INTERDEPENDENCIES BETWEEN OPERATING OPTIONS

by

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Interdependencies Between Operating Options

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Abstract

   This paper presents a computationally feasible technique to value
operating flexibilities in making capital budgeting decisions. We investigate
how the value of a project is affected by the simultaneous introduction of
<u>several</u> operating options. Previous studies have focused on operating options
one at a time.

   A numerical example demonstrates the options to wait to invest, to
abandon, and to temporarily shut down -- first, one at a time and then more
than one at a time. -- As expected, the project value increases with the
introduction of additional options. Adding new options, however, reduces the
value of the previously available options. We also study the impact of adding
new options on the critical boundaries at which existing options are
exercised. These results help sharpen our intuition about the effects of and
interactions between operating options.

   Easiliy implemented on a Personal Computer, the model is sufficiently
general to handle various types of production flexibilities and assumptions
regarding the economic environment. Hence, for the first time, we have
available a quantitative technique that accounts for operating flexibilities
that can be incorporated practically in the capital budgeting process.

1. Introduction

Academics and managers have long been dissatisfied with the inability of conventional capital budgeting techniques, such as discounted cash flow (DCF) methods, to capture the strategic impact of projects.[1] In particular, DCF methods ignore the "operating flexibility" that gives management the option to revise decisions while the project is under way. Examples include decisions to shut down, to abandon, or to change the technology.[2]

In recent years the finance literature has addressed these criticisms by modifying conventional capital budgeting methods to include the impact of operating flexibilities. It is now well known that when investment is irreversible and future market conditions are uncertain, an investment decision must not be based solely on the usual net present value (NPV) rule. An investment expenditure implicitly calls for sacrifice of the option to wait to invest (i.e., to invest instead at a time in the future), so we must treat this lost option value as part of the investment cost.

McDonald and Siegel [1986] show that even with moderate amounts of uncertainty the value of the option to wait to invest can be large, which means that an investment rule ignoring the option value can be grossly in error. Similar adjustments to value are necessary when there are options to abandon (McDonald and Siegel [1986] and Myers and Majd [1983]); options to temporarily shut down (McDonald and Siegel [1985] and Brennan and Schwartz

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1 See, for example, R. Hayes and W. Abernathy, "Managing Our Way to Economic Decline," Harvard Business Review (July-August 1980); and Hayes and D. Garvin, "Managing As If Tomorrow Mattered," Harvard Business Review (May-June 1982).

2 These issues are raised in a series of articles contained in a recent issue of the Midland Corporate Finance Journal, Vol 5, NO. 1 (Spring 1987). _See Myers [1987], Kensinger [1987] and Trigeorges and Mason [1987].

[1985]); options depending on the sequential nature of investment (Majd and Pindyck [1987]); and options to choose capacity (Pindyck [1986]).[3]

All these papers, however, treat options one at a time, thereby precluding possible interaction between their values. This paper provides a general dynamic framework that enables simultaneous evaluation of the effect of many operating options. Each option then becomes a special case of a more general "flexibility" option that allows the firm to choose from an entire menu of available technology modes. Our example illustrates options to wait to invest, to temporarily shut down, and to abandon a project and shows how the value of each option and critical price boundaries are affected by the inclusion of other options.

Section presents a dynamic programming model of flexibility with careful attention paid to the assumptions underlying the exogeneity of uncertainty. In section 3, we show how the flexibility model can be specialized to value a project that offers options to wait to invest, to shut down temporarily, and to abandon. Numerical simulations illustrate these results. Finally, in section 4 we make some concluding remarks.

2. A Model of Flexibility

2.1 Characterizing Exogenous Uncertainty

The value of real options lies in the enhanced ability of the option holder (in this case the firm) to better cope with exogenous uncertainty. Hence, a first step in modeling the flexibility option is to characterize the

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3 A general overview of applying contingent claims valuation models to real investment and production decisions appears in Mason and Merton [1984].

sources and nature of this uncertainty. Traditionally, models of real options treat the project value as the exogenous stochastic variable.[4] The presence of real options, however, influences that project value. If the value of the project without the options follows a particular stochastic process (for example, lognormal), and real options are introduced, the resulting project will no longer follow the same process. In other words, the stochastic dynamics of the project value are endogenously determined by the presence of real options.

Previous researchers handled this problem by assuming that dynamics for a project including an option follow a sufficiently simple stochastic process. Although this procedure can skirt the issue of exogeneity when we consider projects with a single operating option, it is inadequate when there is more than one option involved.

Consider, for example, a project including an option to abandon. In order to value the project with the abandonment option, we characterize the project with the option as following some stochastic process. Now suppose that we add to this project, the option to shut down temporarily. If we characterize the project with the single option (abandonment) as following a lognormal process, the project including both options will no longer follow the same process. Hence, it is not possible to separately identify the effects attributable to the two options.

