A COMPUTATION OF THE ACTION OF THE MAPPING CLASS GROUP ON ISOTOPY CLASSES OF CURVES AND ARCS IN SURFACES

by

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Robert Clark Penner

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ABSTRACT

Let $MC(F_g)$ denote the group of homeomorphisms modulo isotopy of the g-holed torus F_g; let $\mathcal{S}'(F_g)$ denote the collection of isotopy classes of closed one-submanifolds of F_g, no component of which bounds a disc in F_g. Max Dehn gave a finite collection of generators for $MC(F_g)$; he also described a one-to-one correspondence between $\mathcal{S}'(F_g)$ and a certain subset of $\mathbb{Z}^{6g-6}$, denoted Σ_g. We describe the natural action of $MC(F_g)$ on $\mathcal{S}'(F_g)$ by computing the corresponding action of a collection of generators for $MC(F_g)$ on Σ_g. This action has an intricate but tractable description as a map from Σ_g to itself. We use this description to give an algorithm for solving the word problem for $MC(F_g)$. This computation has applications to several problems in low-dimensional topology and dynamics of surface automorphisms.

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Title: Professor of Mathematics

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SECTION 1

Let F_g denote the g-holed torus. We define a multiple curve in F_g to be an isotopy class of (unoriented) one-submanifolds embedded in F_g, no component of which bounds a disc in F_g. In 1922, M. Dehn [5] described a one-to-one correspondence between the collection of multiple curves in F_g and a subset of $\mathbb{Z}^{6g-6}$. Such a correspondence will be called a parametrization of multiple curves, and the corresponding subset of $\mathbb{Z}^{6g-6}$ will be called the collection of parameter values.

The mapping class group of F_g, denoted $MC(F_g)$, is the group of orientation-preserving homeomorphisms of F_g modulo isotopy. An element of $MC(F_g)$ is called a mapping class on F_g. In 1938, Dehn [6] exhibited a finite set of generators for $MC(F_g)$ of a certain geometrical type, which are now called Dehn twists. Thus, a mapping class may be described by a word whose letters are Dehn twists. This word is not uniquely determined as there are relations in $MC(F_g)$ amongst such words.

An orientation-preserving homeomorphism τ of F_g acts on an (unoriented) one-submanifold c embedded in F_g by taking the image of c under τ, and this gives a well-defined action of $MC(F_g)$ on the collection of multiple curves in F_g. A natural problem arises - to compute the action of Dehn twist generators for $MC(F_g)$ on parameter values of multiple curves, thus describing the action of $MC(F_g)$ on multiple curves in F_g.

In this thesis, we will explicitly compute this action for a fixed choice of parametrization. We will obtain a faithful representation of $MC(F_g)$ as a group of transformations on the collection of parameter values, and we will describe an efficient algorithm for solving the word problem for a certain collection of Dehn twist generators for

$MC(F_g)$. This computation also has direct applications to several problems in low-dimensional topology and dynamics of surface automorphisms, as we shall see. In this section, we will briefly describe how this computation is performed for the surface F_2 and indicate some of the techniques employed. We also give a brief account of some of the history.

If c is a simple closed curve in F_g, then the right and left Dehn twists, denoted $\tau_c^{\pm 1}$, are defined by cutting F_g along c, twisting by $\pm 2\pi$ and regluing. The definition of the direction of the twist depends only on the orientation of the surface F_g. (We will give a more detailed definition in Section 3.) Thus, if c and d are the simple closed curves in the surface F_1 indicated in Figure 1.1a, then $\tau_c^{+1}d$ and $\tau_c^{-1}d$ are the curves pictured in Figures 1.1b and 1.1c, respectively. In fact, $MC(F_1)$ is easily shown to be generated by τ_c and τ_d. For surfaces of higher genera, R. Lickorish [11] has independently refined Dehn's original set of generators of $MC(F_g)$ to a more useful set of 3g-1 curves along which to perform Dehn twists. These curves are pictured on the surface F_2 in Figure 1.2. (The extensions to higher genera will be discussed in Section 3.)

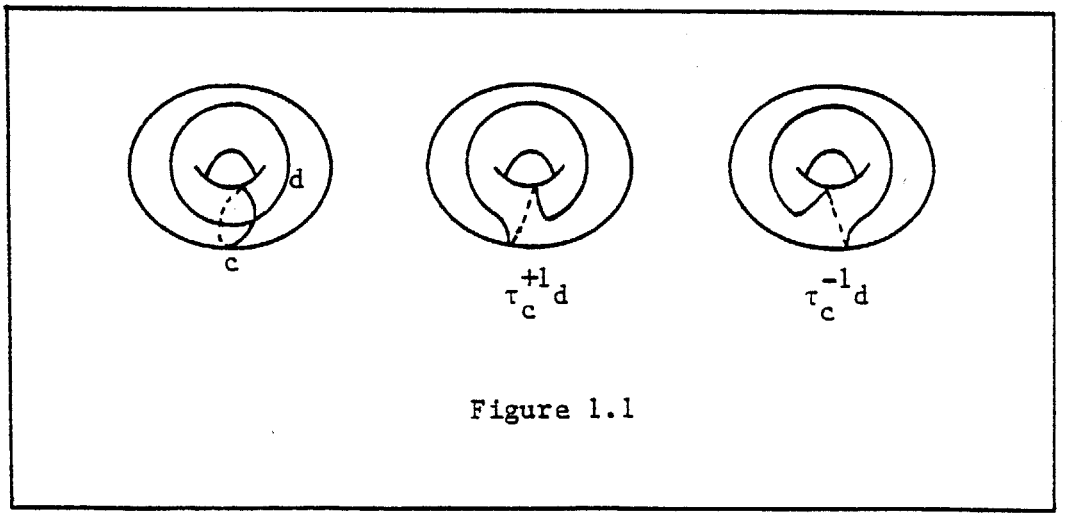


Figure 1.1

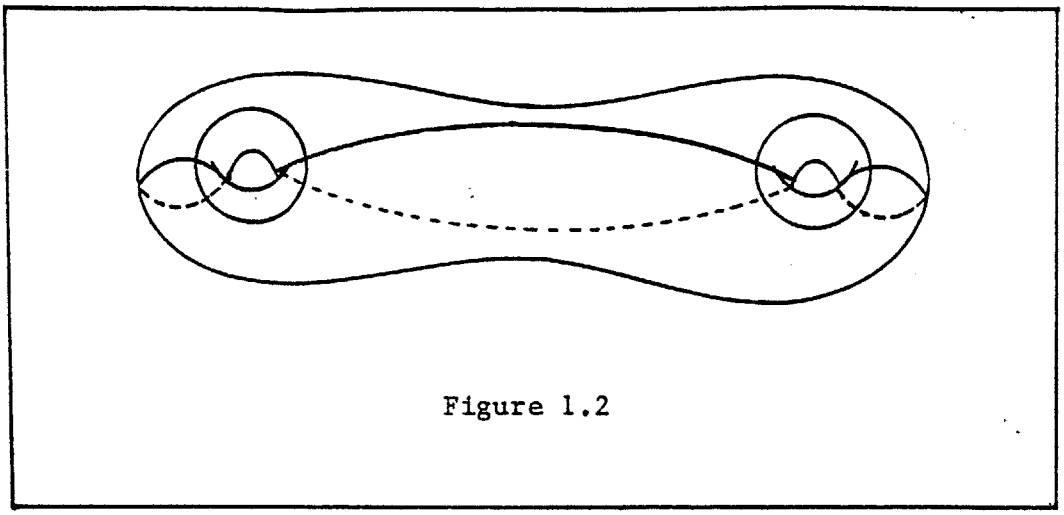


Figure 1.2

$MC(F_1)$ can be shown to be isomorphic to the group of two-by-two integral matrices of determinant one, and one has a complete set of relations between τ_c and τ_d. For a closed surface of genus two, J. Birman and M. Hilden [3] have given a complete set of relations amongst the Lickorish generators. For closed surfaces of arbitrary genus, W. Thurston and A.Hatcher [19] have given an algorithm for constructing a complete set of relations for $MC(F_g)$, but their results are quite complicated.

Just as Lickorish independently refined Dehn's original set of generators, Thurston [17] rediscovered Dehn's parametrization for multiple curves. Dehn's parametrization apparently was not published, but appears in some lecture notes in the Archives of the University of Texas at Austin [5]. (I am grateful to J. Stillwell for supplying me a translation of these notes from the original German.) We will refer to this parametrization as the Dehn-Thurston parametrization.

The Dehn-Thurston parametrization depends on several choices of convention, one of which is a certain decomposition of the surface F_g. We define a <u>pants decomposition</u> of F_g to be a collection $\{K_i\}$ of disjoint simple closed curves in F_g so that each component C of $F_g \setminus \bigcup \{K_i\}$ is topologically a sphere with three disjoint closed discs deleted. Every

surface F_g has a pants decomposition consisting of 3g-3 curves. (In fact, every pants decomposition of F_g has exactly this many curves.) Some examples of pants decompositions of the surface F_2 are indicated in Figure 1.3. A pair of pants is a sphere minus the interiors of three disjoint closed discs; it is a planar surface with boundary having three boundary components. Note that we do not require the closure of the set C to be, topologically, a closed pair of pants; see Figures 1.3b and 1.3c. We only require that C itself be the interior of a pair of pants.

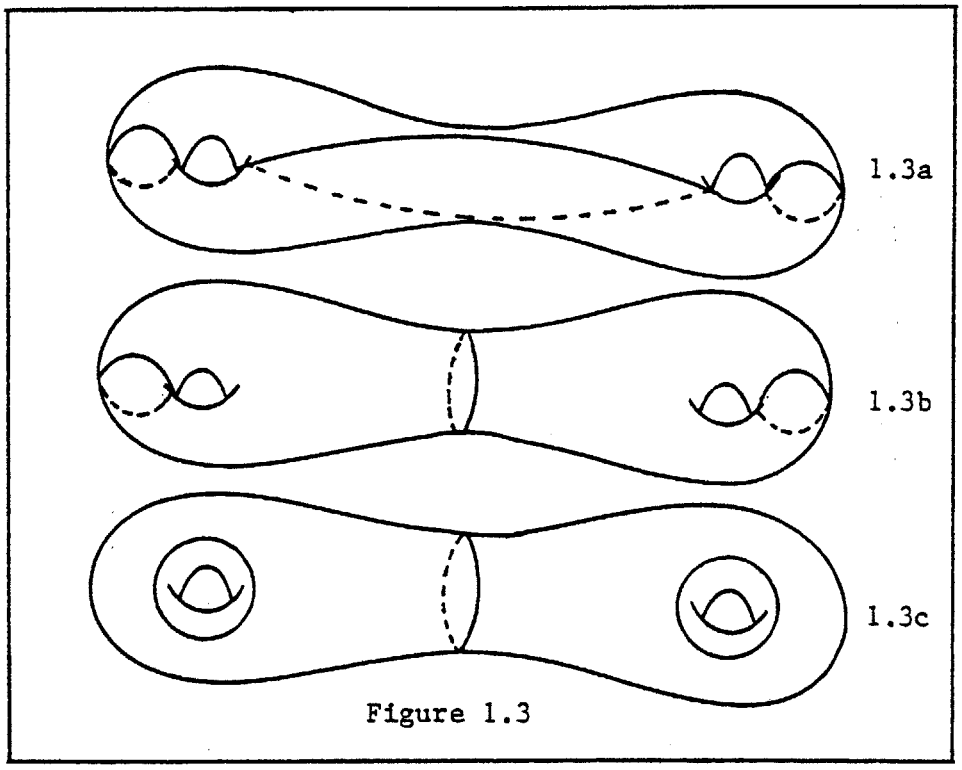


Figure 1.3

Very roughly speaking, the Dehn-Thurston parametrization for multiple curves counts how many times the curve crosses each of the "pants curves" K_i, and how many times it twists around while going from one component of $F_g \setminus \cup\{K_i\}$ to another.

Corresponding to the pants decomposition in Figure 1.3a (plus some other choices of convention), the Dehn-Thurston theorem gives a

parametrization of the collection of multiple curves in F_2. Comparing Figures 1.2 and 1.3a, we note that three of the Lickorish generating curves are curves in the pants decomposition. It turns out that the actions on the Dehn-Thurston parameter values of Dehn twists along these curves are simple linear maps and thus trivial to compute. This fact was noted by Dehn.

However, the action of Dehn twists along the other two curves in the Lickorish generating set are not nearly so simple. To tackle the problem of computing them, we note that these curves are curves in the pants decomposition indicated in Figure 1.3c. If we had a way of computing the Dehn-Thurston parameter values relative to the pants decompositions in Figure 1.3c from the parameter values relative to the pants decomposition in Figure 1.3a and vice-versa, then we would be able to compute the action of each of the Lickorish generators relative to the original pants decomposition in Figure 1.3a. This is in fact what we do. The philosophy comes from linear algebra: if a transformation (a Dehn twist) is hard to compute, change basis (pants decomposition).

We pass from Figure 1.3a to Figure 1.3c by means of two <u>elementary transformations</u>, which we now describe. The first one takes us from the pants decomposition in Figure 1.3a to the one in Figure 1.3b. It may also be described as the transformation pictured in Figure 1.4b; cutting along the right-most and left-most curves in Figures 1.2a and 1.3b gives us the surface pictured in Figure 1.4b. The second transformation takes us from the pants decomposition in Figure 1.3b to the one in Figure 1.3c. It may also be described by two applications of the transformation pictured in Figure 1.4a; cutting along the nullhomologous curves in Figures 1.3b and 1.3c gives us <u>two</u> copies of the surface pictured in Figure 1.4a.

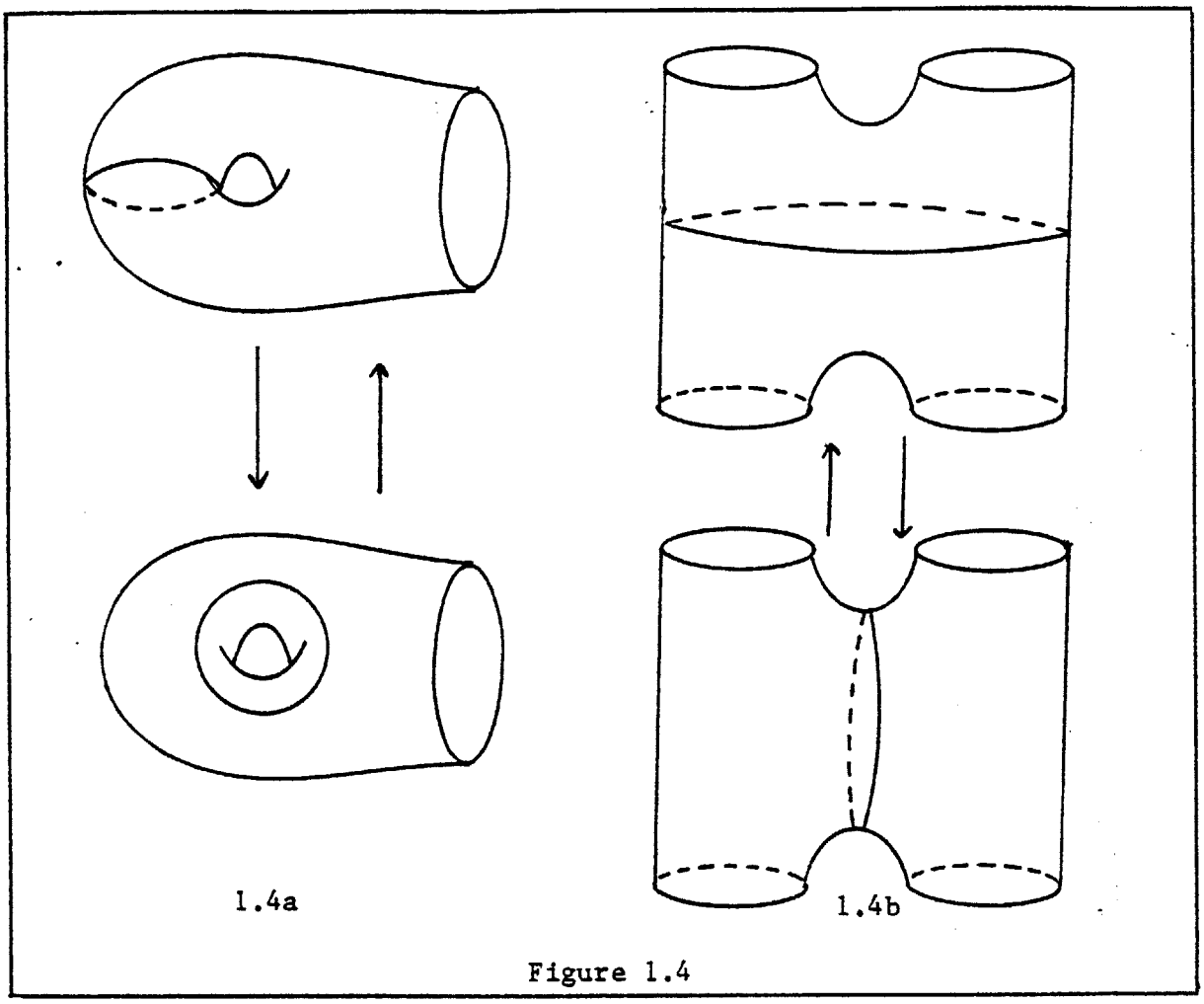


Figure 1.4

We will call the transformation pictured in Figure 1.4a and 1.4b the first and second elementary transformations, respectively. Thus, the computation of the action of $MC(F_2)$ on the collection of multiple curves in F_2 is reduced to the computation of the two elementary transformations. In fact the same procedure works for surfaces of arbitrary genus; there exists a collection of pants decompositions of F_g, all related by sequences of elementary transformations, so that each of the Lickorish generating curves is a pants curve in at least one of the pants decompostions. Our problem reduces in general to the computation of the two elementary transformations. I am grateful to D. Gabai [9] for showing this reduction, which he discovered.

This thesis is concerned with computing the two elementary

transformations. The first elementary transformation is relatively easy and can be done by actually isotoping curves and arcs about on the torus-minus-a-disc. This has been done successfully by Gabai. The second elementary transformation requires much more work, yet the techniques we develop to handle the second elementary transformation also apply to the first elementary transformation.

At this juncture of the exposition, it would be pointless to give the formulas describing the two elementary transformations. The reader wishing to see what the formulas involve is encouraged to read Sections 2 and 3 and then skip to Section 8 where these formulas are presented and some applications discussed. We describe briefly here the nature of the results we obtain.

The subset of $\mathbb{Z}^{6g-6}$ that parametrizes the collection of multiple curves in F_g naturally embeds as a subset of $\mathbb{Z}^n$ in some high-dimensional Euclidean ball B^n. To each element ψ of $MC(F_g)$ corresponds a finite simplicial decomposition K_ψ of B^n. ψ acts simplicially as a map of K_ψ to itself, and, what is more, it acts like an integral matrix on each top-dimensional simplex of K_ψ. We call such a transformation a <u>piecewise-integral</u> transformation. Our main theorem may be stated as follows.

<u>Theorem 1.1: The action of $MC(F_g)$ on the collection of multiple curves in F_g admits a faithful representation as a group of piecewise-integral transformations.</u>

We prove the theorem by actually computing the action of $MC(F_g)$ on the parameter values corresponding to multiple curves relative to a fixed parametrization. We then note that the action is piecewise-integral and check faithfulness.

We remarked previously that our explicit computations in fact

describe the action of words in Lickorish's generators on the collection of multiple curves in F_g, rather than the action of $MC(F_g)$ itself. We use this fact to our advantage by applying our theorem to give a practical algorithm for solving the word problem in Lickorish's generators. Using the Alexander Lemma, one can easily show (see Proposition 8.1) that a mapping class on F_g contains the identity if and only if it fixes the isotopy classes of a certain collection of 2g+1 simple closed curves. This result proves the faithfulness of our respresentation. Moreover, one can check if a word in Lickorish's generators is the identity in $MC(F_g)$ simply by computing whether the word fixes 2g+1 particular multiple curves. This fact immediately gives the following Corollary.

Corollary 1.1: There is a practical algorithm for solving the word problem in Lickorish's generators for $MC(F_g)$.

An outline of this thesis is as follows: In Section 2, we present the Dehn-Thurston parametrization of multiple curves and derive a new parametrization of such curves that is more useful for our purposes, and we give some basic definitions. In Section 3, we give some additional basic definitions, and we reduce our main computation to the computation of the two elementary transformations. These are in turn performed in Sections 6 and 7. Section 5 develops the main technical tools used in Sections 6 and 7. Section 4 contains results about one-submanifolds immersed in surfaces which are used in Section 5; these results have an independent interest as well. In Section 8, we discuss the formulas derived and mention several applications.

I should remark that though we have restricted attention to the action of $MC(F_g)$ on multiple curves in F_g, the computations in this thesis in fact apply more generally to any surface of negative Euler characteristic.

I should also remark that there is a natural generalization of multiple curves called measured train tracks. Furthermore, the (discrete) set of multiple curves sits inside the (connected) space of measured train tracks in a natural way. In fact, our computations can be made to describe the action of $MC(F_g)$ on the space of measured train tracks. Thus, though we restict our attention in this thesis to the essentially combinatorial setting of multiple curves, our results apply more generally to the analytical setting of measured train tracks. This point of view will be discussed in Section 8.

SECTION 2

Let F be a compact oriented surface. If i is an embedding of some one-manifold 0 into F, then we refer to the subset $i(0) \subset F$ as a one-submanifold embedded in F. Thus, a one-submanifold embedded in F is regarded an an unoriented point set. Recall that an embedding i(0) of a one-manifold 0 in F is proper if $i(\partial 0) = \partial F \cap i(0)$. Though we are primarily interested in embeddings of closed one-submanifolds in g-holed tori, the considerations of this section lead to the more general setting of one-submanifolds properly embedded in a compact oriented surface of negative Euler characteristic $\chi(F)$. We thus consider this more general setting from the outset. Note that the requirement of a negative Euler characteristic excludes precisely four surfaces: the torus, sphere, disc and cylinder.

Let c be a one-submanifold properly embedded in F. Note that components of c are either simple closed curves in F, called "closed components", or arcs properly embedded in F, called "arc components". We will say c is essential if no arc component of c can be homotoped (through proper embeddings) into ∂F; furthermore, no closed component of c can bound a disc in F. If a component of c can be homotoped into ∂F, we say that it is boundary-parallel. We will require a notion of isotopy that is slightly stronger than proper isotopy and weaker than isotopy rel ∂F. Choose, once and for all, a point x_i in each boundary component C_i of F. We allow a proper isotopy to move points about in $C_i \setminus x_i$, but it must keep the point x_i fixed. A multiple arc in F is such a proper isotopy class of essential one-submanifolds of F. We define a multiple curve in F to be an isotopy class of essential closed one-submanifolds in F. (Note that boundary-parallel closed components of multiple arcs and curves are allowed.) We denote the collection of multiple arcs in F by $\mathscr{S}(F)$ and describe an explicit parametrization for multiple arcs in this section. A parametrization for multiple arcs is a one-to-one correspondence between

$\mathcal{S}'(F)$ and some subset of $\mathbb{Z}^n$, for some n. We will refer to any subset of $\mathbb{Z}^n$ as an integral lattice.

In order to parametrize multiple arcs in F, we must choose a certain decomposition of the surface F. A pair of pants is a closed disc D minus the interiors of two disjoint closed discs contained in the interior of D. A pants decomposition $\{K_i\}$ of F is an embedded closed one-submanifold in F so that each component C of $F \setminus \cup\{K_i\}$ is the interior of a pair of pants. We do not require the closure of the set C in F to be an embedded pair of pants. Some examples of pants decompositions are given in Figure 2.1. Note that the boundary components of F are necessarily curves in a pants decomposition. One constructs such a pants decomposition $\{K_i\}$ by taking a maximal family of disjointly embedded simple closed curves in F, no two of which are freely homotopic. Two unoriented simple closed curves in F are said to be parallel if they are freely homotopic. One can check that there are $M = |\chi(F)|$ pairs of pants in a pants decomposition and that the number of components of $\{K_i\}$ is given by

$$N = \tfrac{1}{2}(3|\chi(F)| + (\text{number of boundary components of } F)).$$

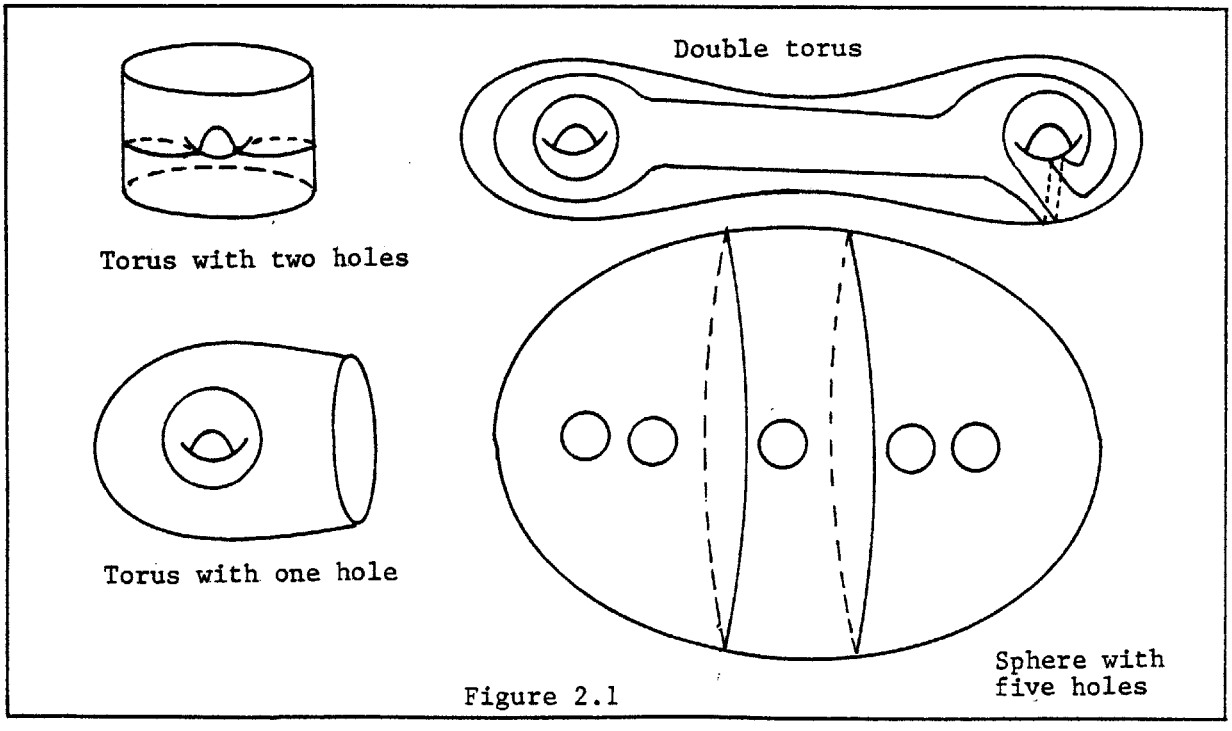


Figure 2.1

We begin with a discussion of the standard pair of pants P, which is regarded as oriented, with the boundary components ∂_k numbered as in Figure 2.2 and with the arcs $w_k \subset \partial_k$ as pictured, k=1,2,3. We refer to these arcs as windows. Two arcs c and d in P with ∂c and ∂d contianed in windows are said to be parallel if there is a disc D embedded in P with ∂D contained in $c \cup d \cup \text{Interior}(w_1 \cup w_2 \cup w_3)$.

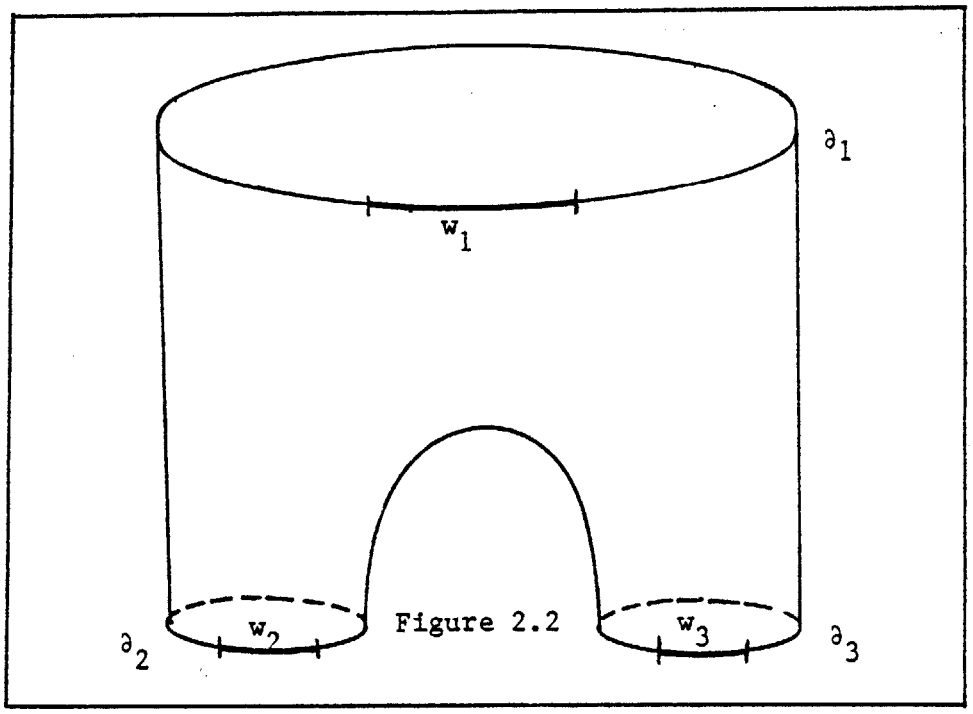


Figure 2.2

It is shown in [8, Expose 2] that two essential one-submanifolds in P with no boundary-parallel components, say c and d, satisfy $\text{card}(c \cap \partial_k) = \text{card}(d \cap \partial_k)$, k=1,2,3, if and only if there is a homeomorphism fixing ∂P component-wise and carrying c to d. Moreover, any such homeomorphism is isotopic rel ∂P to a map that is the identity off an annular neighborhood of ∂P. In general, if γ and δ are multiple arcs, we define the geometrical intersection numbers of γ and δ to be the minimum of $\text{card}(c \cap d)$, where c and d vary among the essential one-submanifolds of F representing γ and δ. The result from [8] just

mentioned suggests that the geometrical intersection numbers of a multiple arc with the (isotopy class of) each of the boundary components ∂_k of P are useful in parametrizing multiple arcs in P. Note that if c is a properly embedded one-submanifold in P, and $m_k = \text{card}(c \cap \partial_k)$, $k=1,2,3$, then $m_1+m_2+m_3$ is even.

Given any triple m_1,m_2,m_3 of non-negative integers with $m_1+m_2+m_3$ even, we now construct a multiple arc attaining the intersection numbers m_1,m_2 and m_3 and meeting ∂_k in w_k (if at all).

Construction 2.1: Let 1_{ij} denote the arc in the standard pair of pants indicated in Figure 2.3, for $i \leq j = 1,2,3$. Note that 1_{ij} connects boundary components ∂_i and ∂_j of P. We call an arc parallel to some 1_{ij} a canonical piece. Note that by definition of parallelism for arcs in the standard pair of pants, the endpoints of an arc parallel to some 1_{ij} always lie in the windows w_k.

For each triple m_1,m_2,m_3 with $m_1+m_2+m_3$ even, we construct a one-submanifold of P with no boundary-parallel components, called a canonical model. We construct these canonical models by taking various disjointly embedded collections of the canonical pieces. There are four cases, as follows:

a) The m_i satisfy all possible triangle inequalities (that is, $m_1 \leq m_2+m_3$, $m_2 \leq m_1+m_3$ and $m_3 \leq m_1+m_2$). In this case, we take a disjointly embedded collection consisting of:

$(m_1+m_2-m_3)/2$ mutually disjoint parallel copies of arc 1_{12}.

$(m_1+m_3-m_2)/2$ mutually disjoint parallel copies of arc 1_{13}.

$(m_2+m_3-m_1)/2$ mutually disjoint parallel copies of arc 1_{23}.

An example when $m_1=3$, $m_2=1$ and $m_3=2$ is given in Figure 2.4 a.

b) $m_1 > m_2+m_3$. In this case we take a disjointly embedded collection consisting of:

m_2 mutually disjoint parallel copies of arc 1_{12}.

m_3 mutually disjoint parallel copies of arc 1_{13}.

$(m_1-m_2-m_3)/2$ mutually disjoint parallel copies of arc 1_{11}.

An example when $m_1=4$, $m_2=1$ and $m_3=1$ is pictured in Figure 2.4b. The next two cases are similar to case b).

c) $m_2 > m_1+m_3$. In this case we take a disjointly embedded collection consisting of:

m_1 mutually disjoint parallel copies of arc 1_{12}.

m_3 mutually disjoint parallel copies of arc 1_{23}.

$(m_2-m_1-m_3)/2$ mutually disjoint parallel copies of arc 1_{22}.

d) $m_3 > m_1+m_2$. In this case we take a disjointly embedded collection consisting of:

m_2 mutually disjoint parallel copies of arc 1_{23}.

m_1 mutually disjoint parallel copies of arc 1_{13}.

$(m_3-m_1-m_2)/2$ mutually disjoint parallel copies of arc 1_{33}.

Remark 2.1: Let $\ell_{ij}(m_1,m_2,m_3)$ be the number of canonical pieces of type 1_{ij} in the canonical model described in Construction 2.1. By counting the number of times each canonical piece intersects the boundary curves ∂_k, one easily verifies that the following identities are valid in all cases of Construction 2.1.

$$m_1 = 2\ell_{11}+\ell_{12}+\ell_{13}$$
$$m_2 = 2\ell_{22}+\ell_{12}+\ell_{13}$$
$$m_3 = 2\ell_{33}+\ell_{13}+\ell_{23}$$

Remark 2.2: It is easy to check that it is possible to choose such

disjointly embedded collections for each case, and moreover that all possible combinations of disjointly embedded canonical pieces are described by these four cases. For instance, l_{11} and l_{33} cannot be embedded disjointly, as is obvious geometrically. (It also follows immediately from the main theorem of Section 4.)

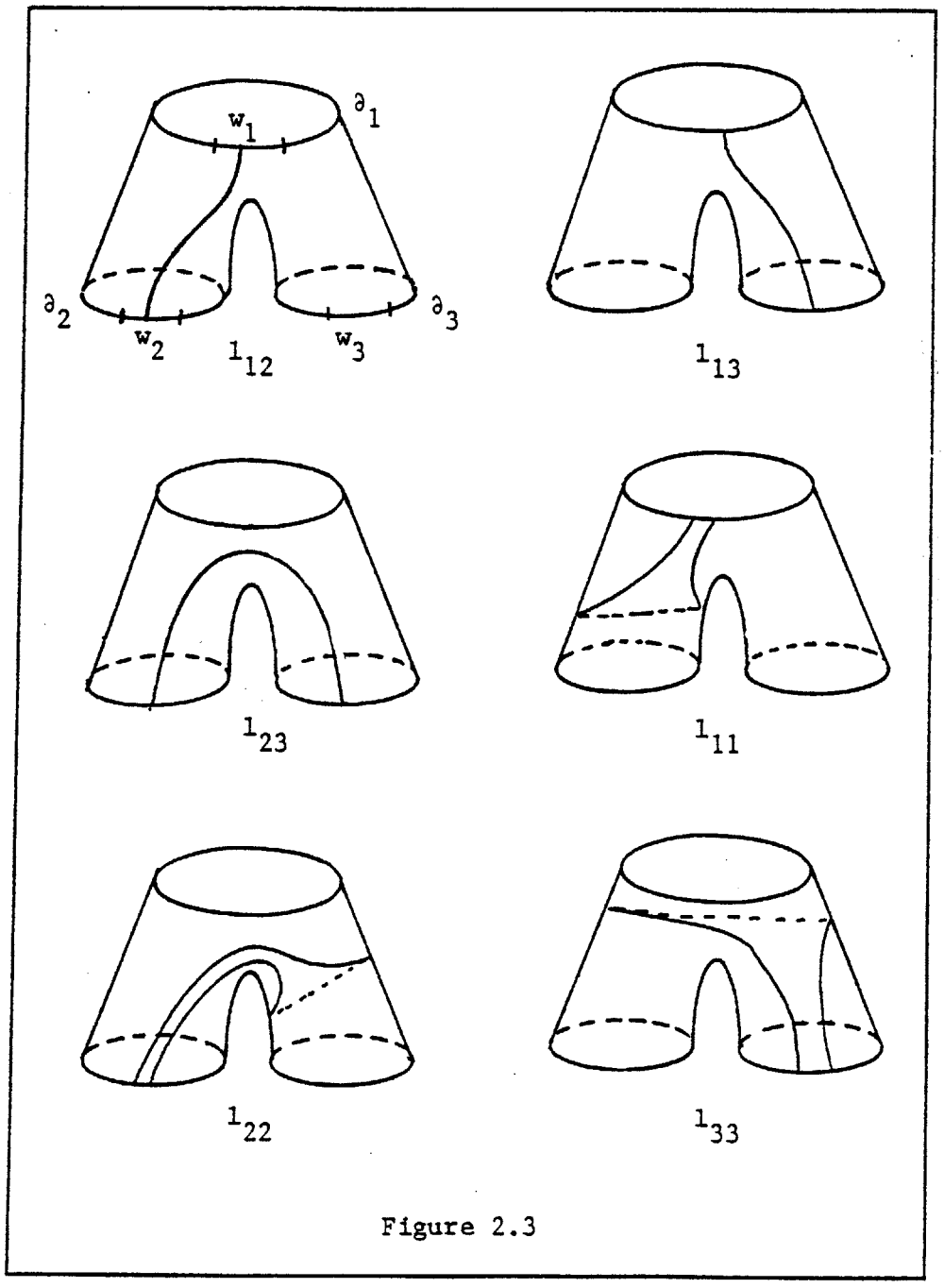
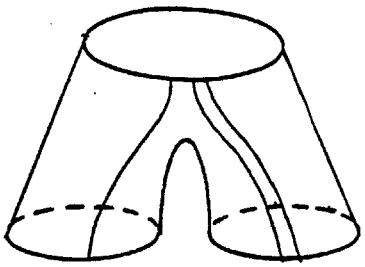
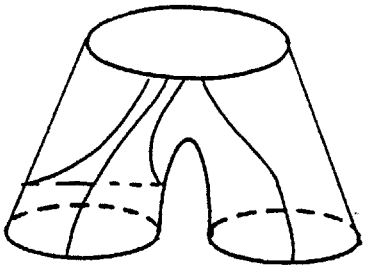


Figure 2.3



$m_1=3,\ m_2=1,\ m_3=2.$

2.4a



$m_1=4,\ m_2=1,\ m_3=1.$

2.4b

Figure 2.4

We next state and prove the main parametrization theorem; it is due to Dehn [5] and Thurston [17] independently. Subsequently, we define a different parametrization, which is the one we will use. The proof that our description does give a parametrization will follow as a corollary to the proof of the Dehn-Thurston theorem.

Theorem 2.1 (Dehn-Thurston): If F is a compact oriented surface with pants decomposition $\{K_i\}_1^N$, then there is a parametrization of $\mathcal{S}(F)$ by a subset of $(\mathbb{Z}^+)^N \times \mathbb{Z}^N$, where $\mathbb{Z}^+$ denotes the non-negative integers. The point $(m_1,\ldots,m_N)\times(t_1,\ldots,t_N) \in (\mathbb{Z}^+)^N \times \mathbb{Z}^N$ corresponds to a multiple arc if and only if the following conditions are satisfied:

a) If $m_i = 0$, then $t_i \geq 0$.

b) If K_i, K_j and K_k bound an embedded pair of pants, then $m_i+m_j+m_k$ is even.

c) If K_i bounds an embedded torus-minus-a-disc, then m_i is even.

We call this parametrization the Dehn-Thurston parametrization; it depends on certain choices which we now describe.

Let A_i be a regular neighborhood of K_i in the surface F. Choose a trivialization υ_i of A_i as follows: it is an orientation-preserving homeomorphism which carries A_i to the standard oriented annulus $A = S^1 \times [-1,1]$ if K_i is in the interior of F, and carries A_i to $S^1 \times [0,1] \subset S^1 \times [-1,1]$ if K_i is a boundary component of F. Let G be a projection of A onto the core $S^1 \times 0 \subset A$. Note that $S^1 \times 0 = \upsilon_i(K_i)$ by definition of a trivialization. Choose an embedded arc $u_i \subset K_i$, called a window, let D_j be the closure of a component of $F \setminus \cup\{A_i\}$, and choose an orientation-preserving homeomorphism f_j of D_j to P carrying $(\upsilon_i^{-1} o G^{-1} o \upsilon_i(u_i)) \cap D_j$ to a window w_1, w_2 or w_3 in the standard pair of pants P.

In practice, we regard F as embedded in S^3 and draw pictures. υ_i is chosen as the trivialization that extends across a disc in S^3 with boundary K_i, and G is taken to be the projection along the fibers in the standard annulus A. We then draw and label $\{K_i\}$, $\{u_i\}$, $f_j^{-1}(1_{12})$ and $f_j^{-1}(1_{13})$ as in Figures 2.5 and 2.6. Since $P \setminus \{1_{12}, 1_{13}\}$ is contractible, some straightening and the Alexander Lemma show that $f_j^{-1}(1_{12})$ and $f_j^{-1}(1_{13})$ determine f_j up to isotopy. For example, we let f and f' be the homeomorphism to the standard pants P corresponding to the choices in Figures 2.6a and 2.6b, respectively. The map $f' o f^{-1}$ from P to itself is isotopic to a right Dehn twist along the boundary component ∂_3 of P.

Proof of Theorem 2.1: We first describe how one computes the Dehn-Thurston parameter values for some $\gamma \in \mathcal{S}'(F)$. Define the intersection numbers m_i to be the geometrical intersection numbers of γ with (the isotopy class of) K_i. Choosing a representative c for γ, we may isotope c to attain these intersection numbers with each component of ∂A_i by [8, Proposition 3.12]. Thus, for any i, a component of $c \cap A_i$ intersects each component of ∂A_i.

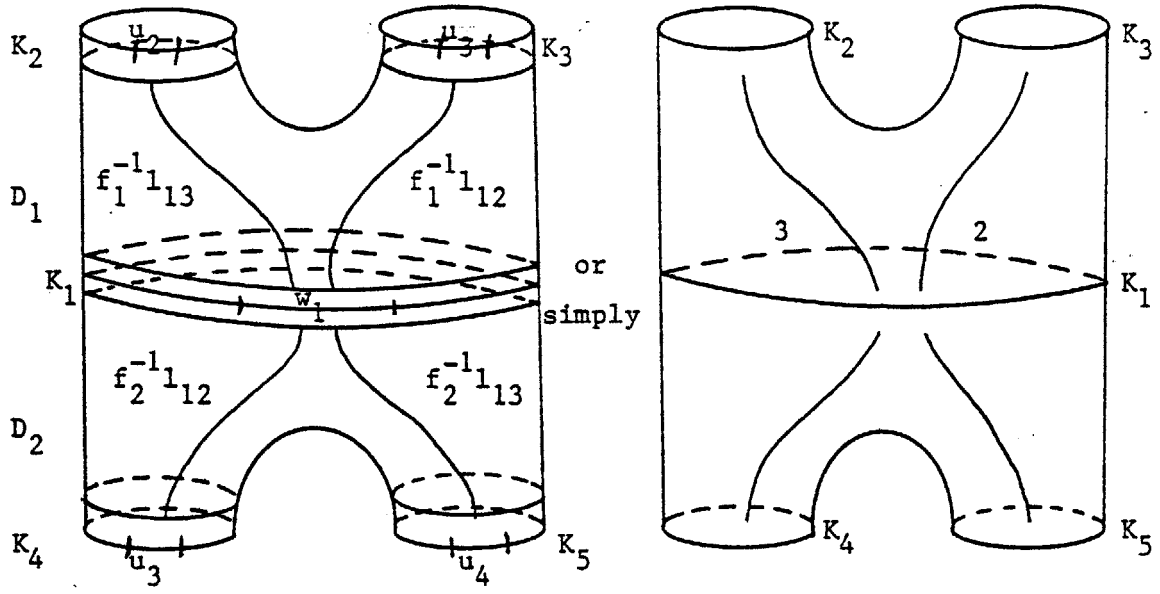
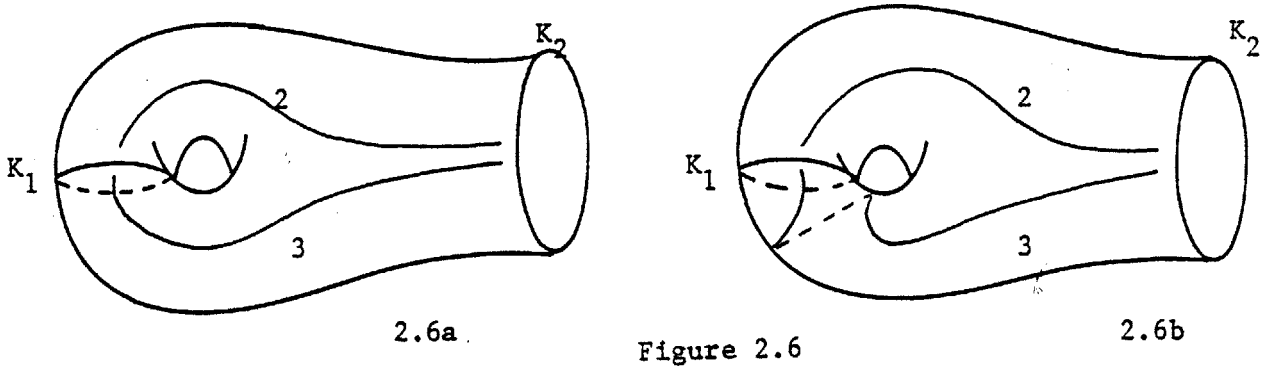


Figure 2.5



2.6a

2.6b

Figure 2.6

Isotope c so that it intersects each component of ∂A_i in $\upsilon_i^{-1} \circ G^{-1} \circ \upsilon_i(u_i)$ and so that $f_j(c \cap D_j)$ is one of the canonical models in Construction 2.1. (This is possible by the result from [8] mentioned before Construction 2.1.) We then define the <u>twisting numbers</u> t_i' as follows:

-If $m_i = 0$, take t_i to be the number of components of $\upsilon_i(c \cap A_i)$.

-If $m_i > 0$, then $|t_i|$ is defined to be the geometrical intersection

number of the isotopy class rel ∂A_i of $\upsilon_i(c \cap A_i)$ and $G^{-1}(x)$, where x is one of the boundary points of the window u_i. The sign of t_i is positive if some component of $\upsilon_i(c \cap A_i)$ twists to the right in the oriented annulus A, and the sign of t_i is negative if some component of $\upsilon_i(c \cap A_i)$ twists to the left in the oriented annulus A. (Note that if one component of $\upsilon_i(c \cap A_i)$ twists to the right, no component can twist to the left.)

It follows immediately from [8, Lemma 4.5] that at most one $\gamma \in \mathcal{S}(F)$ may achieve a particular tuple of intersection numbers and twisting numbers. One shows easily that every parameter value is achieved, using Construction 2.1. The theorem follows. ⊠

Example 2.1: Consider the pants decomposition on the surface F_2 indicated in Figure 2.7. We will draw a representative c of the multiple arc γ with Dehn-Thurston parameter values $(3,1,2)\times(2,-1,0) = (m_1,m_2,m_3) \times(t_1,t_2,t_3)$. There are three components of $c \cap A_1$ since $m_1=3$, and two of these twist to the right since $t_1=+2$. (It is geometrically impossible to have one component twist around 2 times and the other not at all.) Similarly, there is one component of $c \cap A_2$ since $m_2=+1$, and it twists once to the left since $t_2=-1$; there are two components of $c \cap A_3$ since $m_3=2$ and no twisting since $t_3=0$. Thus, we draw our representative c of γ in each of the annuli A_i, $i=1,2,3$, as in Figure 2.8a. We then connect up these arcs uniquely using the pre-images (under f_1 and f_2) of canonical pieces parallel to l_{12}, l_{23}, l_{13} as shown in Figure 2.8b.

Example 2.2: Continuing to use the choice of conventions indicated in Figure 2.7, we will draw a representative c of the multiple arc γ with Dehn-Thurston parameter values $(0,1,1)\times(2,-2,1) = (m_1,m_2,m_3) \times(t_1,t_2,t_3)$. There is one component of each of $c \cap A_2$ and $c \cap A_3$ since

$m_2 = m_3 = 1$. There are two twists to the left in A_2 and one to the right in A_3 since $t_2 = -2$ and $t_3 = 1$. Since $m_1 = 0$, c does not intersect A_1, and since $t_1 = 2$, there are two components in A_1 parallel to K_1. Again we draw our representative c of γ in each of the annuli A_i, $i = 1,2,3$, as in Figure 2.9a, and we connect up the arcs uniquely as in Figure 2.9b.

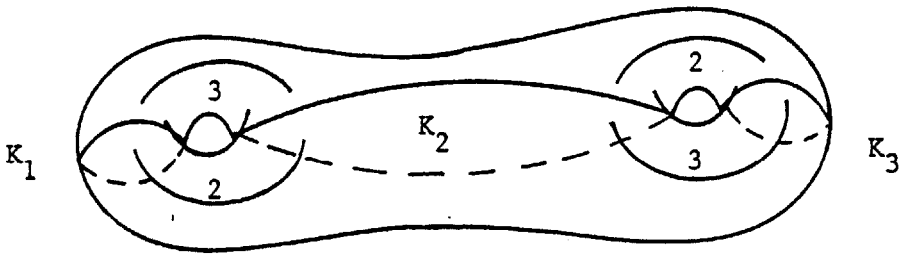
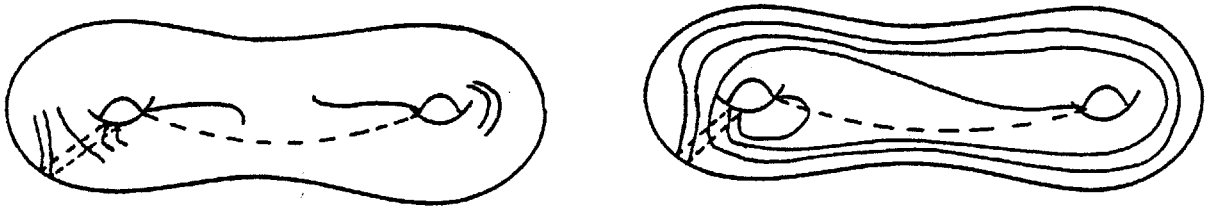
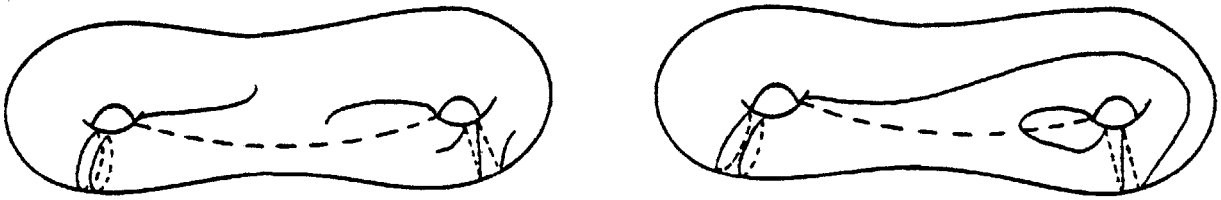


Figure 2.7



2.8a 2.8b

Figure 2.8



2.9a 2.9b

Figure 2.9

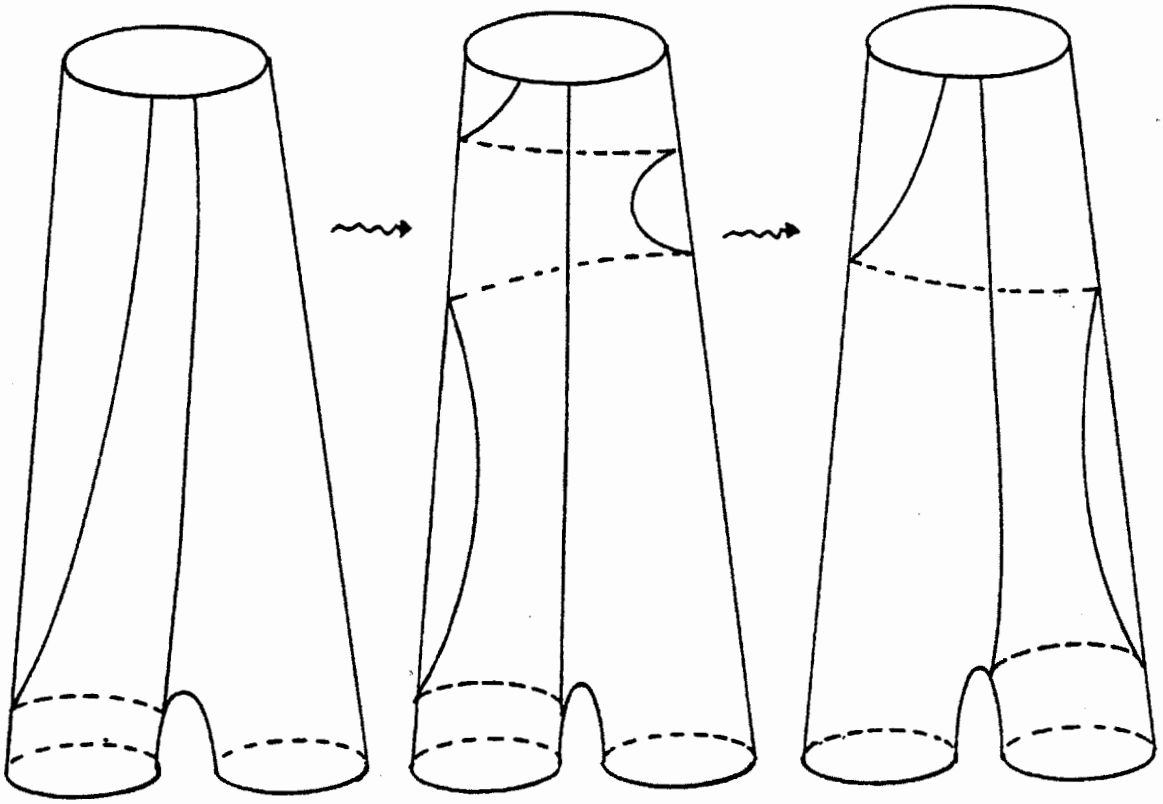
Remark 2.3: As indicated in the examples, sometimes a multiple arc has a connected representative and sometimes not. It is a hard combinatorial problem to compute the number of components of a multiple arc given only its Dehn-Thurston parameter values. Except for a few special cases, this remains an open problem.

