Nonlinear Control Using Linearizing Transformations

by

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Abstract

The application of linearizing transformations for the control of nonlinear mechanical systems with particular emphasis to underactuated systems was investigated. Within the framework of canonical transformation theory a new set of transformations were derived. These transformations, termed orthogonal canonical transformations, also preserve Hamilton's equations and characterize a special class of Hamiltonian systems that admit a linear representation in the transformed coordinate system. Using this approach, the solution to the original nonlinear equations are obtained from the inverse transformation. The general conditions for such a transformation were derived, and an example was presented that illustrates this linearizing property.

The Riemann Curvature Tensor was introduced as a computational tool by which it can be determined whether a given mechanical system admits a coordinate system in which the equations of motion appear linear. It was shown that the curvature tensor can be used to test for the existence of point transformations such that in the transformed coordinates the nonlinear system appears as a double integrator linear model. An example was presented that admits such a coordinate system, and the linearizing transformation was computed.

An existing control design methodology was adopted as an approach to control underactuated nonlinear systems. This approach expands the operating region of linear control designs by constructing a linear approximation about an equilibrium point accurate to second or higher order. A computational method to test for the order of linearization was derived. This approach was applied to an underactuated example problem. Simulation results showed a substantial improvement in the range of operation of the linear control design.

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Chapter 1

Introduction

For the control of systems described by linear, time-invariant dynamics there exist powerful design methods . The field has matured to the point where standard computer-aided design packages now allow the control engineer to design compensators using the latest theoretical developments in linear system theory. In contrast, the development of control design methodologies for nonlinear systems have lagged their linear counterparts. As most physical systems behave in a nonlinear fashion, there exists a strong incentive to develop nonlinear controller design methods.

The usual approach to controlling nonlinear systems is to linearize, about an operating point, the nonlinear dynamics and apply proven linear control design approaches. The design is then verified and validated by exhaustive computer simulations of the response of the nonlinear dynamics with the linear controller over a variety of initial conditions and disturbances. Such an approach is practical for only a small range of operating conditions. However, there are instances where the nonlinear terms cannot be ignored. In high performance applications where a wide range of operating conditions are encountered, linear control design based on local approximations may be inadequate, and in the worst case fail.

Another case when linearized analysis is inadequate is when "hard" or discontinuous nonlinearities are present. It is not unusual to encounter "hard" nonlinearities in practice. Examples are saturating actuators, on-off actuators such as reaction or thruster control systems for spacecraft, unidirectional thrusters where thrust can only be applied in one direction, backlash in geared systems caused by gaps, dry friction where the friction force is dependent on the velocity direction etc. Linearized analysis fails because it is inherently a "continuous" analysis tool and it is not possible to approximate discontinous nonlinearities by linear functions at the points of discontinuity.

At the other extreme are the so called "soft" or continuous nonlinearities. A common source of "soft" nonlinearities is when a system is described in a non-inertial frame. When describing the rotational motion of a rigid body, it is common practice to express the equations of motion relative to a rotational frame attached to the body. Observing the equations of motion from body-fixed coordinate frame results in the well known *Euler's equations* which are quadratic in velocities. Other examples of non-inertial frame induced nonlinearities are the centrifugal and coriolis effects present in rotating systems such as robotic manipulators, planar linkage mechanisms like four-bars, planetary gear-trains etc. Such nonlinearities may be refered to as kinematic since they originate or are induced by the kinematical structure of the system. Another example of "soft" nonlinearity is that of a softening or hardening spring that can be modeled by a polynomial expression of the state.

A further complication arises when the nonlinear system to be controlled is underactuated. When there are less actuators than degrees of freedom the system is defined as underactuated. A common occurence is when a fully actuated system experiences a failure in one or more actuators. For this type of system a general control design methodology does not exist. Most established nonlinear control design methods require "square" or fully actuated systems.

The objective of this thesis is to investigate the problem of continuous nonlinear system control design with particular emphasis to underactuated systems. The general approach with which this problem is addressed is the use of transformation techniques that simplify the nonlinear equations such that existing results in linear control theory can be applied.

1.1 Properties Of Linear Time-Invariant And Nonlinear Systems

In this section, a brief overview of the main properties of linear and nonlinear systems is presented. The objective is to highlight and contrast the complicated behaviour that nonlinear system can exhibit as opposed to the response of linear systems. For this purpose let the linear time-invariant dynamics be defined as,

$$\dot{x} = A\,x + B\,u$$

and the nonlinear time-invariant dynamics as:

$$\dot{x} = f(x, u)$$

First, let us consider the properties of the linear system. The main features to be considered are equilibrium points, stability, and forced response.

(a) **Equilibrium:** For the unforced system (i.e. $u = 0$) the equilibrium point is *unique* if A is nonsingular. Then $x_{eq} = 0$ is the *only* equilibrium point. If A is singular there exist an infinite number of equilibrium points.

(b) **Stability:** Stability about the equilibrium point is *solely* defined by the spectrum of A. The system is stable if all eigenvalues of A have negative real parts. The definition of stability is independent of *initial conditions*, *forcing functions*, and the concepts of *local* or *global* behaviour.

(c) **Forced Response:** Linear systems satisfy the property of superposition,

$$x(\, u_1(t) \, + \, u_2(t)\,) \, = \, x(\, u_1(t)\,) \, + \, x(\, u_2(t)\,)$$

and homogeneity.

$$x(\, \alpha\, u(t)\,) \, = \, \alpha\, x(\, u(t)\,)$$

Additionally, if the impulse response of the system is known the response to *any* input can be constructed using the principle of convolution. This defines linear systems as *integrable.*

For nonlinear systems, no general statements like the above can be made. Their behaviour is much more complicated and cannot be captured by a few simple characteristics.

(a) **Equilibrium:** For the unforced system (i.e. $u = 0$) there may exist *none*, 1, or *multiple* equilibrium points.

(b) **Stability:** Stability about *an* equilibrium point is dependent on *initial conditions* and *forcing functions* as well as having a *local* or *global* property. Furthermore, nonlinear systems may exhibit *limit cycles* which are *closed*, *unique* trajectories or orbits. These equilibrium manifolds may be *attractive* or *repulsive.*

(c) **Forced Response:** Nonlinear systems *do not* satisfy the principles of superposition and homogeneity. The response can also be non-unique, exhibit chaotic behaviour etc. Also, in general they are *non-integrable.*

Another interesting property is that of bifurcation where a change in a system parameter can alter stability and equilibrium points. In conclusion, nonlinear systems exhibit a plethora of behaviors which makes their analysis a difficult task. Usually, each system must be studied separately as there are very few properties that are shared by all nonlinear systems. As a consequence, in general, there does not exist a systematic approach to analyze or predict their response, let alone to alter it. This situation, however, has forced the development of particular nonlinear control design techniques that are applicable to certain systems. In the next section, currently available control synthesis methods are presented.

1.2 Current Nonlinear Control Synthesis Methods

In this section, a brief summary of available nonlinear control design methods is presented. As can be surmized from the previous section, there do not exist any general nonlinear control design methodologies. However, there exist a multitude of powerful methods applicable to certain classes of nonlinear systems. Consequently, this is a thriving research area that has attracted the attention of many researchers from disparate scientific disciplines. The most recent decade has witnessed a rejuvination of interest in nonlinear control techniques and substantial effort has been expended to overcome some of the impediments to designing practical controllers. This new wave has been driven by ever increasing sophisticated applications requiring stringent performance specifications and the advances in computer technology.

Essentially all of the control design tools for nonlinear system control design methods can be thought of as providing in one form or another a way to generate the control action as a function of system states. This mapping may be linear or nonlinear, static or dynamic. One way to classify the controller design approaches is according to the method used in compensator design:

1. Compensator design based on linear methods:

   (a) **Linear Control:** The nonlinear system is linearized about an equilibrium point, and a linear controller using a variety of techniques is designed. Using fecently developed synthesis tools, robustly stable compensators can be designed to account for norm-bounded model uncertainty. However, such designs have a limited operating region where the linear approximation is valid.

(b) **Gain Scheduled Linear Control:** This approach attempts to expand the region of linear control operation by linearizing the dynamics about different operating points and designing linear controllers for each point. In between operating points, the control action or the gains are interpolated or "scheduled". Some of the drawbacks of such an approach are that there are no stability guarantees during transition between operating points, and is computationally intensive if many operating points are considered as well as when the dimension of the nonlinear system is high.

2. Compensator design using Lyapunov stability criteria:

(a) **Sliding Mode Control:** This is an example of a robust nonlinear tracking control design method applicable to systems that can be put in a controllable canonical form (see for example Slotine [41] Chapter 7). This is a powerfull method that provides stability robustness to parametric modeling uncertainty and unmodeled dynamics. The approach is to define a so called "sliding surface" in state space that represents the tracking error. The control action is then chosen such that the system remains on the "sliding surface" in the presence of model uncertainty. The main features are that the undesirable nonlinear dynamics are *robustly cancelled* and desirable linear dynamics inserted. This approach which was originally developed in the Soviet Union, was pioneered by Slotine [39] by eliminating undesirable chattering and large control authority that plagued previous designs. The smoothing is achieved by using proportional control inside an attractive region about the "sliding surface" known as a boundary layer. Inside the boundary layer there are no stability or robustness guarantees. This approach requires that the system can be expressed in control canonical form [41], the number of outputs equals the number of inputs, and exact state measurement information is available.

(b) **Adaptive Control:** This is a control methodology applicable to linear or nonlinear systems with unknown or uncertain parameters [3]. One approach known as indirect adaptive control, is to estimate on-line the unknown system parameters using measurements. Another method is direct adaptive control which adjusts the controller parameters such that a desired closed-loop behaviour is achieved. Note that the uncertain parameters are treated as time-varying. This may cause degradation in performance if the actual parameters are state dependent. A recent development in the field is the method developed by Slotine and Li [40] which exploits the Lagrangian and linear in parameters structure of rigid manipulators. The drawbacks of such an approach for nonlinear as well as linear sytems is the stability issue in the presence of disturbances, measurement noise and unmodeled dynamics.

3. Compensator design using transformation methods:

   (a) **Input-Output Feedback Linearization:** This approach utilizes state and control transformations coupled with feedback to realize an equivalent linear representation of the nonlinear system from the inputs to the outputs and was introduced by Isidori *et al.*, [22]. The main concept in this and the following methods is the use of transformations in the state and control variables to alter the nonlinear dynamics to a nearly linear form such that the remaining nonlinearities can be cancelled by feedback. The state and control transformations must be constructed in such a manner that the remaining terms appear in the path of the control action in order to be cancelled. Like Sliding Mode control, the nonlinear system is put in controllable canonical form [41]. Once the linear input-output relationship is obtained, linear theroy can be used to design a controller. The drawbacks of this approach are the sensitivity to parameter uncertainty and unmodeled dynamics, the requirement of full state measurement and can only be applied to certain nonlinear systems.

   (b) **Input-State or Exact Feedback Linearization:** This is a similar approach to Input-Output linearization except that the linear equivalence is established between the inputs and the complete state. The dimension of the linear equivalent system is identical to the nonlinear one, whereas for the Input-Output approach it is less than or equal to. Originating with the work of Krener [27] (1973), and Brockett [5] (1978), the problem was completely solved by Jacubczyck and Respondek [23] (1980), Su [46] (1982), and Hunt, Su, Meyer [21] (1983). The existence conditions for this approach are fairly restrictive, and evaluating them is computationally intensive requiring symbolic mathematics software. Even when the existence conditions are satisfied, finding a solution requires solving a nontrivial set of partial differential equations. Implementing this approach requires full state information and is also sensitive to modeling errors. The existence conditions usually fail for underactuated systems.

(c) **Approximate Feedback Linearization:** This approach attempts to construct a linear approximation about an equilibrium point accurate to second or higher order as opposed to *all* orders for input-state linearization. This approach was formulated by Krener [28] (1984), [29] (1987). The existence conditions for this approach are similar to those for input-state linearization, yet are much less stringent. The computation of the requisite transformations requires a solution to a set of algebraic equations instead of solving partial differential equations. However, it does require full state information and is sensitive to modeling errors.

It should be noted that most of the above approaches cannot be applied to underactuated problems. They require the system to be fully actuated. As a consequence, one controller design candidate for nonlinear underactuated systems is the extended feedback linearization approach.

1.3 Thesis Contributions

As stated previously, the objective of this thesis is the development of linearizing transformations for the control of nonlinear mechanical systems, with particular emphasis to underactuated systems. This problem was addressed by exploiting the special properties of mechanical systems. Such systems obey the principles of analytical dynamics, for which there exists an extensive and very rich literature. One of the major thrusts of classical dynamics has been the search for general methods of solution to nonlinear differential equations of motion. Since the form of the equations of motion depend on the particular choice of coordinates employed, one approach to simplifying these equations has been the selection of a suitable coordinate system. To achieve this goal a systematic coordinate transformation theory was developed within the framework of Hamiltonian dynamics. These approach is referred to as canonical transformation theory.

The historical development of canonical transformation theory has proceeded from the viewpoint of dynamics, where the possibility of control action was not considered. Cast in the control framework, these transformations attempt to linearize the nonlinear dynamics via coordinate and control transformations *only*. A feedforward signal is not employed to cancel any remaining nonlinearities. It is apparent that such an approach is much more restrictive with more stringent requirements than the feedback linearization approaches. In effect, the nonlinearities are eliminated by proper choice of coordinates. This requires the existence of a coordinate system in which the nonlinear equations appear linear. Thus, given a nonlinear mechanical system, the objective is to determine whether the system admits such a coordinate system, and construct the transformation when this is possible. With this in mind, the main contributions of this thesis can be summarized in the following.

(a) **Orthogonal Canonical Transformations:** These are a new set of transformations that preserve Hamilton's canonical equations and which are not canonical by the classical definition. These transformations lead to a special set of Hamiltonian systems that admit a linear representation in the transformed coordinate system. The general conditions for such a transformation are derived and an example illustrating the linearizing property is presented. It is shown that the solution to the general conditions results in the generation of a linearizing coordinate transformation. Furthermore, the solution to the original nonlinear equations is obtained from the inverse transformation.

(b) **Linearizing Transformations For Mechanical Systems:** The Riemann Curvature Tensor is introduced as a computational tool to test whether a given mechanical system admits a coordinate system in which the equations of motion appear linear. It is shown that the curvature tensor can be used to test for the existence of a point transformation such that in the transformed coordinates the system appears as a double integrator linear model. An example is presented that admits such a coordinate system, and the linearizing transformation is computed.

(c) **Control Of Underactuated Systems:** The approach of Krener [28], extended feedback linearization, is applied to the control of underactuated systems. A computational method to test for the order of linearization is derived from the general existence conditions. This approach is applied to an example problem. Simulation results show a substantial improvement in the range of operation of the linear control design.

1.4 Thesis Organization

In Chapter 2, the Hamiltonian formulation of dynamics is introduced, and the framework for canonical transformations is developed. A general condition for the preservation of Hamilton's canonical equations is derived, and the three most prevalent definitions of canonical transformations appearing in the literature are reviewed. It is shown that all three definitions are equivalent. A new set of canonical transformations that preserve Hamilton's canonical equations are derived, which lead to a special set of Hamiltonian systems that admit a linear representation in the transformed coordinates. The general conditions for such a transformation are derived. For a system defined by two generalized coordinates, an example is presented that illustrates the linearizing property.

In Chapter 3, linearizing point transformations are investigated for mechanical systems. It is shown that one approach to transform a nonlinear system to a double integrator linear model in the transformed coordinates involves the use of point transformations. The well known properties of point transformations, preservation of Lagrange's equations of motion and the fact that *all* point transformations are canonical, are reviewed. An alternative derivation of an existing result on the "square-root" factorization of the inertia matrix in terms of the transformation Jacobian matrix is presented. This factorization leads to a double integrator linear model in the transformed coordinates. To test for the existence of such a factorization, the Riemann Curvature Tensor is introduced as a computational tool. The cart-pole example is shown to satisfy the curvature conditions, and the linearizing transformation is computed. Euler's rotational equations of motion are shown to violate the curvature conditions for an axi-symmetric inertia distribution.

In Chapter 4, the approximate feedback linearization methodology is presented. Feedback equivalence for linear and nonlinear systems is presented. The method of exact feedback linearization is presented, where a transformation in state and control variables is used to generate a linear equivalent system. The approach of extended feedback linearization is reviewed. For approximate linearization, a computational approach to test for the order of the linearization is derived. This method is applied to the cart-pole problem. It is shown that this example is not exactly linearizable, and a second order linear approximation is constructed. Simulation results show that for the same closed loop pole locations a substantial improvement in the range of operation of the linear control design is achieved.

Finally, in Chapter 5 concluding remarks and recommendations for future research are presented.

Chapter 2

Canonical Transformations

In this chapter, the Hamiltonian formulation of dynamics is introduced. Starting from the Lagrangian framework, the canonical equations of Hamilton are derived. The main features of these equations are that they result in first-order differential equations, employ two sets of independent variables instead of one, and allow the development of a systematic coordinate transformation theory. The objective is to integrate the equations of motion by identifying ignorable coordinates or finding a coordinate system in which the equations of motion appear linear. Within the Hamiltonian framework, canonical transformations are investigated with a goal of generating linearizing transformations. The main feature of such transformations is that Hamilton's equations are preserved in the transformed coordinate system.

In Section 2.1, Hamilton's canonical equations are introduced followed by an overview of coordinate transformations. In Section 2.3 the framework for canonical coordinate transformations is developed. A general condition for preservation of the canonical equations is derived, and the three most prevalent definitions appearing in the literature are reviewed. It is shown that all three different definitions are equivalent in that they require the preservation of a skew-symmetric quadratic form.

In Section 2.4 a novel set of transformations that preserve Hamilton's canonical equations is derived. This approach is termed *orthogonal canonical transformation* and leads to a special set of Hamiltonian systems that admit a linear representation in the transformed coordinate system. The general conditions for such a transformation are derived. An example is presented for a system defined by two generalized coordinates that illustrates the linearizing property.

2.1 Hamilton's Canonical Equations

One approach to the Hamiltonian formulation of the equations of motion is to start from the Lagrangian description. By means of a Legendre transformation, a Langrangian system of second-order equations is transformed into a system of first-order equations, referred to as Hamilton's canonical equations. For a dynamical system with n degrees of freedom, the Lagrangian, in general, is expressed as

$$L = L(q, \dot{q}, t)$$

where q is the n-vector of generalized coordinates. By contrast, the Hamiltonian description utilizes $2n$ first-order equations in $2n$ variables. For the Hamiltonian framework a new variable, the momentum *conjugate (dual)* to the generalized coordinate q_i, is defined as:

$$p_i \equiv \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_i}$$

The set (q, p) is usually referred to as *canonical* variables.

The procedure of transforming from the Lagrangian set $(q, \dot{q}, t)$ to the Hamiltonian set (q, p, t) involves the *Legendre* transformation. This transform provides an approach to change the independent variable $\dot{q}$ to the independent variable p without loss of information. The Legendre transform $H(q, p, t)$ of the Lagrangian function $L = L(q, \dot{q}, t)$ with respect to $\dot{q}$ is,

$$H(q, p, t) = \sum_{i=1}^{n} p_i \dot{q}_i \; - \; L(q, \dot{q}, t) \tag{2.1}$$

where the new function $H(q, p, t)$ is the *Hamiltonian* of the system. One approach to deriving Hamilton's equations involves the total differential of the Hamiltonian:

$$dH = \sum_{i=1}^{n} \left[\frac{\partial H(q,p,t)}{\partial q_i} dq_i + \frac{\partial H(q,p,t)}{\partial p_i} dp_i \right] + \frac{\partial H(q,p,t)}{\partial t} dt$$

From (2.1), however, this is equal to:

$$dH = \sum_{i=1}^{n} \left[p_i \, d\dot{q}_i + \dot{q}_i \, dp_i - \frac{\partial L(q, \dot{q}, t)}{\partial q_i} dq_i - \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_i} d\dot{q}_i \right] - \frac{\partial L(q, \dot{q}, t)}{\partial t} dt \tag{2.2}$$

From the definition of $p_i = \dfrac{\partial L(q,\dot{q},t)}{\partial \dot{q}_i}$, the first and fourth term on the right- hand side of (2.2) cancel:

$$dH = \sum_{i=1}^{n} \left[\dot{q}_i \, dp_i - \frac{\partial L(q,\dot{q},t)}{\partial q_i} dq_i \right] - \frac{\partial L(q,\dot{q},t)}{\partial t} dt$$

Equating coefficients results in:

$$\begin{aligned} \dot{q}_i &= \frac{\partial H(q,p,t)}{\partial p_i} \\ \frac{\partial L(q,\dot{q},t)}{\partial \dot{q}_i} &= -\frac{\partial H(q,p,t)}{\partial q_i} \\ \frac{\partial L(q,\dot{q},t)}{\partial t} &= -\frac{\partial H(q,p,t)}{\partial t} \end{aligned}$$

The expression for $\dot{p}_i$ is determined next. This is accomplished by expressing $\dfrac{\partial L(q,\dot{q},t)}{\partial \dot{q}_i}$ in terms of p. To this end, one reverts to the Lagrangian formulation of the equations of motion. In the case of a holonomic, conservative system described by a set of *independent* generalized coordinates q, the Lagrange equations in *standard form* are:

$$\frac{d}{dt}\left(\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}_i}\right) - \frac{\partial L(q,\dot{q},t)}{\partial q_i} = 0 \tag{2.3}$$

Using (2.3), $\dfrac{\partial L(q,\dot{q},t)}{\partial q_i} = \dfrac{d}{dt}\left(\dfrac{\partial L(q,\dot{q},t)}{\partial \dot{q}_i}\right) = \dot{p}_i$ by definition. This leads to the so-called *Hamilton's canonical equations*:

$$\begin{aligned} \dot{q}_i &= \frac{\partial H(q,p,t)}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H(q,p,t)}{\partial q_i} \end{aligned}$$

These equations can be written in compact form using matrix notation. Let x denote the extended state of $2n$ coordinates $x = [q_1, \ldots, q_n, p_1, \ldots, p_n]^T$. Then, the canonical equations can be written as

$$\dot{x} = ZH_x^T$$

where

$$\begin{aligned} H_x &= \left[\frac{\partial H}{\partial q_1}, \ldots, \frac{\partial H}{\partial q_n}, \frac{\partial H}{\partial p_1}, \ldots, \frac{\partial H}{\partial p_n}\right] \\ Z &= \begin{bmatrix} 0_{n\times n} & 1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{bmatrix} \end{aligned} \tag{2.4}$$

and $1_{n\times n}$, $0_{n\times n}$ denote, respectively, the $n \times n$ identity and zero matrices. In most instances the Lagrangian or Hamiltonian functions are not explicit functions of time, and this will be assumed in the following. If the dynamic system in question involves non-conservative generalized forces u_i, the Lagrangian equations in *standard form* become

$$\frac{d}{dt}\left(\frac{\delta L(q,\dot{q})}{\partial \dot{q}_i}\right) - \frac{\partial L(q,\dot{q})}{\partial q_i} = u_i$$

The canonical equations in this case become:

$$\begin{aligned} \dot{q}_i &= \frac{\partial H(q,p)}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H(q,p)}{\partial q_i} + u_i \end{aligned}$$

In the most general case, the impressed action on a system may have the form of a velocity (flow source in bond-graph terminology [20]) rather than a force or torque (effort source). In this case, the most general form of Hamilton's equations is written in the form [20]:

$$\begin{aligned} \dot{q} &= \frac{\partial H(q,p)^T}{\partial p} - F(q,p,t) \\ \dot{p} &= -\frac{\partial H(q,p)^T}{\partial q} + E(q,p,t) \end{aligned} \tag{2.5}$$

Up to this point, no mention has been made about the explicit expression of the Lagrangian function. Let q denote the n-dimensional vector of *independent* generalized coordinates of a holonomic system, in which a kinetic energy $T(q,\dot{q})$ quadratic in generalized velocities, and a potential energy $V(q,t)$ dependent on generalized coordinates is defined. The expression for the kinetic energy is defined:

$$T(q,\dot{q}) = \frac{1}{2}\,\dot{q}^T B(q)\dot{q}$$

where $B(q)$ is the symmetric and positive definite inertia matrix. For a system where all the generalized forces are derivable from a potential function $V(q,t)$, the standard definition of the Lagrangian function is $L(q,\dot{q},t) = T(q,\dot{q}) - V(q,t)$. It should be noted that this is not the primitive form of the Lagrangian because we can dispense with the artifact of finding a potential function from which the generalized conservative forces are derivable. Instead, all the generalized forces, be they conservative or non-conservative, are treated on an equal basis as applied forces. Then, the Lagrangian can be defined $L(q,\dot{q},t) \equiv T(q,\dot{q})$ which results in the *primitive* form of the Lagrange equations [13],

$$\frac{d}{dt}\left(\frac{\partial T(q,\dot{q})}{\partial \dot{q}_i}\right) - \frac{\partial T(q,\dot{q})}{\partial q_i} = u_i$$

where u_i represents the effect of all conservative and nonconservative forces.

2.1.1 Transformed Hamilton's Equations

In this section background material needed in the remainder of this chapter will be presented. Specifically, the general form of Hamilton's equations with respect to coordinate transformations will be derived. This result will be needed when coordinate transformations which preserve Hamilton's equations are considered. To derive the transformed dynamics the most general Hamiltonian description will be considered. Rewriting (2.5) in matrix notation,

$$\dot{x} \;=\; Z\, H_x^T \;+\; \tilde{u} \tag{2.6}$$

where $\tilde{u} \;=\; \left[\, -F^T(q,p)\;\; E^T(q,p)\,\right]^T$. Consider the most general coordinate tranformation where the new variables are:

$$\begin{aligned} Q &= Q(q,p) \\ P &= P(q,p) \end{aligned}$$

The transformed equations can be written compactly using matrix notation. Let X denote the extended state of $2n$ transformed coordinates $X \;=\; [\, Q_1,\, \ldots,\, Q_n,\, P_1,\, \ldots,\, P_n\,]^T$. Differentiating $X(x)$ with respect to time and utilizing (2.6), the equations of motion in the new coordinates assume the form,

$$\dot{X} = N\ Z\ H_x^T + N\ \tilde{u} \tag{2.7}$$

ehere N is the transformation Jacobian matrix given by:

$$N = X_x = \begin{bmatrix} \dfrac{\partial Q(q,p)}{\partial q} & \dfrac{\partial Q(q,p)}{\partial p} \\ \dfrac{\partial P(q,p)}{\partial q} & \dfrac{\partial P(q,p)}{\partial p} \end{bmatrix}$$

To obtain the transformed equations in terms of the new coordinates, H_x^T must be expressed in terms of H_X^T. This can be accomplished using of the following lemma.

Lemma 1 *Consider the real-valued function $H(x) \in \mathcal{R}$, $x \in \mathcal{R}^{2n}$, and the coordinate change $X(x) : \mathcal{R}^{2n} \longrightarrow \mathcal{R}^{2n}$ with nonsingular Jacobian matrix $N(x)$. Then:*

$$H_x^T = N^T H_X^T \tag{2.8}$$

Proof: It is desired to show that the gradient of any real-valued function transforms under a coordinate change according to the rule (2.8). The partial of H with respect to x_i is:

$$\frac{\partial H}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial X_j}{\partial x_i}\frac{\partial H}{\partial X_j} + \sum_{k=n+1}^{2n} \frac{\partial X_k}{\partial x_i}\frac{\partial H}{\partial X_k} \tag{2.9}$$

Substituting generalized coordinates and momenta, (2.9) can be written in matrix form as:

$$\begin{bmatrix} \dfrac{\partial H^T}{\partial q} \\ \dfrac{\partial H^T}{\partial p} \end{bmatrix} = \left[\begin{array}{c|c} \dfrac{\partial Q_j}{\partial q_i} & \dfrac{\partial P_j}{\partial q_i} \\ \hline \dfrac{\partial Q_j}{\partial p_i} & \dfrac{\partial P_j}{\partial p_i} \end{array}\right] \begin{bmatrix} \dfrac{\partial H^T}{\partial Q} \\ \dfrac{\partial H^T}{\partial P} \end{bmatrix} \qquad \forall i,j \ = \ 1,\ 2,\ \ldots,\ n \tag{2.10}$$

where the Jacobian matrix has been partitioned into 4 sub-matrices of dimension $n \times n$. For example, the (j,i)-element of the first sub-matrix is given by $\dfrac{\partial Q_j}{\partial q_i}$. Similarly, the coordinate change $\dot{X} = N(x)\,\dot{x}$ can be written as:

$$\begin{bmatrix} \dot{Q} \\ \dot{P} \end{bmatrix} = \left[\begin{array}{c|c} \dfrac{\partial Q_i}{\partial q_j} & \dfrac{\partial Q_i}{\partial p_j} \\ \hline \dfrac{\partial P_i}{\partial q_j} & \dfrac{\partial P_i}{\partial p_j} \end{array}\right] \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} \qquad \forall i,j \ = \ 1,\ 2,\ \ldots,\ n \tag{2.11}$$

where the (i,j)-element of the first sub-matrix is given by $\frac{\partial Q_i}{\partial q_j}$. Comparing the transformation Jacobian matrix for the gradient (2.10) with the one for the velocity vector (2.11), the Lemma is established, since transposition reverses the order of the elements, i.e. the (i,j)-element becomes the (j,i)-element.

Substituting (2.8) in (2.7) results in the desired form of the transformed Hamilton's equations, where all quantities have been expressed in terms of the new coordinates:

$$\dot{X} = N\, Z\, N^T\, H_X^T + N\, \tilde{u} \tag{2.12}$$

The utility of this result will be evident when coordinate transformations preserving the Hamiltonian framework are investigated. This will lead to necessary and sufficient conditions that must be satisfied by the coordinate transformations and the Hamiltonian functions.

2.2 Coordinate Transformations In Hamiltonian Dynamics

Thus far, an alternative means of expressing the equations of motion through the Hamiltonian formalism has been introduced. It should be noted that this formulation does not contain any more information than the Lagrangian approach. As such, it does not provide a material improvement over the Lagrangian approach in solving the equations of motion. The advantage of the Hamiltonian formalism, however, is not its use as a computational tool but rather in the deeper insight it provides in understanding the structure of dynamical systems. In Arnold's words [1] "Hamiltonian mechanics is geometry in phase space". In this framework the equal status afforded generalized coordinates and conjugate momenta as independent variables allows complete freedom in the choice of "coordinates" and "momenta". This degree of freedom can be exploited to provide a more abstract picture of the underlying dynamics and in search of methods/tools to analyze and simplify the equations of motion.

One of the major thrusts of classical dynamics has been the search for general methods to solve nonlinear equations of motion. Since the form of the equations depend on the coordinates employed, one approach to simplifying these equations has been the selection of a suitable coordinate system that simplifies the problem at hand. Given the original coordinates, it is desired to transform them to a set in which the equations of motion appear in a simpler form, e.g. linear or even constant. Within the Lagrangian framework, a transformation of the generalized coordinates q_i to a new set Q_i is defined by:

$$Q_i = Q_i(q,t) \qquad i = 1,\ldots,n$$

Such transformations are known as *point transformations.* In the Hamiltonian formulation, transformations of both the coordinates q and momenta p to a new set (Q,P) via

$$\begin{aligned} Q &= Q(q,p) \\ P &= P(q,p) \end{aligned}$$

can be considered. These equations define a transformation of *phase space*, whereas point transformations define transformations on *configuration space.*

To characterize coordinate transformations in Hamiltonian dynamics from other forms of transformations, requires the satisfaction of certain proerties. The standard approach is the require the preservation of Hamilton's equations in the transformed coordinates. To derive the requisite conditions, consider the transformed Hamilton's equations which were derived in Section 2.2 (see equation (2.12)) and will be repeated here:

$$\dot{X} \;=\; NZN^T H_X^T \;+\; N\,\tilde{u} \tag{2.13}$$

For (2.13) to satisfy the Hamiltonian framework, however, the transformed equations must appear as:

$$\dot{X} \;=\; Z\,H_X^T \;+\; N\,\tilde{u} \tag{2.14}$$

Setting (2.13) equal to (2.14), leads to the following general condition for the preservation of Hamilton's canonical equations:

$$\Big[\,Z \;-\; N\,Z\,N^T\,\Big]\,H_X^T \;=\; 0_{2n\times 1} \tag{2.15}$$

This result is summarized in the following definition.

Definition 1 *Let the transformation $X(x) : \mathcal{R}^{2n} \longrightarrow \mathcal{R}^{2n}$ be differentiable, with nonsingular Jacobian matrix $N(x)$. The transformation preserves Hamilton's canonical equations if and only if:*

$$\Big[\,Z \;-\; N\,Z\,N^T\,\Big]\,H_X^T \;=\; 0_{2n\times 1} \tag{2.16}$$

A transformation which satisfies (2.16) is defined as a coordinate transformation.

Within the Hamiltonian framework, two such transformation methods exist:

Canonical Coordinate Transformations: These are (usually) time-independent coordinate transformations to a new set of position and momentum variables which exhibit certain desirable properties. A consequence of such a transformation is that in the new coordinates Hamilton's canonical equations are preserved or remain invariant.

