Multiaccess and Fading in Communication Networks

by

Edmund Meng Yeh

B.S., Electrical Engineering
Stanford University (1994)

M.Phil, Electrical Engineering
University of Cambridge (1995)

Submitted to the Department of Electrical Engineering and Computer Science
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Electrical Engineering and Computer Science

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 2001

Author . . . .
Department of Electrical Engineering and Computer Science
August 17, 2001

Certified by . . . .
Robert G. Gallager
Professor
Thesis Supervisor

Accepted by . . . .
Arthur C. Smith
Chairman, Department Committee on Graduate Students

Multiaccess and Fading
in Communication Networks

by

Edmund Meng Yeh

Submitted to the Department of Electrical Engineering and Computer Science
on August 17, 2001, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy in Electrical Engineering and Computer Science

Abstract

Two fundamental issues in the design of wireless communication networks are the interference among multiple users and the time-varying nature of the fading wireless channel. We apply fundamental techniques in information theory and queueing theory to gain insights into the structure of these problems.

In a terrestrial cellular or space network, multi-user interference arises naturally as different users in the same cell or region attempt to transmit to the base station or satellite at the same time and in the same frequency range. We first examine the impact of this interference on the design of error correction codes for reliable data transmission. At the physical layer of the wireless network, the phenomenon of multi-user interference is captured by the multiaccess (many-to-one) channel model. The set of all data rates at which reliable communication can take place over this channel is characterized via information theory by the so-called multiaccess capacity region. A basic problem is developing coding schemes of relatively low complexity to achieve all rates in this capacity region. By exploiting the underlying geometrical structure of the capacity region, we develop a method of reducing the multi-user coding problem to a set of single-user coding problems using the ideas of time-sharing and successive decoding. Next, we investigate the effect of multi-user interference on higher-layer quality-of-service issues such as packet delay. Under certain conditions of symmetry, we find that the structure of the multiaccess capacity region can again be used to obtain a "load-balancing" queue control strategy which minimizes average packet delay for Poisson data sources.

Due to the mobility of users and constantly changing multipath environments, wireless channels are inherently time-varying, or fading. Any sensible design of wireless networks must take into account the nature of this fading and the ability of the system to track channel variations. We consider a wireless system in which a single user sends time-sensitive data over a slowly varying channel. Information regarding the state of the channel is fed back with some delay to the transmitter, while the receiver decodes messages within some fixed and finite amount of time. Under these conditions, we demonstrate a provably optimal transmission strategy which maximizes the average data rate reliably sent across the wireless channel. The strategy is based on the information-theoretic idea of "successive refinement," whereby the decoder decodes at different rates according to the observed channel state.

Thesis Supervisor: Robert G. Gallager
Title: Professor

For Mother and Father

Acknowledgments

In the past few years, I have been granted the extraordinary privilege of being able to spend a few years contemplating a topic of some interest, in a serene and intellectual atmosphere. In the course of doing so, I have learned a few things about the topic of communications, and therefore also further ascertained the extent of my ignorance. It was but another step in my education, but perhaps a particularly piquant one.

One does not go through a place like MIT without encountering an abundance of intellectual brilliance. But personally, I have always been more impressed by kindness and generosity. Here, I have been very fortunate. My thesis supervisor, Professor Robert Gallager, is an epitome of the great scholar/teacher. His amazing intellectual insights are surpassed only by his generosity of spirit. It has been a true honor to be guided on this journey by someone who understands the terrain so well, and who never forgets to remind the traveler of the true meaning of his undertaking.

During the preparation of this thesis, I received thoughtful guidance and mentoring from a number of academic and industrial researchers. Of these, my thanks go first to members of my thesis committee: Professor Muriel Médard, Professor Sanjoy Mitter, and Dr. Philip Whiting. Dr. Whiting played a crucial role in shaping the thesis. It was during my two-summer stay at Bell Labs that many of the problems presented here were first formulated. In particular, Chapter 4 represents our joint work. Dr. Whiting's intellectual curiosity and total dedication to research were great sources of inspiration for me. On a number of topics in this work, Professor Médard offered insightful suggestions for refinement and extension. She also helped me finish the thesis in another way: *everything* seemed trivial next to the task of being junior faculty *and* mother to three young daughters! Finally, Professor Mitter provided a much-needed global perspective throughout my years at LIDS. His wide-ranging interests, infallible taste, and fundamental thinking made him a veritable oracle of wisdom.

My graduate experience benefited greatly from other fortunate associations. At Bell Labs, Professor Emre Telatar and Professor Rudiger Urbanke were sources of invaluable suggestions and ideas. Indeed, Chapter 3 can be seen as a sequel to Professor Telatar's earlier work on combining information theory and queueing theory, and Chapter 2 is another take on the rate-splitting work of Professor Urbanke. I also thank Professor Sem Borst for his careful mentoring at Bell Labs and Dr. Debasis Mitra for his generous offers of summer internships at the Math Center. During the early years of my research effort, interactions with Professor Bixio Rimoldi helped to clarify my understanding of rate-spitting and time-sharing. His pioneering work on both these topics are fundamental to the discussion in Chapter 2. I have also gained much from discussions with Professor David Tse, who while on leave at LIDS, suggested a number of interesting problems.

At LIDS, Professors Eytan Modiano and Professor John Tsitsiklis gave useful advice on the subject of queue control. Professor Modiano also offered excellent practical counsel on the academic profession in general. I thank Professor Vincent Chan for running LIDS so efficiently (even getting air-conditioning installed in the student offices (!), albeit after my graduation) and for being such an accessible mentor. Thanks to Professor G. David Forney for giving me the opportunity to TA a cutting-edge digital communications course, to Professor Mitchell Trott for giving me the first research project at MIT, and to Professor Gregory Wornell for being my academic advisor. Lastly, many thanks to Professor Alvin Drake, who played an instrumental part in convincing me to come to MIT.

The wonderful intellectual atmosphere in LIDS is due in no small part to its extraordinary cast of graduate students. As with a musician, a researcher is most alive amid an orchestra of talents. My follow music-makers included: Professor Randall Berry, Dr. Aaron Cohen, Dr. Sae-Young Chung, Stark Draper, Anand Ganti, Dr. Hisham Kassab, Dr. Andrew Kim, Dr. Thierry Klein, Michael Kneely, Asuman Koksal, Emre Koksal, Nicholas Laneman, Dr. Anant Sahai, Dr. Brett Schein, Dr. Sekhar Tatikonda, Dewey Tucker, and Won Yoon. Of these, I thank Randy Berry for being a great office neighbor for four years, Aaron Cohen for listening to my views on almost anything, Thierry Klein for our many long conversations, and Sae-Young Chung, Thierry Klein and Nick Laneman for the great foosball games.

The inclement weather and the occasional stress I encountered at MIT were largely alleviated by the great friendships I made here. I thank Alvin Fu and Frank Wang for their lasting camaraderie. We have been through much together. To the "salon" members, Sharon Kuo, Fleming Ray, Frank Wang, and Rebecca Xiong, I owe one of the best social experiences at MIT. In addition, I thank Eugene Bae for the many afternoon tea sessions, Tracey Ho and Irina Medvedev for their great sense of fun, and Andy Wang for his good company.

Now I come to the people for whom words like "thanks," "grateful," and "appreciation" lose their significance. There has never been a question of such terms being used. The unconditional love that endlessly flows from my parents is as natural as life's continuity. To learn of my parents' experience is to understand the true value of opportunity and privilege. I have tried to put these to good use. I dedicate the spirit of this work to my mother and father.

Contents

List of Figures

Chapter 1

Introduction

The past two decades have seen dramatic shifts in the development of communication technology, the most significant being the move from point-to-point circuit-switched networks to multiplexed packet-switched networks, and that from analog to digital technology. In the last few years, there has also been much excitement over the progress of wireless communications. In many ways, wireless is hardly a new technology. Marconi pioneered wireless telegraphy more than one hundred years ago. Other wireless technologies such as satellite networks and television broadcasting have existed for many decades. It does seem, however, that the new generation of digital wireless voice and data services (GSM, IS-95, Third Generation), combined with ready access to vast information sources such as the World Wide Web, has captured the popular imagination. In developed telecommunication markets, wireless promises liberation from the mess of electronic tethers, and embodies the essence of a society "on the move." In fledgling telecom markets, wireless offers a means of bypassing the handicap of a poor wired infrastructure. The societal impact of wireless in these developing areas can hardly be overstated.

1.1 Multi-user Interference and Fading

For the communications engineer, wireless poses a number of new technical challenges. The two most fundamental issues are: interference among multiple users, and the time-varying nature of the wireless channel. The phenomenon of multi-user interference arises from the inherent structure of wireless networks. In a typical cellular telephone network, wireless Local Area Network (LAN), or satellite network, a given geographical area is divided into a number of coverage zones, called *cells* (or regions), each served by a *base station* (or a satellite). Wireless subscribers (mobiles) communicate by connecting to their respective assigned base stations, which are in turn connected to the wired network via a Mobile Telephone Switching Office (MTSO).

The wireless link from a base station to a mobile is called the *forward channel*. A base station communicates to many mobiles in its cell over the respective forward channels by multiplexing the messages for the various receivers and sending a composite signal. The receivers must then sift out their individual messages from the composite signal in the presence of interference and background noise. In communication terminology, the collection of forward channels in this one-to-many scenario is called a *broadcast channel*. Correspondingly, the wireless link from a mobile to a base station is called the *reverse channel*. Mobiles in a cell communicate to the base station by sending their respective signals over the reverse channels. The base station, which receives the sum of the mobiles' signals, must then decode each message in the presence of interference and background noise. In communication terminology, the collection of reverse channels is called a *multiaccess channel*. It is clear, then, that multi-user interference is a central characteristic of both the broadcast and multiaccess channels. Any viable network must have some means of resolving this interference.

In addition to contending with multi-user interference and background noise, a wireless system must also deal with the time-varying nature of the wireless channel. In a typical wireless environment, there may be a number of obstacles between the mobile and the base station, possibly precluding line-of-sight communication. The transmitted signal usually reaches the receiver via multiple paths caused by reflections from obstacles. Moreover, the strength of each path may vary in time due to relative motion among the mobile, the obstacles, and the base station. These changes in turn cause changes in the interference pattern of the paths at the receiver, leading to the phenomenon of *multipath fading*.[1] Fading represents another fundamental source of randomness that any sensible design of a wireless network must address.

[1]Multipath fading is sometimes called *fast fading*, in contrast to *slow fading* caused by a phenomenon called *shadowing*, whereby partially absorbing material lying between the sending and receiving antennas attenuate the electromagnetic wave of the transmitted signal.

1.2 Purpose and Outline of Thesis

In this work, we give an analysis of the multi-user interference and fading issues using techniques in information theory and queueing theory. Our aim is to gain insight into the structure of these problems, with the hope that the insights may point the way to sensible architectural designs for practical systems. In the first part of the dissertation (Chapters 2 and 3), we treat the problem of multi-user interference. In particular, we focus on the multiaccess setting encountered on the reverse link of a cellular or satellite network. The problem of communicating among many senders and a common receiver is in fact central to the study of both wired and wireless networks. Mechanisms such as time-division, frequency-division, and code-division, for instance, have been used in practical systems to deal with this issue. Our aim, however, is to examine multiaccess communication at a more fundamental level, both at the physical layer and Medium Access Control (MAC)/network layers of the data network. At the physical layer, we adopt a noisy multiaccess channel model with independent transmitters each receiving a constant stream of source bits and outputting codewords for reception at a common receiver [CT91]. Given this, the natural questions are: what is the set of all data rates at which reliable communication is possible, and how do we design high-performance, low-complexity error correction codes which operate at those rates?

The answers to the above questions fall naturally in the domain of information theory. Indeed, the capacity limits for memoryless multiaccess channels have been known since the 1970's, and are given by the so-called multiaccess capacity region. There has been, however, relatively little work on the design of multi-user error correction codes [RU96]. By contrast, research on the design of codes for the single-user channel has been ongoing for the last fifty years. The development of single-user codes has been especially rapid in the last few years, following the discovery of Turbo codes in 1993. Recently, for example, it has been shown that powerful low-density parity-check codes can get to within 0.0045 dB of Shannon capacity on the additive white Gaussian noise (AWGN) channel [CFRU01]. Given this, a natural question is whether the multi-user coding problem can somehow be reduced to a manageable set of single-user coding problems. In Chapter 2, we investigate how several key ideas may be combined to make such a reduction possible, leading to a potentially useful multiaccess communication technique.

After examining the impact of multiaccess interference on the design of error correction codes at the physical layer, we move to investigate the effect of multiaccess on higher-layer quality-of-service (QOS) issues such as packet delay. Unfortunately, the conventional information-theoretic model of multiaccess, which does a fine job of addressing physical-layer concerns, proves to be inadequate in dealing with QOS issue. One major problem with the information-theoretic model is the implicit assumption that there are steady streams of information bits arriving at all transmitters, so that all users of the channel are always active, in the sense that they always have bits to send. In many actual multiaccess systems (such as wireless data networks), on the other hand, messages arrive at the transmitters in a random manner. By adhering to the conventional source-channel-destination model, information theory ignores the random arrival of messages and thereby renders meaningful analysis of delay impossible [EH98, Gal85]. In contrast to the information theoretic approach, multiaccess network theory (ALOHA, CSMA, *etc.*) and other notions such as effective bandwidth give sophisticated analyses of network layer issues such as source burstiness,[2] network delay, and buffer overflow, but do not adequately address physical layer concerns such as channel modeling, coding, and detection. For instance, the characterization of the multiaccess channel as a collision channel (even with notions of capture) is too simplified and pessimistic from a physical layer viewpoint [EH98, Gal85]. Thus, while the above approaches offer different perspectives on the multiaccess question, each addresses only a part of the overall problem. What is needed, then, is a theory of multiaccess communications that treats issues of noise, interference, randomness of arrivals, delay, and error probability in a more cohesive framework.

In Chapter 3, we take an "inter-layer" approach and examine a problem where physical-layer issues of rate and reliability are considered together with higher-layer issues of source burstiness and delay. We consider a multiaccess model where multiple sources generate packets of variable lengths according to Poisson processes. The packets are queued in the users' respective buffers until being sent by the corresponding transmitters. All packets are then decoded at a common receiver, which receives the sum of the transmitted signals plus noise. It is further assumed that optimal coding can somehow be performed at the physical layer so that all rates in the multiaccess capacity region are achievable. The objective is then

[2]Here, we are referring to burstiness in bit arrivals, not in packet arrivals (which may be homogeneous, as in the Poisson process)

to design the rate allocation policy to minimize the overall delay in the system. We show that under certain conditions of symmetry, this “inter-layer” multiaccess problem admits an elegant solution.

In the second part of the dissertation (Chapter 4), we turn to the problem of time variations in wireless channels. We examine a number of important questions. First, fading introduces a new source of randomness (in addition to background noise) to the coding problem. Any sensible coding scheme must somehow average over both sources of randomness. This raises the important question of how large the decoding delay requirement is relative to the time scale of the fading process. When the encoder/decoder pair has enough time to average over the fading process, traditional information-theoretic notions of ergodic capacity are relevant (in the sense that they provide meaningful benchmarks for performance analysis). If not, alternative measures of performance must be formulated. A second important question is how well can the system track the time variations in the channel. The timeliness and accuracy of the information available to the transmitter and receiver regarding the channel state directly impact system performance. Finally, there is the question of what kind of traffic the wireless system is designed to support. Is it voice traffic, which often requires a constant bit rate, or is it data traffic which can be transmitted at variable rates, so long as the average rate over time meets user requirements? The answer to this question affects the basic architecture of the coding scheme. In Chapter 4, we study a wireless communication scenario in which all three issues mentioned above play significant roles in determining the optimal communication strategy. We consider a wireless system in which a single user sends time-sensitive data over a slowly varying channel. Information regarding the state of the channel is fed back with some delay to the transmitter, while the receiver decodes messages within some fixed and finite amount of time. Under these conditions, we demonstrate an optimal transmission strategy which maximizes the average data rate reliably sent across the wireless channel. The strategy is based on the information-theoretic idea of “successive refinement,” whereby the decoder decodes at different rates according to the observed channel state.

Chapter 2

Time Sharing for Multiaccess Channels

2.1 Introduction

We begin by investigating the effects of multi-user interference encountered on the reverse links of cellular wireless networks, captured by the multiaccess channel model. Our focus in this chapter is on the impact of multiaccess interference on the design of error correction codes at the physical layer of the data network. As such, we use the conventional information-theoretic multiaccess model [CT91], where a number of independent transmitters each receive a constant stream of source bits and send codewords for reception at a common receiver and decoder. Conceptually, this many-to-one communication problem is a specific instance of the general unsolved network information theory problem in which many senders and receivers communicate through a channel transition matrix describing the effects of interference and noise in the network. Fortunately, results concerning communication limits on the multiaccess channel are relatively complete [Ahl71, Lia72, Pol83, HH85, Gal85, CT91]. As mentioned in Chapter 1, our aim is to develop high-performance, low complexity error correction codes which operate near the communication limits. In particular, we would like to reduce the multiaccess coding problem to a set of single-user coding problems, so as to make use of the recent exciting developments in coding for single-user channels.

A pioneering effort in this direction is the work of Rimoldi and Urbanke [RU96] and

Grant *et al.* [GRUW01]. Using a technique called rate-splitting multiple-access (RSMA), they have shown that any point in the asynchronous capacity region of an M-user multiple access channel (MAC) can be achieved using at most $2M-1$ single-user codes, with no user having more than two codes each. The RSMA approach involves splitting each user of a multiaccess channel into at most two virtual users, each having a fraction of the rate and power of the original user. Another approach, pioneered by Grant and Rasmussen [GR95], uses the idea that rates in the capacity region can be achieved via time-sharing of a small number of single-user codes. Indeed, it has been shown by Rimoldi [Rim99] that time-sharing can yield the same $2M-1$ performance for asynchronous M-user MAC's.

We expand on the ideas presented in [GR95, Rim97, Rim99] and provide an alternative view on the time-sharing approach, emphasizing the underlying geometrical structure of the problem. We show that geometrical properties combined with results on parallel channels [Gal68] yields a natural *projective time-sharing* mechanism which achieves any rate tuple in the M-user MAC capacity region with at most $\frac{1}{2}M\log_2 M + M$ single-user codes. Although our approach is inspired by the work in [RU96, GR95, Rim97, Rim99], we will see that geometrically, it differs significantly from previous constructions. We elaborate on this difference in Section 2.5. While most of this section concentrates on the M-user discrete memoryless multiple access channel (DMMAC) with finite input and output alphabets, our analysis extends directly to the case of discrete-time memoryless MAC's with continuous alphabets and energy constraints. Indeed, when appropriate, we shall make comments regarding the continuous cases, with emphasis on the additive Gaussian MAC in particular.

Consider the M-user DMMAC defined in terms of M finite input alphabets $\mathcal{X}_i, i \in \{1,\ldots,M\}$, output alphabet $\mathcal{Y}$, and a stochastic matrix $P : \mathcal{X}_1 \times \ldots \times \mathcal{X}_M \to \mathcal{Y}$ where $P(y \mid x_1,\ldots,x_M)$ is the probability that the channel output is y when the inputs are $x_1,\ldots,x_M$. For each $i, i=1,\ldots,M$, let $Q_{X_i}(x_i)$ be a probability assignment on the input alphabet $\mathcal{X}_i$. Define the *achievable rate region* $\mathcal{R}\left(P, \prod_{i=1}^M Q_{X_i}\right)$ corresponding to the channel transition matrix P and product input distribution $\prod_{i=1}^M Q_{X_i}$ as the set of rate tuples $\boldsymbol{R} \in (\mathbb{R}^+)^M$ such that

$$\sum_{i\in S} R_i \le I(X_S; Y \mid X_{S^c}), \qquad \forall S \subseteq \{1,\ldots,M\} \tag{2.1}$$

where S is any non-empty subset of $\{1,\ldots,M\}$, $X_S \equiv (X_i)_{i\in S}$, $X_{S^c} \equiv \{1,\ldots,M\}\backslash S$. The average mutual information quantity in (2.1) is defined as

$$I(X_S; Y \mid X_{S^c}) \equiv \sum_{x_1,\dots,x_M,y} \prod_{i=1}^{M} Q_{X_i}(x_i) P(y|x_1,\dots,x_M) \log \frac{P(y|x_1,\dots,x_M)}{\sum_{x_S} \prod_{i\in S}^{M} Q_{X_i}(x_i) P(y|x_1,\dots,x_M)}.$$

Now consider the union of rate regions over all product input distributions

$$\mathcal{C} = \bigcup_{Q_{X_1} Q_{X_2} \cdots Q_{X_M}} \mathcal{R}\left(P, \prod_{i=1}^{M} Q_{X_i}\right)$$

It is well known that the closure of $\mathcal{C}$ is the capacity region of the asynchronous M-user DMMAC [HH85, Pol83], while the convex closure of $\mathcal{C}$ is the capacity region of the synchronous M-user DMMAC [Ahl71, Lia72]. Reliable communication is possible for any rate tuple in the interior of the capacity region, while it is impossible outside of the capacity region.