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4 E.g., Myers and Majd [1983].

In this study we assume that the price of an input or of an output, rather than project value, is the exogenous stochastic variable. For example, firms in many industries face uncertainty in competitively determined energy prices, so the real options for an input-pricetaking firm will not significantly affect the market price of the input.[5] Assuming that prices are exogenous allows us to compute the value of the project as the discounted sum of cash flows.

Suppose that the price, Θ, follows the stochastic process:

$$(1) \qquad d\Theta_t = \alpha(\Theta_t, t)\, dt + \sigma(\Theta_t, t)\, dZ_\Theta,$$

where $\alpha(\Theta_t, t)$ and $\sigma(\Theta_t, t)$ can be functions of both Θ_t and t, and dZ_Θ is a standard Gauss Weiner process.[6] For purposes of numerical simulation we assume that Θ follows a mean reverting process:

$$(2) \qquad d\Theta_t = \mu(\Theta_t - \Theta^\circ)\, dt + \sigma\, \Theta_t\, dZ_\Theta,$$

where Θ° is the mean of Θ. The instantaneous drift term, $(\Theta_t - \Theta^\circ)$, acts as an elastic force that produces mean reversion. The stochastic term dZ_Θ, with variance σ, causes continuous fluctuations about Θ°.

At time t the firm observes the realization of Θ_t and fixes it

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5 Flexibility is of some strategic bargaining value to the firm in its negotiations with input suppliers by becuase it represents a credible threat to use an alternative production mode and switch away from the input. See Kulatilaka and Marks [1987].

6 When we model the price by geometric Brownian motion, the value of the flexibility option can be obtained in closed form in terms of cumulative multinomial distribution. (See Kulatilaka and Marcus [1986].)

contractually for a short period.[7] When contracting arrangements are given

exogenously we can study the effects of changing contract duration on the

value of flexibility. When contract length is under the firm's control, we

can approximate the limiting, continuous-time case by setting the time period

equal to an arbitrarily small value.

In order to discretize the problem, Θ is allowed to take N discrete

values; $\Theta^1,\ldots,\Theta^N$. (In this notation $\Theta_t=\Theta^n$ means that the realization of Θ at

time t takes the value Θ^n.) Define the transition probability that $\Theta(t+1)$ is

state Θ^i given that $\Theta(t)$ was state Θ^j as p_{ij}; i.e. $p_{ij}=\text{prob}(\Theta_{t+1}=\Theta^i/\Theta_t=\Theta^j)$.

When the parameters of the stochastic process are known, the matrix of

transition probabilities can be computed.[8]

2.2. Technology Modes and Value of the Project

We can define the various operating modes of the project using profit

functions. The modes may characterize different production processes (e.g.,

one that fires a steam boiler by natural gas and another that uses fuel oil),

as well as states that describe waiting to invest, shutting down, and

abandoning the project.

Suppose a project consists of M modes with profit functions, π^i,

i=1,...,M. The profit functions are defined for the duration of a single time

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7 In many applications, firms write contracts for purchasing inputs and
outputs. This stylization will accurately depict such firms. Quasi-
contractual arrangements, sluggish price adjustments, and transactions in
forward and futures purchases also are common in output goods.

8 The derivation of the transition probabilities is outlined in an Appendix.

period and have as their arguments input and output prices.[9] As we are considering only one source of uncertainty, we will carry only that price as the argument of the profit function: i.e. the profit function of the ith mode at time t is $\pi^i(\theta_t)$.

The value of a <u>fixed technology</u> project of life T periods that is dedicated to technology type i then can be written as:

$$(3) \qquad V_i(\theta_0) = E_0[\sum_{j=0}^{T} \pi^i(\theta_j) \ \phi^j],$$

where E_0 is the expectation operator conditional on information at time 0 (i.e., on the realization of θ_0), and ϕ is the one period discount factor.

Several caveats regarding the discount rate bear mentioning. If the underlying asset follows an equilibrium growth rate, then we can follow the "risk-neutral equilibrium" approach of Cox and Ross [1976] and compute the present value of the "risk-neutral expectation" discounting at the risk free rate. In this approach, actual expected rates of return are replaced by their certainty equivalents as there would be no risk premium in the risk-neutral equilibrium.