We define a basis A for multiple arcs to be a choice of pants decomposition together with a choice of conventions as in the proof of the Dehn-Thurston theorem, including a choice of (parallelism class of) canonical pieces for each embedded pair of pants D_j. Note that the Dehn-Thurston parametrization depends on a choice of basis, and that once a basis is chosen, there is a one-to-one correspondence between $\mathscr{S}'(F)$ and the integral lattice described in Theorem 2.1.

The choice of parallelism class of canonical pieces of type l_{12}, l_{13} or l_{23} is determined by the choices in Construction 2.1. Consider our choice of canonical piece l_{11} drawn in Figure 2.10a. The isotopy indicated in Figure 2.10 shows that the arc in Figure 2.11, denoted $\hat{l}_{11}$, corresponds to a different choice of parallelism class for the canonical piece l_{11}. Similar remarks apply to the canonical pieces l_{22} and l_{33}, and we indicate some alternative choices, denoted $\hat{l}_{22}$ and $\hat{l}_{33}$, in Figure 2.11.

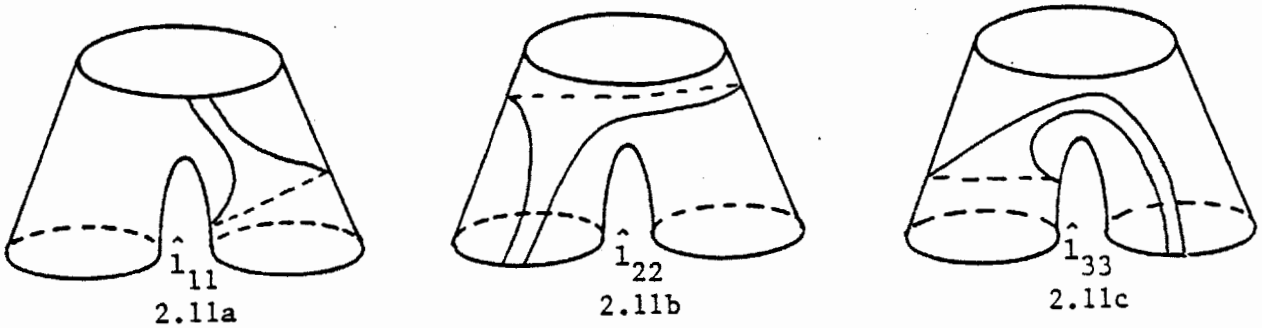
We will say that an essential one-submanifold c is in good position with respect to a basis A if $f_j(c \cap D_j)$ is a canonical model for each j; furthermore, for each i, we require that c intersect a component of ∂A_i exactly m_i times, where m_i is an intersection number of the Dehn-Thurston parametrization.

Instead of keeping track of the intersection number m_i, one might keep track of the numbers ℓ^j_{**} of canonical pieces of a good representative



2.10a 2.10b 2.10c

Figure 2.10



$\hat{1}_{11}$ $\hat{1}_{22}$ $\hat{1}_{33}$

2.11a 2.11b 2.11c

Figure 2.11

parallel to each l_{**} in each embedded pair of pants D_j. This is the idea behind our parametrization of multiple arcs. The corresponding integral lattice with respect to the basis A will be denoted $\mathscr{S}'_A(F)$.

Corollary 2.1: If F is a compact oriented surface with pants decomposition $\{K_i\}_1^N$ into M embedded pairs of pants $\{D_j\}$, then there is a parametrization of $\mathscr{S}'(F)$ by a subset of $((\mathbb{Z}^+)^6)^M \times \mathbb{Z}^N$. The subset is denoted $\mathscr{S}'_A(F)$. To the multiple arc γ, we associate the tuple

$$\prod_{j=1}^{N} (\ell_{11}^j,\ell_{12}^j,\ell_{13}^j,\ell_{22}^j,\ell_{23}^j,\ell_{33}^j) \times (t_1,\ldots,t_N) \in ((\mathbb{Z}^+)^6)^M \times \mathbb{Z}^N.$$

The t_i are the twisting numbers in the Dehn-Thurston parametrization, and the number ℓ_{**}^j denotes the number of components of $f_j c$ parallel to the canonical piece l_{**}, where c is a good representative of γ with respect to the basis A. The corresponding subset $\mathscr{S}'_A(F)$ satisfies the following conditions:

a) $\ell_{12}^j \neq 0$ implies $\ell_{33}^j = 0$, for all j.

b) $\ell_{13}^j \neq 0$ implies $\ell_{22}^j = 0$, for all j.

c) $\ell_{23}^j \neq 0$ implies $\ell_{11}^j = 0$, for all j.

d) $\ell_{11}^j \neq 0$ implies $\ell_{23}^j = \ell_{22}^j = \ell_{33}^j = 0$, for all j.

e) $\ell_{22}^j \neq 0$ implies $\ell_{13}^j = \ell_{11}^j = \ell_{33}^j = 0$, for all j.

f) $\ell_{33}^j \neq 0$ implies $\ell_{12}^j = \ell_{11}^j = \ell_{22}^j = 0$, for all j.

g) If K_i is isotopic to both $f_j^{-1}\partial_1$ and $f_j^{-1}\partial_2$, then $\ell_{11}^j = \ell_{22}^j = 0$.

h) If K_i is isotopic to both $f_j^{-1}\partial_2$ and $f_j^{-1}\partial_3$, then $\ell_{22}^j = \ell_{33}^j = 0$.

i) If K_i is isotopic to both $f_j^{-1}\partial_1$ and $f_j^{-1}\partial_3$, then $\ell_{11}^j = \ell_{33}^j = 0$.

If K_i is isotopic to $f_j^{-1}\partial_k$, we define

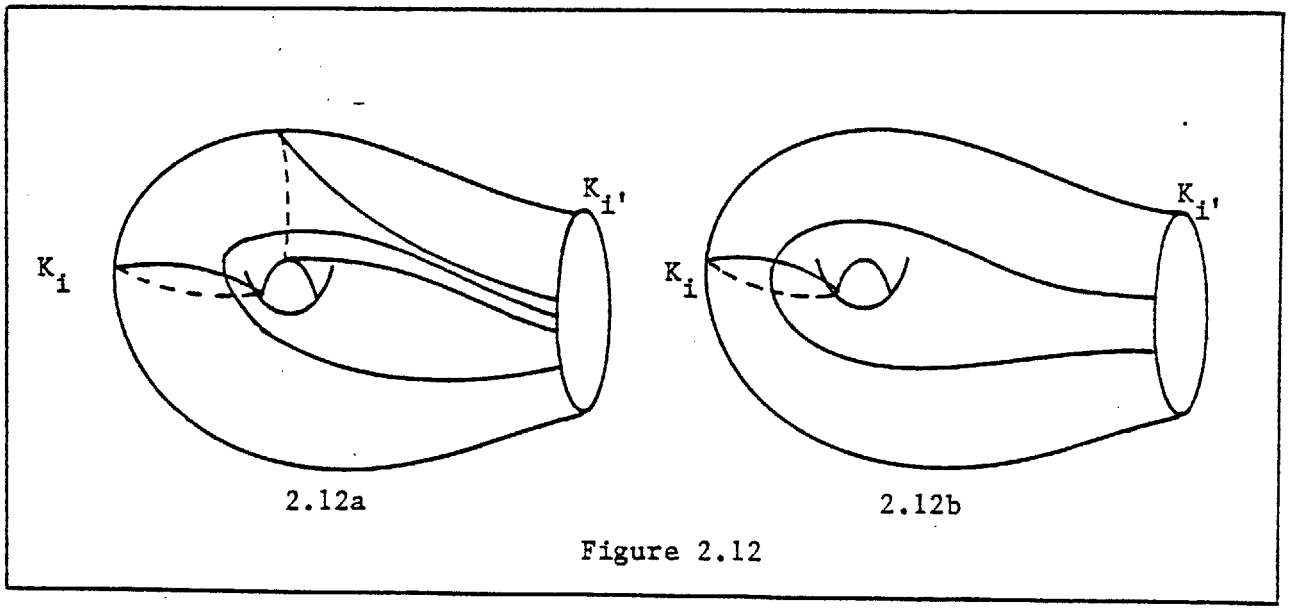
$$m(K_i,j) = \begin{cases} 2\ell_{11}^j + \ell_{12}^j + \ell_{13}^j, & k=1. \\ 2\ell_{22}^j + \ell_{12}^j + \ell_{23}^j, & k=2. \\ 2\ell_{33}^j + \ell_{13}^j + \ell_{23}^j, & k=3. \end{cases}$$

We further require that

j) $m(K_i,j) = 0$ implies that $t_i \geq 0$.

k) If K_i is isotopic to both $f_j^{-1}\partial_k$ and $f_{j'}^{-1}\partial_{k'}$, then $m(K_i,j) = m(K_i,j')$, $j = j'$.

Proof: Restriction a)-f) follow from Remark 2.2. Restriction j) is convention a) of Theorem 2.1, and restriction k) guarantees that the pants glue together properly. Restrictions g)-i) require further comment. Suppose that K_i is isotopic to both $f_j^{-1}\partial_1$ and $f_j^{-1}\partial_2$; suppose, moreover, $f_j^{-1}\partial_3$ is isotopic to $K_{i'}$. Since $f_j^{-1}\partial_1$ and $f_j^{-1}\partial_2$ are isotopic, geometric intersection numbers with these curves are the same for all multiple arcs in F. Thus, either $m_{i'} \leq 2m_i$ or $m_{i'} > 2m_i$, and we are in case a) or d) of Construction 2.1, in which case $\ell_{11} = 0$ and $\ell_{22} = 0$. An example where $m_{i'} = 4$ and $m_i = 1$ is indicated in Figure 2.12a, and an example when $m_{i'} = 2$ and $m_i = 1$ is indicated in Figure 2.12b. Restrictions h) and i) are analogous.



2.12a 2.12b

Figure 2.12

Thus, these restrictions are all necessary. Sufficiency follows by applying Construction 2.1 to give examples for each parameter value satisfying a)-k). The corollary follows.⊠

To be sure, this parametrization is less algebraically clean than the Dehn-Thurston parametrization, but it is geometrically natural and is what we need. In fact, we have embedded the integral lattice of the Dehn-Thurston parametrization in a large-dimensional integral lattice. Note moreover that by the formulas of Remark 2.1 and Construction 2.1, one can pass back and forth between the Dehn-Thurston parametrization and our own parametrization, provided that both are computed relative to the same basis A. We introduce the second parametrization because the action of $MC(F_g)$ on $\mathscr{S}'_A(F_g)$ is simpler than its action on the Dehn-Thurston integral lattice, as we shall see.

In subsequent sections, we will compute the action of $MC(F_g)$ on the collection of multiple arcs in F_g. Corollary 2.1 tells us that given a choice of basis A, there is a one-to-one correspondence between the set $\mathscr{S}'(F_g)$ and the subset $\mathscr{S}'_A(F_g)$ of $\mathbb{Z}^{15g-15}$. Thus, after a choice of basis A, we may describe the action of $MC(F_g)$ on $\mathscr{S}'(F_g)$ <u>in coordinates</u>: we compute the corresponding action on the particular integral lattice $\mathscr{S}'_A(F_g)$.

<u>Remark 2.4</u>: In the sequel, we will regard $\mathscr{S}'_A(F) \subset (\mathbb{Z}^+)^{6M} \times \mathbb{Z}^N$, and we will regard $\{\ell^i_{**}, t_i\}$ as a collection of generators for $\mathscr{S}'(F)$ satisfying certain relations. Thus, if γ and δ are multiple arcs, we may speak of the sum of their corresponding parameter values. If γ and δ have disjointly embedded representatives, then the sum of their parameter values corresponds to the isotopy class of the union of these representatives.

If γ and δ do not have disjointly embedded representatives, then the sum of their parameter values may or may not correspond to some multiple arc, depending on whether restrictions a)-f) of Corollary 2.1 are satisfied for this sum.

The reader wishing to familiarize him or herself with the parametrizations is urged to consult Figures 3.8 and 7.4 for definitions of bases and then verify the computations in Appendix B.

Without too much trouble, one can extend the results of this section to a non-compact and/or non-orientable surface F with negative Euler characteristic and two-sided essential one-submanifolds embedded in F. For our present purposes, it is not worth the effort.

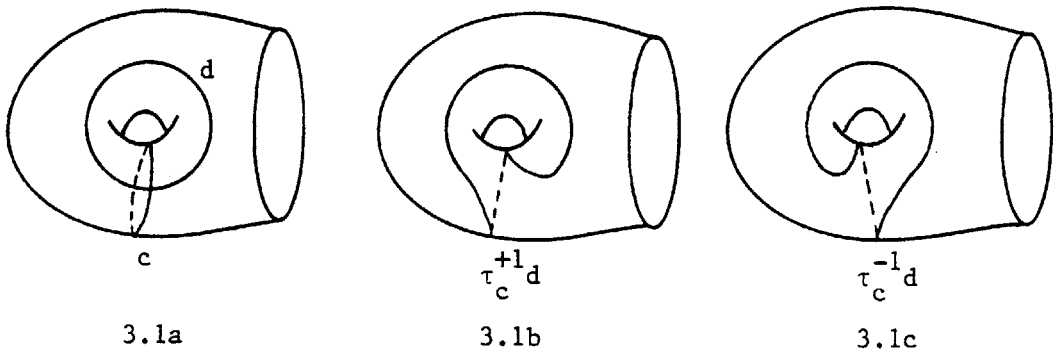
SECTION 3

In the previous section, we gave an explicit parametrization relative to a basis A for the collection of multiple arcs in the g-holed torus by an integral lattice $\mathscr{S}'_A(F_g) \subset \mathbb{Z}^{15g-15}$. Our main goal is to compute the action of $MC(F_g)$ on $\mathscr{S}'(F_g)$ by computing its action in coordinates on the particular integral lattice $\mathscr{S}'_A(F_g)$. The specification of an element of $MC(F_g)$ depends on a result of Lickorish [11] which gives a collection of generators for $MC(F_g)$ provided $g \geq 2$.

Just as Dehn knew of a parametrization for multiple arcs, he also constructed a finite set of generators for $MC(F_g)$ [6] called Dehn twists, whose definition we will recall presently. Lickorish independently refined this result by exhibiting a more useful collection of 3g-1 Dehn twist generators for $MC(F_g)$. More recently, S. Humphries [10] has sharpened this result. In his thesis, Humphries gives a collection of 2g+1 Dehn twist generators for $MC(F_g)$ and shows that 2g+1 is the least possible number of such. (Humphries' generators are contained in Lickorish's set.)

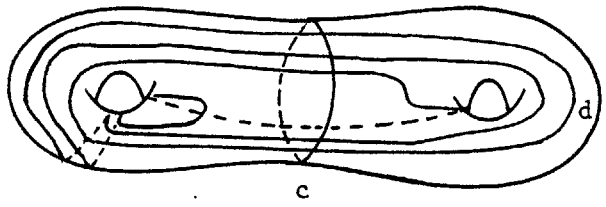
A Dehn twist along a curve c in an oriented surface F is defined as follows. We identify a closed regular neighborhood N of c with the standard oriented annulus $A = S^1 \times [-1,1]$ via an orientation-preserving homeomorphism f_c. On the neighborhood N of c, define the right and left Dehn twists on c, denoted $\tau_c^{\pm 1}$, to be the conjugate by f_c of the map $(\theta,t) \to (\theta \pm \pi(t+1),t)$ on the standard annulus A; define $\tau_c^{\pm 1}$ to be the identity on $F \setminus N$. This construction is independent of the orientation of c, and the mapping class of $\tau_c^{\pm 1}$ depends only on the isotopy class of the unoriented curve c and the orientation of F.

Figure 3.1 illustrates the curves $\tau_c^{\pm 1}d$, where c and d are as pictured. The action of a single Dehn twist on a connected element of $\mathcal{S}_A'(F_g)$ can be quite complicated. A reasonably complicated example of such an action is indicated in Figure 3.2.



3.1a 3.1b 3.1c

Figure 3.1



$\tau_c^{+1}d$

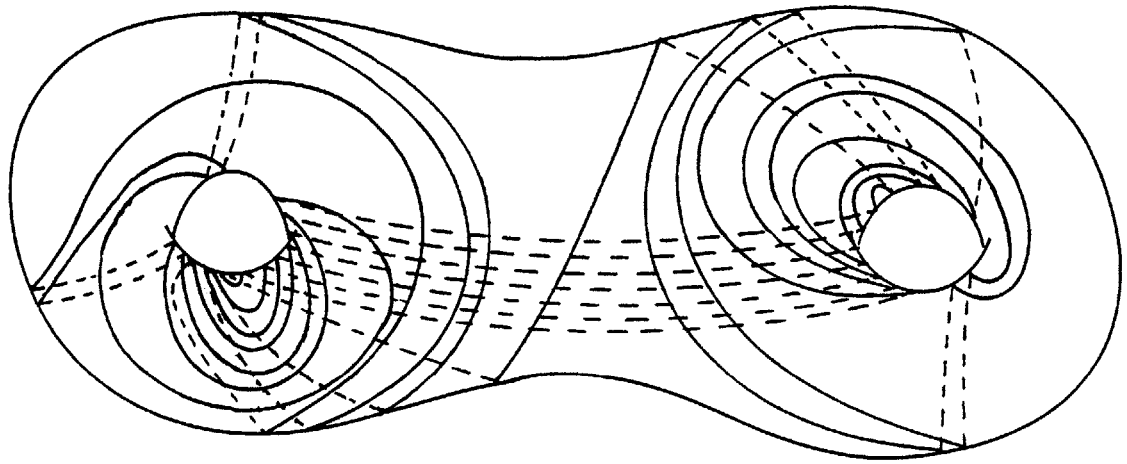


Figure 3.2

A fundamental theorem in the study of surfaces is the theorem of Lickorish mentioned previously.

Theorem 3.1 (Lickorish): If $\mathscr{L}_g$ is the collection of 3g-1 curves on the g-holed torus pictured in Figure 3.3, then $MC(F_g)$ is generated by the Dehn twists along the elements of $\mathscr{L}_g$.

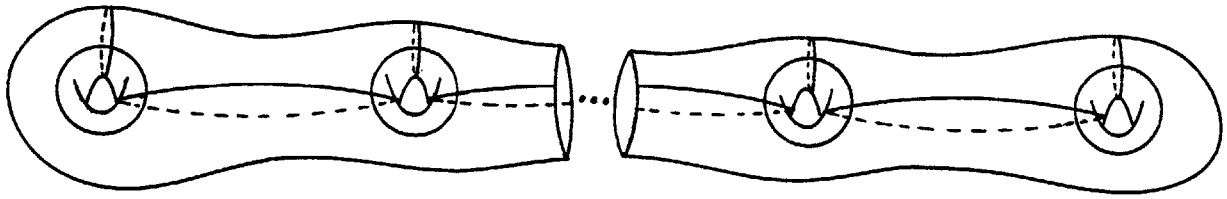


Figure 3.3

It follows that $MC(F_g)$ is a quotient of the free group on Lickorish's generators, and a precise statement of our main problem is:

Problem: For some choice of basis A for $\mathscr{S}'(F_g)$, compute the action on $\mathscr{S}'_A(F_g)$ of Dehn twists along the curves in $\mathscr{L}_g$.

Provided that F_g is oriented and regarded as embedded in S^3, a basis A for the collection of multiple arcs is:

a) A pants decomposition $\{K_i\}_1^{N=3g-3}$ of F_g, and a window $u_i \subset K_i$ for each curve K_i.

b) A homeomorphism $f_j: D_j \to P$ from each embedded pair of pants D_j (which is a component of $F_g \setminus \cup \{A_i\}$, where A_i is a regular neighborhood of K_i) to the standard pants P.

c) A choice of canonical pieces (as in Construction 2.1) for each embedded pair of pants D_j.

Fix some basis A for now, and suppose that we want to compute the action on the Dehn-Thurston parametrization for multiple arcs of $\tau_{K_k}^{\pm 1}$, the Dehn twists on the pants curve K_k of the basis A. If γ is a multiple arc with Dehn-Thurston parameters $\{(m_i,t_i)\}_1^N$, then $\tau_{K_k}^{\pm 1}c$ has Dehn-Thurston parameter values $\{(m_1,t_1),\ldots,(m_{k-1},t_{k-1}),(m_k,t_k\pm m_k),(m_{k+1},t_{k+1}),\ldots,(m_N,t_N)\}$. Note that this result is independent of choices b) and c) above.

Example 3.1: Adopting the basis A on F_2 indicated in Figure 2.7, we compute the action of the word $\tau_{K_1}\tau_{K_2}^2\tau_{K_3}$ on the multiple curve with corresponding Dehn-Thurston parameter values $(3,1,2)\times(2,-1,0)$. (We will read words in Lickorish's generators from right to left. Note, however, that τ_{K_1}, τ_{K_2} and τ_{K_3} commute with one another since the curves K_i are all disjoint.) Let c be the good representative of γ shown in Figure 2.8b. By isotopy, one can arrange that τ_{K_i} is the identity off the annular neighborhood A_i of K_i used in the basis A. Thus, the image of c under $\tau_{K_1}\tau_{K_2}^2\tau_{K_3}$ agrees with c outside of $\{A_i\}$, and $\tau_{K_1}\tau_{K_2}^2\tau_{K_3}c$ is a good representative of its isotopy class with respect to the basis A. This curve is pictured in Figure 3.4 and has Dehn-Thurston parameter values $(3,1,2)\times(5,1,2)$ relative to the basis A.

Similarly simple (but more awkward to state) is the action of $\tau_{K_k}^{\pm 1}$ on the integral lattice $\mathcal{S}_A(F_g)$, where K_k is a pants curve in the basis A. It would thus serve us well to choose a pants decomposition for F_g that overlaps as much as possible with the Lickorish curves $\mathcal{L}_g$.

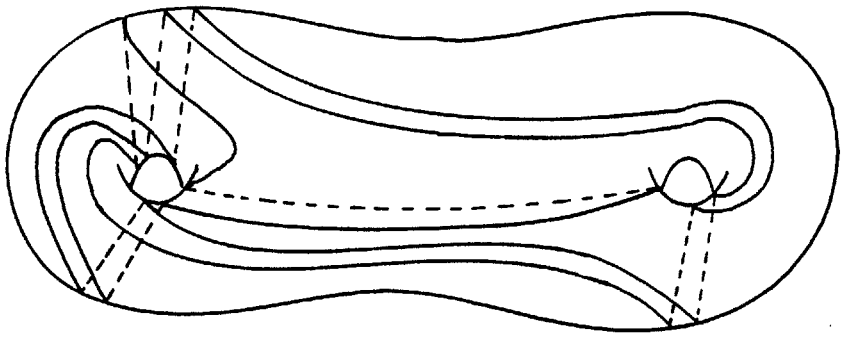
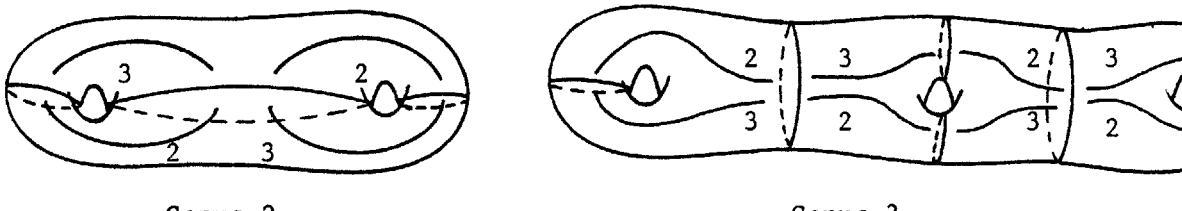
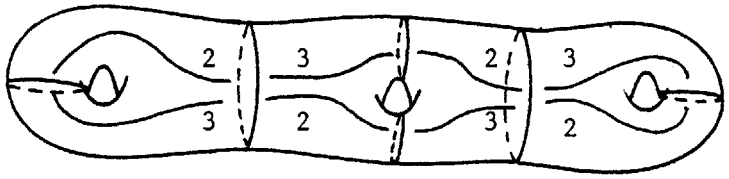


Figure 3.4

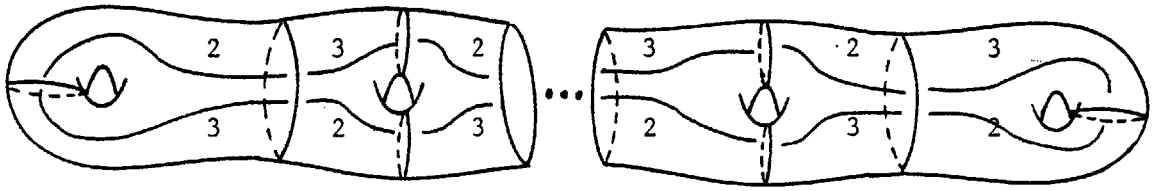
For $F_g \subset S^3$, we choose the <u>standard pants decomposition</u> and the standard homeomorphisms $f_j : D_j \to P$ indicated in Figure 3.5. For the time being, we make choice c) above as in Construction 2.1. As one might expect, these choices give the <u>standard basis</u> A_g on F_g.



Genus 2.



Genus 3.



Genus g > 3 .

Figure 3.5

Let $\mathcal{C}_g$ be the collection of 2g-1 curves in F_g indicated in Figure 3.2. The Lickorish curves $\mathcal{L}_g$ are contained in $\mathcal{C}_g \cup \{K_i\}_1^{3g-3}$. Thus, to solve the main problem, it remains to analyze the twists $\tau_c^{\pm 1}$, for $c \in \mathcal{C}_g$. Note that in fact the containment $\mathcal{L}_g \subset \mathcal{C}_g \cup \{K_i\}_1^{3g-3}$ is proper.

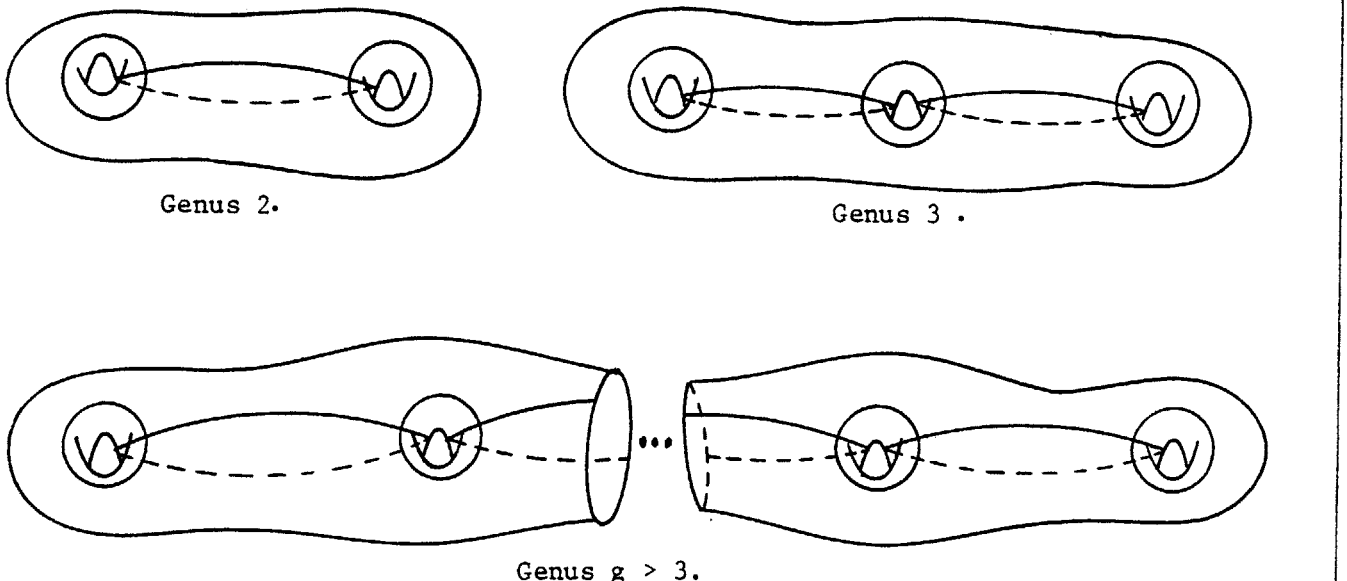


Figure 3.6

Consider the various changes of basis indicated in Figure 3.7 (preserving the choice c) of canonical pieces). This picture shows that for each curve $c \in \mathcal{C}_g$, there is a change of basis $A_g \to B$ so that c is a pants curve in the basis B. Conjugating by this change of basis, we could thus compute as above the action of $\tau_c^{\pm 1}$ on $\mathcal{S}'_A(F_g)$ for $c \in \mathcal{C}_g$. The philosophy comes from linear algebra: if a transformation ($\tau_c^{\pm 1}$ for $c \in \mathcal{C}_g$) is hard to compute, change basis.

Note that in fact computing these changes of basis will allow us to compute the action on $\mathcal{S}'_A(F_g)$ of Dehn twists not only along

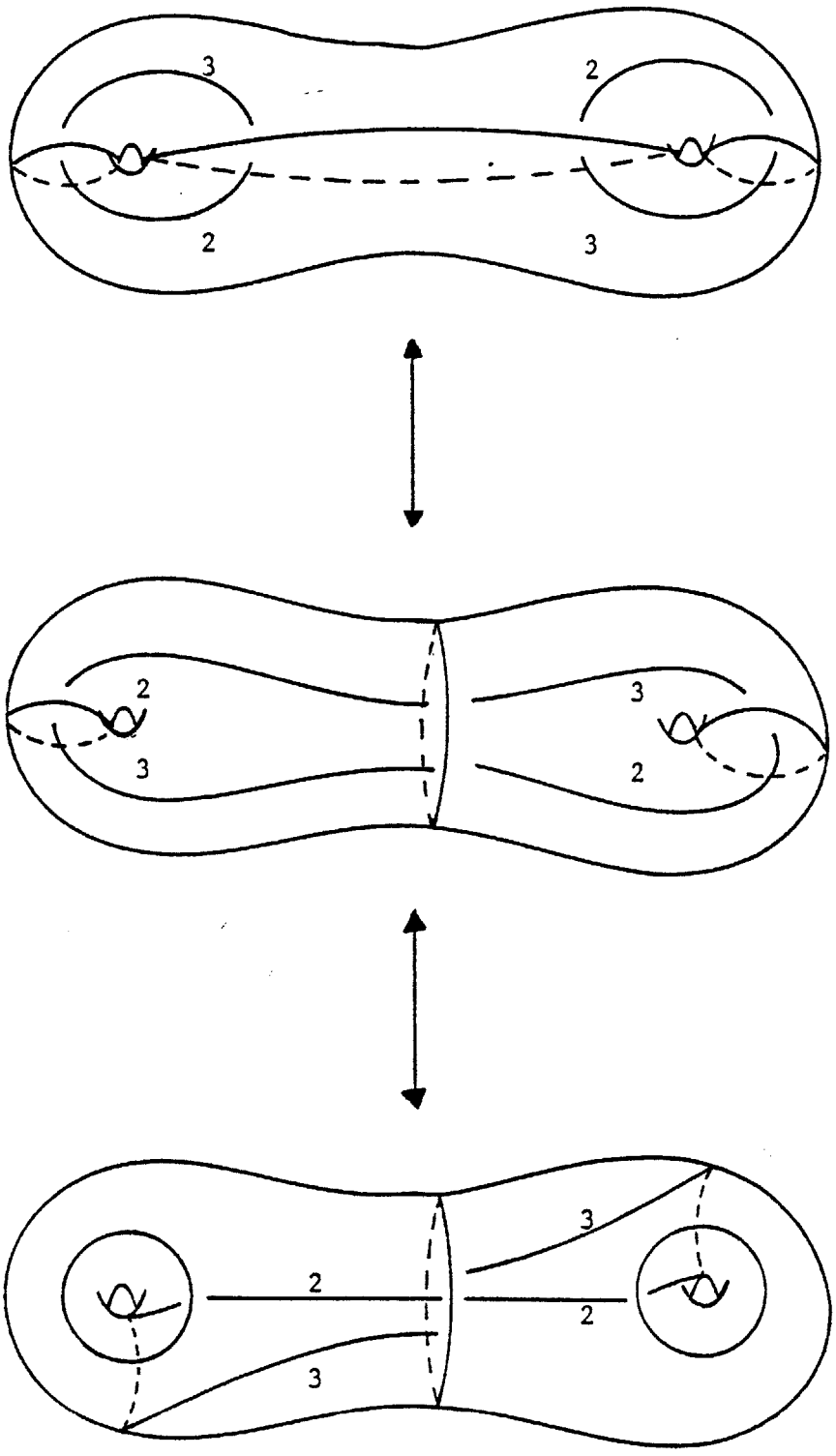


Figure 3.7 (genus=2)

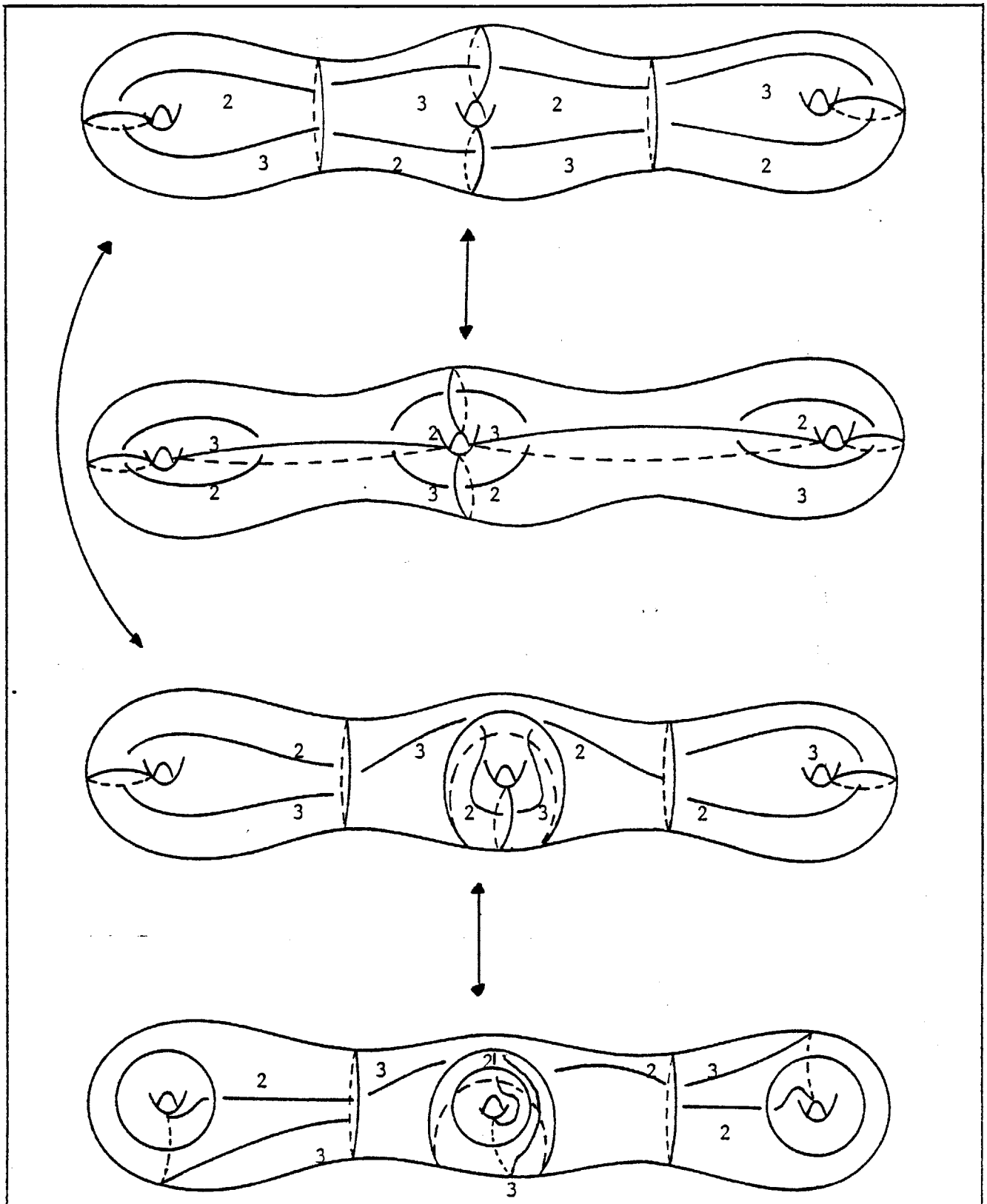


Figure 3.7 (genus=3)

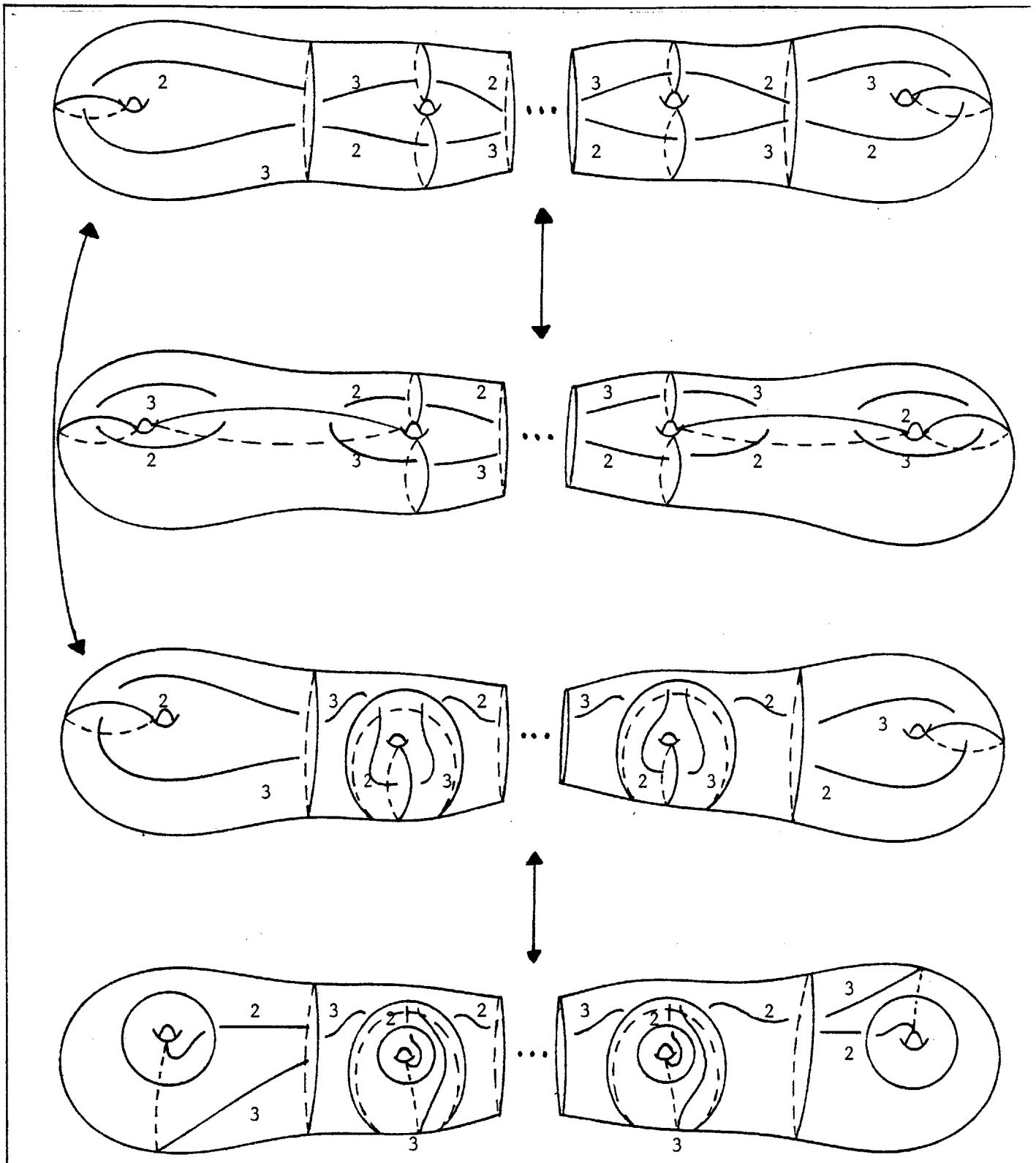


Figure 3.7 (g > 3)

the 3g-1 curves in $\mathscr{L}_g$, but along all 5g-4 curves in $\mathscr{C}_g \cup \{K_i\}_1^{3g-3}$. Though the action of the Humphries Dehn twist generators alone would suffice to describe the action of $MC(F_g)$ on multiple arcs, we use this much larger set for two reasons: we get the extra information for free, and the greater number of generators allows a greater flexibility in specifying mapping classes. Moreover, I do not know of a way to take advantage of considering only the Humphries generators.

Each change of basis in Figure 3.7 can be written as a composition of the elementary transformations pictured in Figure 3.8. We have thus reduced the main problem to the computation of the two elementary transformations. This reduction of the problem was shown to me by Gabai [9].

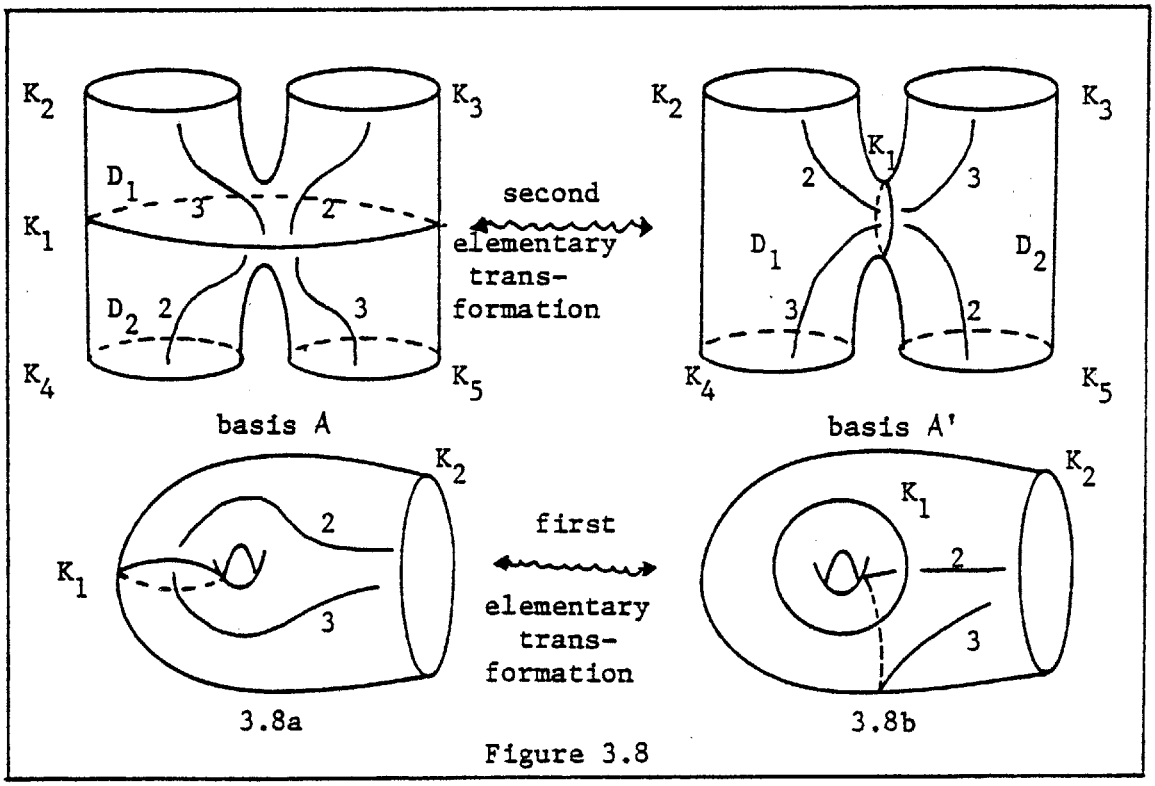


Figure 3.8

The first elementary transformation is comparatively simple, and the computations can be done by hand (by actually isotoping curves

and arcs about on the torus-minus-a-disc), but the second elementary transformation requires some work. However, the machinery we develop to handle the second elementary transformation also applies to the first elementary transformation. We next describe some of this machinery and outline the actual computations.

Let S_1 be the torus-minus-a-disc, and let S_2 be the sphere-minus-four-discs. Let A be the basis on S_i indicated in Figure 3.8a, and let A' be the basis on S_i indicated in Figure 3.8b, i=1,2. Given a multiple arc γ, we choose a good representative c of γ with respect to the basis A, and we orient the components of c arbitrarily. We will choose a lift $\tilde{c}$ of c to a certain regular planar cover $\Pi_i:\tilde{S}_i \to S_i$. We isotope $\tilde{c}$ about in $\tilde{S}_i$ to some $\tilde{\bar{c}}$ so that each component of $f_j(\Pi_i\tilde{\bar{c}} \cap D_j)$ is a canonical piece in the standard pants P, where the f_j are the homeomorphisms of the basis A. Define $\bar{c} = \Pi_i\tilde{\bar{c}}$. The reason for passing to a covering space is that we gain a facility in picturing the homotopy from c to $\bar{c}$ as an isotopy from $\tilde{c}$ to $\tilde{\bar{c}}$. However, we cannot guarantee that the isotopy from $\tilde{c}$ to $\tilde{\bar{c}}$ is Π_i-equivariant, so $\bar{c}$ is not in general embedded. However, $\bar{c}$ is at least homotopic to the embedding c.

We will introduce a combinatorial object, called a symbol, in Section 5. The collection of symbols relative to a basis A forms a semi-group, and we associate to each tuple in $\mathscr{S}'_A(S_i)$ some symbol. More generally, we will associate a symbol to an <u>immersed</u> one-submanifold such as $\bar{c}$.

It turns out that the symbol of $\bar{c}$ is the symbol of an embedding c' homotopic to $\bar{c}$. c and c' are thus homotopic, and, in fact, c' will be in good position with respect to the basis A'. It is well known [7]

that two homotopic embedded one-submanifolds are in fact isotopic. (This also follows from the results of the next section.) c' is thus a good representative of γ with respect to the basis A', and we can compute the parameter value of γ in $\mathscr{S}_{A'}(S_i)$ from the symbol of c' (which is the same as the symbol for $\bar{c}$). Thus, the elementary transformations are given by maps between semi-groups of symbols, which we will call combinatorial homotopies.

The difficult part of this process is showing that the symbol of $\bar{c}$ is in fact the symbol of an embedding c' homotopic to $\bar{c}$. We prove some results about one-submanifolds immersed in surfaces in the next section that are of independent interest. These are applied in Section 5 to show that if the symbol of $\bar{c}$ satisfies a few technical properties, then such a c' exists. Much of the hard combinatorial work of Sections 6 and 7, where we compute the two elementary transformations in turn, is devoted to showing that the particular symbol for $\bar{c}$ satisfies these technical properties.

The planar covers $\Pi_i:\tilde{S}_i \to S_i$, i=1,2, are defined in Sections 6 and 7. They are particularly pleasant to work with. The two groups of covering translations are groups of isometries of $\mathbb{R}^2$ with its usual metric. Thus, the push-forward under Π_i of the usual metric on $\tilde{S}_i \subset \mathbb{R}^2$ gives a Euclidean structure on S_i. For the special case of multiple curves with no boundary-parallel components, the combinatorial homotopies that describe the elementary transformations are closely related to straightening to geodesics in this Euclidean structure.

The reader wishing to skip the explicit computations can proceed directly to Section 8, where we give the formulae for the elementary transformations and discuss some applications.

As usual, though we restrict attention to the case of g-holed tori, the techniques described in this section work for an arbitrary non-compact and/or non-orientable surface F of negative Euler characteristic. In case F is non-orientable, we interpret MC(F) as the group generated by isotopy classes of Dehn twists. (It is not true that the Dehn twists generate the group of homeomorphisms of F modulo isotopy. See [1] and [12].) If F is non-orientable, we interpret $\mathscr{S}'(F)$ as the collection of isotopy classes of two-sided essential one-submanifolds embedded in F. With these more general definitions, the action of MC(F) on $\mathscr{S}'(F)$ can always be computed from the two elementary transformations, as in this section.

A final observation: Thurston and Hatcher [19] have shown that one can pass between any two pants decompositions on F_g by sequences of our two elementary transformations.

SECTION 4

In this section, we prove some results about one-manifolds properly immersed in surfaces. For convenience, we will assume that the surfaces we consider have a fixed smooth structure. As usual, we will also assume that our surfaces have a negative Euler characteristic. Let j be a proper immersion of a (smooth) one-manifold in the surface F. We will consider the image of j as an immersed manifold α, yet we will refer to the the image under j of components of 0 as the "components" of α. We will also refer to a component of α as a "closed component" or "arc component", according to whether the corresponding component of 0 is a circle or an arc. A specific choice of map j will be called a "parametrization" of the immersion α.

By an n-gon in F we mean a (smoothly) embedded open disc (whose closure lies in the interior of F) with embedded piecewise-smooth boundary and n discontinuities in the tangent of the bounding curve. Some examples of n-gons are pictured in Figure 4.1. If there is an n-gon in F with its frontier in an immersion α, then we say that α has a complementary n-gon.

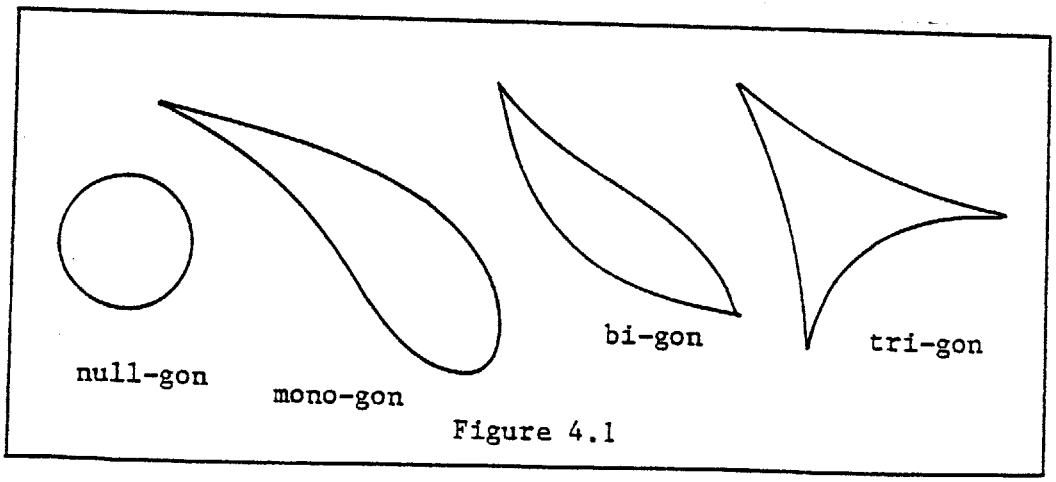


Figure 4.1

If α has a complementary mono-gon or bi-gon, then the homotopies indicated in Figure 4.2 show that α cannot have minimal self-intersection number in its homotopy class rel ∂F. The main result of this section is a converse: <u>if α does not have minimal self-intersection number in its homotopy class rel ∂F, then the application of a finite sequence of the homotopies indicated in Figure 4.2 gives a representative of the homotopy class of α that does</u>.

