

**EXPLORATION IN COMPETITIVE NONRENEWABLE
RESOURCE MARKETS: AN EXTENSION OF
PINDYCK'S PERFECT FORESIGHT MODEL**

by

David Laughton

MIT-EL 86-018WP

August 1986

EXPLORATION
IN
COMPETITIVE NONRENEWABLE RESOURCE MARKETS:

AN EXTENSION
OF
PINDYCK'S PERFECT FORESIGHT MODEL

David Laughton

Center for Energy Policy Research
Energy Laboratory
and
Applied Economics, Finance and Accounting Area
Sloan School of Management
MIT

August 1986

Abstract

Pindyck's model of exploration for, and production of, a non-renewable resource (Pindyck 1978) is extended so that the production cost function may depend separately on concurrently available reserves and on the total amount of past production. A method for obtaining the optimal trajectory of parameterised specifications of the model is tested on elaborations of a corrected version of the parameterised specification used by Pindyck in his paper. The initial price for each simulation is tabulated.

Acknowledgements

The author wishes to acknowledge fellowship support from SSHRC Canada and from the Sloan School of Management, MIT. He wishes also to thank R.M. Solow for bringing the Pindyck model to his attention, and R.S. Pindyck for illuminating discussions and suggestions.

1. Introduction

The first models of nonrenewable resource markets (see Dasgupta and Heal 1978 for references) were simple perfect foresight models. These were designed to show that nonrenewable resource prices should generally rise with time to reflect the scarcity value of the resource as it is depleted. Unfortunately for these models, prices for holding nonrenewable resources are observed not to be monotonically increasing. This may result from a combination of many factors, including the impact of uncertainty and changing levels of competition on the market. However, there is at least one factor that could cause decreasing prices, even within a simple perfect foresight, competitive paradigm. That factor is changes in production costs over time. Such changes may be postulated as an exogenous phenomenon, or as a result of some behaviour of the agents involved. Examples of such behaviour include the search for new lower cost resources or new lower cost production technologies. R. S. Pindyck (Pindyck 1978) analysed a model of the effects of exploration for new resources on production costs and hence on prices.

The essence of Pindyck's model is his specification of unit production cost as a decreasing function of concurrently available reserves. This provides an incentive for producing agents to explore for new reserves. The marginal cost of maintaining a rate of discovery is specified in his model to

be an increasing function of that discovery rate. This means that continuous, rather than pulsed, exploration is a potentially optimal policy. If the initial reserves available to each producing agent are small, early optimal exploration may be large enough to increase reserves and decrease unit production cost enough to cause price to drop over a period of time. Eventually, discovery becomes so costly that the resource is economically (if not actually physically) depleted and price rises. The price path produced by this optimising behaviour has a "U" shape, rather than being monotonically increasing.

Pindyck's model, despite its success at demonstrating the possibility of nonmonotonic price paths, is unnatural in that unit production cost depends only on the amount of reserves concurrently available to the producing agent rather than separately on those reserves and on the amount of past production by that agent. Fortunately, a model in which unit production cost has a separate dependency on each of these two variables is no more complicated to analyse in one important class of parameterised specifications than is Pindyck's model.

In this gloss on Pindyck's model, unit production cost is assumed to be a nondecreasing function of the amount of cumulative past production. It is also assumed to be nonincreasing in concurrently available reserves, and, a fortiori, in cumulative past discoveries. (Reserves are the

difference between the two.) These assumptions are a mock up of the observation that, in a world of uncertainty, exploration may lead to the discovery of resources, some of which may have lower production costs than those already discovered and not yet produced. In the world of certainty of this model, this possibility is made a certainty. They also mock up the observation that reserves with lower production costs are usually more easily discovered, and, once discovered, are produced first.

Exploration in Pindyck's model, and in this gloss, may be viewed as an activity that results with certainty in the discovery of new resources, but with decreasing returns to scale. Moreover, these new resources have a particular distribution of production costs that depends on how much has already been discovered. In reality, exploration is an activity the results of which are uncertain. However, each of these models may be such that each time step in each of its trajectories is the average of many time steps in trajectories of an underlying model in which the results of exploration are uncertain. Arrow and Chang (1982) and Lasserre (1984) have analysed a simple version of such a model. Unfortunately, their simple model is limited by the assumptions that there be no production costs and that infinite rates of exploration be possible. It remains to be determined whether a generalisation of such a model may provide a foundation for Pindyck's model or this gloss.

The model is presented in Section 2 and compared in Section 3 to the Pindyck model. An unimportant error in Pindyck's original paper is pointed out. The exploration model is then analysed in Section 4 to show how it might give a nonmonotonic price path.

In his original paper on this subject, Pindyck went on to provide a concrete functional specification of his model. He used data from a particular Texas oilfield to estimate parameters in this specification. He then used the parameterised specification of the model to simulate the development of the field. The simulation was performed by attempting to solve numerically the boundary value problem that results from the initial, first order, and transversality conditions derived from the hamiltonian version of the optimal control problem. Unfortunately, Pindyck's specification allowed for fixed period costs of exploration. This makes invalid the theoretical analysis he used to develop the boundary value problem that he attempted to solve. It is not clear what his numerical results mean, if they mean anything.

Section 5 is a brief discussion of the relevance of such simulations, if done properly using a set of self-consistent parameterised specifications, for the analysis of the qualitative and the quantitative features of the development of an overall market. The particular methods used to perform

the simulations presented in section 5 are described in Appendix A. The problems in the calculations in Pindyck's paper are discussed there.

Section 6 contains conclusions.

2. The Model

The agents in the model are a number, N , of identical firms, each of which has an identical region in which to explore for and produce a resource. The number N is large enough that each of the firms is a price taker in all of its product and factor markets. Each has perfect foresight about the development of these markets.

Production costs, c , for each agent depend upon the amount of its cumulative past production, Y , and the amount of its cumulative past discoveries, X . The production rate is denoted by q . Exploration effort is denoted by w . Exploration costs, k , and the discovery function, f , are unrelated to production and thus does not depend on the concurrent production rate or on cumulative past production. They do depend upon exploration effort. The discovery function also depends upon the amount of cumulative past discoveries. For simplicity, there is no explicit time dependence in the cost and discovery functions.

Each agent solves the following optimisation problem:

$$\begin{aligned}
 PV = \max_{q, w} \int_0^{\infty} [(p(t) - c(Y(t), X(t)))q(t) - k(w(t))] \delta(t) dt \\
 \text{s.t. } \dot{Y}(t) = q(t), \quad Y(0) = 0, \\
 \dot{X}(t) = f(w(t), X(t)), \quad X(0) = R_0, \quad (2-1) \\
 Y(t) \leq X(t), \quad q(t), w(t) \geq 0 ;
 \end{aligned}$$

where:

PV is the present value of the optimal programme and thus of the firm;

$\delta(t)$, which for simplicity is restricted to have the form e^{-rt} , is the present value of a unit of cash flow occurring at time t ;

$p(t)$ is the unit price of the resource at time t ; and

R_0 is the initial amount of reserves.

