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# Error Exponents for Multipath Fading Channels: A Strong Coding Theorem

by

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B.Sc., University of Melbourne (2001)
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Submitted to the Department of Electrical Engineering and Computer Science
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## Abstract

We derive upper and lower bounds on the probability of error (the exponents of which are the error exponents) with "peaky" signaling — the signaling strategy that achieves the capacity of the multipath fading channel under an average power constraint in the limit of infinite bandwidth. These bounds constitute a strong coding theorem for the channel as they not only delimit the range of achievable rates, but also give us a relationship among the error probability, data rate, bandwidth, "peakiness", and fading parameters such as the coherence time. They can be used to compare peaky signaling to other large bandwidth systems over fading channels, such as ultra-wideband (UWB) radio and wideband CDMA. We first derive an upper bound for general fading, then specialize to the case of Rayleigh fading where we obtain upper and lower bounds that are exponentially tight and therefore yield the reliability function. We study the behavior of the reliability function and the upper and lower error probability bounds numerically.

Thesis Supervisor: Muriel Médard
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# Chapter 1

# Introduction

The channel coding theorem, as derived by Shannon [14, 15], states that there is a *capacity* associated with a communications channel and that transmission can be achieved with an arbitrarily low probability of error only for rates below capacity. A stronger form of the theorem was derived by Fano [3] and Gallager [5, 6] for discrete-time memoryless channels, which gives upper and lower bounds on the minimum probability of error that decay exponentially in the block length of the code for rates below capacity. The exponents with which these bounds vanish with increasing block length are the *error exponents* for the channel. Thus, in addition to a statement regarding achievability, there is a relationship among the error probability, data rate, block length, and channel behavior. The utility of such a result is manifold; for example, by considering the modulation system as part of the channel, the error exponents can be used to yield a more meaningful comparison of various modulation systems for a coding application than the one that could be made on the basis of channel capacity alone. Another useful application of error exponents is for assessing the performance that is achievable with block coding given a constraint on the maximum block length or decoding delay — a constraint that is pertinent in any actual implementation but that is not addressed by Shannon's coding theorem.

Multipath fading channels are the standard channel model for wireless communication. Their capacity is known in the case where there is no bandwidth constraint [17]; so for rates under capacity, the probability of error can be made arbitrarily small by making the transmission bandwidth arbitrarily large. This establishes a weak achievability result. In the present work, we turn our attention to developing a strong coding theorem for the multipath fading channel, i.e. we look at upper and lower bounds on the probability of error that vanish with increasing transmission bandwidth. Note that, unlike discrete-time memoryless channels, it is the transmission bandwidth that is increased to decrease the error probability rather than the decoding delay. Though decoding delay still features, it is not of principal interest.

## 1.1 Problem motivation

The emergence of proposals for systems such as ultra-wideband (UWB) radio and wideband CDMA in recent years has brought about renewed interest in very large bandwidth fading channels. In the very large bandwidth regime, there are a number of information-theoretic results that can be brought to bear.

First, it is known that direct-sequence spread-spectrum signals perform poorly in terms of capacity. Telatar and Tse [17] considered a finite number of time-varying paths and demonstrated that the mutual information for white-like signals (such as those used in DS-CDMA) is inversely proportional to the number of resolvable paths in the wideband limit. It follows that the mutual information is close to zero if the number of resolvable paths is large. Médard and Gallager [9] considered a channel that exhibits time and frequency decorrelation and showed that the mutual information approaches zero with increasing bandwidth if spread-spectrum input signals are used. Spread-spectrum signals were characterized as those whose energy and fourth moment (kurtosis) in a fixed band scale inversely with the total bandwidth and the square of the total bandwidth respectively — loosely speaking, those that spread energy more or less evenly over the entire available band. Subramanian and Hajek [16] instead considered a wide-sense stationary uncorrelated scattering fading channel with a constraint on the fourth moment of the output signal and again found an inverse relation between capacity and bandwidth.

Secondly, given an infinite bandwidth and an average received power constraint, the capacity of the multipath fading channel is the same as that of the AWGN channel. No channel state information is assumed to be available to either the receiver or the transmitter. The capacity can be reached by transmitting at a low duty cycle and using frequency-shift keying — so called peaky signaling (transmission energy is concentrated into narrow regions of time and frequency). This result was presented by Kennedy [8] and by Gallager [6, §8.6] for the case of Rayleigh fading, and most recently by Telatar and Tse [17] for general multipath fading. The peaky transmission scheme proposed in [17], which is described fully in subsequent chapters, will form the basis of our work and will be referred to as capacity-achieving since it achieves capacity in the wideband limit.

In summary, using signals whose energy is spread evenly over a wide band results in poor performance with channels that exhibit time and frequency decorrelation (as is typical for fading channels). This makes sense intuitively since we are essentially transmitting over a large number of independent channels. If energy is spread evenly over the available band, then the ability to measure each channel decreases as the bandwidth increases and performance suffers. Rather, a peaky signaling strategy allows capacity to be achieved in the wideband limit.

It follows that, if the bandwidth is large enough, then spreading energy over that band in an even manner that keeps the fourth moment constrained, for example with direct-sequence or related spread spectrum techniques, is not advisable. In addition, peaky signaling should yield good performance. The bandwidth at which spreading begins to become detrimental, however, is not entirely clear (though the issue is partially addressed in [10]) nor is the bandwidth at which peaky signaling begins to become advantageous. The latter is the issue that will be addressed by a strong coding theorem. In particular, we shall be able to assess the performance of the capacity-achieving peaky transmission scheme for a given, finite bandwidth, thus facilitating comparison with other transmission schemes on broadband fading channels. For example, impulse radio [19, 20, 12] (a form of UWB) is a proposed spread-spectrum system that appears sub-optimal from the point of view of approaching capacity. Without a strong coding result, however, we have little means for comparing the capacity-achieving scheme with the transmission scheme used in impulse radio. The result will moreover bring out the dependence of the error probability on the other characteristics of the scheme, such as its peakiness or peak power. The peak power is an important implementation issue since it is restricted by the physical limitations of the antenna and the power supply, as well as by safety regulations.

## 1.2 Thesis outline

The necessary background is covered in Chapter 2 — we present the channel model and the capacity-achieving peaky transmission scheme expounded by Telatar and Tse in [17]. We show that peaky signaling is indeed capable of achieving capacity in the wideband limit, which constitutes a weak coding theorem for the multipath fading channel. In Chapter 3, we analyze the probability of error with peaky signaling. We upper bound the probability of error and arrive at a strong coding theorem. We shall see that an upper bound that decays much faster as bandwidth increases can be obtained if the fading process is assumed to be Rayleigh, and that, under this additional assumption, a lower bound that is exponentially tight to the upper bound can be found.

# Chapter 2

# Capacity of the multipath fading channel

In this chapter, we provide a summary of one of the main topics addressed in [17]: that of the capacity of the multipath fading channel without bandwidth constraint. This is a “weak” coding theorem in the sense that it only delimits the range of achievable rates. We shall build upon the result in the subsequent chapter.