For any rate tuple $\boldsymbol{R} \in \mathcal{C}$, there exists a product input distribution $\prod_{i=1}^{M} Q_{X_i}$ such that $\boldsymbol{R} \in \mathcal{R}\left(P, \prod_{i=1}^{M} Q_{X_i}\right)$. We assume for the rest of the analysis in the section that the product input distribution is fixed, and focus on $\mathcal{R}\left(P, \prod_{i=1}^{M} Q_{X_i}\right)$. The *dominant face* $\mathcal{D}$ of $\mathcal{R}$ is the subset of rate tuples $\boldsymbol{R} \in \mathcal{R}$ which gives equality in (2.1) for $S = \{1,\dots,M\}$:

$$\sum_{i=1}^{M} R_i = I(X_1 \dots X_M; Y) \tag{2.2}$$

That is, $\mathcal{D}$ is the subset of rate tuples giving the maximal sum rate over all users. Due to the special *polymatroidal* structure of $\mathcal{R}$ (more on this in Section 2.2), it can be shown [HW94, TH98] that any point of $\mathcal{R}$ is dominated componentwise by a point in $\mathcal{D}$. That is, for any rate tuple $\boldsymbol{R} \in \mathcal{R}$, there exists a rate tuple $\boldsymbol{\Phi} \in \mathcal{D}$ such that $R_i \le \Phi_i, i = 1,\dots,M$. Therefore, to examine the achievability of points in $\mathcal{R}$, it is sufficient to restrict attention to the achievability of points in $\mathcal{D}$. The achievable region $\mathcal{R}$ and its dominant face $\mathcal{D}$ are illustrated for the two-user case in Figure 2-1.

For points in $\mathcal{D}$, equations (2.1) and (2.2) give a pair of upper and lower bounds on the

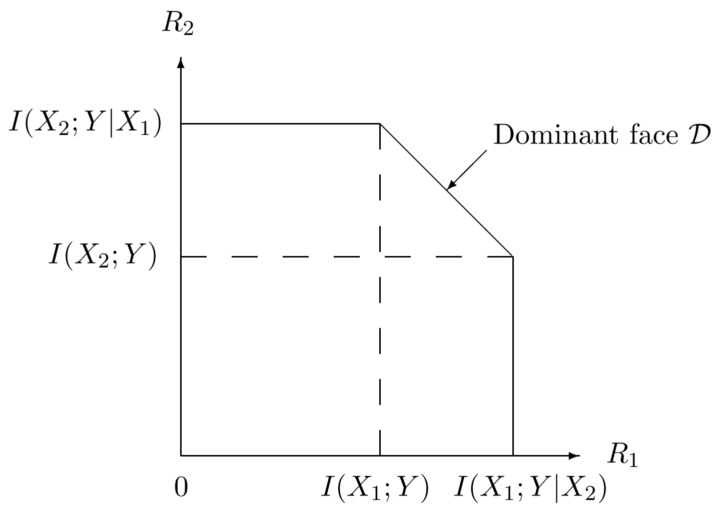

Figure 2-1: Achievable region $\mathcal{R}$ of a two-user DMMAC for a fixed input probability distribution.

combined rate over any non-empty subset $S \subseteq \{1, \ldots, M\}$. Thus, for any $\boldsymbol{R} \in \mathcal{D}$,

$$I(X_S; Y) \leq \sum_{i \in S} R_i \leq I(X_S; Y \mid X_{S^c}), \qquad \forall S \subseteq \{1, \ldots, M\} \tag{2.3}$$

Conversely, any $\boldsymbol{R}$ satisfying (2.3) clearly meets (2.1) and has maximal sum rate (take $S = \{1, \ldots, M\}$ in (2.3)). Hence, (2.3) is an equivalent condition for $\boldsymbol{R} \in \mathcal{D}$.

2.2 The Geometry of $\mathcal{R}$ and $\mathcal{D}$

This section reviews some of the fundamental geometrical structure of the rate region $\mathcal{R}$ and the dominant face $\mathcal{D}$. Most of the properties are discussed in the pioneering work of Grant *et al.* [GRUW01]. Due to the central role of geometry in our work, however, we feel compelled to revisit the landscape. We believe a clear understanding of the geometry makes the main results of this section seem natural.

A *convex polyhedron* is the intersection of a finite family of closed halfspaces of $\mathbb{R}^n$. A *convex polytope* is a bounded polyhedron. A hyperplane $\mathcal{H}$ is said to support a convex polytope $\mathcal{P}$ if $\mathcal{H} \cap \mathcal{P} \neq \emptyset$ and $\mathcal{P}$ is entirely contained in one of the two halfspaces determined by $\mathcal{H}$. A *face* of a convex polytope is the intersection of the convex polytope with a supporting hyperplane. A face of a convex polytope is again a convex polytope.

Given a convex polytope $\mathcal{P}$, the *affine hull* of $\mathcal{P}$, aff($\mathcal{P}$), is the set of all affine combinations formed from all finite subsets of $\mathcal{P}$:

$$\operatorname{aff}(\mathcal{P}) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} = \sum_{i=1}^{k} \lambda_i x_i, \lambda_i \in \mathbb{R}, \sum_{i=1}^{k} \lambda_i = 1; x_i \in \mathcal{P}, i = 1, \dots, k\}.$$

Equivalently, $\operatorname{aff}(\mathcal{P})$ is the smallest affine subspace containing $\mathcal{P}$. The dimension of an affine subspace is the dimension of the corresponding linear subspace. The convex polytope $\mathcal{P}$ is said to have dimension d if $\operatorname{aff}(\mathcal{P})$ has dimension d. A face $\mathcal{F}$ of $\mathcal{P}$ is called a *vertex* or *extreme point* if $\dim(\mathcal{F}) = 0$, an *edge* if $\dim(\mathcal{F}) = 1$, and a *facet* if $\dim(\mathcal{F}) = d - 1$. It is a fundamental fact that every convex polytope as defined above is the convex hull of its vertices.

With the above definitions, a geometric analysis of $\mathcal{R}$ and $\mathcal{D}$ defined in the previous section follows. From (2.1), $\mathcal{R}$ is a convex polytope. A crucial observation is that $\mathcal{R}$ is a very special type of convex polytope called a *polymatroid* [HW94, TH98]. Polymatroids have the convenient property that they are convex polytopes in which a greedy algorithm gives the optimal solution for a linear optimization problem with the polytope as the feasible set [Edm70, GLS85, Wel76]. It turns out that the vertices of a polymatroid are easily enumerated. Let Π_M be the permutation group of $\{1, \dots, M\}$. For any $\pi \in \Pi_M$, let the rate tuple $\boldsymbol{R}^\pi = (R_1^\pi, \dots, R_M^\pi)$ be defined by

$$R_{\pi(i)}^{\pi} = \begin{cases} 0 & \text{for } i = 1, \dots, k \\ I(X_{\pi(i)}; Y | X_{\pi(1)}, \dots, X_{\pi(i-1)}) & \text{for } i = k+1, \dots, M. \end{cases} \tag{2.4}$$

where $0 \leq k \leq M$. Rate tuple $\boldsymbol{R}^\pi$ is then a vertex of $\mathcal{R}$. Moreover, all vertices of $\mathcal{R}$ takes the form in (2.4) [Edm70, GLS85, Wel76]. For instance, for the two-user DMMAC rate region in 2-1, there are five vertices: $(I(X_1;Y), I(X_2;Y|X_1))$, $(I(X_1;Y|X_2), I(X_2;Y))$, $(0, I(X_2;Y|X_1))$, $(I(X_1;Y|X_2), 0)$, and $(0,0)$. For $k = 0$, there are $M!$ vertices of the form in (2.4), corresponding to the $M!$ permutations in Π_M. These vertices have natural interpretations as rate tuples achievable using successive decoding. The vertex $\boldsymbol{R}^\pi$ (for $k = 0$) is achieved with the decoding order π, where $\pi(j)$ is the user which decodes in the jth position of the decoding order. For the two-user case, for example, there are two vertices of this form. The vertex $(I(X_1;Y), I(X_2;Y|X_1))$ is achieved by decoding user 1 first, followed by user 2. The vertex $(I(X_1;Y|X_2), I(X_2;Y))$ is achieved by decoding user 2 first, followed by user 1.

Returning to the dominant face $\mathcal{D}$, it follows from (2.1) and (2.2) that $\mathcal{D}$ is the face of $\mathcal{R}$ determined by the intersection of $\mathcal{R}$ and the supporting hyperplane $\mathcal{H}_{\{1,\dots,M\}}$, where $\mathcal{H}_{\{1,\dots,M\}} \equiv \left\{\boldsymbol{R} \in \mathbb{R}^M : \sum_{i=1}^M R_i = I(X_1 \dots X_M; Y)\right\}$. Thus, $\mathcal{D}$ is a convex polytope and can be represented as the convex hull of its vertices. Due to the polymatroidal structure of $\mathcal{R}$, however, the vertices of $\mathcal{D}$ are the same as the vertices of $\mathcal{R}$ for which $k = 0$ in (2.4). Thus, $\mathcal{D} = conv\{\boldsymbol{R}^\pi : k = 0, \pi \in \Pi_M\}$, where *conv* denotes the convex hull operation.

From equations (2.1) and (2.2), we see that the dimension of the dominant face convex polytope is at most $M-1$. It is not difficult to find examples of DMMAC's where $\dim(\mathcal{D}) < M - 1$ (an easy example is an M-user DMMAC made up of M non-interfering binary noiseless single-user channels, where $\mathcal{D}$ is a single point). Grant *et al.* derive the following test condition for a collapse in the dimension of $\mathcal{D}$.

Lemma 2.1 (See [GRUW01]) *The following statements are equivalent:*

(a) $I(X_S; Y) = I(X_S; Y|X_{S^c})$ *for some set* $S, \emptyset \subset S \subset \{1, \dots, M\}$.

(b) $\dim(\mathcal{D}) < M - 1$.

The lemma is quite intuitive. Condition (a) amounts to saying that the successful decoding of users in set S^c does not make it easier to decode users in set S. Thus, the users in S and S^c may be decoded independently of each other at the receiver. As we see later, this really implies that the M-user DMMAC can be decomposed into at least two independent sub-DMMAC's, one for users in S and another for users in S^c. The lemma says that whenever such a situation occurs, $\mathcal{D}$ must lose dimension. We comment that this cannot occur in any discrete-time memoryless additive Gaussian MAC where each user has nonzero power.[1] This follows from the fact that condition (a) in Lemma 2.1 is never satisfied for the discrete time Gaussian channel: successful decoding of some users always reduces the effective interference for the other users and makes them more easily decodable. In this

[1]A discrete-time memoryless M-user additive Gaussian MAC is defined by

$$Y = \sum_{i=1}^{M} X_i + Z \tag{2.5}$$

where $X_i, i = 1, \dots, M$, and Y each has as its alphabet the set of real numbers, and Z is a zero mean Gaussian random variable with variance σ^2, independent of each $X_i, i = 1, \dots, M$. The inputs $X_i, i = 1, \dots, M$, are each constrained in their mean square values: $E[X_i^2] \le \mathcal{E}_i, i = 1, \dots, M$.

case, there are $M!$ distinct vertices in $\mathcal{D}$ corresponding to distinct permutations in Π_M, and $\mathcal{D}$ is topologically equivalent to a classical polytope called the *permutahedron* [Zie95].

2.2.1 The Many Faces of $\mathcal{D}$

Assume for the following that $\mathcal{D}$ is full-dimensional. That is, $\dim(\mathcal{D}) = M-1$, which implies $\mathcal{D}$ is a facet of $\mathcal{R}$. Consider the set $\mathcal{F}_S = \mathcal{D} \cap \mathcal{H}_S$, where $\mathcal{H}_S \equiv \left\{\boldsymbol{R} : \sum_{i\in S} R_i = I(X_S;Y)\right\}$ is the hyperplane associated with the constraint set S, $\emptyset \subset S \subset \{1,\dots M\}$, $1 \leq |S| = k \leq M-1$ and $|S^c| = M-k$. From (2.3), it follows that $\mathcal{D}$ is entirely contained in one of the two halfspaces created by $\mathcal{H}_S$. Thus, $\mathcal{H}_\mathcal{S}$ is a supporting hyperplane of $\mathcal{D}$ and $\mathcal{F}_S$ is a face of $\mathcal{D}$. Since each $\mathcal{F}_S$ results from the intersection of $\mathcal{D}$ with a hyperplane $\mathcal{H}_S$, $\dim(\mathcal{F}_S) \leq \dim(\mathcal{D}) - 1$. Here, once again, it is possible to find examples of channels where $\dim(\mathcal{F}_S) < \dim(\mathcal{D}) - 1$ for some set $S, \emptyset \subset S \subset \{1,\dots,M\}$ (see [GRUW01]). And once again, it can be shown (see Appendix A.1) that for the discrete-time Gaussian MAC, $\dim(\mathcal{F}_S) = \dim(\mathcal{D}) - 1 = M-2$ ($M \geq 2$) for any $S, \emptyset \subset S \subset \{1,\dots,M\}$, and hence each $\mathcal{F}_S$ is indeed a facet of $\mathcal{D}$.

Figure 2-2 illustrates the dominant face $\mathcal{D}$ and its facets $\mathcal{F}_S$ for a three-user MAC. We have assumed $\dim(\mathcal{D}) = 2$ and $\dim(\mathcal{F}_S) = 1$ for all $\emptyset \subset S \subset \{1,2,3\}$. Here $\mathcal{D}$ is a hexagon with $3! = 6$ vertices, each corresponding to a successive decoding order on the users 1, 2, 3. For instance, (123) denotes the vertex corresponding to decoding user 1 first, then user 2, then user 3. In this case, the facets $\mathcal{F}_S$ are edges connecting two vertices. For instance, the facet $\mathcal{F}_{\{1,2\}}$ connects vertices (213) and (123). Note that each facet (edge) connects two vertices whose labels differ by an adjacent transposition of coordinates.

Now combining the definition of $\mathcal{F}_S$ with (2.2) we have for $\boldsymbol{R} \in \mathcal{F}_S \subset \mathcal{D}$,

$$\sum_{i\in S} R_i = I(X_S;Y), \quad \sum_{i\in S^c} R_i = I(X_{S^c};Y|X_S). \tag{2.6}$$

Hence, rate tuples in $\mathcal{F}_S$ satisfy two equality constraints, one on the sum rate over subset S and the other on the sum rate over subset S^c. As shown in [GRUW01], these two constraints effectively define two new dominant faces of lower dimension, one corresponding to users in S, the other corresponding to users in S^c.

Let $Q_{X_S} = \prod_{i\in S} Q_{X_i}$ and $Q_{X_{S^c}} = \prod_{i\in S^c} Q_{X_i}$ be the joint probability measures over the inputs X_S and X_{S^c}, respectively. Define the DMMAC $P_S : \bigotimes_{i\in S} \mathcal{X}_i \to \mathcal{Y}$ by the stochastic

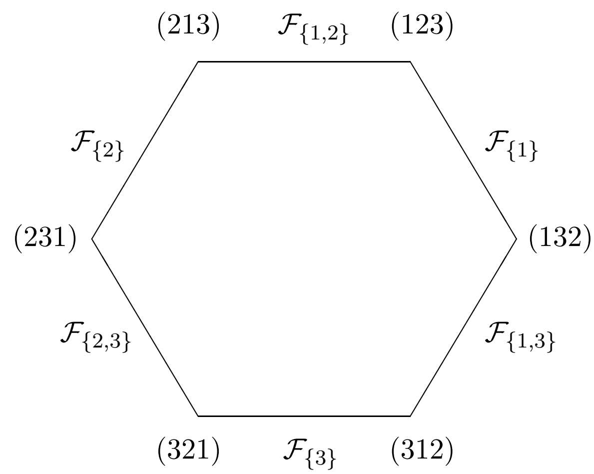

Figure 2-2: Dominant face and its facets for a 3-user MAC.

matrix P_S where

$$P_S(y \mid x_S) = \sum_{x_{S^c}} P(y \mid x_1, \ldots, x_M) Q_{X_{S^c}}(x_{S^c}). \tag{2.7}$$

Here, P is the transition probability matrix for the original M-user DMMAC. Thus, P_S is the channel with inputs indexed by S, averaged over all possible values of the other inputs indexed by S^c. That is, it is the channel with inputs X_S with X_{S^c} regarded as "noise." Next, define the DMMAC (with side information at the receiver) $P_{S^c|S} : \bigotimes_{i \in S^c} \mathcal{X}_i \to \mathcal{Y} \times \bigotimes_{i \in S} \mathcal{X}_i$ by the matrix $P_{S^c|S}$ where

$$P_{S^c|S}(y, x_S \mid x_{S^c}) = Q_{X_S}(x_S) P(y \mid x_1, \ldots, x_M). \tag{2.8}$$

This is the "genie-aided" channel where the inputs are indexed by S^c while X_S is available at the decoder as side information.[2] Clearly, $I(Q_{X_S}; P_S) = I(X_S; Y)$ and $I(Q_{X_{S^c}}; P_{S^c|S}) = I(X_{S^c}; Y, X_S) = I(X_{S^c}; Y|X_S)$ (since the inputs are independent).

Let $\mathcal{D}_{P_S}$ and $\mathcal{D}_{P_{S^c|S}}$ be the dominant faces associated with the channels P_S and $P_{S^c|S}$

[2]If we were considering the Gaussian MAC, $P_{S^c|S}$ is the channel which results from subtracting out the known inputs X_S from the received signal Y.

respectively. Then by (2.3), we have

$$\mathcal{D}_{P_S} = \{\boldsymbol{R}_S \in (\mathbb{R}^+)^k : I(X_W;Y) \leq \sum_{i \in W} R_i \leq I(X_W;Y \mid X_{S \setminus W}), \forall W \subseteq S\}, \tag{2.9}$$

$$\mathcal{D}_{P_{S^c|S}} = \{\boldsymbol{R}_{S^c} \in (\mathbb{R}^+)^{M-k} : I(X_W;Y|X_S) \leq \sum_{i \in W} R_i \leq I(X_W;Y \mid X_{W^c}), \forall W \subseteq S^c\} \tag{2.10}$$

where $k = |S|$. We then have $\dim(\mathcal{D}_{P_S}) \leq k-1$ and $\dim(\mathcal{D}_{P_{S^c|S}}) \leq M-k-1$.

Combining the rate region bounds in (2.1) and the constraints for $\mathcal{F}_S$ in (2.6), as well as (2.9) and (2.10), it can be shown that $\boldsymbol{R} \in \mathcal{F}_S$ implies $\boldsymbol{R}_S$ ($\boldsymbol{R}$ projected onto the S coordinates) $\in \mathcal{D}_{P_S}$ and $\boldsymbol{R}_{S^c}$ ($\boldsymbol{R}$ projected onto the S^c coordinates) $\in \mathcal{D}_{P_{S^c|S}}$. Conversely, $\boldsymbol{R}_S \in \mathcal{D}_{P_S}$ and $\boldsymbol{R}_{S^c} \in \mathcal{D}_{P_{S^c|S}}$ imply $\boldsymbol{R} = (\boldsymbol{R}_S, \boldsymbol{R}_{S^c}) \in \mathcal{F}_S$. We then have the following lemma first stated in [GRUW01]:

Lemma 2.2 ([GRUW01]) *For any* $\emptyset \subset S \subset \{1, \ldots, M\}$,

$$\mathcal{F}_S = \mathcal{D}_{P_S} \times \mathcal{D}_{P_{S^c|S}}$$

where $\times$ *denotes cartesian product.*

PROOF. See Appendix A.2. □

Lemma 2.2 leads to an important observation: rate tuples on a face $\mathcal{F}_S$ of $\mathcal{D}$ can be achieved by splitting the users into two groups S and S^c and then successively decoding the groups. The users indexed by S may first be decoded regarding the remaining users in S^c as noise. Users X_{S^c} may then be decoded with the estimates $\hat{X}_S$ available at the decoder as side information. In this way, the problem of achieving points on $\mathcal{F}_S$ for an M-user channel reduces to the lower-dimensional problems of achieving points on the dominant faces $\mathcal{D}_{P_S}$ and $\mathcal{D}_{P_{S^c|S}}$ of a k-user channel and an $M-k$-user channel, respectively. This *group splitting* concept is a generalization of the familiar successive decoding idea, and is central to the coding techniques to be developed in this section.