Price is determined by a demand function:

$$Q(t) = D(p(t)) ; \quad (2-2)$$

where:

$Q(t)$ is the total production rate of all firms at time t .

For simplicity, demand does not depend upon time and has a finite choke price, p_{choke} .

Because all firms are identical, total production is a multiple of the production of any one firm:

$$Q(t) = N q(t) . \quad (2-3)$$

For notational purposes, the demand function of the firm is defined:

$$d(p) \equiv 1/N D(p) ; \quad (2-4)$$

so that:

$$q(t) = d(p(t)) . \quad (2-5)$$

The common quantity dynamics for each producer is not analysed directly but rather may be inferred through the firm demand function from the market price dynamics. The quantity dynamics is more basic and determines the price dynamics through equilibrium, but the latter is the more interesting economically and can easily be calculated directly after equilibrium is implicitly imposed.

The set of parameterised specifications that will be considered is restricted implicitly to those such that an optimal trajectory exists, is unique, continuous, and piecewise differentiable, and for which there is some production. Necessary and sufficient conditions for these restrictions have not been developed. However, the nonzero production condition is satisfied whenever the choke price level is sufficiently large compared to the production and discovery cost over the relevant trajectories.

The set of specifications to be examined is further restricted to those such that the optimal path is not determined explicitly by the inequality constraint on the state variables. This constraint is enforced implicitly by properties of the cost functions.

The cost and discovery functions have the following properties:

$$\begin{aligned} c(Y, X) &\geq 0, \\ c_Y(Y, X) + c_X(Y, X) &\geq 0, \\ c_X(Y, X) &\leq 0; \quad 0 < Y < X; \end{aligned} \tag{2-6}$$

$$c(X, X) = \infty; \quad X \geq 0; \tag{2-7}$$

$$\begin{aligned} k'(w) &> 0, \\ k''(w) &\geq 0; \quad w \geq 0; \end{aligned} \tag{2-8}$$

$$k(0) = 0; \tag{2-9}$$

$$\begin{aligned} f(w, X) &> 0, \\ f_w(w, X) &> 0, \\ f_X(w, X) &< 0; \quad w > 0, X \geq 0; \end{aligned} \tag{2-10}$$

$$f(0,X) = 0 ; \quad X \geq 0 ; \quad (2-11)$$

$$f(w,X) \rightarrow 0, \quad X \rightarrow \infty \text{ uniformly in } w ; \quad \text{and} \quad (2-12)$$

$$\begin{aligned} (k'/f_w)_w(w,X) &> 0 , \\ (k'/f_w)_X(w,X) &\geq 0 ; \quad w \geq 0, X \geq 0 . \end{aligned} \quad (2-13)$$

The second item in property (2-6) means that production cost is a nondecreasing function of cumulative past production if reserves are held fixed. Property (2-7) implicitly enforces the inequality constraint on the state variables. The last two conditions are the only other properties that require explanation.

Condition (2-12) states that, while the resource may not be exhaustible, the ability to find new discoveries declines to zero as more of the resource has been discovered. This decline, if drastic enough, assures the eventual economic exhaustion of the resource in the sense that price will be driven up to the choke price. While general necessary or sufficient conditions for such behaviour have not been investigated, its existence in some specifications of the model is sure. It is assumed for any parameterised specification under consideration that the decline in the discovery function, to which condition (2-13) alludes, is sufficiently rapid to assure that prices rise toward the choke price at the end of the optimal programme.

Condition (2-13) states that the marginal cost of maintaining the discovery rate, f , increases with exploration effort (or equivalently with the discovery rate) and with cumulative past discoveries. As mentioned in the introduction, the first of these properties in condition (2-13) assures that the optimal path of exploration is continuous rather than pulsed.

The hamiltonian first order conditions are:

$$\dot{\lambda}^Y - \delta c_Y q = 0 ; \quad (2-14)$$

$$\dot{\lambda}^X - \delta c_X q + \lambda^X f_X = 0 ; \quad (2-15)$$

$$\lambda^Y + \delta(p-c) = 0 ; \quad \text{or} \quad (2-16a)$$

$$q = 0 ; \quad \text{and} \quad (2-16b)$$

$$\lambda^X f_w - \delta k' = 0 ; \quad \text{or} \quad (2-17a)$$

$$w = 0 . \quad (2-17b)$$

where λ^Y , λ^X are the costate variables of Y, X .

The first order conditions give the control variable dynamics. Specifically, if production is occurring, equations (2-14) and (2-16a) give the price dynamics:

$$\dot{p} = r(p-c) + c_X f . \quad (2-18a)$$

Otherwise:

$$p(t) \geq p_{choke} . \quad (2-18b)$$

Equations (2-15) and (2-17a) give the exploration dynamics while exploration is occurring:

$$\dot{w} = \frac{r(k'/f_w) + (-f_X)(k'/f_w) - (k'/f_w)_X f - (-c_X)d}{(k'/f_w)_w} \quad (2-19a)$$

Otherwise, obviously:

$$w = 0 . \quad (2-19b)$$

The initial condition and dynamics of the state variables are given by the constraints of the optimal control problem:

$$Y(0) = 0 ; \quad (2-20)$$

$$\dot{Y}(t) = d(p(t)) ; \quad (2-21)$$

$$X(0) = R_0 ; \quad \text{and} \quad (2-22)$$

$$\dot{X}(t) = f(w(t), X(t)) . \quad (2-23)$$

Finally, the initial conditions of the control variables are given implicitly by transversality conditions which determine these initial values and the terminal times for each activity.

Production stops at time, T_p , when both demand and production profit margin go to zero:

$$q(T_p) = 0 ; \quad \text{or} \quad (2-24')$$

$$p(T_p) = p_{choke} ; \quad \text{and} \quad (2-24)$$

$$\lambda^Y(T_p) = 0 ; \quad \text{or} \quad (2-25')$$

$$p(T_p) = c(Y(T_p), X(T_p)) . \quad (2-25)$$

Exploration stops at a time, T_E , not greater than T_p , when the marginal cost of maintaining a negligible rate of discovery is not covered by the present value of the future cost savings resulting from the extra marginal discovery:

$$w(T_E) = 0 ; \quad \text{and} \quad (2-26)$$

$$\lambda^X(T_E) = 0 ; \quad \text{or} \quad (2-27')$$

$$\frac{-k'(0)}{f_w(0, X(T_E))} = \frac{1}{\delta(T_E)} \int_{T_E}^{T_p} c_X(Y(s), X(T_E)) d(p(s)) \delta(s) ds . \quad (2-27)$$