## 2.1 Channel model

The channel output waveform $y(t)$ that results from an input waveform $x(t)$ passed through a multipath fading channel is generally given by

$$y(t) = \sum_{l=1}^{L} a_l(t)x(t - d_l(t)) + z(t), \tag{2.1}$$

where $L$ is the number of paths, $a_l(t)$ and $d_l(t)$ are the gain and delay on the $l$th path at time $t$ respectively, and $z(t)$ is white Gaussian noise with power spectral density $N_0/2$.

We associate with the channel a *coherence time* $T_c$ and *delay spread* $T_d$. The coherence time roughly quantifies the duration of time over which the passband channel is essentially time invariant; for a carrier frequency of $f_c$, the coherence time satisfies

$$\sup_{l,s,t:|s-t|\leq T_c} f_c[d_l(t) - d_l(s)] \ll 1. \tag{2.2}$$

The delay spread or multipath spread is a measure of the range of differences in the path delays; it satisfies

$$\sup_{l,m,t}[d_l(t) - d_m(t)] \leq T_d. \tag{2.3}$$

We use a block fading model in time; i.e. we assume that the processes $\{a_l(t)\}$ and $\{d_l(t)\}$ are constant and i.i.d. over time intervals of $T_c$. Real channels typically vary in a much more continuous manner, and with some statistical correlation over time intervals greater than $T_c$. The block fading assumption, however, is frequently used in the analysis of fading channels (see, for example, [10, 13, 1]) as it greatly increases the tractability of the problem while capturing the essential time-varying quality embodied by channel coherence. We assume, in addition, that the channel is *underspread*, i.e. $T_d \ll T_c$. For wireless channels, delay spreads are usually on the order of microseconds whereas coherence times are on the order of milliseconds. The underspread assumption is therefore not particularly restrictive.

## 2.2 Peaky signaling

We now describe the capacity-achieving scheme. We shall show that the scheme indeed achieves all rates below capacity in the next section.

Suppose that we have a code-book of size $M$. Let $\theta \in (0, 1]$. The $m$th code word is represented at baseband as a complex sinusoid of amplitude $\sqrt{P/\theta}$ at frequency $f_m$, i.e.

$$x_m(t) = \begin{cases} \sqrt{P/\theta}\exp(j2\pi f_m t) & 0 \leq t \leq T_s, \\ 0 & \text{otherwise}; \end{cases} \tag{2.4}$$

where the time duration of the signal $T_s$ is taken to be the coherence time $T_c$. The frequency $f_m$ is chosen such that it is an integer multiple of $1/T'_s$, where $T'_s = T_s - 2T_d$; therefore, the size of the code-book $M$ is directly related to the minimum transmission bandwidth required $W$ by $W = M/T'_s$.

Let us consider the channel output over the interval $[T_d, T_s - T_d]$. If the time axis at the receiver is shifted appropriately, then during this interval, $\{a_l(t)\}$ and $\{d_l(t)\}$ are constant owing to the assumptions of the model, and we denote their values by $\{a_l\}$ and $\{d_l\}$ respectively. Hence by (2.1), the received signal when message $m$ is sent is

$$\begin{aligned} y(t) &= \sum_{l=1}^{L} a_l \sqrt{P/\theta} \exp(j2\pi f_m(t - d_l)) + z(t) \\ &= G\sqrt{P/\theta} \exp(j2\pi f_m(t - d_l)) + z(t) \end{aligned} \tag{2.5}$$

where $G = \sum_{l=1}^{L} a_l \exp(-j2\pi f_m d_l)$ is the complex fading gain. We define signal power in the conventional sense as the received signal power, and thus normalize the channel gain so that $\mathbf{E}[|G|^2] = 1$.

At the receiver, we form the correlator outputs

$$R_k = \frac{1}{\sqrt{N_0 T'_s}} \int_{T_d}^{T_s - T_d} \exp(-j2\pi f_k t) y(t) dt \tag{2.6}$$

for $1 \leq k \leq M$. Therefore,

$$R_k = \delta_{km} G \sqrt{\frac{PT'_s}{\theta N_0}} + W_k, \tag{2.7}$$

where $\{W_k\}$ is a set of i.i.d. circularly-symmetric complex Gaussian random variables, each satisfying $\mathbf{E}[|W_k|^2] = 1$.

The message is then repeated over $N$ disjoint time intervals to obtain time diversity. Hence, denoting the $k$th correlator output at interval $n$ by $R_{k,n}$, we have

$$R_{k,n} = \delta_{km} G_n \sqrt{\frac{PT'_s}{\theta N_0}} + W_{k,n}, \tag{2.8}$$

for $1 \leq k \leq M$ and $1 \leq n \leq N$, where, owing to the block fading assumption, $\{G_n\}$ is a sequence of i.i.d. complex random variables (with no particular distribution) and $\{W_{k,n}\}$ is a set of i.i.d. circularly-symmetric complex Gaussian random variables of unit variance. We construct the decision variables

$$S_k = \frac{1}{N}\sum_{n=1}^{N} |R_{k,n}|^2 = \frac{1}{N}\sum_{n=1}^{N} \left| \delta_{km} G_n \sqrt{\frac{PT'_s}{\theta N_0}} + W_{k,n} \right|^2 \tag{2.9}$$

and use a threshold decoding rule: Let

$$A = 1 + (1-\epsilon)\frac{PT'_s}{\theta N_0} \tag{2.10}$$

(where $\epsilon \in (0,1)$ is an arbitrary parameter) be the threshold. If $S_k$ exceeds $A$ for one value of $k$ only, then we estimate $\hat{m} = k$; otherwise we declare an error. Note that the decoding rule is non-coherent, measuring only the energy of the received signal.

We transmit using the above scheme for a fraction of time $\theta$ and then transmit nothing for the remainder of the time. Hence the average power is $P$. Observe that the scheme transmits $\ln M$ nats in $NT_s/\theta$ seconds, so the rate $R$ is given by

$$R = \frac{\theta}{NT_s}\ln(M). \tag{2.11}$$

## 2.3 The weak coding theorem

The following theorem establishes a weak coding result for the channel.

**Theorem 2.3.1 (Weak coding theorem for multipath fading channels)** *All data rates $R$ that satisfy*

$$R < \left(1 - 2\frac{T_d}{T_c}\right)\frac{P}{N_0} \tag{2.12}$$

*can be achieved with an arbitrarily small probability of error over a multipath fading channel with average power constraint $P$ but no bandwidth constraint.*

*Proof.* Owing to symmetry, we can assume without loss of generality that the message $m = 1$ was sent. An error occurs if $S_1 < A$ or if $S_k \geq A$ for some $2 \leq k \leq M$. Let $B_1$ be the event that $S_1 < A$ given $m = 1$ and let $B_k$ be the event that $S_k \geq A$ given $m = 1$ for $2 \leq k \leq M$. Then, denoting the probability of error by $p_e$, we have

$$\begin{aligned} p_e &= \Pr\left\{\bigcup_{k=1}^{M} B_k\right\} \\ &\leq p_e^{(1)} + M p_e^{(2)}, \end{aligned} \tag{2.13}$$

where, for notational convenience, we have defined

$$p_e^{(1)} \triangleq \Pr\{B_1\}, \tag{2.14}$$

$$p_e^{(2)} \triangleq \Pr\{B_2\}. \tag{2.15}$$

Observe that

$$\begin{aligned} \mathbf{E}[S_1|m = 1] &= \mathbf{E}\left[\left|G_1\sqrt{\frac{PT_s'}{\theta N_0}} + W_{1,1}\right|^2\right] \\ &= A + \epsilon\frac{PT_s'}{\theta N_0}, \end{aligned} \tag{2.16}$$

so $\mathbf{E}[S_1|m = 1] > A$. Hence it is evident that $p_e^{(1)} \to 0$ as $N \to \infty$ since $\{G_n\}$ is an i.i.d. sequence and therefore ergodic (in fact, the weak result still holds if we relax the i.i.d. assumption to an ergodicity assumption). Recalling that the rate is given by (2.11), it follows that for a fixed rate $R$, $p_e^{(1)} \to 0$ as $M \to \infty$.