Hamilton-Jacoby Theory: This approach provides a systematic method whereby the equations of motion can be directly integrated. The solution is expressed as a time-dependent mapping from initial conditions to future states. This approach can be thought of as a special canonical transformation in which the transformed coordinates are constant.

It may be noted that in the above discussion, a rigorous definition of canonical transformations was not presented. The reason for this is that in the mechanics literature, there does not seem to be an agreement as to what constitutes a canonical transformation. Various authors have adopted different and, in general, non-equivalent definitions. For a short summary of alternative definitions (8 definitions to be exact) see Santilli [38]. However, there seem to exist two fundamental and universal properties that apparently permeate all definitions of canonical transformations. These are:

(a) Transformations preserving Hamilton's canonical equations. This means that the equations of motion satisfy Hamilton's canonical equations in the new coordinate system. That is:

$$\dot{X} = Z \, H_X^T$$

(b) Metric preserving transformations. The metric preservation property is defined in terms of a special skew-symmetric quadratic form:

$$\frac{1}{2} \, dx^T \, Z \, dx = \frac{1}{2} \, dX^T \, Z \, dX$$

It should be noted, however, that the two definitions are not equivalent. Metric preserving transformations preserve Hamilton's canonical equations. The converse is not true, as will be shown in the sequel by an example. The metric preserving property is adopted here as definition for canonical transformations, since this is the most primitive characteristic that most definitions satisfy.

2.3 Canonical Coordinate Transformations

In this section, the framework for canonical transformations is developed, and the three most prevalent (in the opinion of this author) definitions appearing in the literature are reviewed. The three different (but to be shown equivalent) definitions are:

(a) Generating Functions

(b) Canonical 2-Form

(c) Poisson Brackets

In the steps leading to the definition of a canonical transformation, two very important ideas are used. The first key concept is that in the transformed coordinate system, Hamilton's equations appear in the canonical form. The second key concept underlying canonical transformations is the requirement that Hamilton's equations are preserved for *all* Hamiltonian functions $H(q,p)$. Stated in another manner, a transformation is *canonical* if for *every* Hamiltonian $H(q,p)$ there exists a transformed Hamiltonian $\tilde{H}(Q,P)$ such that:

$$\dot{Q}_i = \frac{\partial \tilde{H}(Q,P)}{\partial P_i} \tag{2.17}$$

$$\dot{P}_i = -\frac{\partial \tilde{H}(Q,P)}{\partial Q_i} \tag{2.18}$$

Consequently, the gradient of the Hamiltonian must also be arbitrary. Then, to satisfy (2.16) requires that:

$$N\,Z\,N^T = Z$$

This condition is also the requirement for metric preservation. The definition of a canonical transformation follows.

Definition 2 *Let the transformation $X(x) : \mathcal{R}^{2n} \longrightarrow \mathcal{R}^{2n}$ be differentiable, with nonsingular Jacobian matrix $N(x)$. The transformation is canonical if and only if:*

$$N\,Z\,N^T = Z \tag{2.19}$$

It should be noted that the nonsingularity of the transformation Jacobian matrix $N(x)$ is a direct consequence of this definition. This can be seen by taking the determinant of (2.19):

$$\begin{aligned} \det\left[\, N\, Z\, N^T \,\right] &= \det[N]^2 \det[Z] \\ &= \det[Z] \end{aligned}$$

Then:

$$\det[N] = |1|$$

This proves the nonsingularity of $N(x)$. It can actually be shown (see for example Pars [34]) that $\det[N] = +1$. Since the transformation Jacobian matrix, $N(x)$, is nonsingular, its inverse can be explicitly constructed [44]:

$$N^{-1} = -\, Z\, N^T\, Z$$

This identity can be verified by considering it as a left-inverse:

$$N^{-1}\, N = -\, Z\, \left(N^T\, Z\, N \right) = -Z^2 = 1_{n \times n}$$

The definition (2.19) can also be cast in an alternate but equivalent form $N^T\, Z\, N = Z$. To accomplish this, right multiplication by N of N^{-1} results in:

$$1_{n \times n} = -\, Z\, \left(N^T\, Z\, N \right)$$

Finally, left multiplication by Z of the above results in the desired result:

$$N^T\, Z\, N = Z \tag{2.20}$$

This result could have also been obtained from left multiplication of (2.19) by Z, N^T, and right multiplication by N^{-T}, Z. It should be noted that in this context N is a square matrix and as such the two definitions (2.19) and (2.20) are equivalent.

The classical canonical transformation theory has also been extended to allow for the possibility of *increasing* the degrees of freedom in the transformed coordinate system by Stiefel *et al.* [44]. An example in the dynamics literature where a redundant coordinate system is used involves using the 4 element quaternion to represent the orientation of a rigid body which can also be expressed using 3 Euler angles. Finally, it may be noted that (2.20), albeit under the guise of the canonical 2-form (which will be introduced in the following), is adopted as the definition for a canonical transformations by Arnold [1], Dubrovin *et al.* [11], and Rasband [35]. Alternatively, the preservation of Hamilton's canonical equations (2.18) is adopted as the definition of canonical transformations by Landau *et al.* [30], Goldstein [12], and Groesberg [14].

Canonical theory can be further extended by allowing transformations over the complex domain. Such a transformation can be defined as $X(x) = X_R(x) + i\, X_I(x)$ with $X(x) : \mathcal{R}^{2n} \longrightarrow \mathcal{C}^{2n}$. The Jacobian matrix of this transformation can likewise be defined as $N(x) = N_R(x) + i\, N_I(x)$. One possible definition for a complex canonical transformation would be:

$$\mathrm{Re}\left[\, N\, Z\, N^T \right] = Z$$

2.3.1 Generating Function

In this section, the generating function approach is introduced as a means to establish a systematic method of constructing canonical transformations. Once this function is determined, the desired transformation equations can be obtained. This approach is not a definition of a canonical transformation, but rather is the machinery or computational tool used to generate such transformations, hence the name. This approach can be derived from various principles such as metric preservation by Stiefel *et al.* [44], Rasband [35], or preservation of Hamilton's variational principle by Goldstein [12], Landau *et al.* [30], Groesberg [14]. A brief presentation of the variational approach to generating functions follows.

An alternative method for obtaining the equations of motion involves variational principles. The original generalized coordinates satisfy a modified version of Hamilton's principle [12]:

$$\delta \int_{t_1}^{t_2} [\, p\, \dot{q} \, - \, H(q,p)\,]\, dt \, = \, 0 \quad \forall\, t_1,\, t_2 \tag{2.21}$$

For (Q, P) to be canonical variables, they must also satisfy a transformed modified Hamilton's principle:

$$\delta \int_{t_1}^{t_2} \left[\, P\, \dot{Q} \, - \, H(Q,P) \right] dt \, = \, 0 \quad \forall\, t_1,\, t_2 \tag{2.22}$$

To simultaneously satisfy (2.21) and (2.22), the integrands in these equations need not be identical. This is because by definition of Hamilton's principle, the variation of the independent variables is zero at the end points. They can be related up to an additive, arbitrary, exactly differentiable function S:

$$\lambda\, (\, p\, \dot{q} \, - \, H(q,p)\,) \, = \, P\, \dot{Q} \, - \, H(Q,P) \, + \, \frac{dS(q,p,Q,P)}{dt} \tag{2.23}$$

Here λ is a scaling constant which can be interpreted as a simple scale transformation where the magnitude of the units of the independent variables is changed. Such scale transformations will not be considered here, and in the following it will be assumed that by appropriate scaling $\lambda = 1$.

There is no contribution to (2.22) from the additional S term because

$$\delta \int_{t_1}^{t_2} \frac{dS}{dt}\, dt = \delta S\Big|_{t_1}^{t_2} = \frac{\partial S}{\partial q}\,\delta q\Big|_{t_1}^{t_2} + \frac{\partial S}{\partial p}\,\delta p\Big|_{t_1}^{t_2} + \frac{\partial S}{\partial Q}\,\delta Q\Big|_{t_1}^{t_2} + \frac{\partial S}{\partial P}\,\delta P\Big|_{t_1}^{t_2} = 0$$

and the variation of all independent variables is zero at the end points of the interval of integration by definition of Hamilton's principle. Using (2.23) leads to the required transformation equations, and it can be seen that it represents a sufficient condition for a canonical transformation. Four possibilities exist for the form of the generating function relating the old to the new coordinates [14]:

$$\begin{aligned} S_1 &= S_1(q,Q) \\ S_2 &= S_2(q,P) \\ S_3 &= S_3(p,Q) \\ S_4 &= S_4(p,P) \end{aligned} \tag{2.24}$$

To show how the generating function can be used to specify the equations for a canonical transformation, suppose S_2 is chosen as a generating function. Substituting $S = S_2(q,P) - QP$ into (2.23) in order to introduce a term involving $\dot{P}$, (2.23) becomes:

$$p\,\dot{q} - H(q,p) = -Q\,\dot{P} - H(Q,P) + \frac{\partial S_2}{\partial q}\dot{q} + \frac{\partial S_2}{\partial P}\dot{P}$$

Equating coefficients, the following equations are obtained

$$\begin{aligned} p &= \frac{\partial S_2}{\partial q} \\ Q &= \frac{\partial S_2}{\partial P} \\ H(Q,P) &= H(q,p) \end{aligned}$$

An interesting question is whether such transformations are indeed metric preserving. It has been shown by Stiefel *et al.* [44] that generating function transformations satisfy (2.19) and (2.20). This fact allows for an alternate derivation that dispenses with the variational approach. The central component to the proof is the equality of mixed partial derivatives, i.e.

partial differentiation of a function is independent of the order in which the partial derivatives are computed. This is a very important property because of its implication to integration theory that will be encountered in later chapters. Simply stated, if one is presented with the task of integrating a set of first order partial differential equations, this property can be used to test whether the given set is integrable. The set is integrable when there exists a function (the integral) which when differentiated will generate the given equations. Indeed most of the results in this thesis require this property in one form or the other.

2.3.2 Canonical 2-Form

An alternate definition of metric preserving transformations can be given in terms of the so called canonical 2-form. This definition uses the mathematical machinery of *forms.* For a readable survey of the subject from a geometric viewpoint see Tabor [47]. Loosely speaking, a form can be viewed as a functional with certain special properties, i.e. linearity and skew-symmetry. Recall that a functional is defined as a transformation from a vector space into the space of real (or complex) scalars. Similarly, a *k-form* operates on k-vectors to return a real scalar. Differential forms are similar to forms except that they operate on vector fields instead of vectors. A simple visualization of a differential 1-form is the directional (or Lie) derivative operator on a real-valued function. Once a direction or a vector is specified, taking the inner product of the gradient of the function with the vector results in a scalar. For example, a differential *1-form* denoted by ω^1 in two dimensions is given by

$$\omega^1 = a_1(x_1, x_2)\, dx_1 + a_2(x_1, x_2)\, dx_2$$

where a_1, a_2 are the component scalar fields along the basis dx_1, dx_2.

For k-forms there exists a product operation, "$\wedge$", called the exterior (wedge) product to form higher rank forms from lower ones. This product operation is different from the regular product operation in that it is skew-symmetric:

$$dx_i \wedge dx_j = - dx_j \wedge dx_i \tag{2.25}$$

A direct consequence of this definition is that the wedge of a quantity with itself is zero:

$$dx_i \wedge dx_i = 0$$

This product operation may be written without the wedge as long as it is remembered that it satisfies skew-symmetry. For example, (2.25) could have just as well been written as:

$$dx_i\, dx_j = -\, dx_j\, dx_i$$

Now consider forming a 2-form from the product of two 1-forms $\omega^1 = a_1(x_1,x_2)\, dx_1 + a_2(x_1,x_2)\, dx_2$ and $\sigma^1 = b_1(x_1,x_2)\, dx_1 + b_2(x_1,x_2)\, dx_2$. Applying the wedge product:

$$\begin{aligned}\omega^1 \wedge \sigma^1 &= (\, a_1(x_1,x_2)\, dx_1 + a_2(x_1,x_2)\, dx_2 \,) \wedge (\, b_1(x_1,x_2)\, dx_1 + b_2(x_1,x_2)\, dx_2 \,) \\ &= a_1(x_1,x_2)\, b_2(x_1,x_2)\, dx_1 \wedge dx_2 + b_2(x_1,x_2)\, a_1(x_1,x_2)\, dx_2 \wedge dx_1 \\ &= (\, a_1(x_1,x_2)\, b_2(x_1,x_2) - b_2(x_1,x_2)\, a_1(x_1,x_2) \,)\, dx_1\, dx_2\end{aligned}$$

In this context, a metric preserving canonical transformation is defined by Arnold [1]:

Definition 3 *Let Φ be a differentiable mapping of the phase space $\Phi : \mathcal{R}^{2n} = (q,p) \rightarrow \mathcal{R}^{2n} = (Q,P)$. The mapping Φ is called* canonical, *or a* canonical transformation, *if Φ preserves the 2-form:*

$$\omega^2 = \sum_{i=1}^{n} dq_i \wedge dp_i \tag{2.26}$$

The meaning of this definition is that under a transformation to a new coordinate system $Q = Q(q,p)$, $P = P(q,p)$, the canonical 2-form, ω^2, is invariant, that is:

$$\omega^2 = \sum_{i=1}^{n} dq_i \wedge dp_i = \sum_{i=1}^{n} dQ_i \wedge dP_i$$

The geometric interpretation is that the sum of the (q_i,p_i) unit planes is equal to the sum of the (Q_i,P_i) unit planes.

However, this definition can be recast into the somewhat more transparent setting of quadratic forms. To accomplish this, first note that (2.26) can be written as:

$$\omega^2 = \frac{1}{2} \sum_{j=1}^{N} \sum_{i=1}^{n} dq_i \wedge dq_j + z_{i,j}\, dq_i \wedge dp_j + z_{i,j}\, dp_i \wedge dq_j + dp_i \wedge dp_j \tag{2.27}$$

where $z_{i,j}$ is a weighting coefficient that is selected such that (2.27) equal (2.26):

$$z_{i,j}(dq_i, dq_j) = \begin{cases} -1 & \text{for } dp_i \wedge dq_i \\ +1 & \text{otherwise} \end{cases}$$

The sign change on $dp_i \wedge dq_i$ is employed to generate two $dq_i \wedge dp_i$ terms which coupled with the one-half scaling factor results in (2.26) since the remaining terms sum to zero. To show this note that

$$
\begin{aligned}
\sum_{j=1}^{n} \sum_{i=1}^{n} dq_i \wedge dq_j &= 0 \\
\sum_{j=1}^{n} \sum_{i=1}^{n} dp_i \wedge dp_j &= 0
\end{aligned}
$$

from the skew-symmetry property of the wedge product. All repeat terms are zero, i.e. $dq_i \wedge dq_i = 0$ and $dp_i \wedge dp_i = 0 \ \forall i$. The remaining terms sum to zero because e.g. $dq_i \wedge dq_j + dq_j \wedge dq_i = 0$ and $dp_i \wedge dp_j + dp_j \wedge dp_i = 0$. Similarly, for the mixed wedge terms (i.e. $dq_i \wedge dp_j$ etc.) the unlike coefficient terms sum to zero (from skew-symmetry) and the effect of the sign change and the scaling by one-half generates the desired result. That is:

$$
\begin{aligned}
\frac{1}{2} \sum_{i=1}^{n} z_{i,j}\, dq_i \wedge dp_i \; + \; z_{i,j}\, dp_i \wedge dq_i &= \frac{1}{2} \sum_{i=1}^{n} dq_i \wedge dp_i \; - \; dp_i \wedge dq_i \\
&= \frac{1}{2} \sum_{i=1}^{n} dq_i \wedge dp_i \; + \; dq_i \wedge dp_i \\
&= \sum_{i=1}^{n} dq_i \wedge dp_i
\end{aligned}
$$

Finally, (2.26) can now be written as a quadratic form with appropriate weighting weighting coefficients as:

$$
\omega^2 = \sum_{i=1}^{n} dq_i \wedge dp_i \; = \; \frac{1}{2}\, dx^T \, Z \, dx
$$

where Z is the skew-symmetric matrix as in (2.4), x is the extended state of $2n$ coordinates $x = [q_1, \ldots, q_n, p_1, \ldots, p_n]^T$, and $dx = [\, dq_1, \ldots, dq_n, dp_1, \ldots, dp_n \,]^T$. It is evident that the canonical 2-form is just a quadratic form with metric Z. Thus a canonical transformation is just an isometry (metric preserving transformation) of phase space.

Now, consider a transformation to a new coordinate system $Q = Q(q,p)$, $P = P(q,p)$. Let X denote the extended new state of the $2n$ coordinates $X = [Q_1, \ldots, Q_n, P_1, \ldots, P_n]^T$, and $dX = [dQ_1, \ldots, dQ_n, dP_1, \ldots, dP_n]^T$. It may be noted that the coordinate transformation is

$X = X(x)$. The Jacobian matrix $N(x)$ of this transformation is defined by:

$$N(x) = D_x[X(x)] = \begin{bmatrix} \dfrac{\partial Q(q,p)}{\partial q} & \dfrac{\partial Q(q,p)}{\partial p} \\ \dfrac{\partial P(q,p)}{\partial q} & \dfrac{\partial P(q,p)}{\partial p} \end{bmatrix}$$

The invariance condition of the canonical 2-form in the quadratic formalism can now be expressed as:

$$\omega^2 = \frac{1}{2}\, dx^T\ Z\ dx\ = \frac{1}{2}\, dX^T\ Z\ dX$$

From this it can be concluded that a canonical transformation defined by the canonical 2-form is equivalent to metric preservation:

$$N^T(x)\ Z\ N(x) = Z$$

Even though the definition of a canonical transformation using the 2-form initially appears to be different from that of a generating function approach, it is seen that both definitions are equivalent.

An interesting result in the case of a scalar system is the following which shows that the preservation of the 2-form $\omega^2 = dq \wedge dp$ is equivalent to preservation of area in phase space.

Theorem 1 *For $n = 1$, a transformation is canonical if and only if the transformation Jacobian determinant is unity, i.e.*

$$\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = 1 \tag{2.28}$$

Proof: Let $P = P(q,p)$, $Q = Q(q,p)$. For a canonical transformation, we have to show that

$$\omega^2 = dp \wedge dq = dP \wedge dQ$$

By the chain rule of differentiation

$$\begin{aligned} dQ &= \frac{\partial Q}{\partial q} dq + \frac{\partial Q}{\partial p} dp \\ dP &= \frac{\partial P}{\partial q} dq + \frac{\partial P}{\partial p} dp \end{aligned}$$

Substituting, and utilizing the anti-symmetry property of forms (i.e. $dq \wedge dq = 0$, $dq \wedge dp = -dp \wedge dq$)

$$\begin{aligned}
dp \wedge dq &= \left(\frac{\partial P}{\partial q}dq + \frac{\partial P}{\partial p}dp\right) \wedge \left(\frac{\partial Q}{\partial q}dq + \frac{\partial Q}{\partial p}dp\right) \\
&= \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}dq \wedge dp + \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q}dp \wedge dq \\
&= \left(\frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}\right) dp \wedge dq
\end{aligned}$$

The equality is satisfied if and only if (2.28) holds.

Note, however, that this property cannot be extrapolated to higher dimensions. A property of canonical transformations is that they are volume preserving transformations (i.e. $\det[N] = |1|$). However, the converse is not true. Volume preserving transformations are not in general canonical. In conclusion it has been shown that an initially complicated definition of canonical transformations expressed using the machinery of 2-forms can actually be expressed in terms of a quadratic form.

2.3.3 Poisson Brackets

Yet another method to express the conditions for metric preserving transformations is through the use of the so called *Poisson brackets.* This bracket notation proves to be a convenient way of formulating the total time derivative of functions defined in phase space. In the Hamiltonian description of dynamical systems, consider the total time derivative of a function $F(q,p,t)$:

$$\dot{F} = \frac{\partial F}{\partial t} + \sum_{k=1}^{n}\left(\frac{\partial F}{\partial q_k}\dot{q}_k + \frac{\partial F}{\partial p_k}\dot{p}_k\right)$$

Substituting the expressions for $\dot{q}_k$ and $\dot{p}_k$ from Hamilton's canonical equations results in

$$\dot{F} = \frac{\partial F}{\partial t} + \sum_{k=1}^{n}\left(\frac{\partial F}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial H}{\partial q_k}\right) = \frac{\partial F}{\partial t} + \{F,H\}$$

where $\{f,g\}$ is called the *scalar Poisson bracket.* A precise definition is presented in the following.

Definition: *The scalar Poisson bracket of two scalar fields $f_i(q,p), g_j(q,p)$ defined on a 2n-dimensional phase space with respect to a set of canonical variables (q,p) is defined as:*

$$\{f_i, g_j\}(q,p) = \sum_{k=1}^{n} \frac{\partial f_i}{\partial q_k}\frac{\partial g_j}{\partial p_k} - \frac{\partial f_i}{\partial p_k}\frac{\partial g_j}{\partial q_k}$$

In matrix notation using x to denote the extended state, this becomes

$$\{f_i, g_j\}(x) = D_x[f_i(x)]\ Z\ D_x[g_j(x)]^T$$

where $D_x[f_i(x)] = \dfrac{\partial f_i(x)}{\partial x}$ denotes the gradient of the scalar field $f_i(x)$. Using this definition, the scalar bracket is seen to satisfy the following properties [11]:

1. $\{f_i, g_j\} = -\{g_j, f_i\}$ (skew-symmetric)
2. $\{a_1 f_i + a_2 f_j, g_k\} = a_1\{f_i, g_k\} + a_2\{f_j, g_k\}$ (bilinear), where a_1, a_2 are arbitrary constants.
3. $\{f_i, \{g_j, h_k\}\} + \{h_k, \{f_i, g_j\}\} + \{g_j, \{h_k, f_i\}\} = 0$ (Jacobi's identity)
4. $\{f_i g_j, h_k\} = f_i\{g_j, h_k\} + g_j\{f_i, h_k\}$
5. $D\{f_i, g_j\} = -[Df_i, Dg_j]$, where $[,]$ is the operation of taking the commutator (Lie Bracket) of vector fields.

A consequence of the skew-symmetry property is that the bracket of a scalar field with itself is zero, i.e.

$$\{f_i, f_i\} = 0$$

Similarly, the *matrix Poisson bracket* for vector fields is defined in the following.

Definition: *The matrix Poisson bracket of two vector fields $f(q,p), g(q,p) \in \mathcal{R}^{2n}$ defined on a 2n-dimensional phase space with respect to a set of canonical variables (q,p) is defined as:*

$$\{f, g\}(x) = D_x[f(x)]\ Z\ D_x[g(x)]^T$$

This result certainly has appeared in the mechanics literature previously. In lieu of a reference, a brief computation shows that this is indeed the matrix generalization of the scalar Poisson bracket. Carrying out the indicated operations:

$$\begin{aligned}\{f,g\}(x) &= \begin{bmatrix} D_q[f(q,p)] & D_p[f(q,p)] \end{bmatrix} \begin{bmatrix} 0_{n\times n} & 1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{bmatrix} \begin{bmatrix} D_q[g(q,p)]^T \\ D_p[g(q,p)]^T \end{bmatrix} \\ &= D_q[f(q,p)]D_p[g(q,p)]^T - D_p[f(q,p)]D_q[g(q,p)]^T\end{aligned}$$

Now, the i,j-element of this matrix is

$$\begin{aligned}\{f,g\}_{i,j}(x) &= D_q[f_i(q,p)]\, D_p[g_j(q,p)]^T - D_p[f_i(q,p)]\, D_q[g_j(q,p)]^T \\ &= D_x[f_i(x)]\, Z\, D_x[g_j(x)]^T \\ &= \{f_i,g_j\}(x)\end{aligned}$$

which is seen to be identical to the scalar Poisson bracket of $f_i(x), g_j(x)$. For the matrix case, the skew-symmetric property still applies but in the following form:

$$\{f,g\} = -\{g,f\}^T$$

An immediate consequence is that $\{f,f\} \neq 0$. For example, let $Q(q,p) = [Q_1(q,p),\ Q_2(q,p)]^T$ and computing $\{Q,Q\}$ one obtains,

$$\{Q,Q\}(q,p) = \begin{bmatrix} \{Q_1,Q_1\} & \{Q_1,Q_2\} \\ -\{Q_2,Q_1\} & \{Q_2,Q_2\} \end{bmatrix} = \begin{bmatrix} 0 & \{Q_1,Q_2\} \\ -\{Q_2,Q_1\} & 0 \end{bmatrix}$$

where the scalar skew-symmetric property has been used to eliminate the two terms on the main diagonal. It is seen that unless $\{Q_2,Q_1\} = 0$, $\{Q,Q\} \neq 0$.

Metric preserving transformations can also be defined using matrix Poisson brackets [34]. Let $Q = Q(q,p)$, $P = P(q,p)$ denote a new coordinate system in some neighborhood of phase space.

Definition: *The transformation $Q = Q(q,p)$, $P = P(q,p)$, is canonical if and only if it satisfies the Poisson bracket relations:*

$$\{Q_i,Q_j\}(q,p) = 0 \qquad \{P_i,P_j\}(q,p) = 0 \qquad \{Q_i,P_j\}(q,p) = \delta_{i,j} \quad \forall i,j = 1,\ldots,n$$

In matrix notation, using the matrix Poisson bracket, this condition can be written as:

$$\{X,X\}(x,x) = Z$$

From the definition of the matrix Poisson bracket it is seen that this condition is the same as that given in the previous section for the canonical 2-form, i.e.:

$$N\,ZN^T = Z$$

In conclusion, it has been shown that all three different definitions of canonical transformations i.e. Generating Function, Canonical 2-Form, and Poisson Bracket relation are equivalent in that they all preserve a skew-symmetric metric. Even though at first these definitions appear not to be identical, it was shown that the fundamental and unifiying property underlying these definitions is the preservation of a simple quadratic form. This result should help in aleviating the ambiguity in the mechanics literature as to what constitutes a canonical transformation.

2.4 Orthogonal Canonical Transformations

From the previous discussion on metric preserving canonical transformations it was seen that the necessary and sufficient condition for (2.16) to be satisfied for an *arbitrary* Hamiltonian H was the preservation of the metric, i.e. $N\ Z\ N^T = Z$. However, if the requirement for arbitrariness of the Hamiltonian is relaxed then metric preservation is not required. This assumption that the Hamiltonian function is not arbitrary leads to a new set of transformations, termed orthogonal canonical transformations. In this section the implications of this new approach are explored, with a secondary goal of deriving linearizing transformations for nonlinear Hamiltonian systems.

Recall that the general condition for preservation of Hamilton's canonical equations was given by (2.16), which for convenience will be repeated here:

$$\left[Z \ - \ N\ Z\ N^T \right] H_X^T = 0_{2n\times 1} \tag{2.29}$$

The definition of orthogonal canonical transformations follows.

Definition 4 *A differentiable coordinate transformation $X(x)$, even dimensional in both generalized coordinates and momenta and with nonsingular Jacobian matrix $N(x)$, for which $N\ Z\ N^T \neq Z$ is called an orthogonal canonical transformation if it satisfies (2.29) for some Hamiltonian $H(q,p)$.*

To satisfy (2.29) for nontrivial H_X^T, without having $N\ Z\ N^T = Z$ for all x, requires that the characteristic matrix $\left[Z \ - \ N\ Z\ N^T\right]$ be singular. To see this, let $A(x) = \left[Z - NZN^T\right]$ and $y = H_X^T$. Then (2.29) can be written as:

$$A(x)\, y \ = \ 0 \tag{2.30}$$

It is obvious that if A is the zero matrix, then y is an arbitrary vector. On the other hand, from linear algebra (see e.g. Strang [45]), if $A(x)$ is nonsingular for all x then the only solution to (2.30) is the zero solution, $y = 0$. For (2.30) to be satisfied by a nonzero y for all x, $A(x)$ must be singular for all x. Thus, the only way a nonzero y can satisfy (2.30) without having $A(x) = 0_{2n\times 2n}$ is that $A(x)$ is singular for all x. Computing the characteristic matrix using matrix Poisson brackets:

$$\left[Z - N\,Z\,N^T\right] = \begin{bmatrix} \{Q,Q\}(q,p) & 1_{n\times n} - \{Q,P\}(q,p) \\ -1_{n\times n} - \{P,Q\}(q,p) & \{P,P\}(q,p) \end{bmatrix}$$

Using the identity $\{Q,P\}(q,p) = -\{P,Q\}^T(q,p)$, the above reduces to:

$$\left[Z - N\,Z\,N^T\right] = \begin{bmatrix} \{Q,Q\}(q,p) & 1_{n\times n} - \{Q,P\}(q,p) \\ -1_{n\times n} + \{Q,P\}^T(q,p) & \{P,P\}(q,p) \end{bmatrix}$$

It has been established that for a transformation to be canonical, the characteristic matrix must equal the zero matrix. This is accomplished if and only if:

(a) $\{Q,P\}(q,p) = 1_{n\times n}$

(b) $\{P,P\}(q,p) = 0_{n\times n}$

(c) $\{Q,Q\}(q,p) = 0_{n\times n}$

To construct an orthogonal canonical transformation, the transformation Jacobian matrix must be selected in a manner that annihilates the characteristic matrix. Two cases in which the characteristic matrix is singular will be derived.

1. The characteristic matrix $\left[Z - N\,Z\,N^T\right] = \begin{bmatrix} *_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix}$ when:

   (a) $\{Q,P\}(q,p) = 1_{n\times n}$

   (b) $\{P,P\}(q,p) = 0_{n\times n}$

   (c) $\{Q,Q\}(q,p) = *_{n\times n} \neq 0_{n\times n}$

2. The characteristic matrix $\left[Z - N\,Z\,N^T\right] = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & *_{n\times n} \end{bmatrix}$ when:

   (a) $\{Q,P\}(q,p) = 1_{n\times n}$

   (b) $\{Q,Q\}(q,p) = 0_{n\times n}$

   (c) $\{P,P\}(q,p) = *_{n\times n} \neq 0_{n\times n}$

In the above, $*_{n\times n}$ represents an arbitrary *rank-n skew-symmetric even* dimensional matrix. This is because of the skew-symmetry property of the matrix Poisson bracket. It should be noted that the dimension of the arbitrary matrix must be *even*, since odd-dimensional skew-symmetric matrices are singular. To prove this, let $S = -S^T$ be any $n \times n$ skew-symmetric matrix. Then, the skew property can be expressed as $S = -1_{n\times n}\, S^T$, where $-1_{n\times n}$ is the $n \times n$ identity matrix. Taking the determinant of both sides:

$$\begin{aligned} \det S &= \det\left[-1_{n\times n}\, S^T\right] \\ &= \det\left[-1_{n\times n}\right] \det\left[S^T\right] \\ &= (-1)^n \det S \end{aligned}$$

This results in:

$$[\,1 - (-1)^n\,] \det S = 0 \tag{2.31}$$

It is evident that if n is odd, (2.31) is satisfied if and only if $\det S = 0$ or S is singular. When n is even, (2.31) does not impose any constraints on the rank of S.

To show that the above two requirements are necessary and sufficient for the characteristic matrix to be singular, first the expression for the determinant of a partitioned matrix is needed [32].

Definition 5 *If A, D are nonsingular matrices of orders m, n, and B, C are $m \times n$, $n \times m$, respectively, then:*

$$\begin{aligned} \det\begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \det A \, \det\left[D - C\,A^{-1}\,B\right] \\ &= \det D \, \det\left[A - B\,D^{-1}\,C\right] \end{aligned}$$

Using this identity, the determinant of the characteristic matrix becomes:

$$\begin{aligned} \det\left[Z - NZN^T\right] &= \det\{Q,Q\} \, \det\left[\,\{P,P\} - (\{Q,P\}^T - 1_{n\times n})\,\{Q,Q\}^{-1}\,(1_{n\times n} - \{Q,P\})\,\right] \\ &= \det\{P,P\} \, \det\left[\,\{Q,Q\} - (1_{n\times n} - \{Q,P\})\,\{P,P\}^{-1}\,(\{Q,P\}^T - 1_{n\times n})\,\right] \end{aligned}$$

It is evident that the characteristic matrix is singular if either $\{Q,Q\}$ or $\{P,P\}$ is singular. This establishes the sufficiency of the conditions. Another possibility is for the second determinant on the right-hand side to be zero. However, determining general conditions for which the second term is singular is nontrivial due to the complicated nature of these expressions. One approach is to simplify these terms by assuming:

$$\{Q,P\}(q,p) = 1_{n\times n}$$

This assumption establishes the necessity of the conditions. Substituting this in the determinant expression for the characteristic matrix:

$$\begin{aligned} \det\left[Z - N\,Z\,N^T \right] &= \det\{Q,Q\}\ \det\{P,P\} \\ &= \det\{P,P\}\ \det\{Q,Q\} \end{aligned}$$

This is satisfied if either $\det\{Q,Q\} = 0$ or $\det\{P,P\} = 0$ or both. However, since for a canonical transformation both must be zero, one can choose either one to be zero while the other is non-zero. Thus, this leads to two possible cases for which the characteristic matrix is singular and is not identically equal to the zero matrix. This proves that the two cases for the singularity of the characteristic matrix presented previously are indeed necessary and sufficient.