Equation (2-27) results from the combination of:

$$\lambda^X(T_p) = 0 ; \quad (2-28)$$

$$\dot{\lambda}^X(t) = \delta(t) c_X(Y(t), X(T_E)) q(t) ; \quad T_E \leq t \leq T_p ; \quad \text{and} \quad (2-29)$$

$$k'/f_w(0, X(T_E)) = \lambda^X(T_E)/\delta(T_E) . \quad (2-30)$$

Production stops at time, T_p , when both demand and production profit margin go to zero:

$$q(T_p) = 0 ; \quad \text{or} \quad (2-24')$$

$$p(T_p) = p_{choke} ; \quad \text{and} \quad (2-24)$$

$$\lambda^Y(T_p) = 0 ; \quad \text{or} \quad (2-25')$$

$$p(T_p) = c(Y(T_p), X(T_p)) . \quad (2-25)$$

Exploration stops at a time, T_E , not greater than T_p , when the marginal cost of maintaining a negligible rate of discovery is not covered by the present value of the future cost savings resulting from the extra marginal discovery:

$$w(T_E) = 0 ; \quad \text{and} \quad (2-26)$$

$$\lambda^X(T_E) = 0 ; \quad \text{or} \quad (2-27')$$

$$\frac{-k'(0)}{f_w(0, X(T_E))} = \frac{1}{\delta(T_E)} \int_{T_E}^{T_p} c_X(Y(s), X(T_E)) d(p(s)) \delta(s) ds . \quad (2-27)$$

Equation (2-27) results from the combination of:

$$\lambda^X(T_p) = 0 ; \quad (2-28)$$

$$\dot{\lambda}^X(t) = \delta(t) c_X(Y(t), X(T_E)) q(t) ; \quad T_E \leq t \leq T_p ; \quad \text{and} \quad (2-29)$$

$$k'/f_w(0, X(T_E)) = \lambda^X(T_E)/\delta(T_E) . \quad (2-30)$$

Equation (2-28) is the basic hamiltonian transversality condition. Equations (2-29) and (2-30) are rewritten versions of equation (2-15) and (2-17a), valid in the given time periods. The second term in equation (2-15) is zero when there is no exploration, because, from property (2-11), there is no discovery without exploration:

$$f(0, X) = 0 . \quad (2-31)$$

If the marginal cost of discovering new resources is zero at the time, T_E , when it is optimal to cease exploration, and if, at this same time, production costs are a strictly decreasing function of past cumulative discoveries, then it is optimal to explore as long as it is to produce. The cost of maintaining a small exploration rate at any time prior to cessation of production is small compared to the decrease in future production costs effected by that exploration. The simultaneous cessation of production and exploration under these conditions is reflected in equation (2-27). The left hand side of the equation is zero, while the integrand on the right hand side is strictly negative. This forces T_E and T_P , the endpoints of the right hand side integral, to be the same.

If the marginal cost of discovering new resources is strictly positive at time T_E , exploration ceases before

production. The dynamics of the problem, after exploration ceases, is reduced to:

$$\dot{p}(t) = r(p(t) - c(Y(t), X(T_E))) ; \quad (2-32)$$

$$w(t) = 0 ; \quad (2-33)$$

$$\dot{Y}(t) = d(p(t)) ; \quad \text{and} \quad (2-34)$$

$$X(t) = X(T_E) . \quad (2-35)$$

It should be noted that, unless exploration and production cease at the same time, the production-only programme must be solved first to yield the right hand side integral, I , of equation (2-27):

$$I(Y(T_E), X(T_E)) \equiv -\frac{1}{\delta(T_E)} \int_{T_E}^{T_p} c_X(Y(s), X(T_E)) d(p(s)) \delta(s) ds . \quad (2-36)$$

3. A Comparison with Pindyck's Model

The model of Pindyck is a special case of the exploration model presented in Section 2. In Pindyck's model, production costs are forced to depend only upon the difference between cumulative past discoveries and cumulative past production. An examination of equations (2), (3), (9) and (13) of Pindyck's paper shows them to be equivalent to this special case of equations (2-21), (2-23), (2-28a) and (2-29a). The state variable initial conditions are trivially equivalent.

With one exception, the transversality conditions would also be the same except for an error in Pindyck's treatment of them. He apparently assumes (on p.847, para.1 of his paper) that the equivalent of first order condition for exploration (2-17a) must hold even when no exploration is occurring. Exploration is nonnegative, and the first order condition need not hold on the boundary of the set of possible control variables. Therefore, this condition need not be valid during a period of no exploration. This error does not affect the rest of his paper.

The exception mentioned in the previous paragraph is equation (2-27). In Pindyck's model, a simple expression for the integral, I , can be found. Because production costs in his model depend only on the difference of the two state

variables, X and Y , the dynamics of the costate variables are related. First, the derivatives of the production cost function are related:

$$c_Y = -c_X . \quad (3-1)$$

This condition, with equations (2-14) and (2-29), means that:

$$\dot{\lambda}Y(t) = -\dot{\lambda}X(t) ; \quad T_E \leq t \leq T_p . \quad (3-2)$$

Because both costate variables vanish due to transversality conditions at the production terminal time, each must be the negative of the other from the termination of exploration onward. Using equation (2-16a) and equation (2-30), this means that equation (2-27), under the specialised conditions of Pindyck's model, is equivalent to:

$$\frac{-k'(0)}{f_X(0, X(T_E))} = p(T_E) - c(Y(T_E), X(T_E)) . \quad (3-3)$$

The simplicity of the exploration termination condition, if exploration ceases before production, is the one practical advantage that the Pindyck model has over the more general model.

4. Price Paths in the Exploration Model

It has already been stated that, for the parameterised specifications to be considered, new discoveries become difficult so that the resource is economically, if not physically, exhaustible. The price of the resource eventually rises up to the choke price and the market closes.

An argument will now be presented to show that decreases in price are possible, in some parameterised specifications of the exploration model, before price begins to rise to the choke price at the end of the production period. The analysis begins with an expression for the production profit margin:

$$p(t) - c(Y(t), X(t)) = \frac{1}{\delta(t)} \int_t^{T_p} \delta(s) c_Y(Y(s), X(s)) d(p(s)) ds; \quad (4-1)$$

derived from equations (2-20), (2-22a) and (2-31). This expression gives a signing for the price dynamics equation:

$$\dot{p} = r(p-c) - (-c_X)f; \quad (4-2)$$

that shows explicitly that any decrease in price must be due to production cost decreases resulting from exploration.