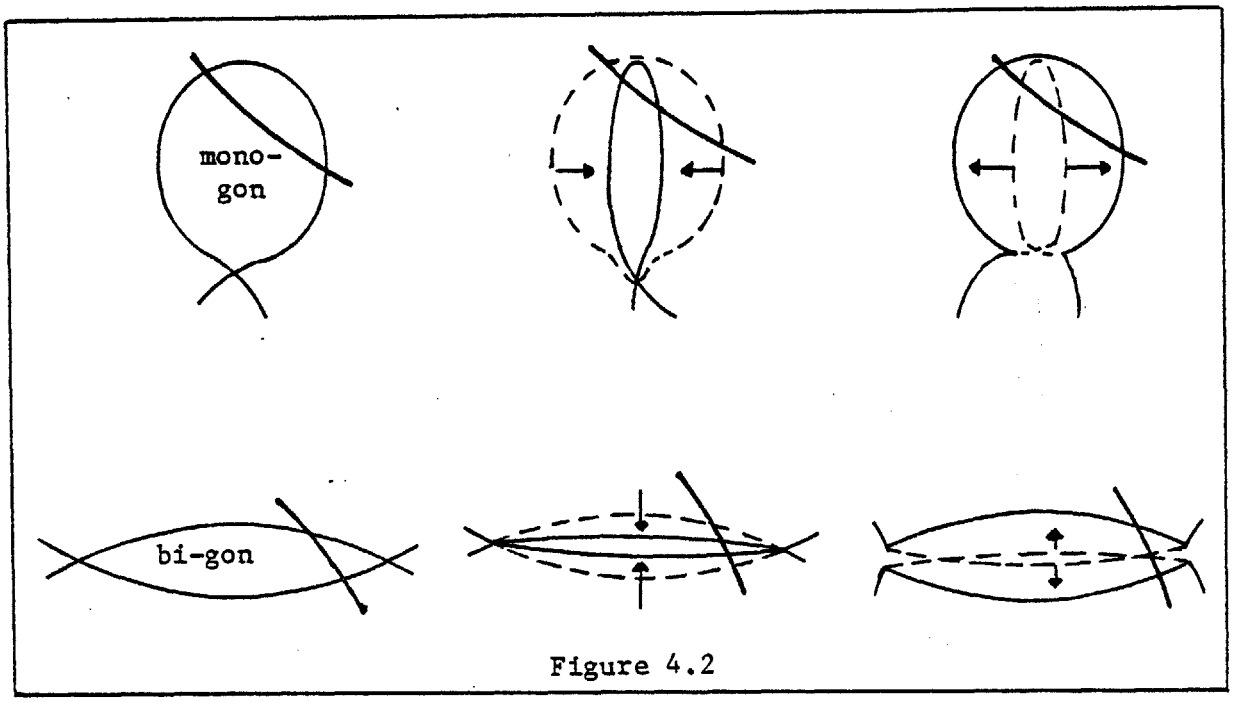


Figure 4.2

A special case (Corollary 4.1) of this result is applied in a rather technical setting in the next section. More generally, this result is useful for determining whether the homotopy class of a given immersion has an embedded representative: remove complementary mono-gons and bi-gons using the homotopies in Figure 4.2. This process terminates and either yields the desired embedding or an immersion with no complementary mono-gons or bi-gons, in which case there can be no embedded representative of the given homotopy class.

If F is a surface of negative Euler characteristic, then it is well known [13] that F supports a Riemannian metric of finite area and of constant -1 curvature so that ∂F is geodesic. The total space of a universal cover $\Pi:\tilde{F} \to F$ is isometric to a contractible subset of the Poincare disc $\mathbb{D}$ with geodesic boundary. There is a natural compactification of the (open) Poincare disc by a circle, and the points in this circle, which is denoted S^1_∞, are called the <u>points at infinity</u>. We regard $\mathbb{D} \subset \mathbb{D} \cup S^1_\infty$, which is homeomorphic to a closed disc, and regard the points at infinity as being infinitely far from any point in $\mathbb{D}$. The closure of $\tilde{F}$ in $\mathbb{D} \cup S^1_\infty$ is topologically a closed disc, which we will denote by K, and $K \cap S^1_\infty$ is called the <u>limit set</u>. The limit set is either all of S^1_∞ or a Cantor set in S^1_∞, depending on whether the surface F is closed or has boundary. We denote by ∂K the frontier of K in $\mathbb{D}$ plus the limit set, so that ∂K is homeomorphic to a circle.

In the 1920's through 1940's, J. Nielsen developed a very beautiful theory of surface automorphisms [13] by studying the natural action of homeomorphisms of the surface on the points of K at infinity. (In fact, Nielsen's work anticipates some of the recent developments in the theory of surface automorphisms. Nielsen had a pretty complete picture of current work in surface automorphisms, but he did not use the machinery of foliations, which had not yet been invented.) We will require in the sequel only a handful of elementary results from the Nielsen theory. We presently recall these facts. We may identify the group $\pi_1(F)$ of covering transformations with a discrete subgroup of the group of isometries of $\mathbb{D}$, so that the subgroup consists entirely of hyperbolic Mobius transformations of $\mathbb{D}$. A hyperbolic transformation

of $\mathbb{D}$ has a simple geometrical picture: geodesics in $\mathbb{D}$ are circles orthogonal to S^1_∞, and a hyperbolic transformation ψ is translation along such a geodesic. The fundamental points of ψ are the endpoints at infinity of this geodesic, called the axis of ψ. The action of ψ on $\tilde{F}$ extends continuously to an action on K, and the fundamental points of ψ are the only fixed points of ψ on ∂K. Finally, if two axes of elements of $\pi_1(F)$ intersect at infinity, then they coincide, since otherwise their commutator would be a parabolic transformation [14]. (Recall that $\pi_1(F)$ consists entirely of hyperbolic transformations.)

Given a component c of the immersion α, we define a complete lift of c to K, denoted $\tilde{\tilde{c}}$, as follows. If c is a properly immersed arc or inessentially immersed closed curve, we define $\tilde{\tilde{c}}$ to be simply a lift of c to $\tilde{F} \subset K$. If c is an essentially immersed closed curve, we define $\tilde{\tilde{c}}$ to be the closure in K of a bi-infinite sequence $\{\tilde{c}_i\}$ of lifts of c to $\tilde{F} \subset K$, where the final point of $\tilde{c}_i$ is the initial point of $\tilde{c}_{i+1}$. Thus, a complete lift of an essential curve component or proper arc component of α is an arc properly immersed in the ball K. A complete lift of an inessential closed curve component of α is an immersed closed curve in $\tilde{F} \subset K$.

We will consider only properly immersed one-submanifolds α in general position in F. We assume that $\alpha \cap \partial F$ is already in general position in F. Thus, α has at most double points, and α is embedded near ∂F. Let $\Delta(\alpha)$ denote the set of double points of α.

Lemma 4.1: Suppose that α is a (smooth) one-submanifold properly immersed in a surface F in general position. α has minimal self-intersection number in its homotopy class rel ∂F if and only if for every pair c and d of components of α (with perhaps c=d), complete lifts

$\tilde{\tilde{c}}$ and $\tilde{\tilde{d}}$ are embedded and satisfy one of

a) $\tilde{\tilde{c}} \cap \tilde{\tilde{d}} = \phi$.

b) $\tilde{\tilde{c}}$ and $\tilde{\tilde{d}}$ intersect transversely in a point.

c) $\tilde{\tilde{c}} \cap \tilde{\tilde{d}} = (\tilde{\tilde{c}} \cap \partial K) \cap (\tilde{\tilde{d}} \cap \partial K)$. (This implies that c and d are closed curve components of α with homotopic powers, provided $\tilde{\tilde{c}} \cap \tilde{\tilde{d}} \neq \phi$.)

d) $\tilde{\tilde{c}} = \tilde{\tilde{d}}$. (This implies that c and d have homotopic powers.)

Proof: ($\Leftarrow$) We begin by counting the double point set of α in case every complete lift of a component of α is embedded. Choose, once and for all, lifts $\tilde{c} \subset \tilde{F} \subset K$ of the components of α. Since α is in general position in F, we may choose our lift $\tilde{c}$ starting at a point in $c \setminus \Delta(\alpha)$.

Since $\pi_1(F)$ act transitively on the fibers of Π, if $p \in c \cap \Delta(\alpha)$, there is some $\tilde{p} \in \tilde{c} \cap \Pi^{-1}(p)$ and some complete lift $\tilde{\tilde{d}}$ of some component d of α with $\tilde{p}$ in $\tilde{\tilde{d}}$, and so that $\tilde{c}$ is not contained in $\tilde{\tilde{d}}$ (since α is in general position). There is thus a one-to-one correspondence between $c \cap \Delta(\alpha)$ and the complete lifts of components of α that intersect $\tilde{c}$. Let $N_{\tilde{c}}(\alpha)$ denote the cardinality of $c \cap \Delta(\alpha)$, so that card $\Delta(\alpha) = \frac{1}{2}\sum_{c} N_{\tilde{c}}(\alpha)$. It is easy to show that any homotopy in F of α rel ∂F lifts to $\tilde{F}$ and extends continuously to a homotopy in K that is constant on ∂K. Thus, if complete lifts are embedded and satisfy one of a)–d), then a homotopy in F of α rel ∂F cannot decrease $N_{\tilde{c}}(\alpha)$ for any component c of α, and the implication follows.

($\Rightarrow$) We first show that complete lifts are embedded. To derive a contradiction, suppose that $\tilde{\tilde{c}}$ is not embedded, where c is an arc or curve component of α. Parametrizing $\tilde{\tilde{c}}$ (by the circle or interval), this means that there are parameter values $t_1 < t_2$ with $\tilde{\tilde{c}}(t_1) = \tilde{\tilde{c}}(t_2)$. Since $\tilde{\tilde{c}}$ is either an arc or a closed curve properly immersed in the

disc K, $\tilde{\tilde{c}}[t_1,t_2]$ is null homotopic in K. Let $\tilde{p} = \tilde{\tilde{c}}(t_1) = \tilde{\tilde{c}}(t_2)$, so that $\Pi\tilde{p}$ is a double point p of c. If c is an arc component of α, then $\tilde{p}$ is in the interior of K by general position. If c is an inessential curve component of α, then $\tilde{p}$ is in the interior of K since $\tilde{\tilde{c}}$ is. If c is an essential curve component of α, then since $\tilde{\tilde{c}}$ has distinct endpoints at infinity, $\tilde{p}$ is in the interior of K. Thus, p has a neighborhood in the interior of F as in Figure 4.3a, which we mdoify as in Figure 4.3b. The dotted lines in Figure 4.3 denote arcs immersed in F, and the mono-gon in Figure 4.3a denotes a disc immersed in F. This move decreases card $\Delta(\alpha)$ by exacly one, and it is the projection by Π of a homotopy in K. This contradicts the minimality of α and proves that complete lifts are embedded.

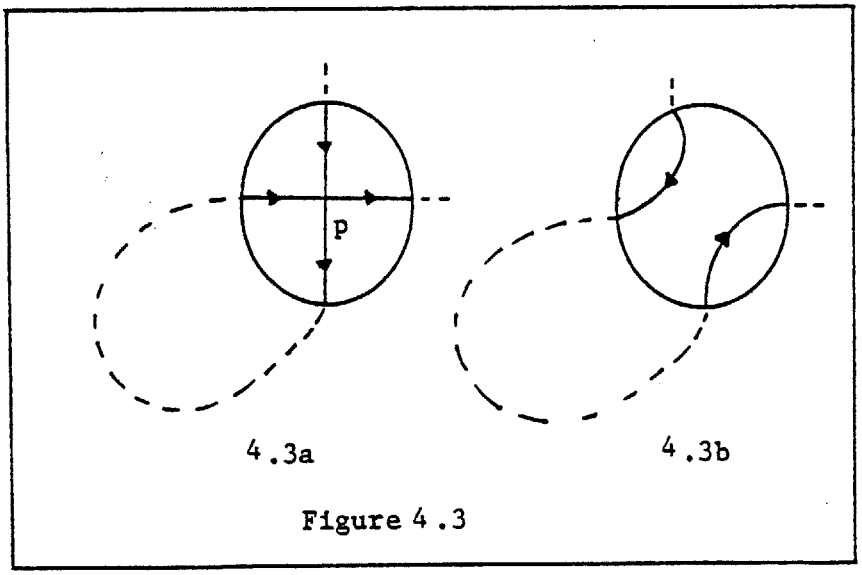


Figure 4.3

Suppose that c is an inessential curve component of α, where α is in minimal position. Any complete lift $\tilde{\tilde{c}}$ must satisfy either a) or d) for any complete lift $\tilde{\tilde{d}}$ of any component d of α. This is because

one can easily homotope these inessential components to embedded circles out of the way. Thus, we may assume that α has no inessential closed curve components.

Now, if c and d are components of α, and $\tilde{c}$ and $\tilde{d}$ have distinct endpoints yet fail to satisfy a), b) or d), then there are parametrizations of $\tilde{c}$ and $\tilde{d}$ (by the unit interval), and there are parameter values $s_1 < s_2$ and $t_1 < t_2$ with $\tilde{c}(s_i) = \tilde{d}(t_i)$, i=1,2. Moreover, $c[s_1,s_2]*d^{-1}[t_1,t_2]$ must bound a disc in $\tilde{F}$, where $*$ denotes concatenation of arcs. Let $\tilde{p} = \tilde{c}(s_1) = \tilde{d}(t_1)$ and $\tilde{q} = \tilde{c}(s_2) = \tilde{d}(t_2)$, and let $p = \Pi\tilde{p}$ and $q = \Pi\tilde{q}$. If p and q are distinct double points of α, then there are neighborhoods of p and q in F as in Figure 4.4a, which we can modify as in Figure 4.4b, contradicting the minimality of α.

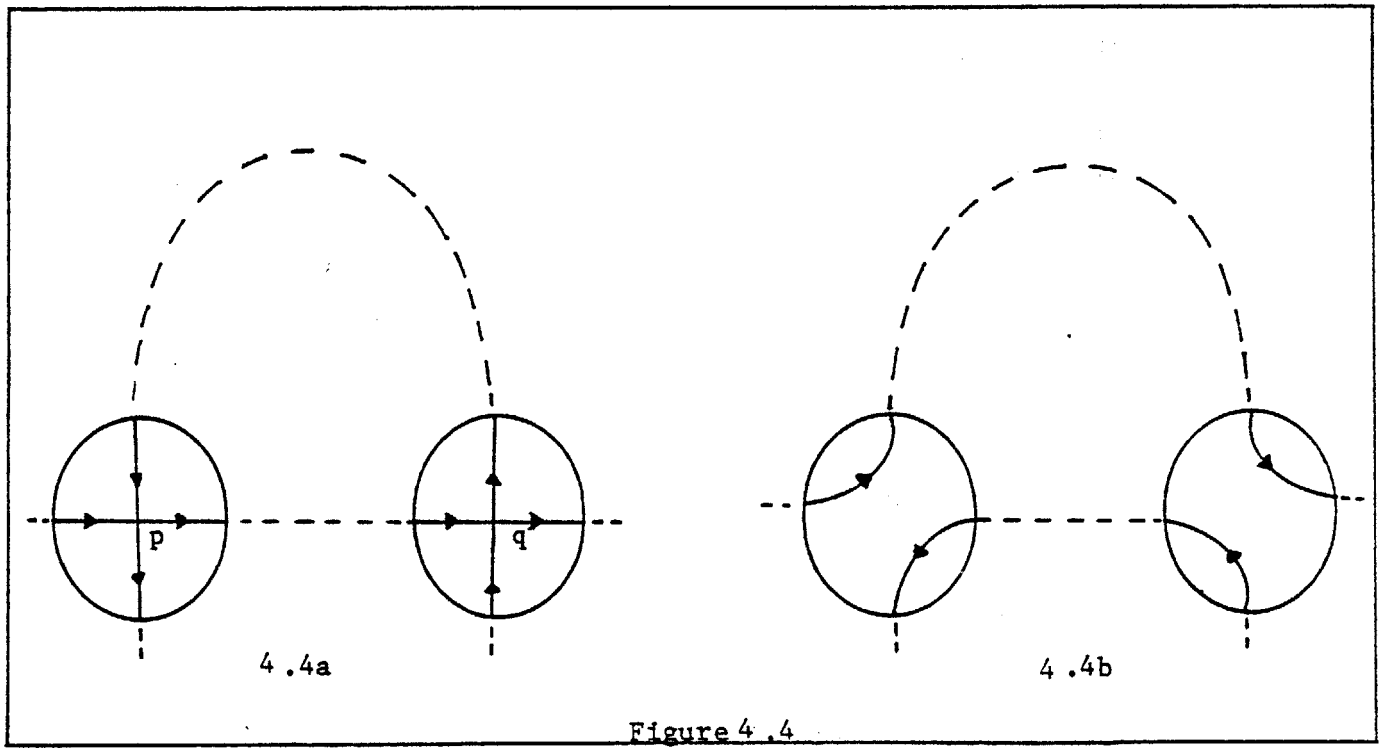


Figure 4 .4

If p = q, then by general position, either c = d is an arc component of α, or $\tilde{c}[s_1,s_2]$ multiply covers the closed curve component

c and $\tilde{\tilde{d}}[t_1,t_2]$ multiply covers the closed curve component d.

In case c = d is an arc component, general position implies that there is a covering translation $\psi \in \pi_1(F)$ carrying $\tilde{\tilde{c}}[s_1,s_2]$ to $\tilde{\tilde{d}}[t_1,t_2]$, and ψ must interchange $\tilde{p}$ and $\tilde{q}$ since it acts without fixed points on $\tilde{F}$. Thus, $1 = \Pi(\tilde{\tilde{c}}[s_1,s_2] * \tilde{\tilde{d}}^{-1}[t_1,t_2]) = (\Pi\tilde{\tilde{c}}[s_1,s_2])^2 \in \pi_1(F,p)$. Since $\pi_1(F,p)$ is without torsion, the loop $\Pi\tilde{\tilde{c}}[s_1,s_2]$ must be inessential, which is absurd because $\tilde{\tilde{c}}[s_1,s_2]$ is not a closed loop.

Suppose that $\tilde{\tilde{c}}[s_1,s_2]$ multiply covers the closed curve component c and $\tilde{\tilde{d}}[t_1,t_2]$ multiply covers the closed curve component d. Since p = q, there is a covering transformation ψ so that $\psi(\tilde{p}) = \tilde{q}$. Both $\psi(\tilde{\tilde{d}})$ and $\tilde{\tilde{d}}$ intersect $\tilde{\tilde{c}}$ at $\tilde{q}$; this violates the uniqueness of lifting unless $\psi(\tilde{\tilde{d}}) = \tilde{\tilde{d}}$. Thus, the fundamental points of ψ coincide with the endpoints of $\tilde{\tilde{d}}$, and symetrically for $\tilde{\tilde{c}}$. This contradicts the assumption that $\tilde{\tilde{c}}$ and $\tilde{\tilde{d}}$ have distinct endpoints at infinity.

Complete lifts of essential curve components have one endpoint in common if and only if both of their endpoints coincide. Thus, it remains to consider only the case of two essential closed curve components c and d so that $\tilde{\tilde{c}}$ and $\tilde{\tilde{d}}$ have the same endpoints at infinity. In this case, powers of c and d are easily shown to be homotopic, say $c^m = d^n$, where $|m| < |n|$. Consider the irregular cover of F by an annulus A corresponding to $\langle c^m \rangle = \mathbb{Z} \subset \pi_1(F,p)$, where p is a point in $c \cap d$. (See [7].) c and d each lift to closed curves in this cover that intersect. We may apply the move indicated in Figure 4.4 in the cover A to reduce card $\Delta(\alpha)$. This contradiction proves the lemma. ⊠

If c is connected and has a double point set $\Delta(c)$, we call $p \in \Delta(c)$ <u>inessential</u> if c may be parametrized so that $c(s_1) = c(s_2) = p$, $s_1 < s_2$,

where $c[s_1,s_2]$ is null homotopic. We say $p \neq q \in \Delta(c)$ are <u>companions</u> if either of conditions a) or b) below are satisfied.

a) There is a parametrization of c with $p = c(s_1) = c(s_3)$ and $q = c(s_2) = c(s_4)$, $s_1 < s_2 < s_3 < s_4$, where $c[s_1,s_2]*c^{-1}[s_3,s_4]$ is null homotopic.

b) There is a parametrization of c with $p = c(s_1) = c(s_4)$ and $q = c(s_2) = c(s_3)$, $s_1 < s_2 < s_3 < s_4$, where $c[s_1,s_2]*c[s_3,s_4]$ is null homotopic.

Figures 4.5a and 4.5b illustrate cases a) and b) of companion double points. If c and d are distinct and connected, we say $p \neq q \in \Delta(c \cap d)$ are <u>companions</u> if there are parametrizations of c and d with $p=c(s_1)=d(t_1)$ and $q=c(s_2)=d(t_2)$, $s_1 < s_2$ and $t_1 < t_2$, where $c[s_1,s_2]*d^{-1}[t_1,t_2]$ is null homotopic.

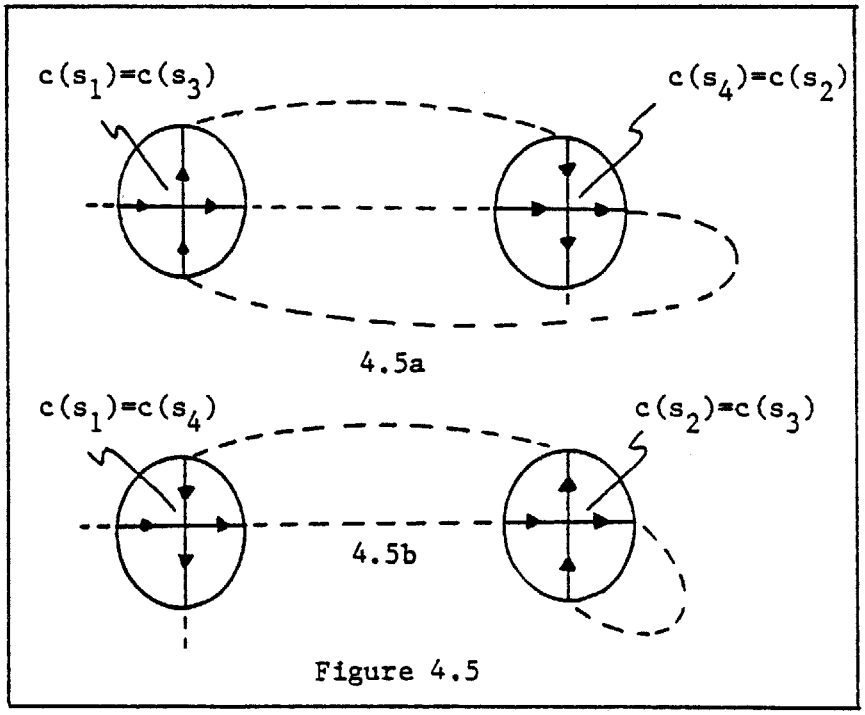


Figure 4.5

An immediate corollary of Lemma 4.1 is the following proposition.

<u>Proposition 4.1</u>: <u>Suppose α is a properly immersed (smooth) one-submanifold in general position in a surface F with negative Euler characteristic</u>.

Under these conditions, α has non-minimal self-intersection number in its homotopy class rel ∂F if and only if there is an inessential double point or there are a pair of companion double points in α. ⊠

The main result of this section is the following theorem.

Theorem 4.1: α as above has non-minimal self-intersection number in its homotopy class rel ∂F if and only if there is a mono-gon or a bi-gon with its boundary in α.

Proof: The implication ($\Leftarrow$) is trivial as in Figure 4.2, and the implication ($\Rightarrow$) takes some work to prove. To start off, suppose that α is connected, and parametrize α once and for all. Let $\mathcal{J}$ be the collection of sub-intervals [a,b] of the arc or curve which parametrizes α so that one of the following three conditions is satisfied.

a) $\alpha(a) = \alpha(b)$ is an inessential double point of α.

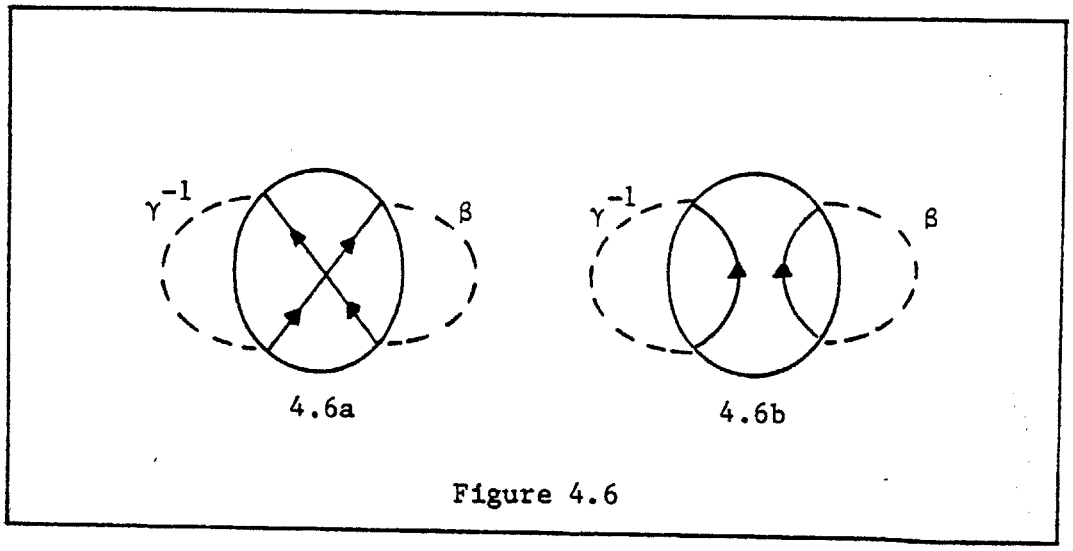
b) $\alpha(a) = \alpha(b)$, and there is a companion q to $\alpha(a)$ so that $\alpha^{-1}(q) \subset [a,b]$.

c) $\alpha(a)$ and $\alpha(b)$ are companion double points and $\alpha^{-1}(\alpha(a))$ $\alpha^{-1}(\alpha(b))$ are contained in [a,b].

By Proposition 4.1, $\mathcal{J} \neq \phi$. Let $[a_0,b_0]$ be an innermost interval in $\mathcal{J}$.

Case a): $\alpha(a_0)=\alpha(b_0)$ is inessential. If $\alpha|_{(a_0,b_0)}$ is an embedding, we have exhibited the mono-gon bounded by $\alpha[a_0,b_0]$. If $\alpha|_{(a_0,b_0)}$ is not an embedding, then choose $a_0 < a_1 < b_1 < b_0$ so that $\alpha(a_1) = \alpha(b_1)$ and so that $\alpha|_{(a_1,b_1)}$ is an embedding. One can arrange this by choosing $[a_1,b_1]$ innermost among sub-intervals of $[a_0,b_0]$ so that $\alpha(a_1) = \alpha(b_1)$. Let $\beta = \alpha[a_1,b_1]$ and $\gamma = \alpha^{-1}[a_0,a_1]*\alpha^{-1}[b_0,b_1]$, so that $\beta = \gamma$ as elements

of $\pi_1(F,\alpha(a_1))$. We modify β and γ in a neighborhood of $\alpha(a_1)$ as in Figure 4.6, and we retain the names β and γ^{-1} for the components of the result.



4.6a 4.6b

Figure 4.6

Now, γ^{-1} is an immersion homotopic to the embedding β^{-1}, and so γ^{-1} and β are disjointly embedded since $[a_0,b_0]$ was chosen to be innermost, using Proposition 4.1. β is not null homotopic since $[a_0,b_0]$ was chosen to be innermost, so γ^{-1} and β bound an annulus in F. Thus, $2\beta = 1$ in $\pi_1(F,\alpha(a_1))$, which is impossible.

In cases b) and c), if the innermost bi-gon is not embedded, then one easily constructs a null homotopic loop that is the composition of two disjointly embedded non null homotopic loops (consult Figure 4.5) and derives a contradiction as above.

This proves the theorem in case α is connected. In general, the same argument is valid provided $\mathcal{J}$ also includes any intervals arising

from companions on different components of α. ⊠

Corollary 4.1: If α is a proper immersion homotopic to an embedding rel ∂F and α is in general position in F, then either α is already embedded or there is a mono-gon or a bi-gon in F with its boundary in α. ⊠

Corollary 4.2: If α and β are proper immersions in general position each with minimal self-intersection number in its homotopy class rel ∂F, then $\alpha \cup \beta$ is in non-minimal position if and only if there is a bi-gon in F with half its boundary in α and half its boundary in β. ⊠

It is not true that if α and β are proper immersions, then $\alpha \cup \beta$ is in non-minimal position if and only if there is a mono-gon or a bi-gon in F with its boundary in $\alpha \cup \beta$ as the example pictured in Figure 4.7 indicates.

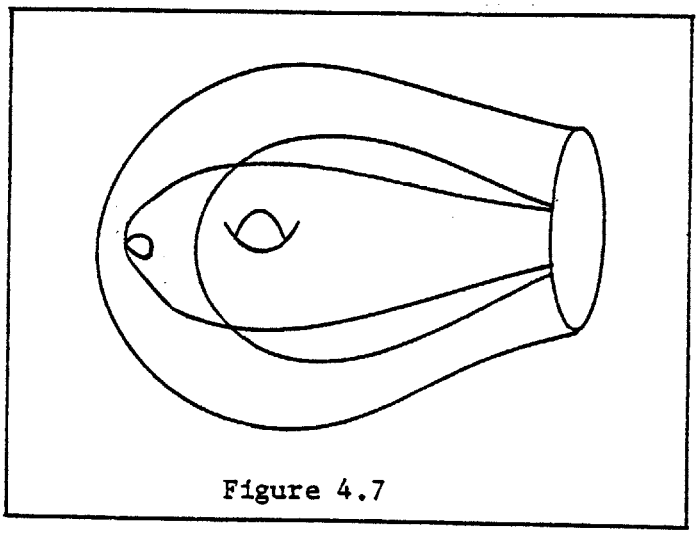


Figure 4.7

This section contains generalizations of some results of [7]. Corollary 4.2 is proved there for α and β embedded curves.

SECTION 5

This section contains a description of some of our technical foundations and a final overview of the computations to be performed in Sections 6 and 7. We will introduce a combinatorial description of curves and arcs immersed in surfaces, called symbols. We will then use the results of Section 4 to show that, under suitable conditions, a symbol for an immersed one-submanifold in fact describes an embedded one-submanifold. With this machinery developed, we distinguish four types of multiple arcs and give a detailed desription of the computation of the two elementary transformations for each type. We then describe the computations of Sections 6 and 7 step by step.

Later in this section, when we apply the results of Section 4, we will require our surfaces to have a fixed smooth structure; for the present, we may work in the topological setting. We adopt the notation defined in Section 2, where we introduced the notion of a basis for the collection of multiple arcs. Fix a choice of basis A, let M be the number of embedded pairs of pants D_j, and let N be the number of curves K_i for the basis A.

Suppose that $\alpha \in \mathscr{S}'(F)$ is a multiple arc in the compact oriented surface F. We choose a good representative of α with components $\{c\}$, and we parametrize each component c as a map from the unit interval or unit circle into F, depending on whether c is an arc component or a closed curve component of α. We have oriented the components of α arbitrarily and once and for all.

There is a finite partition of the unit interval or unit circle by intervals $[t_{k-1}, t_k]$, which is maximal subject to the condition

that $c[t_{k-1},t_k]$ intersect $\bigcup_j \partial D_j$ exactly in $c(t_{k-1})$ and $c(t_k)$. Thus, each $c[t_{k-1},t_k]$ is contained in either an annulus A_i or an embedded pair of pants D_j. If $c[t_{k-1},t_k] \subset A_i$, we formally associate to $[t_{k-1},t_k]$ the symbol st_i^n, where n is the twisting number of $\upsilon_i c[t_{k-1},t_k]$ in the standard annulus A. n is taken to be zero if $\upsilon_i c[t_{k-1},t_k]$ runs directly from window to window with no twisting. If $c[t_{k-1},t_k] \subset D_j$, we formally associate to $[t_{k-1},t_k]$ the symbol $s\ell_{11}^j$, $s\ell_{12}^j$, $s\ell_{13}^j$, $s\ell_{22}^j$, $s\ell_{23}^j$ or $s\ell_{33}^j$, according to which canonical piece 1_{**} in the standard pants $f_j c[t_{k-1},t_k]$ is parallel to. Each symbol in the following set is called a <u>letter</u>.

$$\{st_i^n: n \in \mathbb{Z},\ i=1,\ldots,N\} \cup \{s\ell_{11}^j, s\ell_{12}^j, s\ell_{13}^j, s\ell_{22}^j, s\ell_{23}^j, s\ell_{33}^j: j=1,\ldots,M\}$$

Once the components of α are oriented, it makes sense to distinguish between letters corresponding to canonical pieces with different orientations. It will be convenient to do this in Section 7; where we consider the second elementary transformation. For now, however, letters are to be regarded as "unoriented".

A <u>connected symbol</u> is defined to be an (ordered) sequence of letters. A <u>symbol</u> is defined to be a finite collection of connected symbols, called the <u>components</u> of the symbol. Once a basis A is chosen we define the A-symbol of the component c of α to be the concatenation of the letters associated to $[t_{k-1},t_k]$, in order. If c is an arc component of α, then this symbol is unique (once c is oriented); however, if c is a closed curve component of α, the symbol depends on the choice of a starting point of the parametrization $c:S^1 \to F$ subject to $c(t_k) \in \bigcup_j \partial D_j$, for all k. Define the A-symbol of $\alpha \in \mathscr{S}'(F)$ to be the collection of symbols of components of α. We will call an A-symbol simply a symbol when the choice of basis is clear.

A symbol is thus a finite collection of words in the free semi-group on the letters. When convenient, we will use the semi-group notation. For instance, we will write the symbol $(s\ell^j_{**}\ st^1_i\ s\ell^{j'}_{**})^n$ as shorthand for n concatenations of the symbol $s\ell^j_{**}\ st^1_i\ s\ell^{j'}_{**}$. When convenient, we will also delete a letter st^0_i from a symbol. For instance, we will write the symbol $s\ell^j_{**}\ s\ell^{j'}_{**}$ as shorthand for the symbol $s\ell^j_{**}\ st^0_i\ s\ell^{j'}_{**}$.

We will say a symbol is <u>embedded admissible</u> (with respect to the basis A) if it arises as above from some multiple arc α. Embedded admissible symbols satisfy many properties. In Appendix A, we verify some simple properties of embedded admissible symbols. In particular, in Corollary A.1, we show that if $st^{n_1}_i 1$ and $st^{n_2}_i 2$ are letters in (some components of) an embedded admissible symbol, then $\text{sgn}(n_1) = \text{sgn}(n_2)$. sgn(0) is undefined, but st^0_i can occur only with st^{+1}_i or st^{-1}_i. Any symbol satisfying this condition for each i=1,...,N is said to be a <u>constant parity</u> symbol.

We will call a symbol <u>immersed admissible</u> (with respect to the basis A) if the following conditions are satisfied.

a) There is a (parametrized) immersion $\bar{\alpha}$ of a one-manifold 0 into F which is <u>homotopic rel ∂F to an embeddeding</u>. <u>$\bar{\alpha}$ may be either a proper or improper immersion</u>.

b) There is a partition $\{[t_{k-1},t_k]\}$ of each component of 0, maximal subject to the condition that $\bar{\alpha}[t_{k-1},t_k]$ intersect $\bigcup_j \partial D_j$ exactly in $\bar{\alpha}(t_{k-1})$ and $\bar{\alpha}(t_k)$, for each k. Furthermore, for each k, $f_j\bar{\alpha}[t_{k-1},t_k]$ is a canonical piece in the standard pants for some j, or $\cup_i \bar{\alpha}[t_{k-1},t_k]$ is some number of twists in the standard annulus A for some i.

c) $\bar{\alpha}$ gives rise to the symbol, as above.

We will say an immersion $\bar{\alpha}$ is in good position (with respect to the basis A) if there is a partition of the components of 0 so that conditions b) and c) above are satisfied. Thus, to each (parametrized) immersion in good position, there corresponds a unique symbol. However, it is not true that there is a unique homotopy class rel boundary (of the immersion) of good immersions corresponding to a given symbol.

One can easily give necessary and sufficient conditions for a symbol to arise as above from an immersion, provided that we do not require that the immersion be homotopic to an embedding. Necessary and sufficient conditions for a symbol to be either immersed admissible or embedded admissible are not known. This appears to be a very hard problem, which we will happily be able to avoid.

Note that an embedded admissible symbol is immersed admissible. Moreover, note that if $\alpha \in \mathscr{S}'(F)$, then one can compute the parameter values $\{\ell^j_{**}, t_i\}$ corresponding to α (with respect to the basis A) from the A-symbol corresponding to α. Indeed, if c is a component of a good representative of α, then the parameter values ℓ^j_{**} and t_i corresponding to the isotopy class of c are the exponent sums of the letters $s\ell^j_{**}$ and st_i in the symbol corresponding to c. The parameter values of a disconnected α are the sums of the parameter values of its components. (See Remark 2.4.) We call the various exponent sums on the letters of an arbitrary symbol the coordinates of the symbol. We will denote these exponent sums by ℓ^j_{**} and t_i. This abuse of notation will not cause any confusion.

Suppose that s s' is a pair of adjacent letters in the symbol $\bar{s}$ corresponding to the immersion $\bar{\alpha}$. Suppose that s arises from $[t_{k-1}, t_k]$, and

s' arises from $[t_k, t_{k+1}]$. We say that the symbol $\bar{s}$ is alternating if one of the following conditions holds for every pair s s' of adjacent letters.

a) $\bar{\alpha}[t_{k-1}, t_k]$ and $\bar{\alpha}[t_k, t_{k+1}]$ lie on different sides of some component of $\bigcup_i \partial A_i$.

b) $\bar{\alpha}[t_{k-1}, t_k]$ and $\bar{\alpha}[t_k, t_{k+1}]$ are each twists of the same direction in some annulus A_i.

In particular, an embedded admissible symbol is always alternating. An immersed admissible symbol may fail to be alternating; for instance, this can happen when $D_j = D_{j'}$, where j and j' are superscripts of consecutive letters arising from canonical pieces.

We are mostly interested in alternating, constant parity symbols because an alternating, constant parity, immersed admissible symbol can often be shown to be embedded admissible. More precisely, we will prove the following proposition.

Proposition 5.1: Suppose that $\bar{\alpha}$ is a good proper immersion and the symbol $\bar{s}$ of $\bar{\alpha}$ is alternating, constant parity and immersed admissible. Suppose, moreover, that the coordinates of $\bar{s}$ satisfy restrictions a)-f) of Corollary 2.1. Under these conditions, $\bar{\alpha}$ is homotopic rel ∂F to an embedding that has the same symbol $\bar{s}$.

(Restrictions a)-f) of Corollary 2.1 require simply that in each embedded pair of pants D_j, the canonical pieces that occur may be disjointly embedded simultaneously.)

Before we prove this proposition, we introduce some machinery, called train tracks, due to Thurston [17]. We do not use train track theory in any essential way; train tracks are simply a technical convenience in the proof of Proposition 5.1.

A properly embedded branched one-submanifold T in a compact oriented

surface is called a train track with stops provided $F \setminus T$ contains no null-gons, mono-gons, bi-gons or smooth annuli. (See Section 4.) In the literature, it is customary to require a train track to be a closed branched one-submanifold embedded in F. Our definition is more general for utility.

We construct seven train tracks with stops on the standard pair of pants using the following combinations of canonical pieces: i) (l_{11},l_{12},l_{13}), ii) (l_{22},l_{12},l_{23}), iii) (l_{33},l_{13},l_{23}), iv) (l_{12},l_{13},l_{23}), v) $(\hat{l}_{11},l_{12},l_{13})$, vi) $(\hat{l}_{22},l_{12},l_{23})$, vii) $(\hat{l}_{33},l_{13},l_{23})$. (Recall the alternative choices $\hat{l}_{**}$ of canonical pieces indicated in Figure 2.11.) These train tracks with stops are pictured in Figure 5.1; we indicate two train tracks with stops on the standard annulus in Figure 5.2.

Let M_1 be the number of embedded pants D_j so that no two boundary components of D_j are parallel in F, and let $M_2 = M-M_1$. We will construct $4M_1 + 2(N+M_2)$ train tracks with stops in F, called the standard train tracks with stops (with respect to the basis A).

Construction 5.1: A branched one-submanifold T in the surface F is said to be a standard train track with stops with respect to the basis A if the following conditions are satisfied.

a) $f_j^{-1}T$ is one of the tracks i)-vii) in the standard pair of pants corresponding to the choice of canonical pieces in the pants D_j, for $j=1,\ldots,M$.

b) $\upsilon_i^{-1}T$ is one of the tracks on the standard annulus illustrated in Figure 5.2, for $i=1,\ldots,N$.

c) If D_j is a pair of pants so that $f_j^{-1}\partial_2$ is parallel to $f_j^{-1}\partial_3$ in F, then $f_j^{-1}T$ is one of the tracks i), iv), or v).

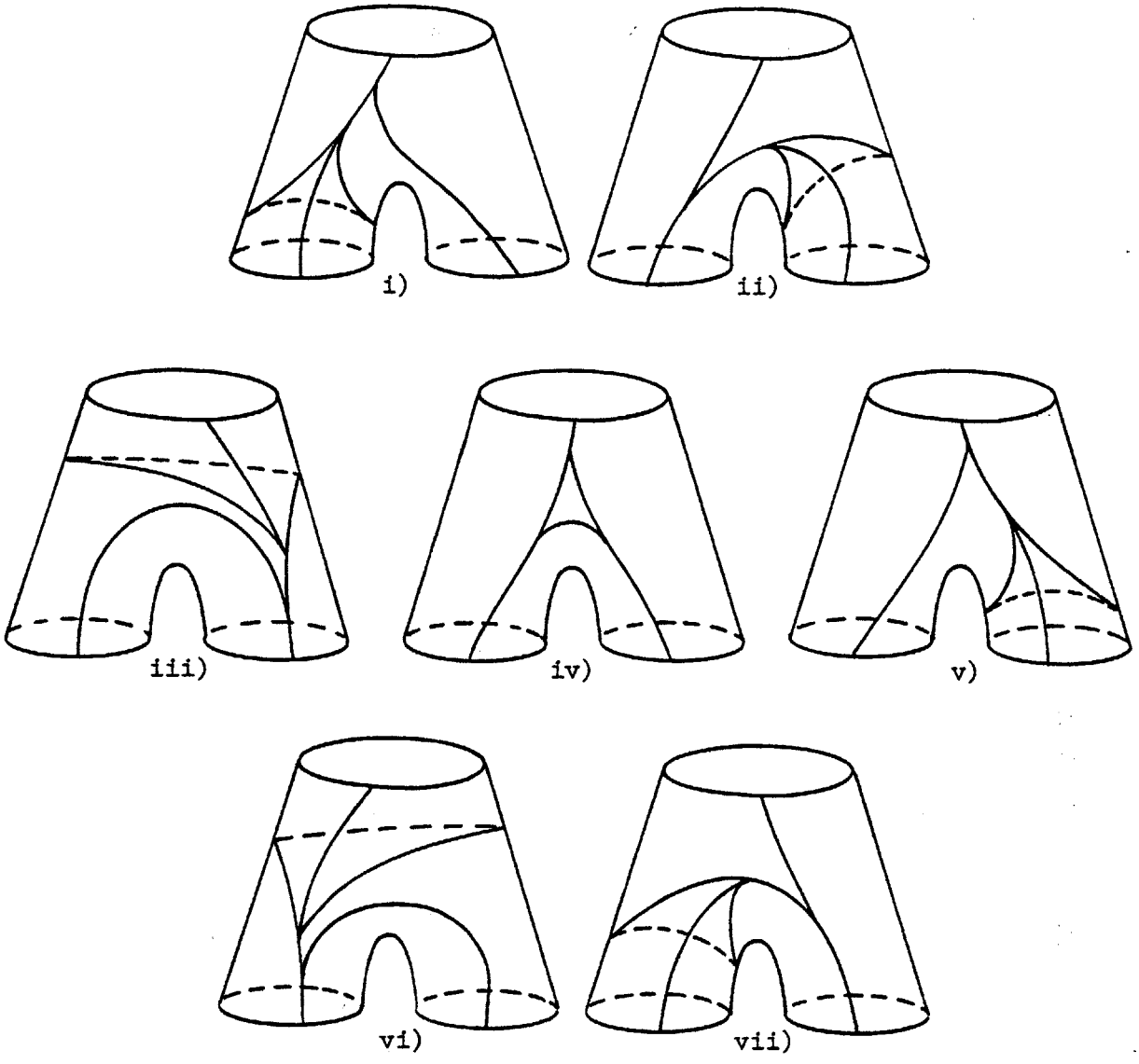


Figure 5.1

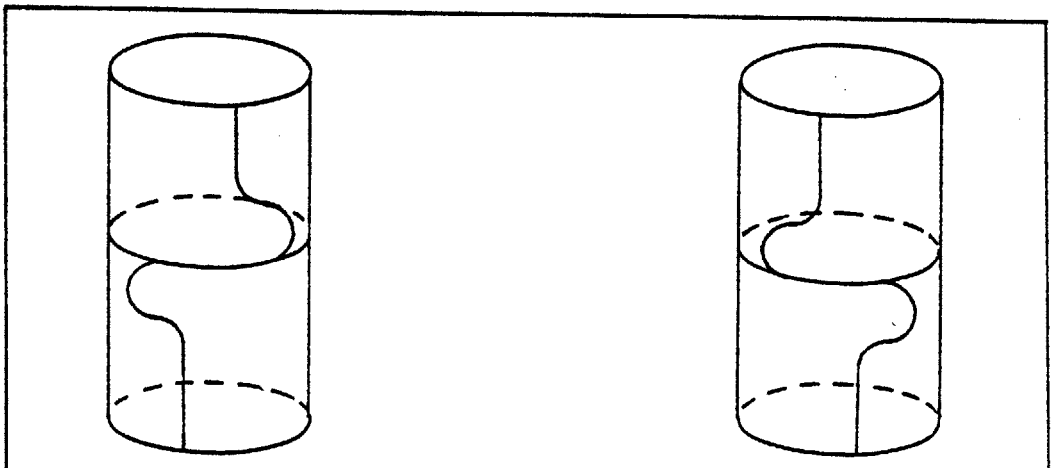


Figure 5.2

d) If D_j is a pair of pants so that $f_j^{-1}\partial_1$ is parallel to $f_j^{-1}\partial_3$ in F, then $f_j^{-1}T$ is one of the tracks ii), iv) or vi).

e) If D_j is a pair of pants so that $f_j^{-1}\partial_1$ is isotopic to $f_j^{-1}\partial_2$ in F, then $f_j^{-1}T$ is one of the tracks iii), iv) or vii).

(It is easy to check that any branched one-submanifold so constructed is actually a train track with stops. The restrictions c)-e) are explained by restrictions g)-i) of Corollary 2.1.)

Proof of Proposition 5.1: Let $\bar{\alpha}$ be the particular immersion giving rise to an immersed admissible, alternating, constant parity symbol $\bar{s}$. Since the coordinates of $\bar{s}$ satisfy restrictions a)-f) of Corollary 2.1 by hypothesis, we may assume that $\bar{\alpha}$ lies in a regular neighborhood N(T) of one of the standard train tracks with stops, say T. Moreover, since $\bar{s}$ is constant parity, we may choose T so that $\bar{\alpha}\cap A_i$ is (smoothly) homotopic into $T\cap A_i$, for each i=1,...,N. Since $\bar{\alpha}$ consists of the pre-images under f_j and υ_i of canonical pieces in the standard pants and twists in the standard annulus, and since $\bar{\alpha}$ has an alternating symbol, there can be no mono-gons complementary to $\bar{\alpha}$. Thus, by Corollary 4.1, either $\bar{\alpha}$ is already embedded which proves the proposition, or there are complementary bi-gons. Let B be a complementary bi-gon. Since $\bar{\alpha}\subset N(T)$, $\bar{\alpha}$ is (smoothly) homotopic into T, and $F\setminus T$ contains no bi-gons, it follows that $B\subset N(T)$. By induction on the number of times B intersects $\cup_i \partial A_i$, one easily homotopes $\bar{\alpha}$ to get rid of the bi-gon B, yielding an immersion homotopic to $\bar{\alpha}$, with fewer double points than $\bar{\alpha}$, and with the same symbol as $\bar{\alpha}$. This proves the proposition.$\boxtimes$

Now that we have developed some of the machinery of symbols, we give a more detailed description of the computation of the two elementary

transformations than the description given in Section 3. Recalling the notation given there, let S_1 be the torus-minus-a-disc, and let S_2 be the sphere-minus-four-discs. Let A be the basis on S_i indicated in Figure 3.8a, and let A' be the basis on S_i indicated in Figure 3.8b. Let $\pi_i:\tilde{S}_i \to S_i$ be the regular planar cover mentioned in Section 3 (and yet to be defined) with group of covering translations Λ_i.

Given $\alpha \in \mathscr{S}'(S_i)$, we may assume without loss (as far as computing the elementary transformations is concerned) that $t_2 = 0$ for $\alpha \in \mathscr{S}'(S_1)$, and $t_2 = t_3 = t_4 = t_5 = 0$ for $\alpha \in \mathscr{S}'(S_2)$. (The pants curves $\{K_i\}$ of the basis A are numbered as in Figure 3.8.)

We begin by considering the elementary transforms of connected multiple arcs. If $\alpha \in \mathscr{S}'(S_i)$ has components {c}, then the elementary transform of α has A' parameter values given by the sums of the A' parameter values of the transforms of the components of c. However, recall that computing the A parameter values of the components {c} (or even the number of components) from the A parameter values of α is an unsolved problem. (See Remark 2.3.)

If s is an A-symbol arising from some component c of the multiple arc α on S_i, we define the A-length of s to be the number of letters in the A-symbol s. The A'-length of s is the number of letters in the A'-symbol corresponding to the isotopy class of c. (Using the semi-group notation, we omit any letters st_i^0 or $st_i'^0$.)

We distinguish four types of connected symbols, and we will presently describe the computations for each case.

Type 1): s arises from a closed component of α; that is, s corresponds to a connected multiple curve.

Type 2): s arises from an arc component of α, and both the A-length and the A'-length of s are at least two.

Type 3): s arises from an arc component of α, and s has A-length one.

Type 4): s arises from an arc component of α, and s has A'-length one. The typical types are 1) and 2), and we refer to connected symbols of Types 3) or 4) as exceptional. Similarly, we say that a disconnected symbol is non-exceptional if none of its components are exceptional.

Type 1): If s is of Type 1), the transform of c will be computed without using the machinery of symbols. We will relate the A- and A'-parameter values of multiple curves with no boundary-parallel components to the "slope" of a geodesic representative with respect to a certain metric.

We will let $\overset{\circ}{\mathscr{S}}'(S_i) \subset \mathscr{S}'(S_i)$ denote the collection of multiple curves with no boundary-parallel components. Similarly, if $\mathcal{B}$ is a basis for multiple arcs, we will let $\overset{\circ}{\mathscr{S}}'_{\mathcal{B}}(S_i) \subset \mathscr{S}'_{\mathcal{B}}(S_i)$ denote the parameter values corresponding to such multiple curves with respect to the basis $\mathcal{B}$.

Type 2): If s is of type 2), suppose $s = s_1 \ldots s_n$ with $n \neq 1$, where the s_i are A-letters. We will consider a lift $\tilde{c}$ of c to $\tilde{S}_i$, and we will describe an isotopy between $\tilde{c}$ and some other arc $\bar{\tilde{c}}$ embedded in $\tilde{S}_i$. $\bar{c} = \Pi_i(\bar{\tilde{c}})$ is thus an immersion homotopic to the embedding c. The isotopy in $\tilde{S}_i$ will be so that we can read off the immersed admissible A'-symbol of the immersion $\bar{c}$. We will check that this A'-symbol satisfies the hypotheses of Proposition 5.1, and we can compute the coordinates of this A'-symbol. This will complete the computation for Type 2) symbols.