A comparison with metric preserving canonical transformations can now be made. From the definition of a canonical transformation, $Z - NZN^T = 0$, and the general condition (2.29) it is seen that the gradient of the Hamiltonian is *arbitrary*. That is, a transformation is *canonical* if for *every Hamiltonian*, $H(q,p)$, there exists a transformed Hamiltonian $\tilde{H}(Q,P)$ such that $N\ Z\ N^T = Z$. From a linear algebra perspective, since $Z\ -\ N\ Z\ N^T = 0$, the characteristic matrix has a $2n$ dimensional nullspace. Since H_X is of the same dimension $(2n \times 1)$ it is completely arbitrary because it is an element of this nullspace. That is, all $2n$ entries or elements of H_X can be assigned arbitrarily.

On the other hand, for the general condition (2.29) to hold when $N\ Z\ N^T \neq Z$ the Hamiltonian is *not arbitrary* and only special Hamiltonians satisfy (2.29). Starting from the form of the transformed Hamiltonian, and the conditions for $N\ Z\ N^T \neq Z$ the original Hamiltonian function can be retrieved. This approach somewhat resembles the inverse of the canonical method. Instead of beginning from a Hamiltonian and computing the coordinate transformation by solving the metric preservation property to generate the transformed Hamiltonian, here one starts with a specific form of transformed Hamiltonian and solves the orthogonality conditions to compute the coordinate transformation from which the original Hamiltonian can be retrieved. With $N\ Z\ N^T \neq Z$ the general condition $\left[\ Z\ -\ N\ Z\ N^T\ \right]\ H_X^T = 0_{2n\times 1}$ can be written in two equivalent alternate forms:

1. The transformed Hamiltonian is a function of P only, i.e.

$$\begin{bmatrix} *_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} \begin{bmatrix} \dfrac{\partial H^T}{\partial Q} \\ \dfrac{\partial H^T}{\partial P} \end{bmatrix} = \begin{bmatrix} 0_{n\times 1} \\ 0_{n\times 1} \end{bmatrix} \tag{2.32}$$

   for which a solution has to have the form $H = H(P)$.

2. The transformed Hamiltonian is a function of Q only, i.e.

$$\begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & *_{n\times n} \end{bmatrix} \begin{bmatrix} \dfrac{\partial H^T}{\partial Q} \\ \dfrac{\partial H^T}{\partial P} \end{bmatrix} = \begin{bmatrix} 0_{n\times 1} \\ 0_{n\times 1} \end{bmatrix} \tag{2.33}$$

   for which a solution has to have the form $H = H(Q)$.

In both of these cases, it is seen that the nullspace of the characteristic matrix is of dimension n and therefore only n elements of H_X^T can be assigned arbitrarily. To see why the Hamiltonian is a function of either the transformed generalized coordinate or momentum, first consider (2.32). Carrying out the indicated multiplication:

$$[\, *_{n\times n} \,] \frac{\partial H^T}{\partial Q} = 0_{n\times 1} \tag{2.34}$$

Since, by assumption, the arbitrary matrix $*_{n\times n}$ is full rank the only solution to (2.34) is:

$$\frac{\partial H^T}{\partial Q} = 0_{n\times 1} \quad \forall\, Q$$

This result implies that H is not a function of Q, since $\dfrac{\partial H}{\partial Q_i} = 0, \ \ i = 1, \ldots, n$ for all values of Q. Using the same arguments, it can be shown that for (2.33) to hold for all P the Hamiltonian must be a function of Q only.

In conclusion, it has been shown that preservation of Hamilton's equations can be accomplished by two distinct set of transformations; canonical or metric preserving transformations which annihilate the characteristic matrix, and orthogonal canonical transformations. It is clear that transformations which preserve Hamilton's equations need not be canonical. In this section a new set of transformations were derived which preserve Hamilton's equations but are not canonical. The key idea in this new transformations is that it has been derived for some special Hamiltonians. That is, the new approach is not valid for all Hamiltonian functions. It has been shown that preservation of Hamilton's equations for a set of special, non-arbitrary Hamiltonian functions can be accomplished using orthogonal canonical transformations. A set of general conditions similar to those for a canonical transformation were derived in terms of matrix Poisson brackets. One consequence of the existence conditions is the restriction to even dimensional systems. That is, the transformation must be even dimensional in both the generalized coordinates and momenta. Another consequence is that the transformed Hamiltonian is only a function of either the transformed generalized coordinate Q or momentum P. However, its form is arbitrary and can be selected at will.

Once an orthogonal transformation has been obtained and the form of the Hamiltonian function in the transformed coordinates has been selected, the original or untransformed Hamiltonian can be obtained by direct substitution of the coordinate transform. Finally, the equations of motion in the original coordinates can be obtained from the untransformed Hamiltonian function. Of course the choice of transformed Hamiltonian influences the form of the equations of motion in the original coordinates. A particular choice for the transformed Hamiltonian is one that results in a set of linear equations of motion in the transformed coordinates. The solution for the transformed variables is particularly simple in this case and by the inverse transformation the solution to the original, in general, nonlinear equations of motion is obtained. Thus, the utility of orthogonal transformations is that they generate linearizing transformations and as a consequence provide a means to integrate the original equations of motion. They also provide a method of parametrizing a class of linearizable Hamiltonian systems.

2.4.1 Orthogonal Transformations For A Second Order System

In this section, the constraint (partial differential) equations resulting from the singularity of the characteristic matrix will be derived for a system defined by two generalized coordinates. This system will be used to illustrate the process of generating orthogonal canonical transformations and the resulting Hamiltonian systems to which they apply.

Consider the case where the transformed Hamiltonian is a function of P only. The derivation for the case when the transformed Hamiltonian is a function of Q only is analogous to this one, and for this reason will not be presented. In this case, the conditions for the characteristic matrix to be singular were previously stated as:

1. The characteristic matrix $\left[Z - N\,Z\,N^T\right] = \begin{bmatrix} *_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} \end{bmatrix}$ when:
2. $\{Q,P\}(q,p) = 1_{2\times2}$
3. $\{P,P\}(q,p) = 0_{2\times2}$
4. $\{Q,Q\}(q,p) = *_{2\times2} \neq 0_{2\times2}$

For a two state system, the first expression is written as:

$$\{Q,P\}(q,p) = \begin{bmatrix} \{Q_1,P_1\}(q,p) & \{Q_1,P_2\}(q,p) \\ \{Q_2,P_1\}(q,p) & \{Q_2,P_2\}(q,p) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The second condition is:

$$\{P,P\}(q,p) = \begin{bmatrix} 0 & \{P_1,P_2\}(q,p) \\ -\{P_1,P_2\}(q,p) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The third condition is:

$$\{Q,Q\}(q,p) = \begin{bmatrix} 0 & \{Q_1,Q_2\}(q,p) \\ -\{Q_1,Q_2\}(q,p) & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Collecting all the conditions together results in:

1. $\{Q_1,P_1\} = 1$
2. $\{Q_1,P_2\} = 0$
3. $\{Q_2,P_1\} = 1$
4. $\{Q_2,P_2\} = 0$
5. $\{P_1,P_2\} = 0$
6. $\{Q_1,Q_2\} \neq 0$

These six Poisson bracket relations must be solved for the four unknown variables Q_1, Q_2, P_1, P_2 in order to determine the orthogonal canonical transformation. Writing out the bracket relations, the following six partial differential equations are obtained:

$$1.\ \{Q_1, P_1\} = \frac{\partial Q_1}{\partial q_1}\frac{\partial P_1}{\partial p_1} - \frac{\partial Q_1}{\partial p_1}\frac{\partial P_1}{\partial q_1} + \frac{\partial Q_1}{\partial q_2}\frac{\partial P_1}{\partial p_2} - \frac{\partial Q_1}{\partial p_2}\frac{\partial P_1}{\partial q_2} = 1$$

$$2.\ \{Q_1, P_2\} = \frac{\partial Q_1}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial Q_1}{\partial p_1}\frac{\partial P_2}{\partial q_1} + \frac{\partial Q_1}{\partial q_2}\frac{\partial P_2}{\partial p_2} - \frac{\partial Q_1}{\partial p_2}\frac{\partial P_2}{\partial q_2} = 0$$

$$3.\ \{Q_2, P_1\} = \frac{\partial Q_2}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial Q_2}{\partial p_1}\frac{\partial P_1}{\partial q_1} + \frac{\partial Q_2}{\partial q_2}\frac{\partial P_1}{\partial p_2} - \frac{\partial Q_2}{\partial p_2}\frac{\partial P_1}{\partial q_2} = 0$$

$$4.\ \{Q_2, P_2\} = \frac{\partial Q_2}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial Q_2}{\partial p_1}\frac{\partial P_2}{\partial q_1} + \frac{\partial Q_2}{\partial q_2}\frac{\partial P_2}{\partial p_2} - \frac{\partial Q_2}{\partial p_2}\frac{\partial P_2}{\partial q_2} = 1$$

$$5.\ \{P_1, P_2\} = \frac{\partial P_1}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial P_1}{\partial p_1}\frac{\partial P_2}{\partial q_1} + \frac{\partial P_1}{\partial q_2}\frac{\partial P_2}{\partial p_2} - \frac{\partial P_1}{\partial p_2}\frac{\partial P_2}{\partial q_2} = 0$$

$$6.\ \{Q_1, Q_2\} = \frac{\partial Q_1}{\partial q_1}\frac{\partial Q_2}{\partial p_1} - \frac{\partial Q_1}{\partial p_1}\frac{\partial Q_2}{\partial q_1} + \frac{\partial Q_1}{\partial q_2}\frac{\partial Q_2}{\partial p_2} - \frac{\partial Q_1}{\partial p_2}\frac{\partial Q_2}{\partial q_2} \neq 0$$

These 6 constraint partial differential equations must be solved for the 16 unkown partials of the transformed coordinates. These equations represent an underconstrained set of nonlinear equations. Thus, there exists a certain freedom in obtaining a solution, that is, to pick 10 of the partials and then solve 6 equations in 6 unknowns. In this manner auxiliary constraints can be appended to fully specify the solution. For example, it may be desirable to require that the kinetic energy in the original coordinates be positive-definite. Once the solution has been obtained and the form of the transformed Hamiltonian has been selected, the original Hamiltonian can be recovered by direct substitution. This approach is illustrated in the following example.

2.4.2 Second Order System Example

In this section a simple example of an orthogonal canonical transformation is presented. Consider the following transformation for a dynamical system defined by two generalized coordinates. This transformation was obtained as *a* solution of the 6 constraint partial differential equations.

$$
\begin{aligned}
Q_1 &= -\, q_2\, p_2 \;-\; \ln(p_1) \\
Q_2 &= q_2\, p_2 \;+\; \ln(q_2) \\
P_1 &= q_1\, p_1 \\
P_2 &= q_2\, p_2
\end{aligned}
$$

In order for this transformation to be well defined, the coordinates q_2, p_1 must be assumed to be strictly positive, since the natural logarithm function is only defined in this range. The transformation Jacobian matrix is computed as:

$$
N(x) = \begin{bmatrix} 0 & -\,p_2 & -\,p_1^{-1} & -\,q_2 \\ 0 & p_2 + q_2^{-1} & 0 & q_2 \\ p_1 & 0 & q_1 & 0 \\ 0 & p_2 & 0 & q_2 \end{bmatrix}
$$

Since $q_2 \neq 0,\;\; p_1 \neq 0$, by assumption, this matrix is nonsingular with determinant $+1$ and thus invertible. This is established by expanding the determinant about the last row:

$$
\det[\, N(x)\,] \;=\; p_2 \left[p_1 \left(-\frac{q_2}{p_1} \right) \right] \;+\; q_2 \left[p_1 \left(p_2 + \frac{1}{q_2} \right) \frac{1}{p_1} \right] \;=\; 1
$$

Since $N(x)$ is nonsingular, from the Implicit Function theorem [37] this is a 1-1 mapping, i.e. to every (Q, P) there corresponds a unique (q, p). The nonsingularity of $N(x)$ also guarantees the existence of a differentiable inverse map with Jacobian matrix $N^{-1}(X)$ since the forward map (or transformation) is differentiable [31]. The Jacobian matrix of the inverse map also has determinant equal to $+1$ since $N(x)\, N^{-1}(x) = 1_{4\times4}$. For this example the inverse transformation is also easy to obtain. From the coordinate transformation, expressions for q_2 and p_1 can be obtained from Q_1, Q_2 first, while the remaining variables are obtained from P_1, P_2:

$$
\begin{aligned}
q_1 &= P_1\, e^{(\,Q_1 + P_2\,)} \\
q_2 &= e^{(\,Q_2 - P_2\,)} \\
p_1 &= e^{-\,(\,Q_1 + P_2\,)} \\
p_2 &= P_2\, e^{-(\,Q_2 - P_2\,)}
\end{aligned}
$$

The Jacobian matrix of the inverse transformation is:

$$N^{-1}(X) = \begin{bmatrix} P_1\, e^{(Q_1+P_2)} & 0 & e^{(Q_1+P_2)} & P_1\, e^{(Q_1+P_2)} \\ 0 & e^{(Q_2-P_2)} & 0 & -\, e^{(Q_2-P_2)} \\ -\, e^{-(Q_1+P_2)} & 0 & 0 & -\, e^{-(Q_1+P_2)} \\ 0 & -\, P_2\, e^{(-Q_2+P_2)} & 0 & e^{(-Q_2+P_2)} + P_2\, e^{(-Q_2+P_2)} \end{bmatrix}$$

Explicitly computing the determinant of $N^{-1}(X)$ it can be verified that indeed it is $+1$ as expected, and expanding the deteriminant about the third column the required result is obtained.

$$\begin{aligned} \det\left[N^{-1}(X)\right] &= e^{(Q_1+P_2)}\left[-1\left\{-e^{-(Q_1+P_2)}\right\}\right. \\ &\quad \left.\left\{e^{(Q_2-P_2)}\left(e^{(-Q_2+P_2)} + P_2 e^{(-Q_2+P_2)}\right) - P_2 e^{(-Q_2+P_2)} e^{(Q_2-P_2)}\right\}\right] \\ &= 1 \end{aligned}$$

Before proceeding any further, first it will be verified that indeed this is an orthogonal canonical transformation. Computing the requisite Poisson bracket relations results in:

(a) $\{Q,P\}(q,p) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\{P,P\}(q,p) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $\{Q,Q\}(q,p) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

It is seen that the orthogonality constraints in terms of Poisson bracket relations are all satisfied and hence this is a valid orthogonal tranformation. It is also seen that this transformation is not canonical (metric preserving) since the condition $\{Q,Q\}(q,p) = 0_{2\times 2}$ is not satisfied. Note that the other two conditions i.e. (a) and (b) are satisfied.

To recover the original Hamiltonian, the form of the transformed Hamiltonian must be selected. Since the candidate transformation satisfies the conditions for an orthogonal transformation of the first kind, the transformed Hamiltonian must be a function of P only, i.e. $H = H(P)$. Choose this Hamiltonian as

$$H(P) = \frac{1}{2}\left\{ P_1^2 + P_2^2 \right\}$$

which would result in linear dynamics in the transformed coordinates. By substituting the coordinate transformation in the transformed Hamiltonian, the original Hamiltonian can be retrieved.

$$H(q,p) = \frac{1}{2}\left\{ q_1^2\, p_1^2 + q_2^2\, p_2^2 \right\}$$

In terms of generalized coordinates, the equivalent Lagrangian can be written as:

$$L(q,\dot{q}) = \frac{1}{2}\left\{ \frac{1}{q_1^2}\, \dot{q}_1^2 + \frac{1}{q_2^2}\, \dot{q}_2^2 \right\}$$

The linearizing property can now be used to solve the equations of motion in the original coordinates. Recall that the particular choice for the transformed Hamiltonian results in linear dynamics in the transformed coordinates. This is evident from Hamilton's equations in the transformed coordinates:

$$\begin{aligned} \dot{Q}_i &= P_i \\ \dot{P}_i &= 0 \end{aligned}$$

However, computing the equations of motion in the original coordinates results in nonlinear dynamics, as can be seen from Hamilton's equations in the original coordinates:

$$\begin{aligned} \dot{q}_i &= q_i^2\, p_i \\ \dot{p}_i &= q_i\, p_i^2 \end{aligned}$$

Comparing the original and transformed dynamics, the utility of the orthogonal transformation is evident. The solution to the orthogonal transformation problem has resulted in the generation of a linearizing coordinate transformation. The original dynamics can be solved very simply by solving the transformed problem and back substituting (i.e. the inverese transform computed previously).

2.4.3 Solution Using Canonical Transformation

In this section, the second order example is solved using a transformation that is canonical. This problem could be solved using the generating function approach and as a byproduct the transformation is known to be canonical. However, the generating function must be known first before the transformation can be constructed. Determining the generating function is tantamount to solving the problem. One approach to finding the generating function is by the Hamilton-Jacobi integration theory [44]. To solve the problem, a novel solution method will be employed. This approach will be developed in detail in the next chapter. The solution method hinges on finding a point transformation where the transformed Lagrangian function is coordinate independent.

For the two state example, the Lagrangian function in the original coordinates was:

$$L(q,\dot{q}) = \frac{1}{2} \begin{bmatrix} \dot{q}_1 & \dot{q}_2 \end{bmatrix} \begin{bmatrix} \frac{1}{q_1^2} & 0 \\ 0 & \frac{1}{q_2^2} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \frac{1}{2}\, \dot{q}^T\, B(q)\, \dot{q}$$

The key concept is to find a coordinate transformation such that in the new coordinate system the transformed inertia matrix is a multiple of the identity matrix, e.g. $B(q) = 1_{2\times 2}$. To this end consider the following transformation:

$$Q_1 = \ln(q_1)$$

$$Q_2 = \ln(q_2)$$

It is seen that in the new coordinate (Q_1, Q_2) system the Lagrangian function becomes:

$$L(\dot{Q}) = \frac{1}{2} \left\{ \dot{Q}_1^2 + \dot{Q}_2^2 \right\}$$

In the transformed coordinates, Lagrange's equations are:

$$\ddot{Q}_1 = 0$$

$$\ddot{Q}_2 = 0$$

This is just a linear set of double integrators that are easy to solve. The solution in the original coordinates is constructed form the inverse transformation:

$$q_1 = e^{Q_1}$$

$$q_2 = e^{Q_2}$$

To show that this transformation is indeed a canonical transformation, the required Poisson brackets must be evaluated. The transformed generalized momentum is obtained by definition:

$$P_1 = \frac{\partial L(\dot{Q})}{\partial \dot{Q}_1} = q_1\, p_1$$

$$P_2 = \frac{\partial L(\dot{Q})}{\partial \dot{Q}_2} = q_2\, p_2 \quad .$$

The requisite Poisson brackets are:

(a) $\{Q,P\}(q,p) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\{P,P\}(q,p) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $\{Q,Q\}(q,p) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

It is seen that the above satisfy the requirements for metric preservation, and thus the candidate transformation is canonical.

2.5 Conclusion

In this chapter canonical transformations were introduced as a systematic coordinate transformation theory. Three different definitions, common in the dynamics literature, for canonical transformations were presented. Even though at first these deffinitions appear not to be identical, it was shown that indeed they are equivalent in that all three preserve the same metric. The general condition for preservation of Hamilton's equations and canonical transformation theory were exploited to derive an alternative, new set of transformations termed orthogonal canonical transformations. This approach leads to a restricted set of Hamiltonian systems that admit a linear representation in the transformed coordinate system. The general conditions for such transformations were derived. Finally, an example of the application of orthogonal canonical transformations for a system defined by two generalized coordinates was presented. It was shown that the solution to the transformation problem resulted in the generation of a linearizing coordinate transformation. The transformed dynamics were linear whereas the original dynamics where nonlinear. The solution to the nonlinear problem was obtained from the inverse transformation.

Chapter 3

Linearizing Transformations

One approach to devising control schemes for nonlinear systems, is the use of transformation techniques that render the nonlinear dynamical equations amenable to existing results in control theory. One such approach is the concept of linearizing transformation or feedback linearization, where a change of coordinates in the state and control space coupled with feedback applied to the nonlinear system results in a controllable linear system. Once this has been achieved, one can apply existing linear controller design methods to the transformed linear system. In this chapter the application of canonical transformation theory to the feedback linearization problem is investigated.

In Section 3.1 linearizing transformations are investigated. These are transformations that generate a linear system in the transformed coordinates. In Sections 3.2 and 3.3 the well known properties of point transformations are reviewed. The two properties are the invariance of Lagrange's equations of motion and the fact that *all* point transformations are canonical.

In Section 3.4 an alternative derivation of a result on the decomposition of the system inertia matrix is presented. To achieve an equivalent linear double integrator representation of the nonlinear system using point transformations, the inertia matrix must be expressed as the "square" of the transformation Jacobian matrix. This decomposition is termed canonical factorization.

In Section 3.5 the condition for the canonical factorization of the inertia matrix is shown to be the same as that for a point transformation from a Riemannian metric to a Euclidean metric. The Riemann Curvature Tensor is introduced as a computational tool to test for existence of such a transformation. The special factorization exists if and only if all components of the curvature tensor are identically zero. In Sections 3.6-3.8 examples illustrating this methodology are presented. In Section 3.6, the cart-pole problem is shown to possess a canonical factorization, and the linearizing transformation is computed. In Section 3.7, it is shown that a 2-link planar manipulator does not admit such a decomposition. In Section 3.8, Euler's rotational equations of motion are investigated and shown to not be linearizable for an axi-symmetric inertia distribution.

3.1 Linearizing Point Transformations

The basis of canonical transformation theory is to provide the ability to obtain alternative generalized coordinates and conjugate momenta which preserve the Hamiltonian form of the equations of motion. Among all possible canonical transfomations, it is desirable to find the ones that result in the simplest transformed Hamilton's equations. The goal is to make the integration of Hamilton's equations as simple as possible. The most elementary form of Hamilton's equations is one in which the transformed coordinates are constant. Starting with initial coordinates (q,p) and Hamiltonian $H(q,p)$, it is desired to find a canonical transformation to yield transformed coordinates (Q,P), such that the transformed Hamiltonian $H(Q,P)$ is constant. Then, the equations of motion reduce to:

$$\begin{aligned} \dot{Q} &= 0_{n\times 1} \\ \dot{P} &= 0_{n\times 1} \end{aligned}$$

This type of transformation can be found using the Hamilton-Jacobi integration theory [12]. This approach generates the solution to the original equations of motion via a time varying transformation between the constant transformed variables and the time varying original coordinates. The time evolution of the original system is generated by the transformation mapping.

At the next level of complexity is the case in which the equations of motion are linear in the transformed coordinates. Consider the most general form of Hamilton's equations [20] in matrix notation,

$$\dot{x} = Z\,H_x^T(x) + u \tag{3.1}$$

where $u = \begin{bmatrix} -F^T & E^T \end{bmatrix}^T$ represents the control action. The requirement that the transformed equations appear linear can be expressed as

$$\dot{X}(x) = A\,X(x) + v \tag{3.2}$$

where $X = X(x) : \mathcal{R}^{2n} \rightarrow \mathcal{R}^{2n}$ is the transformed or new coordinate, A is a constant $2n \times 2n$ coefficient matrix and v is the transformed control action. Differentiating the coordinate transformation, (3.2) can be written as

$$\dot{X}(x) = N(x)\,Z\,H_x^T(x) + N(x)\,u \tag{3.3}$$

where $N(x) = \dfrac{\partial X(x)}{\partial x}$ is the transformation Jacobian matrix. To solve for the transformation $X(x)$ that takes (3.1) to (3.2), set (3.3) equal to (3.2):

$$N(x)\,Z\,H_x^T(x) + N(x)\,u = A\,X(x) + v \tag{3.4}$$

To simplify (3.4), the transformed control action can be selected as $v = N(x)\ u$. Then (3.4) reduces to:

$$N(x)\,Z\,H_x^T(x) = A\,X(x) \tag{3.5}$$

To obtain the linearizing transformation, (3.5) must be solved for $X(x)$. Note that (3.5) represents a set of first order partial differential equations in X with non-constant coefficients. To visualize this, let $f(x) = ZH_x^T(x)$. Then the $i-th$ element of (3.5) is

$$\sum_{k=1}^{2n} \left[f_k(x)\,\frac{\partial X_i}{\partial x_k} - a_{i,k}\,X_k \right] = 0 \qquad i = 1,\ldots,2n \tag{3.6}$$

where $a_{i,k}$ is the i,k element of A. In general, the solution to (3.6) is dependent on the form of $f(x)$ and in general is complicated even in the case of low dimensional systems. For this reason, this approach will not be pursued further.

Another approach is to use canonical transformations to achieve the linearization. In Chapter 2, Section 2.1.1, it was shown that $H_x^T(x) = N^T(x)\ H_X^T(X)$. Using this result, (3.5) becomes:

$$N(x)\,Z\,N^T(x)\,H_X^T(X) = A\,X(x) \tag{3.7}$$

Substituting in (3.7) the metric preserving property of canonical transformations, $N(x)\ Z\ N^T(x) = Z$, (3.7) reduces to:

$$Z\,H_X^T(X) = A\,X(x) \tag{3.8}$$

Multiplying both sides of (3.8) by $-Z$ results in:

$$H_X^T(X) = -ZA\,X(x) \tag{3.9}$$

The $i-th$ element of (3.9) is,

$$\frac{\partial H}{\partial X_i} - \sum_{k=1}^{2n} c_{i,k}\, X_k = 0 \qquad i = 1,\ldots,2n \tag{3.10}$$

where $c_{i,k}$ is the constant weighting coefficient obtained from the i,k element of $C = -ZA$. Furthermore, the matrix C must be symmetric. From (3.10):

$$\frac{\partial^2 H}{\partial X_i\, \partial X_k} = c_{i,k}$$

$$\frac{\partial^2 H}{\partial X_k\, \partial X_i} = c_{k,i}$$

However, from the equality of the mixed partials of H:

$$\frac{\partial^2 H}{\partial X_i\, \partial X_k} = \frac{\partial^2 H}{\partial X_k\, \partial X_i}$$

Thus, $c_{i,k} = c_{k,i}$ which establishes the symmetry property of the C matrix. The solution to equation (3.10) is:

$$H = \frac{1}{2} \sum_{i,k=1}^{2n} c_{i,k}\, X_i\, X_k$$

However, this approach requires a transformation that not only satisfies the canonical constraint $N(x)\; Z\; N^T(x) = Z$ but also solves the gradient transformation rule $H_x^T(x) = N^T(x)\; H_X^T(X)$ which also represents a set partial differential equations. Its $i-th$ element is:

$$\sum_{k=1}^{2n} X_k(x)\, \frac{\partial X_k(x)}{\partial x_i} - \frac{\partial H(x)}{\partial x_i} = 0 \qquad i = 1,\ldots,2n$$

In general, solving this set of partial differential equations is difficult at best and a solution may not exist. Hence, this approach will not be pursued further. It may be observed that a simpler expression for the Hamiltonian in the transformed coordinates can be obtained by setting $A = Z$ in which case the C matrix is diagonal.

To further simplify matters, the form of the target linear system will be specialized to that of a double integrator. To accomplish this, the A matrix is defined in the following manner:

$$A = \begin{bmatrix} 0_{n\times n} & 1_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix}$$

The linear equations in the transformed coordinates appear as:

$$\begin{aligned} \dot{Q} &= P + v_1 \\ \dot{P} &= v_2 \end{aligned} \tag{3.11}$$

Differentiating the coordinate transformation:

$$\begin{aligned} \dot{Q} &= \frac{\partial Q}{\partial q}\frac{\partial H^T}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial H^T}{\partial q} + \frac{\partial Q}{\partial q}u_1 + \frac{\partial Q}{\partial p}u_2 \\ \dot{P} &= \frac{\partial P}{\partial q}\frac{\partial H^T}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial H^T}{\partial q} + \frac{\partial P}{\partial q}u_1 + \frac{\partial P}{\partial p}u_2 \end{aligned} \tag{3.12}$$

Setting (3.11) equal to (3.12) results in the required constraint equations that the coordinate transformation must satisfy. One possible solution approach is to set:

$$\begin{aligned} P &= \frac{\partial Q}{\partial q}\frac{\partial H^T}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial H^T}{\partial q} \\ v_1 &= \frac{\partial Q}{\partial q}u_1 + \frac{\partial Q}{\partial p}u_2 \\ v_2 &= \dot{P} \end{aligned}$$

However, there still remains the task of computing the coordinate transformation. One approach would be to select the generalized coordinate transformation $Q(q,p)$, and the expression for P can then be obtained from the time derivative of Q. The transformed control v_2 is then used to cancel all the dynamics appearing in $\dot{P}$. The number of terms that are cancelled by v_2 depends on the choice of coordinate transformation $Q(q,p)$. A judicious choice of coordinate tranformation would be one that minimizes the number of cancelled terms. A desirable choice is not to cancel any terms, which in turn requires that v_2 is just a control transformation, i.e.

$$v_2 = \frac{\partial P}{\partial q}u_1 + \frac{\partial P}{\partial p}u_2$$

$$\frac{\partial P}{\partial q}\frac{\partial H^T}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial H^T}{\partial q} = 0_{n \times 1}$$

It was seen in the above that state and control transformations play an important role in determining the extent to which nonlinearities are required to be cancelled by the control action. The spectrum of linearizing transformations ranges from one extreme where *all* nonlinearities are cancelled exclusively by state and control transformations. In this case cancellation is not required since there are no nonlinearities that have remained after the transformations. In general such a transformation has the form:

$$\begin{aligned} X &= T(x) \\ v &= T_1(x)\, u \end{aligned}$$

At the other extreme are examples where an identity state transformation is employed, and all the nonlinearities must be cancelled by the control transformation. This is an example where all the nonlinearities appear in the control path and thus can be cancelled. In between these two extremes are transformations where some of the nonolinearities are distributed over the state transformation with the remaining terms cancelled by the control transformation. In general such a transformation has the form:

$$\begin{aligned} X &= T(x) \\ v &= T_1(x)\, u + T_2(x) \end{aligned}$$

It should also be pointed out that the state transformation must be constructed in such a manner that the remaining nonlinear terms appear in the path of the control action in order to be cancelled.

To simplify matters even further, it will be assumed that $u_1 = 0$. The physical significance is that the control variable is a force rather than a velocity source. In many, if not most, real world applications the control action is either a force or a torque and thus this is a reasonable assumption. In the following the notation $u = u_2$ will be employed. In this case, u represents the control variable which has already precompensated forcing terms arising from the system potential function. In the transformed system the equivalent control is:

$$v = \frac{\partial X}{\partial p} u = \begin{bmatrix} \dfrac{\partial Q}{\partial p} \\ \dfrac{\partial P}{\partial p} \end{bmatrix} u \tag{3.13}$$

Note that $v \in \mathcal{R}^{2n}$ while $u \in \mathcal{R}^{n}$. It has been implicitly assumed that an independent control variable in the transformed system is assigned to every element of $\frac{\partial X}{\partial p}u$. From the control perspective, however, the dimension of the transformed control variable cannot be larger than that of the original control because the inverse transformation will not be solvable. Equation (3.13) represents a system of overconstrained equations which cannot be solved for u. That is, given an arbitrary v a solution u cannot be found. Since the matrix $\frac{\partial X}{\partial p}$ is $2n \times n$ and its maximal rank is n, u can be exactly retrieved from an n dimensional subspace of v only. Thus the map from u to v is not *onto* and is an example of an overconstrained set of linear equations. It is apparent that the entries of v cannot be treated as independent variables in control synthesis for the transformed system. The solution is to assign transformed control variables only in the range space of an $n \times n$ full rank submatrix of $\frac{\partial X}{\partial p}$. This can be accomplished by selecting n rows of $\frac{\partial X}{\partial p}$ to be zero such that the remaining n rows are linearly independent for all q,p. Linear independence of this submatrix is required for the mapping to be invertible. From (3.13) it is seen that the tradeoff in the coordinate transformation is the dependence on p. Since the dimension of both $\frac{\partial Q}{\partial p}$ and $\frac{\partial P}{\partial p}$ matrices is n, a similar dimensioned submatrix can be constructed by requiring a total of n transformed variables, Q_i and P_i, to be independent of p.

One approach to generating an invertible control transformation is to choose:

$$\frac{\partial P}{\partial p} = 0_{n \times n}$$

In this case $Q = Q(q,p)$ and $P = P(q)$. Another solution is:

$$\frac{\partial Q}{\partial p} = 0_{n\times n}$$

Thus, Q can *only* be a function of q, i.e. $Q = Q(q)$ a *point* transformation. For the map between the transformed control and original control variable to be invertible the matrix $\dfrac{\partial P(q,p)}{\partial p}$ must be nonsingular for all q,p. For a point transformation, the transformed dynamics appear as:

$$\begin{aligned} \dot{Q} &= P \\ \dot{P} &= v \end{aligned} \tag{3.14}$$

Additionally, if the transformation is canonical, then the transformed equations can be obtained from the Hamiltonian framework. In the following sections, it will be shown that *all* point tranformations are canonical. Thus, the target linear dynamics (3.14) are obtained from the Hamiltonian function:

$$H = \frac{1}{2} P^T P$$

This can be verified by direct application of Hamilton's canonical equations.