To upper bound $p_e^{(2)}$, we use a Chernoff bound:

$$\begin{aligned} p_e^{(2)} &= \Pr\{NS_2 \geq NA|m = 1\} \\ &\leq \exp\left(-N\sup_{r>0}\{rA - \ln(\mathbf{E}[\exp(r|W_{2,1}|^2)])\}\right). \end{aligned} \tag{2.17}$$

Since $W_{2,1}$ is a circularly-symmetric complex Gaussian random variable with unit variance, it follows that $|W_{2,1}|^2$ is an exponentially-distributed random variable with unit mean, and $\mathbf{E}[\exp(r|W_{2,1}|^2)$ is its moment-generating function. Hence

$$\begin{aligned} p_e^{(2)} &\leq \exp\left(-N \sup_{r>0}\{rA - \ln(1-r)\}\right) \\ &= \exp(-N[A - 1 - \ln(A)]). \end{aligned} \tag{2.18}$$

By substituting for the threshold using (2.10) and for $N$ using (2.11), we can write

$$p_e^{(2)} \leq \exp\left(-\ln(M) \cdot \frac{\theta E_2(\theta,\epsilon)}{RT_s}\right), \tag{2.19}$$

where

$$E_2(\theta,\epsilon) = \frac{(1-\epsilon)PT_s'}{\theta N_0} - \ln\left(1 + \frac{(1-\epsilon)PT_s'}{\theta N_0}\right). \tag{2.20}$$

Therefore,

$$Mp_e^{(2)} \leq \exp\left(-\ln(M)\left[\frac{\theta E_2(\theta,\epsilon)}{RT_s} - 1\right]\right). \tag{2.21}$$

Hence $Mp_e^{(2)} \to 0$ as $M \to 0$ as long as $\theta E_2(\theta,\epsilon)/RT_s > 1$, i.e.

$$\frac{(1-\epsilon)PT_s'}{N_0} - RT_s - \theta\ln\left(1 + \frac{(1-\epsilon)PT_s'}{\theta N_0}\right) > 0, \tag{2.22}$$

which is an equivalent condition to

$$R < \frac{(1-\epsilon)PT_s'}{T_s N_0} - \frac{\theta}{T_s}\ln\left(1 + \frac{(1-\epsilon)PT_s'}{\theta N_0}\right). \tag{2.23}$$

Noting that $\theta$ and $\epsilon$ may be taken arbitrarily close to 0 and recalling that $T_s = T_c$, the result follows. □

## 2.4 Converse to the coding theorem

We can view the multipath fading channel as the cascade of two channels: one that causes the multipath fading and a second that adds white Gaussian noise. Since the capacity of the continuous-time Gaussian channel without bandwidth constraint is $P/N_0$ (see, for example, [2, §10.3]), then owing to the data processing inequality (see, for example, [2, §2.8]), it follows that the capacity of the cascaded channel can be no greater than $P/N_0$. Therefore, transmission with an arbitrarily small probability of error cannot take place for rates greater than $P/N_0$.

In summary, the capacity of the multipath fading channel with average power constraint $P$ but no bandwidth constraint is at least $(1-2T_d/T_c)P/N_0$ (which is very close to $P/N_0$ because of the underspread assumption) and at most $P/N_0$. Thus, the peaky transmission scheme from §2.2 is referred to as capacity-achieving, and we equate the capacity of the multipath fading channel to that of the AWGN channel, though we keep in mind that these statements do not apply in an exact sense.

# Chapter 3

# Probability of error with peaky signaling

We now derive upper and lower bounds on the probability of error with peaky signaling; specifically, with the capacity-achieving peaky transmission scheme described in §2.2. These bounds decay to zero with increasing transmission bandwidth for all rates under capacity, thus yielding a "strong" coding theorem that differs from the weak coding theorem of §2.3 in that it not only delimits the range of achievable rates, but also brings out the relationship among the error probability, data rate, and parameters of the transmission scheme. It is worth emphasizing that the lower bound applies to the probability of error *of the capacity-achieving scheme* and not to general transmission schemes over fading channels. Therefore, there may exist transmission schemes that achieve better performance than the lower bound. Nevertheless, the lower bound is useful as it gives us a notion of the tightness of the upper bound.

The capacity and error exponents associated with a fading channel have been studied previously by Telatar [18] though the model used is very different. Telatar built upon Gallager's results for energy-limited channels [7], where the channel is modeled as discrete-time and discrete-input. He showed that, using random block codes and 0-1 signaling, the capacity of the Rayleigh fading channel is the same as that of the AWGN channel in the limit of large bandwidth and large signal-to-noise ratio, and examined the rate at which this limiting behavior is approached. He found that the rate of this approach is very slow and therefore, under the conditions imposed by the energy-limited formulation, inordinately large bandwidths are required for good performance, not to mention very complex encoding and decoding schemes.

## 3.1 The strong coding theorem for multipath fading channels

Recall from the previous chapter that we have the following upper bound on the error probability $p_e$:

$$p_e \leq p_e^{(1)} + M p_e^{(2)} \tag{3.1}$$

where, conditioned upon the message being $m = 1$, $p_e^{(1)}$ is the probability that the decision variable $S_1$ does not exceed the threshold $A$ (the event $B_1$) and $p_e^{(2)}$ is the probability that the decision variable $S_2$ exceeds the threshold (the event $B_2$). An upper bound on $p_e^{(2)}$ was obtained and thus, in order to upper bound $p_e$, it remains only to find an upper bound on $p_e^{(1)}$. As a first step, we can upper bound $p_e^{(1)}$ using the Chebyshev inequality.