Unlike financial assets, however, production inputs and outputs are not necessarily traded in efficient markets and, their growth rates need not equal the fair rate of return. In such cases we can use an insight from McDonald and Siegel [1984], and replace the drift term of the diffusion process is

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9 In the continuous time case π^t can be defined as the instantaneous flow profit.

replaced by its equilibrium rate of appreciation, and then follow risk-neutral discounting. As fair rates of return are defined with the use of an asset pricing model, though, such an adjustment imposes additional constraints.[10]

When the project is flexible, that is, the firm has the options to switch between modes (to switch from mode i to j) in the process incurring costs δ_{ij}, the value of the project includes implicit contributions becuase of a series of nested compound options attributable to the ability to switch back and forth across operating modes. If θ followed a continuous time lognormal, process and if there were only two operating modes with linear profit functions, then the value of the infinite horizon problem would yield closed-form solutions.[11] The finite horizon problem requires the numerical solution of the resulting partial differential valuation equation (Myers and Majd [1983] and Brennan and Schwartz [1985]).

The finite time problem when cast in discrete time can also be solved in closed form in terms of cumulative multivariate normal distributions (Kulatilaka and Marcus [1987]). The finite time solution to the general problem for which the stochastic process is not lognormal and the project has multiple operating modes, though, becomes impossible to solve in closed form. This paper explains the dynamic programming problem and demonstrates a computationally feasible method to handle this general case.

Consider first the project entering the last period. Suppose that technology mode m had been in use during the period immediately preceding.

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10 See Kulatilaka and Marcus [1986] for details.

11 For examples, see the previously cited papers on real options.

The firm wants to compare the profits over the last period under each operating mode and choose the mode yielding the highest profit. We denote this maximum value of profits, net of switching costs, by $F(\Theta_T, m)$,

(4) $F(\Theta_T, m) = \max_i [\pi^i(\Theta_T) - \delta_{mi}]$

where Θ_T refers to the price realization, m is the mode of the previous period, T-1, δ_{mi} is the cost of switching from mode m to mode i (if i≠m), and $\delta_{mm} = 0$ if the firm continues to operate in mode m.

At periods prior to T, however, the future realizations of Θ are unknown to the firm. For example, consider the mode choice at time T-1 and suppose the mode of use during T-2 was m. The value of the flexible system at T-1 is then the profits over the next period using the mode that maximizes that period's net profits (net of switching costs) plus the expectation (at time T-1) of the project value at T; i.e.,

(5) $F(\Theta_{T-1}, m) = \max_i \{ \pi^i(\Theta_{T-1}) - \delta_{mi} + \phi E_{T-1}[F(\Theta_T, i)] \}$,   m=1,...,M.

The conditional expectations, E_T, are taken over the possible realizations of Θ and are weighted by the transition probabilities, p_{nk} (the probability that $\Theta_{t+1} = \Theta^k$ given that $\Theta_t = \Theta^n$). That is, if the Θ_t realization was Θ^n, then the conditional expectation is calculated as:

(6) $E_t[F(\Theta_{t+1}, m)] = \sum_{k=1}^{N} [F(\Theta_{t+1} = \Theta^k, m) \, p_{nk}]$

Generalizing (5), we can write the dynamic programming equations at time t, when the price realization is Θ^n and when the mode in operation at time t-1 is m, as

(7) $F(\Theta_t = \Theta^n, m) = \underset{i}{\text{Max}} \{ \pi^i - \delta_{mi} + \phi E_t[F(\Theta_{t+1}, i)]\}$ $i, m = 1, \ldots, M$
    $t = 1, \ldots, T,$
    $n = 1, \ldots, N.$

These M-simultaneous dynamic programs can be solved numerically to obtain the value functions $F(\Theta_t, m)$ for all t and m. Furthermore, argument maxima will yield the optimal mode choice.

3. Valuation of Projects that include Options to Wait to Invest, to Shut Down and to Abandon

We can illustrate the valuation of the flexibility option by applying it to a project that offers options to wait to invest, to shut down temporarily, and to abandon. Considering each option separately, we investigate its impact on project value, on the value of each option, and on the critical values of Θ (at which the options are exercised), when the other options are introduced. Project characteristics change with the inclusion of each option, so the exogenous project-value assumption made in previous studies does not permit this sort of comparison.

3.1 Definition of Modes

We characterize the firm before it makes the investment as being in mode 1 ("waiting to invest"). Once the investment I is made (i.e., the cost of switching from mode 1 to mode 2, $\delta_{12} = I$), the firm will be in operation (mode

2). While it is in operation, the firm has the ability to shut down temporarily (mode 3). Shutting down the plant will incur some costs (δ_{23}), and while it remains shut down, the firm continues to incur some fixed costs, F. From the shutdown mode, the firm can startup in the production mode (incurring a startup cost), or permanently abandon the project (mode 4) and receive the salvage value. This scenario is described below:

Mode	Description	Flow profit
1	waiting to invest	0
2	production mode	$\pi(\theta)$
3	shut down	-F
4	abandon	0

Switching Costs

δ_{12} = initial investment[12]
δ_{23} = cost of shutting down the plant
δ_{32} = cost of startup from the shutdown mode
δ_{34} = - scrap value[13]

3.2 Numerical Example

In the numerical example, we specify the price dynamics by a mean reverting process (given in equation 2) with the mean set at $\theta^{\circ}=0.5$. We limit the price fluctuations to the range 0 to 1 and divide this interval into 51 discrete values. Hence $\theta^1=0.0$, $\theta^2=0.02,\ldots,\theta^{51}=1.0$.[14] For the base case

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12 The initial investment cost and the scrap value easily can be made contingent on time and usage.