In summary, the process of linearization can be thought as one where first a target or desired transformed representation of the dynamics is selected and then one searches for a transformation that will recover this target system. The constraints imposed by the particular choice of target system are then incorporated in the search for an appropriate transformation.

3.1.1 Point Transformations Using Generating Functions

A canonical point transformation may also be constructed using the generating function approach. It should be noted, however, that usually one has to assume the form of the generating function *a priori*, and, for this reason, the expertise of the practitioner is crucial when applying this approach. This fact notwithstanding, for point tranformations a type-1 (i.e. $S_1(q,Q)$) generating function is not a viable candidate since q and Q are not independent variables. A type-4 (i.e. $S_4(p,P)$) generating function is also not valid. To show this, first note that the dependent variables are q,Q and are obtained from [12]:

$$q_i = -\frac{\partial S_4(p,P)}{\partial p_i}$$

$$Q_i = \frac{\partial S_4(p,P)}{\partial P_i}$$

Then,

$$\begin{aligned}
\dot{q} &= \frac{\partial q}{\partial p}\dot{p} + \frac{\partial q}{\partial P}\dot{P} \\
&= -\frac{\partial q}{\partial p}\frac{\partial H^T}{\partial q} + \frac{\partial q}{\partial p}u + \frac{\partial q}{\partial P}v \\
&= \frac{\partial H(q,p)}{\partial p}
\end{aligned}$$

since

$$\begin{aligned}
\dot{p} &= -\frac{\partial H^T}{\partial q} + u \\
\dot{P} &= v
\end{aligned}$$

For $\dot{q}$ to be independent of the controls (original and transformed), it must not be a function of (p,P), which leads to a contradiction. These results are summarized in the following definition.

Definition 6 *A canonical transformation to the linear form (3.14) can only be realized by a type-2 or type-3 generating function, i.e.*

$$\begin{aligned}
S_2 &= S_2(q,P) \\
S_3 &= S_3(p,Q)
\end{aligned}$$

3.2 Point Transformations Preserve Lagrange Equations

In this section, a study of the properties of point transformations is initiated which will be continued in the next section. A well known fact in mechanics states that the Lagrangian equations of motion are preserved or are invariant under a point transformation. This preliminary result will help pave the way to the ultimate goal of finding linearizing transformations. In lieu of a reference, a proof of this statement is presented in the following.

Let q denote the n-dimensional vector of *independent* generalized coordinates of a system, for which a kinetic energy $T(q, \dot{q})$ quadratic in generalized velocities is defined,

$$T(q, \dot{q}) = \frac{1}{2}\dot{q}^T B(q)\dot{q}$$

where $B(q)$ is the symmetric and positive definite inertia matrix. For the purpose at hand, the original or primitive form of the Lagrange equations [13] will be employed:

$$\frac{d}{dt}\left(\frac{\partial T^T}{\partial \dot{q}}\right) - \frac{\partial T^T}{\partial q} = u \tag{3.15}$$

where, by abuse of notation, u represents the effect of all generalized forces, disturbances and control actuators. Now, consider a point transformation:

$$\begin{aligned} Q &= f(q) \\ \dot{Q} &= J(q)\,\dot{q} \end{aligned}$$

Here, $J(q)$ is the coordinate transformation Jacobian matrix, i.e.

$$\frac{\partial f(q)}{\partial q} = J(q)$$

Under such a transformation the kinetic energy is invariant and can be written in either coordinates as:

$$T = \frac{1}{2}\dot{q}^T B(q)\dot{q} = \frac{1}{2}\dot{Q}^T J^{-T}(q)\, B(q)\, J^{-1}(q)\, \dot{Q}$$

It will be assumed that $f(q)$ is a differentiable mapping with non-singular Jacobian matrix $J(q)$ for all q. These requirements are necessary and sufficient to guarantee the existence of a differentiable inverse [31]. The Jacobian matrix of the inverse transformation $q = f^{-1}(Q)$ is [31]:

$$\frac{\partial q}{\partial Q} = J^{-1}(f^{-1}(Q))$$

Before proceeding further, a preliminary lemma is required.

Lemma 2 *Given* $Q = f(q)$, $\dot{Q} = J(q)\,\dot{q}$, *with* $\det[J(q)] \neq 0, \quad \forall q$. *Then:*

$$\frac{\partial \dot{Q}}{\partial q} = \dot{J}(q)$$

Proof: First note that $\dot{Q} = \dot{Q}(q, \dot{q})$. Differentiating this expression results in:

$$\begin{aligned}\ddot{Q} &= \frac{\partial \dot{Q}}{\partial q}\,\dot{q} + \frac{\partial \dot{Q}}{\partial \dot{q}}\,\ddot{q} \\ &= \dot{J}(q)\,\dot{q} + J(q)\,\ddot{q}\end{aligned}$$

Combining like terms on both sides results in:

$$\left[\frac{\partial \dot{Q}}{\partial q} - \dot{J}(q)\right]\dot{q} = 0_{n\times 1}$$

For the equality to hold for arbitrary $\dot{q}$ requires that $\frac{\partial \dot{Q}}{\partial q} = \dot{J}(q)$ which proves the identity. This is the easiest and quickest approach to prove the identity. Alternatively, the same result can be obtained using a derivation that is more insightfull and which uses a property that will appear repeatedly. This derivation is now presented. From the definition, the i-th element of $\dot{Q}$ is:

$$\dot{Q}_i = \sum_{k=1}^{n} J_{i,k}(q)\,\dot{q}_k$$

Note that $J_{i,k}(q) = \frac{\partial f_i(q)}{\partial q_k}$. The j-th partial of $\dot{Q}_i$ is:

$$\frac{\partial \dot{Q}_i}{\partial q_j} = \sum_{k=1}^{n} \frac{\partial J_{i,k}(q)}{\partial q_j}\,\dot{q}_k \tag{3.16}$$

Now, the time derivative of the (i, j) element of the Jacobian matrix is expressed as:

$$\dot{J}_{i,j}(q) = \sum_{k=1}^{n} \frac{\partial J_{i,j}(q)}{\partial q_k}\,\dot{q}_k \tag{3.17}$$

At this point the equality of mixed partial derivatives is employed [31], i.e. the order of partial differentiation can be reversed. It will be assumed that $f(q)$ is a sufficiently smooth function (i.e. $f(q) \in C^2$ or $f(q)$ has continuous partial derivatives to second order) then:

$$\frac{\partial^2 f_i(q)}{\partial q_j\,\partial q_k} = \frac{\partial^2 f_i(q)}{\partial q_k\,\partial q_j} \qquad \forall\, i, j, k$$

This is the key property that will be evident throughout this document and will appear in a multitude of instances. It is a necessary and sufficient condition for exactness of differential equations; i.e. it is a test of whether a solution exists. Rewriting (3.16) and (3.17) in terms of $f(q)$ results in:

$$\begin{aligned}
\frac{\partial \dot{Q}_i}{\partial q_j} &= \sum_{k=1}^{n} \frac{\partial^2 f_i(q)}{\partial q_j\, \partial q_k}\, \dot{q}_k \\
\dot{J}_{i,j}(q) &= \sum_{k=1}^{n} \frac{\partial^2 f_i(q)}{\partial q_k\, \partial q_j}\, \dot{q}_k
\end{aligned}$$

It is clearly evident that the application of equality of mixed partials results in,

$$\frac{\partial \dot{Q}_i}{\partial q_j} = \dot{J}_{i,j}(q) \quad \forall\, i,j$$

which proves the property:

$$\frac{\partial \dot{Q}}{\partial q} = \dot{J}(q)$$

At this point the question of whether Lagrange's equations are preserved under point transformations is addressed. The goal is to show that in the transformed coordinates Lagrange's equations are of the form,

$$\frac{d}{dt}\left(\frac{\partial T^T}{\partial \dot{Q}}\right) - \frac{\partial T^T}{\partial Q} = \tilde{u} \tag{3.18}$$

where $\tilde{u} = J^{-T}(q)\, u$. The approach is to use (3.15) and compute its entries in terms of the transformed variable and show that the resulting expression is (3.18). As a first step, compute $\frac{\partial T^T}{\partial q}$ using Lemma 2 to obtain:

$$\begin{aligned}
\frac{\partial T^T(Q(q), \dot{Q}(q,\dot{q}))}{\partial q} &= \left(\frac{\partial T}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial T}{\partial \dot{Q}}\frac{\partial \dot{Q}}{\partial q}\right)^T \\
&= J^T(q)\,\frac{\partial T^T}{\partial Q} + \dot{J}^T(q)\,\frac{\partial T^T}{\partial \dot{Q}}
\end{aligned}$$

Proof: First note that $\dot{Q} = \dot{Q}(q, \dot{q})$. Differentiating this expression results in:

$$\begin{aligned} \ddot{Q} &= \frac{\partial \dot{Q}}{\partial q}\,\dot{q} + \frac{\partial \dot{Q}}{\partial \dot{q}}\,\ddot{q} \\ &= \dot{J}(q)\,\dot{q} + J(q)\,\ddot{q} \end{aligned}$$

Combining like terms on both sides results in:

$$\left[\frac{\partial \dot{Q}}{\partial q} - \dot{J}(q)\right]\dot{q} = 0_{n\times 1}$$

For the equality to hold for arbitrary $\dot{q}$ requires that $\frac{\partial \dot{Q}}{\partial q} = \dot{J}(q)$ which proves the identity. This is the easiest and quickest approach to prove the identity. Alternatively, the same result can be obtained using a derivation that is more insightfull and which uses a property that will appear repeatedly. This derivation is now presented. From the definition, the i-th element of $\dot{Q}$ is:

$$\dot{Q}_i = \sum_{k=1}^{n} J_{i,k}(q)\,\dot{q}_k$$

Note that $J_{i,k}(q) = \frac{\partial f_i(q)}{\partial q_k}$. The j-th partial of $\dot{Q}_i$ is:

$$\frac{\partial \dot{Q}_i}{\partial q_j} = \sum_{k=1}^{n} \frac{\partial J_{i,k}(q)}{\partial q_j}\,\dot{q}_k \tag{3.16}$$

Now, the time derivative of the (i,j) element of the Jacobian matrix is expressed as:

$$\dot{J}_{i,j}(q) = \sum_{k=1}^{n} \frac{\partial J_{i,j}(q)}{\partial q_k}\,\dot{q}_k \tag{3.17}$$

At this point the equality of mixed partial derivatives is employed [31], i.e. the order of partial differentiation can be reversed. It will be assumed that $f(q)$ is a sufficiently smooth function (i.e. $f(q) \in C^2$ or $f(q)$ has continuous partial derivatives to second order) then:

$$\frac{\partial^2 f_i(q)}{\partial q_j\,\partial q_k} = \frac{\partial^2 f_i(q)}{\partial q_k\,\partial q_j} \qquad \forall\, i,j,k$$

Next, compute $\dfrac{\partial T}{\partial \dot{q}}$:

$$\begin{aligned}\frac{\partial T^T(Q(q),\dot{Q}(q,\dot{q}))}{\partial \dot{q}} &= \left(\frac{\partial T}{\partial \dot{Q}}\frac{\partial \dot{Q}}{\partial \dot{q}}\right)^T \\ &= J^T(q)\,\frac{\partial T^T}{\partial \dot{Q}}\end{aligned}$$

Note that there is no $\dfrac{\partial T}{\partial Q}\dfrac{\partial Q}{\partial \dot{q}}$ term since $\dfrac{\partial Q}{\partial \dot{q}} = 0_{n\times n}$ because Q is not a function of $\dot{q}$. Now, the time derivative of the i-th row of this quantity is given by:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T^T}{\partial \dot{q}}\right)_i &= \frac{d}{dt}\left(J^T(q)\,\frac{\partial T^T}{\partial \dot{Q}}\right)_i \\ &= \frac{d}{dt}\left(\sum_{k=1}^{n} J_{k,i}(q)\,\frac{\partial T}{\partial \dot{Q}_k}\right) \\ &= \sum_{k=1}^{n}\left[J_{k,i}(q)\,\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}_k}\right) + \dot{J}_{k,i}\,\frac{\partial T}{\partial \dot{Q}_k}\right]\end{aligned}$$

In matrix form, this can be written as:

$$\frac{d}{dt}\left(\frac{\partial T^T}{\partial \dot{q}}\right) = J^T(q)\,\frac{d}{dt}\left(\frac{\partial T^T}{\partial \dot{Q}}\right) + \dot{J}^T(q)\,\frac{\partial T^T}{\partial \dot{Q}}$$

Utilizing the above resuts, Lagrange's equations in the original coordinates becomes:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T^T}{\partial \dot{q}}\right) - \frac{\partial T^T}{\partial q} &= u \\ &= J^T(q)\,\frac{d}{dt}\left(\frac{\partial T^T}{\partial \dot{Q}}\right) + \dot{J}^T(q)\,\frac{\partial T^T}{\partial \dot{Q}} - J^T(q)\,\frac{\partial T^T}{\partial Q} - \dot{J}^T(q)\,\frac{\partial T^T}{\partial \dot{Q}} \\ &= J^T(q)\left(\frac{d}{dt}\left(\frac{\partial T^T}{\partial \dot{Q}}\right) - \frac{\partial T}{\partial Q}\right)\end{aligned}$$

Finally, the desired result is obtained by equating the two right-hand sides of the above:

$$\frac{d}{dt}\left(\frac{\partial T^T}{\partial \dot{Q}}\right) - \frac{\partial T^T}{\partial Q} = J^{-T}(q)\, u$$

Summarizing, an alternative derivation of the invariance of Lagranges equations under point transforamtions has been presented. This result will be useful when linearizing transformations for mechanical systems are considered by providing an alternative approach to deriving linearizing transformations.

3.3 Point Transformations Are Canonical

In this section, the investigation into the properties of point transformations will be further pursued. In the previous section it was shown that point transformations preserve Lagranges equations. Here, these transformations are examined from the point of view of canonical transformations. It will be shown that *all* point transformations $Q = f(q)$ are canonical or metric preserving, i.e. $Z - N\, Z\, N^T = 0_{n\times n}$. This result certainly has appeared in the mechanics literature previously. However, in lieu of a reference, a proof of this result is presented in the following. Recall that the requirements for a canonical transformation in terms of Poisson brackets were:

(a) $\{Q,P\} = 1_{n\times n}$

(b) $\{Q,Q\} = 0_{n\times n}$

(c) $\{P,P\} = 0_{n\times n}$

As a preliminary step, the expression for P is first determined. For a point transformation $Q = f(q)$, $\dot{Q} = J(q)\,\dot{q}$, the expression for the Kinetic energy is:

$$T = \frac{1}{2}\,\dot{q}^T\, B(q)\,\dot{q} = \frac{1}{2}\,\dot{Q}^T\, J^{-T}(q)\, B(q)\, J^{-1}(q)\,\dot{Q}$$

Since mechanical or natural systems are considered here, the momenta in both the original and transformed system are denoted by p and P. Just as the momentum in the original system is defined by,

$$p = \frac{\partial T^T}{\partial \dot{q}} = B(q)\,\dot{q}$$

the momentum in the transformed coordinates is similarly defined by:

$$P = \frac{\partial T^T}{\partial \dot{Q}} = J^{-T}(q)\, B(q)\, \dot{q} = J^{-T}(q)\, p$$

Whereast the expressions for the transformed generalized coordinates and generalized momenta have been explicitly defined, the canonical criteria can be evaluated.

The first canonical requirement, in expanded form, is written as,

$$\begin{aligned}
\{Q,P\} &= \frac{\partial Q}{\partial q}\frac{\partial P^T}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P^T}{\partial q} \\
&= \frac{\partial Q}{\partial q}\frac{\partial P^T}{\partial p} \\
&= J(q)\left[J^{-T}(q)\right]^T \\
&= 1_{n\times n}
\end{aligned}$$

where the property that Q is not a function of p has been used to annihilate the second term on the right-hand side, and the definition of $P = J^{-T}(q)p$ has been used to obtain the final result. It is concluded that the first requirement is satisfied. Similarly, writting out the second canonical requirement the following expression is obtained where the property that Q is not a function of p has been used to annihilate all terms in the right-hand side.

$$\begin{aligned}
\{Q,Q\} &= \frac{\partial Q}{\partial q}\frac{\partial Q^T}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial Q^T}{\partial q} \\
&= 0_{n\times n}
\end{aligned}$$

It is evident that the second requirement is also satisfied. Finally, the third canonical requirement in expanded form is given by:

$$\begin{aligned}
\{P,P\} &= \frac{\partial P}{\partial q}\frac{\partial P^T}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial P^T}{\partial q} \\
&= \frac{\partial P}{\partial q}J^{-1}(q) - J^{-T}(q)\frac{\partial P^T}{\partial q} \qquad (3.19)
\end{aligned}$$

To further simplify this equation, a preliminary result utilizing the chain rule of differentiation in matrix form to P is required to elliminate the Jacobian matrix. First define the inverse map to the point transformation by $g(Q) = f^{-1}(Q) = q$. Let the Jacobian matrix of the inverse transformation be denoted by $G(Q)$, i.e. $\dot{q} \; = \; G(Q)\,\dot{Q}$. From the fact that a necessary and sufficient condition for the existence of a differentiable inverse to $f(q)$ is the nonsingualrity of $J(q)$, and the Jacobian matrix of the inverse transformation is given by $J^{-1}(g(Q))$ [31] the expression for $G(Q)$ is $G(Q) = J^{-1}(g(Q))$. Now, the expression for P can be written as:

$$P \; = \; J^{-T}(q)\,p \; = \; G^{T}(Q)\,p$$

The expression for P_i becomes:

$$P_i \; = \; \sum_{l=1}^{n} G_{l,i}(Q)\,p_l$$

The partial of P_i with respect to Q_k is given by:

$$\frac{\partial P_i}{\partial Q_k} \; = \; \sum_{l=1}^{n} \frac{\partial G_{l,i}(Q)}{\partial Q_k}\,p_l \tag{3.20}$$

Hence, the partial derivative of (3.20) with respect to q_j takes the form

$$\begin{aligned}
\frac{\partial P_i}{\partial q_j} \; &= \; \sum_{l=1}^{n}\sum_{k=1}^{n} \frac{\partial G_{l,i}(Q)}{\partial Q_k}\,\frac{\partial Q_k}{\partial q_j}\,p_l \\
&= \; \sum_{k=1}^{n}\sum_{l=1}^{n} \frac{\partial G_{l,i}(Q)}{\partial Q_k}\,p_l\,\frac{\partial Q_k}{\partial q_j} \\
&= \; \sum_{k=1}^{n} \frac{\partial P_i}{\partial Q_k}\,\frac{\partial Q_k}{\partial q_j} \\
&= \; \left[\frac{\partial P_i}{\partial Q_1}, \; \ldots, \; \frac{\partial P_i}{\partial Q_n} \right] \begin{bmatrix} \dfrac{\partial Q_1}{\partial q_j} \\ \vdots \\ \dfrac{\partial Q_n}{\partial q_j} \end{bmatrix}
\end{aligned}$$

where the order of summation has been exchanged and use has been made of the expression for $\dfrac{\partial P_i}{\partial Q_k}$. Finally, by induction it can be seen that the matrix form expression for $\dfrac{\partial P}{\partial q}$ can be expressed as:

$$\frac{\partial P}{\partial q} \; = \; \frac{\partial P}{\partial Q}(Q,p)\,\frac{\partial Q}{\partial q}$$

$$= \frac{\partial P}{\partial Q}(Q,p)\, J(q) \tag{3.21}$$

This result can now be used to simplify the third canonical requirement. Substituting (3.21) in (3.19):

$$\begin{aligned}\{P,P\} &= \frac{\partial P}{\partial Q}(Q,p)\, J(q)\, J^{-1}(q) \;-\; J^{-T}(q)\, J^{T}(q)\, \frac{\partial P^{T}}{\partial Q}(Q,p) \\ &= \frac{\partial P}{\partial Q} - \frac{\partial P^{T}}{\partial Q} \end{aligned}\tag{3.22}$$

Now, the i,j-th element of (3.22) is:

$$\begin{aligned}\{P,P\}_{i,j} &= \frac{\partial P_i}{\partial Q_j} - \frac{\partial P_j}{\partial Q_i} \\ &= \sum_{l=1}^{n} \frac{\partial G_{l,i}(Q)}{\partial Q_j}\, p_l - \sum_{l=1}^{n} \frac{\partial G_{l,j}(Q)}{\partial Q_i}\, p_l \end{aligned}$$

Since $G_{l,i} = \dfrac{\partial g_l(Q)}{\partial Q_i}$, the i,j-th element becomes:

$$\{P,P\}_{i,j} = \sum_{l=1}^{n} \left[\frac{\partial^2 g_l(Q)}{\partial Q_j\, \partial Q_i} - \frac{\partial^2 g_l(Q)}{\partial Q_i\, \partial Q_j} \right] p_l \tag{3.23}$$

But since by assumption $g(Q) \in C^2$, i.e. it is twice differentiable, its mixed partial derivatives are equal [31]:

$$\frac{\partial^2 g_l(Q)}{\partial Q_j\, \partial Q_i} = \frac{\partial^2 g_l(Q)}{\partial Q_i\, \partial Q_j}$$

Substituting this in (3.23) annihilates the right hand side, which is the desired result. Finally, since $\{P,P\}_{i,j} = 0$ independent of i,j it is evident that:

$$\{P,P\} = 0_{n \times n}$$

Thus, the third requirement for a canonical transformation is also satisfied. In conclusion, it has been shown that *all* point transformations are indeed canonical transformations.

3.4 Canonical Factorization

In this section, the canonical transformation methodology will be applied to the problem of linearizing general nonlinear equations of motion expressed in the Hamiltonian framework. This approach is an example of feedback linearization where a change of coordinates in the state and control space coupled with feedback applied to the nonlinear system results in a controllable linear system.

Starting with the kinetic energy expression and Lagrange's equations as in (3.15), the Hamiltonian is defined as:

$$H = \frac{1}{2}\, p^T\, B^{-1}(q)\, p$$

where

$$p \;=\; \frac{\partial T}{\partial \dot{q}} \;=\; B(q)\, \dot{q}$$

H is the Kinetic energy expression, and p is the generalized or conjugate (i.e. the dual variable to the generalized coordinate) momentum. The Hamiltonian equations of motion become:

$$\begin{aligned} \dot{q} &= B^{-1}(q)\, p \\ \dot{p} &= -\left[\frac{\partial}{\partial q}\, \tfrac{1}{2}\, p^T\, B^{-1}(q)\, p \right]^T + u \end{aligned} \qquad (3.24)$$

The objective is to choose a point tranformation $Q = Q(q)$ to obtain a Hamiltonian that is a function of the generalized momentum only, i.e.

$$H = \frac{1}{2} P^T P$$

which will result in the linear double integrator system in the transformed coordinates. Note that the restriction to point transformations is known *a priori* to be canonical. With $Q = f(q)$, assume that $f \in C^1$ with $\det[f'(q)] \neq 0, \;\; \forall q$ (i.e. f is a diffeomorphism). Differentiating Q results in

$$\dot{Q} \;=\; J(q)\, \dot{q} \;=\; J(q)\, B^{-1}(q)\, p$$

where $J(q)$ is the Jacobian matrix of $f(q)$. In order to have the transformed equations appear linear in their arguments, it is necessary that $\dot{Q} = P$ as in (3.14). Then,

$$P \;=\; J(q)\, B^{-1}(q)\, p$$

and the Hamiltonian becomes:

$$H = \frac{1}{2} P^T J^{-T}(q) B(q) J^{-1}(q) P$$

To obtain the target linear description requires that:

$$J^{-T}(q) B(q) J^{-1}(q) = 1_{n\times n} \tag{3.25}$$

To generate the transformed control variable it is noted that the original control is obtained from the virtual work principle $\delta W = u^T \delta q$. Since $q = f^{-1}(Q)$, this can be written as:

$$\begin{aligned} \delta W &= u^T \frac{\partial f^{-1}(Q)}{\partial Q} \delta Q \\ &= u^T J^{-1}(q) \delta Q \\ &= v^T \delta Q \end{aligned}$$

Therefore, the transformed control variable, v, is given by $v = J^{-T}(q) u$. Satisfaction of (3.25) leads to the desired target linear description.

$$\begin{bmatrix} \dot{Q} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} P \\ J^{-T}(q)u \end{bmatrix} \tag{3.26}$$

This result can be summarized in the following theorem.

Theorem 2 *Define the point transformation $Q = f(q)$ with $f \in C^1$ and $\det[J(q)] \neq 0, \quad \forall q$. This is a linearizing canonical transformation if and only if:*

$$B(q) = J^T(q) J(q) \tag{3.27}$$

The transformed linear dynamics are given by (3.26)

This special factorization of the inertia matrix is refered to as a canonical factorization. It should be noted that this factorization has been independently obtained by Koditschek [26] and by Gu [15]. However, the existence question of this factorization were not addressed in these references. The existence conditions for this factorization are presented in the next section. The above derivation is an alternative to the previous derivations cited in the literature. The meaning of this factorization in physical terms is that there exists a coordinate system in which the inertia matrix is coordinate independent. Also note that the full power of canonical transformations has not been utilized. This is because of the restriction to point transformations which are a subset of canonical transformations. In the most general case the coordinate transformation would be a function of both generalized coordinates and momenta.

An alternative approach to linearize (3.24) is by the so called "computed torque" approach [41] pioneered in the robotics literature. The simplest choice of coordinate transformation that linearizes (3.24) is to choose $z_1 = q, \quad z_2 = \dot{q}$. Then:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= v \end{aligned}$$

The transformed control variable, v, is used to cancel all the nonlinearities that appear in the expression for $\dot{z}_2$. This approach can be expanded to include point transformations $z_1 = f(q)$. For nonredundant manipulators (the number of links equal the dimension of the end-effector) it has been shown [48] that setting $z_1 = f(q)$ where $f(q)$ is the forward kinematics describing the end-effector location, results in decoupled input output linear dynamics. This approach, however, is not an example of a canonical transformation and Hamilton's equations are not preserved.

3.4.1 Lagrangian Feedback Linearization

In this section it will be shown that the special decomposition of the inertia matrix (3.27) can be obtained using the Lagrangian framework without recourse to canonical theory. As was observed previously, the Hamiltonian framework is more general in that canonical transformations that are not restricted to point transformations can be employed. The key concept in the Lagrangian based derivation is the realization that only nonlinear terms arising from the kinetic energy can be simplified; the kinetic energy is the most significant system summarizing quantity, and Lagrange's equations in their most primitive form are derived from the kinetic energy.

Now suppose that there exists a point transformation $Q = f(q)$ such that the kinetic energy in the new coordinate system can be written as:

$$T = \frac{1}{2} \dot{Q}^T \dot{Q} = \frac{1}{2} \dot{q}^T B(q) \dot{q}.$$

Substituting this expression for $T(\dot{Q})$ in Lagrange's equations in transformed coordinates results in

$$\ddot{Q} = v$$

where the transformed control $v = J^{-T}(q)\, u$. Thus in the transformed coordinate and control space the equations of motion appear in a double integrator linear form which is the desired result. Again, the main requirement for realizing such a transformation is the special decomposition of the inertia matrix (3.27).

3.5 Existence Of Canonical Factorization

Up to this point, it has been shown that a canonical transformation to a double integrator linear description is possible if and only if the special (canonical) factorization of the composite inertia tensor exists. Thus, the key to any simplification of the nonlinear control problem rests on the existence of such a factorization. To this end, this section addresses the question of what conditions are required for such a transformation to exist.

From the previous development, the central concept in deriving the canonical factorization is the system kinetic energy. This is an example of an inner product endowed with a metric, in this case the composite inertia tensor, which is an example of a *Riemannian metric.* Canonical transformations are an example of coordinate changes that seek to alter this metric into a particularly useful form: the identity matrix. If such a transformation is possible, the original nonlinear system can be globally transformed to a linear system. To set this discussion in a geometric perspective the following definitions are required [11]:

Definition 7 *A Riemannian metric in a region of the space $\mathcal{R}^n$ is a positive definite quadratic form defined on vectors originating at each point P of the region and depending smoothly on P.*

Consider the vector v, originating from a point P, and the quadratic form

$$\sum_{i,j=1}^{n} g_{i,j}\, v_i\, v_j = v^T G\, v$$

where $g_{i,j}$ depends on P and the co-ordinate system, and G is the matrix $G = [g_{i,j}]$. If one were to choose an arbitrary coordinate system $(y_1, \ldots, y_n)$, then the Riemannian metric is a family of smooth functions $g_{i,j} = g_{i,j}(y_1, \ldots, y_n)$, $i, j = 1, \ldots, n$ such that:

(a) $G > 0$, i.e. G is positive definite.

(b) The Riemann metric G transforms under a co-ordinate change $y_i = y_i(z)$, $i = 1,\ldots,n$ according to $\tilde{G} = J^T(z)\ G\ J(z)$, where $J(z) = \dfrac{\partial y(z)}{\partial z}$ is the Jacobian transformation matrix.

From this definition, the kinetic energy is a Riemannian metric since the inertia tensor is positive definite. Next consider the definition of a Euclidean metric.

Definition 8 *A metric $g_{i,j} = g_{i,j}(z)$ is said to be Euclidean if there exist coordinates $x = [x_1,\ldots,x_n]^T$, $x_i = x_i(z)$, such that*

$$\det J(z) \neq 0, \qquad G = J^T(z)\, J(z)$$

where $J(z) = \dfrac{\partial x(z)}{\partial z}$ is the Jacobian transformation matrix, and $G = [g_{i,j}(z)]$ is the metric expressed in matrix notation.

Relative to the coordinate system $x_1,\ldots,x_n$, the Euclidean metric transforms under a co-ordinate change $z_i = z_i(x)$ to:

$$\tilde{G} = J^{-T} G J^{-1} = J^{-T} J^T J J^{-1} = 1_{n \times n}$$

In these coordinates, the Euclidean metric is the identity matrix.

From the above discussion, it is evident that a canonical factorization is just a Euclidean metric. That is, a new co-ordinate system is desired such that a Riemannian metric is transformed to a Euclidean metric. Under such a transformation the metric remains invariant. This is referred to as an *isometric* transformation [11].

Definition 9 *The transformation $x_i = x_i(z_i,\ldots,z_n)$ is called an* isometry *(or an isometric transformation) if:*

$$\tilde{G}(z_i,\ldots,z_n) = G((x_i(z),\ldots,x_n(z))$$

Returning to the original problem of the existence of the inertia tensor's special factorization, one can summarize the existence question associated with linearizing canonical transformations by the following:

Existence Question: Let $B(q),\ q \in \mathcal{R}^n$ be a Riemannian metric. Under what conditions does there exist a point transformation $Q = f(q)$ such that:

$$\dot{q}^T B(q)\dot{q} = \dot{Q}^T \dot{Q}$$

The objective is to find a transformation $Q = f(q)$ such that $B(q) = J^T(q)J(q)$. This definition can be utilized to provide an approach to answer the existence question. It is noted that the inertia tensor is expressed in terms of the transformation Jacobian matrix elements. But the elements of a Jacobian matrix satisfy certain rules. So the key to the solution of the problem is to find these rules and apply them to $B(q)$ thereby establishing a method to check for existence once a specific inertia tensor has been supplied. The key property that a Jacobian matrix satisfies turns out to be the integrability condition or equality of mixed partials. Writing out an arbitrary element (say i, j) of the inertia tensor in terms of the Jacobian matrices one obtains:

$$B_{i,j}(q) = \sum_{k=1}^{n} \frac{\partial f_k(q)}{\partial q_i} \frac{\partial f_k(q)}{\partial q_j} \tag{3.28}$$

The approach to deriving existence conditions lies in differentiation of the $B_{i,j}(q)$'s in order to express them in terms of second-order mixed partial derivatives of $f(q)$, whence the equality of mixed partials can be applied to obtain the desired conditions.