**Theorem 3.1.1 (Strong coding theorem for multipath fading channels)** *Let a multipath fading channel have coherence time $T_c$, delay spread $T_d$, and Gaussian noise of power spectral density $N_0/2$. Let $\sigma^2 = \mathrm{var}(|G_n|^2)$ be the variance of the complex fading gains. Then there exists a transmission scheme of average power $P$, peak power $P/\theta$, bandwidth $M/T'_s$, and rate $R$ with probability of error upper bounded by*

$$\begin{aligned} p_e \leq \min_{\epsilon \in (0,1)} \Bigg\{ & \frac{R T_s}{\epsilon^2 \ln(M)} \left( \frac{\sigma^2}{\theta} + \frac{2N_0}{P T'_s} + \frac{\theta N_0^2}{P^2 T_s'^2} \right) \\ & + \exp\left( -\frac{\ln(M)}{R T_s} \left[ \frac{(1-\epsilon) P T'_s}{N_0} - R T_s - \theta \ln\left(1 + \frac{(1-\epsilon) P T'_s}{\theta N_0}\right) \right] \right) \Bigg\} \end{aligned} \tag{3.2}$$

*where $\theta$ is in the half-open interval $(0, 1]$, $T_s = T_c$, and $T'_s = T_s - 2T_d$.*

*Moreover, for all rates $R$ less than $(1-2T_d/T_c)P/N_0$, the probability of error can be made to vanish as $M \to \infty$.*

*Proof.* We use the transmission scheme described in §2.2. We observe that, in this scheme,

$$\begin{aligned}\operatorname{var}(S_1|m=1) &= \frac{1}{N}\operatorname{var}\left(\left|G_1\sqrt{\frac{PT_s'}{\theta N_0}}+W_{1,1}\right|^2\right)\\ &= \frac{1}{N}\operatorname{var}\left(\frac{PT_s'}{\theta N_0}|G_1|^2+|W_{1,1}|^2+2\sqrt{\frac{PT_s'}{\theta N_0}}\operatorname{Re}\{G_1W_{1,1}\}\right).\end{aligned} \tag{3.3}$$

Keeping in mind that $G_1$ and $W_{1,1}$ are independent, it is straightforward to verify that

$$\operatorname{cov}\left(\frac{PT_s'}{\theta N_0}|G_1|^2+|W_{1,1}|^2,\ 2\sqrt{\frac{PT_s'}{\theta N_0}}\operatorname{Re}\{G_1W_{1,1}\}\right)=0, \tag{3.4}$$

hence

$$\begin{aligned}\operatorname{var}(S_1|m=1) &= \frac{1}{N}\left[\operatorname{var}\left(\frac{PT_s'}{\theta N_0}|G_1|^2+|W_{1,1}|^2\right)+\operatorname{var}\left(2\sqrt{\frac{PT_s'}{\theta N_0}}\operatorname{Re}\{G_1W_{1,1}\}\right)\right]\\ &= \frac{1}{N}\left[\frac{P^2T_s'^2\sigma^2}{\theta^2N_0^2}+\frac{2PT_s'}{\theta N_0}+1\right],\end{aligned} \tag{3.5}$$

and substituting for $N$ using (2.11) gives

$$\operatorname{var}(S_1|m=1)=\frac{RT_s}{\theta\ln(M)}\left[\frac{P^2T_s'^2\sigma^2}{\theta^2N_0^2}+\frac{2PT_s'}{\theta N_0}+1\right]. \tag{3.6}$$

Recalling (2.16) for the expectation of $S_1$ given that $m=1$, we have

$$\begin{aligned}p_e^{(1)} &= \Pr\{S_1<A|m=1\}\\ &\le \Pr\left\{\left|S_1-\overline{S_1}\right|>\epsilon\frac{PT_s'}{\theta N_0}\middle|m=1\right\}.\end{aligned} \tag{3.7}$$

We now apply the Chebyshev inequality to obtain

$$\begin{aligned} p_e^{(1)} &\leq \operatorname{var}(S_1|m=1)\frac{\theta^2 N_0^2}{\epsilon^2 P^2 T_s'^2} \\ &= \frac{RT_s}{\epsilon^2 \ln(M)}\left(\frac{\sigma^2}{\theta} + \frac{2N_0}{PT_s'} + \frac{\theta N_0^2}{P^2 T_s'^2}\right). \end{aligned} \tag{3.8}$$

The upper bound (3.2) follows by substituting the upper bound on $p_e^{(1)}$ given by (3.8) and the upper bound on $Mp_e^{(2)}$ given by (2.21) combined with (2.20) into (3.1) and minimizing $\epsilon$ over its domain to set the optimal threshold.

The second part of the theorem follows by noting that the first term of the upper bound, being the upper bound on $p_e^{(1)}$, vanishes as $M \to \infty$ for all $R$ while the second term of the upper bound, being the upper bound on $Mp_e^{(2)}$, can be made to vanish as $M \to \infty$ only for $R < (1 - 2T_d/T_c)P/N_0$, as established in §2.3. $\square$

Although Theorem 3.1.1 does indeed give a valid upper bound on the error probability, the bound decays very slowly in $M$: For sufficiently large $M$, the first additive term dominates, which decays as $1/\ln(M)$. The theorem, however, holds for any distribution of $G_n$[1]. If we were given some information on the statistics of $G_n$, we could determine the statistics of $S_1$, which depend on those of $G_n$ through

$$S_1 = \frac{1}{N}\sum_{n=1}^{N}\left|G_n\sqrt{\frac{PT_s'}{\theta N_0}} + W_{1,n}\right|^2, \tag{3.9}$$

and potentially obtain a bound that decays much faster.

## 3.2 The strong coding theorem for Rayleigh fading channels

In wireless channels, the *Rayleigh fading* model is commonly used, where $G_n$ is modeled as a circularly-symmetric complex Gaussian random variable, so $|G_n|$ has a

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[1] We speak here of any $G_n$ for $1 \leq n \leq N$; it does not matter which one, since their distributions are all identical

Rayleigh distribution. The Rayleigh fading model is valid when there is a large number of independent scatterers or reflectors in the channel and no line-of-sight or specular path; wherein application of the central limit theorem leads to the model. Even though there may be a relatively small number of reflectors in typical wireless situations, the Rayleigh fading model is often adopted regardless because of its simplicity.

Making the additional assumption of Rayleigh fading leads to a simple exponential bound on the error probability:

**Theorem 3.2.1 (Strong coding theorem for Rayleigh fading channels)** *Let a Rayleigh fading channel have coherence time $T_c$, delay spread $T_d$, and Gaussian noise of power spectral density $N_0/2$. Then there exists a transmission scheme of average power $P$, peak power $P/\theta$, bandwidth $M/T_s'$, and rate $R$ with probability of error upper bounded by*

$$p_e \leq 2\exp(-\ln(M)\cdot E(R,\theta)), \tag{3.10}$$

*where*

$$\begin{aligned} E(R,\theta) = \frac{\theta}{RT_s}\Bigg\{ & \frac{RT_sN_0}{PT_s'} + \frac{\theta N_0}{PT_s'}\ln\left(1+\frac{PT_s'}{\theta N_0}\right) - 1 \\ & - \ln\left(\frac{RT_sN_0}{PT_s'} + \frac{\theta N_0}{PT_s'}\ln\left(1+\frac{PT_s'}{\theta N_0}\right)\right)\Bigg\}, \end{aligned} \tag{3.11}$$

*$\theta$ is in the half-open interval $(0,1]$, $T_s = T_c$, and $T_s' = T_s - 2T_d$.*

*Moreover, for all rates $R$ less than $(1-2T_d/T_c)P/N_0$, the probability of error can be made to vanish as $M\to\infty$.*

*Proof.* We again use the transmission scheme described in §2.2. Applying the Chernoff bound yields

$$\begin{aligned} p_e^{(1)} &= \Pr\{NS_1 < NA | m=1\} \\ &\leq \exp\left(-N\sup_{r<0}\{rA - \ln(\mathbf{E}[\exp(r|G_1\sqrt{PT_s'/(\theta N_0)} + W_{1,1}|^2)])\}\right). \end{aligned} \tag{3.12}$$