At this point, the reader may object that in decoding the second set of users X_{S^c}, having the estimates $\hat{X}_S$ available at the decoder as side information is not equivalent to having the true inputs X_S (genie-aided) as assumed in the channel model of (2.8). For the purposes of analyzing the overall probability of error (the probability of making a decoding error for any of the users of the DMMAC), however, it has been shown in [RU96] that the genie-aided model and the model representing the actual successive decoding process are equivalent. That argument also applies to the group splitting process discussed here. We shall use such genie-aided analysis throughout the section.

Finally, we elaborate on Lemma 2.1 using our new perspectives. It turns out [GRUW01] that the dominant face $\mathcal{D}$ loses dimension if and only if it can be written as a Cartesian product of lower dimensional dominant faces, so that the DMMAC decomposes into at least two independent sub-DMMAC's.

Lemma 2.3 (See [GRUW01]) *For any $\emptyset \subset S \subset \{1, \ldots, M\}$, the following are equivalent:*

(a) $I(X_S; Y) = I(X_S; Y|X_{S^c})$.

(b) $\mathcal{D} = \mathcal{D}_{P_S} \times \mathcal{D}_{P_{S^c}}$.

In this case, it can be shown using the Irrelevancy Lemma from [GRUW01] that $\mathcal{D}_{P_{S^c|S}} = \mathcal{D}_{P_S}$ and thus $\mathcal{D} = \mathcal{F}_S = \mathcal{D}_{P_S} \times \mathcal{D}_{P_{S^c}}$. Thus, $\mathcal{D}$ collapses into one of its faces.

2.3 Achieving $\mathcal{D}$ Using Time-Sharing

It is well-known that time-sharing can be combined with successive decoding to achieve the non-vertex rate tuples on the dominant face of the MAC capacity region [CT91]. Two difficulties, however, have been cited to discredit this approach [RU96]. First, the number of single-user codes needed is onerous. Conventional time-sharing requires M^2 single-user codes to achieve any point in the capacity region. Second, synchronization is needed for conventional time-sharing, whereas a global time reference is difficult to obtain in practice if there is absolutely no feedback from the receiver back to the transmitters.

The problem of synchronization is the easier of the two to overcome. First, the difficulty in obtaining block synchronization in a discrete-time model may be overstated, since approximate block synchronization is attainable using only mild feedback in a multiple access setting [Gal85]. Usually, it is sufficient for each transmitter to synchronize with the receiver separately. The receiver can then appropriately delay each transmitter's signal to achieve global block synchronization. Thus, a global time reference may not be necessary for global block synchronization. Furthermore, we note that the idea of time-sharing also applies to frequency. By projecting a continuous-time multiple access channel on the appropriate set of orthonormal functions, frequency-sharing schemes in which different coding strategies are adopted for different frequency bands can be analyzed in much the same way as time-sharing schemes. Hence, theoretical analyses for time-sharing and frequency-sharing are the same, both involving independent parallel channels. In practice, however, frequency-sharing may be preferred since synchronization is not required.

As mentioned before, the complexity issue has been addressed successfully in [Rim99], in which it is shown that time-sharing schemes can attain the $2M-1$ performance of rate-splitting schemes [GRUW01, RU96]. Our independent analysis of this issue starts by noting that one single-user code can be used over several time-sharing segments corresponding to different decoding orders. In what follows, we show that the user faces a *deterministically varying channel* (DVC) over those segments, and that the maximum achievable rate over those segments is the average mutual information over the DVC. This in turn implies that the number of single-user codes needed can be dramatically reduced while the achievability of rate tuples on the dominant face is preserved. We begin with an analysis of the two-user case.

2.3.1 The Two-User Channel

We start by re-examining the conventional time-sharing set-up as illustrated in Figure 2-3. Here we consider a two-user DMMAC. We shall assume block synchronism for the moment. Suppose a communication system attempts to achieve the rate pair $\mathbf{\Phi} = (1-\lambda)\mathbf{\Phi}^{(1\ 2)} + \lambda\mathbf{\Phi}^{(2\ 1)} \in \mathcal{D}$, where $0 < \lambda < 1$. Here, $\mathbf{\Phi}^{(1\ 2)} = (I(X_1;Y), I(X_2;Y|X_1))$ and $\mathbf{\Phi}^{(2\ 1)} = (I(X_1;Y|X_2), I(X_2;Y))$ are the two vertices of $\mathcal{D}$ associated with the permutations $\pi_1 = (1\ 2)$ and $\pi_2 = (2\ 1)$. A possible implementation of the time-sharing scheme for a fixed overall block length N divides the interval $T \equiv \{1, \dots, N\}$ into two disjoint subintervals $T^{(1\ 2)}$ and $T^{(2\ 1)}$ which partition T, where $|T^{(1\ 2)}| = (1-\lambda)N$ and $|T^{(2\ 1)}| = \lambda N$ (we ignore the integer constraint for analytical convenience). To implement time-sharing, user one employs code C_{11}, with block length $(1-\lambda)N$, over $T^{(1\ 2)}$ and code C_{12}, with block length λN, over $T^{(2\ 1)}$. User two employs code C_{21}, with block length $(1-\lambda)N$, over $T^{(1\ 2)}$ and code C_{22}, with block length λN, over $T^{(2\ 1)}$. The decoding order G on the codes (an ordered permutation on the set of indices of the codes) is arranged such that user 1 is decoded first

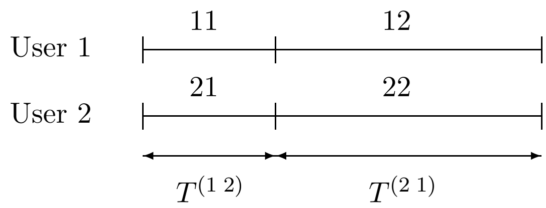

Figure 2-3: Time-sharing coding configuration for conventional time-sharing. The labels above the sub-intervals are the indices of the codes used over the sub-interval. The size of the sub-intervals are $|T^{(1\ 2)}| = (1-\lambda)N$ and $|T^{(2\ 1)}| = \lambda N$. The decoding order $G = (11\ 21\ 22\ 12)$, and the user decoding orders are (1 2) and (2 1) over the respective intervals.

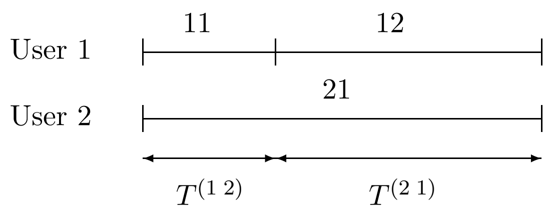

Figure 2-4: Time-sharing coding configuration for the two-user DMMAC. The labels above the sub-intervals are the indices of the codes used over the sub-interval. The size of the sub-intervals are $|T^{(1\ 2)}| = (1-\lambda)N$ and $|T^{(2\ 1)}| = \lambda N$. The decoding order $G = (11\ 21\ 12)$, and the user decoding orders are (1 2) and (2 1) over the respective intervals.

over $T^{(1\ 2)}$ followed by user 2, and user 2 is decoded first over $T^{(2\ 1)}$ followed by user 1. Choosing $G = (11\ 21\ 22\ 12)$, for instance, would accomplish this.

One may now ask whether it is really necessary that both users employ two codes to accomplish time-sharing between user decoding orders (1 2) over $T^{(1\ 2)}$ and (2 1) over $T^{(2\ 1)}$. Suppose user 2 employs only one code C_{21} over the whole interval T. Let the decoding order be $G = (11\ 21\ 12)$. This set-up is illustrated in Figure 2-4. Since C_{11} is decoded first over $T^{(1\ 2)}$, user 2 sees a "clean" channel over $T^{(1\ 2)}$, but sees user 1 as noise over $T^{(2\ 1)}$. Hence, over T, user 2 faces a set of independent channels which vary with channel index n. What is the highest rate that user 2 can achieve in this situation? Intuitively, one would guess that the answer is $(1-\lambda)I(X_2;Y|X_1) + \lambda I(X_2;Y)$.[3] This is indeed correct, as we show in the next section. It follows that any rate pair $\mathbf{\Phi} = (1-\lambda)\mathbf{\Phi}^{(1\ 2)} + \lambda\mathbf{\Phi}^{(2\ 1)}$, $0 \le \lambda \le 1$, is achievable with the configuration in Figure 2-4. Thus, the number of single-user codes needed have been reduced from four to three. To generalize the argument to more than two

[3]We are assuming here that the actual input X_1, rather than just the estimate $\hat{X}_1$, is available at the decoder for code C_{21} over the interval $T^{(1\ 2)}$. For the analysis of overall probability of error, this genie-aided analysis is valid. See previous discussion in Section 2.2.1.

users, we first need to give an analysis of so-called deterministically varying channels.

2.3.2 Deterministically Varying Channels

Definition 2.1 *Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{S} = \{1, \ldots, V\}$ be finite sets. A discrete memoryless deterministically varying channel (DVC) is a set of discrete channels $P_{\boldsymbol{s}}^N = \{P_{s_n}(y|x) : s_n \in \mathcal{S}, n = 1, \ldots, N\}$, where $\boldsymbol{s} = (s_1, \ldots, s_N) \in \mathcal{S}^N$ and the indices s_n are known parameters. $P_{s_n}(y|x)$ is the conditional probability that the channel output is $y \in \mathcal{Y}$ when the channel input is $x \in \mathcal{X}$ and the channel index is $s_n \in \mathcal{S}$. The operation of a DVC on N-tuples $\boldsymbol{x} = (x_1, \ldots, x_N) \in \mathcal{X}^N, \boldsymbol{y} = (y_1, \ldots, y_N) \in \mathcal{Y}^N, \boldsymbol{s} = (s_1, \ldots, s_N) \in \mathcal{S}^N$, is given by*

$$P_{\boldsymbol{s}}^N(\boldsymbol{y}|\boldsymbol{x}) = \prod_{n=1}^{N} P_{s_n}(y_n|x_n). \tag{2.11}$$

Definition 2.2 *Let $Q(x)$ be an arbitrary probability assignment on the channel input alphabet $\mathcal{X}$. An (N, R, Q) random code ensemble is an ensemble of (N, R) block codes where each codeword $\boldsymbol{x}$ is independently chosen according to the product distribution $Q^N(\boldsymbol{x}) = \prod_{n=1}^{N} Q(x_n)$.*

We now consider a sequence (in N) of DVC's such that the fraction of constituent discrete channels belonging to each channel type in $\mathcal{S}$ remains constant as N increases.

Definition 2.3 *A sequence (in N) of DVC's $\{P_{\boldsymbol{s}}^N\}$ is said to have fixed fraction if the fraction of channels of type $v \in \mathcal{S}$, $\theta_v = |\{n : P_{s_n} = P_v\}|/N$, remains constant as $N \to \infty$, where $\sum_{v=1}^{V} \theta_v = 1$.*

The following theorems can be established using the Parallel Channels Result of [Gal68, pp. 149-150] and the classic coding theorems for discrete memoryless channels (DMC's).

Theorem 2.1 *Let $\{P_{\boldsymbol{s}}^N\}$ be a sequence (in N) of fixed-fraction DVC's. Let R be any positive number. Consider a sequence (in N) of (N, R, Q) random code ensembles. For every positive integer N, let $\overline{P}_e$ be the ensemble average probability of decoding error averaged over all messages $l, 1 \leq l \leq \lceil e^{NR} \rceil$, using maximum likelihood (ML) decoding. Then,*

$$\overline{P}_e \leq \exp\left[-NE_r(R, Q)\right] \tag{2.12}$$

where

$$E_r(R,Q) \equiv \max_{0\le\rho\le 1}\left[\sum_{v=1}^{V}\theta_v E_v(\rho,Q) - \rho R\right], \tag{2.13}$$

$$E_v(\rho,Q) \equiv -\ln\sum_y\left\{\sum_x Q(x)P_v(y|x)^{1/(1+\rho)}\right\}^{1+\rho}. \tag{2.14}$$

and P_v denotes a discrete channel of type $v \in \mathcal{S}$.

PROOF. See Appendix A.3. □

Theorem 2.2 *For any sequence of fixed-fraction DVC's $\{P_{\boldsymbol{s}}^N\}$, the random error exponent $E_r(R,Q)$, as defined in (2.13) and (2.14), is a convex, decreasing, and positive function of R for $0 \le R < \sum_{v=1}^{V}\theta_v I(Q;P_v)$.*

PROOF. See Appendix A.4. □

It follows from the proof of Theorem 2.2 in Appendix A.4 that the random coding exponent $E_r(R,Q)$ for the fixed-fraction DVC's can be obtained parametrically through $\rho, 0 \le \rho \le 1$, with $E_r(\rho,Q)$ and the rate $R(\rho,Q)$ being given as the average of the respective quantities $E_{r_v}(\rho,Q)$ and $R_v(\rho,Q)$ for each channel type $v \in \mathcal{S}$:

$$E_r(\rho,Q) = \sum_{v=1}^{V}\theta_v E_{r_v}(\rho,Q), \quad R(\rho,Q) = \sum_{v=1}^{V}\theta_v R_v(\rho,Q).$$

In other words, the $(E_r(\rho,Q), R(\rho,Q))$ curve for the fixed fraction DVC's is formed by the vector average of points *of the same slope* $\rho, 0 \le \rho \le 1$, from the $(E_{r_v}(\rho,Q), R_v(\rho,Q))$ curves for the constituent channel types in $\mathcal{S}$.

Theorems 2.1 and 2.2 show that for every fixed N, the probability of error averaged over the (N,R,Q) code ensemble used over the DVC $P_{\boldsymbol{s}}^N$ is bounded with a positive exponent as in (2.12) for all $R < \sum_{v=1}^{V}\theta_v I(Q;P_v)$. It follows that for each N and every $R < \sum_{v=1}^{V}\theta_v I(Q;P_v)$, there exists at least one code in the (N,R,Q) code ensemble for which the error probability is so bounded. In other words, over a sequence of fixed-fraction DVC's, it is always possible to pick a sequence of codes appropriately to obtain exponentially decreasing error probabilities so long as the rate of the codes stays below the mixture of average mutual informations for the channel types in each DVC. The following definition and theorem summarize the above results.

Definition 2.4 *The rate Φ is said to be achievable over the sequence (in N) of DVC's $\{P_{\boldsymbol{s}}^N\}$ with the input distribution Q if for every $\delta > 0$, and all $R \leq \Phi - \delta$, there exists a sequence of (N, R) block codes, each a member of the (N, R, Q) code ensemble, for which the average probability of decoding error P_e^N tends to zero as $N \to \infty$.*

Theorem 2.3 *Over a sequence of fixed-fraction DVC's $\{P_{\boldsymbol{s}}^N\}$, all rates $\Phi \leq \sum_{v=1}^{V} \theta_v \cdot I(Q; P_v)$ are achievable with the input distribution Q.*

The converse to Theorem 2.3 is very similar to the usual converse for DMC's. Let $U^K = (U_1, \ldots, U_K), U_i \in \mathcal{U}, i = 1, \ldots, K$, be a sequence of K outputs of a discrete source connected to a destination through a sequence of N channel uses of a DVC $P_{\boldsymbol{s}}^N$. Let the sequence $\hat{U}^K = (\hat{U}_1, \ldots, \hat{U}_K)$ be the output of the decoder. The rate of the channel code is $R = K/N$. Let $P_{e,i}^K = \Pr\left(\hat{U}_i(Y^N) \neq U_i\right)$ be the probability of error in the ith bit, and let $P_b^K = \frac{1}{K}\sum_{i=1}^{K} P_{e,i}^K$ be the average error probability over the sequence of K bits, or the bit error rate.

Theorem 2.4 *Let Q be a fixed distribution on the input alphabet $\mathcal{X}$ and N be a fixed positive integer. Then, over a fixed-fraction DVC $P_{\boldsymbol{s}}^N$,*

$$R\left(1 - H(P_b^K) - P_b^K \ln(|\mathcal{U}| - 1)\right) \leq \sum_{v=1}^{V} \theta_v I(Q; P_v)$$

where $H(P_b^K) \equiv -P_b^K \log P_b^K - (1 - P_b^K)\log(1 - P_b^K)$.

PROOF. See Appendix A.5. □

The aim of this section has been to exactly characterize the set of achievable rates over a sequence of fixed-fraction deterministically varying channels using a fixed input distribution. The result is that the highest achievable rate is the appropriate mixture of the average mutual informations between the input distribution and the respective channel transitions. This justifies our assertion with regards to Figure 2-4 in Section 2.3.1 that user 2 can indeed achieve $(1-\lambda)I(X_2; Y|X_1) + \lambda I(X_2; Y)$. To apply this result to DMMAC's with more than two users, we need to generalize the idea of time-sharing.

2.3.3 Time-Sharing Coding Configurations

Rimoldi [Rim97] has given a general formulation in which the ideas of time-sharing and successive decoding are combined. In this set-up, each user of a DMMAC has a number of single-user codes which it chooses to employ over subsets of time specified using a scheduling sequence. The single-user codes are decoded successively according to a decoding order. We will show that the scheduling sequence and the decoding order combine to determine, for each user and over each coding sub-interval, a sequence of fixed-fraction DVC's with a corresponding achievable rate. The following definition closely resembles the one in [Rim97].

Definition 2.5 *A time-sharing coding configuration* $\mathcal{T} = (N, C, \{\boldsymbol{t}_n\}, G)$ *for an* M*-user DMMAC consists of the following:*

(a) Each user $i, 1 \le i \le M$*, employs an overall block length* N*.*

(b) Each user i *has a set of* J_i *single-user codes* $\{C_{ij} : 1 \le j \le J_i\}$*. Each code* C_{ij} *is a* $(N\gamma_{ij}, R_{ij})$ *block code, where for each* i*,* $\sum_{j=1}^{J_i} \gamma_{ij} = 1$*. Let* $C \equiv \{C_{ij} : 1 \le i \le M, 1 \le j \le J_i\}$*.*

(c) A sequence of scheduling M*-tuples,* $\{\boldsymbol{t}_n\}_{n=1}^{N}$*, where* $\boldsymbol{t}_n = (t_{1n}, \ldots, t_{Mn})$*, determines the code to be used by each user at each time. Specifically,* $t_{in} \in \{1, \ldots, J_i\}$ *is the index of the code used by user* i *at time* n*, Let* $T_{ij} \equiv \{n : t_{in} = j\}, 1 \le i \le M, 1 \le j \le J_i$*, be the subset of* $\{1, \ldots, N\}$ *on which user* i *uses the* ij*th code, where* $|T_{ij}| = N\gamma_{ij}$*.*

(d) The decoding order on the single-user codes is determined by G*, an ordered permutation on* $E = \{ij : 1 \le i \le M, 1 \le j \le J_i\}$*.*

The coding process using the configuration $\mathcal{T}$ involves the following.

Encoding User $i, 1 \le i \le M$, forms J_i codewords $w_{ij}, 1 \le j \le J_i$, where the codeword w_{ij} is produced by the encoder corresponding to the code C_{ij}, and has block length $N\gamma_{ij}$. At time $n, 1 \le n \le N$, the ith transmitter sends the next symbol of codeword w_{ij} if $t_{in} = j$. The ith scheduling sequence t_{in} is known only to transmitter i, while the entire scheduling M-tuple sequence $\{\boldsymbol{t}_n\}$ is known to the common receiver of all transmitters.

Decoding The codewords generated by each code in C are decoded successively. Decoding proceeds according to the order G as follows. If $ij \in E$ is the first entry in G, then the decoder for the code C_{ij} observes channel outputs $\{Y_n : t_{in} = j\}$, and decodes the codeword w_{ij} regarding codewords from all other users as noise. The decoded codeword $\widehat{w_{ij}}$ is then available as side information to all other decoders corresponding to codes with indices following ij in G. The decoding proceeds with the decoder for code B_{kl} if kl is the next entry in G. This process continues until all codewords $\{w_{ij}, ij \in E\}$ have been decoded. At each step of the decoding process, all previously decoded codewords are available as side information at the active decoder.