There are families of parameterised specifications which have, at any given time, the same numerical value for the first term and any arbitrary negative second term in equation (4-2). This may be most easily demonstrated at the initial time. Each member of such a family of parameterised

specifications has the same numerical value of each factor in the right hand side of equation (4-2), except for c_x . It is obvious that the term structure factor and the production cost may be left unchanged by such a variation in the production cost function. However, the initial price and discovery rate (or implicitly the initial exploration rate) each depend on some integral of a function of the future trajectory. Any variation in the production cost function will cause that trajectory and thus those integrals to vary. However, these imputed variations may be undone by changes in the cost and discovery functions that change the trajectory, not at the initial time, but over some period of time later in the programme.

This analysis hardly gives any insight into what such a family of parameterised specifications might be. Another approach would be to keep the parameterised specification the same except for variations in the initial reserves available for production.

If these initial reserves were large enough, exploration would at all times be costly and initially not very beneficial. The market would initially look much like a market where new discoveries were impossible, and price at all times would be an increasing function of time.

If initial reserves were zero, initial production would perforce be zero and the initial price would be at or above the choke price. If production were ever to take place, some

exploration would be needed to bring production costs down so that price would be below the choke price over some period of time. Then there would be some part of that period of time during which market price would be decreasing. If the choke price is smaller than the limit of production costs at vanishing initial reserves, then this nonmonotonic behaviour of price would exist in some set of parameterisations for which initial reserves are close to zero.

5. Use of Quantitative Simulations in Perfect Foresight Models

It has been demonstrated in Pindyck's original paper, and reiterated in Section 4, that decreasing prices are possible in the initial stages of the development of a competitive nonrenewable resource market in which the producing agents have perfect foresight. In particular, this is "usually" the case if initial reserves of the resources are "small". More particularly, this can occur if production costs depend separately on past production and current reserves.

The question arises whether there is anything of value that can be extracted from such a model other than the demonstration, in a self-consistent, if counterfactual, model, that exploration can be a price-depressing force in nonrenewable resource markets. An answer to this question might come from numerical simulation of behaviour in a fairly general specification of this model. If the essential features of that behaviour were insensitive to plausible variations of the parameters used, then it is possible that the model might be acceptable as a self-consistent possible description of that behaviour. The important point is that, if this insensitivity of market behaviour to parameter variation be true, the agents would have enough foresight to make the perfect foresight assumption self-consistent.

A specification within which one might carry out such a self-consistency check might be the following:

$$d(p) = \max(h (p_{choke} - p)^{\eta}, 0) ; \quad (5-1)$$

$$c(Y,X) = c_0 \frac{R_0}{X-Y}^{\beta} \exp (\gamma Y/X_0) ; \quad (5-2)$$

$$k(w) = bw ; \quad \text{and} \quad (5-3)$$

$$f(w,X) = A w^{\alpha} \exp (-X/X_0) . \quad (5-4)$$

This is a corrected and expanded version of the specification used for the numerical work in the Appendix of Pindyck's paper (see Appendix A of this comment). The parameter γ tests the effects of the separate dependence of production costs on past production allowed by the expanded exploration model of this comment. The parameter β tests the effects of different choices of the singularity in cost function at zero reserves. The parameter η tests the effects of a more general demand function than the linear demand in Pindyck's specification.

As a preliminary task, simulations, using elaborations of the parameter set given by Pindyck for the oil field analysis in the Appendix of his paper, were used to test the numerical methods outlined in Appendix A. Because the Pindyck parameter set is based on an analysis of a particular deposit rather than a whole market, this collection of parameter sets may not be appropriate for a test of the quantitative usefulness of the model. If a collection of plausible parameter sets for a

real market were developed, such an examination could be made. This has not been done.

Nevertheless, the preliminary calculations are of some interest in their own right. First, the methods can be used to find the initial values of the control parameters that produce, through a solution of the resulting initial value problem, trajectories that solve the maximisation problem and, in a restricted sense, satisfy the transversality conditions. This success is restricted because the trajectories close to the solution have terminal portions that diverge radically from each other. Despite this, enough of the solution trajectory may be defined so that an examination elsewhere of the self-consistency conjecture would be possible.

Second, there is considerable variation in initial price and exploration effort for the collection of parameter sets examined (see Table 5-1). Moreover, as might be expected, initial price is an increasing function of γ (the parameter for the exponential dependence of production costs on past production) and of β (the power of the singularity in the production cost function at zero reserves). Finally, initial exploration effort is an increasing function of β and is ambiguously influenced by differences in γ . While it is gratifying in that the observed variation corresponds in direction with what was expected, it is large enough that, if the collection of plausible parameter sets for a real market were to have as much variation, the self-consistency conjecture might not be tenable.

Table 5-1

Initial Price and Exploration for
Generalizations of the Corrected Pindyck Parameter Set

1) Parameters not varied

r	=	0.05
h	=	20.0
pchoke	=	33.00
η	=	1
c ₀	=	1.250
b	=	0.067
A	=	55.076
α	=	0.599
X ₀	=	4428.7
R ₀	=	7170.0

2) Initial Price and Exploration

β	γ	p ₀	w ₀
1.0	0.0	5.069-5.071	8681-8683
1.0	0.5	7.110-7.115	9405-9440
1.0	1.0	9.500-9.505	9150-9230
0.5	0.0	4.50 -4.53	4649-4676
1.5	0.0	5.50 -5.51	11740-11757

6. Conclusion

Perfect foresight models may be useful in economics for the demonstration of the existence of, and of the qualitative effects of, some economic forces. This may be so only if knowledge of the general trends produced by such forces is all the foresight that the agents need to behave as if, from the point of view of economist performing the analysis, they have perfect foresight. Moreover, it is necessary that such knowledge be all that the economist needs to model adequately the dynamics in which he is interested.

With these considerations as background, a gloss on a model first analysed by Pindyck has been constructed to examine the effects of exploration on the development of the market for a nonrenewable resource. (In the course of this, a minor error in Pindyck's original analysis has been found and corrected.) New discoveries can, in certain parameterised specifications of each model, cause prices to drop before scarcity causes prices to rise. There is an obvious connection between this phenomenon and the condition that the amount of initial reserves be small. More generally, this phenomenon is the result of large initial production costs and the possibility that these costs might be rapidly and relatively inexpensively reduced.

A simple method has been set up to calculate the market trajectories of this model, and tested with a preliminary collection of parameter sets for a particular specification.

(In the course of this, a major problem was found in the numerical calculations performed originally by Pindyck.) If an appropriate collection of parameterised specifications is developed for a real market, the self-consistency of the perfect foresight assumption in this model may then be tested.

The concluding remark in Pindyck's paper suggests that he thought that the perfect foresight assumption might be the most important deficiency in his model. Preliminary calculations here suggest that, at least for the quantitative use of such models, he may have been correct.