Now, since we have assumed that the fading is Rayleigh, the $G_n$ are i.i.d. circularly-symmetric complex Gaussian random variables so the $|G_n\sqrt{PT_s'/(\theta N_0)} + W_{1,n}|^2$ are i.i.d. exponentially-distributed random variables with mean $PT_s'/(\theta N_0) + 1$. Thus, the expectation of $\exp(r|G_1\sqrt{PT_s'/(\theta N_0)} + W_{1,1}|^2)$ is simply the moment-generating function of an exponentially-distributed random variable, and we have

$$\begin{aligned} p_e^{(1)} &\leq \exp\left(-N \sup_{r<0}\{rA - \ln(1 - r[1 + PT_s'/(\theta N_0)])\}\right) \\ &= \exp(-N[A' - 1 - \ln(A')]), \end{aligned} \tag{3.13}$$

where

$$A' = \frac{A}{PT_s'/(\theta N_0) + 1}. \tag{3.14}$$

By substituting (2.10) and (2.11), we can write

$$p_e^{(1)} \leq \exp\left(-\ln(M) \cdot \frac{\theta E_1(\theta, \epsilon)}{RT_s}\right), \tag{3.15}$$

where

$$E_1(\theta, \epsilon) = \frac{-\epsilon PT_s'}{\theta N_0 + PT_s'} - \ln\left(1 - \frac{\epsilon PT_s'}{\theta N_0 + PT_s'}\right). \tag{3.16}$$

For notational convenience, we define $p_{e,u}^{(1)}$ to be the upper bound on $p_e^{(1)}$ given by (3.15) and $p_{e,u}^{(2)}$ to be the upper bound on $p_e^{(2)}$ given by (2.19), viz.

$$p_{e.u}^{(i)} = \exp\left(-\ln(M) \cdot \frac{\theta E_i(\theta, \epsilon)}{RT_s}\right), \tag{3.17}$$

for $i = 1, 2$.

Now, if we choose $\epsilon$ optimally, we have

$$p_e \leq \min_{\epsilon \in (0,1)} \{p_{e,u}^{(1)} + Mp_{e,u}^{(2)}\}. \tag{3.18}$$

It is evident upon differentiation that, as functions of $\epsilon$, $p_{e,u}^{(1)}$ is strictly decreasing whilst $Mp_{e,u}^{(2)}$ is strictly increasing. In addition, $p_{e,u}^{(1)} = Mp_{e,u}^{(2)}$ when

$$\epsilon = \epsilon_0 \triangleq \frac{\theta N_0 + PT_s'}{PT_s'} \left[ 1 - \frac{RT_s N_0}{PT_s'} - \frac{\theta N_0}{PT_s'} \ln \left( 1 + \frac{PT_s'}{\theta N_0} \right) \right], \tag{3.19}$$

which is in the interval $(0,1)$ if

$$0 \leq R < \frac{T_s' P}{T_s N_0} - \frac{\theta}{T_s} \ln \left( 1 + \frac{PT_s'}{\theta N_0} \right). \tag{3.20}$$

Therefore, given that (3.20) is satisfied, we can upper bound (3.18) by

$$\begin{aligned} p_e &\leq 2 \min_{\epsilon \in (0,1)} \{ \max(p_{e,u}^{(1)}, Mp_{e,u}^{(2)}) \} \\ &= 2p_{e,u}^{(1)} \big|_{\epsilon = \epsilon_0} \\ &= 2 \exp(-\ln(M) \cdot E(R, \theta)), \end{aligned} \tag{3.21}$$

where $E(R, \theta) = \theta E_1(\theta, \epsilon_0)/(RT_s)$ is given by equation (3.11).

Observe that we can write $E(R, \theta) = \theta[z - 1 - \ln(z)]/(RT_s)$ where

$$z = \frac{RT_s N_0}{PT_s'} + \frac{\theta N_0}{PT_s'} \ln \left( 1 + \frac{PT_s'}{\theta N_0} \right) > 0. \tag{3.22}$$

Hence, since $\ln(z) < z - 1$ for all $z > 0$, $E(R, \theta) > 0$ over its domain of definition, given by (3.20). The interval (3.20) grows as $\theta$ decreases, approaching the interval that encompasses all rates under capacity as $\theta$ approaches zero. Thus, there exists $\theta \in (0, 1)$ such that $E(R, \theta)$ is positive as long as the rate does not exceed capacity (i.e. (2.12) is satisfied) and therefore such that the bound on $p_e$ given by (3.21) vanishes as $M$ approaches infinity. □

## 3.3 Lower bound on the error probability

We have arrived at an exponential upper bound on the probability of error of the capacity-achieving scheme. There is, however, little information on the tightness of the bound. In a manner analogous to the case of the discrete memoryless channel, we derive a lower bound on the probability of error and compare it to the upper bound.

**Theorem 3.3.1**

*The transmission scheme described in §2.2 has a probability of error that is lower bounded by*

$$\begin{aligned}p_e &> \exp(-N[E_1(\theta,\epsilon)+o_1(N)]) + (M-1)\exp(-N[E_2(\theta,\epsilon)+o_2(N)]) \\ &\quad -(M-1)\exp(-N[E_1(\theta,\epsilon)+E_2(\theta,\epsilon)]) - \frac{(M-1)(M-2)}{2}\exp(-N[2E_2[\theta,\epsilon)]),\end{aligned} \tag{3.23}$$

*where $E_1(\theta,\epsilon)$ and $E_2(\theta,\epsilon)$ are given by (3.16) and (2.20) respectively. The quantities $o_1(N)$ and $o_2(N)$ approach zero with increasing $N$ and can be taken as*

$$o_1(N) = \frac{1}{2N}\ln(2\pi N) + \frac{1}{12N^2}, \tag{3.24}$$

$$o_2(N) = \frac{1}{2N}\ln(2\pi N A^2) + \frac{1}{12N^2}. \tag{3.25}$$

*Proof.* Having made the Rayleigh fading assumption, the probabilities $p_e^{(1)}$ and $p_e^{(2)}$ can in fact be evaluated exactly since, conditioned upon $m = 1$, $S_1$ and $S_2$ are both $\chi^2$ random variables with $2N$ degrees of freedom. Indeed, using the cdf for $\chi^2$ random variables [11, §2.1.4], we have

$$\begin{aligned}p_e^{(1)} &= \Pr\{S_1 < A|m=1\} \\ &= \Pr\left\{\sum_{n=1}^{N}\left|G_n\sqrt{\frac{PT_s'}{\theta N_0}} + W_{1,n}\right|^2 < NA\right\} \\ &= \exp(-NA')\sum_{k=N}^{\infty}\frac{(NA')^k}{k!},\end{aligned} \tag{3.26}$$

where $A'$ is defined by equation (3.14). Similarly,

$$p_e^{(2)} = \exp(-NA)\sum_{k=0}^{N-1}\frac{(NA)^k}{k!}. \tag{3.27}$$

Because $(NA')^k/k!$ and $(NA)^k/k!$ are both positive for all $k$, we arrive at the inequalities

$$\sum_{k=N}^{\infty} \frac{(NA')^k}{k!} \geq \frac{(NA')^N}{N!}, \tag{3.28}$$

and

$$\sum_{k=0}^{N-1} \frac{(NA)^k}{k!} \geq \frac{(NA)^{(N-1)}}{(N-1)!} \tag{3.29}$$

by taking only one of the summation terms. Therefore,

$$p_e^{(1)} \geq \exp\left\{-NA' + \ln\left[\frac{(NA')^N}{N!}\right]\right\} \tag{3.30}$$

and

$$p_e^{(2)} \geq \exp\left\{-NA + \ln\left[\frac{(NA)^{N-1}}{(N-1)!}\right]\right\}. \tag{3.31}$$