A time-sharing coding configuration imposes, at every time n, an order in which the users are decoded. The *user* decoding order $U_n \in \Pi_M$ can be obtained by listing $\{it_{in}, i = 1, \ldots, M\}$ in the order given by G and then removing the second index. Specifically, $U_n(i)$ is the user which decodes in the ith position at time n.

Example 2.1 (Two User Case) To illustrate the idea of time-sharing coding configurations, we interpret the two-user case seen in Figure 2-4 in our new framework. Here, user 1 has two single-user codes ($J_1 = 2$) C_{11} and C_{12}. User 2 has one code ($J_2 = 1$) C_{21}. We have $\gamma_{11} = 1 - \lambda, \gamma_{12} = \lambda, \gamma_{21} = 1$. The scheduling sequences are such that $t_{1n} = 1$ for $n \in \{1, \ldots, (1-\lambda)N\} = T_{11}$, $t_{1n} = 2$ for $n \in \{(1-\lambda)N + 1, \ldots, N\} = T_{12}$, and $t_{2n} = 1$ for all $n \in \{1, \ldots, N\}$. Also, $E = \{11, 12, 21\}$ and $G = (11\ 21\ 12)$. The user decoding order is $U_n = (1\ 2)$ for $n \in T_{11}$ and $U_n = (2\ 1)$ for $n \in T_{12}$. $\square$

For a fixed product distribution $\prod_{i=1}^{M} Q_{X_i}$ on the product of input alphabets $\mathcal{X}^N$, the user decoding orders imposed by a time-sharing configuration create, for each user $i, 1 \leq i \leq M$, a fixed-fraction DVC $P_{\boldsymbol{s}}^{N\gamma_{ij}}$ over the subset $T_{ij}, 1 \leq j \leq J_i$. We can describe the *genie-aided* version of $P_{\boldsymbol{s}}^{N\gamma_{ij}}$, say $\hat{P}_{\boldsymbol{s}}^{N\gamma_{ij}}$, as follows. Here, we assume that all the *actual* inputs corresponding to users decoded before user i at time $n \in T_{ij}$ according to the order U_n are available to the decoder for code C_{ij}. Once again, for consideration of overall decoding error probability, this is sufficient. For a permutation $\pi \in \Pi_M$, let $X_{B(\pi,i)} = \{X_{\pi(1)}, \ldots, X_{\pi(\pi^{-1}(i)-1)}\}$ be the set of inputs decoded *before* X_i under π. Similarly, let $X_{A(\pi,i)} = \{X_{\pi(\pi^{-1}(i)+1)}, \ldots, X_{\pi(M)}\}$ be the set of inputs decoded *after* X_i under π. For convenience, denote $A(U_n, i)$ by A and $B(U_n, i)$ by B. Also, let $Q_{X_A} \equiv \prod_{i \in A} Q_{X_i}$ and $Q_{X_B} \equiv \prod_{i \in B} Q_{X_i}$. Then the nth channel of the genie-aided DVC $\hat{P}_{\boldsymbol{s}}^{N\gamma_{ij}}$ seen by user i, for

$n \in T_{ij}$ is given by

$$\hat{P}_{s_n}(y, x_B|x_i) = \sum_{x_A} Q_{X_A}(x_A) Q_{X_B}(x_B) P(y|x_B, x_i, x_A) \tag{2.15}$$

where P is the stochastic matrix associated with the original M-user DMMAC (*cf.* equations (2.7) and (2.8)). It is clear that $I(Q_{X_i}; \hat{P}_{s_n}) = I(X_i; X_B, Y) = I(X_i; Y|X_B)$.

Achievable Rates

We now define the achievability of a rate tuple using time-sharing of single-user codes in a manner consistent (via the union bound) with the usual definition of achievability for multi-access channels [Gal85].

Definition 2.6 *Over an M-user DMMAC, rate $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_M) \in \mathcal{R}(P, \prod_{i=1}^{M} Q_{X_i})$ is said to be achievable using time-sharing of single-user codes with the product input distribution $\prod_{i=1}^{M} Q_{X_i}$ if for every $\delta > 0$, and all rate tuples $\boldsymbol{R} \in \mathcal{R}$ such that $R_i \leq \Phi_i - \delta, i = 1, \ldots, M$, there exists a sequence (in N) of time-sharing coding configurations $\{\mathcal{T}_N\} = \{(N, C, \{\boldsymbol{t}_n\}, G)\}$ such that*

(a) For every N, $R_i = \sum_{j=1}^{J_i} \gamma_{ij} R_{ij}, i = 1, \ldots, M$, where γ_{ij} is the fraction of time (independent of N) that user i uses the code C_{ij}^N.

(b) The average probabilities of decoding error $P_{e,ij}^N$ for the sequence of $(N\gamma_{ij}, R_{ij})$ codes C_{ij}^N, each a member of the $(N\gamma_{ij}, R_{ij}, Q_{X_i})$ code ensemble, tends to zero on the genie-aided fixed-fraction DVC sequence $\{\hat{P}_{\boldsymbol{s}}^{N\gamma_{ij}}\}$ as $N \to \infty$, for every $ij, 1 \leq i \leq M, 1 \leq j \leq J_i$.

Definition 2.6 implies that a rate tuple $\mathbf{\Phi}$ is achievable using time-sharing with product input distribution $\prod_{i=1}^{M} Q_{X_i}$ if and only if the rates Φ_{ij}, where $\Phi_i = \sum_{j=1}^{J_i} \gamma_{ij} \Phi_{ij}$, are achievable with input distribution Q_{X_i} over the sequence (in N) of fixed fraction genie-aided DVC's $\{\hat{P}_{\boldsymbol{s}}^{N\gamma_{ij}}\}$ corresponding to the configuration sequence $\{\mathcal{T}_N\} = \{(N, C, \{\boldsymbol{t}_n\}, G)\}$, for every ij (see Definition 2.4).

Example 2.2 (Two User Case) For the two-user case in Figure 2-4, any rate pair $\mathbf{\Phi} = (1-\lambda)\mathbf{\Phi}^{(1\,2)} + \lambda \mathbf{\Phi}^{(2\,1)}$, $0 \leq \lambda \leq 1$, is achievable using time-sharing of single-user codes.

This is accomplished with the time-sharing coding configuration given in Example 2.1. Let $\Phi_1 = (1-\lambda)\Phi_{11} + \lambda\Phi_{12}$, and $\Phi_{21} = \Phi_2$. Rate Φ_{11} is achievable over the DVC sequence $\{\hat{P}_{\boldsymbol{s}}^{N(1-\lambda)}\}$, where $\hat{P}_{s_n}(y|x_1) = \sum_{x_2} Q_{X_2}(x_2)P(y|x_1,x_2), n \in \{1,\dots,(1-\lambda)N\}$. Rate Φ_2 is achievable over $\{\hat{P}_{\boldsymbol{s}}^{N}\}$ where $\hat{P}_{s_n}(y,x_1|x_2) = Q_{X_1}(x_1)P(y|x_1,x_2)$ for $n \in \{1,\dots,(1-\lambda)N\}$ and $\hat{P}_{s_n}(y|x_2) = \sum_{x_1} Q_{X_1}(x_1)P(y|x_1,x_2)$ for $n \in \{(1-\lambda)N+1,\dots,N\}$. Rate Φ_{12} is achievable over $\{\hat{P}_{\boldsymbol{s}}^{N\lambda}\}$, $\hat{P}_{s_n}(y,x_2|x_1) = Q_{X_2}(x_2)P(y|x_1,x_2)$, $n \in \{(1-\lambda)N+1,\dots,N\}$. $\square$

Notice that Definition 2.6 is consistent with the usual definition of achievability for DMMAC's. For each N, let $P_{e,ij}^N, 1 \le i \le M, 1 \le j \le J_i$, be the probability of decoding error over a block length of $N\gamma_{ij}$ for the ijth code of user i (a member of the $(N\gamma_{ij}, R_{ij}, Q_{X_i})$ code ensemble) over the genie-aided DVC $\hat{P}_{\boldsymbol{s}}^{N\gamma_{ij}}$. Let $P_{e,i}^N$ be the probability that user i is decoded incorrectly on any part of the overall block length N over the set of genie-aided channels $\hat{P}_{\boldsymbol{s}}^{N\gamma_{ij}}, 1 \le j \le J_i$. Finally, let P_e^N be the overall probability of decoding error, that is, the probability that any user is decoded incorrectly, for block length N. Then by the genie-aided argument [RU96] and by repeated use of the union bound,

$$P_e^N \le \sum_{i=1}^{M} P_{e,i}^N \le \sum_{i=1}^{M}\sum_{j=1}^{J_i} P_{e,ij}^N. \tag{2.16}$$

Hence if $P_{e,ij}^N \to 0$ for every $ij \in E$ as $N \to \infty$, then $P_e^N \to 0$ as $N \to \infty$.

Probability of Error

Next, we use (2.16) to obtain a simple bound on the minimum achievable overall probability of error for any time-sharing coding configuration with a fixed block length N, product input distribution $\prod_{i=1}^M Q_{X_i}$, scheduling sequences $\{\boldsymbol{t}_n\}$, and decoding order G. Given these assumptions, Theorem 2.1 says that for every $ij, 1 \le i \le M, 1 \le j \le J_i$, there exists an $(N\gamma_{ij}, R_{ij})$ code in the $(N\gamma_{ij}, R_{ij}, Q_{X_i})$ code ensemble for which the probability of decoding error over the genie-aided DVC $\hat{P}_{\boldsymbol{s}}^{N\gamma_{ij}}$ using ML decoding is upper bounded as

$$P_{e,ij}^N \le \exp\left[-N\gamma_{ij}E_{r_{ij}}(R_{ij}, Q_{X_i})\right]$$

where

$$E_{r_{ij}}(R_{ij}, Q_{X_i}) \equiv \max_{0\leq\rho\leq 1}\left[\sum_{v=1}^{V_{ij}} \theta_v E_v(\rho, Q_{X_i}) - \rho R_{ij}\right], \tag{2.17}$$

$$E_v(\rho, Q_{X_i}) \equiv -\ln \sum_{y,x_B}\left\{\sum_{x_i} Q_{X_i}(x_i)\hat{P}_v(y, x_B|x_i)^{\frac{1}{1+\rho}}\right\}^{1+\rho}, \tag{2.18}$$

$$\hat{P}_v(y, x_B|x_i) = \sum_{x_A} Q_{X_A}(x_A)Q_{X_B}(x_B)P(y|x_B, x_i, x_A). \tag{2.19}$$

Here, we are using the same notation as in (2.15), with U_n being replaced by U_v, the user decoding order for the genie-aided channels $\hat{P}_v(y, x_B|x_i)$ of type $v \in \mathcal{S}$ in the DVC $\hat{P}_{\boldsymbol{s}}^{N\gamma_{ij}}$. The quantity θ_v is the fraction of genie-aided channels of type v, where $\sum_{v=1}^{V_{ij}} \theta_v = 1$, and V_{ij} is the total number of channel types in $\hat{P}_{\boldsymbol{s}}^{N\gamma_{ij}}$. Once again, the error exponent $E_{r_{ij}}(R_{ij}, Q_{X_i})$ can be obtained parametrically through ρ by averaging the respective error exponents and rates of each type of channel encountered over the DVC $\hat{P}_{\boldsymbol{s}}^{N\gamma_{ij}}$. Now by (2.16),

$$P_e^N \leq \sum_{i=1}^{M}\sum_{j=1}^{J_i} \exp\left[-N\gamma_{ij}E_{r_{ij}}(R_{ij}, Q_{X_i})\right]. \tag{2.20}$$

Hence, there exists an $(N, C, \{\boldsymbol{t}_n\}, G)$ time-sharing coding configuration for which the overall probability of decoding error is upper bounded as in (2.20).

Example 2.3 (Two User Gaussian MAC) We give an error probability analysis for a discrete-time memoryless two-user additive Gaussian MAC as defined in (2.5). Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the average energy constraints of the users and let σ^2 be the background noise variance. We choose the Gaussian distribution $\mathcal{N}(0, \mathcal{E}_i)$ for the code ensemble of user $i, i = 1, 2$. Consider the coding configuration for conventional time-sharing in Figure 2-3. For fixed block length N, let $P_{e,i}^N, i = 1, 2$, be the probability that user i is decoded incorrectly on any part of the overall block length over the corresponding set of genie-aided channels. Then by the genie-aided argument and the union bound,

$$P_{e,1}^N \leq \exp[-N(1-\lambda)E_{r_{11}}(R_{11})] + \exp[-N\lambda E_{r_{12}}(R_{12})], \tag{2.21}$$

where

$$\begin{aligned} E_{r_{11}}(R_{11}) &= \max_{\rho\in[0,1]} \frac{\rho}{2}\ln\left(1+\frac{\mathcal{E}_1}{(\sigma^2+\mathcal{E}_2)(1+\rho)}\right), \\ E_{r_{12}}(R_{12}) &= \max_{\rho\in[0,1]} \frac{\rho}{2}\ln\left(1+\frac{\mathcal{E}_1}{\sigma^2(1+\rho)}\right). \end{aligned}$$

are given by the integral form of (2.17)-(2.19) [Gal85]. Similarly,

$$\begin{aligned} &P_{e,2}^N \le \exp[-N(1-\lambda)E_{r_{21}}(R_{21})] + \exp[-N\lambda E_{r_{22}}(R_{22})], \\ &E_{r_{21}}(R_{21}) = \max_{\rho\in[0,1]} \frac{\rho}{2}\ln\left(1+\frac{\mathcal{E}_2}{\sigma^2(1+\rho)}\right), \\ &E_{r_{22}}(R_{22}) = \max_{\rho\in[0,1]} \frac{\rho}{2}\ln\left(1+\frac{\mathcal{E}_2}{(\sigma^2+\mathcal{E}_1)(1+\rho)}\right). \end{aligned} \tag{2.22}$$

Notice that for large N, and $R_{21} \uparrow I(X_2;Y|X_1), R_{22} \uparrow I(X_2;Y)$, $P_{e,2}^N$ is essentially determined by the second exponential term in (2.22) (with the exponent $\lambda E_{r_{22}}(R_{22})$) corresponding to the more noisy λN channel uses. This is because $P_{e,2}^N$ is lower-bounded by the probability of an error event on the last λN symbols, and moreover the upper bound with the random error exponent is tight (that is, $E_r(R)$ agrees with the sphere packing exponent $E_{sp}(R)$) for rates near capacity [Gal68].

We now turn to the coding configuration in Figure 2-4. Here, it is clear that the bound on $P_{e,1}^N$ is the same as in (2.21). The bound on $P_{e,2}^N$, however, is different. Here, user 2 sees a genie-aided fixed-fraction DVC. Thus, its random error exponent is the parametric (via ρ) convex combination of the error exponent for the clean and noisy channels. We have

$$P_{e,2}^N \le \exp[-NE_{r_2}(R_2)],$$

where

$$E_{r_2}(R_2) = \max_{\rho\in[0,1]}\left[(1-\lambda)\frac{\rho}{2}\ln\left(1+\frac{\mathcal{E}_2}{\sigma^2(1+\rho)}\right)+\lambda\frac{\rho}{2}\ln\left(1+\frac{\mathcal{E}_2}{(\sigma^2+\mathcal{E}_1)(1+\rho)}\right)\right].$$

Notice that $E_{r_2}(R_2)$ is clearly larger than $\lambda E_{r_{22}}(R_{22})$ at all positive rates. We conclude that the coding configuration of Figure 2-4 not only uses fewer single-user codes than the conventional time-sharing configuration of Figure 2-3, but also gives better error probability performance for user 2. □

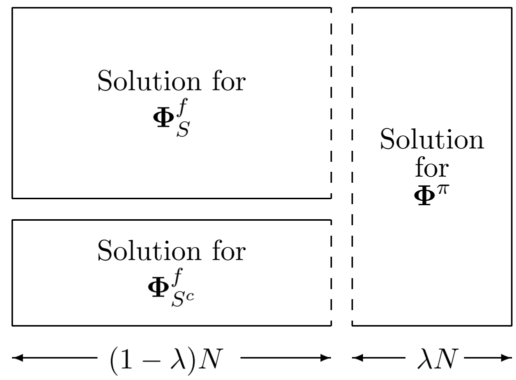

Figure 2-5: Illustration of the main idea of projective time-sharing for a fixed block length N. The coding strategy for $\mathbf{\Phi} \in \mathcal{D}$ is derived from the strategy for a vertex $\mathbf{\Phi}^\pi$ and that for a face point $\mathbf{\Phi}^f \in \mathcal{F}_S$ by time-sharing, where the time sharing parameter here is $0 \leq \lambda \leq 1$. The coding strategy for $\mathbf{\Phi}^f$ is in turn obtained from the coding strategy for $\mathbf{\Phi}_S^f \in \mathcal{D}_{P_S}$ and that for $\mathbf{\Phi}_{S^c}^f \in \mathcal{D}_{P_{S^c|S}}$.

2.4 Projective Time-Sharing

We have now acquired all the essential tools needed to derive our main result. The notions of deterministically varying channels and time-sharing coding configurations will now be combined with the underlying geometry of the dominant face to devise a strategy which achieves any rate tuple in $\mathcal{D}$ using no more than $\frac{1}{2}M\log_2 M + M$ single-user codes.

Towards that end, we consider a specific class of time-sharing coding configurations called *projective time-sharing* configurations. The main idea behind projective time-sharing is that any rate tuple $\mathbf{\Phi} \in \mathcal{D}$ can be expressed, as we show, as the convex combination of any vertex $\mathbf{\Phi}^\pi$ of $\mathcal{D}$ and a point $\mathbf{\Phi}^f$ on some face $\mathcal{F}_S$ (as defined in Section 2.2.1) of $\mathcal{D}$. Hence, the coding strategy for an arbitrary rate-tuple $\mathbf{\Phi}$ can be derived from the strategy for $\mathbf{\Phi}^\pi$ and that for $\mathbf{\Phi}^f$ by time-sharing. Moreover, recall Lemma 2.2 from Section 2.2.1, which says that any face $\mathcal{F}_s$ of $\mathcal{D}$ is the Cartesian product of two lower-dimensional dominant faces $\mathcal{D}_{P_S}$ and $\mathcal{D}_{P_{S^c|S}}$ of the DMMAC's P_S and $P_{S^c|S}$. This implies that the coding strategy for achieving any rate tuple $\mathbf{\Phi}^f$ on some face $\mathcal{F}_S$ of $\mathcal{D}$ can be obtained from the coding strategy for achieving the *projection* of $\mathbf{\Phi}^f$ onto the S-coordinates $\mathbf{\Phi}_S^f$ (belonging to $\mathcal{D}_{P_S}$) and that for the *projection* of $\mathbf{\Phi}^f$ onto the S^c-coordinates $\mathbf{\Phi}_{S^c}^f$ (belonging to $\mathcal{D}_{P_{S^c|S}}$). In this way, the problem of achieving rate tuples on the dominant face $\mathcal{D}$ of an M-user DMMAC reduces naturally to the lower-dimensional problems of achieving points on the dominant faces $\mathcal{D}_{P_S}$ and $\mathcal{D}_{P_{S^c|S}}$. This is illustrated in Figure 2-5. The idea of projective time-sharing is clarified in the following definition.

Definition 2.7 *Let $\mathcal{D}$ be the dominant face of a rate region $\mathcal{R}(P, \prod_{i=1}^{M} Q_{X_i})$ for an M-user DMMAC. Let $\mathbf{\Phi} \in \mathcal{D}$ be such that $\mathbf{\Phi} = \lambda \mathbf{\Phi}^{\pi} + (1-\lambda)\mathbf{\Phi}^{f}, 0 \leq \lambda \leq 1$, where $\mathbf{\Phi}^{\pi}$ is a vertex of $\mathcal{D}$, $\pi \in \Pi_M$, and $\mathbf{\Phi}^{f}$ is a point on some face of $\mathcal{D}$. $\mathbf{\Phi}$ is said to be achievable by projective time-sharing of single-user codes with the product input distribution $\prod_{i=1}^{M} Q_{X_i}$ if for any $\delta > 0$ and any $\boldsymbol{R} \in \mathcal{R}$ such that $R_i \leq \Phi_i - \delta, i = 1, \ldots, M$, there exists a sequence (in N) of time-sharing coding configurations $\{(N, C, \{\boldsymbol{t}_n\}, G)\}$ satisfying the following:*

(a) For every N, there exist disjoint subsets T^f and T^{π} of $\{1, \ldots, N\}$ with $|T^f|/N = 1 - \lambda, |T^{\pi}|/N = \lambda$, and $T^f \cup T^{\pi} = \{1, \ldots, N\}$.

