DUALITY AND RADON TRANSFORM FOR SYMMETRIC SPACES

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Communicated by G. D. Mostow, July 3, 1963

1. **The dual space of a symmetric space.** Let S be a symmetric space (that is a Riemannian globally symmetric space), and let $I_0(S)$ denote the largest connected group of isometries of S in the compact open topology. It will always be assumed that S is of *the noncompact type*, that is $I_0(S)$ is semisimple and has no compact normal subgroup $\neq \{e\}$. Let l denote the rank of S; then S contains flat totally geodesic submanifolds of dimension l. These will be called *planes* in S.

Let o be any point in S, K the isotropy subgroup of $G=I_0(S)$ at o and $\mathfrak{k}_0$ and $\mathfrak{g}_0$ their respective Lie algebras. Let $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$ be the corresponding Cartan decomposition of $\mathfrak{g}_0$. Let E be any plane in S through o, $\mathfrak{a}_0$ the corresponding maximal abelian subspace of $\mathfrak{p}_0$ and A the subgroup $\exp(\mathfrak{a}_0)$ of G. Let C be any Weyl chamber in $\mathfrak{a}_0$. Then the dual space of $\mathfrak{a}_0$ can be ordered by calling a linear function λ on $\mathfrak{a}_0$ positive if $\lambda(H)>0$ for all $H\in C$. This ordering gives rise to an Iwasawa decomposition of G, $G=KAN$, where N is a connected nilpotent subgroup of G. It can for example be described by

$$N = \left\{ z \in G \,\middle|\, \lim_{t\to\infty} \exp(-tH)z\exp(tH) = e \right\},$$

H being an arbitrary fixed element in C. The group N depends on the triple (o, E, C). However, well-known conjugacy theorems show that if N' is the group defined by a different triple (o', E', C') then $N'=gNg^{-1}$ for some $g\in G$.

DEFINITION. A *horocycle* in S is an orbit of a subgroup of the form gNg^{-1}, g being any element in G.

Let $t\to\gamma(t)$ (t real) be any geodesic in S and put $T_t=s_{t/2}s_0$ where s_τ denotes the geodesic symmetry of S with respect to the point $\gamma(\tau)$. The elements of the one-parameter subgroup T_t (t real) are called *transvections* along γ. Two horocycles ξ_1, ξ_2 are called *parallel* if there exists a geodesic γ intersecting ξ_1 and ξ_2 under a right angle such that $T\cdot\xi_1=\xi_2$ for a suitable transvection T along γ. For each fixed $g\in G$, the orbits of the group gNg^{-1} form a parallel family of horocycles.

Let M and M', respectively, denote the centralizer and normalizer of A in K. The group $W=M'/M$, which is finite, is called the Weyl group.

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[1] This work was supported in part by the National Science Foundation, NSF GP-149.

PROPOSITION 1.1. *The group G acts transitively on the set of horocycles in S. The subgroup of G which maps the horocycle $N \cdot o$ into itself equals MN.*

Let $\hat{S}$ denote the set of horocycles in S. Then we have the natural identifications

$$S = G/K, \qquad \hat{S} = G/MN$$

the latter of which turns $\hat{S}$ into a manifold, which we call the *dual space of S*.

PROPOSITION 1.2.

(i) *The mapping*

$$\phi\colon (kM, a) \to kaK$$

is a differentiable mapping of $(K/M)\times A$ onto S and a regular w-to-one mapping of $(K/M)\times A'$ onto S'.

(ii) *The mapping*

$$\hat{\phi}\colon (kM, a) \to kaMN$$

is a diffeomorphism of $(K/M)\times A$ onto $\hat{S}$.

In statement (i) which is well known, w denotes the order of W, A' is the set of regular elements in A and S' is the set of points in S which lie on only one plane through o.

PROPOSITION 1.3. *The following relations are natural identifications of the double coset spaces on the left:*

(i) $K\backslash G/K = A/W$;

(ii) $MN\backslash G/MN = A\times W$.

Statement (i) is again well known; (ii) is a sharpening of the lemma of Bruhat (see [6]) which identifies $MAN\backslash G/MAN$ with W. The proofs of these results use the following lemma.

LEMMA 1.4.

(i) *Let s_0 denote the geodesic symmetry of S with respect to o and let θ denote the involution $g\to s_0 g s_0$ of G. Then*

$$(N\theta(N)) \cap K = \{e\}.$$

(ii) *Let C and C' be two Weyl chambers in $\mathfrak{a}_0$ and $G=KAN$, $G=KAN'$ the corresponding Iwasawa decompositions. Then*

$$(NN') \cap (MA) = \{e\}.$$

2. **Invariant differential operators on the space of horocycles.** For any manifold V, $C^\infty(V)$ and $C_c^\infty(V)$ shall denote the spaces of C^∞