Appendix A: Solving the Boundary Value Problem with Shooting Methods

The mathematical problem generated by the models discussed in this paper is the problem of solving a system of nonlinear ordinary differential equations with boundary conditions at two or more points including one or more free boundary points. There are many methods for considering such problems. The method that Pindyck attempted to use in his paper to solve a specification of his model is called a shooting method. A search is done over sets of the unspecified initial conditions to find a set, the solution of the initial value problem for which is also a solution of the boundary value problem. Thus, the method is divided into an iteration of three steps:

- 1) choosing a trial set of initial values;
- 2) solving the resulting initial value problem; and
- 3) testing the boundary conditions of that solution.

The last of these subproblems is trivial. The second may be solved using one of a number of methods. Pindyck (Pindyck 1985) used a standard Euler difference equation approximation. More general Runge-Kutta methods or multistep methods may also be used (Lambert 1973).

The first subproblem is at the heart of the overall problem. Equations may be formulated to assist in the choice of appropriate sequences of trial initial conditions. They are systems of nonlinear algebraic equation that, in general, must be solved by iterative search. The search pattern may be generated in many ways including combinations of grid searches, steepest descent iteration, Newton iteration, and hybrid (e.g., Powell) iteration. In the region of the actual solution set, the best method is usually a Newton iteration. This would be set up as follows for the simpler class of parameterised specifications in which production and exploration are known to cease simultaneously.

The equations to be solved are simply the terminal conditions (2-24) to (2-26) parameterised by initial price, p_0 , and initial exploration effort, w_0 , and by the common terminal time, T :

$$g_1(p_0, w_0, T) \equiv p_{choke} - p(T; p_0, w_0) = 0 ; \quad (A-1)$$

$$g_2(p_0, w_0, T) \equiv p(T; p_0, w_0) - c(Y(T; p_0, w_0), X(T; p_0, w_0)) = 0 ; \quad \text{and (A-2)}$$

$$g_3(p_0, w_0, T) \equiv w(T; p_0, w_0) = 0 . \quad (A-3)$$

The first order Taylor expansion of these functions, about any choice of their arguments, gives the Newton equations. If these are solved iteratively, they should converge to a point

where the functions vanish. The Newton method and equations are:

$$Z_0 \text{ is close to } Z ; \quad (A-4)$$

$$g_k(Z^i) + \frac{\partial g_k}{\partial Z} (Z^i) (Z^{i+1} - Z^i) = 0 ; \quad i \geq 0, k = 1, 2, 3 ; \quad (A-5)$$

$$Z = Z^\infty . \quad (A-6)$$

where:

$Z \equiv (p_0, w_0, T)$ is the solution;

$Z^i \equiv (p_0^i, w_0^i, T^i) ; \quad i \geq 0$ are the iterates;

$$\frac{\partial g_1}{\partial Z} = - \frac{\partial p}{\partial Z} ; \quad (A-7)$$

$$\frac{\partial g_2}{\partial Z} = \frac{\partial p}{\partial Z} - c_Y \frac{\partial Y}{\partial Z} - c_X \frac{\partial X}{\partial p} ; \quad \text{and} \quad (A-8)$$

$$\frac{\partial g_3}{\partial Z} = \frac{\partial w}{\partial Z} . \quad (A-9)$$

At the solution, Z , it may be easily verified that:

$$\frac{\partial g_k}{\partial T} (Z) = 0 ; \quad k = 1, 2, 3 . \quad (A-10)$$

The transversality conditions, in conjunction with properties of the cost and discovery functions, assure that the time derivatives of the state and control variables vanish at the terminal time. Because of this singularity, the Newton

equations may not be solved directly for this class of problems. Other methods of search must be used. The method used is described next.

The terminal time for any given initial value problem may be defined as the first time at which any of the terminal conditions is satisfied. This would define three new functions of the unknown initial conditions:

$$h_k(p_0, w_0) \equiv g_k(p_0, w_0, T(p_0, w_0)) ; \quad k = 1, 2, 3 ; \quad (A-11)$$

where:

$$T(p_0, w_0) \equiv \min_{k=1,2,3} U \{T \mid g_k(p_0, w_0, T) = 0\} . \quad (A-12)$$

For each point in " p_0 - w_0 " space, at least one of the functions (A-11) is zero. This divides " p_0 - w_0 " space into three "phases". Each phase is defined by the vanishing of a given function (A-11) in it. The phases have phase transition boundaries where two or more of the functions vanish. These boundaries meet at a "tricritical" point where all three functions vanish. The tricritical point is the solution of the search problem.

If the phase boundaries do not meet with any very acute angles, the solution may be found through a series of gradually refined grid searches with each refinement confined to the grid rectangles from the previous search that might contain all three phases.

Unfortunately, the singularity found above makes this search difficult if, as is always the case, the initial value problems cannot be solved exactly. The errors in the simulation introduced by truncation may produce large errors in the terminal time for sets of initial conditions potentially close to the solution. This in itself would not be a problem if it were not that the rest of the Jacobian matrix of the terminal conditions at the solution depends on the following numbers:

$$\frac{\partial s}{\partial Z} (T; Z) ; \quad (A-13)$$

where:

$$s \equiv (Y, X, p, w) ; \quad \text{and}$$

$$Z \equiv (p_0, w_0) .$$

These are the terminal values of the following initial value problem:

$$\begin{aligned} \frac{\partial s}{\partial Z} (t) &= h_s(s(t), t) \frac{\partial s}{\partial Z} (t) , \\ \frac{\partial s}{\partial Z} (0) &= (0_2; I_2) \end{aligned} \quad (A-14)$$

where:

h is defined by $\dot{s}(t) = h(s(t), t)$;

$$O_2 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} ; \quad \text{and} \quad I_2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

If the marginal cost of maintaining a zero discovery rate is zero, then the fact that exploration costs are a strictly increasing function of exploration effort implies that:

$$f_w(w(T), X(T)) = \infty. \quad (A-15)$$

This singularity in the determination of some part of (A-13) combined with errors in the determination of the terminal time can lead implicitly to errors in any search for the optimal initial conditions.

Pindyck attempted to use grid searches to solve a concrete specification of his model of exploration. His combination of specification and solution method apparently contained at least one error. This error will now be analyzed before returning to the main line of the argument.

The specification that Pindyck used is as follows:

$$d(p) = \max (h (p_{\text{choke}} - p), 0) ; \quad (\text{A-16})$$

$$c(Y, X) = c_0 \frac{R_0}{X - Y} ; \quad (\text{A-17})$$

$$k(w) = a + bw ; \quad \text{and} \quad (\text{A-18})$$

$$f(w, X) = A w^{\alpha} e^{-X/X_0} . \quad (\text{A-19})$$

In his parameterisation of this specification, it is crucial that:

$$a > 0 \quad \text{and} \quad 0 < \alpha < 1. \quad (\text{A-20})$$

A parameterised specification with fixed period costs of exploration does not satisfy property (2-9) and make condition (2-27) invalid. Unfortunately, Pindyck attempted to use this condition in trying to solve his problem.