We now apply Stirling's formula [4, §II.9] to bound the factorial function, and obtain the lower bounds

$$\begin{aligned} p_e^{(1)} &> \exp\{-N(A' - 1 - \ln(A') + o_1(N))\} \\ &= \exp(-N[E_1(\theta,\epsilon) + o_1(N)]) \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} p_e^{(2)} &> \exp\{-N(A - 1 - \ln(A) + o_2(N))\} \\ &= \exp(-N[E_2(\theta,\epsilon) + o_2(N)]) \end{aligned} \tag{3.33}$$

where $o_1(N)$ and $o_2(N)$ are given by (3.24) and (3.25) respectively.

By comparing the two lower bounds above with the upper bounds previously obtained, (3.13) and (2.19), we notice that they are exponentially tight to their respective upper bounds in the sense that the exponents are arbitrarily close for $N$

sufficiently large or, equivalently, for $\ln(M)$ sufficiently large. Indeed, we have

$$\exp(-N[E_i(\theta,\epsilon)+o_i(N)]) < p_e^{(i)} \leq \exp(-NE_i(\theta,\epsilon)) \tag{3.34}$$

for $i = 1, 2$.

Having found lower bounds on $p_e^{(1)}$ and $p_e^{(2)}$, we are now in a position to derive a lower bound on the error probability $p_e$. We commence with the following observation:

$$\begin{aligned} p_e &\geq \sum_{k=1}^{M} \Pr\{B_k\} - \sum_{j\neq k} \Pr\{B_j \cap B_k\} \\ &= p_e^{(1)} + (M-1)p_e^{(2)} - (M-1)p_e^{(1)}p_e^{(2)} - \frac{(M-1)(M-2)}{2} p_e^{(2)2} \end{aligned} \tag{3.35}$$

by the independence of the events $B_k$. Using (3.34), we straightforwardly obtain (3.23), thus completing the proof. □

We now turn our attention to the exponential dependence of error probability on $\ln(M)$, which, we recall, is directly related to the bandwidth. We define the *reliability function* of the Rayleigh fading channel using peaky signaling with duty factor $\theta$ as

$$\lim_{M\to\infty} \frac{-\ln(\min_{\epsilon\in(0,1)} p_e)}{\ln(M)}, \tag{3.36}$$

analogously to the treatment of the discrete memoryless channel [6, §5.8]. It represents the true exponential dependence of the error probability on $\ln(M)$ for $M$ sufficiently large.

**Theorem 3.3.2**

*The reliability function of the Rayleigh fading channel using peaky signaling with duty factor $\theta$ is equal to $E(R,\theta)$, the error exponent of the upper bound, for all non-negative rates $R$.*

*Proof.* Applying l'Hôpital's rule to the lower bound (3.23) yields

$$\lim_{N\to\infty} \frac{-\ln(p_e)}{N} \leq \min(E_1(\theta,\epsilon), E_2(\theta,\epsilon) - RT_s/\theta) \tag{3.37}$$

for $0 \leq RT_s/\theta < A - 1 - \ln(A)$. Recall from §3.2 that, before optimization over $\epsilon$, we have the following upper bound on the error probability:

$$p_e \leq \exp(-NE_1(\theta, \epsilon)) + \exp(-N[E_2(\theta, \epsilon) - RT_s/\theta]), \tag{3.38}$$

from which the reverse inequality to (3.37) follows straightforwardly.

For $RT_s/\theta \geq A - 1 - \ln(A)$, we use

$$\begin{aligned} p_e &= 1 - \Pr\left\{\bigcap_{k=1}^{M} B_k^c\right\} \\ &\geq 1 - (1 - p_e^{(1)})(1 - p_e^{(2)})^{(M-1)}, \end{aligned} \tag{3.39}$$

which implies

$$p_e \geq 1 - (1 - p_e^{(2)})^{(M-1)}, \tag{3.40}$$

and get, again by applying l'Hôpital's rule,

$$\lim_{N\to\infty} \ln(p_e) \geq 0. \tag{3.41}$$

By noting that $p_e$ is a probability and is therefore at most 1, the reverse inequality follows. Hence

$$\lim_{N\to\infty} \frac{-\ln(p_e)}{N} = 0. \tag{3.42}$$

Thus we see that the upper and lower bounds on $p_e$ are exponentially tight.

We have now established that $-\ln(p_e)/N$ converges for all $\epsilon \in (0, 1)$ and, coupled with the fact that the function ln is monotonically increasing, it follows that

$$\lim_{N\to\infty} \frac{-\ln(\min_{\epsilon\in(0,1)} p_e)}{N} = \lim_{N\to\infty} \max_{\epsilon\in(0,1)} \frac{-\ln(p_e)}{N} = \max_{\epsilon\in(0,1)} \lim_{N\to\infty} \frac{-\ln(p_e)}{N} = E_1(\theta, \epsilon_0) \tag{3.43}$$

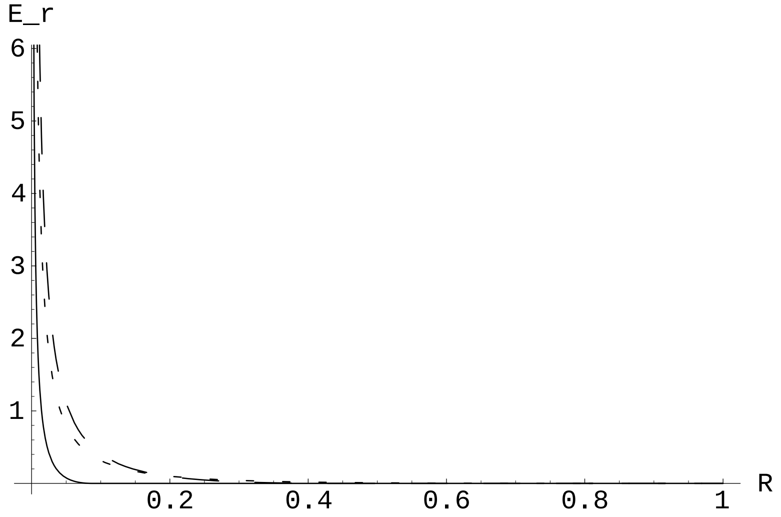

Figure 3-1: Reliability function $E_r(R, \theta)$ as a function of $R$ for $\theta = 10^{-2}$ (solid), $\theta = 10^{-3}$ (dashed), and $\theta = 10^{-4}$ (dotted).

where $\epsilon_0$ is such that $E_1(\theta, \epsilon_0) = E_2(\theta, \epsilon_0) - RT_s/\theta$ and is given by equation (3.19). Hence

$$\lim_{M\to\infty} \frac{-\ln(\min_{\epsilon\in(0,1)} p_e)}{\ln(M)} = \lim_{N\to\infty} \frac{-\theta \ln(\min_{\epsilon\in(0,1)} p_e)}{NRT_s} = \frac{\theta E_1(\theta, \epsilon_0)}{RT_s} = E(R, \theta). \quad (3.44)$$

□

## 3.4 Numerical results

We now proceed to evaluate the quantities derived in the previous sections for particular parameter choices. We choose fading parameters that are typical for very-high frequency transmission in an indoor environment: Let $T_d = 10^{-7}$ s and $T_c = T_s = 2 \times 10^{-3}$ s. Suppose the peak power limitation is $P/\theta \leq 250$; and let $P = N_0 = 1$, so $C \simeq 1$ nat/s (1.44 bits/s). This choice of peak power restriction is not unreasonable; if the average power of the transmitter is 1 mW, for example, then the restriction implies that its peak power will not exceed 0.25 W.