(b) For every N, the scheduling M-tuple $\boldsymbol{t}_n$ is such that $t_{in} = j_i$ for some fixed $j_i \in \{1, \ldots, J_i\}, \forall n \in T^{\pi}, i = 1, \ldots, M$, and the user decoding order over T^{π} is π. That is, $U_{T^{\pi}} = \pi$.

(c) For every N, $R_i = \sum_{j=1}^{J_i} \gamma_{ij} R_{ij}, i = 1, \ldots, M$, where γ_{ij} is the fraction of time (independent of N) that user i uses the code C_{ij}^{N}.

(d) The average probabilities of decoding error $P_{e,ij}^{N}$ for the sequence of $(N\gamma_{ij}, R_{ij})$ codes C_{ij}^{N}, each a member of the $(N\gamma_{ij}, R_{ij}, Q_{X_i})$ code ensemble, tends to zero on the genie-aided fixed-fraction DVC sequence $\{\hat{P}_{\boldsymbol{s}}^{N\gamma_{ij}}\}$ as $N \to \infty$, for every $ij, 1 \leq i \leq M, 1 \leq j \leq J_i$.

In essence, the definition says that $\mathbf{\Phi}$ is achievable using projective time-sharing if there exists a sequence of configurations which time-shares between a strategy for achieving $\mathbf{\Phi}^{f}$ over T^f and a strategy for achieving $\mathbf{\Phi}^{\pi}$ over T^{π}. Over T^{π}, each user uses only one code for a fixed N, and the user decoding order is π associated with the vertex $\mathbf{\Phi}^{\pi}$. The last two conditions follow directly from Definition 2.6.

Example 2.4 (Two User Case) For the two-user case in Figure 2-4, any rate pair $\mathbf{\Phi} = (1-\lambda)\mathbf{\Phi}^{(1\,2)} + \lambda \mathbf{\Phi}^{(2\,1)}$, $0 \leq \lambda \leq 1$, is achievable using projective time-sharing of single-user codes. Here, $\mathbf{\Phi}^{(2\,1)}$ is a vertex of $\mathcal{D}$ and $\mathbf{\Phi}^{(1\,2)}$ is a face point (also a vertex) of $\mathcal{D}$. Also,

$T^f = T^{(1\ 2)}$, $T^\pi = T^{(2\ 1)}$. For each N, $t_{1n} = 2, t_{2n} = 1, \forall n \in T^\pi$ and $U_{T^\pi} = (2\ 1)$. See Examples 2.1 and 2.2 for details. □

2.4.1 The Main Theorem

In this section, we show that any rate tuple $\mathbf{\Phi} \in \mathcal{D}$ can be achieved using projective time-sharing with no more than $\frac{1}{2}M \log_2 M + M$ codes. The main geometric idea of projection onto lower-dimensional dominant faces plays a large role in the proof of the main theorem. That idea, however, is not *by itself* sufficient to limit the number of required single-user codes to $O(M \log_2 M)$. The latter requires additional somewhat subtle arguments to be presented in the proof and the example following. We start with a lemma.

Lemma 2.4 *Let the function $h : \mathbb{Z}^+ \to \mathbb{Z}^+$ be defined by $h(1) = 1$ and for $M \geq 2$,*

$$h(M) = \max_{1 \leq k \leq M-1} h(k) + h(M-k) + \min(k, M-k).$$

Then,

(a) $h(M) = h(\lfloor \frac{M}{2} \rfloor) + h(\lceil \frac{M}{2} \rceil) + \lfloor \frac{M}{2} \rfloor$.

(b) $h(M) \leq \frac{1}{2}M \log_2 M + M$, with equality for $M = 2^r, r \in \mathbb{Z}^+$.

PROOF. See Appendix A.6. □

Theorem 2.5 *Let $\mathcal{D}$ be the dominant face of the rate region $\mathcal{R}(P, \prod_{i=1}^M Q_{X_i})$ for an M-user DMMAC $P(y|x_1, \ldots, x_M)$. Then any rate-tuple $\mathbf{\Phi} \in \mathcal{D}$ can be expressed as the convex combination of an arbitrary vertex $\mathbf{\Phi}^\pi$ of $\mathcal{D}$ and a point $\mathbf{\Phi}^f$ on some face $\mathcal{F}_S$ of $\mathcal{D}$, and is achievable by projective time-sharing using no more than $h(M)$ single-user codes with the product input distribution $\prod_{i=1}^M Q_{X_i}$.*

Before presenting the detailed proof, we illustrate the central concepts of the proof construction with the following example.

Example 2.5 Consider a five-user DMMAC $P(y|x_1, \ldots, x_5)$. Let $\mathcal{R}(P, \prod_{i=1}^5 Q_{X_i})$ be the rate region corresponding to the product input distribution $\prod_{i=1}^5 Q_{X_i}$. Assume the dominant face $\mathcal{D}$ of $\mathcal{R}$ has dimension 4. Let $\mathbf{\Phi} \in \mathcal{D}$. We shall fix an overall block length N,

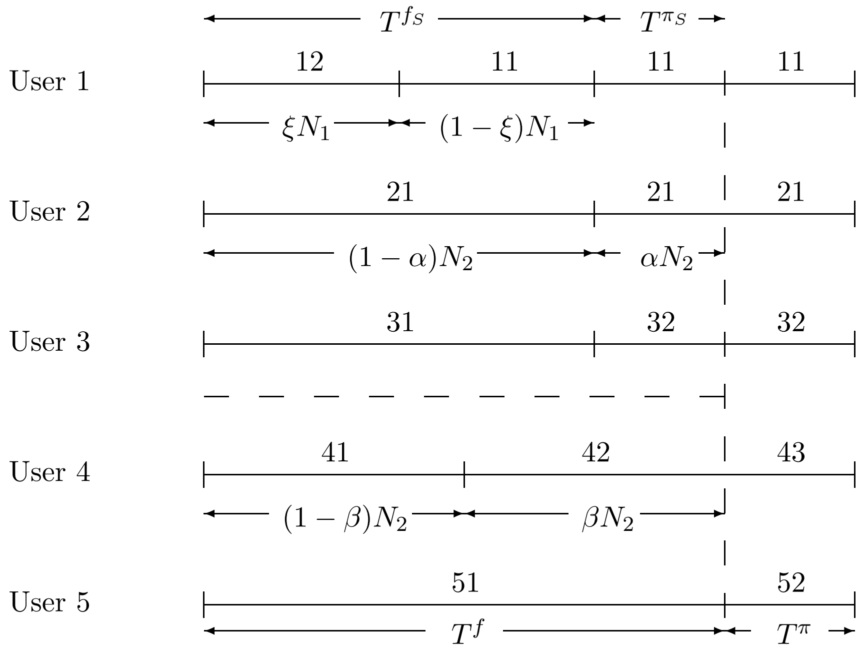

Figure 2-6: Projective time-sharing coding configuration for Example 2.5. The labels above the sub-intervals are the indices of the codes used over the sub-interval. The overall block length is N, where $N_2 = (1-\lambda)N, N_1 = (1-\alpha)N_2$. Also, $|T^f| = (1-\lambda)N = N_2$, $|T^\pi| = \lambda N$, $|T^{f_S}| = N_1 = (1-\alpha)N_2$, and $|T^{\pi_S}| = \alpha N_2$. The overall decoder order is $G = (43\ 11\ 32\ 21\ 52\ 12\ 31\ 41\ 51\ 42)$. The user decoding order among users 1, 2, and 3 over T^{π_S} is $\pi_S = (132)$. The user decoding order is $\pi = (41325)$ over T^π. The vertical dashed line separates the sub-configuration for $\mathbf{\Phi}^\pi$ from that for $\mathbf{\Phi}^f$. The horizontal dashed line separates the sub-configuration for $\mathbf{\Phi}_S^f$ from that for $\mathbf{\Phi}_{S^c}^f$. We have assumed that $\xi, \alpha, \beta, \lambda \in (0,1)$.

and specify the time-sharing coding configuration. We build the configuration in two major steps. First, we successively reduce the dimension of the problem from $M-1=4$ to 0 or 1 using the vertex *selection*, ray *extension*, and rate tuple *projection* (SEP) process. Then, we appropriately supplement and merge the configurations for the lower-dimensional problems to build the overall configuration.

Start by selecting an arbitrary vertex $\mathbf{\Phi}^\pi$ of $\mathcal{D}$ associated with a specific permutation π of the set $\{1,2,3,4,5\}$, say $\pi = (41325)$. Extend a ray $\overrightarrow{\mathbf{\Phi}^\pi\mathbf{\Phi}}$ from $\mathbf{\Phi}^\pi$ through $\mathbf{\Phi}$. Suppose the ray exits $\mathcal{D}$ at some face $\mathcal{F}_S$, where $\emptyset \subset S \subset \{1,\ldots,5\}, 1 \le |S| = k < 5$. Assume without loss of generality that $S = \{1,2,3\}$, so that $|S| = k = 3$, $S^c = \{4,5\}$, $|S^c| = 2$. Note that $|S| \ge |S^c|$. Let $\mathbf{\Phi}^f = \overrightarrow{\mathbf{\Phi}^\pi\mathbf{\Phi}} \cap \mathcal{F}_S$. Then, $\mathbf{\Phi} = \lambda\mathbf{\Phi}^\pi + (1-\lambda)\mathbf{\Phi}^f$ for some $0 \le \lambda \le 1$. Notice that the initial choice of the vertex $\mathbf{\Phi}^\pi$ determines a split of the set $\{1,2,3,4,5\}$ into two subsets $S = \{1,2,3\}$ and $S^c = \{4,5\}$.

Now let $\mathbf{\Phi}_S^f \in (\mathbb{R}^+)^3$ be the result of projecting the rate tuple $\mathbf{\Phi}^f$ onto the $S = \{1,2,3\}$

coordinates, and let $\mathbf{\Phi}^f_{S^c} \in (\mathbb{R}^+)^2$ be the result of projecting $\mathbf{\Phi}^f$ onto the $S^c = \{4,5\}$ coordinates. Then, $\mathbf{\Phi}^f_S \in \mathcal{D}_{P_S}$ and $\mathbf{\Phi}^f_{S^c} \in \mathcal{D}_{P_{S^c|S}}$, where $\mathcal{D}_{P_S}$ (assume dim = 2) and $\mathcal{D}_{P_{S^c|S}}$ (assume dim = 1) are the dominant faces associated with the three-user and two-user DMMAC's P_S and $P_{S^c|S}$, as defined in equations (2.7)-(2.8), respectively. This suggests that $\mathbf{\Phi}^f$ can be achieved by decoding users X_S regarding the other users X_{S^c} as noise, then decoding X_{S^c} with the decoded and reconstructed versions of X_S available as side information at the decoder.

To achieve $\mathbf{\Phi}^f_S$ and $\mathbf{\Phi}^f_{S^c}$ in the lower-dimensional dominant faces, repeat the SEP process described above. For the one-dimensional $\mathcal{D}_{P_{S^c|S}}$, selecting either vertex leads to $\mathbf{\Phi}^f_{S^c} = (1-\beta)\mathbf{\Phi}^{(4\ 5)} + \beta\mathbf{\Phi}^{(5\ 4)}, 0 \leq \beta \leq 1$, where $\mathbf{\Phi}^{(4\ 5)} = (I(X_4;Y|X_1,X_2,X_3),\ I(X_5|X_1,X_2,X_3,X_4))$ and $\mathbf{\Phi}^{(5\ 4)} = (I(X_4;Y|X_1,X_2,X_3,X_5), I(X_5|X_1,X_2,X_3))$. For $\mathbf{\Phi}^f_S$ in the two-dimensional $\mathcal{D}_{P_S}$, select the ray-initiating vertex to be $\mathbf{\Phi}^{\pi_S}$ where $\pi_S = (132)$ is the restriction of the permutation $\pi = (41325)$ (corresponding to the initial vertex choice $\mathbf{\Phi}^\pi$) to the subset $S = \{1,2,3\}$. Note that π_S is a *sub-permutation* of π. Now suppose the ray $\overrightarrow{\mathbf{\Phi}^{\pi_S}\mathbf{\Phi}^f_S}$ hits face $\mathcal{F}_{S_1}$ of $\mathcal{D}_{P_S}$ at the point $\mathbf{\Phi}^{f_S}$, where $S_1 = \{1,2\} \subset S$ and $S \backslash S_1 = \{3\}$. Then $\mathbf{\Phi}^f_S = \alpha\mathbf{\Phi}^{\pi_S} + (1-\alpha)\mathbf{\Phi}^{f_S}$ for some $0 \leq \alpha \leq 1$. After projecting $\mathbf{\Phi}^{f_S}$ onto S_1 and $S\backslash S_1$, we have $\mathbf{\Phi}^{f_S}_{S_1} = (1-\xi)\mathbf{\Phi}^{(1\ 2)} + \xi\mathbf{\Phi}^{(2\ 1)}, 0 \leq \xi \leq 1$, where $\mathbf{\Phi}^{(1\ 2)} = (I(X_1;Y), I(X_2;Y|X_1))$ and $\mathbf{\Phi}^{(2\ 1)} = (I(X_1;Y|X_2), I(X_2;Y))$. Also, $\mathbf{\Phi}^{f_S}_{S\backslash S_1} = I(X_3;Y|X_1,X_2)$.

We have so far reduced the problem of achieving a rate tuple in a four-dimensional dominant face to separate but linked problems of achieving rate tuples in zero- and one-dimensional dominant faces. The zero-dimensional problem is of course, trivial. The projective time-sharing solution to the one-dimensional problem is given in Section 2.3.1. We now merge these zero-dimensional and one-dimensional solutions appropriately, keeping in mind the decoding order on subsets of users implied by the manner in which we performed the SEP process.

Section 2.3.1 gives a configuration requiring at most three single-user codes to achieve $\mathbf{\Phi}^{f_S}_{S_1}$, $S_1 = \{1,2\}$. Figure 2-6 shows this sub-configuration incorporated in the overall coding configuration. The decoding order on the codes of users 1 and 2 (in S_1) is $G_{S_1} = (11\ 21\ 12)$. Since the point $\mathbf{\Phi}^{f_S}$ lies in $\mathcal{F}_{S_1}$, it can be achieved by decoding the users X_{S_1} before user $X_{S\backslash S_1} = X_3$. Hence, the decoding order for $\mathbf{\Phi}^{f_S}$ is $G_{f_S} = (11\ 21\ 12\ 31)$. This is shown in Figure 2-6, where it is assumed that the block length used for $\mathbf{\Phi}^{f_S}$ is N_1. We proceed to build a configuration with the rate tuple $\mathbf{\Phi}^f_S = \alpha\mathbf{\Phi}^{\pi_S} + (1-\alpha)\mathbf{\Phi}^{f_S}$ that is, a configuration with rate $\mathbf{\Phi}^{f_S}$ for a fraction of $1-\alpha$ of the overall block length and rate $\mathbf{\Phi}^{\pi_S}$ for the other α fraction. Accomplish this by first lengthening the block length N_1 by a factor of $\frac{\alpha}{1-\alpha}$ to get $N_2 = \frac{1}{1-\alpha}N_1$ (assume $\alpha \in (0,1)$). Retain the configuration with rate $\mathbf{\Phi}^{f_S}$ over the subset $T^{f_S} \equiv \{1, \dots, N_1\}$. The rate $\mathbf{\Phi}^{\pi_S}$ is attainable over a subset $T^{\pi_S} \equiv \{N_1 + 1, \dots, N_2\}$ if the *user* decoding order over T^{π_S} is precisely $\pi_S = (132)$, a permutation on the set $\{1, 2, 3\}$. To this end, require that code indices 11 and 21 be used by users 1 and 2, respectively. Add a new single-user code, indexed 32, for user 3, and specify that it be used over T^{π_S}. Now insert the new code 32 into decoding order G_{f_S} such that the user decoding order is (132) over T^{π_S}. Choosing the decoding order $G_S = (11\ 32\ 21\ 12\ 31)$ for users in S accomplishes this.

Notice that we were able to add just one more code to attain the required user decoding order (permutation of the set $\{1, 2, 3\}$) over T^{π_S}. This is possible since we already had available to us both permutations of the set $\{1, 2\}$ (user decoding orders (1 2) and (2 1)) from the coding-configuration for rate pair $\mathbf{\Phi}_{S_1}^{f_S}$. Since we are able to insert the new code (32) anywhere in G_{f_S}, any order on users 1, 2, 3 is attainable over T^{π_S}. In general, given a split of $\{1, \dots, M\}$ into two subsets S and S^c (assume $|S| \geq |S^c|$), an arbitrary permutation π of users $\{1, \dots, M\}$ can be obtained by first constructing the sub-permutation π_S (on the elements in the larger subset S) of π, and then inserting code indices for the smaller subset S^c into the appropriate slots in the order imposed by π_S.

Continuing the construction of a configuration for $\mathbf{\Phi}$, we note that the familiar two-user solution can again be used to achieve $\mathbf{\Phi}_{S^c}^{f}$ ($S^c = \{4, 5\}$). This sub-configuration is shown in Figure 2-6, incorporated into the overall coding configuration. The decoding among users in S^c is $G_{S^c} = (41\ 51\ 42)$. Now since $\mathbf{\Phi}^f \in \mathcal{F}_S$, the users X_{S^c} are decoded after X_S. The block length used by all users for rate $\mathbf{\Phi}^f$ is assumed to be N_2. The decoding order for $\mathbf{\Phi}^f$ is $G_f = (11\ 32\ 21\ 12\ 31\ 41\ 51\ 42)$.

Finally, for the rate tuple $\mathbf{\Phi}$, we require a configuration with rate $\mathbf{\Phi}^f$ for a fraction $1-\lambda$ of the overall block length and rate $\mathbf{\Phi}^\pi$ for the other λ fraction of the block length. To accomplish this, lengthen the block length N_2 to N, where $N_2 = (1-\lambda)N$. Retain the configuration with rate $\mathbf{\Phi}^f$ over $T^f \equiv \{1, \dots, N_2\}$. The overall configuration has rate $\mathbf{\Phi}^\pi$ over $T^\pi \equiv \{N_2 + 1, \dots, N\}$ if the user decoding order over T^π is $\pi = (41325)$. As explained above, this is achieved by first obtaining the sub-permutation (132) corresponding to the larger subset S. We require that users 1, 2, and 3 employ the same code indices (11, 21, and 32) over T^{π} as they do over T^{π_S}. Then, the user decoding order over T^{π} among users 1,2,3 is precisely $\pi_S = (132)$. It is now necessary to allocate two new codes with indices 43 and 52, to be used by users 4 and 5 over T^{π}. The indices of the new codes are inserted into decoding order G_f such that the user decoding order over T^{π} is $\pi = (41325)$ as desired. Choosing $G = (43\ 11\ 32\ 21\ 52\ 12\ 31\ 41\ 51\ 42)$ accomplishes this. Notice that the total number of codes used is $10 = h(5)$.

It is easily verified that the configuration established above meets requirements 1-2 of Definition 2.7. By increasing the overall block length N and applying the arguments of Section 2.3.2, we see that any $\mathbf{\Phi} \in \mathcal{D}$ is achievable with a sequence of configurations $\{(N, C, \{\boldsymbol{t}_n\}, G)\}$ in the sense of Definition 2.7. □

From the above example, we see that the reduction in the number of codes needed for achieving $\mathbf{\Phi}$ comes from the interaction of the reduction and merging processes. If, during every stage of the SEP process, we split the larger of two subsets according to the sub-permutations of the original permutation π, then, during the merging stages, it is necessary to add only an extra $\min(|S|, |S^c|) = \min(k, M-k)$ new codes per stage. Thus, the total number of codes can never exceed $h(M)$. Another property evident from the example is that by assigning codes to channel subsets appropriately, it is always possible for any user to use a fixed code over a *continuous interval*. For instance, the code 11 is used by user 1 over the interval $\{\xi N_1 + 1, \ldots, N\}$.

We now turn to the detailed proof of Theorem 2.5.

