $\lambda(\tau)$ such that $\lambda(0) < 0$ and $\lambda(1) > 0$ to obtain G_- and $\lambda(\tau)$ such that $\lambda(0) < 0$ and $\lambda(1) < 0$ to obtain $\tilde{G}_F$.

In this way, the real time matrix propagator is recovered using first quantization.

5.4 Conclusion

To develop ways of dealing with finite temperature phenomena from the point of view of a first quantized theory, we analyzed the case of non-relativistic bosons. We have identified how one can incorporate the effect of the bath on the propagation, in the special case of thermal equilibrium, in analogy with the case of relativistic particles and strings[2]. Clearly, a more general distribution is found in various cosmological and black hole contexts and further work could seek to extend the present analysis to those cases. Moreover, it would be desirablc to derive the weights $A(P)$ and $B(p)$ a priori, based for example of the development of the background geometry.

Bibliography

[1] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, 1982), p. 59.

[2] S.D. Mathur, hep-th 9306090.

[3] J.P. Paz, in *Thermal Field Theories*, ed. H. Ezawa and Y. Hashimoto, (Elsevier Science Publishers, 1991).

[4] K.-C. Chou, Z.-B. Su, B.-L. Hao and L. Yu, *Phys, Rep.* **118**, (1985) 1.

[5] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, 1992), p. 103. Our p_0 is defined to be minus the same quantity of this reference; of course, both choices lead to the usual phase space action $\int dt(p_x \frac{dx}{dt} - \frac{p_x^2}{2m})$ for monotonic functions $t(\tau)$.

[6] J. Govaerts, *Hamiltonian Quantization and Constrained Dynamics* (Leuven University Press, 1991), see p. 337 for a discretization similar to the one used here. The inclusion of ghosts in our case would justify more thoroughly the gauge fixing procedure, but would lead to identical results.

[7] J. Govaerts, *Hamiltonian Quantization and Constrained Dynamics* (Leuven University Press, 1991), p. 246.