We commence by looking at the behavior of the reliability function (3.11) for

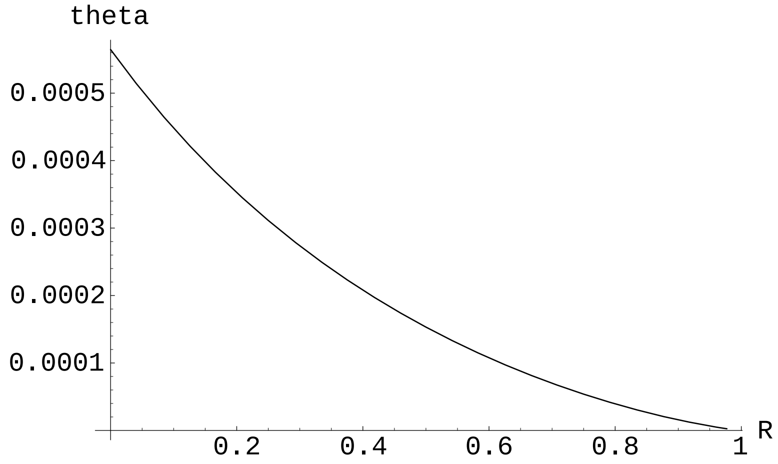

Figure 3-2: Optimal duty factor $\theta$ as a function of $R$.

various values of the duty factor $\theta$, as shown in Figure 3-1. Note the rapid decay of the exponent. We therefore expect that the minimum bandwidth required to achieve a particular performance to increase very rapidly as the rate approaches capacity. It is also evident that smaller values of the duty factor are required to achieve higher rates, though the optimal $\theta$ for a given rate is not immediately apparent. This optimization can be performed numerically and the result is shown in Figure 3-2. As expected, we see that the optimal duty factor gradually decreases to zero as capacity is approached. More surprising, however, is the fact that, even for very low rates, it is necessary that $\theta \simeq 5 \times 10^{-4}$ for a maximal error exponent, which translates to a peak power that is approximately 2000 times larger than the average. Thus, recalling that the peak power limitation is $P/\theta \leq 250$, it follows that, for any rate, the duty factor is optimized over its restricted domain for $\theta = 4 \times 10^{-3}$.

We now turn to investigating the interplay among the physical parameters of interest. The upper and lower bounds are shown as functions of the bandwidth at an $E_b/N_0$ of 13 dB (i.e. a rate of 0.035 nats/s or 0.050 bits/s) in Figure 3-3 and at an $E_b/N_0$ of 15 dB (i.e. a rate of 0.022 nats/s or 0.032 bits/s) in Figure 3-4. These plots allow us to estimate the bandwidth required to achieve a particular performance for a given $E_b/N_0$. In Figures 3-5 and 3-6, we fix the bandwidth at 1 GHz and 10

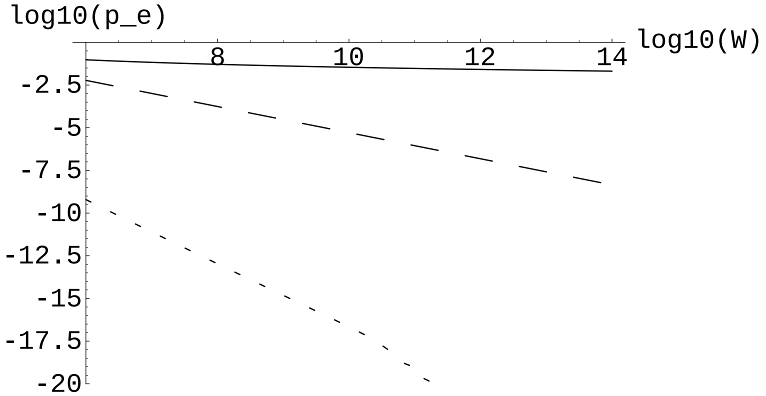

Figure 3-3: Error probability as a function of bandwidth at an $E_b/N_0$ of 13 dB; upper bound for the multipath fading channel (solid), upper bound for the Rayleigh fading channel (dashed), and lower bound for the Rayleigh fading channel (dotted). The bandwidth $W$ is in Hz.

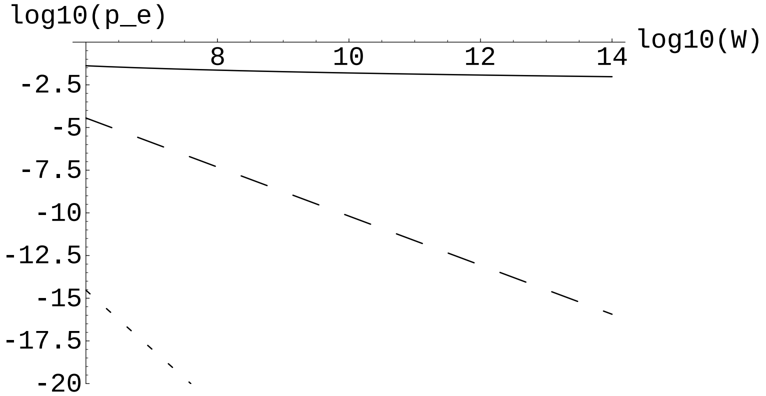

Figure 3-4: Error probability as a function of bandwidth at an $E_b/N_0$ of 15 dB; upper bound for the multipath fading channel (solid), upper bound for the Rayleigh fading channel (dashed), and lower bound for the Rayleigh fading channel (dotted). The bandwidth $W$ is in Hz.

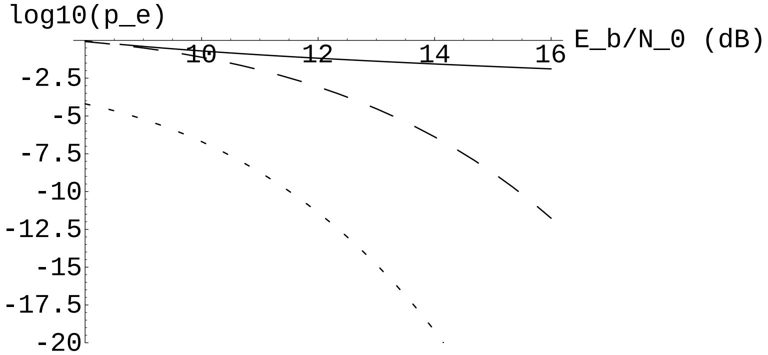

Figure 3-5: Error probability as a function of $E_b/N_0$ at a bandwidth of 1 GHz; upper bound for the multipath fading channel (solid), upper bound for the Rayleigh fading channel (dashed), and lower bound for the Rayleigh fading channel (dotted).