We will describe the isotopy in $\tilde{S}_i$ in the language of symbols after making a definition.

To an embedded admissible symbol $s = s_1 \ldots s_n$, we let correspond a Λ_i-orbit of multiple arcs in $\tilde{S}_i$. We construct a representative of this orbit as follows: choose some good representative c of the multiple arc with symbol s, and take some lift $\tilde{c}$ of c to $\tilde{S}_i$. We may conveniently denote such a lift by $\tilde{s}_1 \ldots \tilde{s}_n$. We call $\tilde{s}_1 \ldots \tilde{s}_n$ a lift of $s_1 \ldots s_n$

to $\tilde{S}_i$. Lifts of symbols are convenient because there is a partition of the interval parametrizing $\tilde{c}$ by $\{[t_{k-1},t_k]: k=1,\ldots,n\}$, so that $f_j\Pi_i\tilde{c}[t_{k-1},t_k]$ is a canonical piece in the standard pants, for some $j=1,\ldots,M$, or $\upsilon_{i'}\Pi_i\tilde{c}[t_{k-1},t_k]$ is a sequence of twists in the standard annulus A, for some $i'=1,\ldots,N$. One can easily formalize this construction into the setting of semi-groups, but this geometric description is better.

We will use the correspondence of the previous paragraph in a more general setting. We will take the lift of any symbol arising from an immersed one-manifold in good position in S_i. O may be improperly embedded in S_i or even improperly immersed.

Conversely, given an arc $\tilde{c}$ (properly or improperly) embedded in $\tilde{S}_i$ with a partition as above, there corresponds an obvious symbol s. There is, however, no guarantee that this symbol s is embedded or immersed admissible in general.

Precisely, then, for symbols of Type 2), we will describe a map from an A-symbol $s = s_1 \ldots s_n$ to an A'-symbol $\bar{s} = \bar{s}_1 \ldots \bar{s}_m$ that describes an isotopy of $\tilde{s}_1 \ldots \tilde{s}_n$ to $\bar{\tilde{s}}_1 \ldots \bar{\tilde{s}}_m$. We will call such a map on symbols a <u>combinatorial homotopy</u>. Suppose for simplicity that $n \geq 3$. The combinatorial homotopy for type 2) symbols will be described in two stages: we first describe a combinatorial homotopy of $s_2 \ldots s_{n-1}$ to some A'-symbol. We then consider how we must modify this map on symbols to describe a combinatorial homotopy of all of $s_1 \ldots s_n$. This second stage, where we compute the effects of considering also the letters s_1 and s_n, is called the computation of the <u>boundary effects</u>.

Needless to say, for small n, stage 1 is not very interesting, and the transformation is governed by the boundary effects. In fact, we

treat the case of $n \leq 4$ separately in Appendix B. We will derive the formulas for the elementary transforms of type 2) symbols of A-length at least five in Sections 6 and 7, and then we will check by hand in Appendix B that these formulas also describe the elementary transforms of non-exceptional symbols of A-length less than five. This avoids considering several special cases in the arguments of the next two sections. I remark that several of the faces of the piecewise-integral structure of positive codimension occur when the symbols have small length, and this fact accounts for the special cases.

I should also remark that the arguments for type 2) components apply to the Type 1) components as well except for some small technical details; in fact, stage one completely describes the combinatorial homotopy for symbols of Type 1). The technical distinction between non-exceptional arcs and curves is simple: the symbols of arcs are well-defined once the arc is oriented; the symbols of curves do not enjoy this property. By the time we have given the argument that legitimizes the computations for arcs, it will be clear how to overcome this technical difficulty for multiple curves. Thus, one could give a unified treatment of Type 1) and Type 2) components. We treat these cases separately to avoid the technical difficulties and to indicate the connections between multiple curves with no boundary-parallel components and a certain Euclidean metric.

Type 3): If s is exceptional of type 3), then it is easy to compute the A'-parameters of the elementary transform of the component c by hand. This is tractable because of the short A-length of s.

Type 4): We require some algorithmic procedure for deciding if

s has A'-length one. More generally, given the A-parameter value of the (potentially) disconnected $\alpha \in \mathscr{S}'(S_i)$, we will need a way to compute the number of components of α that have A'-length one symbols. This is again tractable because if a component c of α has a symbol s with A'-length one, then s has small A-length.

Now, given the A-parameter values of some $\alpha \in \mathscr{S}'(S_i)$ with no boundary-parallel components, the computation of the corresponding A'-parameter values proceeds as follows.

Step 1: Compute the number of components of α with symbols of Type 4). This immediately gives the A'-parameter values corresponding to the A'-length one A'-symbols; let β denote the collection of components of α with type 4) symbols.

Step 2: Compute the parameter values in the basis A of the multiple arc that is α less the components of α with symbols of Types 3) and 4); denote the corresponding multiple arc by $\hat{\alpha}$.

Step 3: Compute the parameter values in the basis A' of the multiple arc $\hat{\alpha}$. (Step 3 has two stages as indicated above.)

Step 4: Compute the parameter values in the basis A' of the components of α with Type 3) symbols. Denote the corresponding multiple arc by γ.

Step 5: The A'-parameter values of α are the sums (see Remark 2.4) of the A'-parameter values corresponding to $\hat{\alpha}$, β, and γ.

In Section 6 and 7, we will compute the first and second elementary transformations, respectively. In each section, we begin by defining the cover $\Pi_i : \tilde{S}_i \to S_i$. We then sketch the computation of the elementary transformations on multiple curves with no boundary-parallel components. (A sketch will suffice, because the computation for multiple curves is

a special case of the computation for Type 2) components, as mentioned above.) Next, we proceed through Steps 1-5 outlined above, and note that these computations agree with the earlier computations for multiple curves. The computations are reasonably intricate, and we will abbreviate the discourse in Sections 6 and 7 by refering to the outline of the computation given here.

We close this section by introducing some notation that will be useful in Sections 6 and 7. If α is a multiple arc in S_i and $\mathcal{B}$ is a basis for $\mathscr{S}'(S_i)$, we will denote the tuple of parameter values corresponding to α with respect to the basis $\mathcal{B}$ by $(\alpha)_{\mathcal{B}} \in \mathscr{S}'_{\mathcal{B}}(S_i)$. Thus, for each multiple arc α in the surface S_i, we will compute $(\alpha)_A$, from $(\alpha)_A$.

SECTION 6

Let S_1 be the torus-minus-a-disc. We begin this section by defining a regular planar cover $\Pi_1:\tilde{S}_1 \to S_1$. Let Λ_1 be the group generated by the integral translations of $\mathbb{R}^2$, a subgroup of the group of isometries of $\mathbb{R}^2$ with its usual metric ρ. The quotient of $\mathbb{R}^2 \setminus \mathbb{Z}^2$ by Λ_1 is a punctured torus, and the cover of the punctured torus by $\mathbb{R}^2 \setminus \mathbb{Z}^2$ with group of translations Λ_1 is the usual cover of the torus by $\mathbb{R}^2$ with a point deleted from each fundamental domain in $\mathbb{R}^2$. Let N be a small, Λ_1-equivariant, square-shaped neighborhood of $\mathbb{Z}^2$ in $\mathbb{R}^2$, as indicated in Figure 6.1. The action of Λ_1 on $\mathbb{R}^2 \setminus N$ gives a cover $\Pi_1:\tilde{S}_1 \to S_1$, and the push-forward of ρ to S_1 by Π_1 gives a Euclidean structure on S_1 with piecewise geodesic (in fact square) boundary. To be explicit, we choose the cover so that vertical lines in $\tilde{S}_1$ cover longitudes, and horizontal lines in $\tilde{S}_1$ cover meridians in S_1. Much of the computation of this section will take place in the total space $\tilde{S}_1$.

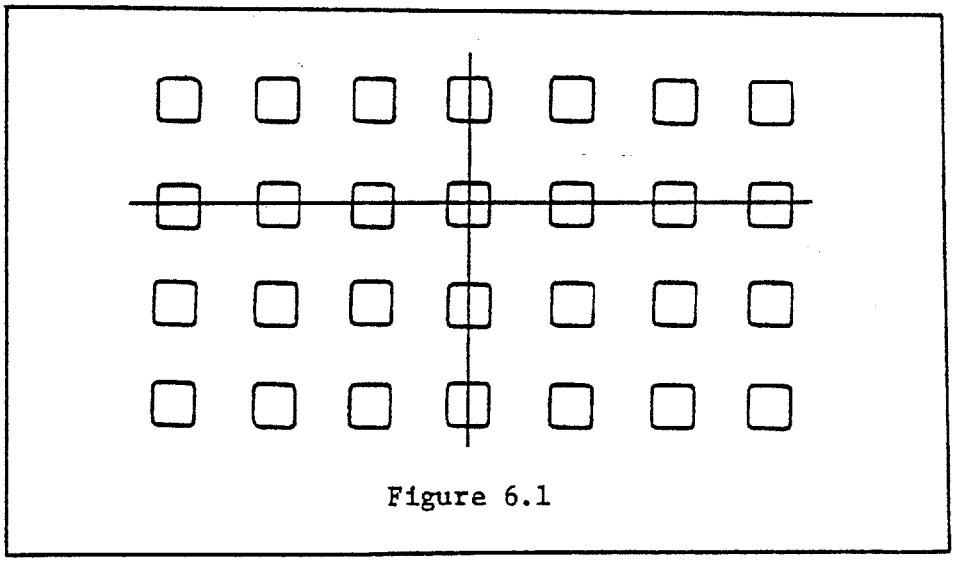
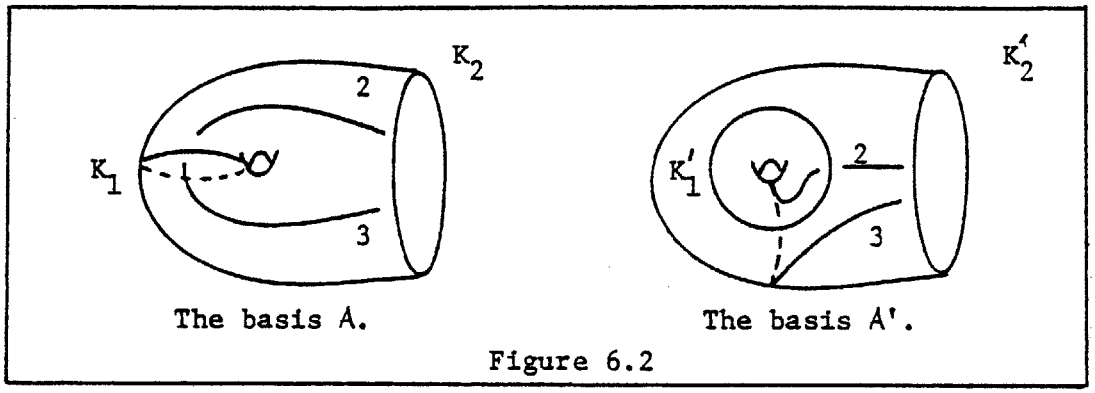


Figure 6.1

Let A and A' be the bases for $\mathcal{S}'(S_1)$ indicated in Figure 6.2, making the choice of canonical pieces as in Construction 2.1. We will adopt

the notation of Section 2 to describe the basis A, but we will delete the sub- and superscript 1 whenever possible since there is only one embedded pair of pants D_j; parameter values for multiple arcs with respect to the basis A are denoted by ℓ_{**} and t_i, i=1,2, the homeomorphism from the embedded pants D to the standard pants is denoted by f, and so on. For convenience, we will denote the corresponding quantities and objects with respect to the basis A' by ℓ'_{**}, t_i, i=1,2, D', f', and so on.



The basis A. The basis A'.

Figure 6.2

Some remarks are in order concerning the parametrizations with respect to the bases A and A'. As in restrictions g)-i) of Corollary 2.1, the two cases of simultaneously embedded canonical pieces are $\{l_{11},l_{12},l_{13}\}$ and $\{l_{12},l_{13},l_{23}\}$. Any (smooth) essential one-manifold properly embedded in S_1 whose corresponding multiple arc has coordinate $t_2 = 0$ is (smoothly) homotopic into one of the train tracks with stops indicated in Figure 6.3a. (See Section 5.) We give the four tracks corresponding to multiple arcs with $t'_2 = 0$ in Figure 6.3b.

Note that a connected, non-exceptional, immersed admissible A-symbol $\bar{s} = \bar{s}_1 \ldots \bar{s}_n$ which is not a closed component is alternating only if one of $\{\bar{s}_1,\bar{s}_n\}$ is an $s\ell_{12}$, and the other of $\{\bar{s}_1,\bar{s}_n\}$ is an $s\ell_{13}$. Thus, the coordinate value ℓ_{12} of such A-symbols is the same as the coordinate value ℓ_{13}, and similarly for the basis A'.

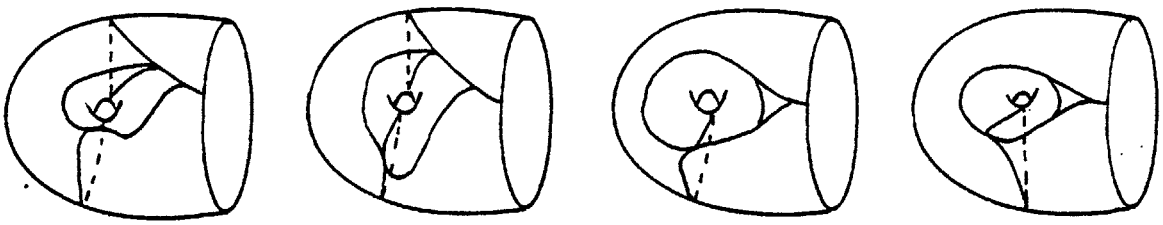


Figure 6.3a

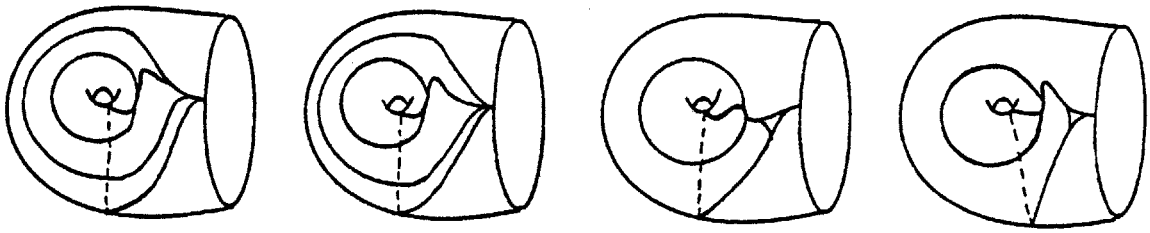


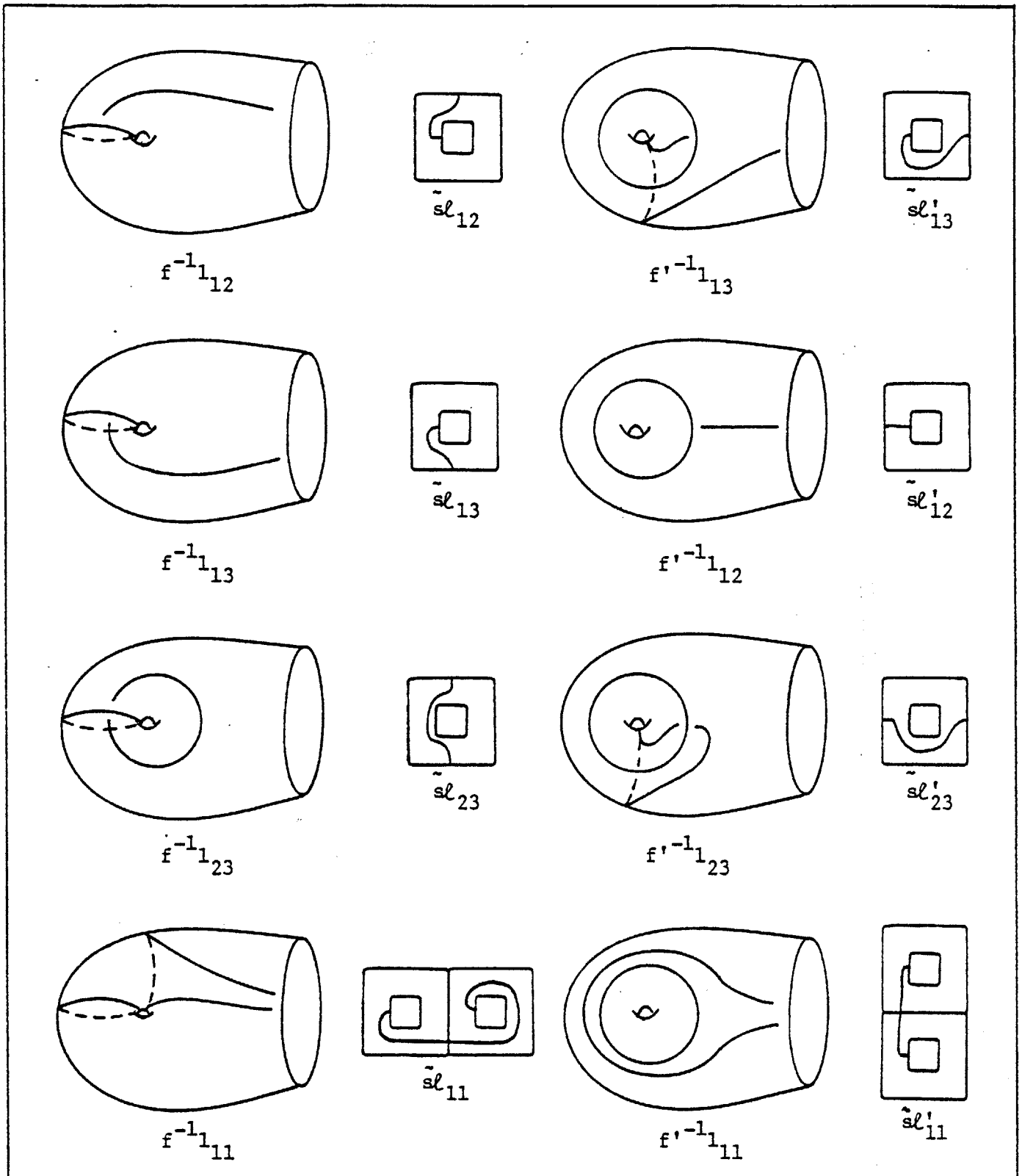
Figure 6.3b

Figure 6.3

To be quite explicit, we give in Figure 6.4a a table of the pre-images of the canonical pieces for the basis A under the homeomorphism f. In Figure 6.4a, we also give lifts (see Section 5) of the various A-letters to $\tilde{S}_1$. In Figure 6.4b, we give all the same data for the basis A'.

We commence the computation of the first elementary transformation by considering first the action of this transformation on $\overset{\circ}{\mathscr{S}}'(S_1)$. (See Section 5.) This is a transformation between $\overset{\circ}{\mathscr{S}}'_A(S_1)$ and $\overset{\circ}{\mathscr{S}}'_{A'}(S_1)$.

Lemma 1: If $\alpha \in \overset{\circ}{\mathscr{S}}'(S_1)$ consists of n components, then α is n parallel copies of a (connected) simple closed curve.



The basis A. The basis A'.

Figure 6.4

Proof: Suppose first that n = 2, and let α have components α_1 and α_2. Cut S_1 along α_1 to get some surface V. We claim that V is connected. For, suppose not, and let V_1 and V_2 be the components. Letting χ denote the Euler characteristic, the following equation holds. $-1 = \chi(S_1) = \chi(V_1) + \chi(V_2)$. Without loss of generality, the number of boundary components of V_1 is two, and the number of boundary components of V_2 is one. Thus $\chi(V_1) = 2g_1 - 1$, and $\chi(V_2) = 2g_2$, where g_i is the genus of the surface V_i, i=1,2. It follows that $g_1 + g_2 = 0$, so V_2 is a disc, which contradicts that α_1 is essential, proving the claim.

Thus, V is a pair of pants, and α_2 is therefore boundary-parallel in V. Since α_2 is not boundary-parallel in S_1, α_2 is parallel to α_1. The general case is similar. ⊠

Consider the parametrization of multiple curves with no boundary-parallel components with respect to the basis A. The only non-zero parameter values of a multiple curve are ℓ_{23} and t_1. Using p,q curves on the torus, it is easy to construct a good connected representative corresponding to each pair of parameter values with ℓ_{23} and $|t_1|$ relatively prime, including $\ell_{23}=0$ and $t_1=1$. It follows that the collection of connected multiple curves with no boundary-parallel components is parametrized by the collection $\overset{\circ}{\mathscr{S}}{}'_A(S_1)$ of parameter values satisfying the following conditions.

a) ℓ_{23} and $|t_1|$ are relatively prime.

b) $\ell_{23}=0$ implies that $t_1=1$.

c) $\ell_{12}=\ell_{13}=\ell_{11}=0=t_2$.

Proposition 6.1: There is a parametrization of connected multiple curves in S_1 with no boundary-parallel components by $\mathbb{Q}\cup\{\infty\}\subset S^1$, i.e., by the rational points on the circle.

Proof: Define a map $\gamma: \mathscr{S}'_A(S_1) \to \mathbb{Q} \cup \{\infty\}$ by $\gamma((\alpha)_A) = \mathrm{sgn}(t_1)\ell_{23}/t_1$. Given $q = \pm n_2/n_1$ with $n_1 > 0$, $n_2 \geq 0$ and $(n_1, n_2) = 1$, $\gamma^{-1}(q)$ is given by $t_1 = \mathrm{sgn}(q)n_1$, $\ell_{23} = n_2$; $\gamma^{-1}(\infty)$ is given by $t_1 = 1$, $\ell_{23} = 0$. ⊠

The rational parametrization in the proposition can be interpreted as the "slope" of α as follows: include $\alpha \subset S_1$ in the punctured torus. Giving the punctured torus the Euclidean structure described previously, any free homotopy class has a geodesic representation, as is easy to show using the previous proposition. The Gauss-Bonnet Theorem shows that the rational slope (in the Euclidean structure) of a geodesic representative is well-defined, and this slope is exactly the rational parametrization above.

Another description of this parametrization is as follows: let μ denote the meridian, and let λ denote the longitude of S_1. Given a connected multiple curve α with no boundary-parallel components, isotope α to have minimal geometric intersection number with the curves μ and λ, and so that α does not hit the point $\mu \cap \lambda$. Then $\gamma((\alpha)_A)$ has absolute value $\mathrm{card}(\alpha \cap \mu)/\mathrm{card}(\alpha \cap \lambda)$. Cut S_1 along μ and λ to get a collection of arcs properly embedded in a disc-minus-a-disc, which inherits an orientation from S_1. One can show that there are four (overlapping) cases as indicated in Figure 6.5; we define the sign of $\gamma((\alpha)_A)$ to be positive in cases one and two and negative in cases three and four. By pursuing this line of reasoning, one can prove Proposition 6.1 without resorting to Theorem 2.1.

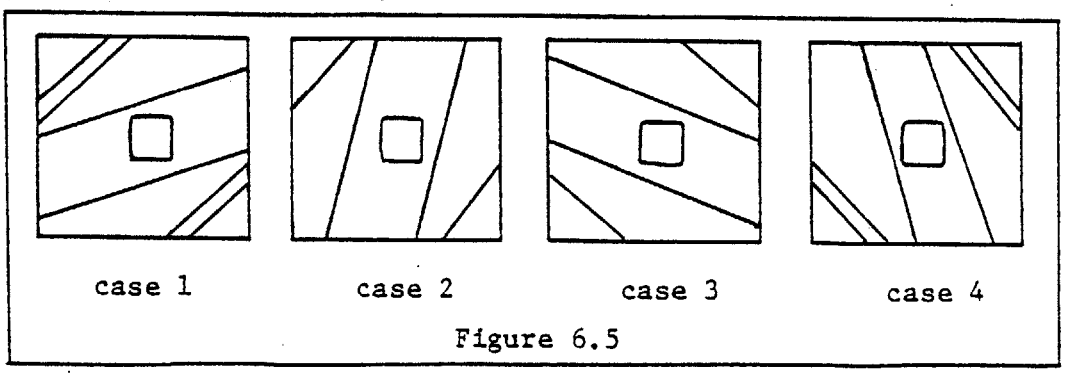


Figure 6.5

Proposition 6.2: i) The first elementary transformation from the basis A to the basis A' on multiple curves with no boundary-parallel components is given by the following formulas.

$$\ell'_{23} = |t_1|$$
$$t'_1 = -\mathrm{sgn}(t_1)\ell_{23}$$

In these formulas, sgn(0)=-1 by definition.

ii) The first elementary transformation from the basis A' to the basis A on multiple curves with no boundary-parallel components is given by the following formulas.

$$\ell_{23} = |t'_1|$$
$$t_1 = -\mathrm{sgn}(t'_1)\ell'_{23}$$

In these formulas, sgn(0)=-1 by definition.

Proof (sketch): First consider the case of a connected $(\alpha)_A \in \overset{\circ}{\mathscr{S}}'_A(S_1)$. By symmetry, there is a map $\gamma' : \overset{\circ}{\mathscr{S}}'_{A'}(S_1) \to \mathbb{Q} \cup \{\infty\}$ given by $\gamma'((\alpha)_{A'}) = \mathrm{sgn}(t'_1)\ell'_{23}/|t'_1|$. We claim that the two rational parametrizations are negative reciprocals of one another. To prove this assertion, one first proves the assertions of the previous paragraph. With this description of the parametrization, it is clear that one passes from one rational parametrization to another by interchanging the curves μ and λ. This amounts to turning one's head by 90^0 as indicated in Figure 6.6. Figure 6.6 also shows that turning one's head by 90^0 corresponds to taking negative reciprocals of rational points on the circle. A computation proves the proposition for connected multiple curves, and the general case follows from Lemma 6.1. The convention that sgn(0)=-1 is the choice consistent with the convention that $m_1=0$ implies $t_1 \geq 0$, where m_1 is the Dehn-Thurston intersection number. ⊠

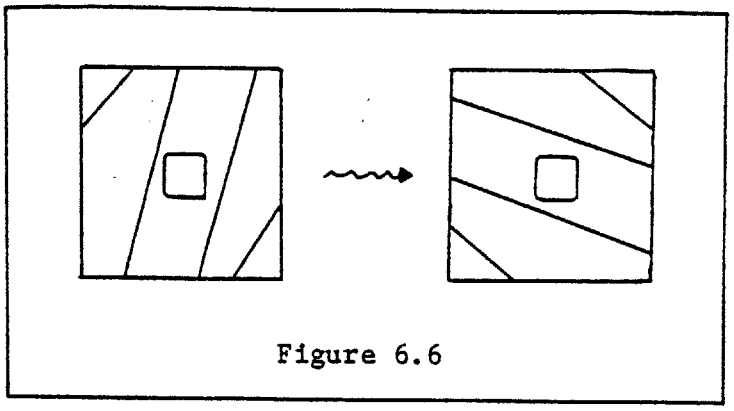


Figure 6.6

Having computed the first elementary transformation on multiple curves with no boundary-parallel components, we return to the general setting of $(\alpha)_A \in \mathscr{S}'_A(S_1)$. For most of this section (until Proposition 6.3), we tacitly assume that the multiple arc α has no closed components. Without loss, we suppose that α has $t_2 = 0$, and we orient the components of α arbitrarily. We proceed through Steps 1-5 of the computation as described at the end of Section 5. The reader should refer there to see the various steps in a wider context.

Step 1 is to compute the number of components of α parallel to $f'^{-1}l_{11}$ (plus two arcs in the annulus A_2 running from window to window with no twisting). As shown in Figure 6.4, such an arc has expression $\ell_{12} + \ell_{13}$ in the basis A. We compute the number of components of α with coordinates $\ell_{12} + \ell_{13}$ as follows: imagine cutting along the boundary of a regular neighborhood A_1 of the pants curve K_1. α twists t_1 times in A_1 and enters and exits ∂A_1 through two windows. These windows are indicated in Figure 6.7, where the label w_{12} indicates, for instance, the region of the window through which $f^{-1}l_{12}$ may pass. Thus, the number of components of α with parameter value $\ell_{12} + \ell_{13}$ is given by $((\ell_{12} - |t_1|) \wedge \ell_{13}) \vee 0$, where $\wedge$ is the infimum, and $\vee$ is the supremum. Note that this expression is equal to $(\ell_{12} - |t_1|) \vee 0$ since the parameter values ℓ_{12} and ℓ_{13} of α are equal.

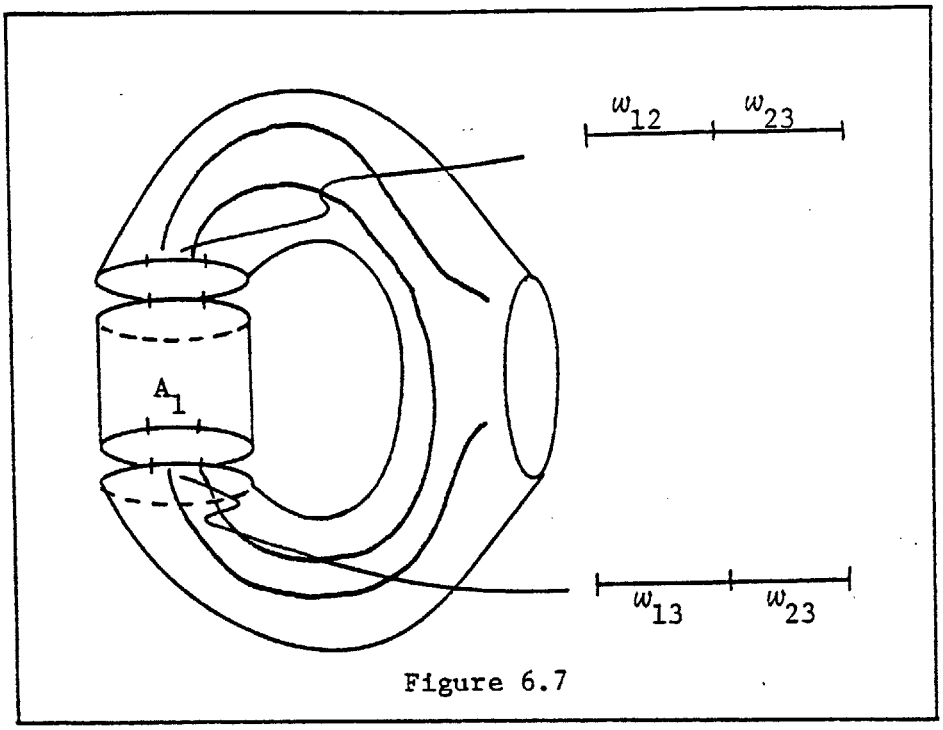


Figure 6.7

Step 2 is to compute the A-parameter values of $\hat{\alpha}$, the non-exceptional part of α. $\hat{\alpha}$ has parameter values given by the following formulas.

$$\hat{\ell}_{12} = \ell_{12} - ((\ell_{12} - |t_1|) \vee 0$$
$$\hat{\ell}_{13} = \ell_{13} - ((\ell_{12} - |t_1|) \vee 0$$
$$\hat{\ell}_{23} = \ell_{23}$$
$$\hat{\ell}_{11} = 0$$
$$\hat{t}_1 = t_1$$
$$\hat{t}_2 = t_2$$

Step 3 is to compute the A' parameter values of α and includes the typical case. For convenience of notation, we assume that α has no exceptional components, whence $(\alpha)_A = (\hat{\alpha})_A \in \mathscr{S}'_A(S_1)$. We assume that α is connected and that there is a unique embedded admissible symbol in the basis A, denoted $s = s_1 \ldots s_n$, corresponding to α, where the s_i are A-letters.

We assume without loss, using the semi-group notation, that each

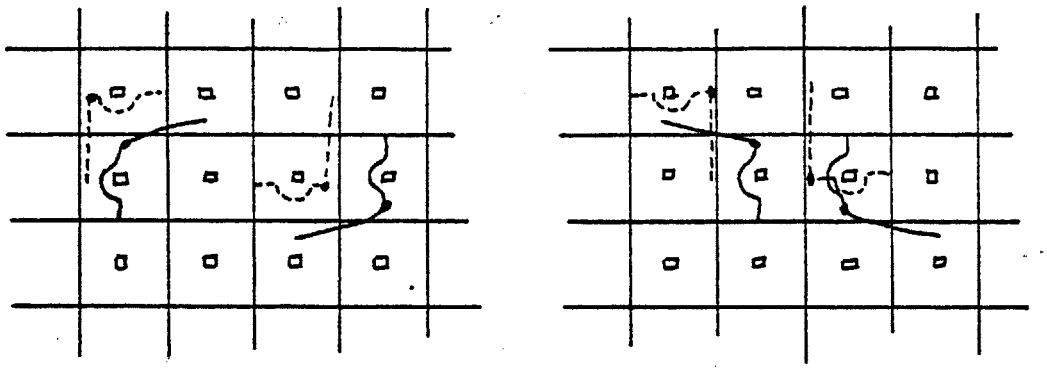
letter st_1^n appearing in s has exponent n equal to either +1 or -1. We assume (temporarily) that the A-length of s is at least three, and we consider the symbol $s_2 \ldots s_{n-1}$. Note that $s_i \in \{s\ell_{23}, st_1^{\pm 1}\}$, for all $i=2,\ldots,n-1$.

Define a map from $\langle s\ell_{23}, st_1^{\pm 1}\rangle$, the free semi-group on the three A-letters, to the free semi-group $\langle s\ell'_{23}, st_1'^{\pm 1}\rangle$ on the A'-letters by extending the map defined below on letters.

$$s\ell_{23} \to st_1'^{-\mathrm{sgn}(t_1)}$$
$$st_1^{\pm 1} \to s\ell'_{23}$$

This map is realized by a homotopy of the lift $\tilde{s}_2 \ldots \tilde{s}_{n-1}$ to some arc, which we will denote δ, improperly embedded in $\tilde{S}_1$. The homotopy translates by $(-\frac{1}{2}, +\frac{1}{2})$ and straightens into the lifts $\tilde{s\ell}'_{23}$ and $\tilde{st}_1'^{\pm 1}$ in $\tilde{S}_1$. Thus, the homotopy is not rel endpoints.

A remark is in order concerning the sign $-\mathrm{sgn}(t_1)$ of the twist that is the image of $s\ell_{23}$. It is obvious from the definition of the homotopy that an $s\ell_{23}$ in s appears as $st_1'^{\pm 1}$ in the symbol of δ. Lifts to $\tilde{S}_1$ of neighborhoods of concatenation points of $f^{-1}l_{23}$ and twists in the annulus A_1 are shown in Figure 6.8; Figure 6.8a depicts $\tilde{s\ell}_{23}\ \tilde{st}_1^{+1}$, and Figure 6.8b depicts $\tilde{s\ell}_{23}\ \tilde{st}_1^{-1}$. The solid lines in Figure 6.8 denote the lifts, and the broken lines denote the arc δ. The figure shows that when t_1 is positive, an $s\ell_{23}$ in s appears as $st_1'^{-1}$ in the symbol of δ, and similarly when the sign of t_1 is negative. Thus, the sign $-\mathrm{sgn}(t_1)$ occurs. Note that the argument above is independent of the choice of orientation of the components of α. We make the convention that $\mathrm{sgn}(0)=-1$; this is again the choice consistent with the convention that $m_1=0$ implies $t_1 \geq 0$. (Note, however, that $m_1=0$ cannot occur for a multiple arc with no closed components.)



6.8a 6.8b

Figure 6.8

To extend this map on symbols to a combinatorial homotopy from embedded admissible A-symbols to immersed admissible A'-symbols, we require the notion of a pair-of-letter expression. If s_1 and s_2 are A-letters, and if s is the A-symbol corresponding to the multiple arc α, then (s_1,s_2) evaluated on s is defined to be the number of times the symbol $s_1\ s_2$ or $s_2\ s_1$ occurs as a subsymbol of s. This does not depend on the orientation of the components of α, and, for symbols on S_1, $(s_1,s_2) = (s_2,s_1)$. (In Section 7, where we will distinguish between the orientations on a given canonical piece, the corresponding pair-of-letter identity will not be valid.)

We describe the boundary effects (see Section 5), and thus extend the map above to a combinatorial homotopy from an embedded admissible non-exceptional A-symbol s(which corresponds to the multiple arc α) to an immersed admissible A'-symbol $\bar{s}$ (which corresponds to the good proper immersion $\bar{\alpha}$) as follows: regard the combinatorial homotopy from $\tilde{s}_2 \ldots \tilde{s}_{n-1}$ to δ as a first approximation to the first elementary transformation from α to $\bar{\alpha}$. If $(s_1,s_2)\neq 0$ on s, where s_1 is one of the letters $s\ell_{12}$ or $s\ell_{13}$, we

modify the symbol of δ according to some rule that corresponds to an isotopy of $\tilde{s}_1\ \tilde{s}_2 \ldots \tilde{s}_{n-1}$ in $\tilde{S}_1$. We adjust the symbol of δ accordingly; we will write $(s_1,s_2) \equiv \Sigma \pm x_i'$ if this adjustment alters the coordinates of the symbol of δ by adding the linear combination $\Sigma \pm x_i'$ of A'-parameter values x_i' to the coordinates of the symbol of δ.

As can be seen in Figure 6.9, the boundary effects are described by the following formulas. In Figure 6.9, the solid lines indicate the lift $\tilde{s}$ before the homotopy to δ; the broken lines denote the improperly embedded arc δ; the crossed lines denote the image of $\tilde{s}$ under our combinatorial homotopy.

i) $(s\ell_{12}, st_1^{+1}) \equiv -\ell_{23}' + t_2'^{+1} + \ell_{13}' + t_1'^{-1}$

ii) $(s\ell_{12}, st_1^{-1}) \equiv \ell_{12}' + t_1'^{+1}$

iii) $(s\ell_{12}, s\ell_{23}) \equiv \ell_{12}' + t_1'^{+1}$

iv) $(s\ell_{13}, st_1^{+1}) \equiv \ell_{12}'$

v) $(s\ell_{13}, st_1^{-1}) \equiv -\ell_{23}' + \ell_{13}'$

vi) $(s\ell_{13}, s\ell_{23}) \equiv \ell_{12}'$

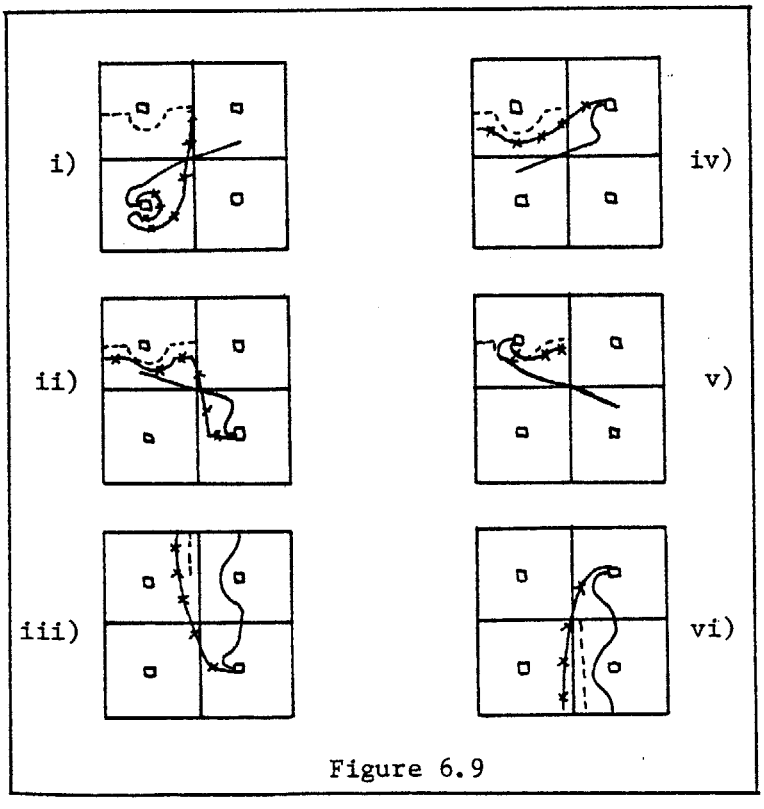


Figure 6.9

Lemma 6.2: If α is a multiple arc with non-exceptional symbol, then α is homotopic rel ∂S_1 to a good immersion $\bar{\alpha}$, where the A'-symbol of $\bar{\alpha}$ has A'-coordinates given by the following formulas.

$$\ell'_{23} = |t_1| - (s\ell_{12}, st_1^{+1}) - (s\ell_{13}, st_1^{-1})$$
$$t'_1 = -\mathrm{sgn}(t_1)\ell_{23} - (s\ell_{12}, st_1^{+1}) + (s\ell_{12}, st_1^{-1}) + (s\ell_{12}, s\ell_{23})$$
$$t'_2 = t_2 + (s\ell_{12}, st_1^{+1})$$
$$\ell'_{13} = (s\ell_{12}, st_1^{+1}) + (s\ell_{13}, st_1^{-1})$$
$$\ell'_{12} = (s\ell_{12}, st_1^{-1}) + (s\ell_{12}, s\ell_{23}) + (s\ell_{13}, st_1^{+1}) + (s\ell_{13}, s\ell_{23})$$
$$\ell'_{11} = 0$$

Proof: Suppose first that α is connected. The computations above prove the lemma in case the A-length of the symbol of α is at least four since the boundary effects influence only the letters adjacent to the boundary. In Appendix B, we check by hand that the formulas above are valid for α not exceptional of length less than five.

Suppose finally that $\alpha \in \mathscr{S}'(S_1)$ is disconnected. It is immediate that the formulas of Lemma 6.2 also apply, provided only that the symbol of α is non-exceptional. This proves the lemma. ⊠

Lemma 6.3: The symbol of $\bar{\alpha}$ given by Lemma 6.2 is alternating and constant parity.

Proof: We first show that the symbol $\bar{s}$ of $\bar{\alpha}$ is alternating. Let $s = s_1 \ldots s_n$, $n \geq 4$, be the A-symbol of α. It is geometrically obvious that the combinatorial homotopy on $s_2 \ldots s_{n-1}$ yields an alternating symbol $\bar{\bar{s}}_2 \ldots \bar{\bar{s}}_{n-1}$. Moreover, the boundary effects are seen only in the letters adjacent to the boundary, and a glance at Figure 6.9i)-vi) shows that the effect of a single $s\ell_{12}$ or $s\ell_{13}$ letter does not destroy the property of being alternating. Thus, to prove that $\bar{s}$ is alternating, it remains to

show that s begins with an $s\ell'_{12}$ if and only if it ends with an $s\ell'_{13}$. To this end, since s is embedded admissible and hence alternating and constant parity, the possible pairs of non-zero pair-of-letter expressions involving the letters $s\ell_{13}$ and $s\ell_{12}$ are the following: $\{(s\ell_{12},st_1^{+1}), (s\ell_{13},st_1^{+1})\}$, $\{(s\ell_{12},st_1^{+1}),(s\ell_{13},s\ell_{23})\}$, $\{(s\ell_{12},st_1^{-1}),(s\ell_{13},st_1^{-1})\}$, $\{(s\ell_{12},s\ell_{23}),(s\ell_{13},st_1^{-1})\}$, $\{(s\ell_{12},s\ell_{23}),(s\ell_{13},s\ell_{23})\}$, $\{(s\ell_{12},st_1^{-1}), (s\ell_{13},s\ell_{23})\}$, $\{(s\ell_{12},s\ell_{23}),(s\ell_{13},st_1^{+1})\}$.

The first four pairs yield an alternating symbol $\bar{s}$, as the formulas for the boundary effects show. The last three cannot in fact occur, as shown in Figure 6.10; that is, since s is embedded admissible, $(s\ell_{12},s\ell_{23}) \neq 0$ implies that $(s\ell_{13},st_1^{+1}) = 0 = (s\ell_{13},s\ell_{23})$, and $(s\ell_{13},s\ell_{23}) \neq 0$ implies that $(s\ell_{12},st_1^{-1}) = 0 = (s\ell_{12},s\ell_{23})$. Thus, $\bar{s}$ is alternating.

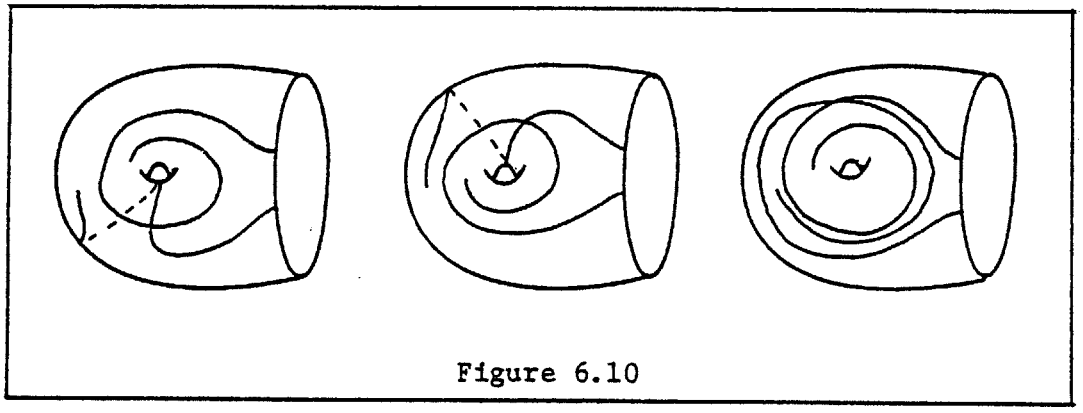


Figure 6.10

To see that $\bar{s}$ is constant parity, note that once again $\bar{\bar{s}}_2 \ldots \bar{\bar{s}}_{n-1}$ is obviously constant parity, and the sign of the t'_1 coordinate of $\bar{\bar{s}}_2 \ldots \bar{\bar{s}}_{n-1}$ is $-\text{sgn}(t_1)$ by definition. Suppose first that $\text{sgn}(t_1) \geq 0$; thus, $(s\ell_{12},st_1^{+1}) = 0$ since s is constant parity, and the formulas show that the boundary effects contribute only positive twists $t_1'^{+1}$ to the coordinates of $\bar{s}$, preserving the property of constant parity. Finally, if $\text{sgn}(t_1) \geq 0$,

then $(s\ell_{12}, st_1^{-1}) = 0 = (s\ell_{13}, st_1^{-1})$, and the only possible problem in preserving the property of constant parity is if $(s\ell_{12}, s\ell_{23}) \neq 0$. Figure 2.11 shows that this cannot occur since the arc α must then spiral indefinitely around $f_1^{-1}l_{23}$; that is, $t_1 \geq 0$ implies that $(s\ell_{12}, s\ell_{23}) = 0$.

This proves the lemma provided the A-length of s is at least five. The cases where the A-length of s is less than five are handled separately in Appendix B. ⊠

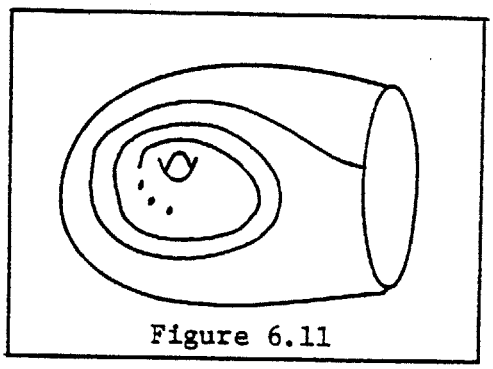


Figure 6.11

Proposition 6.3: Suppose that $\alpha \in \mathscr{S}'(S_1)$ is a multiple arc with non-exceptional components; the first elementary transform $(\alpha)_{A'}$ of $(\alpha)_A$ has A'-coordinates given by the following formulas.

$$\ell'_{11} = 0$$
$$\ell'_{23} = (|t_1| - r) \nabla 0$$
$$\ell'_{12} = \ell'_{13} = r$$
$$t'_2 = t_2 + (r \Delta t_1) \nabla 0$$
$$t'_1 = -\mathrm{sgn}(t_1)(\ell_{23} + r)$$

In these formulas, $r = \ell_{12} = \ell_{13}$, and $\mathrm{sgn}(0) = -1$.

Proof: Just as in Step 1, one can compute the pair-of-letter expressions in Lemma 6.2 from the parameter value of $(\alpha)_A$ in $\mathscr{S}'_A(S_1)$ as follows.

$$(s\ell_{12}, st_1^{+1}) = (\ell_{12} \Delta t_1) \nabla 0$$

$$(s\ell_{13}, st_1^{-1}) = (\ell_{13} \Delta -t_1) \nabla 0$$

$$(s\ell_{12}, st_1^{-1}) = (\ell_{12} \Delta (-t_1 - \ell_{23})) \nabla 0$$

$$(s\ell_{12}, s\ell_{23}) = (-t_1 \Delta \ell_{12} \Delta (t_1 + \ell_{23} + \ell_{12})) \nabla 0$$

Formally substituting the pair-of-letter values above into the formulas of Lemma 6.2 gives the following formulas.

$$\ell'_{11} = 0$$

$$\begin{aligned}\ell'_{23} &= |t_1| - (\ell_{12} \Delta t_1) \nabla 0 - (\ell_{13} \Delta -t_1) \nabla 0 \\ &= |t_1| - (r \Delta t_1) \nabla 0 - (r \Delta -t_1) \nabla 0 \\ &= |t_1| - (r \Delta |t_1|) = (|t_1| - r) \nabla 0\end{aligned}$$

$$\ell'_{12} = \ell'_{13} = (\ell_{12} \Delta t_1) \nabla 0 + (\ell_{13} \Delta -t_1) \nabla 0 = r \Delta |t_1|$$

$$t'_2 = t_2 + (\ell_{12} \Delta t_1) \nabla 0 = t_2 + (r \Delta t_1) \nabla 0$$

$$\begin{aligned}t'_1 &= -\mathrm{sgn}(t_1)\ell_{23} - (\ell_{12} \Delta t_1) \nabla 0 + (\ell_{12} \Delta (-t_1 - \ell_{23})) \nabla 0 \\ &\quad + (-t_1 \Delta \ell_{12} \Delta (t_1 + \ell_{23} + \ell_{12})) \nabla 0 \\ &= -\mathrm{sgn}(t_1)\ell_{23} - (r \Delta t_1) \nabla 0 + (r \Delta (-t_1 - \ell_{23})) \nabla 0 \\ &\quad + (-t_1 \Delta r \Delta (t_1 + \ell_{23} + r)) \nabla 0\end{aligned}$$

It remains to do the algebra to show that the two expressions for $\ell'_{12} = \ell'_{13}$, ℓ'_{23} and t'_1 are equal.

We first consider the coordinate ℓ'_{23}.

$$\ell'_{23} = \begin{cases} t_1 - (r \Delta t_1), & t_1 \geq 0 \\ -t_1 - (r \Delta -t_1), & t_1 \leq 0 \end{cases}$$

$$= \begin{cases} t_1 - r, & t_1 \geq 0 \text{ and } r \leq t_1 \\ 0, & t_1 \geq 0 \text{ and } t_1 \leq r \\ -t_1 - r, & t_1 \leq 0 \text{ and } r \leq -t_1 \\ 0, & t_1 \leq 0 \text{ and } -t_1 \leq r \end{cases}$$

$$= (|t_1| - r) \nabla 0$$

Next we consider the coordinate $\ell'_{12} = \ell'_{13}$. The argument at the end of Lemma 6.3 shows that $t_1 > 0$ implies that $(s\ell_{12}, s\ell_{23}) = 0$. Since the components of α are non-exceptional, an $s\ell_{12}$ is adjacent to either an $s\ell_{23}$ or $st_1^{\pm 1}$; it follows that $t_1 > 0$ implies that $r \Delta |t_1| = r \Delta t_1 = r$. Similarly, $t_1 < 0$ implies that $r \Delta |t_1| = r \Delta -t_1 = r$. Finally, $t_1 = 0$ cannot happen for a non-exceptional component. This justifies that $\ell'_{12} = \ell'_{13} = r \Delta |t_1| = r = \ell_{12} = \ell_{13}$, as one would expect.