To illustrate how the equality of mixed partial derivatives can be used to generate conditions for the existence of solutions to a set of linear partial differential equations, consider the following example. Suppose one is interested in determining whether a solution exists to the following system of first order partial differential equations

$$\frac{\partial \phi_i}{\partial x_j} = f_{i,j}(x, \phi(x)) \tag{3.29}$$

where $x \in \mathcal{R}^n$, and $\phi \in \mathcal{R}^m$. Equation (3.29) can also be written as a system of *total* differential equations

$$d\phi_i = f_{i,j}(x, \phi(x))\, dx_j$$

or

$$\left[\frac{\partial \phi_i}{\partial x_j} - f_{i,j}(x, \phi(x)) \right] dx_j = 0$$

Now, differentiate (3.29) with respect to x_k to obtain:

$$\frac{\partial^2 \phi_i}{\partial x_j\, \partial x_k} = \frac{\partial f_{i,j}(x,\phi(x))}{\partial x_k} + \sum_{l=1}^{m} \frac{\partial f_{i,j}(x,\phi(x))}{\partial \phi_l(x)} f_{l,k}(x,\phi(x))$$

Interchanging the order of differentiation results in:

$$\frac{\partial^2 \phi_i}{\partial x_k\, \partial x_j} = \frac{\partial f_{i,k}(x,\phi(x))}{\partial x_j} + \sum_{l=1}^{m} \frac{\partial f_{i,k}(x,\phi(x))}{\partial \phi_l(x)} f_{l,j}(x,\phi(x))$$

Applying the equality of mixed partials principle requires that

$$\frac{\partial^2 \phi_i}{\partial x_j\, \partial x_k} = \frac{\partial^2 \phi_i}{\partial x_k\, \partial x_j}$$

which results in the necessary and sufficient conditions [7] for integrability of (3.29):

$$\frac{\partial f_{i,j}}{\partial x_k} - \frac{\partial f_{i,k}}{\partial x_j} + \sum_{l=1}^{m} \left[\frac{\partial f_{i,j}}{\partial \phi_l} f_{l,k} - \frac{\partial f_{i,k}}{\partial \phi_l} f_{l,j} \right] = 0$$

It turns out, however, that the answer to the existence question was derived by Riemann. In an unpublished paper, submitted to the Paris Academy in 1861 associated with a heat conduction problem, Riemann considered transformations $y(x)$ such that:

$$\sum_{i,j=1}^{n} g_{i,j}(y)\, dy_i\, dy_j = \sum_{i,j=1}^{n} dx_i\, dx_j$$

The necessary and sufficient conditions for the existence of such a transformation are given in terms of the so called Riemann Curvature Tensor, denoted R_{ijkl}, by the following theorem [42](Chapter 4C):

Theorem 3 *A Riemannian metric can be transformed to a Euclidean metric if and only if:*

$$R_{ijkl} = 0 \qquad \forall\, i,j,k,l = 1,\ldots,n$$

where the Riemann Curvature Tensor is defined by

$$R_{ijkl} = \frac{\partial^2 B_{i,k}(q)}{\partial q_l\, \partial q_j} + \frac{\partial^2 B_{j,l}(q)}{\partial q_k\, \partial q_i} - \frac{\partial^2 B_{i,l}(q)}{\partial q_k\, \partial q_j} - \frac{\partial^2 B_{j,k}(q)}{\partial q_l\, \partial q_i} + \frac{1}{2} \sum_{r,s=1}^{n} B^{-1}_{r,s}(q)\, [c_{rjl}\, c_{sik} - c_{ril}\, c_{sjk}]$$

where

$$c_{ijk} = \frac{\partial B_{i,j}(q)}{\partial q_k} + \frac{\partial B_{i,k}(q)}{\partial q_j} - \frac{\partial B_{j,k}}{\partial q_i}$$

are known as *Christoffel symbols* of the first kind, and $B^{-1}_{r,s}(q)$ denotes the (r,s) element of $B^{-1}(q)$. Note that this definition of the Christoffel symbols is different from that commonly found in the literature. Using the common definition for these symbols, Γ_{ijk} [25], $c_{ijk} = 2\,\Gamma_{jki}$. Based on this theorem, the existence question about canonical factorizations is answered in the following.

Corollary 1 *A linearizing canonical transformation exists, if and only if all elements of the Riemann Curvature Tensor are identically zero.*

If all elements of the curvature tensor for a mechanical system are zero, then that system is defined to be *Euclidean* in that there exists a coordinate system in which the equations of motion appear linear. The fact that the equations appeared nonlinear is due to an unfortunate choice of coordinates.

It can be seen that, without a significant reduction, the number of curvature tensors elements that need to be computed to test for existence of a canonical factorization quickly expand according to n^4, where n indicates the system degrees of freedom. To reduce the burden of computing extraneous curvature tensor components, use is made of the symmetry properties that this tensor satisfies [25]:

1. $R_{ijkl} = -R_{jikl}$ (first skew-symmetry)

2. $R_{ijkl} = -R_{ijlk}$ (second skew-symmetry)

3. $R_{ijkl} = R_{klij}$ (block symmetry)

4. $R_{ijkl} + R_{iklj} + R_{iljk} = 0$ (Bianchi's identity)

A consequence of the first and second skew-symmetry properties is that $R_{ijkl} = 0$ if $i = j$ or $k = l$. Using the above symmetry properties the number of non-redundant or non-zero curvature tensor components can be reduced to a minimal set. It is shown in [25] that for an n-dimensional configuration space there exist a total of $n^2\,(n^2-1)\,/12$ independent and non-zero components of the curvature tensor. For example, for a 2-dimensional configuration space one only needs to compute one curvature component R_{1212}. Similarly, for a 3-dimensional space the number increases to 6 components R_{1212}, R_{1213}, R_{1223}, R_{1313}, R_{1323}, R_{2323}. Finally, to solve for the transformation requires integrating the $n(n+1)/2$ partial differential equations in (3.28).

Another application of the curvature tensor is in addressing the problem of optimal design of physical systems. Since this tensor is a function of not only the system states but also of the system parameters, annihilation of the tensor can also be attempted by a proper choice of physical parameters. Such an approach would generate dynamic design criteria as opposed to kinematic criteria. For example, one area where optimal design has been considered is in the field of robotic manipulators. The optimal dynamic design of manipulators has been investigated by a number of researchers. The main approach has been to redesign the manipulator structure and redistribute the mass in order to make the inertia matrix invariant [2], [51]. Link inertia redistribution was employed to minimize the configuration dependency of the system's kinetic and potential energies in [50]. Finally, [16] proposed the decomposition of the inertia metric into a Euclidean and a constant part using the concept of an imaginary robot model. Essentially all these design approaches have defined an optimal design as one that is "closest" to behaving in a linear fashion. In general, one can define the distance of a transformation from an isometry to be a measure of the defect of a mechanism from behaving in a linear fashion. This distance can be defined as:

$$\| J^{-T}(q)\, B(q)\, J^{-1}(q) \; - \; 1_{n\times n} \|$$

Minimizing this measure over the system parameters would result in an optimal design. Optimality can also be defined in terms of the curvature tensor. The optimal choice of parameters is defined as one for which all the components of the Riemann curvature tensor are zero. This requirement can also be written as the minimization of $\|R\|$, where $R = [R_{1212}, \ldots, R_{ijkl}]$ i.e. R is a vector of all the independent non-zero curvature elements. The advantages of such an approach is that it results in relaxed conditions on the inertia matrix, when compared to existing mass balancing or inertia redistribution methods, by not requiring diagonalization or invariance. In the following section, the concepts introduced in this and previous sections are illustrated via examples.

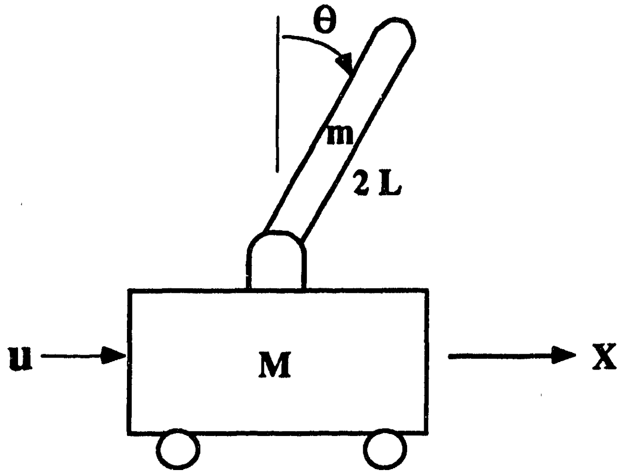

Figure 3.1: The cart-pole

3.6 Example: The Cart-Pole

In this section the methodology of linearizing point transformations is illustrated with a simple example that admits a solution. Consider the inverted pole on a cart, referred to as the cart-pole problem shown in Figure 3.1. The pole is modeled as a uniform density beam with center of mass at the pole center (i.e. at l). For this system the expression for the kinetic energy is:

$$T = \frac{1}{2}\,(\,M\,+\,m\,)\,\dot{x}^2\,+\,\frac{1}{2}\left(\frac{4}{3}\,m\,l^2\right)\dot{\theta}^2\,+\,m\,l\,\cos(\theta)\,\dot{x}\dot{\theta}$$

The inertia matrix $B(x,\theta)$ is:

$$B(x,\theta) = \begin{bmatrix} M+m & ml\,\cos(\theta) \\ ml\,\cos(\theta) & \frac{4}{3}\,ml^2 \end{bmatrix}$$

The equations of motion are as follows:

$$\begin{aligned} (\,M\,+\,m\,)\,\ddot{x}\,+\,ml\,\cos(\theta)\,\ddot{\theta}\,-\,ml\,\sin(\theta)\,\theta^2 &= u \\ \frac{4}{3}\,l\,\ddot{\theta}\,+\,\cos(\theta)\,\ddot{x}\,-\,g\,\sin(\theta) &= 0 \end{aligned}$$

Note that this is an example of an underactuated system since there is only one control action for two degrees of freedom.

The first requirement is to test for existence of a canonical factorization via the Riemann Curvature Tensor. For computational purposes, the curvature tensor was implemented in the symbolic mathematics software $Mathematica^{TM}$. Since the system is 2 dimensional there exists only one non-zero curvature tensor element, R_{1212}. Computing this component for the given inertia matrix it can be shown that:

$$
\begin{aligned}
R_{1212}[B(x,\theta)] &= \frac{\partial^2 B_{1,1}}{\partial \theta^2} + \frac{\partial^2 B_{2,2}}{\partial x^2} - \frac{\partial^2 B_{1,2}}{\partial x\, \partial \theta} - \frac{\partial^2 B_{2,1}}{\partial \theta\, \partial x} + \\
&\quad \frac{1}{2}\Big[(c_{122}\, c_{111} - c_{112}\, c_{121})\, B^{-1}_{1,1} + (c_{122}\, c_{211} - c_{112}\, c_{221})\, B^{-1}_{1,2} + \\
&\quad (c_{222}\, c_{111} - c_{212}\, c_{121})\, B^{-1}_{2,1} + (c_{222}\, c_{211} - c_{212}\, c_{221})\, B^{-1}_{2,2} \Big] \\
&= 0
\end{aligned}
$$

In the above expression the generalized coordinates have been numbered according to $q_1 = x_1$, $q_2 = \theta$. It should be noted that the Christoffel symbols are not all zero as would be the case if the inertia matrix was constant. Indeed, the only nonzero Christoffel symbol is $c_{122} = -2ml\sin(\theta)$. This result is obtained regardless of system constant parameter choices, i.e. cart mass, length of pole, and pole center of mass location. This result guarantees the existence of a canonical factorization or a coordinate system in which the inertia matrix is invariant. Note that linearization is achieved without any mass redistribution. The linearization methods based on mass or inertia redistribution e.g. [2] or [50] would require that the center of mass of the pole be placed at the pivot. In principle, this could be achieved by an appropriate selection of a counterweight.

To construct the point transformation that will result in the canonical factorization of the cart-pole inertia matrix, let the new coordinates Q_1 and Q_2 be defined by:

$$
\begin{aligned}
Q_1 &= f_{1,1}(x) + f_{1,2}(\theta) \\
Q_2 &= f_2(\theta)
\end{aligned}
$$

The reason for this particular choice is that since the inertia matrix has only 3 independent components, setting $B(x,\theta) = J^T(x,\theta)\, J(x,\theta)$ results in only three independent equations. Then, one can solve for at most three unknown transformation components, in this case $f_{1,1}(x)$, $f_{1,2}(\theta)$, and$f_2(\theta)$. The Jacobian matrix for this choice is:

$$
J(x,\theta) = \begin{bmatrix} D_x[f_{1,1}] & D_\theta[f_{1,2}] \\ 0 & D_\theta[f_2] \end{bmatrix}
$$

The expression for $J^T(x,\theta)\,J(x,\theta)$ becomes:

$$J^T(x,\theta)\,J(x,\theta) = \begin{bmatrix} D_x[f_{1,1}]^2 & D_x[f_{1,1}]\;D_\theta[f_{1,2}] \\ D_x[f_{1,1}]\;D_\theta[f_{1,2}] & D_\theta[f_{1,2}]^2 + D_\theta[f_2]^2 \end{bmatrix}$$

Setting $B(x,\theta) = J^T(x,\theta)J(x,\theta)$ results in:

(a) $D_x[f_{1,1}]^2 = M + m$

(b) $D_x[f_{1,1}]\;D_\theta[f_{1,2}] = ml\,\cos(\theta)$

(c) $D_\theta[f_{1,2}]^2 + D_\theta[f_2]^2 = \frac{4}{3}\,ml^2$

The first condition is solved to yield $f_{1,1}(x) = \sqrt{M+m}\;x$. Substituting this result in the second condition and integrating results in $f_{1,2}(\theta) = \dfrac{ml}{\sqrt{M+m}}\sin(\theta)$. Combining these two terms results in the first coordinate transformation:

$$Q_1(x,\theta) = \sqrt{M+m}\,x + \frac{ml}{\sqrt{M+m}}\sin(\theta)$$

The physical interpretation of the transformed coordinate Q_1 is that it represents, modulo a scaling constant, the center of mass of the cart-pole in the x direction. The center of mass in the x direction is:

$$x_{cm} = x + \frac{ml}{M+m}\sin(\theta)$$

Dividing Q_1 by $\sqrt{M+m}$ results in x_{cm}:

$$\begin{aligned} \frac{Q_1(x,\theta)}{\sqrt{M+m}} &= x + \frac{ml}{M+m}\sin(\theta) \\ &= x_{cm} \end{aligned}$$

The third condition, however, does not have a closed form solution. It is an elliptic integral of the second kind and must be evaluated numerically. The second coordinate transformation is then:

$$Q_2(\theta) = \int_{\theta_0}^{\theta}\sqrt{\frac{4}{3}ml^2 - \frac{m^2l^2}{M+m}\cos^2(\theta)}\;d\theta$$

The physical interpretation of Q_2 is more complicated. The term under the square root sign can be written as:

$$\frac{4}{3}ml^2 - \frac{m^2 l^2}{M+m} \cos^2(\theta) = \frac{l}{M+m} \det[B(x,\theta)]$$

The physical significance of the second coordinate transformation is apparent if one considers its time derivative $\dot{Q}_2$.

$$\dot{Q}_2 = \sqrt{\frac{4}{3}ml^2 - \frac{m^2 l^2}{M+m} \cos^2(\theta)} \ \dot{\theta}$$

The expression for $\dot{Q}_1$ is:

$$\dot{Q}_1 = \sqrt{M+m}\, \dot{x} + \frac{ml}{\sqrt{M+m}} \cos(\theta)\, \dot{\theta}$$

Then, the expression for the kinetic energy can be written in terms of $\dot{Q}_1$ as:

$$\begin{aligned} T &= \frac{1}{2}\dot{Q}_1^2 + \frac{1}{2}\left[\frac{4}{3}ml^2 - \frac{m^2 l^2}{M+m}\cos^2(\theta)\right]\dot{\theta}^2 \\ &= \frac{1}{2}\dot{Q}_1^2 + \frac{1}{2}\dot{Q}_2^2 \end{aligned}$$

It is seen that the expression for $\dot{Q}_2$ accounts for the difference $T - \frac{1}{2}\dot{Q}_1^2$.

The Jacobian matrix of this transformation

$$J(x,\theta) = \begin{bmatrix} \sqrt{M+m} & \frac{ml}{\sqrt{M+m}} \cos(\theta) \\ 0 & \sqrt{\frac{4}{3}ml^2 - \frac{m^2 l^2}{M+m} \cos^2(\theta)} \end{bmatrix}$$

is non-singular since

$$\begin{aligned} \det[J(x,\theta)] &= \sqrt{(M+m)\left(\frac{4}{3}ml^2 - \frac{m^2 l^2}{M+m}\cos^2(\theta)\right)} \\ &= \sqrt{\det[B(x,\theta)]} \end{aligned}$$

and $\det[B(\theta)] > 0$ by positive definiteness of inertia matrices for physical systems. The non-singularity of $J(x,\theta)$ immediately establishes the fact that the mapping $Q_1(x,\theta)$, $Q_2(\theta)$ is 1-1, i.e. for every Q_1, Q_2 there exists a unique x, θ. Finally, in the new coordinate system the equations of motion are linear

$$\ddot{Q} = v$$

where:

$$\begin{aligned} v &= \frac{1}{\det[J(x,\theta)]} J^{-T}(x,\theta) \begin{bmatrix} u \\ 0 \end{bmatrix} \\ &= \frac{1}{\det[J(x,\theta)]} \begin{bmatrix} \sqrt{\frac{4}{3}ml^2 - \frac{m^2l^2}{M+m}\cos^2(\theta)} & 0 \\ -\frac{ml}{\sqrt{M+m}}\cos(\theta) & \sqrt{M+m} \end{bmatrix} \begin{bmatrix} u \\ 0 \end{bmatrix} \\ &= \frac{1}{\det[J(x,\theta)]} \begin{bmatrix} \sqrt{\frac{4}{3}ml^2 - \frac{m^2l^2}{M+m}\cos^2(\theta)} \\ -\frac{ml}{\sqrt{M+m}}\cos(\theta) \end{bmatrix} u \end{aligned} \tag{3.30}$$

It is apparent that the transformed control variable v is 2-dimensional whereas the original control u was 1-dimensional. Also, for an independent choice of v_1 and v_2 there does not exist a solution for u since (3.30) represents an overconstrained set of equations. In general, the matching condition $dim\ v \leq dim\ u$ must be met for solvability of the inverse transformation. When a compensator is designed for the transformed system (Q, v) using linear design without taking into account the fact that v_1 and v_2 are not independent control actions, this compensator cannot be implemented because the actual control signal cannot be retrieved from the linear design. For this reason this approach cannot be applied to underactuated systems such as the cart-pole.

3.7 Example: Planar Manipulator

In this section, planar manipulators are investigated for the possibility of linearizing point transformations. Consider a 2-link planar manipulator with generalized coordinates defined by the absolute link angles x_1, x_2. The manipulator parameters are; m_1, m_2 link masses, I_1, I_2 link inertias, l_1, l_2 link lengths, l_{c1}, l_{c2} link center of mass locations. The inertia matrix for this manipulator is:

$$B(x_1, x_2) = \begin{bmatrix} I_1 + m_1 l_{c1}^2 + m_2 l_1^2 & l_1 l_{c2}^2 m_2 \cos(x_2 - x_1) \\ l_1 l_{c2}^2 m_2 \cos(x_2 - x_1) & I_2 + m_2 l_{c2}^2 \end{bmatrix} \tag{3.31}$$

Note that the diagonal entries of the inertia matrix are constant as a consequence of the employment of absolute instead of relative angles.

The curvature tensor for this inertia matrix is:

$$R_{1212} =$$

$$-\frac{\left(2I_1 I_2 l_1 m_2 + 2I_2 l_1 l_{c1}^2 m_1 m_2 + 2I_2 l_1^3 m_2^2 + 2I_1 l_1 l_{c2}^2 m_2^2 + 2l_1 l_{c1}^2 l_{c2}^2 m_1 m_2^2\right) l_{c2} \cos(x_2 - x_1)}{I_1 I_2 + I_2 l_{c1}^2 m_1 + I_2 l_1^2 m_2 + I_1 l_{c2}^2 m_2 + l_{c1}^2 l_{c2}^2 m_1 m_2 + l_1^2 l_{c2}^2 m_2^2 \sin^2(x_2 - x_1)}$$

It is evident that for arbitrary values of system parameters, the curvature tensor is not zero. The denominator term is the determinant of the positive definite inertia matrix and thus is nonzero. It is seen that $R_{1212} = 0, \ \forall \ x$ if and only if $l_{c2} = 0$, i.e. the center of mass of the second link is located at the second joint. Such a choice makes the inertia matrix constant since its state dependent off-diagonal terms are annihilated by this choice. The only other solution for the curvature tensor to be zero is $x_1 = x_2$ when $l_{c2} \neq 0$. Physically this corresponds to a single-link manipulator and the inertia matrix is seen to be constant. It is obvious that a general 2-link planar manipulator cannot be linearized by point transformations only.

Since the 2-link manipulator does not possess zero curvature except for a special inertia distribution or configuration, an alternative is to approximately annihilate it. The objective is to find an inertia matrix that has zero curvature and is a good approximation for the system inertia. As an example, consider the inertia matrix of the form

$$B(x_2) = \begin{bmatrix} b_{11} & b_{12}(x_2) \\ b_{12}(x_2) & b_{22}(x_2) \end{bmatrix}$$

where $b_{12}(x_2)$ and $b_{22}(x_2)$ are *arbitrary* functions of x_2, and b_{11} is an arbitrary constant. It can be shown that $R_{1212}[B(x_2)] = 0$. Such an inertia matrix is a good approximation for the 2-link manipulator. This can be seen when (3.31) is expressed in a coordinate system where the orientation of link 2, θ_2, is expressed relative to link 1:

$$\begin{aligned} B(x_1, x_2) &= B(\theta_2) \\ &= \begin{bmatrix} I_1 + I_2 + m_1 l_{c1}^2 + m_2 \left[l_1^2 + l_{c2}^2 + 2 l_2 l_{c2} \cos(x_2)\right] & I_2 + I_2 + m_2 l_{c2}^2 + m_2 l_1 l_{c2} \cos(x_2) \\ I_2 + I_2 + m_2 l_{c2}^2 + m_2 l_1 l_{c2} \cos(x_2) & m_2 l_{c2}^2 \end{bmatrix} \end{aligned}$$

The only difference is the $\cos(x_2)$ term in the $1,1$ element of (3.31). In conclusion, when this analysis is extended to spatial manipulators the curvature tensor elements will in general be non-zero.

3.8 Example: Euler's Rotational Equations

Consider the rotational motion of a rigid body. The orientation of the body is expressed in terms of Euler angles, and the body fixed frame of reference is aligned with the principal inertia axes. Let x represent the vector of Euler angles (generalized coordinates, in yaw, pitch, roll sequence) and ω the angular rate vector expressed in body coordinates. The control torque u is expressed along the generalized coordinates. The Lagrangian identical to the kinetic energy of the body is given by

$$L = T = \frac{1}{2}\,\omega^T I_B\,\omega$$

where $I_B = diag(I_{ii})$ is the constant diagonal inertia matrix. The angular rates in terms of the Euler angles are given by:

$$\omega = A(x)\,\dot{x} \tag{3.32}$$

where

$$A(x) = \begin{bmatrix} 1 & 0 & -\sin(x_2) \\ 0 & \cos(x_1) & \sin(x_1)\cos(x_2) \\ 0 & -\sin(x_1) & \cos(x_1)\cos(x_2) \end{bmatrix}$$

It is implicitly assumed that the system is restricted from attaining the orientation $x_2 = \pm\,\pi/2$ for which $A(x)$ is singular. Now the kinetic energy can be expressed in terms of generalized coordinates,

$$T = \frac{1}{2}\, \dot{x}^T B(x)\, \dot{x}$$

where $B(x) = A^T(x) I_B\, A(x)$ is the composite inertia matrix in terms of generalized coordinates. The complete equations of motion for this system are

$$\begin{aligned} \dot{x} &= A^{-1}(x)\,\omega & &\text{Kinematic equations} \\ I_B\,\dot{\omega} &= I_B\,\omega \times \omega \;+\; \tau & &\text{Dynamic equations} \end{aligned}$$

where τ represent the external forces on the body.

Applying the results obtained in the previous section, it is required that B can be factored as:

$$B(x) = J^T(x) J(x) \tag{3.33}$$

One possible choice for J could be obtained from the decomposing $I_B = K_B^T K_B$, such that:

$$J(x) = K_B A(x) \tag{3.34}$$

However, the factorization (3.34) cannot be accomplished because $A(x)$ is not a Jacobian of any function. This can be shown by applying the equality of mixed partials test. For example, consider the case $j = 1,\ i = 2,\ k = 3$:

$$\frac{\partial A_{1,2}(x)}{\partial x_3} = 0 \neq \frac{\partial A_{1,3}(x)}{\partial x_2} = -\cos(x_2)$$

except when $x_2 = \pm\, \pi/2$.

To test for the existence of a canonical factorization the Riemann Curvature Tensor test is applied to $B(x)$. Since the inertia matrix is 3 by 3, there exist 6 non-zero and independent tensor components. These six components are presented in the following.

$$\begin{aligned} &\mathbf{R1212} * (I22\ I33) = \\ &\quad -0.5 I11^2 I22 \cos(x(1))^2 + \left(1. I11\, I22^2 - 1. I11\, I22\, I33\right) \cos(x(1))^4 + \\ &\quad \left(-0.5 I22^3 - 1. I22^2 I33 + 1.5 I22\, I33^2\right) \cos(x(1))^6 - 0.5 I11^2 I33 \sin(x(1))^2 + \\ &\quad \left(1. I11\, I22^2 - 2. I11\, I22\, I33 + 1. I11\, I33^2\right) \cos(x(1))^2 \sin(x(1))^2 + \\ &\quad \left(-1. I22^3 - 0.5 I22^2 I33 + 2 I22\, I33^2 - 0.5 I33^3\right) \cos(x(1))^4 \sin(x(1))^2 + \\ &\quad \left(-1. I11\, I22\, I33 + 1. I11\, I33^2\right) \sin(x(1))^4 + \end{aligned}$$

$$\left(-0.5I22^3 + 2I22^2 I33 - 0.5I22\,I33^2 - 1.I33^3\right)\cos(x(1))^2\sin(x(1))^4 +$$
$$\left(1.5I22^2 I33 - 1.I22\,I33^2 - 0.5I33^3\right)\sin(x(1))^6$$

R1213 $*\,(I22\ I33)\ =$

$$\left(-0.5I11^2 I22 + 0.5I11^2 I33\right)\cos(x(1))\cos(x(2))\sin(x(1)) +$$
$$\left(1.I11\,I22^2 - 1.I11\,I33^2\right)\cos(x(1))^3\cos(x(2))\sin(x(1)) +$$
$$\left(-0.5I22^3 - 2.5I22^2 I33 + 2.5I22\,I33^2 + 0.5I33^3\right)\cos(x(1))^5\cos(x(2))\sin(x(1)) +$$
$$\left(1.I11\,I22^2 - 1.I11\,I33^2\right)\cos(x(1))\cos(x(2))\sin(x(1))^3 +$$
$$\left(-1.I22^3 - 5.I22^2 I33 + 5.I22\,I33^2 + 1.I33^3\right)\cos(x(1))^3\cos(x(2))\sin(x(1))^3 +$$
$$\left(-0.5I22^3 - 2.5I22^2 I33 + 2.5I22\,I33^2 + 0.5I33^3\right)\cos(x(1))\cos(x(2))\sin(x(1))^5$$

R1223 $*\,(I22\ I33)\ =$

$$-0.5I11^2 I22\cos(x(1))^2\sin(x(2)) + \left(1.I11\,I22^2 - I11\,I22\,I33\right)\cos(x(1))^4\sin(x(2)) +$$
$$\left(-0.5I22^3 - I22^2 I33 + 1.5I22\,I33^2\right)\cos(x(1))^6\sin(x(2)) -$$
$$0.5I11^2 I33\sin(x(1))^2\sin(x(2)) +$$
$$\left(1.I11\,I22^2 - 2I11\,I22\,I33 + 1.I11\,I33^2\right)\cos(x(1))^2\sin(x(1))^2\sin(x(2)) +$$
$$\left(-1.I22^3 - 0.5I22^2 I33 + 2.I22\,I33^2 - 0.5I33^3\right)\cos(x(1))^4\sin(x(1))^2\sin(x(2)) +$$
$$\left(-(I11\,I22\,I33) + 1.I11\,I33^2\right)\sin(x(1))^4\sin(x(2)) +$$
$$\left(-0.5I22^3 + 2.I22^2 I33 - 0.5I22\,I33^2 - 1.I33^3\right)\cos(x(1))^2\sin(x(1))^4\sin(x(2)) +$$
$$\left(1.5I22^2 I33 - I22\,I33^2 - 0.5I33^3\right)\sin(x(1))^6\sin(x(2))$$

R1313 $*\,(I22\ I33)\ =$

$$-0.5I11^2 I33\cos(x(1))^2\cos(x(2))^2 - \left(1.I11\,I22\,I33 - 1.I11\,I33^2\right)\cos(x(1))^4\cos(x(2))^2 +$$
$$\left(1.5I22^2 I33 - 1.I22\,I33^2 - 0.5I33^3\right)\cos(x(1))^6\cos(x(2))^2 -$$
$$0.5I11^2 I22\cos(x(2))^2\sin(x(1))^2 +$$
$$\left(1.I11\,I22^2 - 2.I11\,I22\,I33 + 1.I11\,I33^2\right)\cos(x(1))^2\cos(x(2))^2\sin(x(1))^2 +$$
$$\left(-0.5I22^3 + 2I22^2 I33 - 0.5I22\,I33^2 - 1.I33^3\right)\cos(x(1))^4\cos(x(2))^2\sin(x(1))^2 +$$
$$\left(1.I11\,I22^2 - 1.I11\,I22\,I33\right)\cos(x(2))^2\sin(x(1))^4 +$$

$$\left(-1.I22^3 - 0.5I22^2 I33 + 2I22\,I33^2 - 0.5I33^3\right)\cos(x(1))^2\cos(x(2))^2\sin(x(1))^4 +$$
$$\left(-0.5I22^3 - 1.I22^2 I33 + 1.5I22\,I33^2\right)\cos(x(2))^2\sin(x(1))^6$$

R1323 $*\,(I22\ \ I33)\ =$

$$\left(-0.5I11^2 I22 + 0.5I11^2 I33\right)\cos(x(1))\cos(x(2))\sin(x(1))\sin(x(2)) +$$
$$\left(1.I11\,I22^2 - 1.I11\,I33^2\right)\cos(x(1))^3\cos(x(2))\sin(x(1))\sin(x(2)) +$$
$$\left(-0.5I22^3 - 2.5I22^2 I33 + 2.5I22\,I33^2 + 0.5I33^3\right)\cos(x(1))^5\cos(x(2))\sin(x(1))\sin(x(2))$$
$$+\left(1.I11\,I22^2 - 1.I11\,I33^2\right)\cos(x(1))\cos(x(2))\sin(x(1))^3\sin(x(2)) +$$
$$\left(-1.I22^3 - 5.I22^2 I33 + 5.I22\,I33^2 + 1.I33^3\right)\cos(x(1))^3\cos(x(2))\sin(x(1))^3\sin(x(2)) +$$
$$\left(-0.5I22^3 - 2.5I22^2 I33 + 2.5I22\,I33^2 + 0.5I33^3\right)\cos(x(1))\cos(x(2))\sin(x(1))^5\sin(x(2))$$

R2323 $*\,(I11\ \ I22\ \ I33)\ =$

$$1.5I11^2 I22\,I33\cos(x(1))^4\cos(x(2))^2 +$$
$$\left(-1.I11\,I22^2 I33 - 1.I11\,I22\,I33^2\right)\cos(x(1))^6\cos(x(2))^2 +$$
$$\left(-0.5I22^3 I33 + 1.I22^2 I33^2 - 0.5I22\,I33^3\right)\cos(x(1))^8\cos(x(2))^2 +$$
$$3.I11^2 I22\,I33\cos(x(1))^2\cos(x(2))^2\sin(x(1))^2 +$$
$$\left(-3.I11\,I22^2 I33 - 3.I11\,I22\,I33^2\right)\cos(x(1))^4\cos(x(2))^2\sin(x(1))^2 +$$
$$\left(-2.I22^3 I33 + 4.I22^2 I33^2 - 2.I22\,I33^3\right)\cos(x(1))^6\cos(x(2))^2\sin(x(1))^2 +$$
$$1.5I11^2 I22\,I33\cos(x(2))^2\sin(x(1))^4 +$$
$$\left(-3.I11\,I22^2 I33 - 3.I11\,I22\,I33^2\right)\cos(x(1))^2\cos(x(2))^2\sin(x(1))^4 +$$
$$\left(-3.I22^3 I33 + 6.I22^2 I33^2 - 3.I22\,I33^3\right)\cos(x(1))^4\cos(x(2))^2\sin(x(1))^4 +$$
$$\left(-1.I11\,I22^2 I33 - 1.I11\,I22\,I33^2\right)\cos(x(2))^2\sin(x(1))^6 +$$
$$\left(-2.I22^3 I33 + 4.I22^2 I33^2 - 2.I22\,I33^3\right)\cos(x(1))^2\cos(x(2))^2\sin(x(1))^6 +$$
$$\left(-0.5I22^3 I33 + 1.I22^2 I33^2 - 0.5I22\,I33^3\right)\cos(x(2))^2\sin(x(1))^8 -$$
$$0.5I11^3 I22\cos(x(1))^2\sin(x(2))^2 +$$
$$\left(1.I11^2 I22^2 - 1.I11^2 I22\,I33\right)\cos(x(1))^4\sin(x(2))^2 +$$
$$\left(-0.5I11\,I22^3 - 1.I11\,I22^2 I33 + 1.5I11\,I22\,I33^2\right)\cos(x(1))^6\sin(x(2))^2 -$$
$$0.5I11^3 I33\sin(x(1))^2\sin(x(2))^2 +$$

$$\left(1.I11^2 I22^2 - 2.I11^2 I22\, I33 + 1.I11^2 I33^2\right) \cos(x(1))^2 \sin(x(1))^2 \sin(x(2))^2 +$$
$$\left(-1.I11\, I22^3 - 0.5 I11\, I22^2 I33 + 2.I11\, I22\, I33^2 - 0.5 I11\, I33^3\right) \cos(x(1))^4 \sin(x(1))^2 \sin(x(2))^2 +$$
$$\left(-1.I11^2 I22\, I33 + 1.I11^2 I33^2\right) \sin(x(1))^4 \sin(x(2))^2$$
$$+ \left(-0.5 I11\, I22^3 + 2.I11\, I22^2 I33 - 0.5 I11\, I22\, I33^2 - 1.I11\, I33^3\right) \cos(x(1))^2 \sin(x(1))^4 \sin(x(2))^2$$
$$+ \left(1.5 I11\, I22^2 I33 - 1.I11\, I22\, I33^2 - 0.5 I11\, I33^3\right) \sin(x(1))^6 \sin(x(2))^2$$

It is readily apparent that the computation of the curvature components is far from trivial. Note, however, that these expressions already reflect a significant simplifying assumption of a diagonal inertia matrix. For a non-diagonal inertia the expressions are even more complicated, i.e. there are more terms in the expression for each tensor element. It is evident that the tensor elements are non-zero and thus a linearizing canonical factorization does not exist for this general diagonal inertia distribution.