For this paper, a series of simulations was performed using the corrected and expanded version of Pindyck's specification outlined in Section 5, and using parameter sets based on Pindyck's original parameters. To solve each boundary value problem, a series of phase space grid searches was performed as described above. The initial value problems were solved using the Euler difference method with a varying step size adapted to bound at each time step the relative change in each dependent variable. Because of the uncertain

validity of the phase analysis described above, the value of the discounted sum of the combined consumer and producer surpluses was calculated for the solution of each trial initial value problem. (A Riemann sum approximation was used to do the numerical integration.) This objective functional should have a maximum at the market equilibrium trajectory of any parameterised specification.

The phase diagram for each of the problems examined has the properties shown in Figure A-1. Thus, a lower bound for the equilibrium initial price in these specifications of the model, calculated using phase analysis, is any p_0 for which there exists a w_0 such the point given by the pair is in phase 2.

An upper bound on the equilibrium initial price may also be found. If the cessation of exploration is not used as a terminal condition for the initial value problem, and a production only programme follows the main programme, terminating when either one of the other two terminal conditions is reached, then phase 3 is split into two subphases, 3-1 where production ceases and 3-2 where production margin vanishes. Figure A-2 shows a typical phase diagram with phase 3 split up. In these circumstances, an upper bound on the equilibrium initial price is p_0 for which there exists a w_0 such that the point given by the pair is in phase 3-1.

Figure A-1

A Typical Phase Diagram

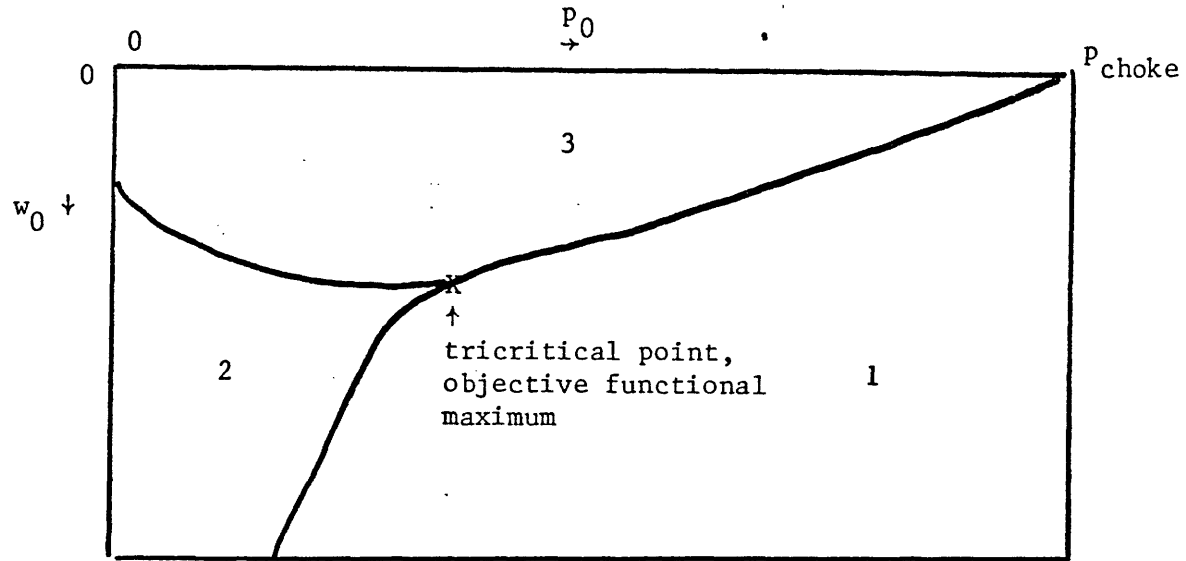
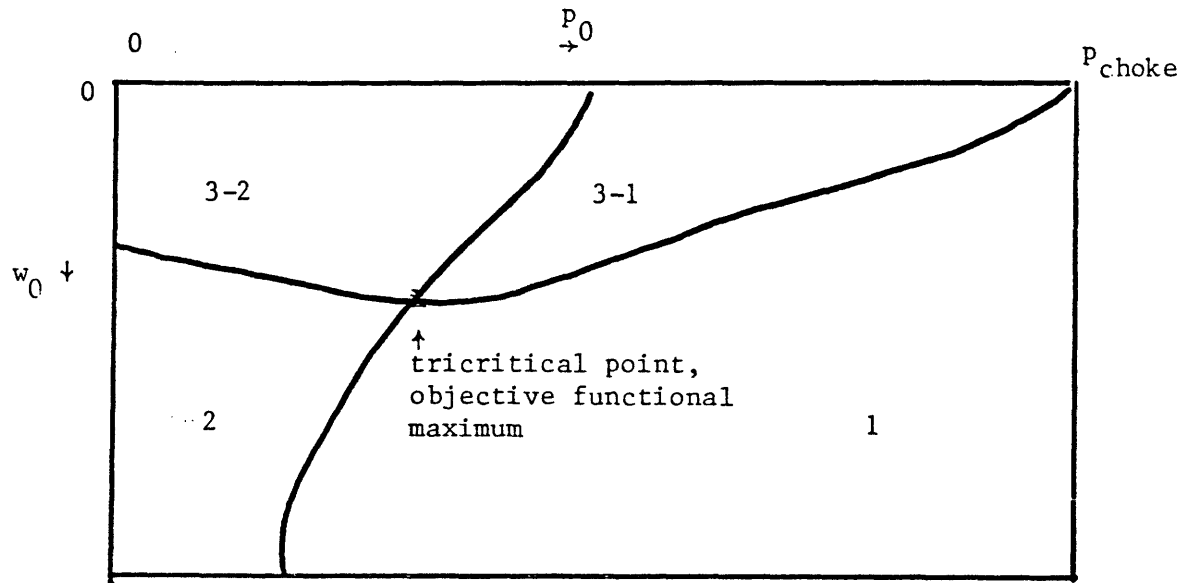


Figure A-2

Subdivision of Phase 3



An examination of the objective functional showed in each case that it is indeed maximised in region of the tricritical point. This is evidence that this phase directed search for solutions to the boundary value problem does lead to a solution.