Finally, note that the coordinate t'_1 for a non-exceptional A-symbol corresponding to an arc with no closed components is non-zero. Consider the expression for t'_1.

$$\begin{aligned} t'_1 &= -\mathrm{sgn}(t_1)\ell_{23} - (r \Delta t_1) \nabla 0 + (r \Delta (-t_1 - \ell_{23})) \nabla 0 \\ &\quad + (-t_1 \Delta r \Delta (t_1 + \ell_{23} + r)) \nabla 0 \\ &= -\mathrm{sgn}(t_1)\ell_{23} \\ &\quad + \begin{cases} -r, & t_1 > 0 \\ r, & t_1 < 0 \text{ and } r + \ell_{23} \leq -t_1 \\ r, & t_1 < 0 \text{ and } \ell_{23} \leq -t_1 \leq \ell_{23} + r \\ r, & t_1 < 0 \text{ and } -t_1 \leq \ell_{23} \end{cases} \\ &= -\mathrm{sgn}(t_1)(\ell_{23} + r) \end{aligned}$$

Thus, the proposition holds for multiple arcs with no closed components and no exceptional components. A computation shows that the formulas above agree with the previous computations for multiple curves, and the proposition follows. ⊠

The proof of the proposition applies also to closed curve components by ignoring boundary effects and allows a unified treatment of multiple arcs and multiple curves. We computed the action on multiple curves separately to indicate the connection with straightening in the Euclidean

metric on the punctured torus.

Finally, we consider Step 4. A glance at Figure 6.4 shows that an $f^{-1}l_{11}$ (plus two arcs in A_2 running from window to window with no twisting) has an expression $\ell'_{12} + \ell'_{13} + t_2'^{+1}$ in the basis A'.

Theorem 6.1: The first elementary transformation from the basis A to the basis A' is given by the following formulas.

$$\ell'_{11} = (r - |t_1|) \vee 0$$
$$r' = \ell'_{12} = \ell'_{13} = (r - \ell'_{11}) + \ell_{11}$$
$$\ell'_{23} = (|t_1| - (r - \ell'_{11}))$$
$$t'_2 = t_2 + \ell_{11} + ((r - \ell'_{11}) \wedge t_1) \vee 0$$
$$t'_1 = -\mathrm{sgn}(t_1)(\ell_{23} + (r - \ell'_{11}))$$

In these formulas, $r = \ell_{12} = \ell_{13}$, and $\mathrm{sgn}(0)=-1$.

Proof: The proof is Step 5 and is the combination of the previous proposition, the sentence before this theorem and Step 1. ⊠

Corollary 6.1: The first elementary transformation from the basis A' to the basis A is given by the following formulas.

$$\ell_{11} = (r' - |t'_1|) \vee 0$$
$$r = \ell_{12} = \ell_{13} = (r' - \ell_{11}) + \ell'_{11}$$
$$\ell_{23} = (|t'_1| - (r' - \ell_{11}))$$
$$t_2 = t'_2 - ((r' - \ell_{11}) \wedge -t'_1) \vee 0 - \ell_{11}$$
$$t_1 = -\mathrm{sgn}(t'_1)(\ell'_{23} + (r' - \ell_{11}))$$

In these formulas, $r' = \ell'_{12} = \ell'_{13}$, and $\mathrm{sgn}(0)=-1$.

Proof: It suffices to check that the transformations in the theorem and the corollary are inverses, which we leave as an algebraic exercise. ⊠

One can of course prove the corollary directly by mimicking the proof of the theorem with (A,A') replacing (A',A). This approach is about as much work as the algebraic exercise that proves the corollary.

We close this section with two final observations.

One easily derives the interesting identity $\ell'_{11} - \ell'_{23} = r - |t_1|$. If $t_1 \leq 0$, then one has the following identities.

$$t'_2 = t_2 + \ell_{11}$$
$$t'_1 = \ell_{23} + (r - \ell'_{11})$$

If $t_1 \geq 0$, then one has the following identities.

$$t'_2 = t_2 + \ell_{11} + r - \ell'_{11}$$
$$t'_1 = -\ell_{23} - r + \ell'_{11}$$

Introducing the parameter $r - |t_1|$, this gives a convenient description of the first elementary transformation as a piecewise-integral map.

The action of $MC(S_1)$ on $\overset{o}{\mathscr{S}}'(S_1)$ admits a faithful representation as an action of the two-by-two integral matrices of determinant one on $\overset{o}{\mathscr{S}}'_A(S_1) \subset \mathbb{Z}^+ \times \mathbb{Z}$. This is because $MC(S_1)$ is a certain central extension of the two-by-two integral matrices of determinant one by $\mathbb{Z}$. The action is a twisted right action, and will be described in Section 8.

SECTION 7

In this section, we compute the second elementary transformation; we will follow closely the outline of Section 6. Let S_2 denote the sphere-minus-four-discs. We begin by defining a regular planar cover $\Pi_2 : \tilde{S}_2 \to S_2$.

Let Λ_2 be the group generated by rotations-by-π about the integral points $\mathbb{Z}^2$ in $\mathbb{R}^2$. Λ_2 is a group of isometries of $\mathbb{R}^2 \setminus \mathbb{Z}^2$ with respect to the usual metric. This action describes a cover of the four-times punctured sphere by $\mathbb{R}^2 \setminus \mathbb{Z}^2$, and the push-forward of the usual metric by the covering projection gives a Euclidean structure on the four-times punctured sphere.

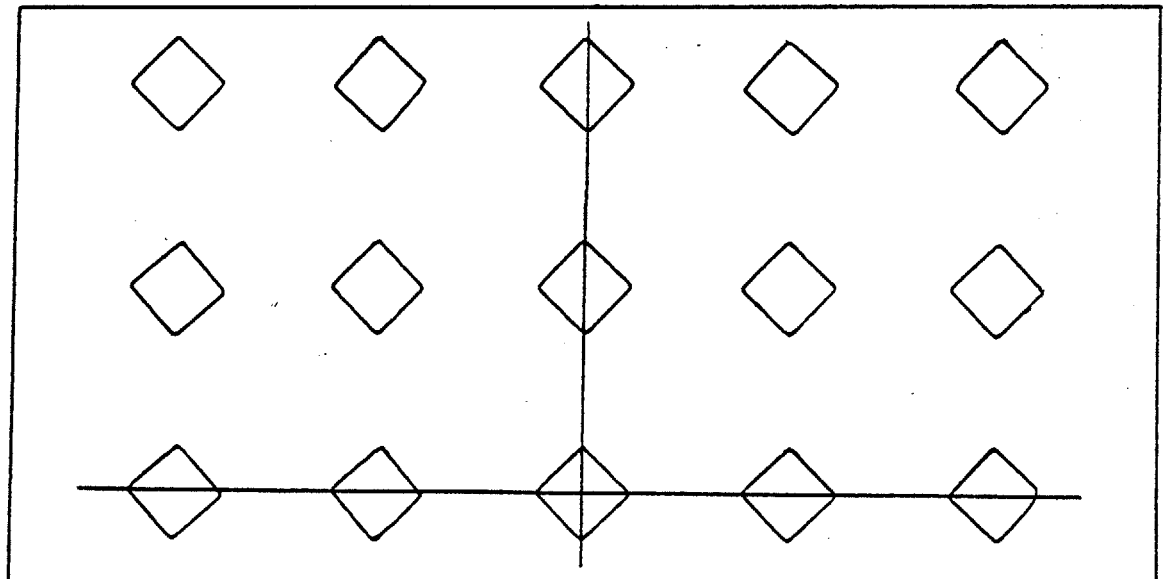


Figure 7.1

Let N be a small, Λ_2-equivariant, diamond-shaped, open neighborhood of $\mathbb{Z}^2$ in $\mathbb{R}^2$, as indicated in Figure 7.1. The action of Λ_2 on $\mathbb{R}^2 \setminus N$ gives a cover of S_2 by $\mathbb{R}^2 \setminus N$, denoted $\tilde{S}_2$. Cuttine S_2 along the arcs $a_1, \ldots, a_4$ in S_2 indicated in Figure 7.2 decomposes S_2 as two

octagons, labeled f and b in Figure 7.2. The lifts to $\tilde{S}_2$ of these octagons give a tiling of $\tilde{S}_2$; if we are careful in the choice of the geodesic arcs a_i, then we can guarantee that the associated tiling is regular. This regular tiling of $\tilde{S}_2$ by octagons is indicated in Figure 7.3; it can be seen in the Park Street Subway Station in Boston as a tiling of $\mathbb{R}^2$ by squares and octagons.

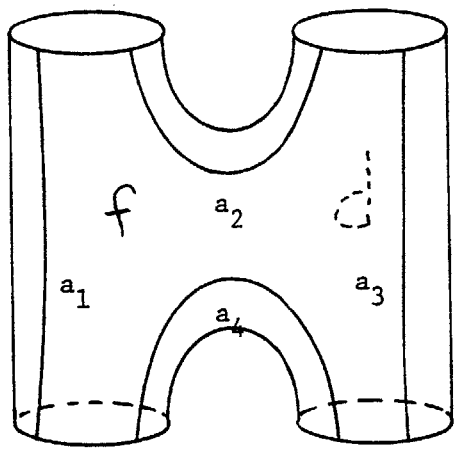


Figure 7.2

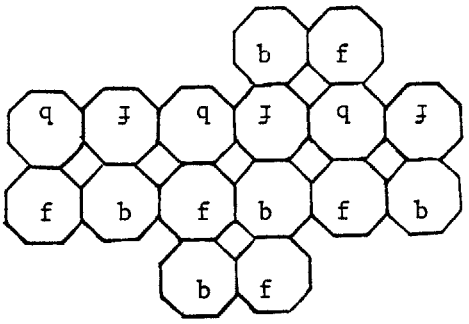
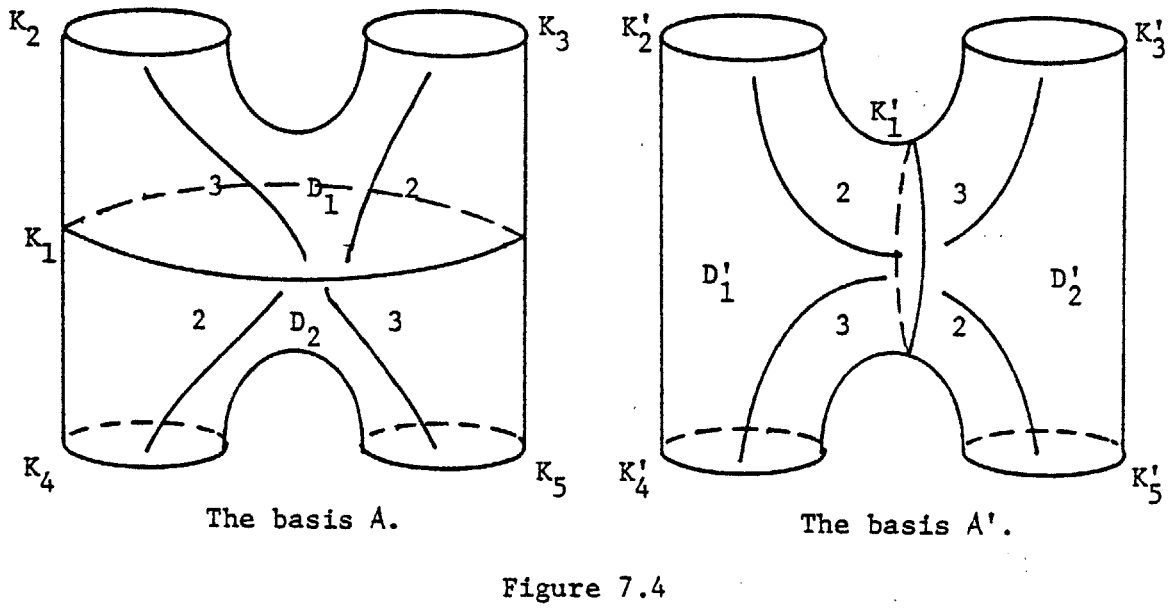


Figure 7.3

Let A and A' be the bases on S_2 shown in Figure 7.4, making the choice of canonical pieces as in Construction 2.1. In contrast to Section 6, all four cases in Construction 2.1 of compatibly embedded canonical pieces can occur in each embedded pants D_j, j=1,2. Thus, there are $512 = 4^{2\times 2^4}$ standard train tracks with stops (see Section 5) on S_2 for each basis A or A': four for each embedded D_j, j=1,2,

and two for each annulus A_i, i=1,...,5.



The basis A. The basis A'.

Figure 7.4

We define a new basis A" on S_2 as follows: the bases A' and A" differ only in that A" uses the canonical pieces $\hat{1}_{11}$ and $\hat{1}_{33}$ (defined in Section 2) instead of the canonical pieces 1_{11} and 1_{33} used in the basis A'. Our goal is to compute the second elementary transformation between the bases A and A'; the basis A" is a technical convenience.

We introduce the following notation <u>for this section only</u>. We will denote the parameter values ℓ^1_{**} with respect to the basis A by ℓ_{**}, and we will denote the parameter values ℓ^2_{**} with respect to the basis A by k_{**}. Similarly, the parameter values $\ell^{1,}_{**}$ and $\ell^{2,}_{**}$ with respect to the basis A' will be denoted ℓ'_{**} and k'_{**}, respectively, and similarly for the basis A". Moreover, A-symbols will be written as words in the the letters st_i and $s\ell_{\sigma\sigma'}$ and $sk_{\sigma\sigma'}$, $\sigma \leq \sigma' = 1,2,3$; we define the analogous notation for A'-symbols and A"-symbols.

Remark 7.1: Since the bases A' and A" differ only in the choice of canonical pieces, the transformation between the corresponding parametrizations is easily computed from Proposition A.1. The transformation from the basis A' to the basis A" is given by the following formulas.

$$t_1'' = t_1' + \ell_{11}' + k_{11}'$$
$$t_3'' = t_3' + k_{33}'$$
$$t_4'' = t_4' + \ell_{33}'$$

The other parameter values are unchanged (i.e., replace ' by "). The transformation from the basis A" to the basis A' is given by the following formulas.

$$t_1' = t_1'' - \ell_{11}'' - k_{11}''$$
$$t_3' = t_3'' - k_{33}''$$
$$t_4' = t_4'' - \ell_{33}''$$

The other parameter values are unchanged (i.e., replace " by ').

We will want to distinguish between the orientations on $s\ell_{11}$ and sk_{11}. Define 1_{11}^+ and 1_{11}^- in the standard pants to be oriented as in Figure 7.5, and define the letters $s\ell_{11}^{\pm}$ and $sk_{11}^{\pm}$ so that $f_1\Pi_2\tilde{s\ell}_{11}^{\pm} = 1_{11}^{\pm}$ and $f_2\Pi_2\tilde{sk}_{11}^{\pm} = 1_{11}^{\pm}$. We will only need to worry about the orientations of these letters for the basis A, and we modify the notion of A-symbol to include the four letters $s\ell_{11}^{\pm}$ and $sk_{11}^{\pm}$. (Of course, we omit the letters $s\ell_{11}$ and sk_{11}.)

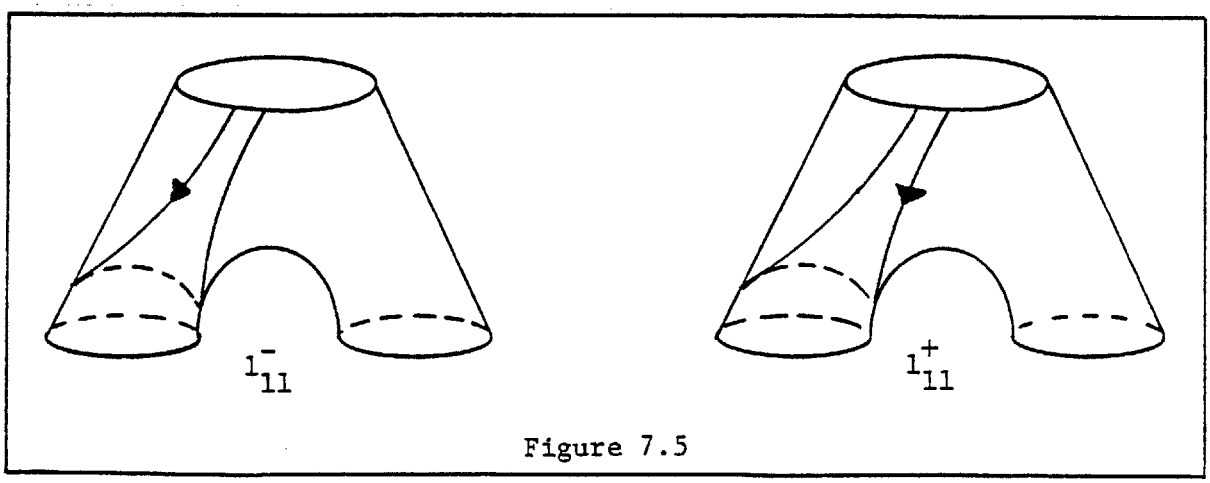


Figure 7.5

To be quite explicit, we give in Figures 7.6a and 7.6b the lifts to $\tilde{S}_2$ of the various letters for the bases A and A", respectively. The numbers in the deleted diamonds in Figure 7.6 indicate the boundary component twice covered by the bounding piecewise-geodesic curve in $\tilde{S}_2$; the curves are numbered as in Figure 7.4.

Note that a connected, constant parity, non-exceptional, immersed admissible symbol in the basis A is alternating if and only if the letters arising from canonical pieces (instead of twists) alternate from $s\ell_{**}$ to sk_{**}, and similarly for the bases A' and A". Moreover, a multiple curve has $m_1 = 2\ell_{11} = 2k_{11}$ in the basis A, where m_1 is the Dehn-Thurston intersection number with the pants curve K_1, and similarly for the bases A' and A".

We first compute the second elementary transformation between the bases A and A' on multiple curves with no boundary-parallel components using the Euclidean structure on S_2. We then perform Steps 1-5 for multiple arcs without closed curve components between the bases A and A", and finally, after some algebraic manipulation, give the general form of the second elementary transformation between the bases A and A'.

Lemma 7.1: If $\alpha \in \mathring{\mathscr{S}}'(S_2)$ consists of n components, then α is n parallel copies of a (connected) simple closed curve.

Proof: Suppose first that $n = 2$, and let α_1 and α_2 be the components of α. Cut S_2 along α_1 to get a surface V. We claim that V is disconnected, for otherwise $\chi(V) = \chi(S_2)$, genus (V) = genus(S_2) and the number of boundary components of V is two greater than the number of boundary components of S_2, which is impossible.

Let V have components V_1 and V_2, and note that $\chi(S_2) = -2 = \chi(V_1) + \chi(V_2)$.

The basis A

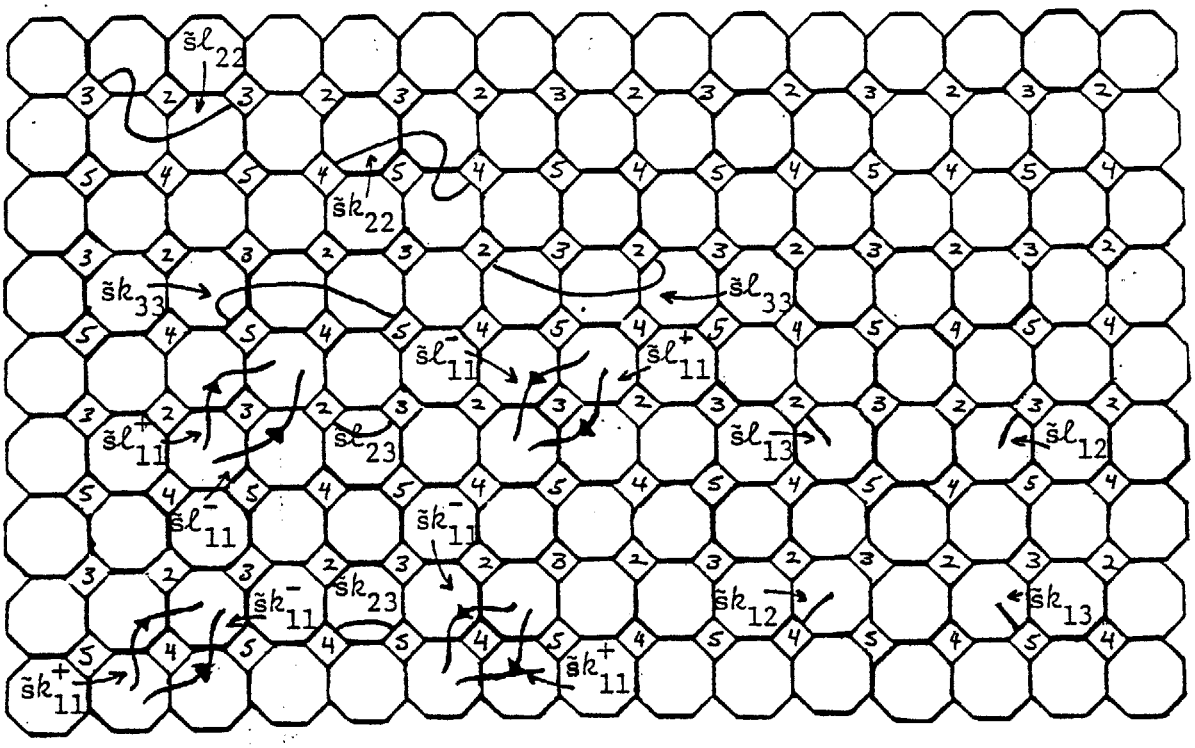


Figure 7.6a

The basis A"

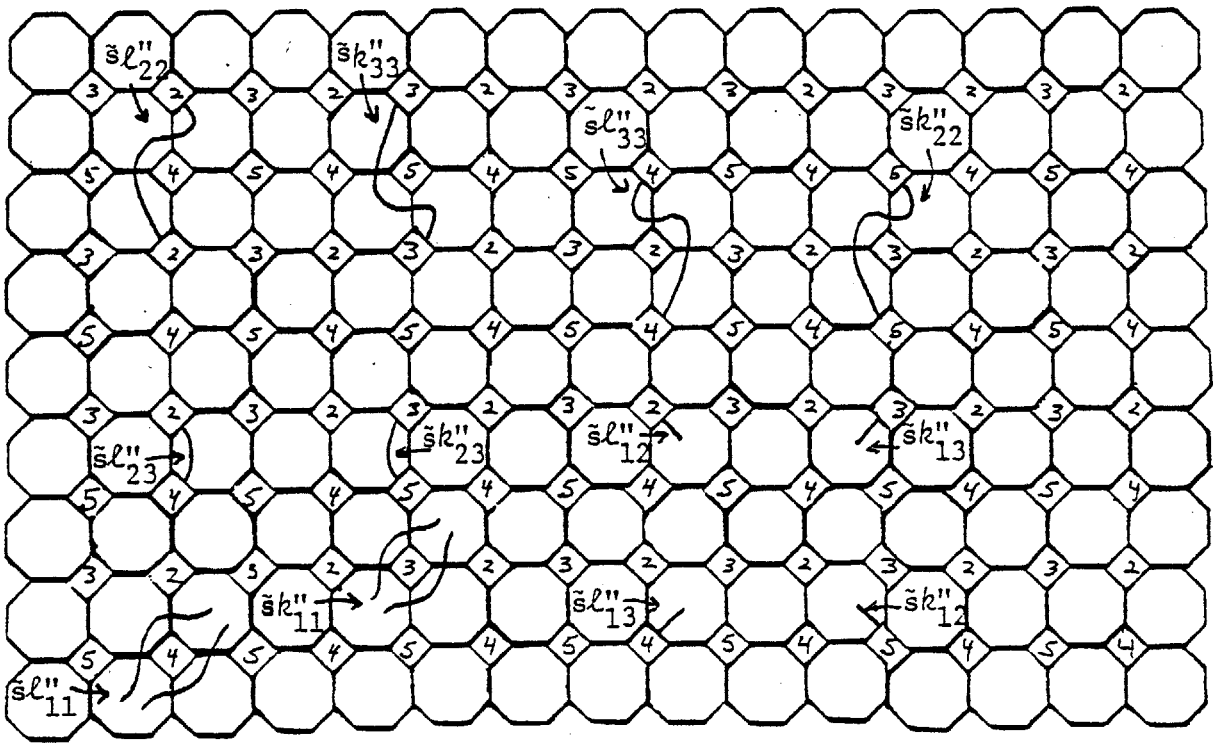


Figure 7.6b

Furthermore, if $\chi(V_i) = 1$, then V_i is a disc, while if $\chi(V_i) = 0$, then V_i is an annulus. Thus, since the components of α are not null-homotopic and not boundary-parallel in S_2, V_1 and V_2 are each a pair of pants. $\alpha_2 \subset V_1$, say, implies that α_2 is parallel to one of the boundary components of V_1, whence α_2 is parallel to α_1. The general case is similar. ⊠

One can easily construct a (connected) simple closed curve corresponding to each pair of parameter values ℓ_{11} and t_1, where ℓ_{11} and $|t_1|$ are relatively prime, including the case $\ell_{11} = 0$ and $t_1 = 1$. Thus, the subset of $\overset{o}{\mathscr{S}}{}'_A(S_2)$ corresponding to connected multiple curves is the subset of $\mathscr{S}'_A(S_2)$ so that the following conditions are satisfied.

a) $\ell_{12} = \ell_{13} = k_{12} = k_{13} = 0$,
$\ell_{22} = \ell_{23} = \ell_{33} = k_{22} = k_{23} = k_{33} = 0$,
and $t_2 = t_3 = t_4 = t_5 = 0$.

b) $\ell_{11} = k_{11}$.

c) ℓ_{11} and $|t_1|$ are relatively prime.

Proposition 7.1: There is a parametrization of connected multiple curves in S_2 by $Q \cup \{\infty\} \subset S^1$, i.e., by the rational points of S^1.

Proof: Define a map $\gamma: \overset{o}{\mathscr{S}}{}'_A(S_2) \to Q \cup \{\infty\}$ by $\gamma((\alpha)_A) = \operatorname{sgn}(t_1+\ell_{11})\ell_{11}/|t_1+\ell_{11}|$. Given $q = \pm n_2/n_1$ with $n_1 > 0$, $n_2 \geq 0$ and $(n_1,n_2) = 1$, $\gamma^{-1}(q)$ is $t_1 = \operatorname{sgn}(q)n_1 - n_2$, $\ell_{11} = n_2$, and $\gamma^{-1}(\infty)$ is $t_1 = -1$, $\ell_{11} = 1$. ⊠

Just as in Section 6, we can show that this parametrization is given by including $\alpha \subset S_2 \subset S^2$-minus-four-points and taking the rational slope of a geodesic representative of the free homotopy class of α.

Again there is a more geometrical description of this parametrization, and one can prove the proposition without resorting to the main parametrization

theorem: connect the four boundary components of S_2 by disjointly embedded simple arcs a_i as indicated in Figure 7.2. Isotope a representative c of a connected $\alpha \in \mathring{\mathscr{S}}'(S_2)$ so that it has minimal geometrical intersection number with each a_i. $|\gamma(q)|$ is given by $\text{card}(c \cap a_2)/\text{card}(c \cap a_1)$. There are four cases indicated in Figure 7.7 for the intersection of this representative with the octagon f, which is indicated in Figure 7.2. The sign of $\gamma(q)$ is defined to be positive in cases one and two and negative in cases three and four. Checking that this indeed gives a parametrization for connected elements of $\mathring{\mathscr{S}}'(S_2)$ is a combinatorial exercise; checking that this is the parametrization described in Proposition 7.1 is case checking on the ration t_1/m_1. We illustrate the result of this case checking diagramatically in Figure 7.8.

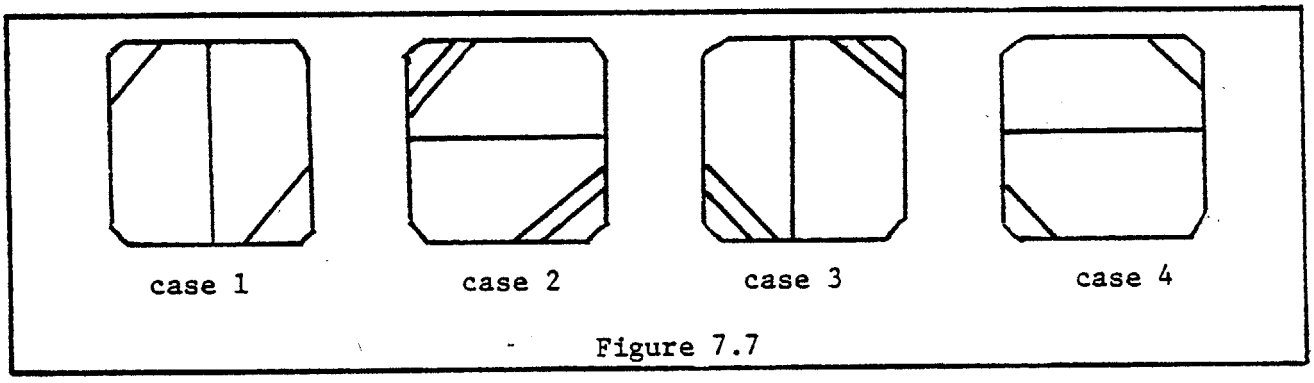


Figure 7.7

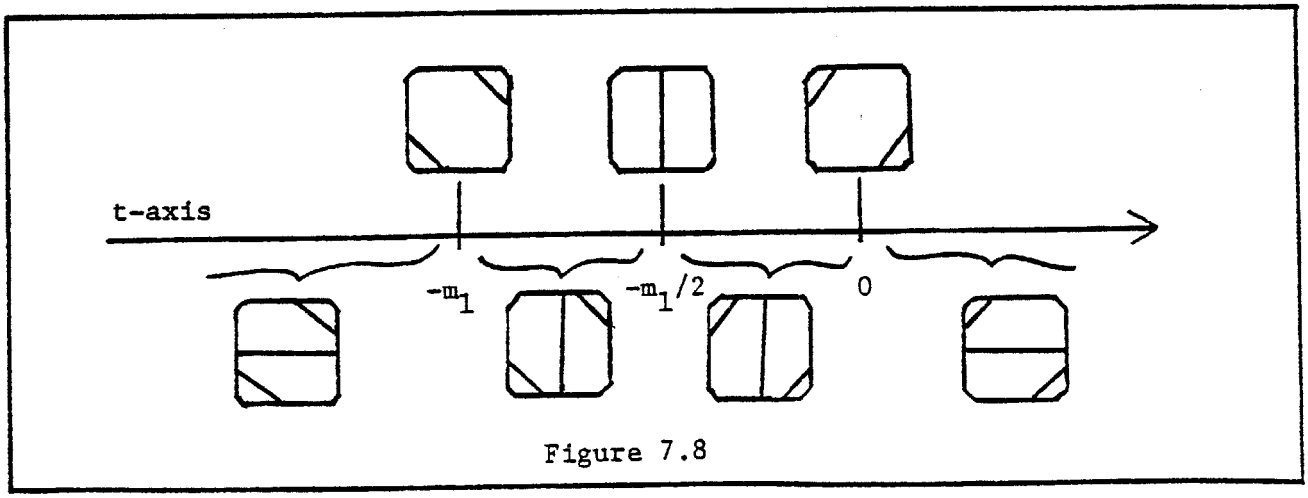


Figure 7.8

Proposition 7.2: i) The second elementary transformation from the basis A to the basis A' on multiple curves with no boundary-parallel components is given by the following formulas.

$$m_1' = 2|m_1/2 + t_1|$$

$$t_1' = -\mathrm{sgn}(m_1/2 + t_1)\ (m_1 + t_1)$$

In these formulas, $\mathrm{sgn}(0) = -1$.

ii) The second elementary transformation from the basis A' to the basis A on multiple curves with no boundary-parallel components is given by the following formulas.

$$m_1 = 2|m_1'/2 + t_1'|$$

$$t_1 = -\mathrm{sgn}(m_1'/2 + t_1')\ (m_1' + t_1')$$

In these formulas, $\mathrm{sgn}(0) = -1$.

Proof (sketch): First consider the case of a connected $\alpha \in \overset{\circ}{\mathscr{S}}'(S_2)$. By symmetry, there is a one-to-one onto map $\gamma' : \overset{\circ}{\mathscr{S}}'_{A'}(S_2) \to Q \cup \{\infty\}$ given by $\gamma'((\alpha)_{A'}) = \mathrm{sgn}(t_1'+\ell_{11}')\ell_{11}'/|t_1'+\ell_{11}'|$, and the two rational parametrizations are negative reciprocals of one another. A computation completes the proof for connected multiple curves with no boundary-parallel components, and the general case follows from Lemma 7.1. ☒

We now consider the general setting of $\alpha \in \mathscr{S}'(S_2)$, a multiple arc, and we compute the second elementary transformation between the bases A and A". Without loss of generality, suppose α has A-coordinates $t_2 = t_3 = t_4 = t_5 = 0$, and orient the components of α arbitrarily.

Step 1 of the computation is to compute the number of components of α isotopic to $f_1''^{-1}1_{22}$, $f_1''^{-1}1_{23}$, $f_1''^{-1}1_{33}$ and $f_2''^{-1}1_{22}$, $f_2''^{-1}1_{23}$, $f_2''^{-1}1_{33}$ in the embedded pants D_1 and D_2, respectively (plus arcs running from window to window with no twisting in the annuli $A_2,\ldots,A_5$). We first consider $f_1''^{-1}1_{22}$ and $f_1''^{-1}1_{23}$. As shown in Figure 7.6, an $f_1''^{-1}1_{22}$ has A"-parameter

$\ell_{13} + k_{11} + \ell_{13}$, and an $f_1''^{-1}l_{23}$ has parameter value $\ell_{13} + k_{12}$. We compute the number of components of α with coordinates $\ell_{13} + k_{11} + \ell_{13}$ and $\ell_{13} + k_{12}$ as follows: imagine cutting along the boundary of the regular neighborhood A_1 of K_1 in the basis A. α enters and exits through the windows in ∂A_1. These windows are as indicated in Figure 7.9, where the label u_{12} (w_{12}) indicates, for instance, the region of the window through which $f_1^{-1}l_{12}$ ($f_2^{-1}l_{12}$) may pass. One can use these decompositions of the windows to derive the following formulas.

$$\ell''_{22} = ((k_{11} + t_1) \Delta k_{11} \Delta (\ell_{13} - k_{11} - t_1 - k_{12})) \nabla 0$$

$$\ell''_{23} = (\ell_{13} \Delta k_{12} \Delta (\ell_{13} - k_{11} - t_1) \Delta (k_{12} + k_{11} + t_1)) \nabla 0$$

The symmetries in Figure 7.10 immediately give the following formulas.

$$k''_{23} = (k_{13} \Delta \ell_{12} \Delta (k_{13} - \ell_{11} - t_1) \Delta (\ell_{12} + \ell_{11} + t_1)) \nabla 0$$

$$k''_{22} = ((\ell_{11} + t_1) \Delta \ell_{11} \Delta (k_{13} - t_1 - \ell_{12} - \ell_{11})) \nabla 0$$

$$\ell''_{33} = ((-t_1 - k_{11}) \Delta \ell_{11} \Delta (k_{12} - \ell_{13} + t_1 + k_{11})) \nabla 0$$

$$k''_{33} = ((-t_1 - \ell_{11}) \Delta k_{11} \Delta (\ell_{12} - k_{13} + t_1 + \ell_{11})) \nabla 0$$

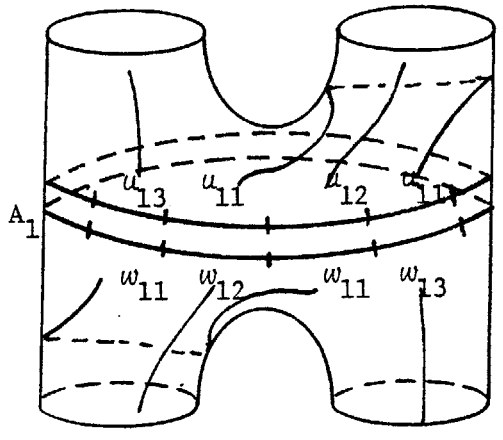


Figure 7.9

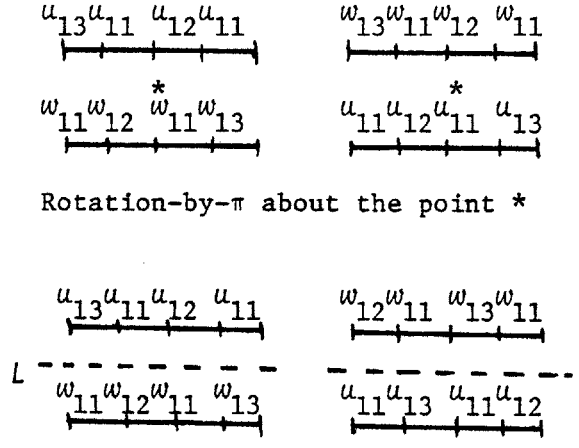


Figure 7.10

Step 2 is to compute the A-parameter values of $\hat{\alpha}$, the non-exceptional part of α. $\hat{\alpha}$ has parameter values given by the following formulas.

$$\hat{\ell}_{13} = \ell_{13} - \ell''_{23} - 2\ell''_{22}$$
$$\hat{k}_{12} = k_{12} - \ell''_{23} - 2\ell''_{33}$$
$$\hat{\ell}_{12} = \ell_{12} - k''_{23} - 2k''_{33}$$
$$\hat{k}_{13} = k_{13} - k''_{23} - 2k''_{22}$$
$$\hat{k}_{11} = k_{11} - k''_{33} - \ell''_{22}$$
$$\hat{\ell}_{11} = \ell_{11} - \ell''_{33} - \ell''_{22}$$
$$\hat{t}_1 = t_1 + \ell''_{33} + k''_{33}$$
$$\hat{t}_2 = \hat{t}_3 = \hat{t}_4 = \hat{t}_5 = 0$$
$$\hat{\ell}_{22} = \hat{\ell}_{33} = \hat{k}_{22} = \hat{k}_{33} = 0 = \hat{\ell}_{23} = \hat{k}_{23}$$

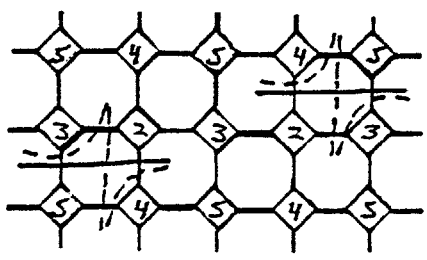
Step 3 is to compute the A"-parameter values of $\hat{\alpha}$ and includes the generic case. For convenience of notation, we assume without loss that α is connected and non-exceptional; thus, $(\alpha)_A = (\hat{\alpha})_A$. Suppose that $s = s_1 \ldots s_n$ is the corresponding embedded admissible A-symbol, and for now assume that the A-length of s is at least three. Note that $s_i \in \{s\ell_{11}^{\pm}, sk_{11}^{\pm}, st_1^{\pm 1}\}$, for each $i=2,\ldots,n-1$.

Define a map from $\langle s\ell_{11}^{\pm}, sk_{11}^{\pm}, st_1^{\pm 1}\rangle$, the free semi-group on the six A-letters, to the free semi-group $\langle s\ell''_{11}, sk''_{11}, st''^{\pm 1}_1\rangle$ on the four A"-letters by extending the following map defined on letters. For the time being, regard sgn(0) as being undefined, and ignore the sign ε in the image of twists.

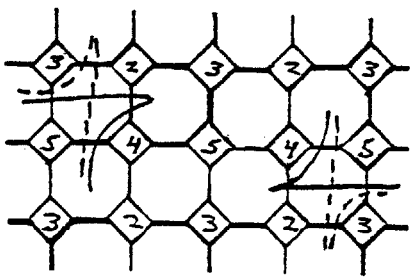
$$st_1^{\pm 1} \to s\ell''_{11}\ st''^{\varepsilon}_1\ sk''_{11} \text{ or } sk''_{11}\ st''^{\varepsilon}_1\ s\ell''_{11}$$
$$s\ell_{11} \to sk''_{11} \qquad\qquad sk_{11} \to s\ell''_{11}$$
$$(s\ell_{11}^{+}, st_1^{-1}) \equiv -2sk''_{11} \qquad\qquad (sk_{11}^{+}, st_1^{-1}) \equiv -2s\ell''_{11}$$
$$\varepsilon = \begin{cases} +1, & t_1 \geq 0. \\ -\mathrm{sgn}(\ell_{11}+k_{11}+2t_1), & \text{else.} \end{cases}$$

Pair-of-letter expressions and the notation for the pair-of-letter adjustments were defined in Section 6. (In Section 6, we adjusted only the coordinates of a symbol. Here we adjust the symbol itself.) The expression (s_i,s_j) evaluated on the A-symbol s is defined exactly as in Section 6 provided neither of the letters s_i or s_j is one of the letters $s\ell_{11}^{\pm}$ or $sk_{11}^{\pm}$. The expression $(s\ell_{11}^{+},s_j)$ evaluated on the A-symbol s, for instance, is defined to be the number of times $s\ell_{11}^{+}\ s_j$ or $s_i\ s\ell_{11}^{-}$ appears as a sub-symbol of s. There is an analogous definition for pair-of-letter expressions involving the letter $sk_{11}^{\pm}$. Thus, $(s\ell_{11}^{+},st_1^{-1}) = (st_1^{-1},s\ell_{11}^{-})$ and $(sk_{11}^{+},st_1^{-1}) = (st_1^{-1},sk_{11}^{-})$ by definition.

We claim that this combinatorial map describes a homotopy rel endpoints from some lift $\tilde{s}_2 \ldots \tilde{s}_{n-1}$ of $s_2 \ldots s_{n-1}$. This can be seen as follows: begin by noting that an $s\ell_{11}^{\pm}$ $(sk_{11}^{\pm})$ has coordinate k''_{11} (ℓ''_{11}) in the basis A". (This explains the technical facility gained by considering first the transformation between the bases A and A".) The homotopy rel endpoints of the lift $\tilde{st}_1^{\pm 1}$ to $\tilde{s\ell}''_{11}\ \tilde{st}_1''^{+1}\ \tilde{sk}''_{11}$ or $\tilde{sk}''_{11}\ \tilde{st}_1''^{+1}\ \tilde{s\ell}''_{11}$ is indicated in Figure 7.11a. The adjustments for the pair-of-letter expression (sk_{11}^{+},st_1^{-1}) are indicated in Figure 7.11b. As usual, the solid lines denote the lifts, and the broken lines denote the image of the homotopy. The adjustments for the expression $(s\ell_{11}^{+},st_1^{-1})$ are similar to the adjustments in Figure 7.11b.



7.11a



7.11b

Figure 7.11

Note that the combinatorial homotopy above has the property that the only way to produce a $st_1''^{\pm 1}$ letter in the A''-symbol image is from an $st_1^{\pm 1}$ letter in the A-symbol. Moreover, the pair-of-letter adjustments never affect $st_1''^{\pm 1}$ letters; however, the sign ε of the t_1'' coordinate does depend on the pair-of-letter adjustments, as we shall see.

We say that a non-twist letter is "stable" if it is unaffected by the pair-of-letter adjustments. Note that for s an embedded admissible sequence of letters $s\ell_{11}^{\pm}$ and $sk_{11}^{\pm}$, the image A''-symbol is necessarily alternating since s is. Furthermore, if s is an embedded admissible sequence of letters $st_1^{\pm 1}$, then the image A''-symbol is an alternating and constant +1 parity $(s\ell_{11}''\ st_1''^{+1}\ sk_{11}'')^m$ or $(sk_{11}''\ st_1''^{+1}\ s\ell_{11}'')^m$.

Claim 7.1: If $t_1 \geq 0$, then the combinatorial homotopy above on $s_2 \ldots s_n$ has an alternating and constant parity A''-symbol image.

Proof: By the previous paragraph, consecutive sequences of letters st_1^{+1} and consecutive sequences of $s\ell_{11}^{\pm}$ and $sk_{11}^{\pm}$ letters each have an alternating and constant +1 parity image. A neighborhood of a concatenation point of $\tilde{s}\ell_{11}^{\pm}$ and $\tilde{st}_1^{+1}$ is shown in Figure 7.12; the lift is denoted by a solid line and the image under the combinatorial homotopy by a broken line. Concatenation points of $\tilde{sk}_{11}^{\pm}$ and $\tilde{st}_1^{+1}$ are similar. Thus, in a neighborhood of a concatenation point, the homotopy preserves the alternating character, and the claim follows. Note that there are no pair-of-letter adjustments. ☒

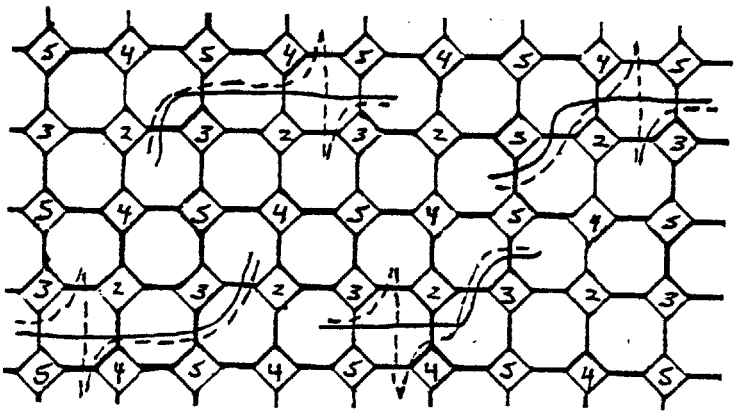


Figure 7.12

Claim 7.2: If $t_1 < 0$, then the combinatorial homotopy above has an alternating A"-symbol image.

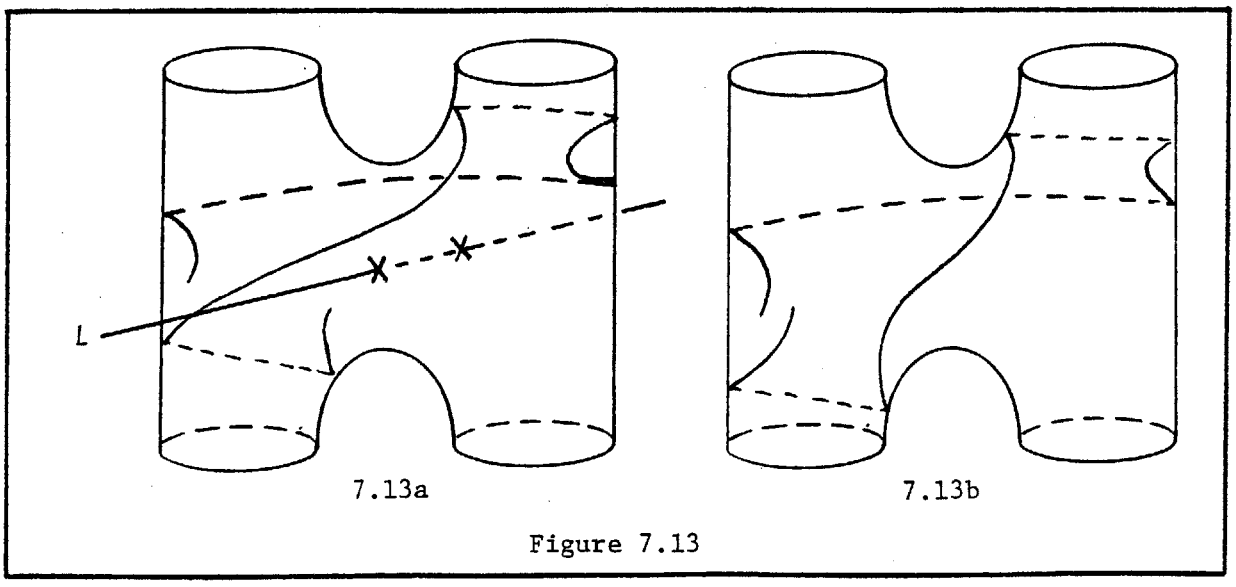
Proof: Denote by $s''_1 \ldots s''_{n''}$ the A"-symbol image of $s_2 \ldots s_{n-1}$ under the homotopy, where the s''_i are A"-letters, and suppose not. Without loss of generality (by rotation-by-π about the line L in Figure 7.13a), there is some $s''_r = s\ell''_{11}$, $s''_{r+m} = s\ell''_{11}$, with $s''_{r+1},\ldots,s''_{r+m-1}$ twists.

There are two cases.

Case 1): s''_r arose from some $s_k = sk^{\pm}_{11}$.

Case 2): s''_r arose from some $s_k = st_1^{\pm 1}$.

In Case 1), since s is alternating, $s_{k+1} = st_1^{-1}$, or $s_{k+1} = s\ell^{\pm}_{11}$. If $s_{k+1} = st_1^{-1}$, then, by Proposition A.3, the homotopy would have erased $s''_r = s\ell''_{11}$ as in Figure 7.11b, which is absurd. Thus, $s_{k+1} = s\ell^{\pm}_{11}$. If $s_{k+2} = sk^{\pm}_{11}$, then $s''_{r+1} = sk''_{11}$ and s''_{r+1} is stable, which is contradictory to hypothesis. Thus, $s_{k+2} = st_1^{-1}$, and so we must have $s_{k+1} = s\ell^{+}_{11}$ by Proposition A.2. In case $s_k = sk^{+}_{11}$, we must have $s_{k+3} = sk^{-}_{11}$ as indicated in Figure 7.13a, so that $s''_{r+1} = st''^{-1}_1$. If $s_{k+4} = s\ell^{-}_{11}$, then $s''_{r+2} = sk''_{11}$ and s''_{r+2} is stable, which is contradictory to hypothesis; thus, $s_{k+4} = s\ell^{+}_{11}$, $s_{k+5} = st_1^{-1}$, $s_{k+6} = sk^{-}_{11}$ and α is forced to spiral around $\Pi_2\tilde{s}k^{-}_{11}\ \Pi_2\tilde{s}\ell^{+}_{11}\ \Pi_2\tilde{s}t_1^{-1}$ indefinitely, which is absurd.



7.13a 7.13b

Figure 7.13

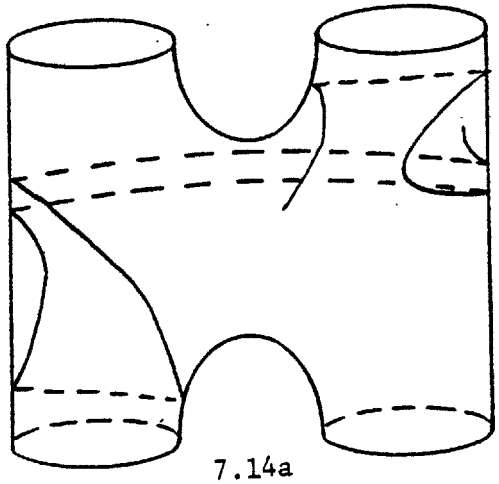
In case $s_k = sk''_{11}$, we must have $s_{k+3} = sk''_{11}$ as indicated in Figure 7.13b, so that $s''_{r+1} = st''^{-1}_1$. If $s_{k+3} = sk^+_{11}$, then $s_{k-1} = st^{-1}_1$, and the homotopy would have erased s_k, which is absurd. If $s_{k+4} = s\ell^-_{11}$, then $s''_{r+2} = sk''_{11}$ and s''_{r+2} is stable, which is contrary to hypothesis; thus, $s_{k+4} = s\ell^+_{11}$, $s_{k+5} = st^{-1}_1$, $s_{k+6} = sk^-_{11}$, and α is forced to spiral indefinitely around $\tilde{\Pi}_2 sk^-_{11}\ \tilde{\Pi}_2 s\ell^+_{11}\ \tilde{\Pi}_2 st^{-1}_1$. Thus, Case 1) cannot lead to a non-alternating image A''-symbol.

In Case 2), there are two sub-cases.

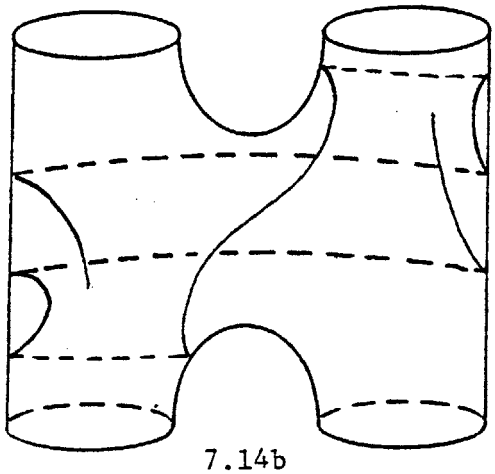
Sub-case a): s_k leads to $sk''_{11}\ \ st''^{+1}_1\ s\ell''_{11}$.

Sub-case b): s_k leads to $s\ell''_{11}\ \ st''^{+1}_1\ sk''_{11}$.