Finally, consider the simplified problem of axi-symmetric inertia distribution, i.e. diagonal inertia matrix with all diagonal entries of the same magnitude say K. The 6 curvature components for this case reduce to:

$$\begin{aligned}
R_{1212} &= -0.5\, K \\
R_{1213} &= 0 \\
R_{1223} &= -0.5\, \sin(x_2)\, K \\
R_{1313} &= -0.5\, \cos^2(x_2)\, K \\
R_{1323} &= 0 \\
R_{2323} &= -0.5\, K
\end{aligned}$$

It is obvious that the curvature components are not all zero. Also, there does not exist a constant choice of K that would annihilate all curvature components. It is interesting to note, however, that such a choice of inertia distribution results in linear Euler's dynamical equations in body frame. To see this, recall that the dynamics expressed in the body frame which is aligned with the principal axes are [8]:

$$\begin{aligned}
I_1\, \dot{\omega}_1 + (I_3 - I_2)\, \omega_2\, \omega_3 &= \tau_1 \\
I_2\, \dot{\omega}_2 + (I_1 - I_3)\, \omega_3\, \omega_1 &= \tau_2 \\
I_3\, \dot{\omega}_3 + (I_2 - I_1)\, \omega_1\, \omega_2 &= \tau_3
\end{aligned}$$

Setting $I_1 = I_2 = I_3 = K$ results in the annihilation of all Euler torques (e.g. $(I_3 - I_2)\ \omega_2\, \omega_3$) which leads to linear dynamics. However, the kinematics are still nonlinear. It is somewhat surprising that the key (or obstacle) to linearizing the rotational equations of a rigid body via point transformations lie in the nonintegrable relationship between body angular rates and generalized coordinate rates. Indeed, it is not possible to find a set of attitude parameters which specify the orientation of a rigid-body and simultaneously have ω as their time derivative [13].

3.9 Conclusion

In this chapter linearizing (canonical) point transformations were introduced as a means to simplify nonlinear dynamical systems for control system design. This change of coordinates in the state and control space coupled with feedback applied to the nonlinear systems results in a linear system to which existing results in linear control theory can be applied. One approach to achieve a target double integrator linear design was through the use of point transformations. The well known properties of point transformations, preservation of Lagrange's equations of motion and the fact that *all* point transformations are canonical, were reviewed. An alternative derivation of an existing result on the special decomposition of the inertia matrix to achieve the target double integrator linear system was presented. The Riemann Curvature Tensor was introduced as a computational tool to test for the existence of the special decomposition. Finally, this approach was illustrated by three examples. For the cart-pole problem it was shown that such a decomposition was possible and the linearizing transformation was computed. Since this problem was an example of an underactuated system, this approach cannot be used to design compensators because it requires a fully actuated model. It was shown that a general 2-link planar manipulator does not admit such a decomposition. For the rotational equations of motion of a rigid body it was shown that the curvature condition was violated for a constant diagonal inertia distribution.

Chapter 4

Approximate Linearization

In the design of control systems there are situations in which nonlinear terms cannot be ignored. A common approach is to linearize the nonlinear dynamics and apply linear control theory to design appropriate controllers. However, the operating region of a linearized design is limited. When the system is required to operate over a wide range of conditions and meet high performance requirements linear controller designs are inadequate. These type of conditions may be encountered in 6 degree-of-freedom underwater vehicle control, sattelite attitude control, high angle-of-attack aircraft control, and magnetic field applications.

A further complication arises when the system to be controlled is underactuated. This situation occurs when there are fewer control actuators than degrees of freedom. For these types of systems a general control design methodology does not exist. Most established design methods require "square" systems [41], that is systems where the number of control actuators equals the degrees of freedom of the system. Examples of "square" design methodologies are Sliding Mode control, Lyapunov based Adaptive control, and "computed-torque" control [41]. For gain-scheduled linear controllers there exist potential stability problems when the scheduling variable is "fast" and are limited to systems which are nonlinear in only a few states.

A recent development in nonlinear control design is that of feedback linearization which transforms the nonlinear system to an equivalent controllable linear system. This is accomplished via a nonlinear state and control transformation. Subsequently, linear control theory can be applied to the equivalent linear system to design appropriate compensation. However, when this approach is applied to underactuated systems, existence conditions are usually not satisfied. One approach is to extend the operating region of linear designs by constructing linear approximations accurate to second or higher order [28]. Another approach is to approximate the nonlinear dynamics by a linearizable nonlinear system [28], and this approach will be used to solve the underactuated control problem.

In Section 4.1, the concept of feedback equivalence is introduced as a precursor to exact feedback linearization. The operations of state and control transformations are used to define an equivalence class of linearizable nonlinear systems. Preliminary mathematical concepts required for exposition of exact linearization are presented in Section 4.2. In Section 4.3, the method of exact feedback linearization is reviewed in abridged form.

In Section 4.4, the method of extended feedback linearization is reviewed. A computational approach to test for the order of an involutive distribution is derived. In Section 4.5, approximate feedback linearization is reviewed. Instead of solving for an exactly linearizing output function, an approximate output function is computed that transforms the nonlinear system to a certain order linear system. In section 4.6 this approach is applied to the cart-pole problem. Simulation results show a substantial improvement in the range of the linear control design.

4.1 Feedback Equivalence

Before proceeding to the subject of feedback linearization, the notion of feedback equivalence is introduced. Feedback equivalence is intimately connected with linearization of nonlinear systems in that the concept central to feedback linearization is the generation of solutions to the nonlinear system via the solution to an equivalent linear system. The most general application of feedback equivalence involves nonlinear transformations of the state and control variables. First, feedback equivalence for linear systems will be reviewed, and then extended to nonlinear systems. It is hoped that by taking this approach to introducing feedback linearization the subject will become transparent to the reader.

For linear systems, it is known that any controllable linear system can be transformed into a controller canonical form via a linear coordinate change of the system state. Further, if a linear coordinate change in the input variable is coupled with linear state feedback, any controllable linear system can be transformed to a special form where all poles of the tranformed system are at the origin [24]. Such a representation is usually refered to as a *Brunovsky canonical form* after its originator [6]. To establish notation, let the state and input transformations and feedback be defined by

$$\begin{aligned} y &= T\,x \\ v &= \alpha\,x + \beta\,u \end{aligned} \tag{4.1}$$

where $y \in \mathcal{R}^n$ and $v \in \mathcal{R}$ are respectively the new state and control variables, T is a nonsingular $n \times n$ constant real matrix, α is a $1 \times n$ row vector and β is a nonzero real constant. This result can be summarized in the following theorem [24].

Theorem 4 *Consider the single input, time-invariant controllable linear system:*

$$\dot{x} = A\,x + B\,u$$

Using the transformation and feedback (4.2), this linear system can be transformed to:

$$\begin{aligned} \dot{y} &= \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v \\ &= \tilde{A}\,y + \tilde{b}\,v \end{aligned} \tag{4.2}$$

Note that the characteristic polynomial is det $\left[sI - \tilde{A}\right] = s^n$, i.e. all poles of the transfomed system are at the origin and there are no system zeros. This resulting decoupled set of integrators is by no means restricted to single input linear systems [24].

This notion of feedback equivalence can naturally be extended to nonlinear systems. The objective is to characterize all nonlinear systems that are feedback equivalent to controllable linear systems [46]. Starting from the single-input nonlinear system

$$\dot{x} = f(x,u)$$

assume that it can be transformed via

$$\begin{aligned} y &= T(x) \\ v &= \gamma(x,u) \end{aligned}$$

to the linear system (4.2). Further, assume that $T(x)$ is a differentiable function with nonsingular Jacobian matrix for all x. Such a transformation is also refered to as a diffeomorphism, i.e. it is differentiable with differentiable inverse. Then, the following result establishes the class of feedback linearizable nonlinear systems [46].

Theorem 5 *If a nonlinear system is in the feedback equivalent class of linear controllable systems, then it has the form*

$$\dot{x} = f(x) + g(x)\,\phi(x,u)$$

where $f(0) = 0$, and ϕ is a scalar function with $\phi(0,0) = 0$ and $\dfrac{\partial \phi}{\partial u} \neq 0 \;\forall x$.

The reason for the requirement $f(0) = 0$ and $\phi(0,0) = 0$ is that the origin (in the state and control space) is an equilibrium point for both the linear and nonlinear systems. It is just an origin matching condition. The final condition $\dfrac{\partial \phi}{\partial u} \neq 0 \;\forall x$ is required for retrieving u from ϕ. It is just the conditiont for the inverse function theorem [37]. In particular, this result can be specialized to the following useful and practical defintion:

Definition 10 *Consider the single-input nonlinear systems affine in the control variable*

$$\dot{x} = f(x) + g(x)\,u \tag{4.3}$$

where $f(x)$ and $g(x)$ are smooth vector fields, $x \in \mathcal{R}^n$, $u \in \mathcal{R}$, and $f(0) = 0$. Then, (4.3) is feedback equivalent to a controllable linear system if there exist a region $\Omega \in \mathcal{R}^n$ containing the origin, state and input transformations and feedback

$$\begin{aligned} y &= T(x) \\ v &= \alpha(x) + \beta(x)\,v \end{aligned} \tag{4.4}$$

where $T(x)$ is a diffeomorphism and $\beta(x) \neq 0$ for $x \in \Omega$, such that the transformed coordinates satisfy (4.2).

If such a transformation exists, (4.3) is said to be *feedback linearizable.* Note that the assumption $f(0) = 0$ is not necessary. It is an origin matching condition, i.e. the origin is an equilibrium point for both systems. In practice the only requirement for $T(x)$ is that it be a differentiable mapping with nonsingular Jacobian matrix. It is seen that (4.5) is just a nonlinear generalization of (4.2).

Before proceeding to the next section, it may be worthwhile to pause and reflect on this notion of feedback equivalence. First, Definition 1 and Theorem 2 above specify the class of nonlinear systems which are feedback equivalent to controllable linear systems. Second, the mechanism by which this is accomplished is the following. The transformation $T(x)$ can be thought of as a nonlinear coordinate transformation where the remaining nonlinearities after athe transformation are shifted or "pushed" such that they only appear in the derivative of the last transformed variable. Note that the nonlinearities remaining after the state transformation have been placed in the path of the control action and thus can be cancelled. Another property of the state transformation is that the control variable does not appear except in the derivative of the last transformed variable. Once this has been accomplished, the original control action can cancel these nonlinearities and inject the transformed control variable for compensator design in the transformed linear domain.

4.2 Preliminary Mathematical Concepts

Before proceeding with the development of feedback linearization, a few preliminary mathematical tools are introduced in this section. Only the essential definitions required for the exposition of feedback linearization will be considered here. In the following, various operations on scalar (e.g. $h(x) : \mathcal{R}^n \to \mathcal{R}$) and vector functions (e.g. $f(x) : \mathcal{R}^n \to \mathcal{R}^n$) will be defined. Such functions are also refered to as *fields* in that a scalar or vector assignment (or map) is made at every point x. Also, it will be usual to assume that these fields are smooth, that is they admit continuous partial derivatives of arbitrary higher order.

The first concept is that of the Lie derivative or directional derivative.

Definition 11 *For a smooth scalar field $h(x) : \mathcal{R}^n \to \mathcal{R}$ and a smooth vector field $f(x) : \mathcal{R}^n \to \mathcal{R}^n$, the Lie derivative of h with respect to f, denoted by $L_f h$, is a new scalar field defined by:*

$$\begin{aligned} L_f\, h &= \nabla h(x)\, f(x) \\ &= \sum_{i=1}^{n} \frac{\partial h(x)}{\partial x_i}\, f_i(x) \end{aligned}$$

The notation $\nabla h(x)$ indicates the gradient of $h(x)$. It is evident that the Lie derivative is just the directional derivative of the scalar function $h(x)$ in the direction $f(x)$. Repeated Lie derivatives are defined recursively by:

$$\begin{aligned} L_f^0\, h &= h \\ L_f^i\, h &= L_f\, \left(L_f^{i-1}\, h \right) \end{aligned}$$

Next consider an operator on vector fields called the Lie bracket.

Definition 12 *For two vector fields $f(x) : \mathcal{R}^n \rightarrow \mathcal{R}^n$ and $g(x) : \mathcal{R}^n \rightarrow \mathcal{R}^n$, the Lie bracket of f and g, denoted by $[\, f,\, g\,]$, is a new vector field defined by:*

$$[\, f,\, g\,] = D_x[\, g\,]\, f - D_x[\, f\,]\, g$$

The notation $D_x[\,]$ is used to denote the derivative operator, i.e. $D_x[\, f(x)\,] = \dfrac{\partial f(x)}{\partial x}$. The Lie bracket satisfies the skew-symmetric property, i.e. $[\, f,\, g\,] = -[\, g,\, f\,]$ and the Jacobi identity [41]:

$$L_{ad_f\, g}\, h = L_f\, L_g\, h - L_g\, L_f\, h \tag{4.5}$$

A geometric interpretation of the Lie bracket is presented in Section 4.2.1. Iterated or repeated bracket operations are defined in terms of the notation $ad_f\, g = [\, f,\, g\,]$:

$$\begin{aligned} ad_f^0\, g &= g \\ ad_f^i\, g &= \left[\, f,\, ad_f^{i-1}\, g \right] \end{aligned}$$

Finally, the last mathematical construct is the notion of involutive vector fields which is required in order to establish integrability of certain vector fields.

Definition 13 *A linearly independent set of vector fields $\{\, X_1(x),\, \ldots,\, X_m(x)\,\}$ is said to be involutive if and only if there are scalar fields $\alpha_{ijk}(x)$ such that*

$$[\, X_i,\, X_j\,](x) = \sum_{k=1}^{m} \alpha_{ijk}(x)\, X_k(x)$$

for all i, j.

Since the Lie bracket is skew-symmetric, when testing for involutivity one need only consider values of $i < j$. A simple example of involutivity is that of constant vector fields. The Lie bracket of such vectors is zero, and a trivial solution for the weighting coefficients is $\alpha_{ijk}(x) = 0$. Integrability of vector fields is established by the following definition [46].

Definition 14 *A linearly independent set of vector fields* $\{ X_1(x), \ldots, X_m(x) \}$ *on* $\mathcal{R}^n$ *is said to be completely integrable if and only if there exist* $n - m$ *linearly independent scalar functions* $h_1(x), \ldots, h_{n-m}(x)$ *such that:*

$$L_{X_j(x)}\, h_i(x) \;=\; 0 \qquad \textit{for } 1 \le i \le n-m,\; 1 \le j \le m$$

The connection between involutivity and complete integrability of linearly independent vector fields (which will be required in the construction of the linearizing transformation) is given by the classical Frobenius theorem [4].

Theorem 6 *A set of linearly independent vector fields is completely integrable if and only if it is involutive.*

To provide an intuitive understanding of the involutivity condition, the equivalence of the equality of mixed partial derivatives and involutivity as integrability criteria is highlighted by a simple example in Section 4.2.2.

4.2.1 Geometrical Interpretation Of The Lie Bracket

In this section a geometrical interpretation of the Lie bracket is presented with the purpose of providing an intuitive understanding. In the context of differential equations, it will be shown that the bracket represents a direction in state space that the solution of these equations can move. To give a geometrical interpretation to the Lie bracket, consider the following two input dynamical system:

$$\dot{x} \;=\; u_1(t)\, f_1(x) \;+\; u_2(t)\, f_2(x) \tag{4.6}$$

It is evident that starting from any point $x(0) = x_0$, the system can move in any direction spanned by the vectorfields $f_1(x)$ and $f_2(x)$, denoted $F(x_0) = span\,\{f_1(x_0), f_2(x_0)\}$. However, it may be possible to move in other directions by switching between inputs. Suppose that starting from $x(0)$ the system moves along $f_1(x)$ for t units of time, then along $f_2(x)$ for t units, then along $-f_1(x)$ for t units and finally along $-f_2(x)$ for t units of time. This path is shown in Figure 4.1. After this circuit has been made, is the difference between the starting and terminal points $x(4t) - x(0) \in F(x(0))$? If this direction is not a linear combination of $f_1(x_0)$ and $f_2(x_0)$ then it represents a new direction in which the solution can move.

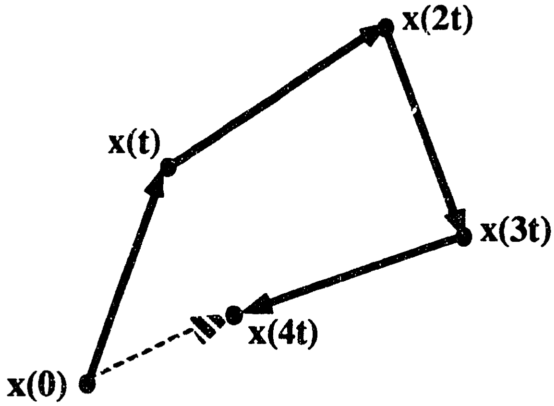

Figure 4.1: Solution trajectory for Lie bracket interpretation

To compute the terminal point along this circuit, first set $u_1 = 1,\ u_2 = 0$ for t units of time. Using a Taylor series expansion, the solution to (4.6) can be written as:

$$\begin{aligned} x(t) &= x(0) + \dot{x}(0)\,t + \frac{1}{2}\ddot{x}(0)\,t^2 + \mathcal{O}^3(t) \\ &= x(0) + f_1(x(0))\,t + \frac{1}{2}\frac{\partial f_1(x(0))}{\partial x} f_1(x(0))\,t^2 + \mathcal{O}^3(t) \end{aligned}$$

To reduce the notational complexity, in the following all functions are evaluated at $x(0)$ unless stated otherwise. Next, set $u_1 = 0,\ u_2 = 1$ for t units of time to obtain (using a Taylor series expansion):

$$x(2t) = x(t) + f_2(x(t))\,t + \frac{1}{2}\frac{\partial f_2(x(t))}{\partial x} f_2(x(t))\,t^2 + \mathcal{O}^3(t)$$

It is desired to express $x(2t)$ in terms of the starting point. To accomplish this expand the vector fields evaluated at t in a series using the previous series expression for $x(t)$:

$$\begin{aligned} f_2(x(t)) &= f_2(x(0)) + \frac{\partial f_2(x(0))}{\partial x}(x(t) - x(0)) + \mathcal{O}^2(x(t) - x(0)) \\ &= f_2 + \frac{\partial f_2}{\partial x} f_1\,t + \mathcal{O}^2(t) \\ \frac{\partial f_2(x(2t))}{\partial x} &= \frac{\partial f_2}{\partial x} + \mathcal{O}(t) \end{aligned}$$

Substituting the above in the expression for $x(2t)$ and using the series expansion for $x(t)$ obtained previously:

$$
\begin{aligned}
x(2t) &= x(t) + \left[f_2 + \frac{\partial f_2}{\partial x} f_1\, t \right] t + \frac{1}{2}\frac{\partial f_2}{\partial x} f_2\, t^2 + \mathcal{O}^3(t) \\
&= x(0) + f_1\, t + \frac{1}{2}\frac{\partial f_1}{\partial x} f_1\, t^2 + \left[f_2 + \frac{\partial f_2}{\partial x} f_1\, t \right] t + \frac{1}{2}\frac{\partial f_2}{\partial x} f_2\, t^2 + \mathcal{O}^3(t) \\
&= x(0) + [\, f_1 + f_2 \,]\, t + \left[\frac{1}{2}\frac{\partial f_1}{\partial x} f_1 + \frac{\partial f_2}{\partial x} f_1 + \frac{1}{2}\frac{\partial f_2}{\partial x} f_2 \right] t^2 + \mathcal{O}^3(t)
\end{aligned}
$$

Next, set $u_1 = -1$, $u_2 = 0$ for t units of time and using the series expansion for $x(2t)$ obtained previously:

$$
\begin{aligned}
x(3t) &= x(2t) - f_1(x(2t))\, t + \frac{1}{2}\frac{\partial f_1(x(2t))}{\partial x} f_1(x(2t))\, t^2 + \mathcal{O}^3(t) \\
&= x(2t) - \left[f_1 + \frac{\partial f_1}{\partial x}(f_1 + f_2)\, t \right] t + \frac{1}{2}\frac{\partial f_1}{\partial x} f_1\, t^2 + \mathcal{O}^3(t) \\
&= x(0) + f_2\, t + \left[\frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2 + \frac{1}{2}\frac{\partial f_2}{\partial x} f_2 \right] t^2 + \mathcal{O}^3(t)
\end{aligned}
$$

Finally, set $u_1 = 0$, $u_2 = -1$ for t units of time using the same approach:

$$
\begin{aligned}
x(4t) &= x(3t) - f_2(x(3t))\, t + \frac{1}{2}\frac{\partial f_2(x(3t))}{\partial x} f_2(x(3t))\, t^2 + \mathcal{O}^3(t) \\
&= x(3t) - \left[f_2 + \frac{\partial f_2}{\partial x} f_2\, t \right] t + \frac{1}{2}\frac{\partial f_2}{\partial x} f_2\, t^2 + \mathcal{O}^3(t) \\
&= x(0) + \left[\frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2 \right] t^2 + \mathcal{O}^3(t) \\
x(4t) &= x(0) + [\, f_1,\, f_2 \,](x_0)\, t^2 + \mathcal{O}^3(t)
\end{aligned}
$$

It is seen that the difference between the initial and terminal points to second order is:

$$
x(4t) - x(0) = [\, f_1,\, f_2 \,](x_0)\, t^2 \tag{4.7}
$$

Note that the Lie bracket is evaluated at the starting point $x(0)$. Thus, $[\, f_1,\ f_2\,](x_0)$ represents the direction in which the system can move and if it is not an element of $F(x_0)$ it represents a new direction in which the solution can move. It is evident that higher order brackets can also be defined for example by including the direction (4.7) in the switching sequence. It is seen that the Lie bracket is a measure of the commutativity of the vector fields. A common example of non-commutativity in dynamics are finite rotations of a rigid body.

The relevance of the Lie brackets in nonlinear control theory is apparent when one considers such issues as controllability and integrability of vector fields. For controllability analysis, the bracket represents a direction the solution may move along even though it may not be in the linear span of the vector fields. Another important application is in determining integrability of vector fields. That is, given a set of arbitrary vector fields is it possible to find a curve (the integral curve) such that at each point its tangent space is spanned by the given vector fields.

4.2.2 Involutivity And The Equality Of Mixed Partials

In this section, the connection between the involutivity condition and the equality of mixed partial derivatives is highlighted using a simple example of first order partial differential equations. It will be shown that the equality of mixed partials condition for integrability is identical to the involutivity requirement. This example is adapted from [43], and the presentation is developed in a similar manner. Consider the set of equations:

$$\begin{aligned} \frac{\partial z}{\partial x} &= f(x,y,z) \\ \frac{\partial z}{\partial y} &= g(x,y,z) \end{aligned} \tag{4.8}$$

A solution $z = \phi(x,y)$ is desired for this set of partial differential equations. From the equality of mixed partial derivatives, it is known that the necessary and sufficient conditions for a solution are:

$$\frac{\partial^2 \phi}{\partial x\, \partial y} = \frac{\partial^2 \phi}{\partial y\, \partial x} \iff \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

An alternative approach to establish conditions for integrability is to use the involutivity criteria. As a first step, appropriate vector fields must be determined for this problem. To accomplish this, consider the solution $z = \phi(x,y)$ as defining a surface in $\mathcal{R}^3$ with coordinates (x, y, z). This surface can be characterized by the function $\Phi(x,y) : \mathcal{R}^2 \rightarrow \mathcal{R}^3$ [43]:

$$\Phi(x,y) = \begin{bmatrix} x \\ y \\ \phi(x,y) \end{bmatrix}$$

Then, the tangent plane at each point (x, y) is spanned by the partial derivatives of Φ [43]:

$$\frac{\partial \Phi}{\partial x} = X_1 = \begin{bmatrix} 1 \\ 0 \\ f(x,y,\phi(x,y)) \end{bmatrix}$$

$$\frac{\partial \Phi}{\partial y} = X_2 = \begin{bmatrix} 0 \\ 1 \\ g(x,y,\phi(x,y)) \end{bmatrix}$$

In geometric terms, solving (4.8) is equivalent to finding a surface, the integral surface or manifold, such that at each (x, y) its tangent plane is spanned by X_1 and X_2. This requirement can be established by using the involutivity condition. For X_1 and X_2 to be involutive vector fields requires that they satisfy:

$$[\, X_1(x,y,z),\ X_2(x,y,z)\,] = \alpha_1(x,y,z)\, X_1(x,y,z) + \alpha_2(x,y,z)\, X_2(x,y,z)$$

Note that for the computation of the Lie bracket X_1 and X_2 are assumed to be functions of $\mathcal{R}^3$, that is (x, y, z). Let $w = [\, x \ \ y \ \ z\,]^T$. Computing the indicated Lie bracket:

$$\begin{aligned}
[\, X_1(x,y,z),\ X_2(x,y,z)\,] &= D_w[\, X_2(w)\,]\, X_1(w) - D_w[\, X_1(w)\,]\, X_2(w) \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ f \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ g \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \end{bmatrix}
\end{aligned}$$

Applying the involutivity condition:

$$\begin{bmatrix} 0 \\ 0 \\ \dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_1 \, f \, + \, \alpha_2 \, g \end{bmatrix} \tag{4.9}$$

From (4.9) the constraint equations are:

$$\begin{aligned} \alpha_1 &= 0 \\ \alpha_2 &= 0 \\ \alpha_1 \, f \, + \, \alpha_2 \, g &= \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \end{aligned}$$

It is evident that to satisfy involutivity, $\alpha_1 = \alpha_2 = 0$ which requires:

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$

This is exactly the equality of mixed partial derivatives. Thus, the equivalence of the involutivity condition and the equality of mixed partials is established for this simple example.

The solution to (4.8) can also be obtained by an alternative approach that has a more geometric flavor. Recall that solving (4.8) was equivalent to finding a solution surface such that its tangent plane is spanned by X_1 and X_2 at each point. Since the solution can be written as $z = \phi(x, y)$, the solution surface (or level surface) is parametrized by the function $h(x, y, z) = 0$ where:

$$h(x, y, z) \; = \; z \; - \; \phi(x, y)$$

One property of such surfaces is that its gradient or normal derivative is orthogonal to the surface at each point. Thus, at each point (x, y) the gradient of $h(x, y, z)$ is orthogonal to the tangent plane defined at that point. Since the tangent plane is spanned by X_1 and X_2:

$$\begin{aligned} \nabla h \, X_1 &= 0 \\ \nabla h \, X_2 &= 0 \end{aligned} \tag{4.10}$$

Substituting $\nabla h = [\, f \;\; g \;\; -1 \,]$ in (4.10), it is evident that it is satisfied. The existence of $h(x, y, z)$ is a necessary and sufficient condition to guarantee the integrability of the vector fields $\{ X_1, X_2 \}$ [46]. To show the existence of such a function, (4.10) can be arranged in matrix form as:

$$\begin{bmatrix} \leftarrow & X_1^T & \rightarrow \\ \leftarrow & X_2^T & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \nabla^T h \\ \downarrow \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, the required gradient function is the nullspace vector of the matrix with rows X_1 and X_2. This nullspace vector can be explicitly computed from the cross product of X_1 and X_2 since it would be orthogonal to both X_1 and X_2. Computing this cross product:

$$\nabla h = (X_1 \times X_2)^T = [-f \;\; -g \;\; 1]$$

The existence of $h(x, y, z)$ hinges on whether the expression for ∇h can be integrated. This requires solution to:

$$\frac{\partial h}{\partial x} = -f$$

$$\frac{\partial h}{\partial y} = -g$$

A solution exists if the mixed partials agree:

$$\frac{\partial^2 h}{\partial x \, \partial y} = \frac{\partial^2 h}{\partial y \, \partial x} \iff \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

Thus, it is seen that the integrability of the vector fields X_1 and X_2 is established by the application of the equality of mixed partials.

4.3 Exact Feedback Linearization

In this section the feedback linearization problem is further explored. In particular the conditions for the existence of such transformations are examined. In the case where they exist, the construction of the transformations is presented. The development of this section is restricted to single-input nonlinear systems. The main reason for this is that all of the underlying concepts and machinery are exhibited in this case, and as such, provide the simplest and most transparent introduction to the subject. The reason for the choice of the title to this section is that only transformations are considered that *exactly* render linear the map from the input to the *full* state in some region of the state space. Feedback linearization has received much attention in the last decade and has emerged as a mature nonlinear control design methodology. Originating with the work of Krener [27] (1973), and Brockett [5] (1978), this problem was completely solved by Jacubczyck and Respondek [23] (1980), Su [46] (1982), and by Hunt, Su, Meyer [21] (1983).