PROOF. For $M = 1$, $\mathcal{D} = \mathcal{R} = I(Q; P)$, where $I(Q; P)$ is the average mutual information between the input distribution Q on the alphabet $\mathcal{X}$ and the DMC $P(y|x)$. Thus, the theorem holds trivially. For $M = 2$, the proof is given by the analysis in Section 2.3.1 and in Examples 2.1-2.4. That discussion shows that in the two user case, the time-sharing scheme presented in Section 2.3.1 is just an instance of projective time-sharing. Furthermore, the number of codes required is no more than $h(2) = 3$.

Now assume the theorem holds for any m-user DMMAC, for $1 \leq m \leq M$. Consider an $M+1$-user DMMAC $P(y|x_1, \ldots, x_{M+1})$. Let $\mathcal{D}$ be the dominant face of the rate region $\mathcal{R}(P, \prod_{i=1}^{M+1} Q_{X_i})$ corresponding to the product input distribution $\prod_{i=1}^{M+1} Q_{X_i}$ for this DMMAC. We must now consider two cases.

Case 1: $\dim(\mathcal{D}) < M$. By Lemmas 2.1 and 2.3, there exists some set $S, \emptyset \subset S \subset \{1, \ldots, M+1\}$ such that $\mathcal{D} = \mathcal{F}_S = \mathcal{D}_{P_S} \times \mathcal{D}_{P_{S^c}}$, where $\mathcal{D}_{P_S}$ and $\mathcal{D}_{P_{S^c}}$ are the dominant faces associated with *independent* DMMAC's P_S and P_{S^c}. That is, the $M+1$-user DMMAC decomposes into at least two independent sub-DMMAC's. Let $\mathbf{\Phi} \in \mathcal{D}$ be given. Since $\mathcal{D} = \mathcal{F}_S$, $\mathbf{\Phi} = \lambda \mathbf{\Phi}^\pi + (1-\lambda)\mathbf{\Phi}^f$ with $\mathbf{\Phi}^f \in \mathcal{F}_S$ holds trivially, with $\lambda = 0$.

Let $1 \leq |S| = k \leq M$. Let $\mathbf{\Phi}_S \in \mathcal{D}_{P_S}$ and $\mathbf{\Phi}_{S^c} \in \mathcal{D}_{P_{S^c}}$ be the projections of $\mathbf{\Phi}$ onto the S-coordinates and S^c-coordinates, respectively. By the inductive assumption, $\mathbf{\Phi}_S$ and $\mathbf{\Phi}_{S^c}$ are achievable by projective time-sharing using no more than $h(k)$ and $h(M+1-k)$ single-user codes, respectively. Therefore, $\mathbf{\Phi} = (\mathbf{\Phi}_S, \mathbf{\Phi}_{S^c})$ is achievable using no more than $h(k) + h(M+1-k) < h(M+1)$ single-user codes.

Case 2: $\dim(\mathcal{D}) = M$. Let $\mathbf{\Phi} \in \mathcal{D}$ be given. We build the configuration sequence to achieve $\mathbf{\Phi}$ by first reducing the dimension of the problem using a vertex *selection*, ray *extension*, and rate tuple *projection* (SEP) process, and then appropriately supplementing and merging the configurations for the lower-dimensional problems to form the overall configuration.

Vertex Selection, Ray Extension, Rate Tuple Projection. Given any rate tuple $\mathbf{\Phi} \in \mathcal{D}$, pick an arbitrary vertex $\mathbf{\Phi}^\pi$ of $\mathcal{D}$, associated with the permutation $\pi \in \Pi_{M+1}$ on the set $\{1, \ldots, M+1\}$, and extend a ray $\overrightarrow{\mathbf{\Phi}^\pi \mathbf{\Phi}}$ from $\mathbf{\Phi}^\pi$ through $\mathbf{\Phi}$. Since $\mathcal{D}$ is a convex polytope bounded as in (2.3), the ray must exit at some face of $\mathcal{D}$ corresponding to $\mathcal{F}_S$ of $\mathcal{D}$, as defined in Section 2.2, associated with a constraint set $\emptyset \subset S \subset \{1, \ldots, M+1\}, 1 \leq |S| = k \leq M$. Let $\mathbf{\Phi}^f = \overrightarrow{\mathbf{\Phi}^\pi \mathbf{\Phi}} \cap \mathcal{F}_S$. Then,

$$\mathbf{\Phi} = \lambda \mathbf{\Phi}^\pi + (1-\lambda)\mathbf{\Phi}^f \tag{2.23}$$

for some $0 \leq \lambda \leq 1$. The vertex selection and ray extension process is illustrated for a three-user case in Figure 2-7. Now let $\mathbf{\Phi}_S^f \in (\mathbb{R}^+)^k$ be the result of projecting $\mathbf{\Phi}^f$ onto the S coordinates, and let $\mathbf{\Phi}_{S^c}^f \in (\mathbb{R}^+)^{M+1-k}$ be the result of projecting $\mathbf{\Phi}^f$ onto the S^c coordinates. From Lemma 2.2, we have $\mathbf{\Phi}_S^f \in \mathcal{D}_{P_S}$ and $\mathbf{\Phi}_{S^c}^f \in \mathcal{D}_{P_{S^c|S}}$.

Let $\delta > 0$ be given. Let $\boldsymbol{R} \in (\mathbb{R}^+)^{M+1}$ be any rate tuple in $\mathcal{R}(P, \prod_{i=1}^{M+1} Q_{X_i})$ such that $R_i \leq \Phi_i - \delta, \forall i = 1, \ldots, M+1$. Choose rate tuple $\boldsymbol{R}^\pi \in (\mathbb{R}^+)^{M+1}$ with $R_i^\pi \leq \Phi_i^\pi - \delta, \forall i = 1, \ldots, M+1$, and choose $\boldsymbol{R}^f \in (\mathbb{R}^+)^{M+1}$ with $R_i^f \leq \Phi_i^f - \delta, \forall i = 1, \ldots, M+1$, so that $\boldsymbol{R} = \lambda \boldsymbol{R}^\pi + (1-\lambda)\boldsymbol{R}^f$. Let $\boldsymbol{R}_S^f \in (\mathbb{R}^+)^k$ be the result of projecting $\boldsymbol{R}^f$ onto the S coordinates,

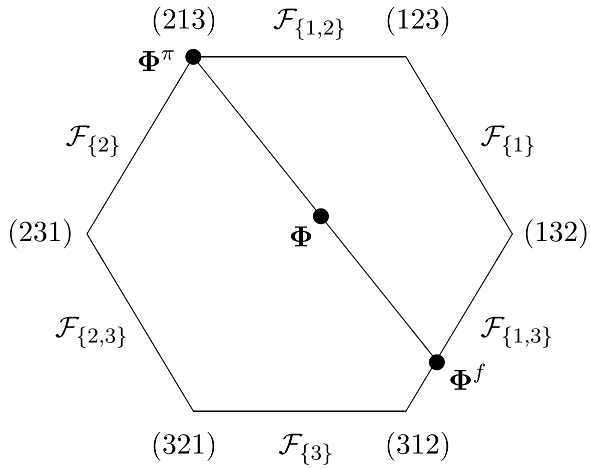

Figure 2-7: Illustration of the vertex selection and ray extension process for a three-user DMMAC. Here, the dominant face $\mathcal{D}$ is a hexagon (not necessarily regular). The vertices are labeled by the permutations associated with them. The initial vertex corresponds to the permutation (213). The ray exits $\mathcal{D}$ at the face $\mathcal{F}_{\{1,3\}}$.

and let $\boldsymbol{R}^f_{S^c} \in (\mathbb{R}^+)^{M+1-k}$ be the result of projecting $\boldsymbol{R}^f$ onto the S^c coordinates.

Next, let π_S and π_{S^c} be the restrictions of π to the coordinates in S and S^c, respectively. For instance, (41325) is a permutation of the set $\{1,2,3,4,5\}$. If $S = \{1,2,3\}$ and $S^c = \{4,5\}$, then $\pi_S = (132)$ and $\pi_{S^c} = (45)$. We call π_S and π_{S^c} *sub-permutations* of the permutation π.[4] The permutations π_S and π_{S^c} correspond to vertex $\mathbf{\Phi}^{\pi_S}$ of $\mathcal{D}_{P_S}$ and to vertex $\mathbf{\Phi}^{\pi_{S^c}}$ of $\mathcal{D}_{P_{S^c|S}}$, respectively.

Reduction to Lower Dimensions. Assume that $k = \max(k, M+1-k)$ (the other case is treated in the same way). By the inductive assumption (since $1 \le |S| = k \le M$), $\mathbf{\Phi}^f_S \in \mathcal{D}_{P_S}$ can be expressed as a convex combination of the vertex $\mathbf{\Phi}^{\pi_S}$ of $\mathcal{D}_{P_S}$ (corresponding to the sub-permutation π_S of π) and a point $\mathbf{\Phi}^{f_S}$ on some face of $\mathcal{D}_{P_S}$:

$$\mathbf{\Phi}^f_S = \alpha \mathbf{\Phi}^{\pi_S} + (1-\alpha)\mathbf{\Phi}^{f_S} \tag{2.24}$$

[4]Formally, assume without loss of generality that $S = \{s_1, s_2, \ldots, s_k : s_1 < s_2 < \ldots < s_k\}$ and $S^c = \{s^c_1, s^c_2, \ldots, s^c_{M+1-k} : s^c_1 < s^c_2 < \ldots < s^c_{M+1-k}\}$ are ordered sets. Let Π_S and Π_{S^c} be the permutation groups of the sets S and S^c, respectively. Given $\pi \in \Pi_{M+1}$, let $j_1 < j_2 < \ldots < j_k$ be the indices for which $\pi(j_i) \in S, i = 1, \ldots k$. Similarly, let $l_1 < l_2 < \ldots < l_{M+1-k}$ be the indices for which $\pi(l_i) \in S^c, i = 1, \ldots M+1-k$. Then let $\pi_S \in \Pi_S$ be such that $\pi_S(s_1) = \pi(j_1), \pi_S(s_2) = \pi(j_2), \ldots, \pi_S(s_k) = \pi(j_k), s_1 < \ldots < s_k$. Also, let $\pi_{S^c} \in \Pi_{S^c}$ be such that $\pi_{S^c}(s^c_1) = \pi(l_1), \pi_{S^c}(s^c_2) = \pi(l_2), \ldots, \pi_{S^c}(s^c_{M+1-k}) = \pi(l_{M+1-k}), s^c_1 < \ldots < s^c_{M+1-k}$.

for some $0 \le \alpha \le 1$. Moreover, $\mathbf{\Phi}^f_S$ is achievable using projective time-sharing with no more than $h(k)$ codes. Notice that the selection of $\mathbf{\Phi}^{\pi_S}$ is *not* arbitrary, but corresponds to the sub-permutation π_S of the π for the initially chosen vertex $\mathbf{\Phi}^{\pi}$. We shall explain later why this selection is crucial to our overall argument. Now by the definition of projective time-sharing, since $R^f_i \le \Phi^f_i - \delta, i \in S$, there exists a sequence of coding configurations $\{(N_f, C_S, \{\boldsymbol{t}^S_n\}, G_S)\}$ ($C_S = \{C_{ij} : ij \in E_S$, $E_S = \{ij : i \in S, 1 \le j \le J^S_i\}$, where J^S_i is the total number of codes used by ith user) with $|C_S| \le h(k)$ such that (*cf.* Definition 2.7)

(a) for every N_f, there exist disjoint subsets T^{f_S} and T^{π_S} with $|T^{f_S}|/N_f = 1 - \alpha$, $|T^{\pi_S}|/N_f = \alpha$, and $T^{f_S} \cup T^{\pi_S} = \{1, \dots, N_f\}$,

(b) for every N_f, the scheduling k-tuple $\boldsymbol{t}^S_n$ is such that $t^S_{in} = j^{\pi_S}_i$ for some fixed $j^{\pi_S}_i \in \{1, \dots, J^S_i\}$, $\forall n \in T^{\pi_S}, i \in S$, and the user decoding order over T^{π_S} is π_S. That is, $U_{T^{\pi_S}} = \pi_S$.

(c) for every N_f, $R^f_i = \sum_{j=1}^{J^S_i} \gamma_{ij} R^f_{ij}, ij \in E_S$, where γ_{ij} is the fraction of time that user i uses the ijth code over block length N_f.

(d) The average probabilities of decoding error $P^{N_f}_{e,ij}$ for the sequence of $(N_f\gamma_{ij}, R^f_{ij})$ codes $C^{N_f}_{ij}$, each a member of the $(N_f\gamma_{ij}, R^f_{ij}, Q_{X_i})$ code ensemble, tends to zero on the genie-aided fixed-fraction DVC sequence $\{\hat{P}^{N_f\gamma_{ij}}_{\boldsymbol{s}}\}$ as $N_f \to \infty$, for every $ij \in E_S$.

Again by the inductive assumption, since $|S^c| = M + 1 - k \le M$, $\mathbf{\Phi}^f_{S^c} \in \mathcal{D}_{P_{S^c|S}}$ can be expressed as a convex combination of an *arbitrary* vertex $\mathbf{\Phi}^{\sigma_{S^c}}$ ($\sigma_{S^c} \in \Pi_{S^c}$) of $\mathcal{D}_{P_{S^c|S}}$ and a point $\mathbf{\Phi}^{f_{S^c}}$ on a face of $\mathcal{D}_{P_{S^c|S}}$:

$$\mathbf{\Phi}^f_{S^c} = \beta\mathbf{\Phi}^{\sigma_{S^c}} + (1-\beta)\mathbf{\Phi}^{f_{S^c}} \tag{2.25}$$

for some $0 \le \beta \le 1$. $\mathbf{\Phi}^f_{S^c}$ is achievable using projective time-sharing with no more than $h(M+1-k)$ codes. Therefore, since $R^f_i \le \Phi^f_i - \delta, i \in S^c$, there exists a sequence of coding configurations $\{(N_f, C_{S^c}, \{\boldsymbol{t}^{S^c}_n\}, G_{S^c})\}$ ($C_{S^c} = \{C_{ij} : ij \in E_{S^c}$, $E_{S^c} = \{ij : i \in S^c, 1 \le j \le J^{S^c}_i\}$, where $J^{S^c}_i$ is the number of codes used by ith user) with $|C_{S^c}| \le h(M+1-k)$ such that (*cf.* Definition 2.7)

(a) for every N_f, there exist disjoint subsets $T^{f_{S^c}}$ and $T^{\sigma_{S^c}}$ with $|T^{f_{S^c}}|/N_f = 1 - \beta, |T^{\sigma_{S^c}}|/N_f = \beta$, and $T^{f_{S^c}} \cup T^{\sigma_{S^c}} = \{1, \dots, N_f\}$,

(b) for every N_f, the scheduling k-tuple $\boldsymbol{t}_n^{S^c}$ is such that $t_{in}^{S^c} = j_i^{\sigma_{S^c}}$ for some fixed $j_i^{\sigma_{S^c}} \in \{1, \ldots, J_i^{S^c}\}, \forall n \in T^{\sigma_{S^c}}, i \in S^c$, and the user decoding order over $T^{\sigma_{S^c}}$ is σ_{S^c}. That is, $U_{T^{\sigma_{S^c}}} = \sigma_{S^c}$.

(c) for every N_f, $R_i^f = \sum_{j=1}^{J_i^{S^c}} \gamma_{ij} R_{ij}^f, ij \in E_{S^c}$, where γ_{ij} is the fraction of time that user i uses the ijth code over block length N_f.

(d) The average probabilities of decoding error $P_{e,ij}^{N_f}$ for the sequence of $(N_f\gamma_{ij}, R_{ij}^f)$ codes $C_{ij}^{N_f}$, each a member of the $(N_f\gamma_{ij}, R_{ij}^f, Q_{X_i})$ code ensemble, tends to zero on the genie-aided fixed-fraction DVC sequence $\{\hat{P}_{\boldsymbol{s}}^{N_f\gamma_{ij}}\}$ as $N_f \to \infty$, for every $ij \in E_{S^c}$.

We re-emphasize that while the choice of the original vertex $\boldsymbol{\Phi}^{\pi}$ constrains the choice $\boldsymbol{\Phi}^{\pi_S}$, it does not constrain the choice of $\boldsymbol{\Phi}^{\sigma_{S^c}}$. That is, σ_{S^c} need *not* be a sub-permutation of π. This is due to the assumption that $k = |S| = \max(k, M+1-k)$. As we show later, such an arrangement leads to the reduction of complexity desired.

We have so far reduced the problem of achieving a rate tuple $\boldsymbol{\Phi}$ in an M-dimensional dominant face to separate but linked problems of achieving the rate tuples $\boldsymbol{\Phi}_S^f$ and $\boldsymbol{\Phi}_{S^c}^f$ in lower-dimensional dominant faces. We now supplement and merge these lower-dimensional solutions appropriately to form the solution for $\boldsymbol{\Phi}$. Refer to Figure 2-5 for an overall picture of the process.

Supplementing and Merging Configurations. We first merge the two coding configuration sequences $\{(N_f,\, C_S,\, \{\boldsymbol{t}_n^S\},\, G_S)\}$ (with rate tuple $\boldsymbol{R}_S^f$) and $\{(N_f,\, C_{S^c},\, \{\boldsymbol{t}_n^{S^c}\},\, G_{S^c})\}$ (with rate tuple $\boldsymbol{R}_{S^c}^f$) to create a sequence of configurations $\{(N_f,\, C_f,\, \{\boldsymbol{t}_n^f\},\, G_f)\}$ with rate tuple $\boldsymbol{R}^f = (\boldsymbol{R}_S^f, \boldsymbol{R}_{S^c}^f)$. For every fixed block length N_f, construct the configuration $(N_f,\, C_f,\, \{\boldsymbol{t}_n^f\},\, G_f)$ from $(N_f,\, C_S,\, \{\boldsymbol{t}_n^S\},\, B_S)$ and $(N_f,\, C_{S^c},\, \{\boldsymbol{t}_n^{S^c}\},\, G_{S^c})$ as follows. The set of codes C_f is the union of the two sets C_S and C_{S^c} : $C_f = C_S \cup C_{S^c} = \{C_{ij} : 1 \le i \le M, 1 \le j \le J_i^f\}$, where the total number of codes J_i^f of user i equals J_i^S for $i \in S$ and equals $J_i^{S^c}$ for $i \in S^c$. The decoding order G_f on the indices E_f is formed by appending the decoding order G_{S^c} to the end of the decoding order G_S, *i.e.*, $G_f = (G_S\ G_{S^c})$. Thus, codewords for $i \in S$ are first successively decoded according to G_S while treating all codeword sequences for $i \in S^c$ as noise. The S^c sequences are then successively decoded according to G_{S^c}, with the decoded codewords for users $i \in S$ available as side information at the active decoder. This is the essence of "group splitting." The scheduling $M+1$-tuple sequence $\{\boldsymbol{t}_n^f\}$ follows directly from the k-tuple sequence $\{\boldsymbol{t}_n^S\}$ and the $M+1-k$-tuple sequence $\{\boldsymbol{t}_n^{S^c}\}$: $\boldsymbol{t}_n^f = (\boldsymbol{t}_n^S, \boldsymbol{t}_n^{S^c})$. The above merging operation allows the configuration sequence $\{(N_f, C_f, \{\boldsymbol{t}_n^f\}, G_f)\}$ to have rate $\boldsymbol{R}_S^f$ for the users in S, and $\boldsymbol{R}_{S^c}^f$ for the users in S^c. Hence, the sequence has rate $\boldsymbol{R}^f = (\boldsymbol{R}_S^f, \boldsymbol{R}_{S^c}^f)$.

Before constructing a new sequence of coding configurations with the desired rate tuple $\boldsymbol{R} = \lambda\boldsymbol{R}^\pi + (1-\lambda)\boldsymbol{R}^f$, it is necessary to consider two degenerate cases. For if $\lambda = 0$, then $\boldsymbol{R} = \boldsymbol{R}^f$ and the configuration sequence $\{(N_f, C_f, \{\boldsymbol{t}_n^f\}, G_f)\}$ achieving $\boldsymbol{R}$ has $|C_f| = |C_S| + |C_{S^c}| \leq h(k) + h(M+1-k) < h(M+1)$ single-user codes, by the inductive assumption and Lemma 2.4. If $\lambda = 1$, the $\boldsymbol{R} = \boldsymbol{R}^\pi$ and the usual successive decoding argument can be applied to show that a mere $M+1$ codes are needed. We shall assume for the rest of proof that $0 < \lambda < 1$.