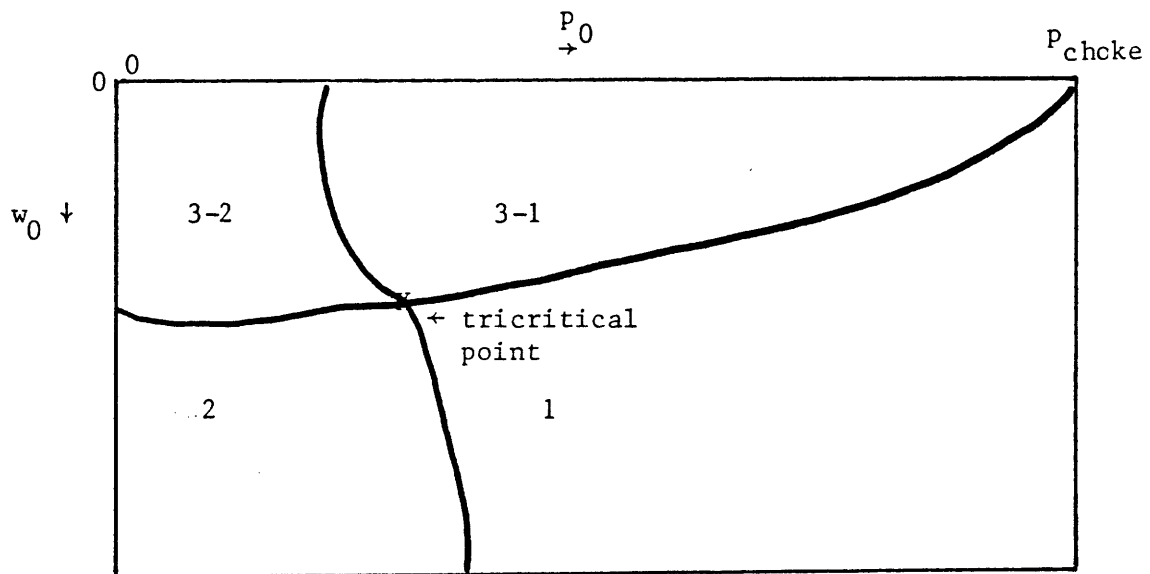
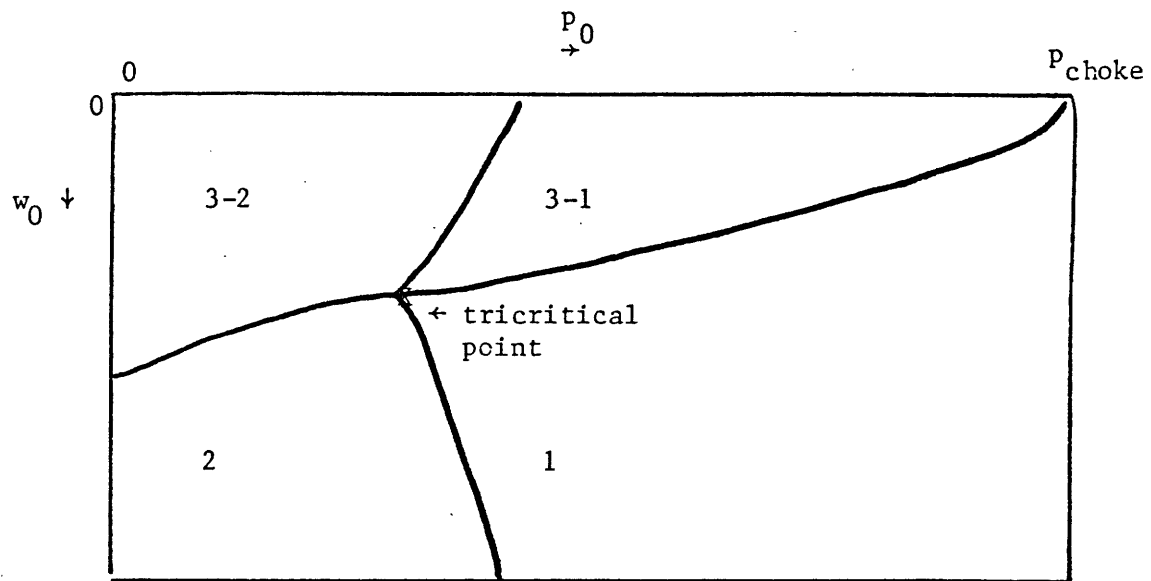
This success does not guarantee that all other parameterised specifications have exactly the same type of phase diagram. Three characteristics are probably universal:

- 1) for a given p_0 , all points in phase 3 have a smaller w_0 than do those points in phases 1 or 2;
- 2) for a given w_0 , all points in phases 2 or 3-2 have a smaller p_0 than those points in phases 1 or 3-1; and
- 3) phases 1 and 3-2 and phases 2 and 3-1 never share a boundary.

These characteristics do not guarantee the validity of the two bounding search heuristics mentioned above (see Figure A-3). Despite this, the characteristics of the phase diagram for any particular instance may be determined by a fairly gross search and equivalent heuristics developed for that case.

Finally, as was expected, the terminal behaviour of the programmes in the tricritical area for each parameter set, particularly at the boundary between phases 1 and 3-1 on the one side and phases 2 and 3-2 on the other, is very sensitive

Figure A-3
Other Types of Phase Diagram



to changes in initial conditions. Figures A-4 and A-5 show this sensitivity, particularly for the production cost and the exploration functions, in simulations using the parameter set most closely linked with the original set of Pindyck. A more precise and more stable method for solving the initial value problem might ameliorate this divergence somewhat.