functions on V (respectively, C^∞ functions on V with compact support). Let $\boldsymbol{D}(S)$ and $\boldsymbol{D}(\hat{S})$, respectively, denote the algebras of all G-invariant differential operators on S and $\hat{S}$. Let $S(\mathfrak{a}_0)$ denote the symmetric algebra over $\mathfrak{a}_0$ and $J(\mathfrak{a}_0)$ the set of W-invariants in $S(\mathfrak{a}_0)$. There exists an isomorphism Γ of $\boldsymbol{D}(S)$ onto $J(\mathfrak{a}_0)$ (cf. [**7**, Theorem 1, p. 260], also [**9**, p. 432]). To describe $\boldsymbol{D}(\hat{S})$, consider $\hat{S}$ as a fibre bundle with base K/M, the projection $p: \hat{S} \to K/M$ being the mapping which to each horocycle associates the parallel horocycle through 0. Since each fibre F can be identified with A, each $U \in S(\mathfrak{a}_0)$ determines a differential operator U_F on F. Denoting by $f|F$ the restriction of a function f on $\hat{S}$ to F we define an endomorphism D_U on $C^\infty(\hat{S})$ by

$$(D_U f) \mid F = U_F(f \mid F) \qquad f \in C^\infty(\hat{S}),$$

F being any fibre. It is easy to prove that the mapping $U \to D_U$ is a homomorphism of $S(\mathfrak{a}_0)$ into $\boldsymbol{D}(\hat{S})$.

THEOREM 2.1. *The mapping $U \to D_U$ is an isomorphism of $S(\mathfrak{a}_0)$ onto $\boldsymbol{D}(\hat{S})$. In particular, $\boldsymbol{D}(\hat{S})$ is commutative.*

Although G/MN is not in general reductive, $\boldsymbol{D}(\hat{S})$ can be determined from the polynomial invariants for the action of MN on the tangent space to G/MN at MN (cf. [**8**, Theorem 10]). It is then found that the algebra of these invariants is in a natural way isomorphic to $S(\mathfrak{a}_0)$, whereupon Theorem 2.1 follows. Let $\hat{\Gamma}$ denote the inverse of the mapping $U \to D_U$.

3. **The Radon transform.** Let ξ be any horocycle in S, ds_ξ the volume element on ξ. For $f \in C_c^\infty(S)$ put

$$\hat{f}(\xi) = \int_\xi f(s) ds_\xi, \qquad \xi \in \hat{S}.$$

The function $\hat{f}$ will be called the *Radon transform* of f.

THEOREM 3.1. *The mapping $f \to \hat{f}$ is a one-to-one linear mapping of $C_c^\infty(S)$ into $C_c^\infty(\hat{S})$.*

Now extend $\mathfrak{a}_0$ to a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$; of the corresponding roots let P_+ denote the set of those whose restriction to $\mathfrak{a}_0$ is positive (in the ordering defined by C). Put $\rho = \frac{1}{2}\sum_{\alpha \in P_+} \alpha$ and let $p \to {}'p$ denote the unique automorphism of $S(\mathfrak{a}_0)$ given by ${}'H = H - \rho(H)$ $(H \in \mathfrak{a}_0)$ (cf. [**7**, p. 260]).

THEOREM 3.2. *Let ${}'\boldsymbol{D}(\hat{S})$ be given by*

$${}'\boldsymbol{D}(\hat{S}) = \{E \in \boldsymbol{D}(\hat{S}) \mid {}'(\hat{\Gamma}(E)) \in J(\mathfrak{a}_0)\},$$

and let $D \rightarrow \hat{D}$ denote the isomorphism of $\boldsymbol{D}(S)$ onto $'\boldsymbol{D}(\hat{S})$ such that

$$'(\hat{\Gamma}(\hat{D})) = \Gamma(D), \qquad D \in \boldsymbol{D}(S).$$

Then

$$(Df)\hat{} = \hat{D}\hat{f} \quad \text{for} \quad f \in C_c^\infty(S).$$

In view of the duality between points and horocycles there is a natural dual to the transform $f \rightarrow \hat{f}$. This dual transform associates to each function $\psi \in C^\infty(\hat{S})$ a function $\check{\psi} \in C^\infty(S)$ given by

$$\check{\psi}(p) = \int_{\xi \cap p = p} \psi(\xi)\, dm(\xi), \qquad p \in S,$$

where the integral on the right is the average of ψ over the (compact) set of horocycles passing through p. We put

$$I_f = (\hat{f})\check{}, \qquad f \in C_c^\infty(S)$$

and wish to relate f and I_f.

THEOREM 3.3. *Suppose the group $G = I_0(S)$ is a complex Lie group. Then*

$$(1) \qquad \square I_f = cf, \qquad f \in C_c^\infty(S),$$

where c is a constant $\neq 0$ and $\square$ is a certain operator in $\boldsymbol{D}(S)$, both independent of f.