Starting with a nonlinear system of the form,

$$\dot{x} = f(x) + g(x)\,u \tag{4.11}$$

the objective is to transform it using a coordinate and control transformation

$$\begin{aligned} z &= T(x) \\ v &= \alpha(x) + \beta(x)\,u \end{aligned}$$

to a linear system in Brunovsky form:

$$\begin{aligned} \dot{T}_1 &= T_2 \\ \dot{T}_2 &= T_3 \\ \vdots &\;\;\vdots\;\;\; \vdots \\ \dot{T}_n &= v \end{aligned} \tag{4.12}$$

To derive the necessary and sufficient conditions for the existence of such a transformation, the approach of Su [46] is used in the following. Substituting the coordinate trasformation in (4.12):

$$\begin{aligned} \dot{T}_1 &= \frac{\partial T_1}{\partial x}\,\dot{x} \\ &= \frac{\partial T_1}{\partial x}\,f + \frac{\partial T_1}{\partial x}\,g\,u = T_2 \\ \dot{T}_2 &= \frac{\partial T_2}{\partial x}\,\dot{x} \\ &= \frac{\partial T_2}{\partial x}\,f + \frac{\partial T_2}{\partial x}\,g\,u = T_3 \end{aligned}$$

$$
\begin{aligned}
\vdots \quad & \vdots \quad \vdots \\
\dot{T}_n &= \frac{\partial T_n}{\partial x}\,\dot{x} \\
&= \frac{\partial T_n}{\partial x}\, f + \frac{\partial T_n}{\partial x}\, g\, u = v
\end{aligned}
$$

Because the state transformation is independent of the control u, to satisfy the above equations requires:

$$
\begin{aligned}
L_g\, T_i &= 0 & & \\
L_f\, T_i &= T_{i+1} & \text{for } i = 1, \ldots, n-1 \\
L_g\, T_n &\neq 0 & &
\end{aligned}
\tag{4.13}
$$

Using the Jacobi identity for Lie brackets (4.5), the conditions (4.13) can be expressed in terms of T_1 only. To see this, first consider:

$$
\begin{aligned}
L_{ad_f g}\, T_1 &= L_f\, (\, L_g\, T_1\,) - L_g\, (\, L_f\, T_1\,) \\
&= -\, L_g\, T_2 \\
&= 0
\end{aligned}
$$

In the above use is made of $L_g\, T_1 = 0$, $L_f\, T_1 = T_2$ and $L_g\, T_2 = 0$. Similarly:

$$
\begin{aligned}
L_{ad_f^2 g}\, T_1 &= L_f\, \left(L_{ad_f g}\, T_1 \right) - L_{ad_f g}\, (\, L_f\, T_1\,) \\
&= -\, L_{ad_f g}\, T_2 \\
&= -\, L_f\, (\, L_g\, T_2\,) + L_g\, (\, L_f\, T_2\,) \\
&= L_g\, T_3 \\
&= 0
\end{aligned}
$$

Continuing in this manner, the general form of this iteration is:

$$
L_{ad_f^k g}\, T_1 = (-1)^k\, L_g\, T_{k+1} \qquad \text{for } k = 0, 1, \ldots, n-2
$$

Thus, the conditions (4.13) can be imbedded in:

$$
L_{ad_f^k g}\, T_1 = 0 \qquad k = 0, \ldots, n-2 \tag{4.14}
$$

$$
L_{ad_f^{n-1} g}\, T_1 \neq 0 \tag{4.15}
$$

Equations (4.14) and (4.15) represent the necessary and sufficient conditions for feedback linearization to the Brunovsky canonical form [46]. A consequence of (4.14) is that the vector fields $\left\{ g,\ ad_f g,\ \ldots,\ ad_f^{n-1} g \right\}$ are linearly independent. To see why this must be true, first consider the case when $ad_f^{n-1} g$ is linearly dependent on the other vector fields:

$$ad_f^{n-1} g \;=\; \sum_{k=0}^{n-2} \alpha_k(x)\, ad_f^k\, g$$

Then, computing

$$L_{ad_f^{n-1} g}\, T_1 \;=\; \sum_{k=0}^{n-2} \alpha_k(x)\, L_{ad_f^k g}\, T_1 \;=\; 0$$

since $L_{ad_f^k g}\, T_1 = 0$ for $0 \le k \le n-2$ from (4.14), and thus condition (4.15) is violated. For any other vector field to be dependent, let:

$$ad_f^j g \;=\; \sum_{k=0,\ k\neq j}^{n-1} \alpha_k(x)\, ad_f^k\, g \qquad 0 \le j \le n-2$$

Then, computing

$$L_{ad_f^j g}\, T_1 \;=\; \sum_{k=0,\ k\neq j}^{n-1} \alpha_k(x)\, L_{ad_f^k g}\, T_1 \;=\; L_{ad_f^{n-1} g}\, T_1 \;\neq 0$$

since $L_{ad_f^{n-1} g}\ T_1 \neq 0$ by (4.14), and thus condition (4.15) is violated. It is concluded that to satisfy conditions (4.14) and (4.15) the vector fields $\left\{ g,\ ad_f g,\ \ldots,\ ad_f^{n-1} g \right\}$ must be linearly independent. Finally, the existence of T_1 satisfying (4.14) is guaranteed if and only if $\left\{ g,\ ad_f g,\ \ldots,\ ad_f^{n-2} g \right\}$ is involutive. Summarizing, the necessary and sufficient conditions for the existence of a feedback linearizing transformation are presented in the following theorem by Su [46]:

Theorem 7 *The nonlinear system (4.11) is input-state linearizable if, and only if, there exists a region Ω such that:*

(a) *The vector fields $\left\{ g,\ ad_f\, g,\ \ldots,\ ad_f^{n-1}\, g \right\}$ are linearly independent for $x \in \Omega$.*

(b) *The vector fields $\left\{ g,\ ad_f\, g,\ \ldots,\ ad_f^{n-2}\, g \right\}$ are involutive for $x \in \Omega$.*

To construct the linearizing transformation requires solving for $T_1(x)$ from (4.14) or:

$$\nabla T_1\, ad_f^i\, g \;=\; 0 \quad i = 0, 1, \ldots, n-2 \tag{4.16}$$

Once a solution for $T_1(x)$ is obtained, the other transformed variables can be obtained in a recursive fashion via:

$$\begin{aligned} T_k(x) &= L_f^{k-1}\, T_1(x) \quad \text{for } k = 2, \ldots, n \\ v &= L_f^k\, T_1(x) + L_g\, L_f^{k-1}\, T_1(x)\, u \end{aligned}$$

When this approach is applied to underactuated control systems, however, the involutivity condition is usually not satisfied. In general, this method is more amenable to fully actuated systems. Also, for higher dimensional nonlinear systems, evaluating the existence conditions is computationally intensive usually requiring symbolic mathematics software. Even when the existence conditions are satisfied, finding a solution requires solving a set of partial differential equations. Implementing a control design based on this methodology also requires full state information. Finally, due to the complicated transformations which may have no physical meaning, the linear compensator design process is complicated. An alternative is not to globally linearize the dynamics, but rather, to expand the region of applicability of a linear model. This approach is presented in the next section.

4.4 Approximate Feedback Linearization: Version I

In this section, the topic of approximate feedback linearization is presented. As opposed to exact feedback linearization which attempts to transform a nonlinear system to a linear system in some region without any error, extended feedback linearization, as formulated by Krener in [28] (1984), [29] (1987) attempts to construct a linear approximation about an equilibrium point accurate to second or higher order. This method can be implemented in two distinct ways, since the proof of the method [28] provides another approach by which the approximation can be constructed. For this reason, in this presentation the higher order linear approximation is refered to as Version I, while the method of the proof which constructs a linearizable nonlinear approximation is refered to as Version II. A benefit of this approach is that the necessary and sufficient conditions are similar to those for exact linearization, yet are much less stringent, and the computation of the requisite transformation requires a solution to a set of linear algebraic equations instead of solving first order partial differential equations. Another advantage is that the target linear system is not the Brunovsky form but rather the first order linearization of the system about an equilibrium point. This method can be thought of as bridging the gap between standard linearization (i.e. first order approximation) and exact linearization.

For this presentation of approximate linearization only single-input nonlinear systems affine in the control variable will be considered. It should be noted that the theory has been developed for multi-input nonlinear systems [28]. The restriction to single-input systems is made to simplify this presentation. Consider the following nonlinear system

$$\dot{x} \; = \; f(x) \; + \; g(x) \, u$$

where $x \in \mathcal{R}^n$. The system is assumed to be in equilibrium for $x^*, u^* = 0$, i.e. $\dot{x}(x^*, u^*) = 0 \implies f(x^*) = 0$. The restriction of the equilibrium control action $u^* = 0$ can be relaxed to include non-zero steady state control action when it is desired to apply this approach about a non-equilibrium point. Towards this end, consider the state and control variable transformation

$$\begin{aligned} z &= T(x) \\ v &= \alpha(x) \; + \; \beta(x) \, u \end{aligned}$$

which transforms the nonlinear system to the order p approximate linear system

$$\dot{z} \; = \; A \, z \; + \; b \, v \; + \; \mathcal{O}^{p+1}(\tilde{x}, u) \tag{4.17}$$

where:

$$A \; = \; \left. \frac{\partial f(x)}{\partial x} \right|_{x=x^*} \qquad b \; = \; g(x^*) \qquad \tilde{x} \; = \; x \; - \; x^*$$

Here the order of the approximation error $\mathcal{O}^{p+1}(\tilde{x}, u)$ implies errors of $\mathcal{O}^{p+1}(\tilde{x})$ or $\mathcal{O}^{p}(\tilde{x}) \, u$.

As formulated by Krener [28], the general approach to realizing the linear system is to expand the state and control transformations in a Taylor series about the equilibrium point. The next step is to compute the constant coefficients in these series expansions by direct substitution in the equations of motion. As an example, consider a second-order approximation for a scalar, single-input nonlinear system of the form

$$\dot{x} \; = \; f(x) \; + \; g(x) \, u \tag{4.18}$$

with $f(x^*) = 0$, $u^* = 0$ and $x, u \in \mathcal{R}$. It is desired to transform (4.18) to the order 2 approximate linear system,

$$\dot{z} \; = \; a \, z \; + \; b \, v \; + \; \mathcal{O}^3(\tilde{x}, u) \tag{4.19}$$

where:

$$a = \left.\frac{\partial f(x)}{\partial x}\right|_{x=x^*} \qquad b = g(x^*)$$

The coordinate and control transformation is approximated as,

$$z(x) = \tilde{x} + \frac{1}{2} z_{xx}\, \tilde{x}^2 + \mathcal{O}^3(\tilde{x})$$

$$v(x,u) = u + \frac{1}{2} v_{xx}\, \tilde{x}^2 + v_{xu}\, \tilde{x}\, u + \mathcal{O}^3(\tilde{x},u)$$

where:

$$z_{xx} = \left.\frac{\partial^2 z(x)}{\partial x^2}\right|_{x=x^*} \qquad v_{xx} = \left.\frac{\partial^2 v(x,u)}{\partial x^2}\right|_{x=x^*} \qquad v_{xu} = \left.\frac{\partial^2 v(x,u)}{\partial x\, \partial u}\right|_{x=x^*}$$

To compute the unknown constant coefficients in the series expansions requires differentiating the coordinate transformation and equating it to the linearized dynamics. Differentiating the coordinate transformation and expanding(4.18) to second order results in:

$$\begin{aligned}
\dot{z} &= \dot{x} + z_{xx}\, \tilde{x}\, \dot{x} \\
&= f(x) + g(x)\, u + f(x)\, z_{xx}\, \tilde{x} + g(x)\, z_{xx}\, \tilde{x}\, u \\
&= a\tilde{x} + g(x^*)u + \left[\frac{1}{2}\frac{\partial^2 f(x^*)}{\partial x^2} + \frac{\partial f(x^*)}{\partial x} z_{xx}\right]\tilde{x}^2 + \left[\frac{\partial g(x^*)}{\partial x} + g(x^*) z_{xx}\right]\tilde{x}u + \mathcal{O}^3(\tilde{x},u)
\end{aligned}$$

Substituting the coordinate and control transformation in (4.19):

$$\dot{z} = a\tilde{x} + g(x^*)u + \frac{1}{2}\left[\frac{\partial f(x^*)}{\partial x} z_{xx} + g(x^*) v_{xx}\right]\tilde{x}^2 + g(x^*) v_{xu}\tilde{x}u + \mathcal{O}^3(\tilde{x},u)$$

Setting these two equations equal to each other, the unknown constant coefficients of the series expansions are determined such that the second order errors (i.e. $\tilde{x}u$ and $\tilde{x}^2$) vanish.

$$\text{For } \tilde{x}^2: \quad \frac{\partial^2 f(x^*)}{\partial x^2} + \frac{\partial f(x^*)}{\partial x} z_{xx} - g(x^*)\, v_{xx} = 0 \tag{4.20}$$

$$\text{For } \tilde{x}u: \quad \frac{\partial g(x^*)}{\partial x} + g(x^*)\, z_{xx} - g(x^*)\, v_{xu} = 0 \tag{4.21}$$

Note that (4.20) and (4.20) are constant algebraic expressions. The procedure for constructing the transformation has been reduced to a linear algebra problem as opposed to a problem involving partial differential equations. Equations (4.20) and (4.21) can be written in matrix form as,

$$A\,c = b$$

where:

$$A = \begin{bmatrix} -\dfrac{\partial f(x^*)}{\partial x} & g(x^*) & 0 \\ -g(x^*) & 0 & g(x^*) \end{bmatrix} \qquad b = \begin{bmatrix} \dfrac{\partial^2 f(x^*)}{\partial x^2} \\ \dfrac{\partial g(x^*)}{\partial x} \end{bmatrix} \qquad c = \begin{bmatrix} z_{xx} \\ v_{xx} \\ v_{xu} \end{bmatrix}$$

Since both the A matrix and b vector are constants, it is easy to solve for the unknown coefficients z_{xx}, v_{xx}, v_{xu} using standard linear algebra techniques. This example highlights another desirable characteristic of this approach in that there may be a possibility for multiple solutions. For this example, if $rank[A] = 2$ there exists a one dimensional parametrization of the family of solutions. The general solution is $c = c_p + c_h$ where c_p is the particular solution and c_h is the homogeneous solution. Since only the direction of c_h is constrained, its magnitude is arbitrary and can be used to parameterize the one dimensional family of solutions. The general solution can then be tailored to achieve secondary objectives such as minimizing $T(x)$, $\alpha(x)$, $\beta(x)$ etc. For a more realistic example, the reader is refered to [9] where this method is applied to a continuous stirred tank reactor and for which a robust linear controller is designed.

Before proceeding to the conditions for extended feedback linearization, a few preliminary definitions [28] are required. Essentially these definitions are the approximate versions of the ones presented for exact linearization which are required in order to define approximate controllability and involutivity conditions about an equilibrium point. The definition for order p controllability is fairly simple. Let $\mathcal{D}_C = \left\{ g,\ ad_f\, g,\ \ldots,\ ad_f^{n-1}\, g \right\}$ denote the controllability distribution or the vectorspace spanned by all linear combinations of its entries. Also, let its entries be denoted by $X_i(x)$, $i = 0, \ldots, n-1$, that is $X_i = ad_f^i\, g$. The definition of approximate controllability is [28]:

Definition 15 *$\mathcal{D}_C$ has an order p local basis around x^* if it is full rank at x^* and for every*

$Y \in \mathcal{D}_C$ *there exist functions* $c_k(x)$ *such that:*

$$Y = \sum_{k=0}^{n-1} c_k(x)\, X_k(x) + \mathcal{O}^{p+1}(\tilde{x}) \tag{4.22}$$

The full rank requirement at x^* is identical to linear controllability. To see this evaluate $\mathcal{D}_c$ at x^*. The first column is just $g(x^*) = b$. The second column is,

$$[\, f,\, g\,](x^*) = D_x[\, g\,](x^*)f(x^*) - D_x[\, f\,](x^*)g(x^*) = -D_x[\, f\,](x^*)g(x^*) = -A\, b$$

since $f(x^*) = 0$. The third column is,

$$\begin{aligned} [\, f,\, [\, f,\, g\,]\,](x^*) &= D_x[\,[\, f,\, g\,]\,](x^*)f(x^*) - D_x[\, f\,](x^*)\, [\, f,\, g](x^*) \\ &= -D_x[\, f\,](x^*)\, [\, f,\, g\,](x^*) \\ &= -A^2 b \end{aligned}$$

because $f(x^*) = 0$. It is seen that a pattern is established for the successive columns of the controllability matrix given by;

$$ad_f^k\, g(x^*) = -D_x[\, f\,](x^*)\, ad_f^{k-1}\, g(x^*)$$

Thus, the expression for the linearized controllability matrix is,

$$\mathcal{D}_c(x^*) = \left[\, b,\ -A\, b,\ -A^2 b,\ -A^3 b,\ \ldots,\ -A^{n-1} b\,\right]$$

which when premultiplied by a full rank matrix to remove the negative signs becomes:

$$\mathcal{D}_c(x^*) = \left[\, b,\ A\, b,\ A^2 b,\ A^3 b,\ \ldots,\ A^{n-1} b\,\right]$$

This is the linear controllability matrix associated with the order 1 linearized system. It is seen that the linearized system must be controllable in order for $\mathcal{D}_c$ to have an order p basis around x^*. For exact feedback linearization, it is seen that the error term $\mathcal{O}^{p+1}(\tilde{x})$ in (4.22) must be zero for $x \in \Omega$.

Similarly, let $\mathcal{D}_I = \left\{ g,\ ad_f\, g,\ \ldots,\ ad_f^{n-2}\, g \right\}$ be the involutivity distribution, and recall that $X_i = ad_f^i\, g$. The definition for approximate involutivity is [28]:

Definition 16 *$\mathcal{D}_I$ is said to be order p involutive at x^* if there exist functions $c_{ijk}(x)$ such that:*

$$[X_i, X_j](x) = \sum_{k=0}^{n-2} c_{ijk}(x)\, X_k(x) + \mathcal{O}^p(\tilde{x}) \tag{4.23}$$

Finally, the classical Frobenius integrability theorem can also be stated in an approximate version [28].

Theorem 8 *(Frobenius with remainder) Let $\mathcal{D}$ be a distribution with order p basis* $\{ X_1, X_2, \ldots, X_l\}$ *at x^*. $\mathcal{D}$ is order p integrable at x^* if and only if $\mathcal{D}$ is order p involutive at x^*.*

The necessary and sufficient conditions for extended feedback linearization of single-input nonlinear systems can now be stated [28].

Theorem 9 *The nonlinear system (4.3) can be transformed into the order p linear system (4.17) where (A, b) is a controllable pair if and only if:*

(a) $\mathcal{D}_C$ has an order p local basis at x^*.

(b) $\mathcal{D}_I$ is order p involutive at x^*.

The multi-input version of this can be found in [28]. Comparing the above requirements with the conditions for exact feedback linearization it is seen that they are the same except for the error term. The term exact linearization arises from the fact that there is no error in the linearization. Note that it is always possible to solve the approximation problem for $p = 1$. It is just the common practice of linearizing the nonlinear dynamics. In this case the transformation is just $z = \tilde{x}$, and $v = u$. In practice, if the linearization about the equilibrium point is controllable the controllability test is not required since it is known [41] that the nonlinear system is also controllable and thus $\mathcal{D}_C$ will possess a higher order local basis. The converse of this statement, however, is not true. The nonlinear system may be controllable while its linear approximation is uncontrollable. However, to *a priori* determine the order of the linearization the order of approximate involutivity must be determined. A method to accomplish this is presented in the next section.

4.4.1 Computational Test For Involutivity Order

In this section, a computational test to determine the approximate involutivity order is presented. To that end, consider the meaning of (4.23) in terms of the error from exact involutivity. Let this error, e_{ij}, be defined as:

$$e_{ij}(x) = [X_i, X_j](x) - \sum_{k=0}^{n-2} c_{ijk}(x)\, X_k(x)$$

To simplify matters, introduce the following notation. Let the vector field $X_{ij} = [X_i, X_j](x)$, and the weighting coefficient vector $c_{ij} = [\, c_{ij,1}(x), \ldots, c_{ij,n-2}(x)\,]^T$. Then, the error equation can be written as:

$$e_{ij}(x) = X_{ij} - \mathcal{D}_I(x)\, c_{ij}(x)$$

Expanding the error term in a Taylor series about the equilibrium point x^*, say to first order, results in:

$$e_{ij}(x) = e_{ij}(x^*) + \sum_{k=1}^{n} \frac{\partial e_{ij}(x^*)}{\partial x_k}\, \tilde{x}_k + \mathcal{O}^2(\tilde{x}) \tag{4.24}$$

This expansion can now be expressed in terms of its constituent elements.

$$\begin{aligned} e_{ij}(x) &= X_{ij}(x^*) - \mathcal{D}_I(x^*)\, c_{ij}(x^*) + \\ &\sum_{k=1}^{n} \left[\frac{\partial X_{ij}(x^*)}{\partial x_k} - \mathcal{D}_I(x^*)\, \frac{\partial c_{ij}(x^*)}{\partial x_k} - \frac{\partial \mathcal{D}_I(x^*)}{\partial x_k}\, c_{ij}(x^*) \right] \tilde{x}_k + \mathcal{O}^2(\tilde{x}) \end{aligned}$$

For order 1 involutivity, there must exist coefficients $c_{ij}(x^*)$ such that the constant term in (4.24) is annihilated, i.e. solve

$$X_{ij}(x^*) - \mathcal{D}_I(x^*)\, c_{ij}(x^*) = 0_{n\times 1}$$

for $c_{ij}(x^*)$. Note that this is just a linear algebra problem since all terms are just constants. Thus, a solution exists if and only if

$$rank\,[\, \mathcal{D}_I(x^*)\,] = rank \left[\, \mathcal{D}_I(x^*) \;\vdots\; X_{ij}(x^*) \,\right]$$

Similarly, for order 2 involutivity, the constant and first order terms must be annihilated. In this case a solution to,

$$X_{ij}(x^*) - \mathcal{D}_I(x^*)\, c_{ij}(x^*) = 0_{n\times 1} \tag{4.25}$$

$$\frac{\partial X_{ij}(x^*)}{\partial x_k} - \mathcal{D}_I(x^*)\, \frac{\partial c_{ij}(x^*)}{\partial x_k} - \frac{\partial \mathcal{D}_I(x^*)}{\partial x_k}\, c_{ij}(x^*) = 0_{n\times 1} \tag{4.26}$$

must be obtained in terms of the constant coefficients $c_{ij}(x^*)$ and $\dfrac{\partial c_{ij}(x^*)}{\partial x_k}$. Note that (4.26) need only be computed when $\tilde{x}_k \neq 0$. Also, (4.25) and (4.26) are necessary and sufficient for order 2 involutivity since $\tilde{x}_k$ are independent of each other for all k and thus the first order term in (4.24) can only be zero if and only if the coefficients of the non-zero $\tilde{x}_k$ are annihilated. Note that these two equations can be solved successively, that is, (4.25) can be solved independently of (4.26) for each $c_{ij}(x^*)$. The result can then be substituted in (4.26) to solve for $\dfrac{\partial c_{ij}(x^*)}{\partial x_k}$. The requirement for existence of a solution to (4.26) is:

$$rank\,[\,\mathcal{D}_I(x^*)\,] \;=\; rank \left[\mathcal{D}_I(x^*) \;\vdots\; \frac{\partial X_{ij}(x^*)}{\partial x_k} - \frac{\partial \mathcal{D}_I(x^*)}{\partial x_k}\, c_{ij}(x^*) \right]$$

This analysis can be extended in similar fashion to higher orders.

4.5 Approximate Feedback Linearization: Version II

The approach of Krener can be applied to obtain an approximate version of the exact linearization approach. Indeed, in the proof of the necessary and sufficient conditions for the existence of approximate linearization, Krener used the Brunovsky form [28]. Although the proof of Theorem 6 will not be presented here, the key point in the derivation will be highlighted. In effect, the proof offers an alternative method for constructing the approximate transformation. Because the choice for the target linear system is the Brunovsky form, this approach is refered to as approximate feedback linearization. Approximate linearization methods for nonlinear systems have received considerable attention recently. The method of pseudolinearization [36], [49] attempts to find a transformation such that in the transformed coordinates its linearization is independent of the operating point. For tracking problems, an approximate input-output linearization approach was proposed in [19]. Also, an approach based on the approximate version of exact linearization was used in the Acrobot example [18], where the approximation is in terms of a linearizable nonlinear system.

As mentioned above, the key point in the proof of approximate linearization is the existence of approximate output functions that annihilate the involutivity distribution to certain order. That is, there exists an approximate output $\tilde{T}_1(x)$ such that it annihilates $\mathcal{D}_I(x)$ to order p:

$$\nabla \tilde{T}_1(x)\;\, ad^{i}_{f(x)}\, g(x) \;=\; \mathcal{O}^{p+1}(\tilde{x}) \qquad i = 0, 1, 2, \ldots, n-2$$

Here, the inner product is expanded in a Taylor series about the equilibrium point x^*. For contrast, one can compare this with the requirement for exact linearization:

$$\nabla T_1(x)\ ad^i_{f(x)}\, g(x) \;=\; 0 \qquad i = 0, 1, 2, \ldots, n-2$$

Note that the requirement for exact linearization must hold for all values of x, whereas the approximate version relaxes this condition. It is noted that exact linearization corresponds to the case where $p \longrightarrow \infty$ and $\mathcal{O}^{p+1}(\tilde{x}) \longrightarrow 0$.

To construct this approximate output function $\tilde{T}_1(x)$ requires the annihilation of as many higher order terms in the Taylor series expansion of $\mathcal{D}_I(x)$ about the equilibrium point. This implies that the solution can be constructed from the nullspace of the expansion. Note that the approximate version of the Frobenius theorem guarantees the existence of such an output function. The series expansion of the involutivity distribution can be expressed as:

$$\mathcal{D}_I(x) \;=\; \mathcal{D}_I(x^*) + \sum_{k=1}^{n} \frac{\partial \mathcal{D}_I(x^*)}{\partial x_k}\, \tilde{x}_k + \frac{1}{2} \sum_{k,l=1}^{n} \frac{\partial^2 \mathcal{D}_I(x^*)}{\partial x_k\, \partial x_l}\, \tilde{x}_k\, \tilde{x}_l + \mathcal{O}^3(\tilde{x})$$

The approximate output function is computed from the $p - term$ annihilator of $\mathcal{D}_I$, i.e.

$$[\, \mathcal{D}_0 + \mathcal{D}_1 + \ldots + \mathcal{D}_p\,]^T\ \nabla \tilde{T}_1 \;=\; 0_{n-1\times 1}$$

where

$$\begin{aligned}
\mathcal{D}_0 &= \mathcal{D}_I(x^*) \\
\mathcal{D}_1 &= \sum_{k=1}^{n} \frac{\partial \mathcal{D}_I(x^*)}{\partial x_k}\, \tilde{x}_k \\
\mathcal{D}_2 &= \frac{1}{2} \sum_{k,l=1}^{n} \frac{\partial^2 \mathcal{D}_I(x^*)}{\partial x_k\, \partial x_l}\, \tilde{x}_k\, \tilde{x}_l \\
\vdots\ &\ \vdots\ \ \vdots
\end{aligned}$$

Once $\tilde{T}_1(x)$ has been computed, the remaining state and control transformation variables are obtained from [28]:

$$\begin{aligned}
\tilde{T}_k(x) &= L_f^{k-1}\, \tilde{T}_1(x) \\
v &= L_f^k\, \tilde{T}_1(x) + L_g\, L_f^{k-1}\, \tilde{T}_1(x)\, u
\end{aligned}$$

In these coordinates, the transformed equations appear as [28]:

$$\begin{aligned} \dot{\tilde{T}}_i &= \tilde{T}_{i+1} + \mathcal{O}^{p+1}(\tilde{x}, u) \\ \dot{\tilde{T}}_n &= v + \mathcal{O}^{p+1}(\tilde{x}, u) \end{aligned}$$

In the following section, this control design approach is applied to the underactuated cart-pole problem.

4.6 Example: The Cart-Pole

In this section, the feedback linearization methods presented in this chapter will be applied to the cart-pole problem introduced in Chapter 3.6. The objective here is to control this underactuated system over large regions of the state space. First exact linearization is attempted. It will be shown that this approach fails and hence approximate feedback linearization will be applied. For comparison purposes, the standard linearization based design will be used as a benchmark.

As a first step, the state space representation of this system is required to cast the problem in the standard form (4.3). Recall that the equations of motion for this system were:

$$(M + m)\,\ddot{x} + ml\,\cos(\theta)\,\ddot{\theta} - ml\,\sin(\theta)\,\dot{\theta}^2 = u \tag{4.27}$$

$$\frac{4}{3}\,l\,\ddot{\theta} + \cos(\theta)\,\ddot{x} - g\,\sin(\theta) = 0 \tag{4.28}$$

Letting $x_1 = x$, $x_2 = \theta$, $x_3 = \dot{x}$, and $x_4 = \dot{\theta}$, the equations of motion can be written as

$$\dot{x} = f(x) + g(x)\,u$$

where

$$f(x) = \begin{bmatrix} x_3 \\ x_4 \\ \dfrac{4}{a}\left\{ ml\,x_4^2\,\sin(x_2) - 0.75\,mg\,\cos(x_2)\,\sin(x_2) \right\} \\ \dfrac{3}{a}\left\{ \dfrac{1}{l}(M+m)g\,\sin(x_2) - m\,x_4^2\,\cos(x_2)\,\sin(x_2) \right\} \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{4}{a} \\ -\dfrac{3}{al}\cos(x_2) \end{bmatrix}$$

and $a = 4(M+m) - 3m\ \cos^2(x_2)$. Note that $\dfrac{ml}{3}a$ is just the determinant of the inertia matrix, and thus $a > 0,\ \forall x_2$. It can be verified that $(x_1, 0, 0, 0)$ is the equilibrium set (i.e the system is in equilibrium when the pole is vertical and this is independent of cart location). To test whether exact linearization can be applied requires checking the controllability and involutivity criteria. However, for the given f and g, computation of both of these distributions results in extremely complicated expressions that are prohibitively difficult to test. To test controllability, however, an alternative exists in that controllability of the linearized system guarantees controllability for the nonlinear system. However, the testing of the involutivity criteria is still required. The culprit for the complexity is the presence of the trigonometric and quadratic terms in f and g which breed further complicated expressions with successive differentiation.

To adress the problem of the computations encountered in testing for the controllability and involutivity criteria, feedback will be employed to reduce the complicated equations of motion. That is, the system will be first precompensated via feedback to cancel as many nonlinear terms as possible in order to obtain a simplified model. This is accomplished in the following manner. First not that (4.28) is a constraint equation that relates $\ddot{x}$ with $\ddot{\theta}$, and thus can be used to elliminate $\ddot{x}$ from (4.27). From (4.28):

$$\ddot{x} = g\ \tan(\theta) - \frac{4l}{3\cos(\theta)}\ddot{\theta} \tag{4.29}$$

Substituting (4.29) in (4.27), $\ddot{x}$ can be eliminated resulting in an expression that is a function of θ and u only

$$\ddot{\theta} = \frac{-1}{a\,l}\left[3ml\ \cos(\theta)\ \sin(\theta)\,\dot{\theta}^2 - 3(M+m)g\ \sin(\theta) + 3\ \cos(\theta)\,u\right] \tag{4.30}$$

Canceling all the nonlinearities from (4.30) results in a double integrator relation. This is accomplished by choosing the control action

$$u = \frac{1}{3 \cos(\theta)} \bar{u}$$

where:

$$\bar{u} = -3ml \cos(\theta) \sin(\theta) \dot{\theta}^2 + 3(M+m)g \sin(\theta) - a\, l\, v$$

Of course, the relationship between $\bar{u}$ and u is unbounded at $\theta = \pm\pi/2$. Using this transformation, the system is not controllable at $\theta = \pm\pi/2$ since the pole is horizontal and the line of control action passes through the pole center of mass and thus is unable to exert a torque on the pole. This fact will be apparent in the subsequent controllability analysis. Using this expression for u, equations (4.29) and (4.30) become:

$$\begin{aligned} \ddot{\theta} &= v \\ \ddot{x} &= g \tan(\theta) - \frac{4l}{3 \cos(\theta)} v \end{aligned}$$

It is apparent that this set of equations is much simpler than the original, and should result in substantial reduction of the computational overhead. The state space equations for this simplified model are:

$$\bar{f}(x) = \begin{bmatrix} x_3 \\ x_4 \\ g \tan(x_2) \\ 0 \end{bmatrix} \qquad \bar{g}(x) = \begin{bmatrix} 0 \\ 0 \\ -\dfrac{4l}{3 \cos(x_2)} \\ 1 \end{bmatrix}$$

The simplified equations of motion appear as:

$$\dot{x} = \bar{f}(x) + \bar{g}(x)\, v$$

Note that in the simplified model the equilibrium set has remained invariant.