In Sub-case a), since $t_1 < 0$, $s_{k+1} = sk^{\pm}_{11}$. If $s_{k+1} = sk^-_{11}$, then the homotopy erases s''_r, which is absurd; thus, we assume that $s_{k+1} = sk^+_{11}$. By Proposition A.2, $s_{k+2} = st^{-1}_1$, so $s_{k+3} = s\ell^-_{11}$. This case is indicated in Figure 7.14a. Thus, $s_{k+4} = sk^+_{11}$, $s_{k+5} = st^{-1}_1$, and α must spiral indefinitely about $\tilde{\Pi}_2 sk^+_{11}\ \tilde{\Pi}_2 st^{-1}_1\ \tilde{\Pi}_2 s\ell^-_{11}$, which is absurd.



7.14a



7.14b

Figure 7.14

In Sub-case b), since $t_1 < 0$, $s_{k+1} = s\ell''^{\pm}_{11}$. If $s_{k+1} = s\ell''^{+}_{11}$, then the sk''_{11} arising from s_k is stable, contradictory to hypothesis; thus, we assume that that $s_{k+1} = s\ell''^{-}_{11}$. If $s_{k+2} = st_1^{-1}$, then the sk''_{11} arising from s_{k+2} is stable, contradictory to hypothesis; thus $s_{k+2} = sk''^{\pm}_{11}$ and $s_{k+3} = st_1^{-1}$, so that $s_{k+2} = sk''^{+}_{11}$ by Proposition A.2. This configuration is indicated in Figure 7.14b. If $s_{k+4} = s\ell''^{+}_{11}$, then the sk''_{11} arising from s_{k+4} would be stable, contradictory to hypothesis; thus, $s_{k+4} = s\ell''^{-}_{11}$, and α must spiral indefinitely about $\Pi_2\tilde{s}\ell''^{-}_{11}\ \Pi_2\tilde{s}k''^{+}_{11}\ \Pi_2\tilde{s}t_1^{-1}$, which is absurd. Thus, Case 2), cannot lead to a non-alternating image A"-symbol, proving the claim. ⊠

Claim 7.3: The A"-symbol image of the combinatorial homotopy above has all its twists in the same direction.

Proof: By the previous claim, the sign of a consecutive block of twists of $s''_1 \ldots s''_{n''}$ is well-defined. Thus, it suffices to prove that if $s''_r = st''^{\pm 1}_1$ and $s''_{r+m} = st''^{\pm 1}_1$, with $s''_{r+1},\ldots,s''_{r+m-1} \in \{s\ell''_{11}, sk''_{11}\}$, then s''_r and s''_{r+m} have the same sign, $m \neq 1$. Suppose then that s''_r arose from s_k leading to $sk''_{11}\ st''^{+1}_1\ s\ell''_{11}$. In this case, either $s_{k+1} = st_1^{-1}$, which is acceptable (and m=3 with $\mathrm{sgn}(s''_r) = \mathrm{sgn}(s''_{r+m}) = +1$), or $s_{k+1} = sk''^{\pm}_{11}$, by Proposition A.2. There are then two cases.

Case 1): s''_{r+m} arose from s_j leading to $sk''_{11}\ st''^{+1}_1\ s\ell''_{11}$.

Case 2): s''_{r+m} arose from s_j leading to $s\ell''_{11}\ st''^{+1}_1\ sk''_{11}$.

In Case 1), $s_{j-1} = s\ell''^{\pm}_{11}$ by Proposition A.4, which is acceptable (and $\mathrm{sgn}(s''_r) = \mathrm{sgn}(s''_{r+m}) = -1$), and in Case 2), $s_{j-1} = sk''^{\pm}_{11}$ by Proposition A.4, which is acceptable (and again $\mathrm{sgn}(s''_r) = \mathrm{sgn}(s''_{r+m}) = -1$).

The case where s''_{r+m} arises from s_j leading to $s\ell''_{11}\ st''^{+1}_1\ sk''_{11}$ is similar, proving the claim. ⊠

Claim 7.4: The sign ε'' of the t_1'' coordinate of $s_1'' \ldots s_{n''}''$ is given by the following formula

$$\varepsilon'' = \begin{cases} -\mathrm{sgn}(2t_1+\ell_{11}+k_{11}), & t_1 < 0 \\ +1, & t_1 \geq 0 \end{cases}$$

This formula is valid unless $s_1'' \ldots s_{n''}''$ is a sequence of twists, in which case ε'' is undefined and $s_2 \ldots s_{n-1}$ is given by $(s\ell_{11}^{+}\ st_1^{-1}\ sk_{11}^{-})^m$ or $(sk_{11}^{+}\ st_1^{-1}\ s\ell_{11}^{-})^m$.

Proof: If a letter st_1^{-1} is not surrounded in s on both sides by $s\ell_{11}^{\pm}$ and $sk_{11}^{\pm}$, then the corresponding t_1'' has a positive sign, and the first part of the claim follows from Claim 7.3. That $s_2 \ldots s_{n-1}$ is $(s\ell_{11}^{+}\ st_1^{-1}\ sk_{11}^{-})^m$ or $(sk_{11}^{+}\ st_1^{-1}\ s\ell_{11}^{-})^m$ in case ε'' is undefined is an easy combinatorial argument. ⊠

The previous four claims prove the following proposition.

Proposition 7.3: The combinatorial homotopy above on $s_2 \ldots s_{n-1}$ gives an alternating and constant parity A"-symbol with $\varepsilon'' = -\mathrm{sgn}(\ell_{11}+k_{11}+2t_1)$ if $t_1 < 0$, $\varepsilon'' = +1$ if $t_1 \geq 0$, and sgn(0) (as yet) undefined. ⊠

(The disparity between the cases $t_1 < 0$ and $t_1 \geq 0$ is reflected in Figure 7.8. The connection between the ε'' just computed and the ratio t_1/m_1 is explained in Remark 7.1.)

We describe the boundary effects and extend the combinatorial homotopy above to a combinatorial homotopy from embedded admissible non-exceptional A-arcs to immersed admissible non-exceptional A"-arcs. Figure 7.15 depicts the homotopies described by the following pair-of-letter adjustments. As usual, the solid lines in Figure 7.15 depict the lifted letters, and the broken lines in Figure 7.15 depict the image under the combinatorial homotopy.

a) $(s\ell_{12}, st_1^{+1}) \equiv k''_{13}$

b) $(s\ell_{12}, st_1^{-1}) \equiv -t''_3 - k''_{11} + k''_{13}$

c) $(s\ell_{12}, sk_{11}^{-}) \equiv k''_{13}$

d) $(s\ell_{12}, sk_{11}^{+}) \equiv k''_{13}$

e) $(s\ell_{13}, st_1^{+1}) \equiv \ell''_{12} + t''_2 - \ell''_{11} - t''_1$

f) $(s\ell_{13}, st_1^{-1}) \equiv \ell''_{12}$

g) $(s\ell_{13}, sk_{11}^{+}) \equiv \ell''_{12}$

h) $(s\ell_{13}, sk_{11}^{-}) \equiv -\ell''_{11} + t''_2 - t''_1 + \ell''_{12}$

i) $(sk_{12}, st_1^{+1}) \equiv \ell''_{13}$

j) $(sk_{12}, st_1^{-1}) \equiv \ell''_{13} - t''_4 - \ell''_{11}$

k) $(sk_{12}, s\ell_{11}^{+}) \equiv \ell''_{13}$

l) $(sk_{12}, s\ell_{11}^{-}) \equiv \ell''_{13}$

m) $(sk_{13}, st_1^{-1}) \equiv -t''_1 + t''_5 - k''_{11} + k''_{12}$

n) $(sk_{13}, st_1^{-1}) \equiv k''_{12}$

o) $(sk_{13}, s\ell_{11}^{-}) \equiv k''_{12} - k''_{11} + t''_5 - t''_1$

p) $(sk_{13}, s\ell_{11}^{+}) \equiv k''_{12}$

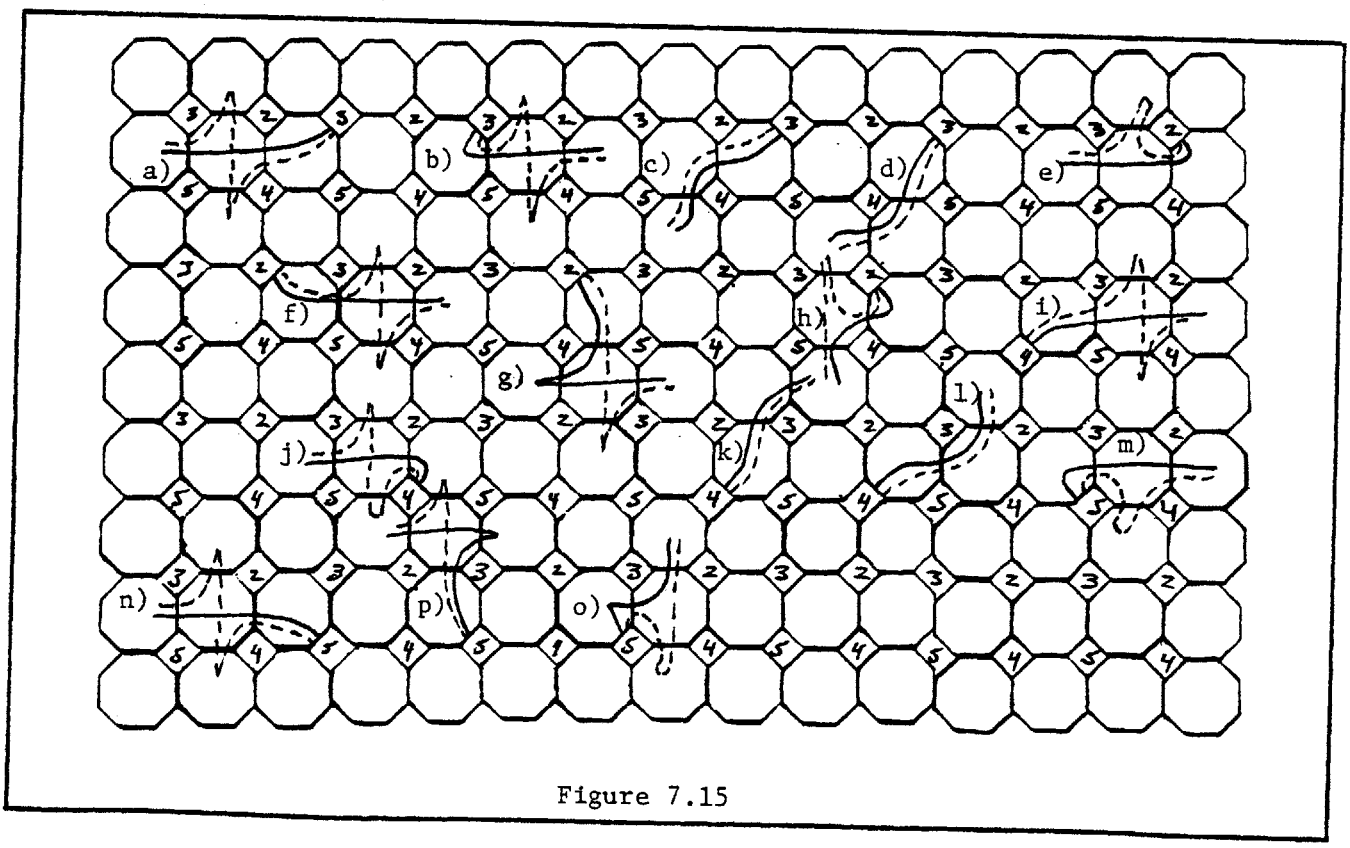


Figure 7.15

Proposition 7.4: If α is not a closed component and its symbol is non-exceptional, then α is homotopic rel ∂S_2 to an immersion $\bar{\alpha}''$, where $\bar{\alpha}''$ has an A''-symbol with the following A'' coordinates.

$$t_1'' = -(s\ell_{13}, st_1^{+1}) - (s\ell_{13}, sk_{11}^-) - (sk_{13}, st_1^{+1}) - (sk_{13}, s\ell_{11}^-) + \operatorname{sgn}(\ell_{11}+k_{11}+2t_1)\, t_1$$

$$t_2'' = (s\ell_{13}, st_1^{+1}) + (s\ell_{13}, sk_{11}^-)$$

$$t_3'' = -(s\ell_{12}, st_1^{-1})$$

$$t_4'' = -(sk_{12}, st_1^{-1})$$

$$t_5'' = (sk_{13}, s\ell_{11}^-) + (sk_{13}, st_1^{+1})$$

$$\ell_{12}'' = \ell_{13} \qquad k_{12}'' = k_{13}$$

$$\ell_{13}'' = k_{12} \qquad k_{13}'' = \ell_{12}$$

$$\ell_{23}'' = \ell_{22}'' = \ell_{33}'' = k_{23}'' = k_{22}'' = k_{33}'' = 0$$

$$k_{11}'' = |t_1| + \ell_{11} - 2(s\ell_{11}^+, st_1^{-1}) - (s\ell_{12}, st_1^{-1}) - (sk_{13}, st_1^{+1}) - (sk_{13}, s\ell_{11}^-)$$

$$\ell_{11}'' = |t_1| + k_{11} - 2(sk_{11}^+, st_1^{-1}) - (s\ell_{13}, st_1^{+1}) - (s\ell_{13}, sk_{11}^-) - (sk_{12}, st_1^{-1})$$

In these formulas, the convention for sgn(0) is the following.

$$\operatorname{sgn}(0) = \begin{cases} +1, \text{ if } \alpha \text{ has symbol } s\ell_{12}\ (sk_{11}^+\ st_1^{-1}\ s\ell_{11}^-)^m\ sk_{12}. \\ -1, \text{ if } \alpha \text{ has symbol } s\ell_{13}\ (sk_{11}^+\ st_1^{-1}\ s\ell_{11}^-)^m\ sk_{13}. \end{cases}$$

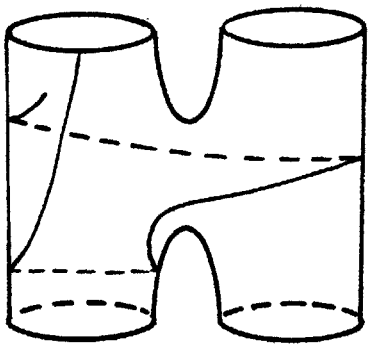
Moreover, the symbol of $\bar{\alpha}''$ is alternating and constant parity.

Proof: We check by hand in Appendix B that the proposition holds when the A-symbol of α has A-length four or less; we will assume here that the A-length of the symbol s of α is at least five. Let $\bar{s}''$ be the symbol of $\bar{\alpha}''$.

Claims 7.1-7.4 above imply that the image of $s_2 \ldots s_{n-1}$ under the map above is alternating and constant parity, and we begin by considering the boundary effects in the total space $\tilde{S}_2$. In this setting, $(s\ell_{12}, st_1^{\pm 1})$, $(sk_{13}, st_1^{\pm 1})$, $(s\ell_{13}, st_1^{\pm 1})$, and $(sk_{12}, st_1^{\pm 1})$ are all obviously alright.

Moreover, $(s\ell_{12}, sk_{11}^-)$ and $(sk_{12}, s\ell_{11}^-)$ are both alright by Proposition A.2 ii) and i), respectively.

In the case of $(s\ell_{13}, sk_{11}^-)$ (and the case of $(sk_{13}, s\ell_{11}^-)$ by symmetry), either s is of A-length three (and perhaps exceptional) or sk_{11}^- is stable, so that $\bar{s}''$ is alternating. To see that $\bar{s}''$ is constant parity, we must show that the t_1'' coordinate of $\bar{s}''$ is negative if the pair-of-letter expression $(s\ell_{13}, sk_{11}^-)$ is non-zero. If $s_1 = s\ell_{13}$, $s_2 = sk_{11}^-$, and $s_3 = st_1^{+1}$, then s is of A-length four as indicated in Figure 7.16a. We assume then that $s_3 = s\ell_{11}^{\pm}$. In case $s_3 = s\ell_{11}^-$, it is geometrically obvious from Figure 7.16b that s has t_1 coordinate value equal to zero. In case $s_3 = s\ell_{11}^+$, either s is of A-length four, $s_4 = st_1^{-1}$ and s has negative twisting by Claim 7.3 as in Figure 7.16c, or $s_4 = sk_{11}^+$. In this last case, either s is of length five and the proposition holds, as shown in Figure 7.16d, or $s_5 = t_1^{-1}$ and $\bar{s}''$ has negative twisting by Claim 7.3, as in Figure 7.16e.



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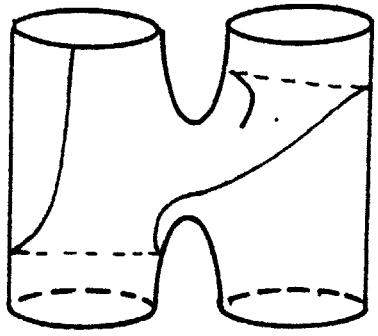

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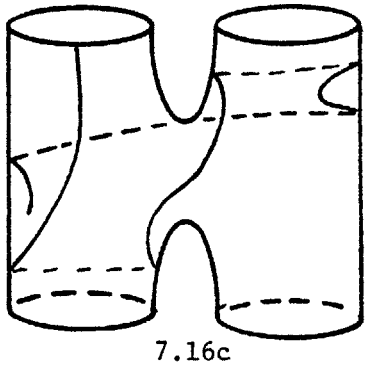

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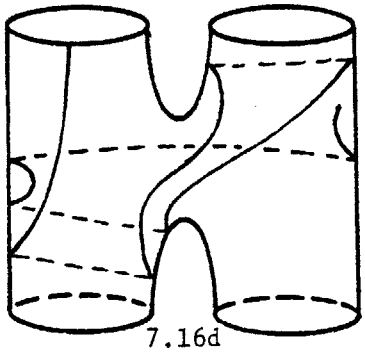

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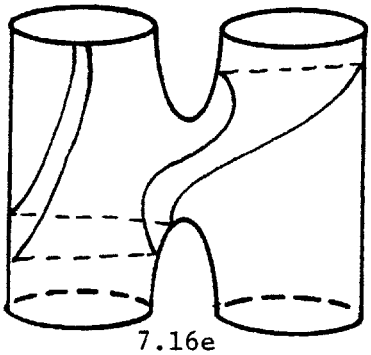
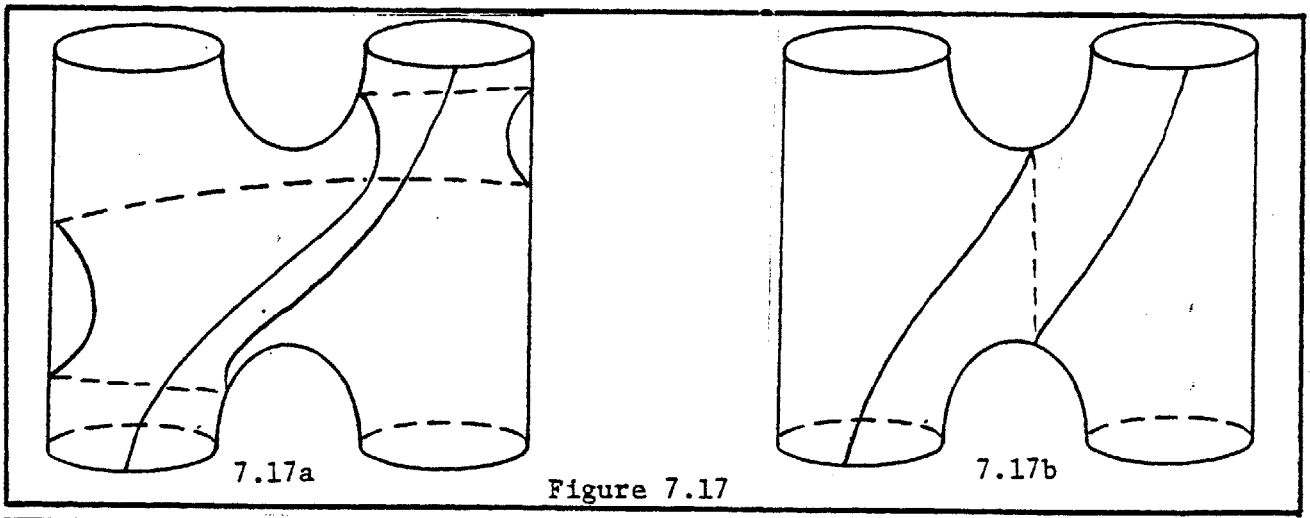


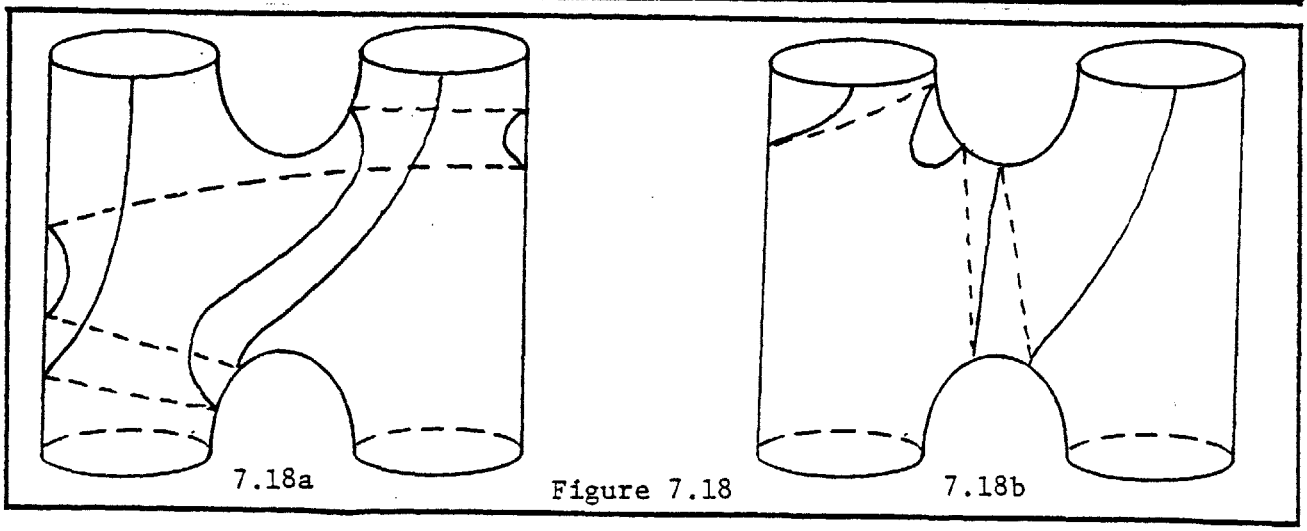
7.16e

Figure 7.16

In the case of $(s\ell_{12}, sk_{11}^{+})$ (and the case of $(sk_{12}, s\ell_{11}^{+})$ by symmetry), "$\bar{s}$" is alternating and constant parity if sk_{11}^{+} is stable; thus, we suppose that $s_1 = s\ell_{12}$, $s_2 = sk_{11}^{+}$, and $s_3 = st_1^{-1}$. Either s is of A-length four, or $s_4 = s\ell_{11}^{-}$. Thus, either $s_5 = sk_{12}$, and s is of A-length five with a negative twisting number, as indicated in Figure 7.17, or $s_5 = sk_{11}^{\pm}$. If $s_5 = sk_{11}^{+}$, then either $s_6 = s\ell_{13}$, which gives a negative twisting number, as indicated in Figure 7.18, or $s_6 = st_1^{-1}$. If $s_5 = sk_{11}^{-}$, then sk_{11}^{-} is stable, and "$\bar{s}$" is alternating and constant parity. The general case is identical to the above with α winding about $\Pi_2\tilde{s}k_{11}^{+}$ $\Pi_2\tilde{s}t_1^{-1}$ $\Pi_2\tilde{s}\ell_{11}^{-}$ followed by $\Pi_2\tilde{s}k_{11}^{-}$, $\Pi_2\tilde{s}k_{12}$, or $\Pi_2\tilde{s}k_{11}^{-}$ $\Pi_2\tilde{s}\ell_{13}$. The former is alternating and constant parity since sk_{11}^{-} is stable, and the latter cases are alternating and constant parity as indicated in Figure 7.17 and 7.18, respectively. Note that sgn(0) = +1 is the appropriate convention here.



7.17a Figure 7.17 7.17b



7.18a Figure 7.18 7.18b

Finally, in case of $(s\ell_{13}, sk_{11}^{+})$ (and the case of $(sk_{13}, s\ell_{11}^{+})$ by symmetry), either $s_1 = s\ell_{13}$, $s_2 = sk_{11}^{+}$, and $s_3 = s\ell_{13}$, so that s is exceptional, or $s_3 = st_1^{-1}$. If $s_4 = s\ell_{12}$, then s has A-length four as indicated in Figure 7.19. If $s_4 = s\ell_{11}^{+}$, then $s_5 = st_1^{-1}$, and the sk_{11}'' arising from s_3 is stable. If $s_4 = s\ell_{11}^{-}$ and $s_5 = sk_{13}$, then $\bar{s}''$ is alternating and constant parity, as indicated in Figure 7.19. If $s_5 = st_1^{-1}$, then $s_6 = sk_{11}^{-}$, and the sk_{11}'' arising from s_5 is stable, and $\bar{s}''$ is alternating and constant parity. Thus, suppose that $s_4 = s\ell_{11}^{-}$ and $s_5 = sk_{11}^{+}$; this implies that $s_6 = st_1^{-1}$. The general case is identical to the above with α winding about $\Pi_2\tilde{s}k_{11}^{+}\ \Pi_2\tilde{s}t_1^{-1}\ \Pi_2\tilde{s}\ell_{11}^{-}$ followed by $\Pi_2\tilde{s}k_{13}$, as indicated in Figure 7.20; or followed by $\Pi_2\tilde{s}k_{11}^{+}\ \Pi_2\tilde{s}t_1^{-1}\ \Pi_2\tilde{s}\ell_{12}$ as in Figure 7.19; or followed by $\Pi_2\tilde{s}k_{11}^{+}\ \Pi_2\tilde{s}t_1^{-1}\ \Pi_2\tilde{s}\ell_{11}^{+}$, and the sk_{11}'' arising from this last twist is stable; or followed by $\Pi_2\tilde{s}k_{11}^{-}$, and the sk_{11}'' arising from the last twist is stable. Note that sgn(0) = -1 is the appropriate convention here. ⊠

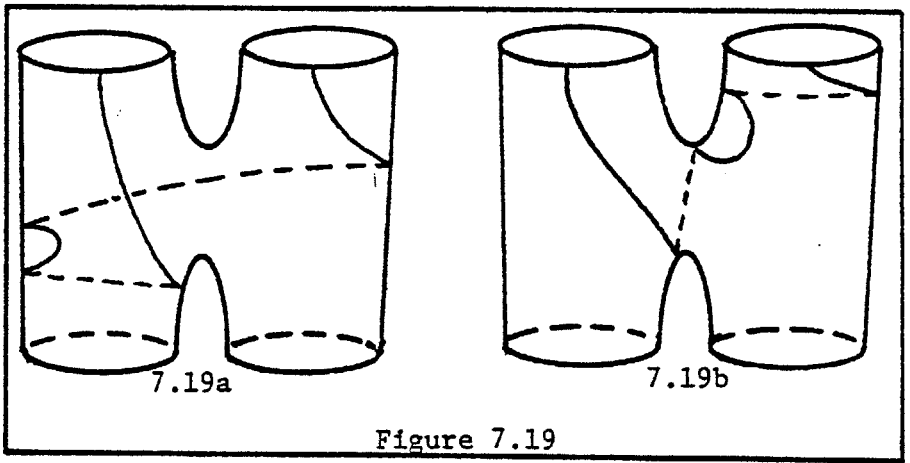


Figure 7.19

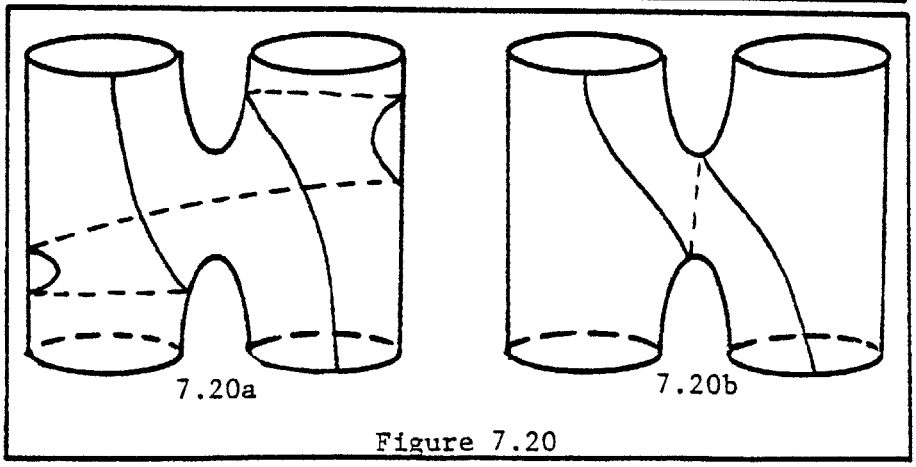


Figure 7.20

Proposition 7.5: Suppose that α is a non-exceptional multiple arc on S_2. The parameter values $(\alpha)_{A''}$ are determined from the parameter values $(\alpha)_A$ by the following formulas.

$$\ell''_{12} = \ell_{13} \qquad \ell''_{13} = k_{12}$$

$$k''_{12} = k_{13} \qquad k''_{13} = \ell_{12}$$

$$\ell''_{23} = \ell''_{22} = \ell''_{33} = k''_{23} = k''_{22} = k''_{33} = 0$$

$$t''_2 = (\ell_{13} \Delta (k_{11} + t_1)) \nabla\ 0 + t_2$$

$$t''_3 = ((\ell_{11} + t_1) \nabla -\ell_{12}) \Delta 0 + t_3$$

$$t''_4 = ((k_{11} + t_1) \nabla -k_{12}) \Delta 0 + t_4$$

$$t''_5 = (k_{13} \Delta (\ell_{11} + t_1)) \nabla 0 + t_5$$

$$t''_1 = \mathrm{sgn}(2t_1+\ell_{11}+k_{11})\ t_1 - (t''_2 - t_2) - (t''_5 - t_5)$$

$$k''_{11} = (t_1 - k_{13} + \ell_{11}) \nabla 0 + (-t_1 - \ell_{11} - \ell_{12}) \nabla 0$$

$$\ell''_{11} = (t_1 - \ell_{13} + k_{11}) \nabla 0 + (-t_1 - k_{11} - k_{12}) \nabla 0$$

In these formulas, sgn(0) is defined as follows.

$$\mathrm{sgn}(0) = \begin{cases} +1, \text{ if } \ell_{12} \neq 0. \\ -1, \text{ else}. \end{cases}$$

Proof: Just as in Step 1, one can compute the pair-of-letter expressions in Proposition 7.4 from the parameter values $(\alpha)_A$ as in the following formulas.

$$(s\ell_{13}, st_1^{+1}) = (\ell_{13} \Delta t_1) \nabla 0$$

$$(sk_{13}, st_1^{+1}) = (k_{13} \Delta t_1) \nabla 0$$

$$(s\ell_{13}, sk_{11}^{-}) = (\ell_{13} \Delta k_{11} \Delta (\ell_{13} - t_1) \Delta (k_{11} + t_1)) \nabla 0$$

$$(sk_{13}, s\ell_{11}^{-}) = (k_{13} \Delta \ell_{11} \Delta (k_{13} - t_1) \Delta (\ell_{11} + t_1)) \nabla 0$$

$$(s\ell_{11}^{+}, st_1^{-1}) = (\ell_{11} \Delta -t_1) \nabla 0$$

$$(sk_{11}^{+}, st_1^{-1}) = (k_{11} \Delta -t_1) \nabla 0$$

$$(s\ell_{12}, st_1^{-1}) = ((-t_1 - \ell_{11}) \Delta \ell_{12}) \nabla 0$$

$$(sk_{12}, st_1^{-1}) = ((-t_1 - k_{11}) \Delta k_{12}) \nabla 0$$

We begin by considering the case where α has no closed curve components, and we plug the formulas above for the pair-of-letter expressions into the formulas of Proposition 7.4. All of the formulas of this proposition follow immediately except for the expressions for the parameters t_2'', t_5'', ℓ_{11}'' and k_{11}''. We must do some algebraic manipulation for these parameter values. We will perform the computations for the parameters t_2'' and k_{11}''; the algebra for the parameters t_5'' and ℓ_{11}'' is similar.

$$
\begin{aligned}
t_2'' &= (\ell_{13} \Delta t_1) \nabla 0 + (\ell_{13} \Delta k_{11} \Delta (\ell_{13} - t_1) \Delta (k_{11} + t_1)) \nabla 0 \\
&= \begin{cases} \ell_{13} \Delta t_1 + ((\ell_{13} - t_1) \Delta k_{11}) \nabla 0, & t_1 \geq 0 \\ (\ell_{13} \Delta (k_{11} + t_1)) \nabla 0, & t_1 \leq 0 \end{cases} \\
&= \begin{cases} k_{11} + t_1, & t_1 \geq 0 \ \& \ k_{11} \leq (\ell_{13} - t_1) \\ t_1 + \ell_{13} - t_1, & t_1 \geq 0 \ \& \ 0 \leq (\ell_{13} - t_1) \leq k_{11} \\ \ell_{13}, & t_1 \geq 0 \ \& \ (\ell_{13} - t_1) \leq 0 \\ (\ell_{13} \Delta (k_{11} + t_1)) \nabla 0, & t_1 \leq 0 \end{cases} \\
&= (\ell_{13} \Delta (k_{11} + t_1)) \nabla 0, \text{ as desired.}
\end{aligned}
$$

$$
\begin{aligned}
k_{11}'' &= |t_1| + \ell_{11} - 2((\ell_{11} \Delta -t_1) \nabla 0) - (((-t_1 - \ell_{11}) \Delta \ell_{12}) \nabla 0) \\
&\quad - ((k_{13} \Delta t_1) \nabla 0) - ((k_{13} \Delta \ell_{11} \Delta (k_{13} - t_1) \Delta (\ell_{11} + t_1)) \nabla 0) \\
&= \begin{cases} t_1 + \ell_{11} - (k_{13} \Delta t_1) - (((k_{13} - t_1) \Delta \ell_{11}) \nabla 0), & t_1 \geq 0 \\ -t_1 + \ell_{11} - 2(\ell_{11} \Delta -t_1) & \\ -(((-t_1 - \ell_{11}) \Delta \ell_{12}) \nabla 0) - ((k_{13} \Delta (\ell_{11} + t_1)) \nabla 0), & t_1 \leq 0 \end{cases}
\end{aligned}
$$

$$= \begin{cases} t_1 + \ell_{11} - k_{13},\ t_1 \geq 0 \ \&\ t_1 \geq k_{13} \\ t_1 + \ell_{11} - t_1 - k_{13} + t_1,\ t_1 \geq 0 \ \&\ k_{13} - \ell_{11} \leq t_1 \leq k_{13} \\ t_1 + \ell_{11} - t_1 - \ell_{11},\ t_1 \geq 0 \ \&\ t_1 \leq k_{13} - \ell_{11} \\ t_1 + \ell_{11} - k_{13},\ t_1 \leq 0,\ -t_1 \leq \ell_{11} \ \&\ k_{13} \leq \ell_{11} + t_1 \\ t_1 + \ell_{11} - \ell_{11} - t_1,\ t_1 \leq 0,\ -t_1 \leq \ell_{11} \ \&\ \ell_{11} + t_1 \leq k_{13} \\ -t_1 - \ell_{11} - \ell_{12},\ t_1 \leq 0,\ -t_1 \geq \ell_{11} \ \&\ \ell_{12} \leq -t_1 - \ell_{11} \\ -t_1 - \ell_{11} + t_1 + \ell_{11},\ t_1 \leq 0,\ -t_1 \geq \ell_{11} \ \&\ -t_1 - \ell_{11} \leq \ell_{12} \end{cases}$$

$= (t_1 - k_{13} + \ell_{11}) \nabla 0 + (-t_1 - \ell_{11} - \ell_{12}) \nabla 0$, as desired.

To complete the proof of the proposition, one checks that the formulas of Proposition 7.5 agree with the earlier computations for multiple curves using Remark 7.1. We leave this as an exercise. ☒

Finally, we consider Step 4. We abuse notation slightly and describe the A"-coordinate values of the various exceptional arcs of A-length one using the notation for the pair-of-letter adjustments.

$$sk_{22} \equiv 2\ell''_{13} + t''_1 + k''_{11}$$
$$sk_{33} \equiv 2k''_{12} + \ell''_{11} + t''_5$$
$$s\ell_{22} \equiv 2k''_{13} + \ell''_{11} + t''_1$$
$$s\ell_{33} \equiv 2\ell''_{12} + t''_2 + k''_{11}$$
$$s\ell_{23} \equiv \ell''_{12} + k''_{13}$$
$$sk_{23} \equiv \ell''_{13} + k''_{12}$$

At last, we are in a position to give the formulas that describe the second elementary transformation. Note that we have performed Steps 1-5 between the bases A and A".

Theorem 7.1: The second elementary transformation from the basis A to the basis A' is given by the following formulas.

$$k'_{11} = k_{22} + \ell_{33} + (L - k_{13}) \nabla 0 + (-L - \ell_{12}) \nabla 0$$

$$k'_{22} = (L \Delta \ell_{11} \Delta (k_{13} - \ell_{12} - L)) \nabla 0$$

$$k'_{33} = (-L \Delta k_{11} \Delta (\ell_{12} - k_{13} + L)) \nabla 0$$

$$k'_{23} = (k_{13} \Delta \ell_{12} \Delta (k_{13} - L) \Delta (\ell_{12} + L)) \nabla 0$$

$$k'_{12} = -2k'_{22} - k'_{23} + k_{13} + k_{23} + 2k_{33}$$

$$k'_{13} = -2k'_{33} - k'_{23} + \ell_{12} + \ell_{23} + 2\ell_{22}$$

$$\ell'_{11} = k_{33} + \ell_{22} + (K - \ell_{13}) \nabla 0 + (-K - k_{12}) \nabla 0$$

$$\ell'_{22} = (K \Delta k_{11} \Delta (\ell_{13} - k_{12} - K)) \nabla 0$$

$$\ell'_{33} = (-K \Delta \ell_{11} \Delta (k_{12} - \ell_{13} + K)) \nabla 0$$

$$\ell'_{23} = (\ell_{13} \Delta k_{12} \Delta (\ell_{13} - K) \Delta (K + k_{12})) \nabla 0$$

$$\ell'_{12} = -2\ell'_{22} - \ell'_{23} + \ell_{13} + \ell_{23} + 2\ell_{33}$$

$$\ell'_{13} = -2\ell'_{33} - \ell'_{23} + k_{12} + k_{23} + 2k_{22}$$

$$t'_2 = \ell_{33} + ((\ell_{13} - \ell'_{23} - 2\ell'_{22}) \Delta (K + \ell'_{33} - \ell'_{22})) \nabla 0 + t_2$$

$$t'_3 = -k'_{33} + ((L + k'_{33} - k'_{22}) \nabla -(\ell_{12} - k'_{23} - 2k'_{33})) \Delta 0 + t_3$$

$$t'_4 = -\ell'_{33} + ((K + \ell'_{33} - \ell'_{22}) \nabla -(k_{12} - \ell'_{23} - 2\ell'_{33})) \Delta 0 + t_4$$

$$t'_5 = k_{33} + ((k_{13} - k'_{23} - 2k'_{22}) \Delta (L + k'_{33} - k'_{22})) \nabla 0 + t_5$$

$$t'_1 = k_{22} + \ell_{22} + k_{33} + \ell_{33} - (\ell'_{11} + k'_{11} + (t'_2 - t_2) + (t'_5 - t_5)) + \operatorname{sgn}(L+K+\ell'_{33}-\ell'_{22}+k'_{33}-k'_{22})\ (t_1 + \ell'_{33} + k'_{33})$$

In these formulas, K is defined to be $k_{11} + t_1$, and L is defined to be $\ell_{11} + t_1$. Furthermore, sgn(0) is defined by the following formula.

$$\operatorname{sgn}(0) = \begin{cases} +1, & \ell_{12} - 2k'_{33} - k'_{23} \neq 0 \\ -1, & \text{else} \end{cases}$$

Proof: The proof is Step 5, an application of Remark 7.1 and some algebraic manipulation. In fact all the formulas follow immediately

from Steps 1-4 and Remark 7.1 except those for ℓ'_{11} and k'_{11}. For these two formulas, we must do some algebraic manipulation.

Consider the expression for k'_{11}. One first verifies that $k'_{22} + k'_{23} + k'_{33}$ is equal to the following expression.

$$Y = \begin{cases} 0, & L \geq 0 \ \& \ L \geq k_{13} \\ k_{13} - L, & L \geq 0 \ \& \ k_{13} - \ell_{12} \leq L \leq k_{13} \\ \ell_{12} + (L \Delta \ell_{11} \Delta (k_{13} - \ell_{12} - L)), & L \geq 0 \ \& \ L \leq k_{13} - \ell_{12} \\ 0, & L \leq 0 \ \& \ -L \geq \ell_{12} \\ \ell_{12} + L, & L \leq 0 \ \& \ \ell_{12} - k_{13} \leq -L \leq \ell_{12} \\ k_{13} - (L \Delta k_{11} \Delta (\ell_{12} - k_{13} + L)), & L \leq 0 \ \& \ -L \leq \ell_{12} - k_{13} \end{cases}$$

One then computes that $k'_{11} - k_{22} - \ell_{33} = (L - k_{13} + Y) \nabla 0 + (-L - \ell_{12} + Y) \nabla 0$ is equal to $(L - k_{13}) \nabla 0 + (-L - \ell_{12}) \nabla 0$, as desired. The algebra for ℓ'_{11} is similar. ⊠

Corollary 7.1: The second elementary transformation from the basis A' to the basis A is given by the following formulas.

$$\ell_{11} = k'_{22} + \ell'_{33} + (L' - k'_{13}) \nabla 0 + (-L' - \ell'_{12}) \nabla 0$$
$$\ell_{22} = (L' \Delta \ell'_{11} \Delta (k'_{13} - \ell'_{12} - L')) \nabla 0$$
$$\ell_{33} = (-L' \Delta k'_{11} \Delta (\ell'_{12} - k'_{13} + L')) \nabla 0$$
$$\ell_{23} = (k'_{13} \Delta \ell'_{12} \Delta (k'_{13} - L') \Delta (\ell'_{12} + L')) \nabla 0$$
$$\ell_{12} = -2\ell_{22} - \ell_{23} + k'_{13} + k'_{23} + 2k'_{33}$$
$$\ell_{13} = -2\ell_{33} - \ell_{23} + \ell'_{12} + \ell'_{23} + 2\ell'_{22}$$
$$k_{11} = k'_{33} + \ell'_{22} + (K' - \ell'_{13}) \nabla 0 + (-K' - k'_{12}) \nabla 0$$
$$k_{22} = (K' \Delta k'_{11} \Delta (\ell'_{13} - k'_{12} - K')) \nabla 0$$
$$k_{33} = (-K' \Delta \ell'_{11} \Delta (k'_{12} - \ell'_{13} + K')) \nabla 0$$

$$k_{23} = (\ell'_{13} \Delta k'_{12} \Delta (\ell'_{13} - K') \Delta (K' + k'_{12})) \nabla 0$$

$$k_{12} = -2k_{22} - k_{23} + \ell'_{13} + \ell'_{23} + 2\ell'_{33}$$

$$k_{13} = -2k_{33} - k_{23} + k'_{12} + k'_{23} + 2k'_{22}$$

$$t_2 = -\ell_{33} + ((L' + \ell_{33} - \ell_{22}) \nabla -(\ell'_{12} - \ell_{23} - 2\ell_{33})) \Delta 0 + t'_2$$

$$t_3 = k'_{33} + ((k'_{13} - \ell_{23} - 2\ell_{22}) \Delta (L' + \ell_{33} - \ell_{22})) \nabla 0 + t'_3$$

$$t_4 = \ell'_{33} + ((\ell'_{13} - k_{23} - 2k_{22}) \Delta (K' + k_{33} - k_{22})) \nabla 0 + t'_4$$

$$t_5 = -k_{33} + ((K' + k_{33} - k_{22}) \nabla -(k'_{12} - k_{23} - 2k_{33})) \Delta 0 + t'_5$$

$$t_1 = \ell'_{33} + k'_{33} + \ell'_{22} + k'_{22} - (\ell_{11} + k_{11} + (t_3 - t'_3) + (t_4 - t'_4)) + \mathrm{sgn}(L' + K'' + k_{33} - k_{22} + \ell_{33} - \ell_{22}) (t'_1 + k_{33} + \ell_{33})$$

In these formulas, K' is defined to be $k'_{11} + t'_1$, and L' is defined to be $\ell'_{11} + t'_1$. Furthermore, sgn(0) is defined by the following formula.

$$\mathrm{sgn}(0) = \begin{cases} +1, \text{ if } \ell'_{12} - 2\ell_{33} - \ell_{23} \neq 0. \\ -1, \text{ else}. \end{cases}$$

Proof: The symmetry of S_2 indicated in Figure 7.21 implies that the formulas for ℓ_{**} (k_{**}) are the formulas for k'_{**} (ℓ'_{**}), replacing ℓ_{**} (k_{**}) by ℓ'_{**} (k'_{**}) and ℓ'_{**} (k'_{**}) by k_{**} (ℓ_{**}). The formula for t_* is the formula for $t_{\sigma^{-1}(*)}$, with the symbol replacement as above and $t'_{\sigma(*)}$ replacing t_*, where σ fixes 1 and is the cyclic permutation (2,4,5,3) on the other pants curves. ⊠

One can of course prove the corollary directly by mimicking the proof of the theorem with (A',A) replacing (A,A'). To prove this corollary by checking that the transformations in the corollary and the theorem are inverses (as in Section 6) is a very difficult computation.

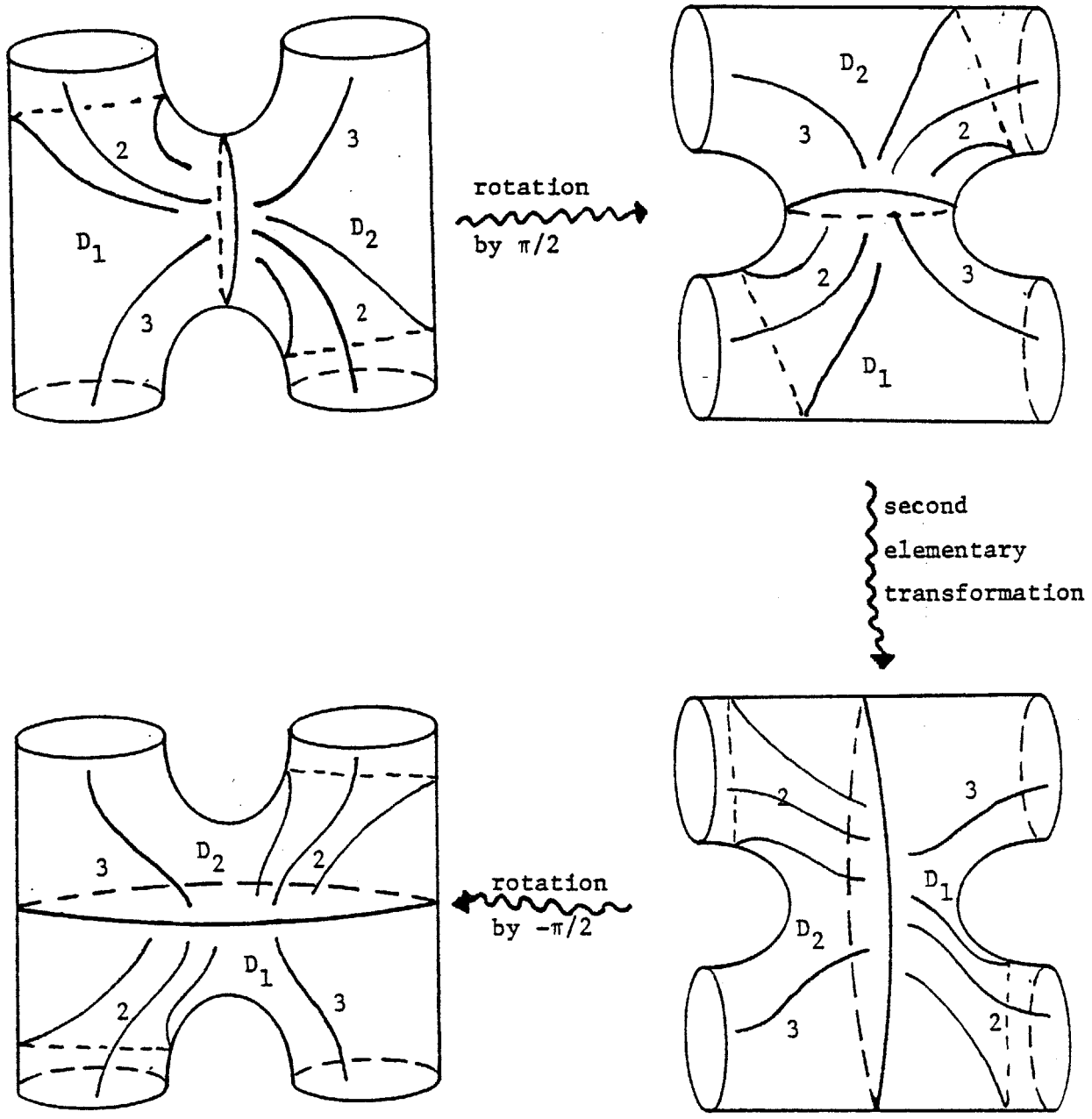

2
3
D_1
D_2
3
2
rotation
by π/2
D_2
3
2
2
3
D_1
second
elementary
transformation
3
D_2
2
3
2
D_1
rotation
by -π/2
2
3
D_1
D_2
3
2

Figure 7.21

SECTION 8

Having computed the first and second elementary transformations (and their inverses) in the previous two sections, we state the results of Theorems 6.1 and 7.1 here for the convenience of the reader.