13 In the experiments where the firm has the option to abandon but does not have an option to shut down, we assume that it can abandon from the operating mode. Then δ_{24} is set equal to the scrap value.

14 Details on the computation of the transition matrix are given in the appendix.

calculations we set the mean reversion parameter to zero ($\mu=0$) and let the price follow a random walk with a 16% variance: i.e. $d\theta/\theta = .4\ dZ_\theta$.

We consider a project with a ten-period life span, setting the profit function of the operating mode to $\pi(\theta)=-0.5+\theta$, so that at the mean price the profits are zero.[15] The initial investment cost, abandonment value, and the shutdown and startup cost are all set at zero. While the project remains temporarily shut down, the firm incurs a fixed cost of 0.05 per period. The discount rate is set at 5% (i.e., $\phi=1/1.05$).

Figure 1 plots the behavior of time zero project values against θ for the following cases taken separately; (a) with no real options (i.e., the project remains in force over the entire life), (b) with the option to wait to invest, (c) with the option to temporarily shut down, and (d) with the option to abandon. The value of the project increases with increasing θ and is positive for $\theta>0.5$. Hence, under an NPV rule that ignores real options, the project will be undertaken for any θ_o realization that is greater than 0.5. When the "wait to invest" option is included, the project is worth more with the live option (i.e., wait to invest) than when the investment becomes committed, in the range θ_o less than 0.84. For higher θ_o values, the option to wait will be forgone and the investment committed. These results corroborate the findings of McDonald and Siegel [1986].

If we include the abandonment option, instead of the waiting-to-invest option, the project will not be in operation for values of θ_o less than 0.16. In this range the project will be abandoned. In other words, with ten periods

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15 This <u>unit</u> profit function corresponds to a single output technology with output price θ and input costs = 0.5.

remaining in the life of the project, the abandonment option will be exercised only if Θ falls below the critical price of 0.16. For any higher values of Θ_o, the plant will be operated (even incurring losses for $\Theta<0.5$). This is because the value of the option to wait to abandon is worth more live than the abandonment value.

Finally, the top line in Figure 1 plots the value of the project with the option to temporarily shut down. Because the only cost to shut down is the fixed cost of 0.05 for at least one period while remaining shut down, the firm will shut down if Θ is less than the critical value of 0.45. In contrast to the case including the abandonment option, the project can be started up from the shutdown mode if conditions improve. Hence, the gross project value with the shutdown option is always greater than its value with the abandonment option.

Our results so far confirm those in previous studies using our numerical technique. We will now consider the value of each option and study the impact of introducing the other options on these option values. It is the ability to capture such interactions between and among operating options that gives richness to our model.

To illustrate the impact of the addition of successive options, Figure 2 plots the value of the project against the initial price realization for four cases -- a project with no options, a project with the option to wait to invest, a project with both options to wait to invest and to abandon, and a project with the three options to wait to invest, to abandon, and shut down. As expected, the value of the project unambiguously increases with the addition of each new option. Although the net contribution is positive, the addition of further options reduces the effect of each option.

Figure 3 plots the value of the option to wait to invest. This is denoted by W and computed as the difference between the project value when the initial mode is 1 and the value when the initial mode is 2, where the latter is the value of the non-flexible project.[16] The option to wait to invest is greatest when θ_o is at a minimum. The option value falls as θ increases reaching zero when θ reaches the critical value 0.84, at which point the option will be forgone and the project will be undertaken.

Now let us include also the option to abandon. The difference in $F(\theta_o,1)$ values between this case and the case above gives the new value of the wait to invest option. Even though the project value is increased by including additional options, the value of the wait to invest option is reduced when we add the abandonment option. The reason for this result is that, now, even if the irreversible investment is committed, the firm has the option to abandon the project if prices fall drastically. That is, the abandonment option increases the incentive to invest now.

When we add a shutdown option instead of the abandonment option, the value of waiting to invest is reduced even more dramatically. Obviously when the firm can shut down temporarily at a small cost, it will do so during bad times (when operating profits become negative enough to offset the shutdown cost), so committing to invest now is more easily revised.