We build a new sequence of configurations $\{(N, C, \{\boldsymbol{t}_n)\}, G)\}$ with rate $\boldsymbol{R}$ by "extending" the configurations $\{(N_f, C_f, \{\boldsymbol{t}_n^f\}, G_f)\}$ and introducing enough new codes so that the resulting new configuration sequence satisfies all four conditions of Definition 2.7. For every fixed N_f; let $N_\pi = N_f(\frac{\lambda}{1-\lambda})$ and $N = N_f + N_\pi$, so that $N_f = (1-\lambda)N$ and $N_\pi = \lambda N$. Let T^f and T^π be distinct subsets of $T \equiv \{1, \ldots, N\}$ which form a partition of T, where $|T^f| = N_f = (1-\lambda)N$ and $|T^\pi| = N_\pi = \lambda N$. For instance, we can choose $T^f = \{1, \ldots, N_f\}$, and $T^\pi = \{N_f + 1, \ldots, N\}$ so that T^f and T^π are "intervals." Now "extend" the configuration $(N_f, C_f, \{\boldsymbol{t}_n^f\}, G_f)$ to T^π by defining the new scheduling $M+1$-tuple $\{\boldsymbol{t}_n\}_{n=1}^N$ as follows. Let $t_{in} = t_{in}^f, \forall n \in T^f, i = 1, \ldots, M+1$. That is, over T^f, the scheduling sequence $\{\boldsymbol{t}_n\}$ is the same as $\{\boldsymbol{t}_n^f\}$. For $n \in T^\pi$ and $i \in S$, let $t_{in} = j_i^{\pi_S} \in \{1, \ldots, J_i^f\}$. That is, users in S use the same code over T^π as they do over T^{π_S} in configuration $(N_f, C_S, \{\boldsymbol{t}_n^S\}, G_S)$ and $(N_f, C_f, \{\boldsymbol{t}_n^f\}, G_f)$. Hence, users in S have the same total number J_i of codes in the new configuration sequence as in the old one ($J_i = J_i^S, i \in S$). For $n \in T^\pi$ and $i \in S^c$, let $t_{in} = J_i$, where $J_i = J_i^f + 1$ index the new codes to be introduced for users in S^c. Thus, users in S^c each uses one more code in the new configuration sequence as in the old one.

Inserting New Code Indices. Let E and E_f be the index sets for the codes in C and C_f, as in Definition 2.5. Incorporating the new code indices into E_f, we have $E = E_f \cup \{iJ_i\}_{i \in S^c}$, where $J_i = J_i^f + 1, i \in S^c$. It remains to form the overall decoding order G (a permutation on E) by appropriately inserting the new indices $\{iJ_i\}_{i \in S^c}$ into G_f. The new indices must be inserted into G_f so that the resulting user decoding order over T^π is the permutation

π associated with the vertex $\mathbf{\Phi}^\pi$ chosen at the start of the proof. That is, we must insert each iJ_i into the list G_f such that when the set $\{it_{in} : n \in T^\pi, i = 1, \ldots, M\}$ is listed in the order of the resulting G and the second indices are removed, the resulting list is precisely π.

To illustrate this procedure, let $j_1 < \ldots < j_k$ be the indices for which $\pi(j_i) \in S, i = 1, \ldots, k$ (the ordered decoding positions of users in S), and $l_1 < \ldots < l_{M+1-k}$ be the indices for which $\pi(l_i) \in S^c, i = 1, \ldots, M+1-k$ (the ordered decoding positions of users in S^c). Insert $\pi(l_1)J_{\pi(l_1)}$ into G_f such that it follows all $\pi(j_i)k$ for all $j_i < l_1$, where $t_{\pi(j_i)n} = k, \forall n \in T^\pi$, and precedes $\pi(j_i)k$ for all $j_i > l_1, t_{\pi(j_i)n} = k, \forall n \in T^\pi$. The placement of $\pi(l_2)J_{\pi(l_2)}$ satisfies the same rules as those for $\pi(l_1)J_{\pi(l_1)}$ except for the additional requirement that it must come after $\pi(l_1)J_{\pi(l_1)}$ in G. The other insertions are accomplished in the same way, with $\pi(l_{M+1-k})J_{\pi(l_{M+1-k})}$ placed according to the requirements for $\pi(l_1)J_{\pi(l_1)}$, plus the additional requirement that it must also follow $\pi(l_1)J_{\pi(l_1)}, \pi(l_2)J_{\pi(l_2)}, \ldots, \pi(l_{M-k})J_{\pi(l_{M-k})}$ in G.

For example, consider the five-user DMMAC in Example 2.5. Here, $\pi = (41325)$ and $S = \{1, 2, 3\}$, $S^c = \{4, 5\}$. Thus, $j_1 = 2$, $j_2 = 3, j_3 = 4, l_1 = 1, l_2 = 5$. We have $G_f = (11\ 32\ 21\ 12\ 31\ 41\ 51\ 42)$. Let $t_{1n} = 1, t_{2n} = 1, t_{3n} = 2\ \forall n \in T^\pi$ and let $J_4 = 3, J_5 = 2$. Following the procedure above, we insert 43 into G_f such that it precedes 11, 32, and 21 in G. Next, insert 52 such that it comes after 11, 32, 21, and after 43. Thus, the resulting G is (43 11 32 21 52 12 31 41 51 42).

The main point of the insertion procedure is that in general, given a split of $\{1, \ldots, M+1\}$ into two subsets S and S^c (assume $k = |S| \geq |S^c| = M+1-k$), an arbitrary permutation π of users $\{1, \ldots, M+1\}$ can be obtained by first constructing the sub-permutation π_S (on the elements in the larger subset S) of π, and then inserting code indices for the smaller subset S^c into the appropriate slots in the order imposed by π_S. The new decoding order G which results from this insertion procedure guarantees that the user decoding order on T^π, U_{T^π}, is precisely $\pi \in \Pi_M$. Note that the insertion procedure can produce many decoding orders (on the codebooks) which meet the requirements presented above, and thus there are many possible resulting configurations which give the desired user decoding order on T^π. Any of these valid configurations will suffice for our purposes.

Constructing New Codes. The first two of the four conditions in Definition 2.7 have now been met. It is necessary now to construct a set of codes C for each N such that the sequence of codes $\{C_{ij}^N\}$ give an overall rate tuple of $\boldsymbol{R}$ and have exponentially decreasing error probabilities for their respective genie-aided fixed-fraction DVC's. To achieve this, we first fix N, we add $M+1-k$ new codes (to be specified) $C_{iJ_i}^N, i \in S^c$, to the set C_f, one for each user $i \in S^c$. The new codes will be used over T^π. We also replace k of the codes in C_f, one for each user $i \in S$, with new codes to be used over a subset of $\{1, \ldots, N\}$ which includes $T^{\pi_S} \cup T^\pi$. Thus, for a fixed N, the number of codes needed over all increases by $M+1-k = \min(k, M+1-k)$.

Let $i \in S^c$. Over T^π, user i sees a genie-aided DMC with an average mutual information of $I(X_i; Y|X_{B(\pi,i)})$, where $X_{B(\pi,i)}$ is the set of inputs decoded before X_i under π. Now since $R_i^\pi \leq \Phi_i^\pi - \delta, i \in S^c$, there exists a sequence of (N_π, R_i^π) block codes $C_{iJ_i}^N$, each a member of the $(N_\pi, R_i^\pi, Q_{X_i})$ code ensemble, such that the corresponding error probability sequence (over the genie-aided DMC) $P_{e,iJ_i}^N \to 0$ as $N \to \infty$, for each $i \in S^c$. Since the average rate for user $i \in S^c$ over T^f is R_i^f, its overall average rate over $T \equiv T^f \cup T^\pi$ becomes

$$\frac{N_f}{N} \cdot R_i^f + \frac{N_\pi}{N} \cdot R_i^\pi = (1-\lambda)R_i^f + \lambda R_i^\pi = R_i.$$

Now let $i \in S$. For notational simplicity, let $j^* \equiv j_i^{\pi_S}$ index the code used by user i over T^{π_S}. Let T_{ij*}^f be the subset of $\{1, \ldots, N_f\}$ (including, but necessarily equal to T^{π_S}) over which $t_{in}^f = j^*$. Let $P_{\boldsymbol{s}}^{N_f \gamma_{ij*}}$ be the genie-aided fixed-fraction DVC channel seen by X_i over T_{ij*}^f. By the inductive assumption, $\{C_{ij*}^{N_f}\}$ is a sequence of $(N_f \gamma_{ij*}, R_{ij*}^f)$ block codes, each a member of the $(N_f \gamma_{ij*}, R_{ij*}^f, Q_{X_i})$ code ensemble, for which the error probabilities (over the DVC sequence $\{P_{\boldsymbol{s}}^{N_f \gamma_{ij*}}\}$) $P_{e,ij*}^{N_f} \to 0$ as $N_f \to \infty$. By Theorem 2.4, we must have

$$R_{ij*}^f \leq \sum_{v=1}^{V_{ij*}} \theta_v I(X_i; Y|X_{B(U_v,i)}) \tag{2.26}$$

where $\theta_v = |\{n : P_{s_n}^{N_f \gamma_{ij*}} = P_v\}|/N_f \gamma_{ij*}$ is the fraction of channels in $\{P_{\boldsymbol{s}}^{N_f \gamma_{ij*}}\}$ of type $v, 1 \leq v \leq V_{ij}^*$, for every N_f. Let $T_{ij*} \equiv T_{ij*}^f \cup T^\pi$. For convenience, let $N_{ij}^f \equiv N_f \gamma_{ij}$, and $N_{ij*} = N_{ij*}^f + N_\pi$. Let $P_{\boldsymbol{s}}^{N_{ij*}}$ be the genie-aided DVC seen by X_i over T_{ij*}. The average mutual information over $P_{\boldsymbol{s}}^{N_{ij*}}$ is

$$\frac{1}{N_{ij*}} \sum_{n \in T_{ij*}} I(X_i; Y|X_{B(U_n,i)}) = \frac{N_{ij*}^f}{N_{ij*}} \sum_{v=1}^{V_{ij*}} \theta_v I(X_i; Y|X_{B(U_v,i)}) + \frac{N_\pi}{N_{ij*}} I(X_i; Y|X_{B(\pi,i)}) \tag{2.27}$$

Let $R_{ij*} \equiv \frac{N_{ij*}^f}{N_{ij*}} \cdot R_{ij*}^f + \frac{N_\pi}{N_{ij*}} \cdot R_i^\pi$. It follows from (2.26) and from the fact that $R_i^\pi \leq \Phi_i^\pi - \delta, i \in S$ that R_{ij*} is strictly less than the RHS of (2.27). Moreover, this is true for every N, since N_{ij*}^f and N_π are both fixed fractions of N for given λ. Therefore, by Theorem 2.3, there exists a sequence of (N_{ij*}, R_{ij*}) block codes $\{C_{ij*}^N\}$, each a member of the $(N_{ij*}, R_{ij*}, Q_{X_i})$ code ensemble, such that the corresponding error probabilities (over the genie-aided DVC sequence $\{P_{\boldsymbol{s}}^{N_{ij*}}\}$) $P_{e,ij*}^N \to 0$ as $N \to \infty$.

If user $i \in S$ replaces the code $C_{ij*}^{N_f}$ with the code C_{ij*}^N in the set of codes C, and uses C_{ij*}^N over T_{ij*} for each N, its average rate over the entire interval T is

$$
\begin{aligned}
\sum_{j \neq j*} \frac{N_{ij}^f}{N} R_{ij}^f + \frac{N_{ij*}}{N} R_{ij*} &= \sum_{j \neq j*} \frac{N_{ij}^f}{N} R_{ij}^f + \frac{N_{ij*}}{N} \left(\frac{N_{ij*}^f}{N_{ij*}} \cdot R_{ij*}^f + \frac{N_\pi}{N_{ij*}} \cdot R_i^\pi \right) && (2.28) \\
&= \sum_{j \neq j*} \frac{N_{ij}^f}{N} R_{ij}^f + \frac{N_{ij*}^f}{N} \cdot R_{ij*}^f + \frac{N_\pi}{N} \cdot R_i^\pi \\
&= \frac{N_f}{N} \sum_{j=1}^{J_i^f} \frac{N_{ij}^f}{N_f} R_{ij}^f + \frac{N_\pi}{N} R_i^\pi \\
&= (1-\lambda) R_i^f + \lambda R_i^\pi && (2.29) \\
&= R_i.
\end{aligned}
$$

Equation (2.28) follows from the definition of R_{ij*}. Equation (2.29) follows from $R_i^f = \sum_{j=1}^{J_i^f} \frac{N_{ij}^f}{N_f} \cdot R_{ij}^f$, $N_f = (1-\lambda)N$ and $N_\pi = \lambda N$.

For a fixed N, let the code set C_f' be the result of replacing the codes $C_{ij*}^{N_f}$ in B_f by the codes C_{ij*}^N. Let $C \equiv C_f' \cup \{C_{iJ_i}^N : i \in S^c\}$. Then, it has been shown that the sequence of configurations $\{(N, C, \{\boldsymbol{t}_n\}, G)\}$ has rate tuple $\boldsymbol{R}$ and that the associated set of code sequence have error probabilities exponentially decreasing in N over their respective genie-aided fixed-fraction DVC's. All four conditions in Definition 2.7 have now been met.

Total Number of Codes. For a fixed N, the number of single-user codes used to accomplish the above is

$$
\begin{aligned}
|C| &= |C_S| + |C_{S^c}| + M + 1 - k && (2.30) \\
&\leq h(k) + h(M+1-k) + \min(k, M+1-k) && (2.31) \\
&\leq h(M+1). && (2.32)
\end{aligned}
$$

Equation (2.30) follows from $C^f = C_S \cup C_{S^c}$ and the fact that $M+1-k$ new codes (one for each user in S^c) were added to C^f to form C. Equation (2.31) follows from the inductive assumption and the assumption that $M+1-k \leq k$. Equation (2.32) follows directly from Lemma 2.4.

The above arguments assume $|S| = k = \max(k, M+1-k)$. The analysis is similar if $|S^c| = M+1-k = \max(k, M+1-k)$. The main point of the proof is that the reduction in the number of codes needed for achieving $\mathbf{\Phi}$ comes from the interaction of the reduction and merging processes. If, during every stage of the SEP process, we split the larger of two subsets according to the sub-permutations of the original permutation π, then, during the merging stages, it is necessary to add only an extra $\min(|S|, |S^c|) = \min(k, M+1-k)$ new codes per stage. Thus, the total number of codes can never exceed $h(M+1)$. $\square$

2.4.2 Projective Time-Sharing and Group Splitting

We can interpret the projective time-sharing configuration in terms of a tree structure where the set of users $\{1, \ldots, M\}$ is split repeatedly at every tree node into two groups until the sizes of the subsets decrease to one or two. The process of splitting the set of users corresponds to the geometric process of projection described in the proof of Theorem 2.5.

At the root node, the set of users is $\{1, \ldots, M\}$. As in the proof of the theorem, the initial choice of a vertex $\mathbf{\Phi}^\pi$, corresponding to the permutation π on the set $\{1, \ldots, M\}$, determines the subsets S and S^c at the child nodes. If both subsets have size greater than two, they are further split into four smaller subsets. One of these two splits is determined by the sub-permutation of the original permutation π. In particular, if S is the larger subset of $\{1, \ldots, M\}$, then it must be split according to the sub-permutation π_S of π. The smaller set S^c, however, can be split using an arbitrary permutation $\pi_{S^c} \in \Pi_{S^c}$. This process continues until the tree hits the base cases involving sets of one or two users. These base cases then form the leaves of the tree. The tree structure corresponding to the group splitting process is illustrated in Figure 2-8 for the five-user channel considered in Example 2.5.

Since the splits at half of the nodes (those corresponding to the larger subsets) are determined by the sub-permutations of the original permutation π chosen at the root, we can view π as the "seed" which partially determines a particular split of the set $\{1, \ldots, M\}$ and its corresponding tree. Geometrically, a particular tree corresponds to a particular

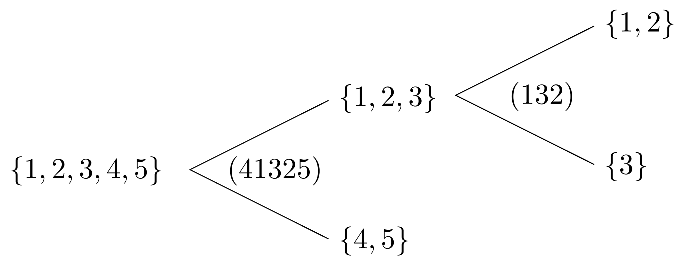

Figure 2-8: The tree structure for the group splitting process in the five-user case of Example 2.5. The bracketed quantity at each splitting node denotes the permutation corresponding to the vertex chosen at that split. Notice that the set $\{1, 2, 3\}$ is split using (132), a subpermutation of (41325).

series of projections of the desired rate tuple $\mathbf{\Phi}$ onto lower dimensions until the projections lie in some zero-dimensional or one-dimensional subset of the dominant face.

Given the splitting tree, the projective time-sharing configuration for achieving $\mathbf{\Phi}$ can be constructed backwards from the leaves of the tree. The time-sharing configurations for achieving a rate tuple in a one-dimensional subset of $\mathcal{D}$ is well-known from Section 2.3.1. The configuration for achieving the rate tuple corresponding to the node at the next higher level can be found by "merging" the configurations for each of the leaves and adding an additional $\min(|S_1|, |S_2|)$ (for two subsets S_1 and S_2 at the leaves) new codes, as described in the proof of the theorem. This process is continued until an overall configuration is obtained for the rate tuple $\mathbf{\Phi}$.

The relatively large number of possibilities ($M!$ for M users) for the initial choice of vertices and the freedom in choosing vertices corresponding to the smaller subset from any split implies that there are many possible splitting trees and different projective time-sharing configurations. For instance, it is possible that with judicious choices for vertices at the root and intermediate nodes, the number of actual single-user codes required to achieve a given rate in $\mathcal{D}$ is much smaller than the upper bound $\frac{1}{2}M \log_2 M + M$ given in Theorem 2.5. In fact, if it were possible to generate a tree in which at every splitting node, the subset corresponding to one of the two baby nodes is a singleton, then in the merging operation, only one new code is added at each splitting node, leading to a total of at most $2M - 1$ single-user codes. This performance is the same as that of [GRUW01, RU96, Rim99]. On top of that, one can verify that in the configuration corresponding to such a tree, no single user would employ more than two codes.

2.5 Summary and Discussion

The projective time-sharing coding schemes presented here are naturally derived from the underlying geometrical structure of $\mathcal{R}$ and $\mathcal{D}$ as well as results on parallel channels. We have addressed the two major concerns which undermine the use of time-sharing for achieving rates in the capacity region of the M-user MAC. First, global block synchronization should be achievable in the multiaccess setting using only mild feedback, and becomes altogether unnecessary if the sharing schemes described are implemented in frequency rather than in time. Second, the number of single-user codes needed is reduced from M^2 to no more than $\frac{1}{2}M \log_2 M + M$.

We conclude with a few observations. First, Theorem 2.5 merely gives an upper bound on the number of single-user codes needed to achieve a general point in $\mathcal{D}$. With a judicious choice of projective time-sharing configurations, the actual number of codes needed may be much smaller than $h(M)$. Improving the upper bound, however, seems to require a much deeper understanding of the geometry of convex polytopes. Second, the geometry of projective time-sharing is fundamentally different from that of rate splitting presented in [RU96]. In particular, the convex polytope of concern here has dimension $M - 1$, while the one for [RU96] has dimension $2M - 2$. Finally, we have seen that projective time-sharing may spawn many possible distributions of codes among the users, depending on the relative balance of the user splitting tree corresponding to the particular time-sharing scheme. This suggests that projective time-sharing is a flexible and potentially viable option for multiple access communications.

Chapter 3

Multiaccess Queue Control: An Inter-layer View

3.1 Introduction

In the previous chapter, we used information theoretic tools to analyze problems of multiaccess communication at the physical layer of the data network. We saw that this approach adequately modeled the noise and interference aspects of problem, and yielded significant insights with respect to coding for multiple users. As discussed in Chapter 1, however, information theoretic models do not allow for meaningful analysis of higher-layer QOS issues such as packet delay. The primary problem is that information theory ignores the random arrival of messages at the transmitter. While this approach may be reasonable for point-to-point channels, it is very problematic for multiaccess situations. This is clearly explained by Gallager [Gal85]:

> For a point to point channel, one normally assumes an infinite reservoir of data to be transmitted. The reason for this is that it is a minor practical detail to inform the receiver when there is no data to send: furthermore, there is no other use for the channel, so potential lack of data might as well be left out of the model. For multiaccess channels, on the other hand, most transmitters have nothing to send most of the time, and only a few are busy. The problem is then to share the channel between the busy users, and this is often the central technical problem in multiaccess communication.