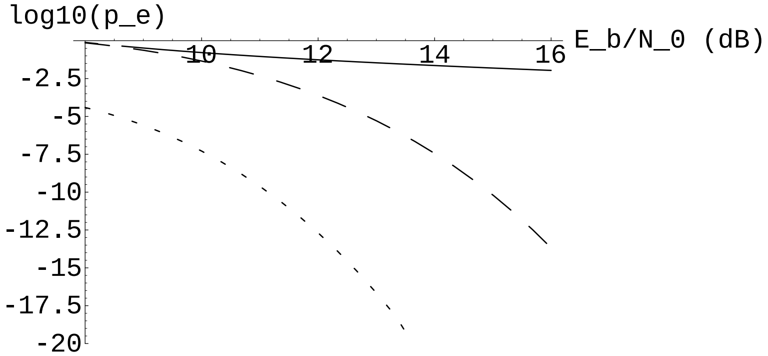

Figure 3-6: Error probability as a function of $E_b/N_0$ at a bandwidth of 10 GHz; upper bound for the multipath fading channel (solid), upper bound for the Rayleigh fading channel (dashed), and lower bound for the Rayleigh fading channel (dotted).

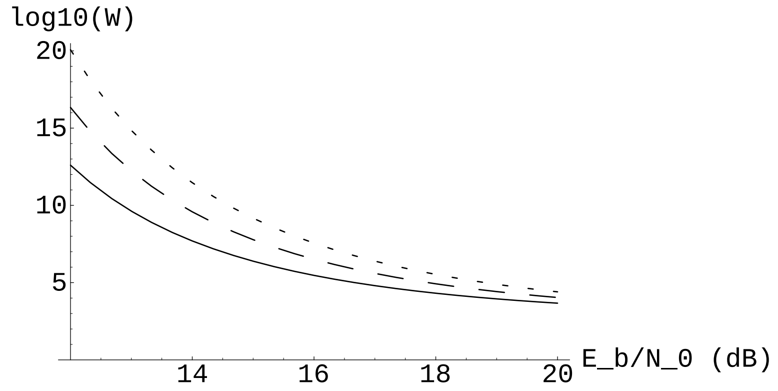

Figure 3-7: Bandwidth required as a function of $E_b/N_0$ for $\theta = 4 \times 10^{-3}$ and error probability bounds of $10^{-5}$ (solid), $10^{-7}$ (dashed), and $10^{-9}$ (dotted). The bandwidth $W$ is in Hz.

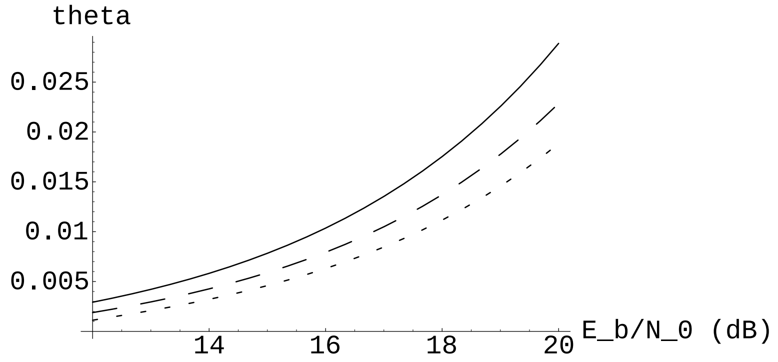

Figure 3-8: Duty factor $\theta$ as a function of $E_b/N_0$ for a bandwidth of 10 GHz and error probability bounds of $10^{-5}$ (solid), $10^{-7}$ (dashed), and $10^{-9}$ (dotted).

GHz respectively, and show the upper and lower bounds as functions of $E_b/N_0$. The general upper bound (3.2) is shown to quantify exactly how much looser it is than that (3.10) under the additional assumption of Rayleigh fading. For the lower bound, we use (3.39). The minimizations over $\epsilon$ for the general bound and for the lower bound are performed numerically.

Of the three bounds, it is the upper bound under the Rayleigh fading assumption that is the most interesting. The general upper bound is too loose to allow us to consider any remotely feasible settings, and lower bounds on the error probability relate to the best possible performance, which is usually of less interest than the worst possible performance (related to upper bounds on the error probability). In addition, the upper bound under the Rayleigh fading assumption has a simple exponential expression that the other two bounds do not. Thus, we concentrate on this upper bound and examine the relationship between bandwidth and $E_b/N_0$ (Figure 3-7) and between the duty factor $\theta$ and $E_b/N_0$ (Figure 3-8) for fixed target error probabilities. The plot of the duty factor as a function of $E_b/N_0$ tells us how peaky the signal needs to be to achieve a particular probability of error for a given $E_b/N_0$ and bandwidth, and therefore the peak power required. For all of the preceding plots, we have taken the peak power limitation to be $P/\theta \leq 250$; this plot gives us a notion of the increase in $E_b/N_0$ (or decrease in rate) that would be necessary to maintain performance if the peak power limitation were lower (and conversely how much $E_b/N_0$ could be decreased if the peak power limitation were higher).

# Chapter 4

# Conclusion

Explicit upper and lower bounds on the probability of error of a peaky transmission scheme that achieves the capacity of the multipath fading channel in the limit of infinite bandwidth — the exponents of which are the error exponents of the channel — were calculated. The upper bounds can be made to decay to zero as the bandwidth goes to infinity for all rates below capacity, thus yielding a strong coding theorem similar to those derived by Gallager for discrete-time memoryless channels. Specifically, an upper bound for general fading and upper and lower bounds under the additional assumption of Rayleigh fading were obtained. It was shown that, in the latter case, the upper bound decays much faster as bandwidth increases than in the former, and that the upper and lower bounds are exponentially tight, hence yielding the true exponential dependence of the error probability, or the reliability function. These bounds allow us to assess the performance of the scheme for a given finite bandwidth and peak power constraint, and give us a notion of how quickly the error probability decays to zero as the bandwidth approaches infinity and of the importance of the various parameters relevant to determining this rate of decay. The interaction among the error probability, bandwidth, data rate, and peakiness of the transmission scheme for specific numerical cases was investigated.

Finally, note that this scheme and the analysis of the error probability can be straightforwardly extended to a multiple access scenario. Since transmission takes place at a low duty cycle, multiple users can be multiplexed by time-division. If the users are co-operating, then it is clear that $\lfloor 1/\theta \rfloor$ non-interfering users can be supported for a given value of the duty factor $\theta$. If they are not co-operating, then a term due to interference from other users can be incorporated into our existing expressions for the upper bound on the probability of error. The probability of two time-slots overlapping, however, is dominated by $2(m-1)\theta$ as $\theta$ approaches zero when there are $m$ users. Hence, unless the duty factor $\theta$ is very minute (at most on the order of the target error probability), such a naïve non-co-operative scheme is unlikely to yield good performance.

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