An even greater reduction in value of the waiting to invest option occurs when we include both the abandonment and the shutdown options. At the grid

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16 The value of waiting to invest, $C(\theta_o) = F(\theta_o,1)-F(\theta_o,2)$.

sizes and the other parameter values chosen for these numerical computations, the incremental effect of adding the abandonment option, with the shutdown option already available, is very small.[17]

Similar evaluations of the behavior of the abandonment and the shutdown options appear in Figures 4 and 5.[18] Table 1 shows the behavior of the critical price at which the project is undertaken with the inclusion of additional options.

Table 1

The Impact of Other Options on the Critical Price to Invest

	critical θ
With no operating options	0.50
waiting to invest alone	0.84
waiting to invest and abandonment	0.70
waiting to invest and shut down	0.58
waiting to invest, shut down and abandonment	0.58

If there are no operating options, the project will be undertaken when NPV > 0, which occurs when θ is equal to 0.5. When there is an option to wait

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17 Assuming a zero scrap value. This effect is heightened if we either increase the abandonment value, reduce the shutdown cost, increase the periodicity, or increase the price variance.

18 Comparative statics were performed with respect to changing variance, mean reversion parameter, profit functions, and switching costs. Although the particular option values were affected, the essence of the results remains unchanged.

to invest, this critical value is increased to 0.84. Becuase undertaking an investment now involves giving up the option to wait to invest, the firm must wait until the NPV increases to compensate for the forgone option value.

When a further operating option, that of abandoning the project, is available, the critical value drops to 0.70. The intuition here is that if the irreversible investment is committed, the firm is insured against drastically low prices. The abandonment option acts as an insurance policy to guarantee a minimum payment.

If, instead of the abandonment option, we add a shutdown option, the critical value at which the project will be undertaken falls to 0.58. At this reduced critical value, the probability of a price realization that would generate negative operating profits increases. Becuase the firm's cash flows are bounded below (at -0.05) by the shutdown option, it still will commit the investment at this lower critical value.

Finally, when all three options (waiting to invest, abandonment, and shutdown) are available, the critical value remains at 0.58. With finer grid sizes we would have noticed a further slight reduction in the critical value. However, becuase the scrap value and switching costs are set at zero, and, the fixed cost of remaining shut down is small, the incremental value of the abandonment option (when the project already includes the shutdown option and has a remaining life of 120 periods) is negligible.

4. Concluding Remarks

We have presented a computationally feasible technique to investigate how the value of a project is affected by the simultaneous introduction of several operating options.

In contrast to previous studies that consider operating options one at a time, we treat them jointly. This process makes the interdependence between the operating options explicit. Our numerical example including the operating options to wait to invest, to abandon, and to shut down temporarily shows that the project value increases with the introduction of additional options. At the same time, addition of other options reduces the value of the previously available options.

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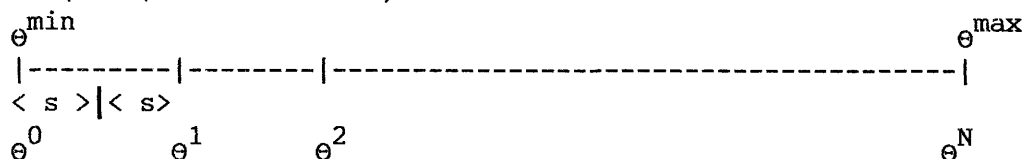
Appendix 1

The Discrete Transition Probability Matrix for a Mean Reverting Process

Consider the mean reverting stochastic process

(A1) $\quad d\theta_t = \mu (\theta° - \theta_t) dt + \sigma dZ_\theta$

where dZ_θ is a standard Wiener process. The first step is to determine the relevant range of price fluctuations, $\{\theta^{min}, \theta^{max}\}$, within which we discretize θ. Depending on the required precision, this range is divided into N discrete states (i.e., N-1 intervals).

```
 min                                                      max
θ                                                        θ
|----------|--------|-------------------------------------|
< s >|< s>
 0          1        2                                    N
θ          θ        θ                                    θ
```

where $\quad s = (\theta^{min} - \theta^{max})/2(N-1)$.

Without loss of generality we can let the discrete time interval, Dt, be equal to 1. In order to bring about a transition from state i to i+1 (i.e., θ^i to θ^{i+1}) the following conditions must be satisfied:

(A2) $\quad D\theta > s \qquad ===> \quad DZ > [s-\mu(\theta°-\theta_i)]/\sigma$

and

(A3) $\quad D\theta < 3 \ s \qquad ===> \quad DZ < [3 \ s-\mu(\theta°-\theta_i)]/\sigma$

where $D\theta$ and DZ are the discrete changes in θ and Z, respectively. Hence the transition probability P(i,i+1) is computed as[19]

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19 P(i,j) is the probability that $\theta_{t+1} = \theta^j$ given that $\theta_t = \theta^i$. In the text we refer to this as p_{ij}.