Theorem 8.1: The first elementary transformation from the basis A to the basis A' is given by:

$$\ell'_{11} = (\ell_{12} - |t_1|) \nabla 0$$

$$\ell'_{12} = \ell'_{13} = (\ell_{12} - \ell'_{11}) + \ell_{11}$$

$$\ell'_{23} = |t_1| - (\ell_{12} - \ell'_{11})$$

$$t'_2 = t_2 + \ell_{11} + ((\ell_{12} - \ell'_{11}) \Delta t_1) \nabla 0$$

$$t'_1 = -\mathrm{sgn}(t_1)\ (\ell_{23} + (\ell_{12} - \ell'_{11}))$$

In these formulas, Δ denotes the infimum, ∇ denotes the supremum, sgn denotes the sign function, and sgn(0) is defined to be -1. ⊠

Theorem 8.2: The second elementary transformation from the basis A to the basis A' is given by:

$$\ell^{2,}_{11} = \ell^2_{22} + \ell^1_{33} + (L^1 - \ell^2_{13}) \nabla 0 + (-L^1 - \ell^1_{12}) \nabla 0$$

$$\ell^{2,}_{22} = (L^1 \Delta \ell^1_{11} \Delta (\ell^2_{13} - \ell^1_{12} - L^1)) \nabla 0$$

$$\ell^{2,}_{33} = (-L^1 \Delta \ell^2_{11} \Delta (\ell^1_{12} - \ell^2_{13} + L^1)) \nabla 0$$

$$\ell^{2,}_{23} = (\ell^2_{13} \Delta \ell^1_{12} \Delta (\ell^2_{13} - L^1) \Delta (\ell^1_{12} + L^1)) \nabla 0$$

$$\ell^{2,}_{12} = -2\ell^{2,}_{22} - \ell^{2,}_{23} + \ell^2_{13} + \ell^2_{23} + 2\ell^2_{33}$$

$$\ell^{2,}_{13} = -2\ell^{2,}_{33} - \ell^{2,}_{23} + \ell^1_{12} + \ell^1_{23} + 2\ell^1_{22}$$

$$\ell'^{1}_{11} = \ell^{2}_{33} + \ell^{1}_{22} + (L^2 - \ell^{1}_{13})\nabla 0 + (-L^2 - \ell^{2}_{12})\nabla 0$$

$$\ell'^{1}_{22} = (L^2 \Delta \ell^{2}_{11} \Delta (\ell^{1}_{13} - \ell^{2}_{12} - L^2))\nabla 0$$

$$\ell'^{1}_{33} = (-L^2 \Delta \ell^{1}_{11} \Delta (\ell^{2}_{12} - \ell^{1}_{13} + L^2))\nabla 0$$

$$\ell'^{1}_{23} = (\ell^{1}_{13} \Delta \ell^{2}_{12} \Delta (\ell^{1}_{13} - L^2) \Delta (L^2 + \ell^{2}_{12}))\nabla 0$$

$$\ell'^{1}_{12} = -2\ell'^{1}_{22} - \ell'^{1}_{23} + \ell^{1}_{13} + \ell^{1}_{23} + 2\ell^{1}_{33}$$

$$\ell'^{1}_{13} = -2\ell'^{1}_{33} - \ell'^{1}_{23} + \ell^{2}_{12} + \ell^{2}_{23} + 2\ell^{2}_{22}$$

$$t'_2 = \ell^{1}_{33} + ((\ell^{1}_{13} - \ell^{1}_{23} - 2\ell'^{1}_{22}) \Delta (L^2 + \ell'^{1}_{33} - \ell'^{1}_{22}))\nabla 0 + t_2$$

$$t'_3 = -\ell'^{2}_{33} + ((L^1 + \ell'^{2}_{33} - \ell'^{2}_{22})\nabla - (\ell^{1}_{12} - \ell'^{2}_{23} - 2\ell'^{2}_{33}))\Delta 0 + t_3$$

$$t'_4 = -\ell'^{1}_{33} + ((L^2 + \ell'^{1}_{33} - \ell'^{1}_{22})\nabla - (\ell^{2}_{12} - \ell'^{1}_{23} - 2\ell'^{1}_{33}))\Delta 0 + t_4$$

$$t'_5 = \ell^{2}_{33} + ((\ell^{2}_{13} - \ell'^{2}_{23} - 2\ell'^{2}_{22}) \Delta (L^1 + \ell'^{2}_{33} - \ell'^{2}_{22}))\nabla 0 + t_5$$

$$t'_1 = \ell^{1}_{22} + \ell^{2}_{22} + \ell^{1}_{33} + \ell^{2}_{33} - (\ell'^{1}_{11} + \ell'^{2}_{11} + (t'_2 - t_2) + (t'_5 - t_5))$$
$$+ \operatorname{sgn}(L^1 + L^2 + \ell'^{1}_{33} - \ell'^{1}_{22} + \ell'^{2}_{33} - \ell'^{2}_{22})\ (t_1 + \ell'^{1}_{33} + \ell'^{2}_{33})$$

In these formulas, Δ and ∇ are as in Theorem 8.1, L^1 denotes the quantity $\ell^{1}_{11} + t_1$, and L^2 denotes the quantity $\ell^{2}_{11} + t_1$. Furthermore, sgn denotes the sign function, and sign(0) is defined as follows:

$$\operatorname{sgn}(0) = \begin{cases} +1, \text{ if } \ell^{1}_{12} - 2\ell'^{2}_{33} - \ell'^{2}_{23} \neq 0. \\ -1, \text{ else.} \end{cases} \boxtimes$$

These theorems give explicit formulas for the action on $\mathscr{S}'_A(F_g)$ of Lickorish's generators for $MC(F_g)$ as described in Section 3. The piecewise-integral character of the action is directly implied by Theorems 8.1 and 8.2. Unfortunately, the formulas are rather cumbersome, insofar as several of the Lickorish generators act as linear maps conjugated by compositions of the elementary transformations.

One's first reaction to the complexity of the situation is panic, and an appropriate response is to write a computer code to perform the algebra of the computations. The formulas of the elementary transformations are particularly amenable to computerization, since they are essentially sums of infs and sups of linear maps. The notable exception to this is the sign that appears in the expression for the twisting number t_1 in either transformation.

A FORTRAN code has been written to compute the action of $MC(F_g)$ on the collection of multiple arcs, as described in this thesis. Several hundred thousand cases of the computation have been run, checking that a transformation followed by its inverse yields the identity in each case. Moreover, many trends predicted by Thurston's theory of surface automorphisms are exhibited by experimenting with this code. (See Subsection 2 below for a brief description of the Thurston theory.) The code is great fun to play with, and a source listing with documentation is contained in Appendix C. (The reader interested in the bookkeeping details of the computation outlined in Section 3 should refer to Appendix C.)

A more optimistic reaction to the complexity of the formulas derived is to begin computing the various compositions that arise in the hopes of simplifications of the sort that occur in the proof of Theorem 7.1. This is not, I think, an unrealistic optimism. The sort of computations that this involves are a rather pleasant blend of combinatorics and linear algebra, and some progress has been made. I will briefly describe some work in this direction.

Consider the pants decomposition shown in Figure 8.1 on the n-times punctured sphere, $n \geq 4$, denoted $S^2 \backslash n^*$. We distinguish the subgroup

of $MC(S^2\backslash n^*)$ corresponding to the homeomorphisms fixing each of the punctures. This subgroup of $MC(S^2\backslash n^*)$ is called the pure mapping class group of $S^2\backslash n^*$, and is denoted $PMC(S^2\backslash n^*)$. It is well-known [2] that $PMC(S^2\backslash n^*)$ is isomorphic to the pure n-braid group on S^2 modulo its center. $PMC(S^2\backslash n^*)$ is generated by Dehn twists along the nullhomologous curves in Figure 8.2. Just as in Section 3, to compute the action of $PMC(S^2\backslash n^*)$ on $\mathscr{P}'(S^2\backslash n^*)$, it suffices to compute the transformations indicated in Figure 8.3. These transformations are easily derived from the formulas of Theorem 8.2; in fact, the piecewise-integral action of PMC(S^2-minus-n-points-minus-(4-n)-discs) is a restriction of the piecewise-integral action of PMC(S^2-minus-four-discs).

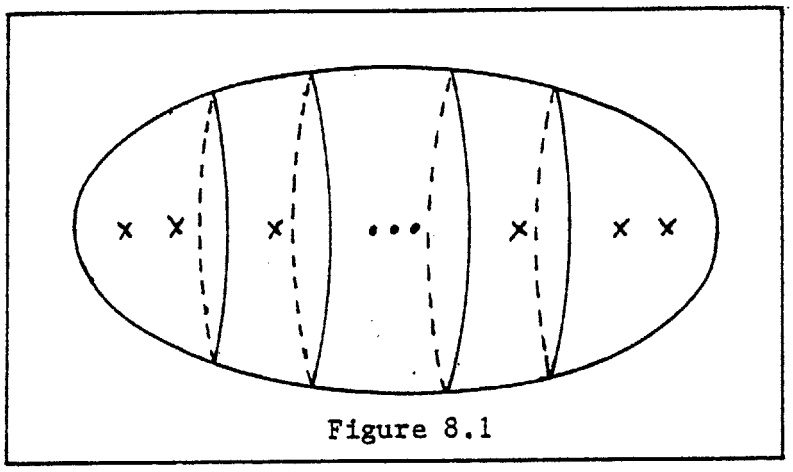


Figure 8.1

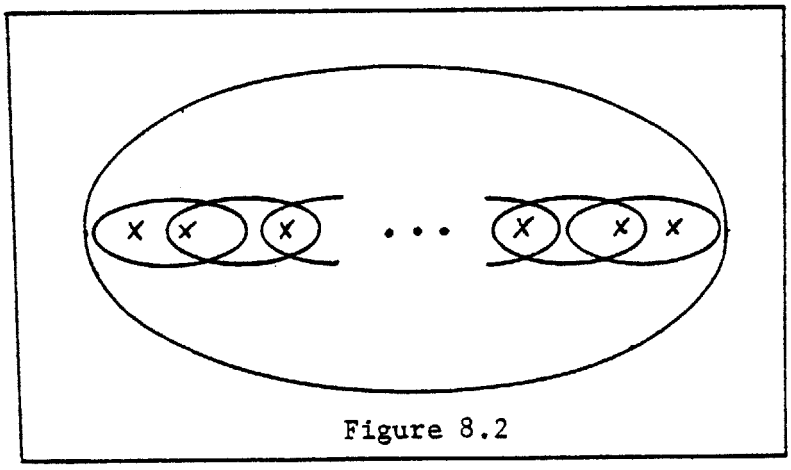


Figure 8.2

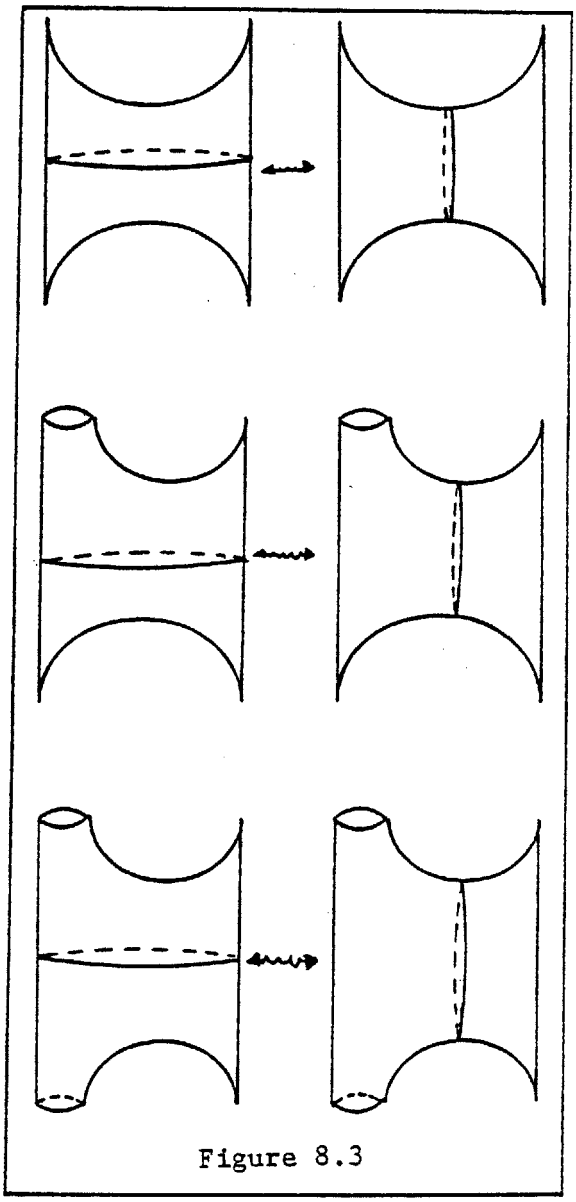


Figure 8.3

For spheres with n punctures, I have explicitly computed several of the compositions of transformations that arise in the action of $PMC(S^2\backslash n^*)$, and there is a concise description of the action in several cases. A particularly simple case is the action of $PMC(S^2\backslash 4^*)$ on the subset of $\mathcal{I}'(S^2\backslash 4^*)$ corresponding to (necessarily closed) multiple arcs with no boundary-parallel components. This action is faithfully represented by an action on our parameter values of the group of invertible integral matrices generated by $\begin{pmatrix} 1 & -2 \\ 0 & -3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. The action is a twisted right action given by:

$$(\ell^1_{11}, t_1)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (|a\ell^1_{11} + ct_1|,\ \mathrm{sgn}(a\ell^1_{11} + ct_1)\ (b\ell^1_{11} + dt_1))$$

This description of the action of the pure mapping class group of $S^2\backslash 4^*$ on the collection described above will be useful in our subsequent discussions of applications.

We will discuss several applications of our computations in turn in the following five subsections. When convenient, we will assume our surfaces are supplied with a fixed smooth structure.

1) The Word Problem for Lickorish's Generators of $MC(F_g)$

Using the Alexander trick, one easily proves the following Proposition.

Proposition 8.1: Let ψ be a mapping class on F_g, and let $\{c_k\}$ be a collection of simple closed curves so that

a) $F_g\backslash\cup\{c_k\}$ consists of discs.

b) $c_k \cap c_{k'}$ is either empty or a single transverse intersection, for $k \neq k'$.

ψ is the identity in $MC(F_g)$ if and only if ψ fixes the isotopy class of each c_k. ⊠

Thus, the computations of this thesis give an efficient algorithm for solving the word problem for Lickorish's generators of $MC(F_g)$: one simply evaluates a given word on some collection $\{c_k\}$ as in Proposition 8.1 and checks that the isotopy class of each c_k is fixed.

It is well-known [2] that $MC(F_g)$ is isomorphic to the group of "orientation-preserving" outer automorphisms of $\pi_1(F_g)$. Thus, a mapping class is the identity in $MC(F_g)$ if and only if each of its representatives acts like an inner automorphism of $\pi_1(F_g)$. This could be regarded as giving an "algorithm" for the word problem in Lickorish's generators: one checks that the word acts like an inner automorphism on a set of generators for $\pi_1(F_g)$. This is an unwieldy computation for words of large length.

Isotopy classes are the same as free homotopy classes of curves embedded in surfaces [7]; furthermore, free homotopy classes of curves are the same as conjugacy classes in $\pi_1(F_g)$ modulo orientation. Thus, our formulas describe the action of Lickorish's generators on embedded conjugacy classes in $\pi_1(F_g)$ modulo orientation.

2) Thurston's Classification of Surface Automorphisms

Let F be a surface of negative Euler characteristic. We will say that a mapping class ψ on F is pseudo-Anosov if, for every iterate ψ^n and for every free homotopy class $[\gamma]$ of non-boundary-parallel, not necessarily embedded connected curve, $[\psi^n\gamma] \neq [\gamma]$. (Note that some iterate of ψ always fixes all boundary-parallel curves.) We will say ψ is periodic if some iterate ψ^n is the identity in MC(F). We will say ψ is reducible if there is a closed multiple arc α, no component of which is boundary-parallel, so that ψ permutes the components of α. α is called a reducing curve for ψ.

Simply stated, Thurston's classification says that a mapping class is one of periodic, pseudo-Anosov or reducible. A mapping class may be both periodic and reducible, and this is the only overlap in the classification. A natural problem is the classification of words in Lickorish's generators into periodic, reducible and pseudo-Anosov mapping classes.

The periodic, pseudo-Anosov or reducible character of a mapping class $\psi \in MC(F)$ has a natural description in terms of its piecewise-integral action on $\mathscr{PL}'_A(F)$. For instance, suppose that ψ fixes a curve whose parameter values lie in a top-dimensional simplex σ of the decomposition K_ψ of the piecewise-integral structure. Under these conditions, there is an eigenvector in $\mathscr{PL}'_A(F_g)$ with eigenvalue 1 for the integral matrix which corresponds to σ in the piecewise-integral structure of the action of ψ.

One reason that this problem is of interest is that one can often find a description of the monodromy of a fibred link in terms of Dehn twists, and the periodic, reducible or pseudo-Anosov character of the monodromy is connected with geometrical structures on the link complement. In many examples, the monodromy is the lift of some map to a branched cover of the disc; hence the action of $MC(S^2\backslash n^*)$ on $\mathscr{PL}'(S^2\backslash n^*)$ is of interest here.

Another reason that this classification problem is of interest pertains to the following theorem, which will be proved elsewhere [15].

Theorem 8.3: Let γ and δ each be multiple curves in a surface F with negative Euler characteristic. Let $\{c_i\}$ and $\{d_j\}$ be the components of γ and δ, respectively. Furthermore, assume the two conditions below.

a) c_i and d_j intersect minimally, for all i and all j.

b) The components of $F \setminus \cup(\{c_i\} \cup \{d_j\})$ are all discs.

Let w be any composition of the Dehn twists $\tau_{c_i}^{+1}$ and $\tau_{d_j}^{-1}$ so that, for each c_i or d_j, $\tau_{c_i}^{+1}$ or $\tau_{d_j}^{-1}$ appears at least once in w. Under these conditions, w represents a pseudo-Anosov mapping class. ⊠

This recipe for constructing pseudo-Anosov mapping classes generalizes known constructions of such. I can prove that this recipe gives all pseudo-Anosov mapping classes in a few cases, and I conjecture that this is true for g-holed tori, at least up to iteration of the map and composition with maps of finite order.

For the special case of a mapping class ψ on $S^2 \setminus 4^*$, there is some iterate ψ^n of ψ that is a pure mapping class. The periodic, reducible or pseudo-Anosov character of ψ is determined by the trace of the matrix corresponding to ψ^n. (See Proposition 8.2.)

3) The Action of $MC(F_g)$ on Thurston's Boundary for Teichmuller Space

The Teichmuller space of F_g, denoted $\mathcal{T}(F_g)$, is defined to be the space of Riemannian metrics (with the natural toplogy) of constant curvature -1, modulo push-forward by diffeomorphisms isotopic to the identity. O. Teichmuller [16] showed that $\mathcal{T}(F_g)$ is homeomorphic to an open 6g-6 disc. There are several classical compactifications of $\mathcal{T}(F_g)$, and Thurston [17] has given a beautiful compactification of $\mathcal{T}(F_g)$ by a 6g-7 sphere. We will presently describe Thurston's compactification.

By an n-gon in a surface F we mean a smoothly embedded open disc in the interior of F, with piecewise smooth frontier and n discontinuities in the tangent of the bounding curve. Some examples of n-gons are pictured in Figure 8.4. A subspace $X \subset F$ is said to have a complementary n-gon if some component of $F \setminus X$ is an n-gon in F; X is said to have a complementary annulus if the closure of some component of $F \setminus X$ is a smooth annulus in F.

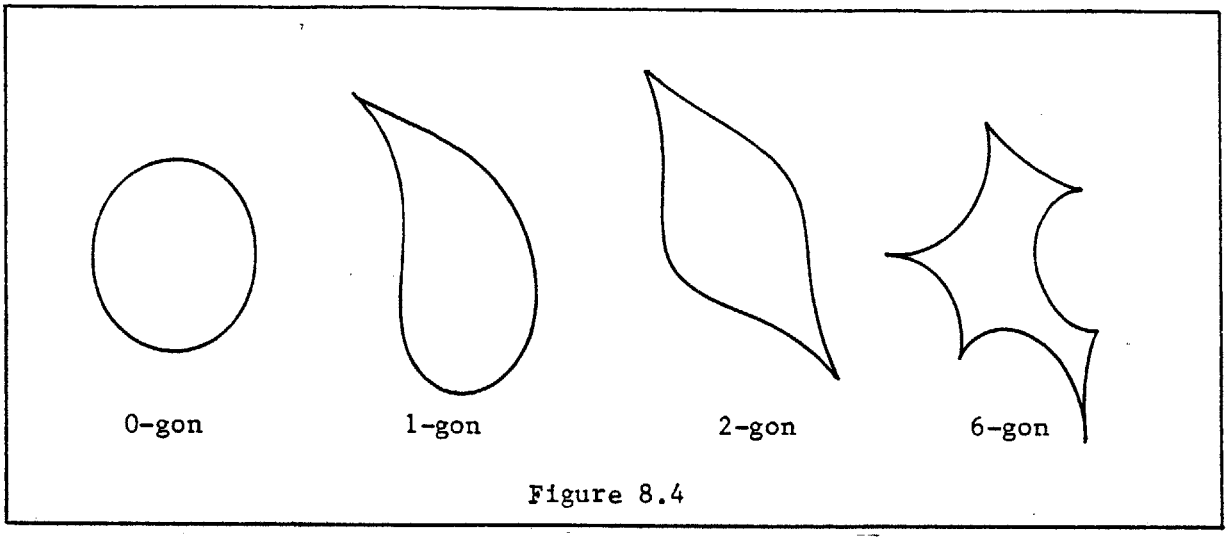


Figure 8.4

A train track T in the surface F is a closed branched one-submanifold embedded in F, so that T has no complementary null-gons, mono-gons, bi-gons or annuli. Some examples of train tracks are pictured in Figure 8.5. A train track T is a one-complex in a natural way; the 0-simplexes are called the branch points of T, and the 1-simplexes are called the branches of T. A train track T in F is said to be transversely recurrent if, for each branch b_i of T, there is a simple closed curve d intersecting b_i transversely in a point; furthermore, there are no bi-gons complimentary to dVT. Transverse recurrence is a technical condition that we will require shortly.

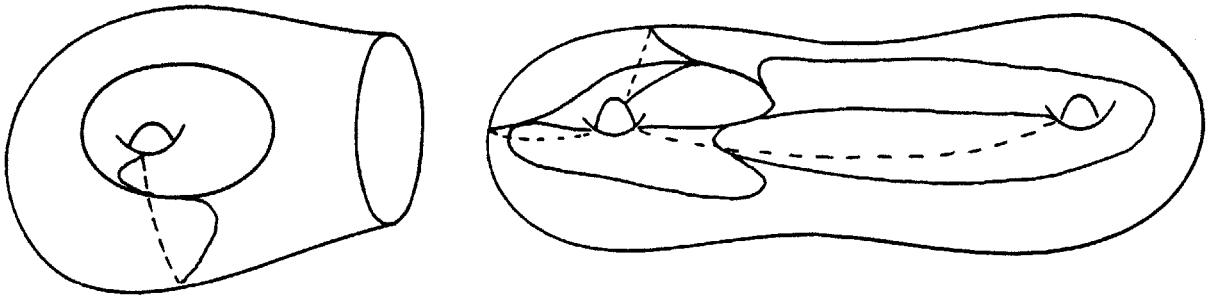


Figure 8.5

A measure m on a train track is an assignment of a positive real number $m(b_i)$ to each branch b_i of the train track T. The number $m(b_i)$ is called the weight of the branch b_i. The weights are required to satisfy a single relation for each branch point; for instance, whenever the branches b_i, b_j and b_k are as indicated in Figure 8.6, the weights must satisfy $m(b_i) = m(b_j) + m(b_k)$. We require an analogous additivity relation when more than three branches have a branch point in common. A measured train track is a natural generalization of a closed multiple arc: closed multiple arcs correspond to measured train tracks with integral weights.

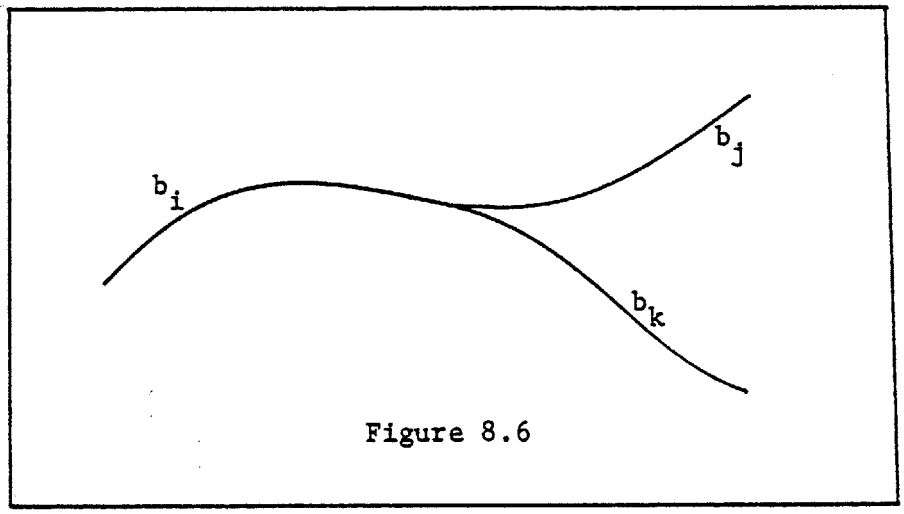


Figure 8.6

Let c be a simple closed curve in F_g, and let (T,m) be a measured train track. Isotope c so that it misses the branch points of T and there are no bi-gons complementary to $c \cup T$. If $\text{card}(c \cap b_i) = t_i$, we define the length of c to be $\Sigma t_i m(b_i)$. The length of the isotopy class of c is well-defined, and we extend the definition of length to closed multiple arcs by requiring length to be additive on components.

We will say that two measured train tracks are equivalent if they

define the same length function on closed multiple curves. Two measured train tracks with length functions L_1 and L_2 are said to be projectively equivalent if there is some positive real number r so that $L_1 = rL_2$. Thurston [17] shows that the collection of projective equivalence classes of transversely recurrent train tracks (with a suitable topology) forms a 6g-7 sphere that compactifies $\mathcal{T}(F_g)$.

$MC(F_g)$ acts on $\mathcal{T}(F_g)$ by push-forward of metrics, and this action extends to the natural action on Thurston's boundary: the action of $MC(F_g)$ on (isotopy classes of) measured train tracks. Our computations describe the action of $MC(F_g)$ on standard (see Construction 5.1) measured train tracks, and it seems almost certain that any measured train track is projectively equivalent to a standard one. In any case, the collection of (projective equivalence classes of) measured train tracks with integral measures is dense in Thurston's boundary for $\mathcal{T}(F_g)$; thus, we have already computed the action of $MC(F_g)$ on a dense subset of Thurston's boundary for $\mathcal{T}(F_g)$.

4) Linear Representations of Mapping Class Groups.

Let F be some surface. The goal is to exhibit a faithful representation of MC(F) as a group of invertible matrices or prove that such a representation cannot exist. Such representations would be useful in better understanding the mapping class groups.

We have derived a faithful representation of MC(F) as a group of piecewise-integral transformations provided the Euler characteristic of F is negative. Furthermore, we have remarked previously that the pure mapping class group of $S^2\backslash 4^*$ admits a faithful representation as a subgroup of $SL_2\mathbb{Z}$. It was mentioned that this action is twisted. Using the results of this thesis, one might hope for analogous twisted linear representations of the pure mapping class groups of $S^2\backslash n^*$, $n \geq 4$. It

is possible that our formulas even describe such a representation as they stand, but the twisting in the action renders this unrecognizable.

5) Dynamics of Surface Homeomorphisms

Suppose that f is a homeomorphism from the open two-disc D^{o2} to itself, and suppose that x is a periodic point of f of period $n \geq 3$. Let $O(x)$ denote the orbit $\{x, f^1x, \ldots, f^{n-1}x\}$. f restricts to a homeomorphism of $D^{o2} \backslash O(x)$, and we may consider the mapping class of this homeomorphism. Identifying $D^{o2} \backslash O(x)$ with $S^2 \backslash (n+1)^*$, this mapping class is given by a coset of the (full) (n+1)-braid group of S^2 by its center. We will call this coset the topological type of f with respect to the orbit $O(x)$. The topological type of f varies from one periodic orbit to another, yet there is an obvious compatibility between topological types of f with respect to various orbits.

The reason that our computations are applicable is that we can compute topological types. The topological type of f with respect to $O(x)$ is determined by its action on $\mathcal{S}'(D^{o2} \backslash O(x))$. Note that the topological type of f^n with respect to $O(x)$ is a pure mapping class. In fact, we can determine the topological type of f^n with respect to $O(x)$ by computing its action on a finite collection of simple closed curves in D^{o2}. (One proves an analogue of Proposition 8.1.)

More crudely, one may simply consider whether the topological type of f with respect to $O(x)$ is periodic, pseudo-Anosov or reducible. There is evidence [4] to suggest that the existence of orbits of certain periods puts restrictions on which of periodic, pseudo-Anosov or reducible topological types can occur.

In the case of period three points, we can use the representation of $PMC(S^2 \backslash 4^*)$ described above to prove the following proposition.

Propostion 8.2: Suppose $f:D^{o_2} \to D^{o_2}$ is a homeomorphism of the open two-disc, and suppose the $x \in D^{o_2}$ is a period three point. The topological type $\psi \in PMC(S^2 \setminus 4^*)$ of f^3 with respect to the orbit of x is described by an invertible two-by-two integral matrix B. B is determined by the action of f^3 on two simple closed curves in D^{o_2}. Moreover,

a) ψ is reducible if and only if $|trB| = 2$.

b) ψ is periodic if and only if $|trB| \leq 2$.

c) ψ is pseudo-Anosov if and only if $|trB| \geq 2$. ⊠

In conclusion, there are many interesting problems associated with the action computed herein. What is lacking as a good qualitative understanding of the formulas that we have derived. The setting in which to begin developing this understanding is the setting of punctured spheres, and some progress has been made in this direction.

APPENDIX A

In this appendix, we prove several technical results about symbols that are used in Sections 6 and 7. We adopt the notation of Sections 6 and 7.

Proposition A.1: Let $\mathcal{B}$ and $\mathcal{B}'$ be two bases on the standard pants P that differ only in the choice of canonical piece l_{11} or $\hat{l}_{11}$ (see Section 2). The transformation between $\mathscr{S}'_{\mathcal{B}}(P)$ and $\mathscr{S}'_{\mathcal{B}'}(P)$ is described by the following formula.

$$(m_1,m_2,m_3)\times(t_1,t_2,t_3) \to (m_1,m_2,m_3)\times(t_1\pm\ell_{11},t_2,t_3)$$

Proof: The simple and very useful isotopy that proves this proposition is indicated in Figure A.1. ⊠

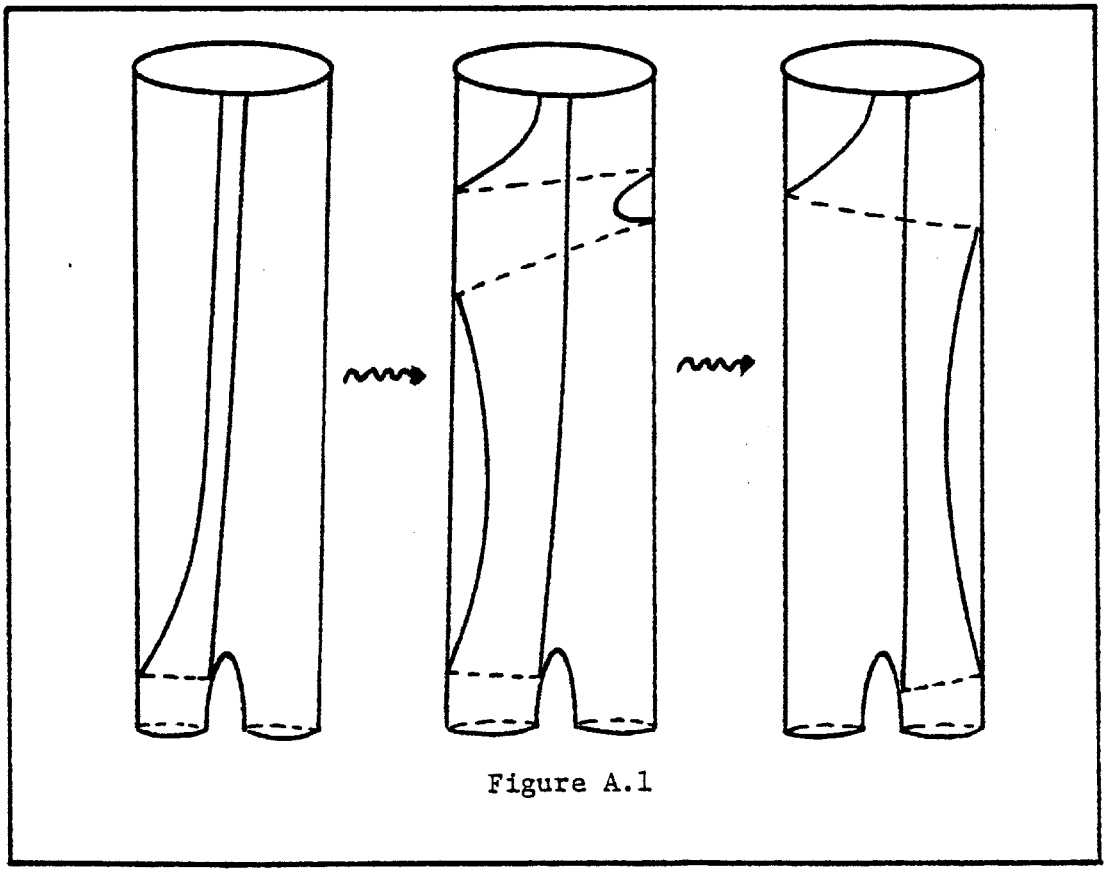


Figure A.1

Proposition A.2: If $st_i^{n_1}$ and $st_i^{n_2}$ both occur as sub-symbols of (some components of) an embedded admissible symbol, where $|n_1|$ and $|n_2|$ are maximal, then $|n_1-n_2| \leq 1$.

Proof: Let t^m denote an arc twisting m times in the standard annulus A; $A_i \setminus \upsilon_i^{-1}t^{n_1}$ is a disc in which $\upsilon_i^{-1}t^{n_2}$ is a properly embedded arc. ☒

Corollary A.1: If $st_i^{n_1}$ and $st_i^{n_2}$ both occur as sub-symbols of (some components of) an embedded admissible symbol, then $\text{sgn}(n_1) = \text{sgn}(n_2)$, provided $n_1 \neq 0 \neq n_2$. ☒

Proposition A.3: a) If $st_1^{-1}\, s\ell_{11}^+$ occurs as a sub-symbol in some embedded admissible A-symbol s on S_2, then an $s\ell_{11}^+$ in s is always followed by a st_1^{-1}.

b) If $st_1^{-1}\, sk_{11}^+$ occurs as a sub-symbol in some embedded admissible A-symbol s on S_2, then an sk_{11}^+ in s is always followed by a st_1^{-1}.

Proof: These are the only embedded possibilities as indicated in Figure A.2. ☒

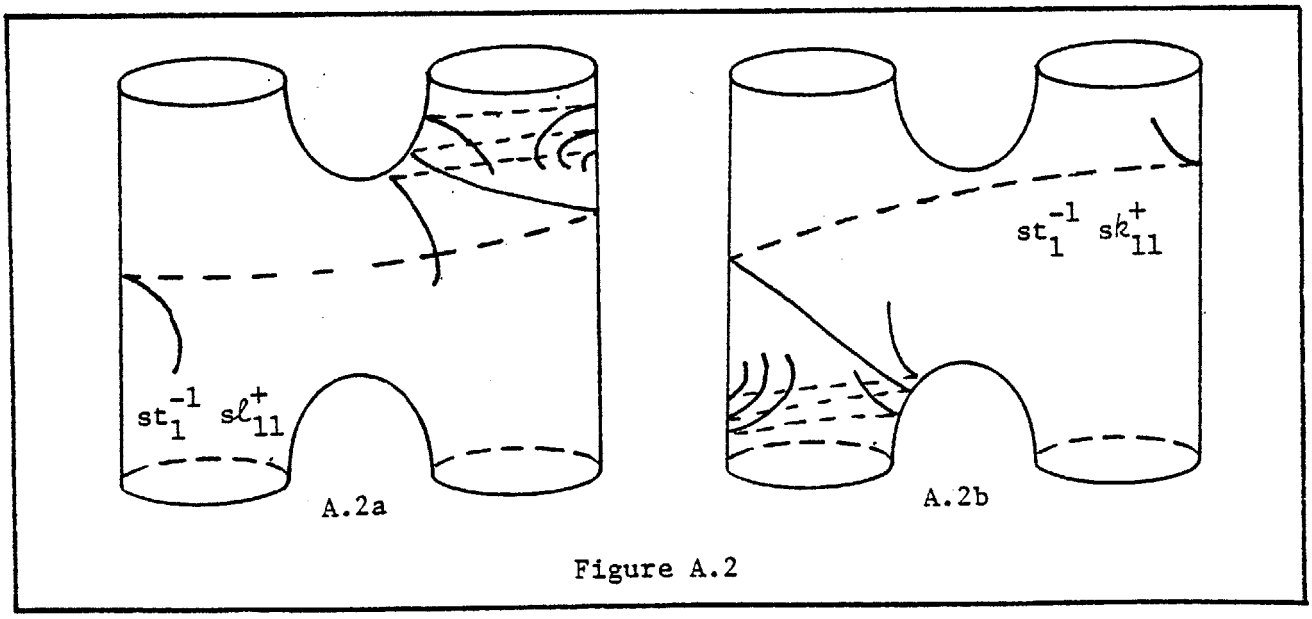
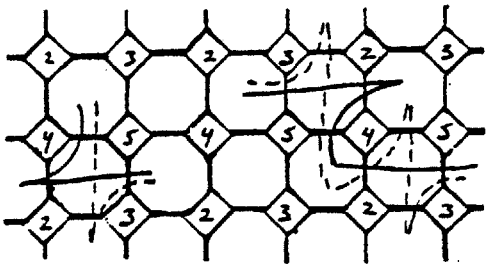


Figure A.2

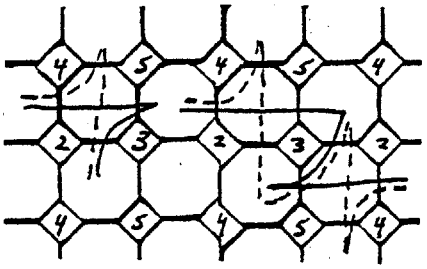
Proposition A.4: a) The image of $s\ell_{11}^{\pm}\ st_1^{-1}$ in an A-symbol on S_2 under the combinatorial homotopy without boundary effects in Section 7 is $st_1''^{+1} s\ell_{11}''$.

b) The image of $sk_{11}^{\pm}\ st_1^{-1}$ in an A-symbol on S_2 under the combinatorial homotopy without boundary effects in Section 7 is $st_1''^{+1}\ sk_{11}''$.

Proof: In Figure A.3, we illustrate the combinatorial homotopy in $\tilde{S}_2$ for cases a) and b). We use the results of Proposition A.3 to guarantee that the lifts are as indicated in Figure A.3. The solid lines indicate the lifts, and the broken lines indicate the image of the homotopy. ⊠



A.3a



A.3b

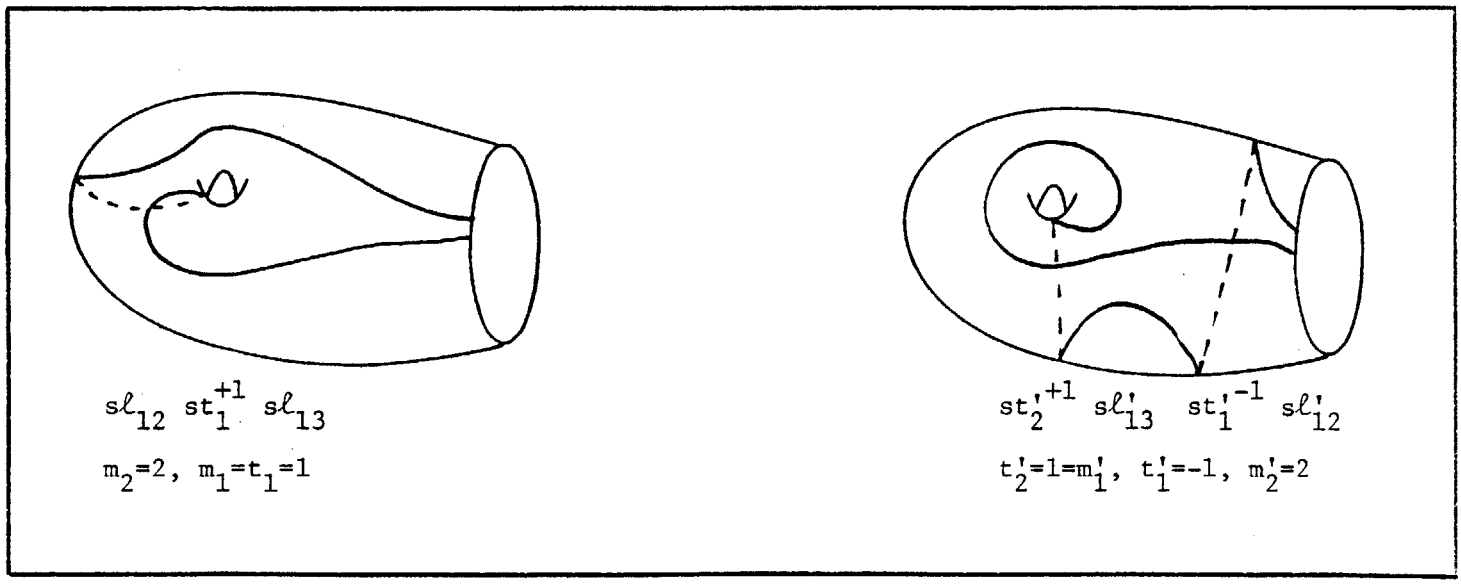
Figure A.3

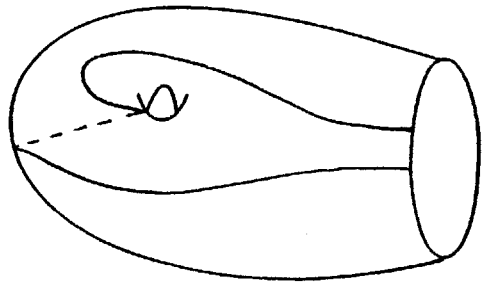
APPENDIX B

In this appendix, we check by hand the formulas for the two elementary transformations on connected, non-exceptional A-symbols of A-length less than five. There are several special cases here that we wished to avoid in the combinatorial arguments of Sections 6 and 7. Since these special cases all have small A-length, it seems easiest to exhaustively check these few cases.

These computations are a good exercise for the reader wishing to familiarize him or herself with the parametrizations in Section 2 and the bases used in Sections 6 and 7.

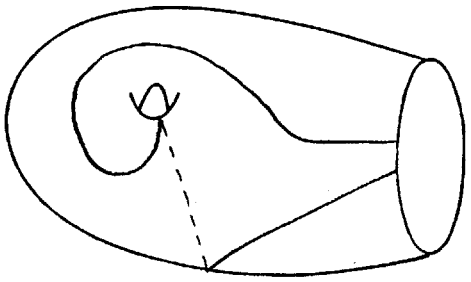
For the first elementary transformation, there are six cases. In the following diagrams, we indicate the A-coordinates and depict a good representative on the left; on the right, we give the same data for the basis A'. We omit mention of any parameter values that are equal to zero.





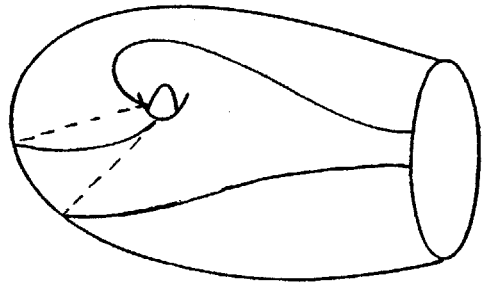
$s\ell_{12}\ st_1^{-1}\ s\ell_{13}$

$m_2=2,\ m_1=1,\ t_1=-1$



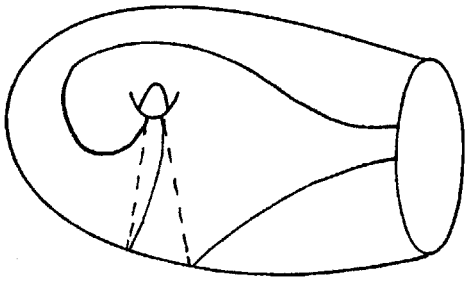
$s\ell'_{12}\ st_1'^{+1}\ s\ell'_{13}$

$t'_1=m'_1=1,\ m'_2=2$



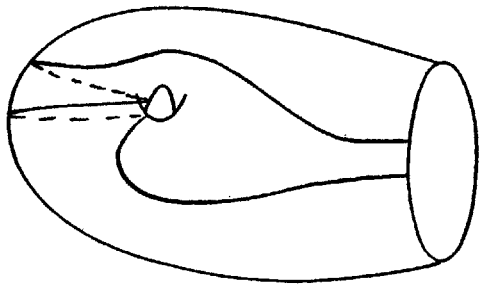
$s\ell_{12}\ st_1^{-1}\ st_1^{-1}\ s\ell_{13}$

$m_2=2,\ m_1=1,\ t_1=-2$



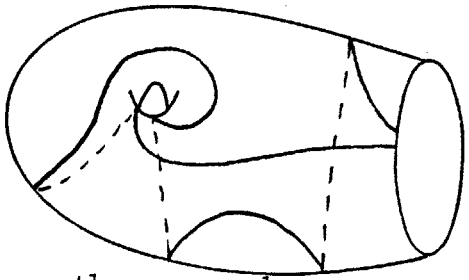
$s\ell'_{12}\ st_1'^{+1}\ s\ell'_{23}\ s\ell'_{13}$

$m'_2=m'_1=2,\ t'_1=1$



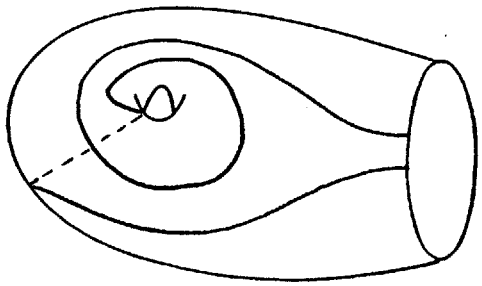
$s\ell_{12}\ st_1^{+1}\ st_1^{+1}\ s\ell_{13}$

$m_2=t_1=2,\ m_1=1$



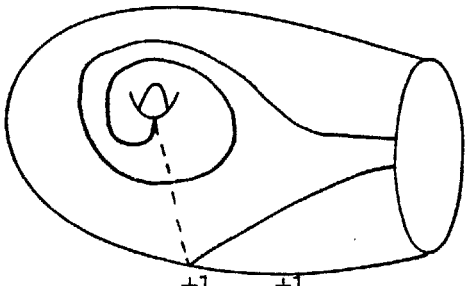
$st_2'^{+1}\ s\ell'_{13}\ st_1'^{-1}\ s\ell'_{23}\ s\ell'_{12}$

$t'_2=1,\ t'_1=-1,\ m'_1=m'_2=2$



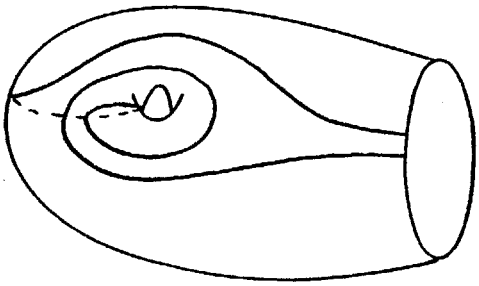
$s\ell_{12}\ s\ell_{23}\ st_1^{-1}\ s\ell_{13}$

$m_1=m_2=2,\ t_1=-1$



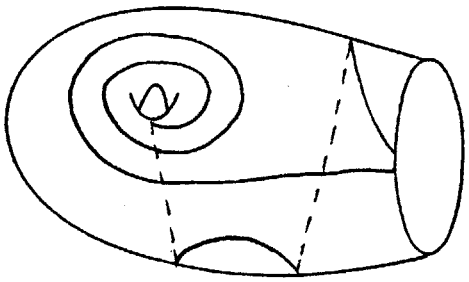
$s\ell'_{12}\ st_1'^{+1}\ st_1'^{+1}\ s\ell'_{13}$

$t'_1=m'_2=2,\ m'_1=1$



$s\ell_{12}\ st_1^{+1}\ s\ell_{23}\ s\ell_{13}$

$t_1=1,\ m_2=m_1=2$



$s\ell'_{12}\ st_1'^{-1}\ st_1'^{-1}\ s\ell'_{13}\ st_2''^{+1}$

$m'_1=t'_2=1,\ t'_1=-2,\ m'_2=2$

For the second elementary transformation, we check eight cases of A-length three and seven cases of A-length four. We use the symmetry of rotation-by-π about the line L in Figure B.1 to avoid considering cases whenever possible. Moreover, in case of A-length four, we do not consider multiple arcs with $|t_1| = 2$ since the considerations of Section 7 apply to this setting.

In the following diagrams, we indicate the A-coordinates and depict a good representative on the left; on the right, we give the same data

for the basis A". We omit mention of any parameter values that are equal to zero.