The next step is to check the controllability and involutivity conditions for the simplified model. The nonlinear controllability distribution is much simpler when compared to the original equations, however, it will not be presented here since the conditions for controllability can be ascertained from linear analysis. Computing the linear controllability matrix:

$$\left[\, b,\ A\,b,\ A^2\,b,\ A^3\,b \,\right] = \begin{bmatrix} 0 & 0 & -\dfrac{4l}{3\,\cos(x_2)} & 1 \\ -\dfrac{4l}{3\,\cos(x_2)} & 1 & 0 & 0 \\ 0 & 0 & \dfrac{g}{\cos^2(x_2)} & 0 \\ \dfrac{g}{\cos^2(x_2)} & 0 & 0 & 0 \end{bmatrix}$$

It is evident that this matrix is unbounded at $x_2 = \pm\,\pi/2$ and thus the system is controllable locally everywhere except at this point. The next requirement that needs to be verified is the involutivity condition. This distribution, $\mathcal{D}_I = \left\{\, \bar{g},\ ad_{\bar{f}}\,\bar{g},\ ad_{\bar{f}}^2\,\bar{g} \,\right\}$ is:

$$\mathcal{D}_I = \begin{bmatrix} 0 & \dfrac{4l}{3\,\cos(x_2)} & \dfrac{8l\,\sin(x_2)}{3\,\cos^2(x_2)}\,x_4 \\ 0 & -1 & 0 \\ -\dfrac{4l}{3\,\cos(x_2)} & -\dfrac{4l\,\sin(x_2)}{3\,\cos^2(x_2)}\,x_4 & \dfrac{3g\,\cos(x_2) - 8l\,x_4^2 + 4l\,\cos^2(x_2)\,x_4^2}{3\,\cos^3(x_2)} \\ 1 & 0 & 0 \end{bmatrix}$$

For this system to be involutive requires that $\left[\,\bar{g},\ ad_{\bar{f}}\,\bar{g}\,\right]$, $\left[\,\bar{g},\ ad_{\bar{f}}^2\,\bar{g}\,\right]$, and $\left[\,ad_{\bar{f}}\,\bar{g},\ ad_{\bar{f}}^2\,\bar{g}\,\right]$ can be expressed as linear combinations of the columns of $\mathcal{D}_I$. Computing the first bracket expression:

$$\left[\,\bar{g},\ ad_{\bar{f}}\,\bar{g}\,\right] = \begin{bmatrix} 0 \\ 0 \\ -\dfrac{8l\,\sin(x_2)}{3\,\cos^2(x_2)} \\ 0 \end{bmatrix}$$

It can be verified by inspection that $\left[\,\bar{g},\ ad_{\bar{f}}\,\bar{g}\,\right] \notin span\ \mathcal{D}_I$. This is determined from:

$$rank\,[\,\mathcal{D}_I\,] \neq rank\left[\,\mathcal{D}_I \,\vdots\, \left[\,\bar{g},\ ad_{\bar{f}}\,\bar{g}\,\right]\,\right]$$

It is evident that the distribution $\mathcal{D}_I$ is not involutive, and as a consequence, this system cannot be exactly feedback linearized. Computing the remaining bracket expressions it can also be shown that,

$$\left[\,\bar{g},\, ad_{\bar{f}}^2\,\bar{g}\,\right] \;=\; \begin{bmatrix} \dfrac{8l\,\sin(x_2)}{3\,\cos^2(x_2)} \\ 0 \\ \dfrac{8l\,(\,\cos^2(x_2)-2\,)}{3\,\cos^2(x_2)}\,x_4 \\ 0 \end{bmatrix} \;\not\in\; span\,\mathcal{D}_I$$

and:

$$\left[\,ad_{\bar{f}}\,\bar{g},\, ad_{\bar{f}}^2\,\bar{g}\,\right] \;=\; \begin{bmatrix} \dfrac{8l\,(\,\cos^2(x_2)-2\,)}{3\,\cos^2(x_2)}\,x_4 \\ 0 \\ \dfrac{-\,2\,\sin(x_2)\,(\,3g\cos(x_2)-12lx_4^2+2l\cos^2(x_2)x_4^2\,)}{3\,\cos^4(x_2)} \\ 0 \end{bmatrix} \;\not\in\; span\,\mathcal{D}_I$$

Since the conditions for exact linearization were violated, it is logical to attempt the approximate version of this method. Because the linearized controllability matrix is full rank (except when the pole is horizontal) it is known that the nonlinear controllability matrix (or distribution) is nonsingular (or controllable) [41]. This implies that the system is order p controllable for arbitrary p and thus does not present a constraint on the order of linearization. Next, the involutivity condition is investigated. Computing the first order Taylor series expansion of $\mathcal{D}_I$:

$$\mathcal{D}_I(x) \;=\; \mathcal{D}_I(x^*) \;+\; \sum_{k=1}^{4} \left.\frac{\partial \mathcal{D}_I(x)}{\partial x_k}\right|_{x=x^*} \tilde{x}_k \;+\; \mathcal{O}^2(\tilde{x})$$

The first order term is found to be zero, i.e.

$$\sum_{k=1}^{4} \left.\frac{\partial \mathcal{D}_I(x)}{\partial x_k}\right|_{x=x^*} \tilde{x}_k \;=\; 0_{4\times 3}$$

To first order, the involutivity distribution is constant, i.e. $\mathcal{D}_I(x) = \mathcal{D}_I(x^*) + \mathcal{O}^2(\tilde{x})$ and therefore it is order 1 involutive. The reason for this is that for order 1 involutivity, it must be shown that the constant term in (4.24) can be annihilated by proper choice of constant coefficients. The Lie bracket operation involves a differentiation, so at most only the first order term in the Taylor series expansion of $\mathcal{D}_I$ need be considered, since it will contribute a constant term to the error equation after differentiation. But $\mathcal{D}_I$ to first order is constant, so a Lie bracket of any combinations of its columns will be zero and thus it will be involutive to first order. Note that in this case the solution for the weighting coefficients is $c_{ij}(x^*) = [\,0\ \ 0\ \ 0\,]^T$. Next, it is desirable to determine whether the system is order 2 involutive. Since $c_{ij}(x^*) = 0_{3\times 1}$, the condition (4.26) reduces to:

$$\frac{\partial X_{ij}(x^*)}{\partial x_k} - \mathcal{D}_I(x^*)\,\frac{\partial c_{ij}(x^*)}{\partial x_k} = 0_{n\times 1} \qquad k = 2,\ldots,4 \tag{4.31}$$

Note that in the above, the index k begins at 2. This is because $\tilde{x}_1 = 0$ and so the first order term associated with $\tilde{x}_1$ is annihilated regardless of its coefficient. For computational purposes, let:

$$\begin{aligned} X'_{ij}(x^*) &= \left[\frac{\partial X_{ij}(x^*)}{\partial x_2}, \ldots, \frac{\partial X_{ij}(x^*)}{\partial x_4} \right] \\ C'_{ij}(x^*) &= \left[\frac{\partial c_{ij}(x^*)}{\partial x_2}, \ldots, \frac{\partial c_{ij}(x^*)}{\partial x_4} \right] \end{aligned}$$

Then, (4.31) can be put in the form:

$$\begin{aligned} \mathcal{D}_I(x^*)\,C'_{12}(x^*) &= X'_{12}(x^*) \\ \mathcal{D}_I(x^*)\,C'_{13}(x^*) &= X'_{13}(x^*) \\ \mathcal{D}_I(x^*)\,C'_{23}(x^*) &= X'_{23}(x^*) \end{aligned}$$

Computing $\mathcal{D}_I(x^*)$:

$$\mathcal{D}_I(x^*) = \begin{bmatrix} 0 & \dfrac{4l}{3} & 0 \\ 0 & -1 & 0 \\ -\dfrac{4l}{3} & 0 & g \\ 1 & 0 & 0 \end{bmatrix}$$

The expressions for the bracketed terms are:

$$X'_{12}(x^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\dfrac{8l}{3} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X'_{13}(x^*) = \begin{bmatrix} \dfrac{8l}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\dfrac{8l}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

$$X'_{23}(x^*) = \begin{bmatrix} 0 & 0 & -\dfrac{8l}{3} \\ 0 & 0 & 0 \\ -2g & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For a solution to exist, the rank condition can be used.

$$\begin{aligned} rank\,[\,\mathcal{D}_I(x^*)\,] &= rank\left[\mathcal{D}_I(x^*) \,\vdots\, X'_{12}(x^*)\right] \\ rank\,[\,\mathcal{D}_I(x^*)\,] &\neq rank\left[\mathcal{D}_I(x^*) \,\vdots\, X'_{13}(x^*)\right] \\ rank\,[\,\mathcal{D}_I(x^*)\,] &\neq rank\left[\mathcal{D}_I(x^*) \,\vdots\, X'_{23}(x^*)\right] \end{aligned}$$

It is evident that the condition for order 2 involutivity is not satisfied, and it is concluded that $\mathcal{D}_I$ is order 1 involutive. However, note that there exists a quality to the degree of violation of the involutivity condition. That is, one can also assign a degree of violation to the order p involutivity condition according to the number of distinct permutations of (4.26) that were violated. For the example at hand, one can define the system as order 1, degree 1 involutive since one of the three order 2 conditions was not violated. If two were not violated, it could be referred to as order 1, degree 2 involutivity etc. A linear system would then be defined as an order 1, degree 0 system. In general, one would expect better performance from an approximate linearization design of the same order but higher degree involutivity than for the corresponding linear design.

The next step is to compute the approximate output function from which the linearizing transformation is obtained. Recall that it is required to solve for an output function that approximately annihilates the involutivity distribution to first order,

$$[\, \mathcal{D}_0 + \mathcal{D}_1 \,]^T \; \nabla \tilde{T}_1^T = 0_{3\times 1}$$

where:

$$\begin{aligned} \mathcal{D}_0 &= \mathcal{D}_I(x^*) \\ \mathcal{D}_1 &= \sum_{k=1}^{4} \frac{\partial \mathcal{D}_I(x^*)}{\partial x_k}\, \tilde{x}_k \end{aligned}$$

However, since $\mathcal{D}_1 = 0_{4\times 3}$ the construction of $\nabla \tilde{T}_1$ only requires the computation of the nullspace of $\mathcal{D}_I^T(x^*)$. Note that this is just a linear algebra problem. Computing this nullspace:

$$\nabla \tilde{T}_1^T = \left[\frac{3}{4l}\; 1\; 0\; 0 \right]^T$$

Integrating this expression results in the approximate output function $\tilde{T}_1$.

$$\tilde{T}_1(x) = \frac{3}{4l} x_1 + x_2$$

It should be noted that $\nabla \tilde{T}_1$ is orthogonal to $\bar{g}(x)$ for all x. Thus, this output function annihilates the first column of $\mathcal{D}_I(x)$ for arbitrary order p. The approximate coordinate transformation is:

$$\begin{aligned} \tilde{T}_1(x) &= \frac{3}{4l} x_1 + x_2 \\ \tilde{T}_2(x) &= \frac{3}{4l} x_3 + x_4 \\ \tilde{T}_3(x) &= \frac{3g}{4l} \tan(x_2) \\ \tilde{T}_4(x) &= \frac{3g}{4l} \sec^2(x_2)\, x_4 \end{aligned}$$

The approximate control transformation is:

$$v = \frac{3g}{4l} \tan(x_2) \sec^2(x_2)\, x_4^2 + \frac{3g}{4l} \sec^2(x_2)\, u$$

In the transformed coordinates the equations of motion appear as:

$$\begin{aligned} \dot{\tilde{T}}_1 &= \tilde{T}_2 \\ \dot{\tilde{T}}_2 &= \tilde{T}_3 \\ \dot{\tilde{T}}_3 &= \tilde{T}_4 \\ \dot{\tilde{T}}_4 &= v \end{aligned}$$

Before proceeding further, consider the effect of the neglected terms in the transformed coordinate system. It can be shown that differentiating the coordinate transformation results in the following set of transformed equations

$$\begin{bmatrix} \dot{\tilde{T}}_1 \\ \dot{\tilde{T}}_2 \\ \dot{\tilde{T}}_3 \\ \dot{\tilde{T}}_4 \end{bmatrix} = \begin{bmatrix} L_{\bar{f}}\, \tilde{T}_1 \\ L_{\bar{f}}^2\, \tilde{T}_1 \\ L_{\bar{f}}^3\, \tilde{T}_1 \\ L_{\bar{f}}^4\, \tilde{T}_1 + L_{\bar{g}}\, L_{\bar{f}}^3\, \tilde{T}_1\, u \end{bmatrix} + \begin{bmatrix} 0 \\ L_{\bar{g}}\, L_{\bar{f}}\, \tilde{T}_1\, u \\ L_{\bar{f}}\, L_{\bar{g}}\, L_{\bar{f}}\, \tilde{T}_1\, u \\ L_{\bar{f}}^2\, L_{\bar{g}}\, L_{\bar{f}}\, \tilde{T}_1\, u + L_{\bar{g}}\, L_{\bar{f}}\, L_{\bar{g}}\, L_{\bar{f}}\, \tilde{T}_1\, u^2 \end{bmatrix} \tag{4.32}$$

as it can be verified by direct computation that:

$$\begin{aligned} L_{\bar{g}}\, \tilde{T}_1 &= 0 \\ L_{\bar{g}}\, L_{\bar{f}}^2\, \tilde{T}_1 &= 0 \\ L_{\bar{g}}^2\, L_{\bar{f}}\, \tilde{T}_1 &= 0 \end{aligned}$$

The second column in the right hand-side of (4.32) represents the neglected terms in the transformed system. The neglected terms are:

$$\begin{aligned} L_{\bar{g}}\, L_{\bar{f}}\, \tilde{T}_1 &= 1 - \frac{1}{\cos(x_2)} \\ L_{\bar{f}}\, L_{\bar{g}}\, L_{\bar{f}}\, \tilde{T}_1 &= -\frac{\sin(x_2)}{\cos^2(x_2)}\, x_4 \\ L_{\bar{f}}^2\, L_{\bar{g}}\, L_{\bar{f}}\, \tilde{T}_1 &= -\frac{x_4^2}{\cos(x_2)} - \frac{2\sin^2(x_2)\, x_4^2}{\cos^3(x_2)} \end{aligned}$$

$$L_{\tilde g}\, L_{\tilde f}\, L_{\tilde g}\, L_{\tilde f}\, \tilde T_1 \;=\; -\frac{\sin(x_2)}{\cos^2(x_2)}$$

A further improvement in the approximation can be accomplished by redefinition of the new control. From the expression for $\dot{\tilde T}_4$ the original control appears linearly in the additional term $L_{\tilde f}^2\, L_{\tilde g}\, L_{\tilde f}\, \tilde T_1\, u$ that is not accounted for in v. Thus, the new control variable could be defined as:

$$v \;=\; L_{\tilde f}^4\, \tilde T_1 \;+\; L_{\tilde g}\, L_{\tilde f}^3\, \tilde T_1\, u \;+\; L_{\tilde f}^2\, L_{\tilde g}\, L_{\tilde f}\, \tilde T_1\, u$$

Since not just the order but also the form of the remainder is important, this redefinition may allow for ellimination of undesirable dynamics. For some problems it may be apparent that certain states have significant influence on the behaviour of the system, while others do not. This observation leads to assigning a quality to the error term, where certain errors may be more significant than others. Even though an improvement in the error order cannot be accomplished by redefining the new control variable, an improvement in quality of error may be possible.

To third order, $\mathcal{O}^3(\tilde x)$ or $\mathcal{O}^2(\tilde x)u$, the full equations of motion in the transformed coordinates appear as:

$$\begin{bmatrix} \dot{\tilde T}_1 \\ \dot{\tilde T}_2 \\ \dot{\tilde T}_3 \\ \dot{\tilde T}_3 \end{bmatrix} = \begin{bmatrix} \tilde T_2 \\ \tilde T_3 \\ \tilde T_4 \\ v \end{bmatrix} - \begin{bmatrix} 0 \\ x_2^2\, u\, /2 \\ x_2\, x_4\, u \\ x_4^2\, u \,+\, x_2\, u^2 \end{bmatrix}$$

It is evident that the approximation is second order since the error terms, $\mathcal{O}^2(\tilde x)u$, are third order in the combined state and control coordinates. For comparison purposes, consider the third order linearization of the simplified equations of motion,

$$\dot x \;=\; A\, x \,+\, b\, u \,+\, \begin{bmatrix} 0 \\ 0 \\ \dfrac{g}{3} x_2^3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\dfrac{2l}{3} x_2^2 \\ 0 \end{bmatrix} u$$

where:

$$A = \frac{\partial \bar{f}(x^*)}{\partial x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad b = \bar{g}(x^*) = \begin{bmatrix} 0 \\ 0 \\ -\frac{4l}{3} \\ 1 \end{bmatrix}$$

The main difference between the two representations is the fact that the linearized system is third order in the pole angle whereas the approximate linearization is second order in the same term multiplied by the control. For large pole angles (i.e. > 1 radian) the effect of this third order term in the linearized dynamics will dominate whereas in the transformed dynamics this effect is absent. Both systems are second order linear, however, with significant differences in how the third order terms manifest themselves. Here, the crucial point is that the decomposition of the higher order error terms may be more important than just merely the order of the error. In the following simulation, the profound difference in the response of the two systems will serve to highlight this point.

Finally, a simple simulation of the closed loop behaviour for initial condition response follows. A compensator was designed to place the closed-loop poles of both the linearized and approximate linearized models at $(-3,-3,-3,-3)$. Note that the choice of compensator and closed loop poles was dictated by the requirement of highlighting the differences in the response rather that optimizing the closed-loop behaviour. In Figure 4.2 the response of the linear model is presented for initial condition $(40°,1,0,0)$. For an initial pole angle of approximately 43° the linear system fails. Now, consider the response of the approximately linearized system for initial condition $(80°,1,0,0)$, shown in Figure 4.3. It is readily apparent that this design has increased the range of operation of the linear control design by 100%! In Figure 4.4 the response of the linear and approximate designs is presented for initial pole angles ranging from 20° to 60° and initial cart displacement of 1m. It is apparent that the approximate design smoothly interpolates the linear response over an expanded operating region. Also, it is apparent that the approximate design delivers superior performance (in terms of overshoot) over the entire operating range (including the region where the linear design is valid). It is interesting to look at the response of both designs via animation of the cart-pole response. The initial and final conditions for the simulation are shown in Figure 4.5. The animation of the linear response

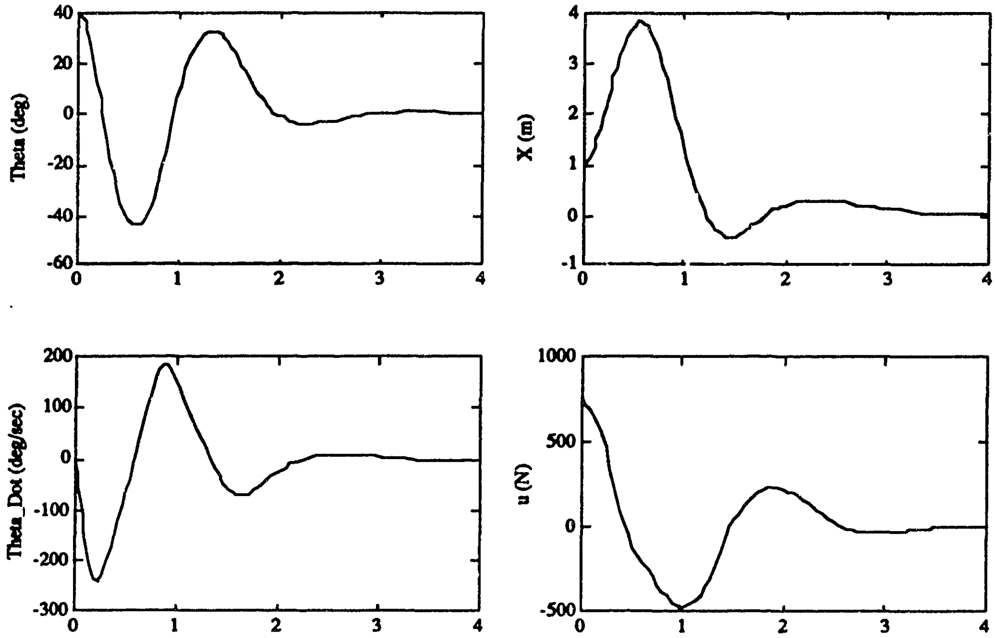

Figure 4.2: Linear design

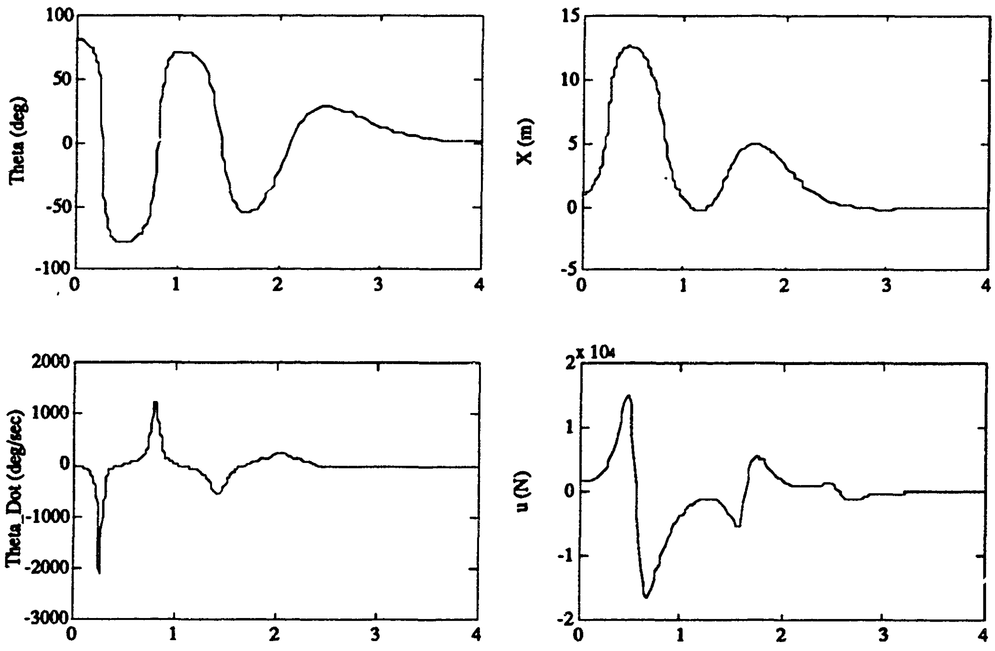

Figure 4.3: Approximate feedback linearization design

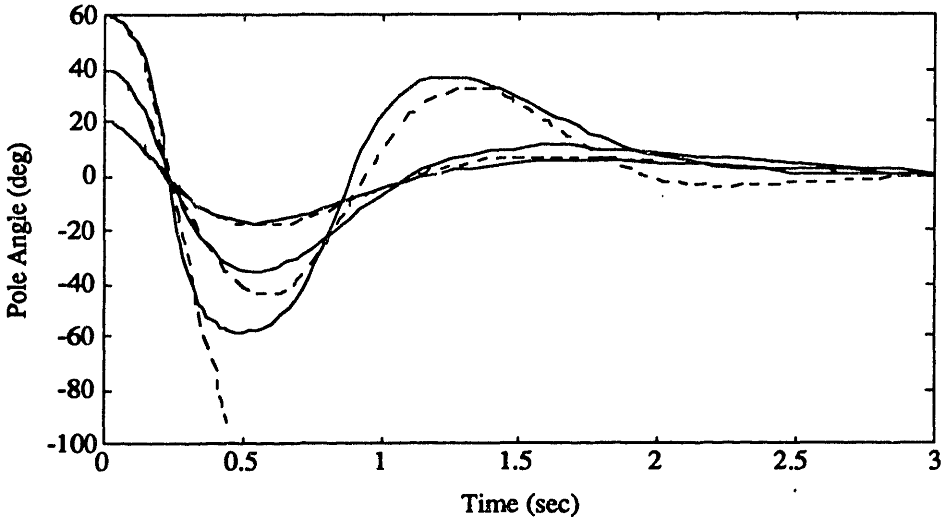

Figure 4.4: Comparison of linear and approximate designs (Solid line = Approximate design, Dashed line = Linear design

is shown in Figure 4.6. Similarly, the animation of the extended linearization design is shown in Figure 4.7. It is evident from Figure 4.7 that to bring the pole back to vertical, from such a large initial deviation, requires a complicated sequence of maneuvers. The linear controller for the approximately linearized system is able to smoothly adjust the system gain in order to accomplish the task whereas it fails for the linear system.

4.7 Conclusion

In this chapter, Krener's [28] extended feedback linearization was adopted as a design methodology for underactuated systems. This approach expands the operating region of linear control designs by approximately linearizing the original nonlinear dynamics. This is in contrast to the exact feedback linearization technique that attempts to globally transform the nonlinear dynamics which is much more difficult to apply in practice due to the stringent existence conditions. For approximate linearization, a computational approach to test for the order of the involutivity distribution was derived. For the cart-pole with one actuator it was shown that it is not possible to exactly linearize the system. This is due to the fact that only one input was available. If a second input were available the system would be exactly linearizable. It

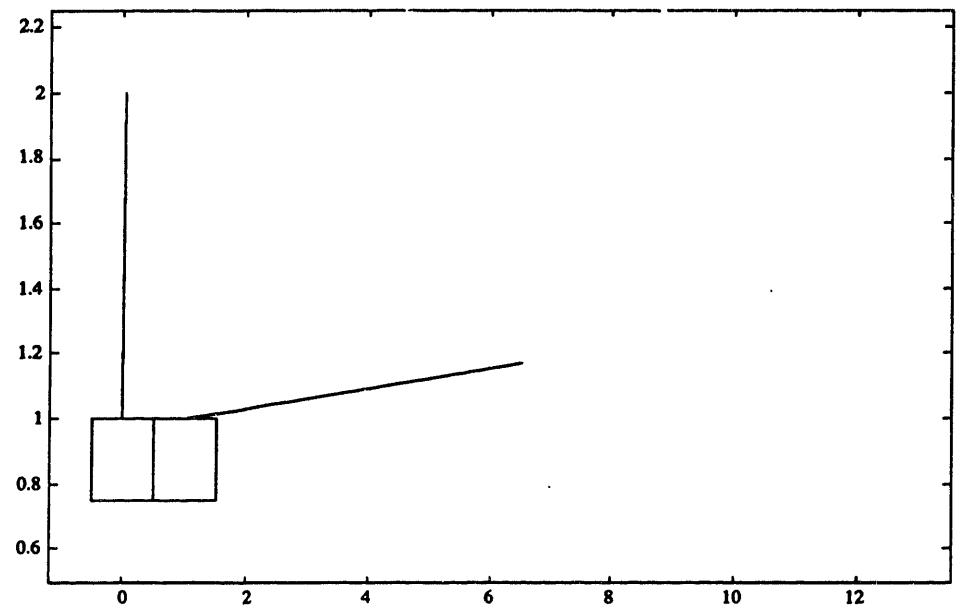

Figure 4.5: Initial and final configuration for approximate design

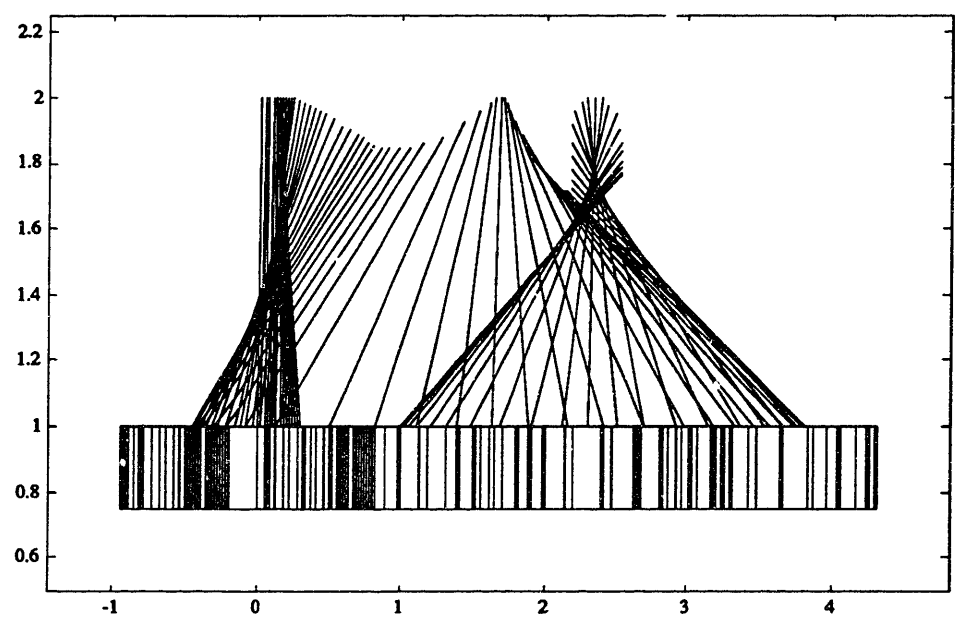

Figure 4.6: Animation of linear design

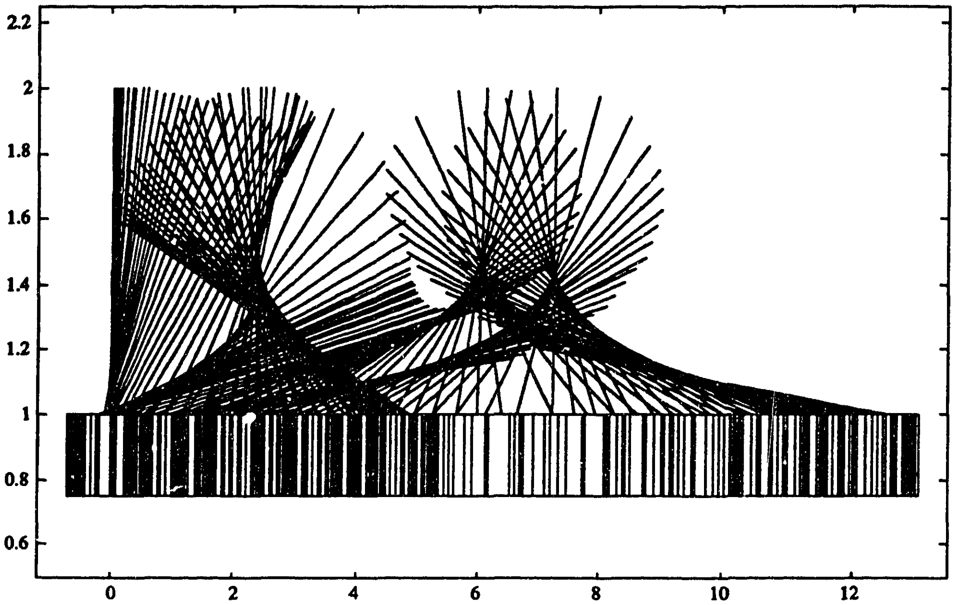

Figure 4.7: Animation of approximate linearization design

was also shown that this example was order 1 involutive. An approximately linear system was constructed and its response was compared to the linear model based design via computer simulation. It was shown by example that for the same closed loop pole locations the approximate design increased the range of operation of the linear compensator by 100%.

Chapter 5

Conclusions And Recommendations

The objective of this thesis was the development of linearizing transformations using methods from analytical dynamics for the control of nonlinear mechanical systems with particular emphasis to underactuated systems. Using coordinate and control transformations, the special properties of mechanical systems were exploited to derive an equivalent linear representation of the nonlinear equations of motion. Once a linear equivalent model is obtained, existing results in linear control theory can be applied to design appropriate controllers. Within the framework of canonical transformation theory, which preserve Hamilton's canonical equations of motion, a new set of transformations were derived. These transformations, termed orthogonal canonical transformations, also preserve Hamilton's equations and parametrize a class of Hamiltonian systems that admit a linear representation in the transformed coordinate system. The Riemann Curvature Tensor was introduced as a computational tool by which it can be determined whether a given mechanical system admits a coordinate system in which the equations of motion appear linear. It was shown that the curvature tensor can be used to test for the existence of a point transformation such that in the transformed coordinates the system appears as a double integrator linear model. Finally, an existing control design methodology, extended feedback linearization, was adopted for the control of underactuated systems. A computational test for the order of linearization was derived from the general existence conditions. An example was presented that highlights the efficacy of this approach.

In Chapter 2, the framework for canonical coordinate transformations was introduced. A general condition for the preservation of Hamilton's equations was derived, and it was shown that three standard definitions for canonical transformations were equivalent. A new set of transformations, termed orthogonal canonical transformations were derived that characterize a special class of Hamiltonian systems that admit a linear representation in the transformed coordinates. With this approach, the solution to the original nonlinear Hamiltonian equations of motion are obtained from the inverse transformation. The general conditions for such a transformation were derived, and an example was presented that illustrates this linearizing property.

In Chapter 3, linearizing transformations were investigated for mechanical systems. It was shown that one approach to transform a nonlinear mechanical system to a double integrator linear model in the transformed coordinates involves the use of point transformations. An alternative derivation of an existing result on the "square-root" factorization of the inertia matrix in terms of the transformation Jacobian matrix was presented. This factorization leads to a double integrator linear model in the transformed coordinates. The Riemann Curvature Tensor was introduced as a computational tool to test for the existence of such a factorization. The cart-pole problem was shown to satisfy the curvature conditions, and the linearizing transformation was computed. Euler's rotational equations of motion were shown not to violate the curvature conditions for an axi-symmetric inertia distribution. This approach was shown to be applicable to fully actuated control systems only.

In Chapter 4, Krener's [28] methodology of extended feedback linearization is adopted as an approach to control underactuated systems. This approach expands the operating region of linear control designs by constructing a linear approximation about an equilibrium point accurate to second or higher order. The notion of feedback equivalence and the methods of exact and extended feedback linearization were reviewed. For approximate feedback linearization, a computational method to test for the order of the linearization was derived. Approximate feedback linearization was applied to the underactuated cart-pole problem. It was shown that this problem is not exactly feedback linearizable. A second order linear approximation was constructed. Simulation results showed that for the same closed loop pole locations a substantial improvement in the range of operation of the linear control design was achieved.

5.1 Recommendations For Further Research

Computing the existence conditions for orthogonal canonical transformations for a two state system, it was shown that these resulted in an underconstrained set of nonlinear equations. For this example, 6 constraint partial differential equations must be solved for the 16 unkown coordinate transformation partials. Thus, there exists a freedom in picking a solution and auxiliary constraints can be appended to fully specify the solution. Since the orthogonal transformation approach represents a class of linearizable Hamiltonian systems, it would be desirable to parametrize the family of solutions to the existence conditions. In this manner, a class of linearizable Hamiltonian systems can be specified. Also, efficient means of solving the constraint partial differential equations are required to further explore the implications of this approach.

With respect to linearizing transformations, the double integrator linear model in the transformed dynamics can be obtained without the restriction to point transformations. One avenue for further research is to derive general conditions for non-point transformations realizing the double-integrator dynamics. It appears, however, that pursuing linearizing canonical transformations may be more useful for its theoretical insight in light of the increasingly practical results being developed in feedbaack linearization. One area in which research using the Riemann Curvature Tensor may prove fruitful is in constructing linearizable nonlinear approximations to given nonlinear mechanical systems. As it appears that most mechanical systems do not satisfy the zero curvature condition, approximate nonlinear systems may be constructed that satisfy this condition. The application of the curvature tensor to the optimal design of mechanical systems appears attractive. A recent result by Brockett *et all* [33] uses harmonic maps to measure the distortion caused by mapping spaces of different curvature to derive optimal link lengths for planar manipulators. However, the application of the curvature tensor to optimal mechanism design is presently limited to lower dimensional systems due to its complexity.

A promising approach to controller design for nonlinear systems, including underactuated systems, appears to be the extended feedback linearization method of Krener [28]. However, further work is required to account for plant uncertainty, and when full state measurement is not available. The effect of the neglected terms in any approximation require further research in that certain state variables may have a substantial impact on the system response while others may not. The application to tracking controller design for nonlinear systems also requires further research. Nonlinear system approximation whether it be higher order linear approximants or nonlinear linearizable approximations appears to offer a profitable approach to nonlinear control as evidenced by recent research activity. For example the recent work of Hauser [17] on uniform system approximation attempts to find a linearizable nonlinear system that approximates a nonlinear system uniformly on an equilibrium manifold.

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