(A4) $P(i,i+1) = \text{Prob} [DZ \, \epsilon \, \{(\, s-\mu(\theta°-\theta_i))/\sigma \, , \, (3 \, s-\mu(\theta°-\theta_i))/\sigma\}]$

Define $Z_0 = -\mu(\theta°-\theta_i)/\sigma$ and $Z_d = s/\sigma$. Then A4 can be rewritten as

(A5) $P(i,i+1) = \Phi[Z_0+3Z_d] - \Phi[Z_0+Z_d]$

where $\Phi[.]$ is the cumulative normal distribution. In general, the transition from state i to j is given by

(A6) $P(i,j) = \Phi[Z_0+\{2(j-i)+1\}Z_d] - \Phi[Z_0+\{2(j-i)-1\}Z_d]$.

Special care must be taken with the end points P_0 and P_N. Lumping all exterior values to the boundary we obtain the transition probabilities

(A7) $P(i,N) = 1 - \Phi[Z_0+\{2(n-i)-1\}Z_d]$

and

(A8) $P(i,1) = \Phi[Z_0+\{2(1-i)+1\}Z_d]$.

Note that for dt $\neq$ 1 we must set $\mu = \mu \, Dt$ and $\sigma = \sigma \, (Dt)^{\frac{1}{2}}$.

Once the discrete probabilities above are available, the expected values (such as those encountered in the dynamic programming problems discussed in sections 2 and 3) are obtained as probability weighted sums. For example, if $\theta_{t-1} = \theta^j$ then $E_{t-1}[V(\theta_t)]$ is

(A9) $E_{t-1}[V(\theta_t)] = \sum_{i=1}^{N} V(\theta^i) \, P_{j,i}$.

Figure 1
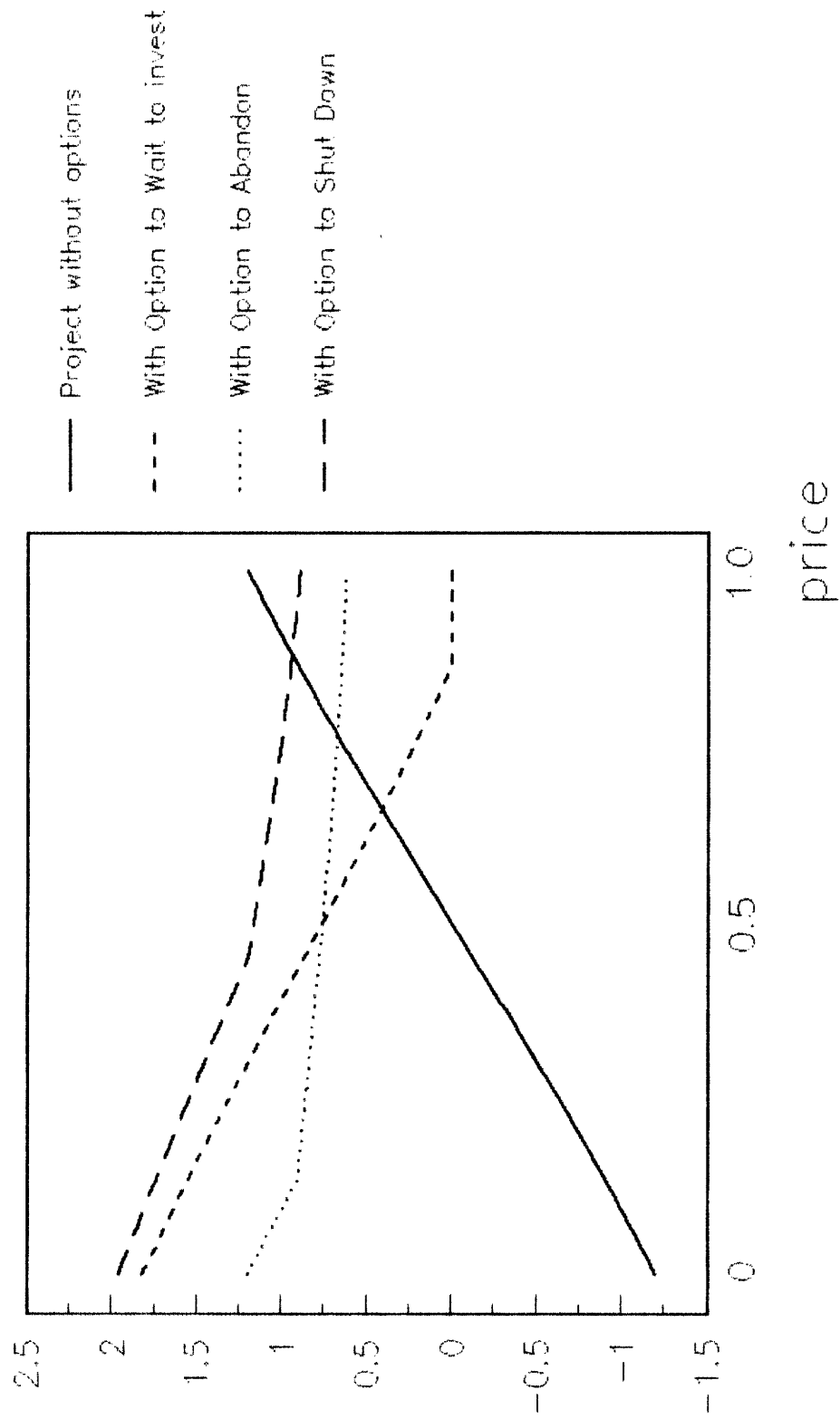
Project Value with One Option At A TIme