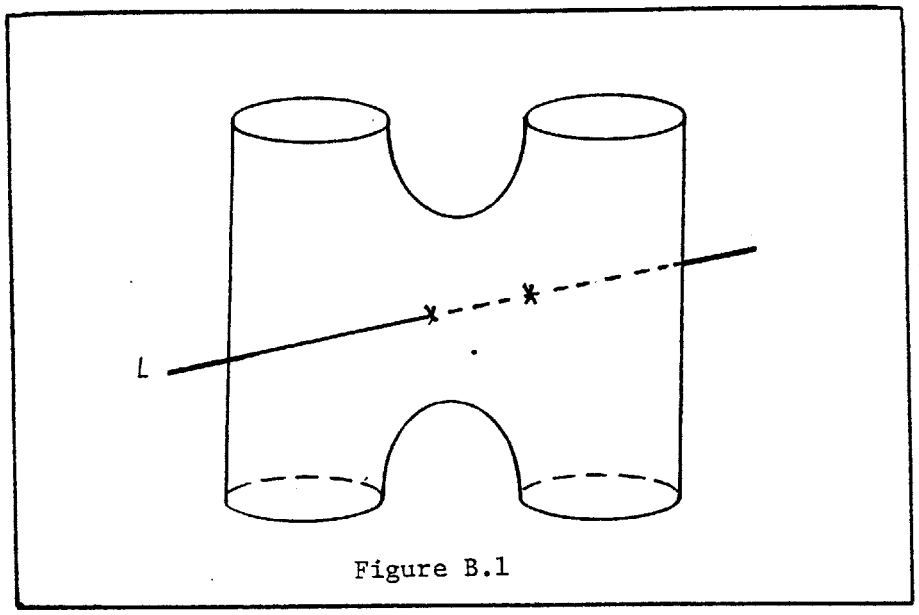
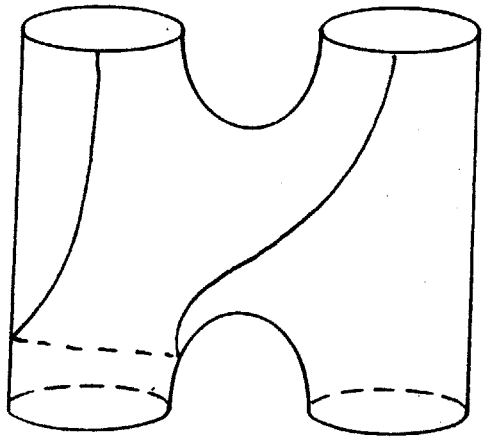
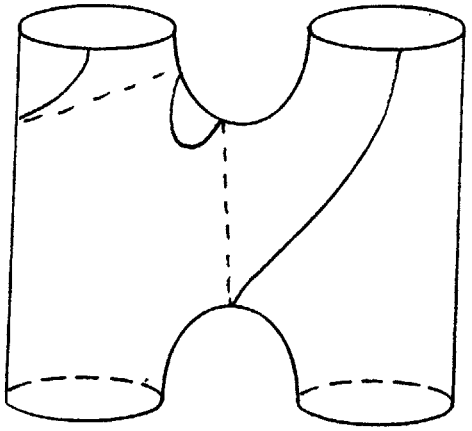


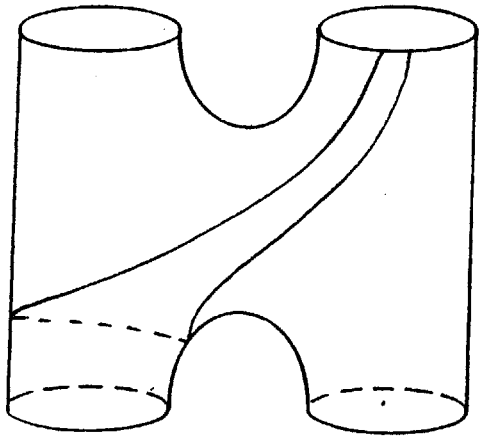
Figure B.1



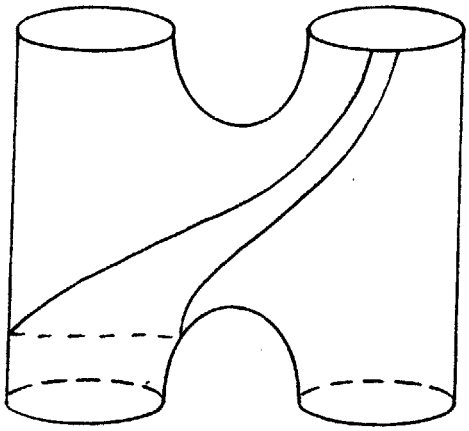
$s\ell_{13}\ sk_{11}^{-}\ s\ell_{12}$
$m_1=2,\ m_2=m_3=1$



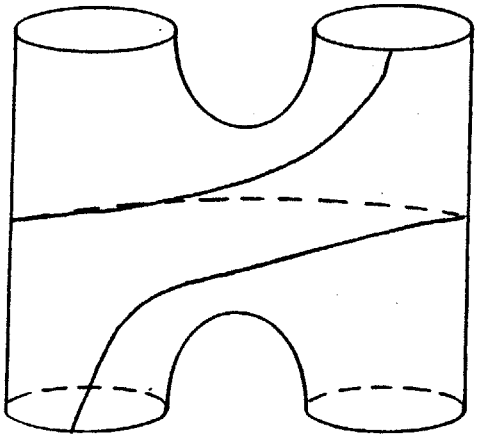
$st_2''^{+1}\ s\ell_{12}''\ st_1''^{-1}\ sk_{13}''$
$t_1''=-1,\ t_2''=m_1''=m_2''=m_3''=1$



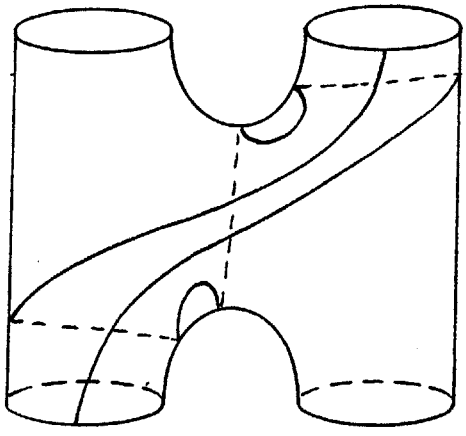
$s\ell_{12}\ sk_{11}^{\pm}\ s\ell_{12}$
$m_1=m_3=2$



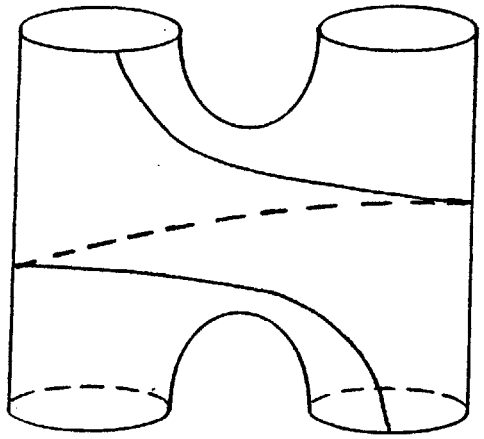
$sk''_{13}\ s\ell''_{11}\ sk''_{13}$
$m''_1=m''_3=2$



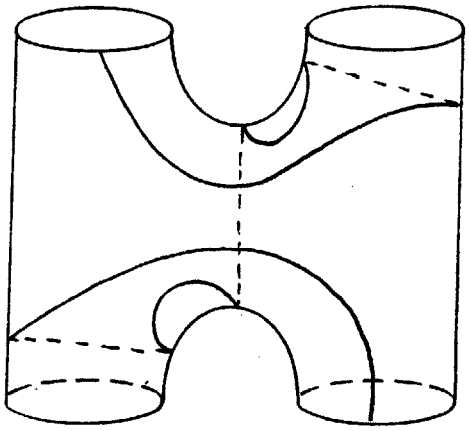
$s\ell_{12}\ st_1^{+1}\ sk_{12}$
$m_1=m_3=m_4=t_1=1$



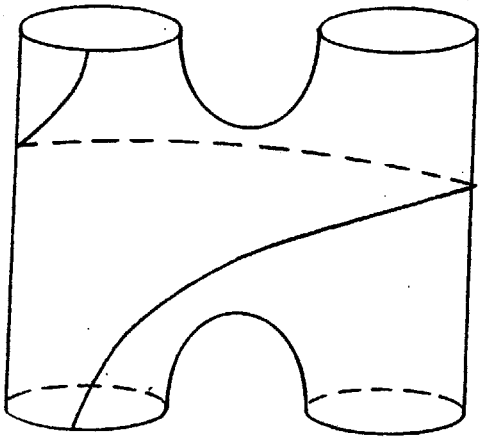
$sk''_{13}\ s\ell''_{11}\ st''^{+1}_1\ sk''_{11}\ s\ell''_{13}$
$m''_1=3,\ m''_3=m''_4=t''_1=1$



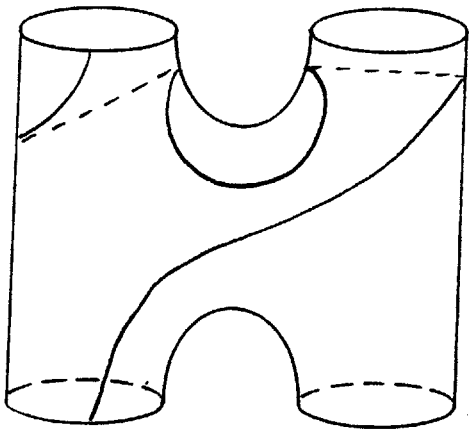
$s\ell_{13}\ st_1^{-1}\ sk_{13}$
$m_1=m_2=m_5=1,\ t_1=-1$



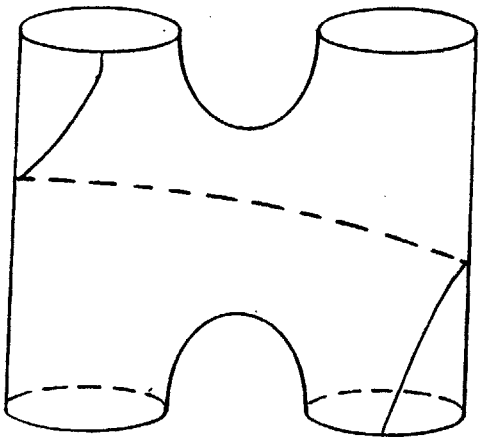
$s\ell''_{12}\ sk''_{11}\ st''^{+1}_1\ s\ell''_{11}\ sk''_{12}$
$m''_1=3,\ t''_1=m''_2=m''_5=1$



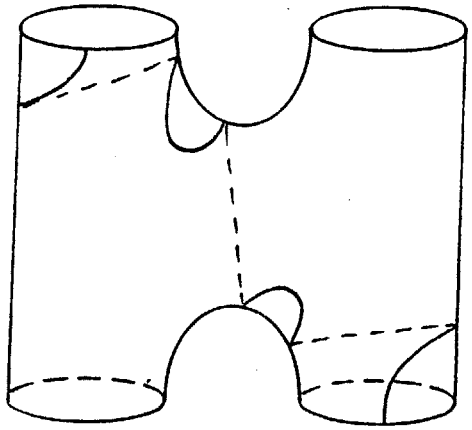
$s\ell_{13}\ st_1^{+1}\ sk_{12}$
$m_1=m_2=m_4=1,\ t_1=1$



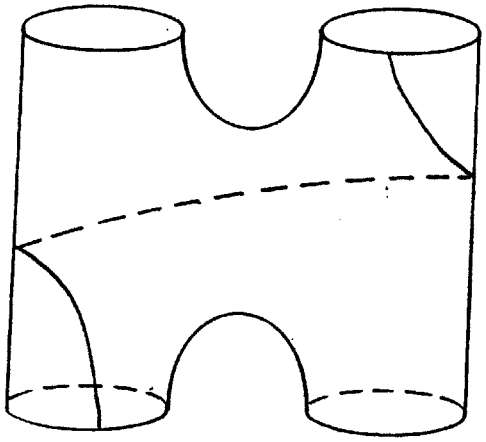
$st''^{+1}_2\ s\ell''_{12}\ sk''_{11}\ s\ell''_{13}$
$m''_2=m''_4=t''_2=1,\ m''_1=2$



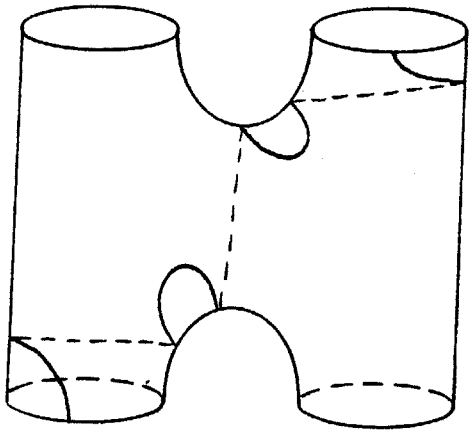
$s\ell_{13}\ st_1^{+1}\ sk_{13}$
$m_1=m_2=m_5=t_1=1$



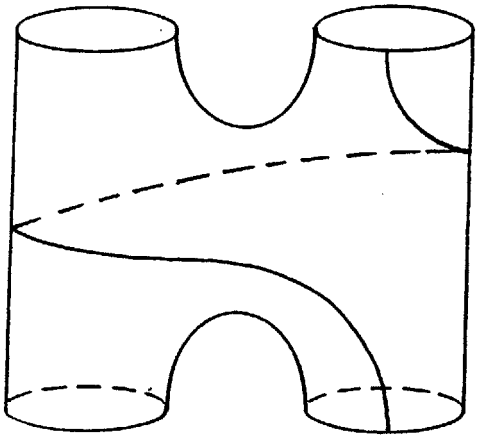
$st_2''^{+1}\ s\ell_{12}''\ t_1''^{-1}\ sk_{12}''\ st_5''^{+1}$
$m_1''=m_2''=m_5''=t_2''=t_5''=1,\ t_1''=-1$



$s\ell_{12}\ st_1^{-1}\ sk_{12}$
$m_1=m_3=m_4=1,\ t_1=-1$

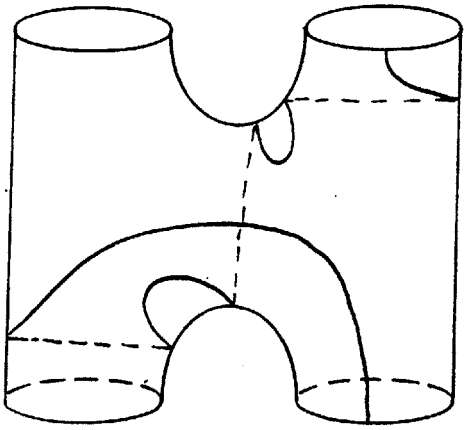


$st_3''^{-1}\ sk_{13}''\ st_1''^{+1}\ s\ell_{12}''\ st_4''^{-1}$
$t_1''=m_1''=m_3''=m_4''=1,\ t_3''=t_4''=-1$



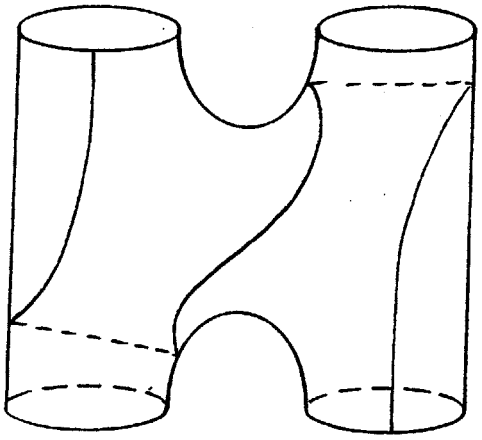
$s\ell_{12}\ st_1^{-1}\ sk_{13}$

$m_1=m_3=m_5=1,\ t_1=-1$



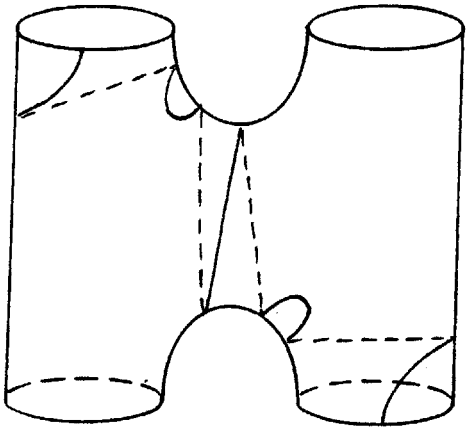
$st''^{-1}_3\ sk''_{13}\ st''^{+1}_1\ s\ell''_{11}\ sk''_{12}$

$m''_3=m''_5=t''_1=1,\ m''_1=3,\ t''_3=-1$



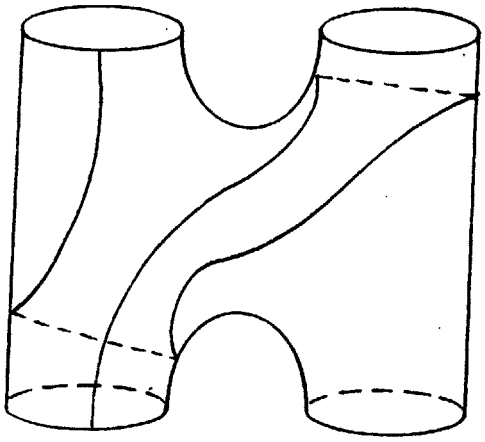
$s\ell_{13}\ sk^{-}_{11}\ s\ell^{+}_{11}\ sk_{13}$

$m_1=3,\ m_2=m_5=1$

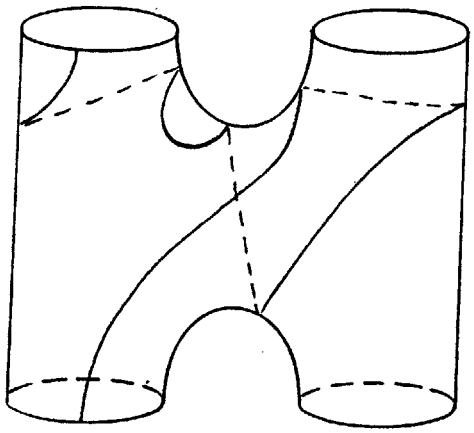


$st''^{+1}_2\ s\ell''_{12}\ st''^{-1}_1\ st''^{-1}_1\ sk''_{12}\ st''^{+1}_5$

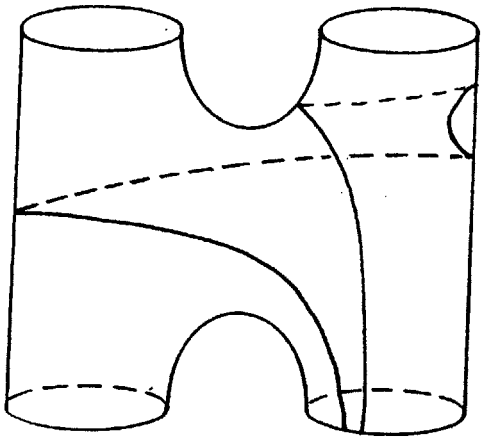
$m''_2=m''_5=t''_2=t''_5=m''_1=1,\ t''_1=-2$



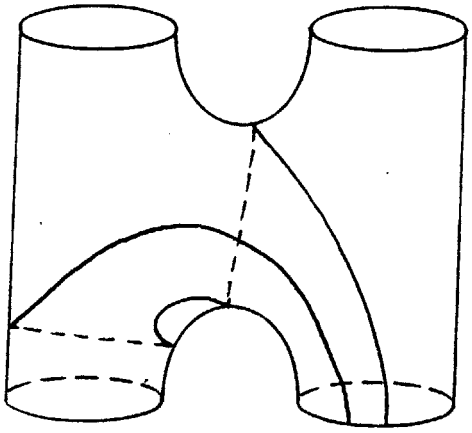
$s\ell_{13}\ sk_{11}^{-}\ s\ell_{11}^{-}\ sk_{12}$
$m_1=3,\ m_2=m_4=1$



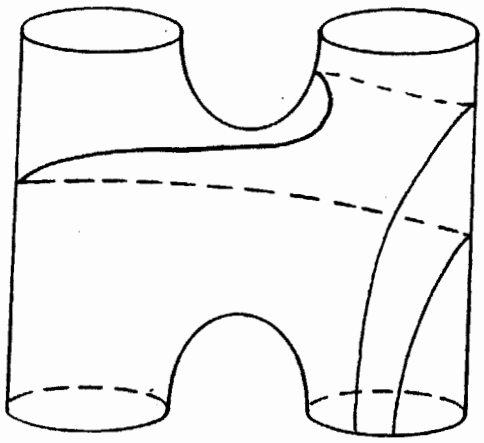
$st_2''^{+1}\ s\ell_{12}''\ st_1''^{-1}\ sk_{11}''\ s\ell_{13}''$
$t_2''=m_2''=m_4''=1,\ m_1''=2,\ t_1''=-1$



$sk_{13}\ st_1^{-1}\ s\ell_{11}^{-}\ sk_{13}$
$t_1=-1,\ m_1=2,\ m_5=2$

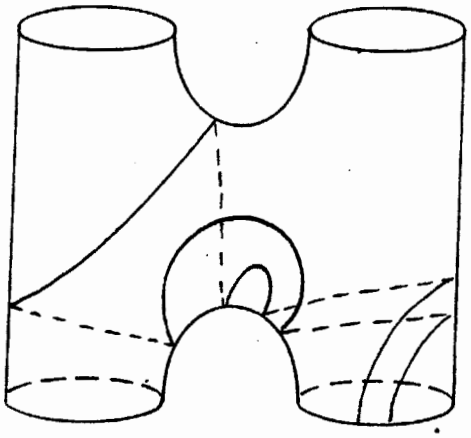


$sk_{12}''\ s\ell_{11}''\ st_1''^{-1}\ sk_{12}''$
$m_5''=m_1''=2,\ t_1''=1$



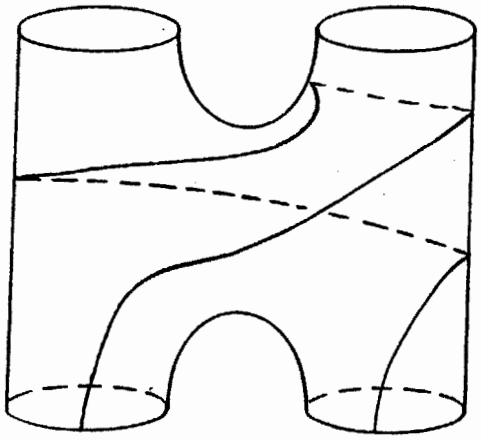
$sk_{13}\ s\ell_{11}^{-}\ st_1^{+1}\ sk_{13}$

$m_1=m_5=2,\ t_1=1$



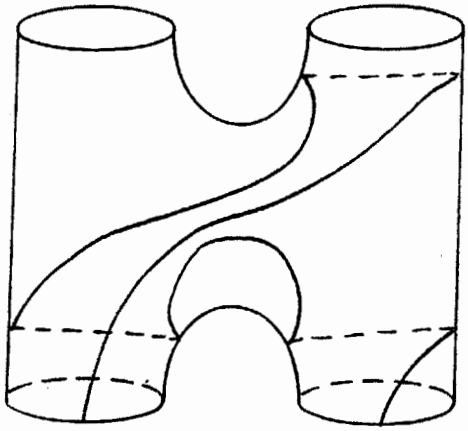
$t_5''^{+1}\ sk''_{12}\ s\ell''_{11}\ st_1''^{-1}\ sk''_{12}\ st_5''^{+1}$

$m_1''=m_5''=t_5''=2,\ t_1''=-1$



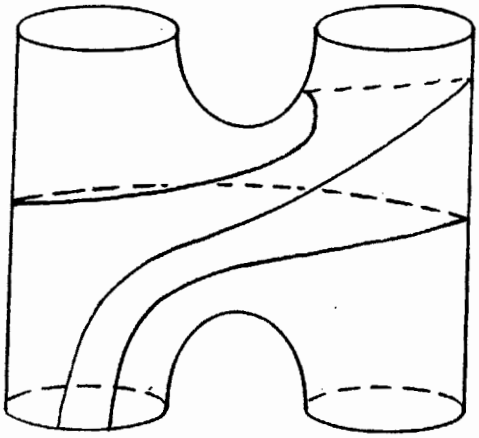
$sk_{12}\ s\ell_{11}^{-}\ st_1^{+1}\ sk_{13}$

$m_1=t_1=m_4=m_5=1$

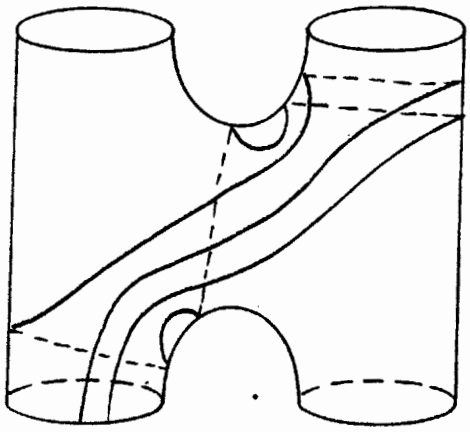


$s\ell''_{13}\ sk''_{11}\ s\ell''_{11}\ sk''_{12}\ st_5''^{+1}$

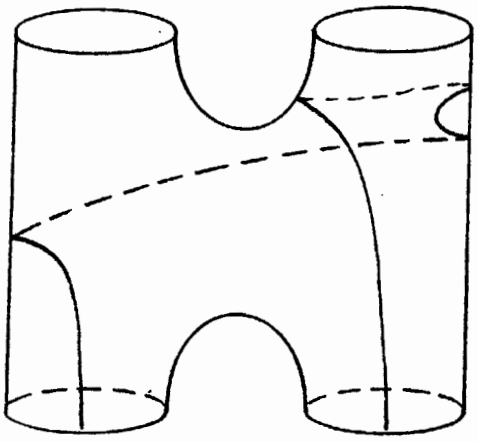
$t_5''=m_4''=m_5''=1,\ m_1''=3$



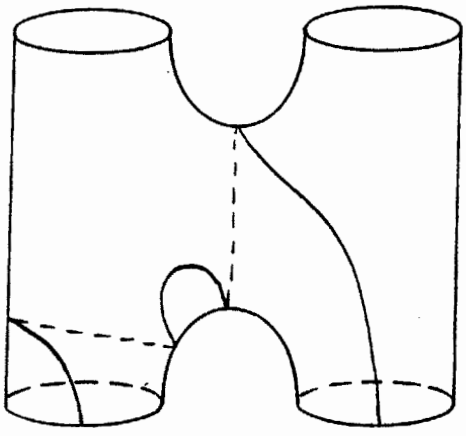
$sk_{12}\ s\ell_{11}^{-}\ t_1^{+1}\ sk_{12}$
$m_1=m_4=2,\ t_1=1$



$s\ell''_{13}\ sk''_{11}\ s\ell''_{11}\ st''_1{}^{+1}\ sk''_{11}\ s\ell''_{13}$
$m''_4=2,\ t''_1=1,\ m''_1=4$



$sk_{12}\ st_1^{-1}\ s\ell_{11}^{-}\ sk_{13}$
$m_1=2,\ m_4=m_5=1$



$st''_4{}^{-1}\ s\ell''_{13}\ st''_1{}^{+1}\ sk''_{12}$
$m''_4=m''_5=m''_1=t''_1=1,\ t''_4=-1$

APPENDIX C

```
$CONTROL USLINIT,NOSOURCE
      PROGRAM TWIST
C ***********************************************************************
C        THIS CODE PERFORMS THE COMPUTATION OF THE ACTION
C  OF MAPPING CLASSES ON CURVES AND ARCS IN CLOSED
C  ORIENTED SURFACES AS DESCRIBED IN MY THESIS.  THE
C  PARAMETRIZATION OF CURVES AND ARCS DEPENDS ON
C  VARIOUS CHOICES, INCLUDING THE CHOICE OF A PANTS
C  DECOMPOSITION OF THE SURFACE.  THE STANDARD PANTS
C  DECOMPOSITIONS ARE INDICATED BY THE CURVES NUMBERED
C  1,...,3*GENUS-3 IN FIGURE 1.  I REFER TO MY THESIS
C  FOR THE OTHER CHOICES OF
C  CONVENTION REQUIRED FOR A PARAMETRIZATION.  BRIEFLY,
C  THE CONVENTIONS ARE THAT ZERO INTERSECTION
C  NUMBER IMPLIES NON-NEGATIVE TWISTING NUMBER, AND
C  ARCS IN THE STANDARD PANTS RUNNING FROM A COMPONENT
C  OF THE BOUNDARY BACK TO THE SAME COMPONENT LOOP
C  AROUND TO THE RIGHT.
C        WE CONSIDER THE ACTION OF DEHN TWISTS ALONG
C  THE 5*GENUS-4 CURVES PICTURED IN FIGURE 1.  THESE
C  ARE OF FOUR GEOMETRICAL TYPES:
C     1) TWISTS ALONG A PANTS CURVE
C     2) TWISTS ALONG A CURVE IN THE TORUS MINUS
C        TWO DISCS INTERSECTING TWO PANTS CURVES
C     3) TWISTS ALONG A CURVE IN THE TORUS MINUS
C        ONE DISC INTERSECTING ONE PANTS CURVE
C     4) TWISTS ALONG A CURVE IN THE SPHERE MINUS
C        FOUR DISCS INTERSECTING ONE PANTS CURVE.
C  THE CURVES IN FIGURE 1 ARE NUMBERED SO THAT
C  TWISTS 1,...,3*GENUS-3 ARE OF TYPE 1, TWISTS
C  3*GENUS-2,...,4*GENUS-5 ARE OF TYPE 2, TWISTS
C  4*GENUS-4 AND 4*GENUS-3 ARE OF TYPE 3, AND
C  TWISTS 4*GENUS-2,...,5*GENUS-4 ARE OF TYPE
C  4.  IN GENUS 2, TWISTS 1,2 AND 3 ARE OF TYPE
C  1 AND TWISTS 4 AND 5 OF TYPE 2.
C        THE CODE PROMPTS THE USER FOR THE INPUT
C  DATA AS WILL BE EXPLAINED, AND THE DIMENSTIONS
C  OF ARRAYS ARE SET TO HANDLE GENERA 2,3 AND 4.
C  TO SET DIMENSIONS FOR HIGHER GENERA, ONE NEED
C  ONLY CHANGE THE FIRST DIMENSION OF THE ARRAY
C  NCURVE IN MAIN AND THE DATA VALUE OF MXDIM
C  TO 3*GENUS-3.
C        NCURVE HOLDS THE PARAMETER VALUE OF THE
C  CURVE UNDER CONSIDERATION AT ANY GIVEN STAGE OF
C  THE COMPUTATION.  AT VARIOUS POINTS, THIS PARAMETER
C  VALUE IS RELATIVE TO VARIOUS PANTS DECOMPOSITIONS,
C  AND WE STORE ONLY ONE SET OF PARAMETERS AT A TIME.
C  THUS, WHEN CALLING ONE OF THE ELEMENTARY
C  TRANFORMATIONS, NCURVE HOLDS THE PARAMETER VALUES
C  RELATIVE TO A CERTAIN PANTS DECOMPOSITION, AND
C  UPON RETURN FROM ONE OF THE ELEMENTARY TRANSFORMATIONS,
C  NCURVE HOLDS THE PARAMETER VALUES RELATIVE TO A
C  DIFFERENT PANTS DECOMPOSITION.
C ***********************************************************************
```

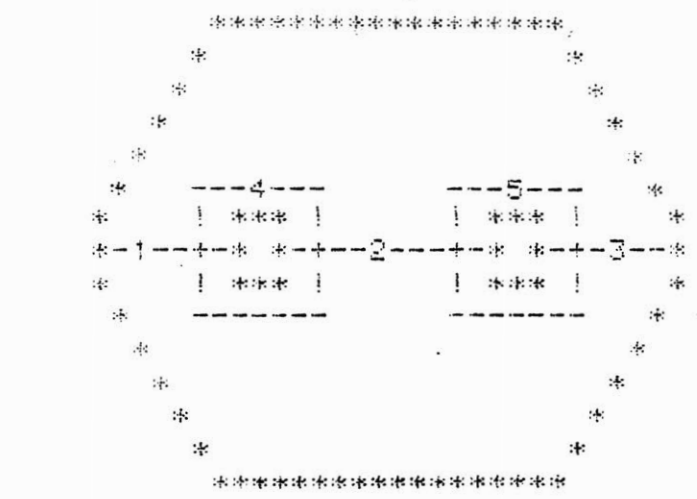


FIGURE 1A

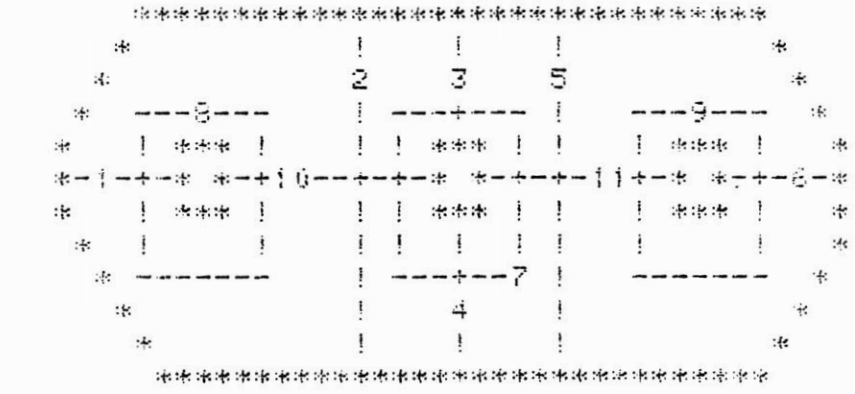


FIGURE 1B

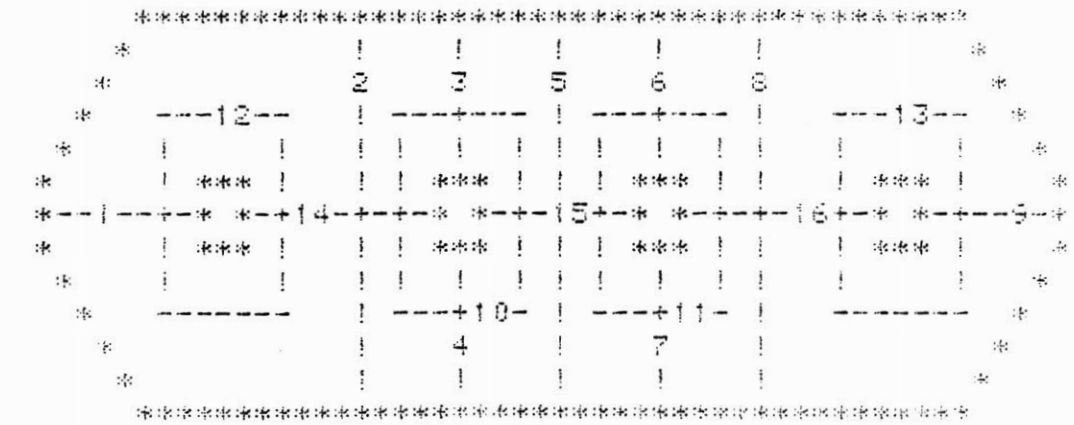


FIGURE 1C

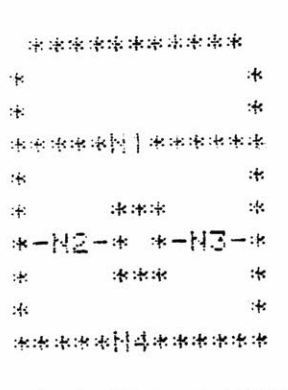


FIGURE 2A

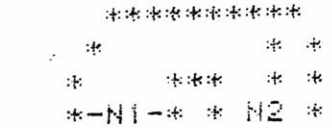


FIGURE 2B

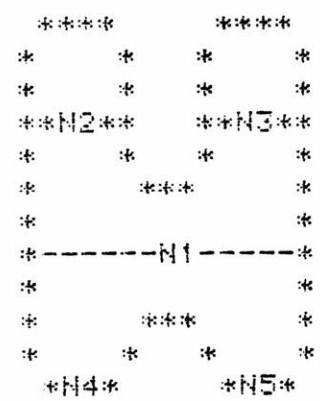


FIGURE 2C

```
C*************************************************************
      COMMON/SENSE/NGENUS,NDIM
      DIMENSION NTAV(25,2)
      DIMENSION NCURVE(9,2)
      CHARACTER*4 CNTRL,STP,ITR,CNT
      DATA STP,ITR,CNT/4HSTOP,4HITER,4HCONT/,MXDIM/9/
C*************************************************************
C  PROMPT THE USER FOR THE GENUS OF THIS RUN.
C*************************************************************
      WRITE(6,56)
      ACCEPT NGENUS
      NDIM=3*NGENUS-3
C*************************************************************
C  PROMPT THE USER FOR THE COORDINATES OF THE CURVE
C  OR ARC UNDER CONSIDERATION.  THE COORDINATES ARE
C  GIVEN BY A (3*GENUS-3)-TUPLE OF (INTERSECTION
C  NUMBER, TWISTING NUMBER), AND THE PANTS CURVES
C  ARE NUMBERED AS IN FIGURE 1.
C*************************************************************
   99 WRITE(6,57)
      ACCEPT ((NCURVE(I,J),J=1,2),I=1,NDIM)
C*************************************************************
C  PROMPT THE USER FOR THE NUMBER OF LETTERS IN THE
C  WORD UNDER CONSIDERATION.  THE MAXIMUM WORD LENGTH
C  IS 25 LETTERS.
C*************************************************************
      WRITE(6,58)
      ACCEPT NTWIST
```

```
C  ***************************************************
C   PROMPT THE USER FOR THE WORD.  WORDS ARE READ FROM
C   LEFT TO RIGHT, AND ARE GIVEN BY A (NTWIST)-TUPLE
C   OF (CURVE NUMBER, EXPONENT), WHERE NTWIST IS THE
C   NUMBER OF LETTERS IN THE WORD.
C  ***************************************************
      WRITE(6,59)
      ACCEPT ((NTAV(I,J),J=1,2),I=1,NTWIST)
C  ***************************************************
C   THIS IS THE MAIN LOOP OVER THE LETTERS IN THE WORD.
C   THE GEOMETRICAL TYPE OF THE LETTER IS DETERMINED,
C   AND THE CALL TO THE APPROPRIATE SUBROUTINE IS
C   MADE.
C  ***************************************************
  100 DO 01 I=1,NTWIST
      NTAU=NTAV(I,1)
      NEXP=NTAV(I,2)
      IF(NTAU.LE.NDIM) CALL TWIST1(NTAU,NEXP,MXDIM,NCURVE)
      IF((NTAU.LE.4*NGENUS-5.AND.NTAU.GE.3*NGENUS-2)
     1.OR.(NGENUS.EQ.2.AND.(NTAU.EQ.4.OR.NTAU.EQ.5)))
     1CALL TWIST2(NTAU,NEXP,MXDIM,NCURVE)
      IF((NTAU.EQ.4*NGENUS-4.OR.NTAU.EQ.4*NGENUS-3)
     1.AND.(NGENUS.NE.2))
     1CALL TWIST3(NTAU,NEXP,MXDIM,NCURVE)
      IF(NTAU.GE.4*NGENUS-2.AND.NTAU.LE.5*NGENUS-4)
     1CALL TWIST4(NTAU,NEXP,MXDIM,NCURVE)
   01 CONTINUE
C  ***************************************************
C   OUTPUT THE COODINATES OF THE IMAGE CURVE AS A
C   (3*GENUS-3)-TUPLE OF (INTERSECTION NUMBER,
C   TWISTING NUMBER).
C  ***************************************************
      WRITE(6,55) (I,(NCURVE(I,J),J=1,2),I=1,NDIM)
C  ***************************************************
C   PROMPT THE USER FOR THE NEXT OPERATION.  THERE ARE
C   THREE OPTIONS: STOP, ITERATE, OR CONTINUE.  THE
C   STOP OPTION ENDS THE RUN.  THE ITERATE OPTION
C   APPLIES THE WORD PREVIOUSLY ENTERED TO THE IMAGE
C   OF THE CURVE UNDER THE PREVIOUS APPLICATION OF THIS
C   WORD.  THE CONTINUE OPTION ALLOWS THE USER TO
C   PERFORM A NEW COMPUTATION, AND THE USER WILL BE
C   PROMPTED FOR THE COORDINATES AND THE WORD, AS
C   BEFORE.  NOTE THAT THE CONTINUE OPTION ASSUMES
C   THE SAME GENUS AS THE PREVIOUS COMPUTATION.
C  ***************************************************
      WRITE(6,62)
      ACCEPT CNTRL
   62 FORMAT(28H CONTINUE, STOP, OR ITERATE?  )
   61 FORMAT(A4)
      IF (CNTRL.EQ.STP) GO TO 101
      IF (CNTRL.EQ.CNT) GO TO 99
      IF (CNTRL.EQ.ITR) GO TO 100
   52 FORMAT(I5,I5)
   53 FORMAT(I5)
   54 FORMAT(I5,I5)
   55 FORMAT(I5,15H                ,I5,5H      ,I5)
   56 FORMAT(7H GENUS?)
   57 FORMAT(8H COORDS?)
   58 FORMAT(27H NUMBER OF LETTERS IN WORD?)
   59 FORMAT(19H (LETTER,EXPONENT)?)
  101 STOP
      END
```

```
      SUBROUTINE TWIST1(NTAU,NEXP,MXDIM,NCURVE)
C ***************************************************
C  THIS SUBROUTINE COMPUTES THE LINEAR ACTION OF A
C  TWIST ALONG A PANTS CURVE.
C ***************************************************
      COMMON/SENSE/NGENUS,NDIM
      DIMENSION NCURVE(MXDIM,2)
      NCURVE(NTAU,2)=NCURVE(NTAU,2)+NEXP*NCURVE(NTAU,1)
      RETURN
      END
      SUBROUTINE TWIST2(NTAU,NEXP,MXDIM,NCURVE)
C ***************************************************
C  THIS SUBROUTINE COMPUTES THE ACTION OF A TWIST
C  OF GEOMETRICAL TYPE TWO.
C ***************************************************
      COMMON/SENSE/NGENUS,NDIM
      DIMENSION NCURVE(MXDIM,2)
C ***************************************************
C  THE VALUE OF NTAU, WHICH IS THE NUMBER OF THE CURVE
C  ON WHICH TO PERFORM A DEHN TWIST, DETERMINES THE
C  NUMBERS OF THE PANTS CURVES INVOLVED IN THE
C  TRANSFORMATION.  WE FIRST COMPUTE THE NUMBERS
C  OF THE PANTS CURVES IN THE TORUS MINUS TWO DISCS,
C  WHICH ARE STORED IN N1, N2, N3 AND N4, ORDERED
C  FROM TOP TO BOTTOM AND LEFT TO RIGHT, AS IN FIGURE 2A.
C ***************************************************
      NP=(NTAU-NDIM-1)*3
      N1=2+NP
      N2=4+NP
      N3=3+NP
      N4=5+NP
      IF(NGENUS.NE.2) GO TO 01
      IF(NTAU.EQ.5) GO TO 02
      N1=3
      N2=1
      N3=2
      N4=3
      GO TO 01
   02 N1=1
      N2=2
      N3=3
      N4=1
   01 CONTINUE
C ***************************************************
C  THE DEHN TWIST IS THE CONJUGATE OF THE LINEAR
C  MAP IN TWIST1 BY A COMPOSITION OF THE ELEMENTARY
C  TRANSFORMATIONS.  THE ARGUMENTS OF ELT1 AND ELT2
C  ARE EXPLAINED BELOW.
C ***************************************************
      CALL ELT2(N2,N4,N1,N3,N3,-1,MXDIM,NCURVE)
      CALL ELT1(N3,N2,+1,MXDIM,NCURVE)
      CALL TWIST1(N3,NEXP,MXDIM,NCURVE)
      CALL ELT1(N3,N2,-1,MXDIM,NCURVE)
      CALL ELT2(N2,N4,N1,N3,N3,+1,MXDIM,NCURVE)
      RETURN
      END
```

```
      SUBROUTINE TWIST3(NTAU,NEXP,MXDIM,NCURVE)
C ***************************************************
C  THIS SUBROUTINE COMPUTES THE ACTION OF A TWIST
C  OF GEOMETRICAL TYPE THREE.
C ***************************************************
      COMMON/SENSE/NGENUS,NDIM
      DIMENSION NCURVE(MXDIM,2)
C ***************************************************
C  WE FIRST COMPUTE THE PANTS CURVES INVOLVED IN
C  THE TRANSFORMATION FROM THE VALUE OF NTAU.  N1
C  STORES THE NUMBER OF THE PANTS CURVE INTERIOR
C  TO THE TORUS MINUS A DISC, AND N2 STORES THE
C  PANTS CURVE THAT IS THE BOUNDARY COMPONENT OF
C  THE TORUS MINUS A DISC, AS IN FIGURE 2B.
C ***************************************************
      IF(NTAU.NE.4*NGENUS-4) GO TO 02
      N1=1
      N2=2
      GO TO 01
   02 N1=NDIM
      N2=NDIM-1
   01 CONTINUE
C ***************************************************
C  THE DEHN TWIST IS THE CONJUGATE OF THE LINEAR
C  MAP IN TWIST1 BY THE FIRST ELEMENTATY
C  TRANSFORMATION.
C ***************************************************
      CALL ELT1(N1,N2,+1,MXDIM,NCURVE)
      CALL TWIST1(N1,NEXP,MXDIM,NCURVE)
      CALL ELT1(N1,N2,-1,MXDIM,NCURVE)
      RETURN
      END
      SUBROUTINE TWIST4(NTAU,NEXP,MXDIM,NCURVE)
C ***************************************************
C  THIS SUBROUTINE COMPUTES THE ACTION OF A TWIST
C  OF GEOMETRICAL TYPE FOUR.
C ***************************************************
      COMMON/SENSE/NGENUS,NDIM
      DIMENSION NCURVE(MXDIM,2)
C ***************************************************
C  WE FIRST COMPUTE THE PANTS CURVES INVOLVED IN
C  THE TRANSFORMATION.  N1 STORES THE PANTS CURVE
C  INTERIOR TO THE SPHERE MINUS FOUR DISCS, AND
C  N2, N3, N4 AND N5 STORE THE PANTS CURVES THAT
C  ARE BOUNDARY COMPONENTS OF THE SPHERE MINUS
C  FOUR DISCS FROM LEFT TO RIGHT AND TOP TO BOTTOM,
C  AS IN FIGURE 2C.
C ***************************************************
      NP=3*(NTAU-4*NGENUS+1)
      N1=5+NP
      N2=4+NP
      N3=3+NP
      N4=7+NP
      N5=6+NP
      IF(NTAU.EQ.4*NGENUS-2) GO TO 02
      IF(NTAU.EQ.5*NGENUS-4) GO TO 03
      GO TO 01
   02 N1=2
      N2=1
      N3=1
      N4=4
      N5=3
```

```
      GO TO 01
   03 N1=NDIM-1
      N2=NDIM-2
      N3=NDIM-3
      N4=NDIM
      N5=NDIM
   01 CONTINUE
C ***************************************************
C  THE DEHN TWIST IS THE CONJUGATE OF THE LINEAR
C  MAP IN TWIST1 BY THE SECOND ELEMENTARY
C  TRANSFORMATION.
C ***************************************************
      CALL ELT2(N1,N2,N3,N4,N5,+1,MXDIM,NCURVE)
      CALL TWIST1(N1,NEXP,MXDIM,NCURVE)
      CALL ELT2(N1,N2,N3,N4,N5,-1,MXDIM,NCURVE)
      RETURN
      END
      SUBROUTINE ELT1(N1,N2,ISGN,MXDIM,NCURVE)
C ***************************************************
C  ELT1 PERFORMS THE COMPUTATION OF THE FIRST
C  ELEMENTARY TRANSFORMATION ON THE TORUS MINUS
C  A DISC.  N1 IS THE PANTS CURVE INTERIOR TO
C  THE TORUS MINUS A DISC, N2 IS THE PANTS CURVE
C  THAT IS THE BOUNDARY COMPONENT OF THE TORUS
C  MINUS A DISC (SEE FIGURE 2B), AND ISGN IS A FLAG THAT DETERMINES
C  WHETHER TO COMPUTE THE FIRST ELEMENTARY
C  TRANSFORMATION OR ITS INVERSE.  ISGN=1 MEANS
C  COMPUTE THE FIRST ELEMENTARY TRANSFORMATION, AND
C  ISGN=-1 MEANS COMPUTE THE INVERSE.
C ***************************************************
      COMMON/SENSE/NGENUS,NDIM
      DIMENSION NCURVE(MXDIM,2)
      DIMENSION L(6)
      NT=NCURVE(N1,2)
      CALL PARM(N2,N1,N1,L,MXDIM,NCURVE)
      NEW1=MAX0(L(2)-IABS(NT),0)
      NEW2=L(2)-NEW1
      NCURVE(N1,2)=ISIGN(1,-NT)*(L(5)+NEW2)
      IF(ISGN.EQ.+1) NCURVE(N2,2)=NCURVE(N2,2)+L(1)
     1+MAX0(0,MIN0(NEW2,NT))
      IF(ISGN.EQ.-1)
     1NCURVE(N2,2)=NCURVE(N2,2)-NEW1-MAX0(0,MIN0(-NT,NEW2))
      L(2)=L(1)+NEW2
      L(1)=NEW1
      L(3)=L(2)
      L(4)=0
      L(5)=IABS(NT)-NEW2
      L(6)=0
      NCURVE(N1,1)=L(2)+L(5)
      NCURVE(N2,1)=2*L(1)+L(2)+L(3)
      RETURN
      END
      SUBROUTINE PARM(M1,M2,M3,L,MXDIM,NCURVE)
C ***************************************************
C  THIS SUBROUTINE STUFFS THE ARRAY L WITH THE NUMBER
C  OF ARCS PARALLEL TO THE VARIOUS CANONICAL PIECES
C  IN THE PAIR OF PANTS WITH BOUNDARY COMPONENTS
C  M1,M2,M3.  L(6) IS STUFFED WITH L11,L12,L13,L22,L23,
C  L33, IN THIS ORDER.
C ***************************************************
```

```
      DIMENSION L(6)
      COMMON/SENSE/NGENUS,NDIM
      DIMENSION NCURVE(MXDIM,2)
      N1=NCURVE(M1,1)
      N2=NCURVE(M2,1)
      N3=NCURVE(M3,1)
      P1=N1-N2-N3
      P2=N2-N1-N3
      P3=N3-N1-N2
      IF(P1.GT.0) GO TO 01
      IF(P2.GT.0) GO TO 02
      IF(P3.GT.0) GO TO 03
      L(1)=0
      L(2)=(N1+N2-N3)/2
      L(3)=(N1+N3-N2)/2
      L(4)=0
      L(5)=(N2+N3-N1)/2
      L(6)=0
      GO TO 04
   01 L(1)=(N1-N2-N3)/2
      L(2)=N2
      L(3)=N3
      L(4)=0
      L(5)=0
      L(6)=0
      GO TO 04
   02 L(1)=0
      L(2)=N1
      L(3)=0
      L(4)=(N2-N1-N3)/2
      L(5)=N3
      L(6)=0
      GO TO 04
   03 L(1)=0
      L(2)=0
      L(3)=N1
      L(4)=0
      L(5)=N2
      L(6)=(N3-N1-N2)/2
   04 CONTINUE
      RETURN
      END
C ***************************************************
C  THE FOLLOWING THREE FUNCTIONS ARE A CONVENIENCE IN
C  THE FORMULAE OF THE SUBROUTINE ELT2.
C ***************************************************
      FUNCTION NTRIP(I,J,K)
      NTRIP=MAX0(0,MIN0(I,MIN0(J,K)))
      RETURN
      END
      FUNCTION NQUAD(I,J,K,L)
      NQUAD=MAX0(0,MIN0(J,MIN0(I,MIN0(K,L))))
      RETURN
      END
      FUNCTION NDUB(I,J)
      NDUB=MAX0(0,MIN0(I,J))
      RETURN
      END
```

```
      SUBROUTINE ELT2(N1,N2,N3,N4,N5,ISGN,MXDIM,NCURVE)
C ***************************************************
C  ELT2 PERFORMS THE COMPUTATION OF THE SECOND
C  ELEMENTARY TRANSFORMATION ON THE SPHERE MINUS FOUR
C  DISCS.  N1 IS THE PANTS CURVE INTERIOR TO THE SPHERE
C  MINUS FOUR DISCS, AND N2,N3,N4,N5 ARE THE BOUNDARY
C  COMPONENTS OF THE SPHRE MINUS FOUR DISCS FROM LEFT
C  TO RIGHT AND TOP TO BOTTOM (SEE FIGURE 2C).  ISGN IS A FLAG THAT
C  DETERMINES WHETHER TO PERFORM THE SECOND ELEMENTARY
C  TRANSFORMATION OR ITS INVERSE.  ISGN=1 MEANS COMPUTE
C  THE SECOND ELEMENTARY TRANSFORMATION, AND ISGN=-1
C  MEANS COMPUTE THE INVERSE.
C  IN CASE ISGN=1, THE ARRAYS L AND K HOLD
C  THE CANONICAL PIECES IN THE UPPER AND LOWER PAIRS
C  OF PANTS, RESPECTIVELY.  THE ARRAYS NL AND NK
C  HOLD THE CANONICAL PIECES IN THE LEFT AND RIGHT
C  PAIRS OF PANTS, RESPECTIVELY.
C ***************************************************
      COMMON/SENSE/NGENUS,NDIM
      DIMENSION NCURVE(MXDIM,2)
      DIMENSION L(6),K(6),NK(6),NL(6)
      IF(ISGN.EQ.-1) GO TO 21
      CALL PARM(N1,N3,N2,L,MXDIM,NCURVE)
      CALL PARM(N1,N4,N5,K,MXDIM,NCURVE)
      GO TO 22
   21 CALL PARM(N1,N2,N4,K,MXDIM,NCURVE)
      CALL PARM(N1,N5,N3,L,MXDIM,NCURVE)
   22 CONTINUE
      NT=NCURVE(N1,2)
      LAM=L(1)+NT
      KAP=K(1)+NT
      NK(1)=K(4)+L(6)+MAX0(0,LAM-K(3))+MAX0(0,-LAM-L(2))
      NK(4)=NTRIP(LAM,L(1),K(3)-L(2)-LAM)
      NK(6)=NTRIP(-LAM,K(1),L(2)-K(3)+LAM)
      NK(5)=NQUAD(K(3),L(2),K(3)-LAM,L(2)+LAM)
      NL(1)=K(6)+L(4)+MAX0(0,KAP-L(3))+MAX0(0,-KAP-K(2))
      NL(4)=NTRIP(KAP,K(1),L(3)-K(2)-KAP)
      NL(6)=NTRIP(-KAP,L(1),K(2)-L(3)+KAP)
      NL(5)=NQUAD(L(3),K(2),L(3)-KAP,K(2)+KAP)
      NT2=L(6)+NDUB(L(3)-NL(5)-2*NL(4),KAP+NL(6)-NL(4))
      NT3=-NK(6)-NDUB(-LAM-NK(6)+NK(4),L(2)-NK(5)-2*NK(6))
      NT4=-NL(6)-NDUB(-KAP-NL(6)+NL(4),K(2)-NL(5)-2*NL(6))
      NT5=K(6)+NDUB(K(3)-NK(5)-2*NK(4),LAM+NK(6)-NK(4))
      IIEPS=-LAM-KAP-NL(6)+NL(4)-NK(6)+NK(4)
      IEPS=ISIGN(1,IIEPS)
      IF (IIEPS.EQ.0.AND.L(2)-2*NK(6)-NK(5).NE.0) IEPS=-IEPS
      NT1=K(4)+L(4)-(NL(1)+NK(1))-IEPS*(NT+NL(6)+NK(6))+
     1L(6)+K(6)
      IF (ISGN.EQ.-1) GO TO 31
      NCURVE(N2,2)=NT2+NCURVE(N2,2)
      NCURVE(N3,2)=NT3+NCURVE(N3,2)
      NCURVE(N4,2)=NT4+NCURVE(N4,2)
      NCURVE(N5,2)=NT5+NCURVE(N5,2)
      NCURVE(N1,2)=NT1-NT2-NT5
      GO TO 32
   31 NCURVE(N2,2)=NT4+NCURVE(N2,2)
      NCURVE(N3,2)=NT2+NCURVE(N3,2)
      NCURVE(N4,2)=NT5+NCURVE(N4,2)
      NCURVE(N5,2)=NT3+NCURVE(N5,2)
      NCURVE(N1,2)=NT1-NT2-NT5
   32 CONTINUE
```

```
      L3=L(3)
      L4=L(4)
      L5=L(5)
      L6=L(6)
      L2=L(2)
      K2=K(2)
      K3=K(3)
      K4=K(4)
      K5=K(5)
      K6=K(6)
      L(2)=-2*NL(4)-NL(5)+L3+L5+2*L6
      K(2)=-2*NK(4)-NK(5)+K3+K5+2*K6
      L(3)=-2*NL(6)-NL(5)+K2+K5+2*K4
      K(3)=-2*NK(6)-NK(5)+L2+L5+2*L4
      K(1)=NK(1)
      K(4)=NK(4)
      K(5)=NK(5)
      K(6)=NK(6)
      L(1)=NL(1)
      L(4)=NL(4)
      L(5)=NL(5)
      L(6)=NL(6)
      IF (ISGN.EQ.-1) GO TO 41
      NCURVE(N1,1)=2*L(1)+L(2)+L(3)
      NCURVE(N2,1)=2*L(4)+L(2)+L(5)
      NCURVE(N3,1)=K(3)+K(5)+2*K(6)
      NCURVE(N4,1)=L(3)+2*L(6)+L(5)
      NCURVE(N5,1)=K(2)+2*K(4)+K(5)
      GO TO 42
   41 NCURVE(N1,1)=2*L(1)+L(2)+L(3)
      NCURVE(N2,1)=2*L(6)+L(3)+L(5)
      NCURVE(N3,1)=2*L(4)+L(2)+L(5)
      NCURVE(N4,1)=2*K(4)+K(2)+K(5)
      NCURVE(N5,1)=2*K(6)+K(3)+K(5)
   42 RETURN
      END
```

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