We shall now indicate the definition of $\square$. Let J denote the complex structure of the Lie algebra $\mathfrak{g}_0$. Then the Cartan subalgebra $\mathfrak{h}_0$ above can be taken as $\mathfrak{a}_0 + J\mathfrak{a}_0$ and can then be considered as a complex Cartan subalgebra of $\mathfrak{g}_0$ (considered as a complex Lie algebra). Let Δ' denote the corresponding set of nonzero roots and for each $\alpha \in \Delta'$ select H'_α in $\mathfrak{h}_0$ such that $B'(H'_\alpha, H) = \alpha(H)$ $(H \in \mathfrak{h}_0)$ where B' denotes the Killing form of the complex algebra $\mathfrak{g}_0$. Then $H'_\alpha \in \mathfrak{a}_0$ and the element $\prod_{\alpha \in \Delta'} H'_\alpha$ in $S(\mathfrak{a}_0)$ is invariant under the Weyl group W. Then $\square$ is the unique element in $\boldsymbol{D}(S)$ such that

$$\Gamma(\square) = \prod_{\alpha \in \Delta'} H'_\alpha.$$

The proof of Theorem 3.3 is based on Theorem 3 in Harish-Chandra [5] (see also Gelfand-Naĭmark [4, p. 156]), together with the Darboux equation for S ([9, p. 442]). In the case when S is the space of positive definite Hermitian $n \times n$ matrices a formula closely related to (1) was given in Gelfand [1]. Radon's classical problem of representing a function in $\mathbf{R}^n$ by means of its integrals over hyperplanes was solved by Radon [13] and John [10]. Generalizations to Riemannian manifolds of constant curvature were given by Helgason [8], Semyanistyi [15] and Gelfand-Graev-Vilenkin [3].

4. Applications to invariant differential equations. We shall now indicate how Theorem 3.3 can be used to reduce any G-invariant differential equation on S to a differential equation with *constant* coefficients on a Euclidean space. The procedure is reminiscent of the method of plane waves for solving homogeneous hyperbolic equations with *constant* coefficients (see John [11]).

DEFINITION. A function on S is called a *plane wave* if there exists a parallel family Ξ of horocycles in S such that (i) $S=\bigcup_{\xi\in\Xi}\xi$; (ii) For each $\xi\in\Xi$, f is constant on ξ.

Theorem 3.3 can be interpreted as a decomposition of an arbitrary function $f\in C_c^\infty(S)$ into plane waves.

Now select $g\in G$ such that Ξ is the family of orbits of the group gNg^{-1}. The manifold $gAg^{-1}\cdot o$ intersects each horocycle $\xi\in\Xi$ orthogonally. A plane wave f (corresponding to Ξ) can be regarded as a function f^* on the Euclidean space A. If $D\in\mathbf{D}(S)$, then Df is also a plane wave (corresponding to Ξ) and $(Df)^*=D_Af^*$, where D_A is a differential operator on A. Using the fact that $aNa^{-1}\subset N$ for each $a\in A$ it is easily proved (cf. [7, Lemma 3, p. 247] or [12, Theorem 1]) that D_A is invariant under all translations on A. Thus an invariant differential equation in the space of plane waves (for a fixed Ξ) amounts to a differential equation with constant coefficients on the Euclidean space A. Using Theorem 3.3, and the fact that $\square$ commutes elementwise with $\mathbf{D}(S)$, an invariant differential equation for arbitrary functions on S can be reduced to a differential equation with constant coefficients (and is thus, in principle, solvable).

EXAMPLE: THE WAVE EQUATION ON S. For an illustration of the procedure above we give now an explicit global solution of the wave equation on S ($I_0(S)$ assumed complex).

Let Δ denote the Laplacian on S and let $f\in C_c^\infty(S)$. Consider the differential equation

$$(1)\qquad \Delta u = \frac{\partial^2 u}{\partial t^2}$$

with initial data

$$(2)\qquad u(p,0)=0;\quad \left\{\frac{\partial}{\partial t}u(p,t)\right\}_{t=0}=f(p)\qquad (p\in S).$$

Let Δ_A denote the Laplacian on A (in the metric induced by E), $\|\rho\|$ the length of the vector ρ in §3. Given $a\in A$, let log a denote the unique element $H\in\mathfrak{a}_0$ for which exp $H=a$. For simplicity, let e^ρ denote the function $a\to e^{\rho(\log a)}$ on A. Let ξ denote the horocycle $N\cdot o$.

Given $x\in G$, $k\in K$, consider the function

$$F_{k,x}(a) = \int_\xi f(xka\cdot s)ds_\xi \qquad (a\in A)$$

and the differential equation on $A\times R$,

$$(\Delta_A - \|\rho\|^2)V^t_{k,x} = \frac{\partial^2}{\partial t^2}V^t_{k,x}, \tag{3}$$

with initial data

$$V^0_{k,x} = 0; \qquad \left\{\frac{\partial}{\partial t}V^t_{k,x}\right\}_{t=0} = e^\rho F_{k,x}.$$

Equation (3) is just the equation for damped waves in the Euclidean space A and is explicitly solvable (see e.g. [**14**, p. 88]). The solution of (1) is now given by

$$u(p, t) = c \quad \Box_p(V(p, t)),$$

where

$$V(xK, t) = \int_K V^t_{k,x}(e)dk. \tag{4}$$

Here dk is the normalized Haar measure on K and c is the same constant as in Theorem 3.3. It is not hard to see that the integral in (4) is invariant under each substitution $x\to xu$ $(u\in K)$ so the function $V(p, t)$ is indeed well defined.

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