Legend:
- Project without options
- With Option to Wait to invest
- With Option to Abandon
- With Option to Shut Down

price

Figure 2
Project Value with Several Options

With no options
With Option to Wait
With Option to Abandon
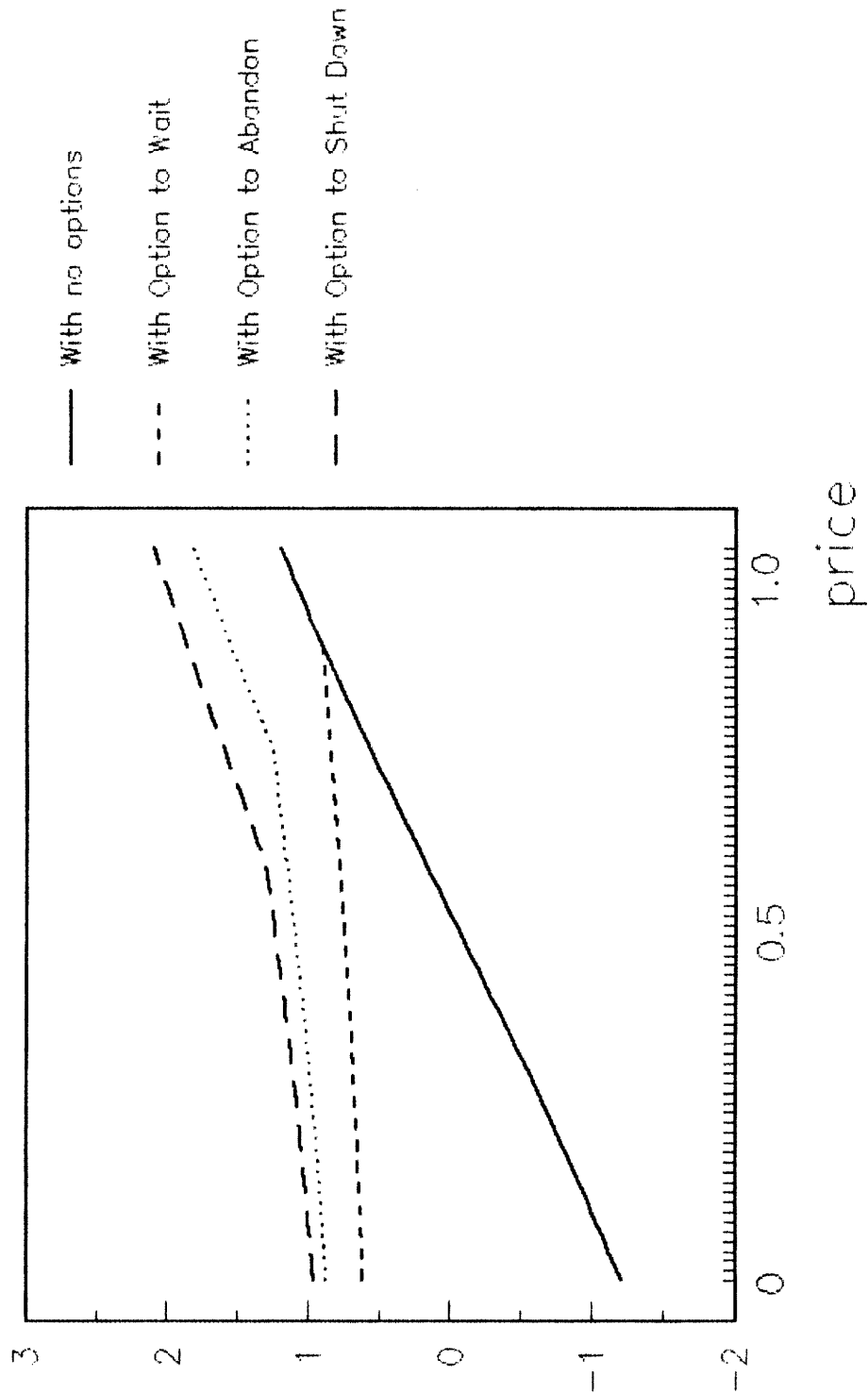With Option to Shut Down