Figure A-4

Price and Production Cost Trajectories for
the Corrected Pindyck Parameter Set

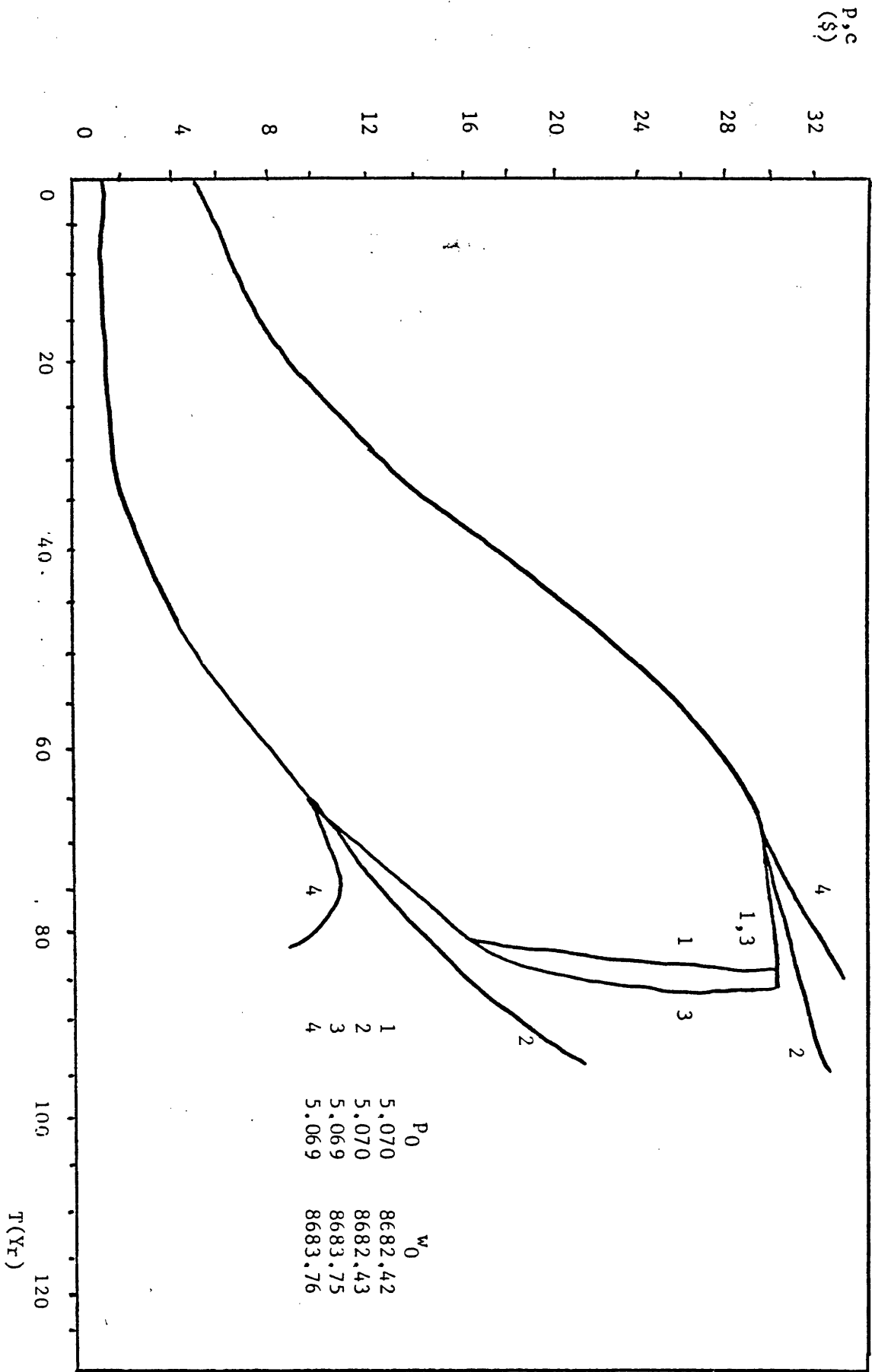
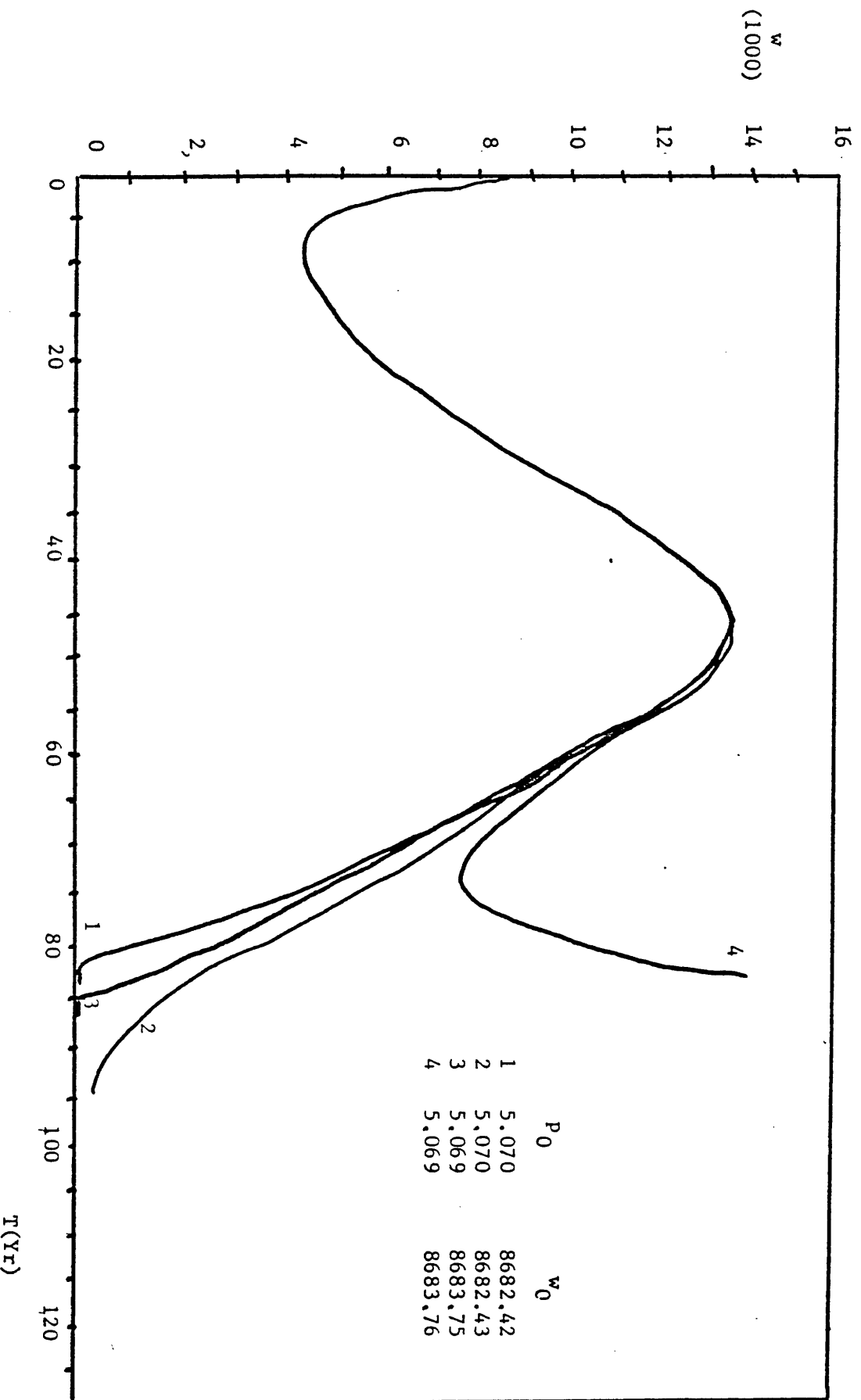


Figure A-5

Exploration Trajectories for
the Corrected Pindyck Parameter Set



References

- Arrow, K.J. and S. Chang, Journal of Environmental Economics and Management 9 (1982) 1-10
- Dasgupta, P.S. and G.M. Heal, Economic Theory and Exhaustible Resources, Cambridge (1978)
- Kamien, M. and W.L. Schwartz, Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management, North-Holland (1981)
- Lambert, J.D., Computational Methods in Ordinary Differential Equations, Wiley (1973)
- Lasserre, P., Journal of Environmental Economics and Management 11 (1984) 191-201
- Pindyck, R.S., Journal of Political Economy 86 (1978) 841-861
- Pindyck, R.S., Private Communication (1985)