Figure 3
Value of the Option to Wait to Invest
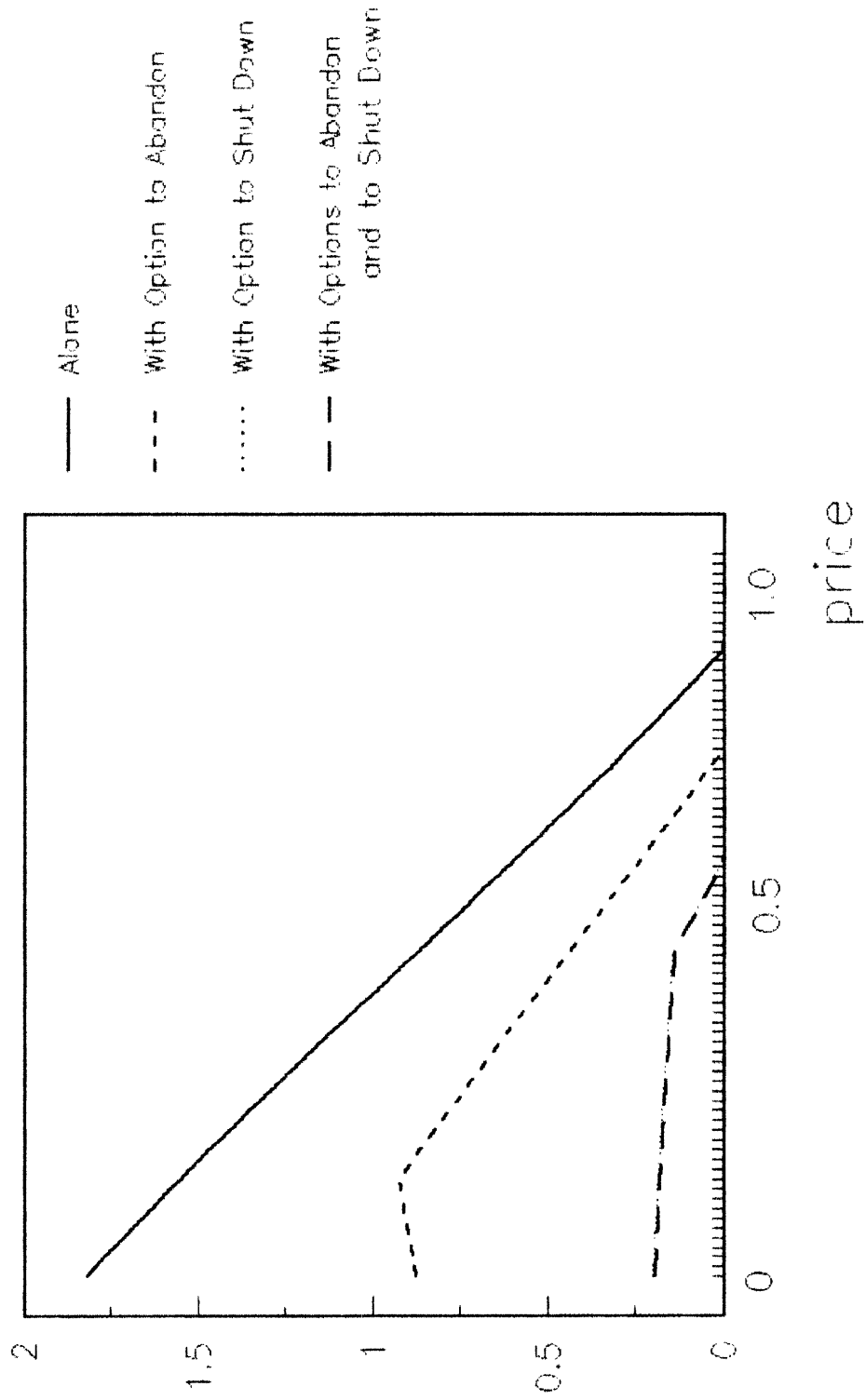

Alone

With Option to Abandon

With Option to Shut Down

With Options to Abandon
and to Shut Down

Figure 4
The Option to Abandon

Legend:
- Option to Abandon alone
- Abandon with w/i
- Abandon with s/d
- Abandon with w/i and

price