Thus, the issues of random message arrivals and multiaccess contention are inherently not separable, and there is a clear need for an analytical approach that combines elements of queueing theory and information theory for multiaccess.

It was Telatar and Gallager [TG95] who took the lead in this effort. Their paper studies communication over continuous time bandlimited additive Gaussian multiaccess channels (MAC's) in which each transmitter has power P. Packets of varying length arrive according to a Poisson process at a given transmitter. The transmitter encodes a given packet into a time signal of infinite duration and transmits at power P until the service demand of the packet (involving the number of total possible messages and the desired error probability) is met by cumulative service units (given in terms of appropriate random coding error exponents), at which time the decoder instructs the transmitter to stop sending (via a feedback link). An infinite node assumption is used in the analysis, whereby each arriving packet is transmitted by a different (virtual) transmitter, so that there are no packet queues at the actual transmitters. It is also assumed that the decoder decodes each packet regarding all other transmissions as noise. The multiaccess system is analyzed as a reversible processor-sharing system where the total service rate depends on the state of the system through the number of transmitted packets competing for service. This permits the calculation of the steady-state number of packets in the system and the average packet transmission duration.

While [TG95] analyzes the delay and throughput performance of a particular multiaccess communication scheme, the study in [Tel95] seeks an optimal control policy to minimize the average delay of packets, with the control space being the information theoretic multiaccess capacity region. Again, a continuous-time Gaussian MAC is considered, where each transmitter has power P. It is assumed that each user has a *fixed* pool of bits to send (assumed to be present at time 0). By making an analogy between the additive Gaussian MAC and a "multi-processor queue" (in which multiple processors with respective service rates are used to accomplish multiple jobs with respective service requirements), Telatar [Tel95] shows that the optimal policy assigns rates in a greedy manner, where at each stage the highest possible rate is assigned to the packet with the smallest remaining service requirement (in untransmitted bits). It is noted in [Tel95], however, that this Shortest-Remaining-Service-Requirement-Highest-Rate (SRSR-HR) policy does not in general minimize the expected time jobs spend in the system when packets are not all present at the beginning, but arrive one by one, according to a Poisson process for instance (although SRSR-HR is optimal in the trivial case of one user [GM80]). Notice it is assumed here that the controller assigning rates knows the actual remaining service requirement of each packet in the system.

We shall take the viewpoint in [Tel95] and seek to minimize the average delay of packets with an appropriate rate allocation policy where the control space is given by multiaccess information theory. Unlike [TG95], we allow for queueing of packets at the transmitters, and unlike [Tel95], we do not assume that all packets are present at time 0 but that they arrive one by one, according to Poisson processes.

We focus on the M-user additive Gaussian noise channel with noise density $N_0/2$ and two-sided bandwidth $2W$.[1] We assume that all transmitters must transmit at power P whenever they are active.[2] The M data sources generate packets according to independent Poisson processes with a common rate λ. The lengths of the packets generated by all sources are assumed to be i.i.d. according to some distribution function $F_Z(z)$ satisfying $\mathbb{E}[Z] < \infty$, independent of the arrival processes. We are implicitly making the approximation here (for analytical convenience) that the number of bits can be taken as a real number. Notice that in case of long packets, this approximation is fairly reasonable. Next, assume that each source i, $i = 1, \dots, M$, has its own buffer (buffer i) into which its packets arrive. For the purposes of analyzing packet delay, we assume that these buffers have infinite capacity. Packets for the ith source are stored in the ith buffer until they are served by transmitter i, whose transmission rate at time $t \geq 0$ is $r_i(t)$, given in bits per second. It is required that at any time $t \geq 0$, the transmission rates assigned to queues 1 to M as a vector $\boldsymbol{r}(t) \equiv (r_1(t), \dots, r_M(t))$ must belong to the information theoretic Gaussian MAC capacity region region $\mathcal{C}$ [CT91], where $\mathcal{C}$ is the set of $\boldsymbol{r} \in \mathbb{R}^M$ such that $r_i \geq 0, \forall i$ and

$$\sum_{i \in S} r_i \leq W \log \left(1 + \frac{|S|P}{N_0 W} \right), \quad \forall S \subseteq \{1, \dots, M\}. \tag{3.1}$$

Thus, we assume in our model that the bursty bit streams of the users may be transmitted, for each $t \geq 0$, at any rate tuple from the multiaccess capacity region $\mathcal{C}$.[3]

[1] There is the technical issue here that strictly band-limited signals cannot be time-limited. In practice, of course, transmitted signals are approximately limited in both time and frequency. Since our main concern in this work is with queueing delay and not with decoding delay, this is not a crucial problem.

[2] The general control problem of joint power and rate allocation as a function of queue state subject to an *average* power constraint is much more complex, and we do not treat it here.

[3] We are implicitly assuming that the transmission rate can be changed instantaneously. Again, for long packets, this is a reasonable assumption.

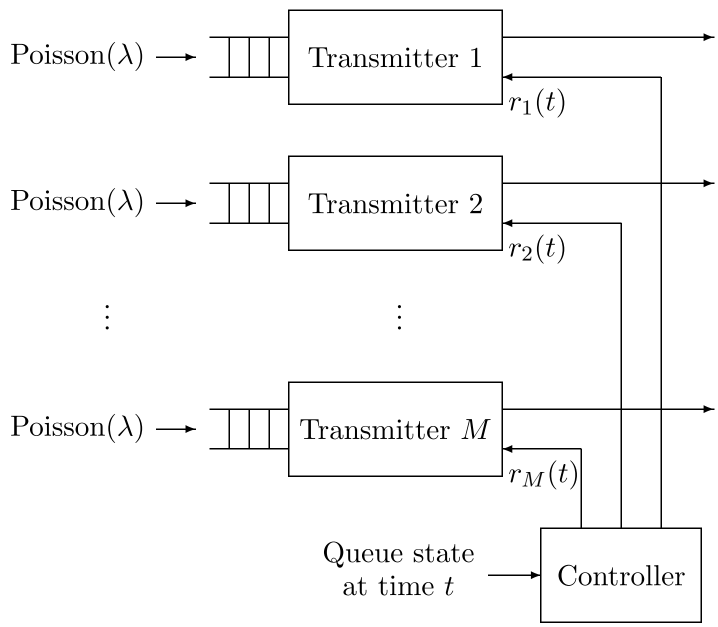

Figure 3-1: Illustration of the multiaccess queue control problem. At any time $t \geq 0$, the allocated rate tuple $\boldsymbol{r}(t)$ must belong to the capacity region $\mathcal{C}$.

For each i, $i = 1, \ldots, M$, we refer to the combination of buffer i and transmitter i as queue i. Our goal is to allocate rates from the capacity region $\mathcal{C}$ to the transmitters as a function of the joint state of the queues, so as to optimize two QOS criteria: average packet delay and average bit delay.[4] With this formulation, we turn the multiaccess communication problem into a queue control problem where the control space is given by multiaccess information theory. The problem setup is illustrated in Figure 3-1.

We study two versions of the delay minimization problem. The first version, analyzed in Section 3.2, assumes that the arrival processes from all sources are independent Poisson with a common parameter λ, and the lengths of packets from all sources have i.i.d. *exponential* distributions with common parameter μ, independent of the arrival processes. Here, let $X_i(t)$ be the *number of packets* in queue i at time t. We refer to $\boldsymbol{X}(t) \equiv (X_1(t), \ldots, X_M(t))$ as the vector of queue lengths at time t, or alternatively as the joint *queue state* at time t (since packets are exponentially distributed). Now consider a controller for which at each time $t \geq 0$, the input is $\boldsymbol{X}(t)$ and the output is a set of rate allocations $r_i(t), i = 1, \ldots, M$, to transmitters 1 to M. We assume here that even though the controller knows $\boldsymbol{X}(t)$, it does

[4]Here, we are analyzing queueing delay. Propagation delay, decoding delay, and other processing delays are not considered.

not know the actual lengths of the packets which have entered queues 1 to M by time t.[5] We wish to design the controller to minimize a certain cost function in terms of $\boldsymbol{X}(t)$. More specifically, let $\mathcal{X} = \mathbb{Z}_+^M$ be the space of $\boldsymbol{X}(t)$ and consider the space of admissible rate allocation policies $G \equiv \{g : \mathcal{X} \mapsto \mathcal{C}\}$ where for $g \in G$ and $t \geq 0$, $g(\boldsymbol{X}(t)) = \boldsymbol{r}(t)$. We would like to find a policy $g^* \in G$ to minimize

$$\lim_{t\to\infty} \mathbb{E}\left[\sum_{i=1}^{M} X_i(t)\right] \tag{3.2}$$

This is equivalent (by Little's Law) to minimizing the average system delay of packets in steady state. In Section 3.2, it is shown that a policy giving Longer Queues Higher Rates (LQHR) minimizes a class of cost functions of which $\mathbb{E}\left[\sum_{i=1}^{M} X_i(t)\right]$ is a specific example, for all $t \geq 0$. The idealizing assumptions made here lead to a simple and appealing proof of optimality based on the concepts of majorization and stochastic coupling.

In the second version of the delay minimization problem, we continue to consider independent Poisson arrival processes with a common parameter λ. However, we now allow the lengths of the packets from all sources to be i.i.d. according to some distribution function $F_Z(z)$ satisfying $\mathbb{E}[Z] < \infty$, independent of the arrival processes. In this case, $\boldsymbol{X}(t) \equiv (X_1(t), \ldots, X_M(t))$ no longer constitutes the queue state at time t. Instead, we focus on the notion of *unfinished work*. For each $i, i = 1, \ldots, M$, let $U_i(t)$ denote the amount of unfinished work (the total number of untransmitted bits) in queue i at time t. Then the vector of unfinished work $\boldsymbol{U}(t) \equiv (U_1(t), \ldots, U_M(t))$ constitutes a joint queue state at time t. Now consider a controller which at time $t \geq 0$ takes as input $\boldsymbol{U}(t)$ and outputs rate allocations $r_i(t), i = 1, \ldots, M$, to transmitters 1 to M, where $\boldsymbol{r}(t) = (r_1(t), \ldots, r_M(t))$ must belong to the capacity region $\mathcal{C}$. Here, we are implicitly assuming that the controller knows the actual lengths of all packets arriving into the system.[6] Let $\mathcal{U} = \mathbb{R}_+^M$ be the space of $\boldsymbol{U}(t)$ and consider the space of admissible stationary rate allocation policies $H \equiv \{h : \mathcal{U} \mapsto \mathcal{C}\}$ where for $h \in H$ and $t \geq 0$, $h(\boldsymbol{U}(t)) = \boldsymbol{r}(t)$. We wish to find a policy $h^* \in H$ to minimize

$$\lim_{t\to\infty} \mathbb{E}\left[\sum_{i=1}^{M} U_i(t)\right]. \tag{3.3}$$

[5]This may be caused by a communication bandwidth constraint between the controller and the respective queues. Later, in Section 3.3, we look at the case where the packet lengths are known.

[6]This would require more communication bandwidth between the controller and the respective queues.

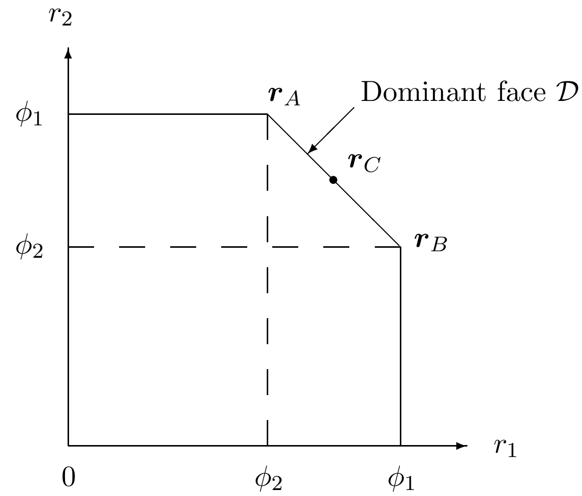

Figure 3-2: The two-user Gaussian multiaccess capacity region $\mathcal{C}$. The extreme point $\boldsymbol{r}_A = (\phi_2, \phi_1)$ corresponds to decoding user 1 first, and then user 2. The extreme point $\boldsymbol{r}_B = (\phi_1, \phi_2)$ corresponds to decoding user 2 first, and then user 1. The rate tuple $\boldsymbol{r}_C = ((\phi_1 + \phi_2)/2, (\phi_1 + \phi_2)/2)$.

This is equivalent (by Little's Law) to minimizing the average system delay of bits in steady state. In Section 3.3, we show using dynamic programming techniques that the LQHR policy, suitably modified, minimizes (3.3) for the *two-user* case.

3.2 Average Delay of Exponential Packets

We first examine the version of the delay minimization problem where the arrival processes are independent Poisson with parameter λ, and packets from all sources have i.i.d. exponential distributions with common parameter μ. As outlined in the introduction, we wish to find a policy $g^* \in G \equiv \{g : \mathcal{X} \mapsto \mathcal{C}\}$ to minimize (3.2).

We begin our quest for the optimal queue control policy by examining the feasible set $\mathcal{C}$. Notice that (3.1) has the same form as (2.1). Thus by the discussion in Chapter 2, the continuous-time Gaussian MAC capacity region is a polymatroid defined by $2^M - 1$ linear inequalities given in (3.1), as well as M non-negativity constraints. Figure 3-2 illustrates $\mathcal{C}$ for the two-user case. As in Chapter 2, it follows from the polymatroidal property [HW94, TH98] that for any $\boldsymbol{r} \in \mathcal{C}$, there exists some $\boldsymbol{r}' \in \mathcal{D} \subset \mathcal{C}$ such that

$r_i \leq r'_i, \forall i = 1, \ldots, M$, where

$$\mathcal{D} = \left\{ \boldsymbol{r} \in \mathcal{C} : \sum_{i=1}^{M} r_i = W \log \left(1 + \frac{MP}{N_0 W} \right) \right\}.$$

is the dominant face of $\mathcal{C}$.[7] From these observations, it should be clear that any policy $g^* \in G$ minimizing (3.2) must allocate rates in $\mathcal{D}$ at all times. Thus, we can restrict our attention to the set of policies $G_{\mathcal{D}} \equiv \{g : \mathcal{X} \mapsto \mathcal{D}\}$ in our optimization.[8]

At this point, we can start to appreciate the peculiarities of the queue control problem at hand. Here, unlike traditional situations, it is not possible to define a clear notion of work conservation. When all M queues are non-empty, allocating any rate in $\mathcal{D}$ causes the total amount of unfinished work (total number of untransmitted bits) to decrease at the maximum sum transmission rate $W \log \left(1 + \frac{MP}{N_0 W} \right)$. When any queue becomes empty, however, the upper bounds in (3.1) corresponding to subsets S other than $\{1, \ldots, M\}$ imply that the total unfinished work can no longer decrease at the maximum sum rate, even if the control policy continues to operate on $\mathcal{D}$. In some ways, the problem here bears resemblance to that considered in [TE93] concerning dynamic server allocation to parallel queues with randomly varying connectivity.

It should be clear that in order to minimize (3.2), we should attempt to maximize the rate at which the overall unfinished work is processed at any given time. On the other hand, we have seen that due to the nature of the region $\mathcal{C}$, the rate at which the system processes overall unfinished work decreases whenever some queue goes empty. This suggests that the optimal policy should somehow minimize the probability that any queue becomes empty, while simultaneously maximizing the rate at which work is done. One way of accomplishing this is to adopt an "equalizing" or "load-balancing" approach to queue control. A load-balancing policy tries to keep the queue lengths distributed as evenly as possible, thus making it unlikely that any queue empties. In the following, we show that a control policy following a *Longer Queue Higher Rate* (LQHR) strategy minimizes (within the set of policies $G_{\mathcal{D}}$) a class of cost functions of which $\mathbb{E}\left[\sum_{i=1}^{M} X_i(t)\right]$ is a specific example, for all $t \geq 0$. The

[7]It can be shown using arguments from Chapter 2 that for the continuous-time Gaussian MAC, $\dim(\mathcal{D}) = M - 1$, and $\dim(\mathcal{F}_S) = M - 2$ ($M \geq 2$), for any $\emptyset \subset S \subset \{1, \ldots, M\}$.

[8]Notice that even though $g \in G_{\mathcal{D}}$ allocates rates in $G_{\mathcal{D}}$ at all times, the departure rates from the queues need not lie in $G_{\mathcal{D}}$. This is due to the simple fact that there are no departures from an empty queue even when the service rate is positive.

LQHR policy implements the load-balancing strategy by assigning higher rates to longer queues, which are less likely to empty out than shorter queues.

Consider the *decreasing* sequence of numbers[9]

$$\phi_i = W \log \left(1 + \frac{P}{(i-1)P + N_0 W}\right) \quad i = 1, \dots, M. \tag{3.4}$$

It can be verified that the vector $\boldsymbol{\phi} = (\phi_1, \dots, \phi_M)$ is an extreme point of the dominant face $\mathcal{D}$. In fact, $\{\boldsymbol{\phi}\mathsf{P} | \mathsf{P}$ a permutation matrix$\}$ is the set of all extreme points of $\mathcal{D}$. Moreover, for each $i = 1, \dots, M$,

$$\phi_i = \max_{\mathcal{D}_i} r_i$$

where $\mathcal{D}_i = \{\boldsymbol{r} \in \mathcal{D} | r_j = \phi_j, \forall 1 \leq j < i\}$. That is, ϕ_i is the maximum rate in $\mathcal{D}$ that can be feasibly allocated to transmitter i, given that ϕ_j has been allocated to transmitter j, for all $1 \leq j < i$. In other words, ϕ_i is the rate assigned to user i in the ith stage of a greedy rate allocation procedure. This follows directly from the polymatroidal property of $\mathcal{C}$. Finally, from (3.1), observe that for any $\boldsymbol{r} \in \mathcal{D}$ and any subset of k indices $\{i_1, \dots, i_k\} \subseteq \{1, \dots, M\}$, where $1 \leq k \leq M$,

$$\sum_{j=1}^{k} r_{i_j} \leq W \log \left(1 + \frac{kP}{N_0 W}\right) = \sum_{i=1}^{k} \phi_i \tag{3.5}$$

with equality for $k = M$.

Next, consider the stationary policy g_{LQHR} which at any time $t \geq 0$ and for each $i = 1, \dots, M$, assigns the rate ϕ_i to the transmitter for the ith largest component of $\boldsymbol{X}(t) \equiv (X_1(t), \dots, X_M(t))$, where $\boldsymbol{X}(t)$ is the queue length vector as a function of time under policy g_{LQHR}.[10] Given the above discussion of the ϕ_i's, it is clear that for any $t \geq 0$, $g_{LQHR}(\boldsymbol{X}(t)) \in \mathcal{D}$ and thus $g_{LQHR} \in G_{\mathcal{D}}$. At each time t, g_{LQHR} assigns rates from $\mathcal{D}$ in a greedy way such that *longer queues receive higher rates*. Moreover, from (3.5), we see that at any time t, the sum rate assigned to the k largest components of the queue length vector under g_{LQHR} is at least as large as the sum rate assigned to the k largest components of

[9]The same numbers were considered in [Tel95] where they correspond to the service rates of the processors in decreasing order.

[10]If the lengths of queue i and queue j are equal and $i < j$, let g_{LQHR} assign the higher rate to queue i and the lower rate to queue j.

the queue length vector under any other policy $g \in G_{\mathcal{D}}$, for all $k = 1, \ldots, M$, with equality for $k = M$. This property turns out to be the linchpin of our optimality argument below.

Let us now give a coding interpretation to the LQHR policy. Notice that in the coding context, $\phi_i, 1 \leq i \leq M$, is simply the maximum rate that can be assigned to a user which is decoded in the $(M-i+1)$th place in a successive decoding scheme. Thus, at any time $t \geq 0$, the LQHR policy in effect decodes all M users successively, with the ith largest component of $\boldsymbol{X}(t)$ being decoded in the $(M-i+1)$th place. Since the order of decoding depends on $\boldsymbol{X}(t)$, LQHR is implementing *adaptive* successive decoding. For instance, for $M = 2$, the policy assigns the extreme point $\boldsymbol{r}_A = (\phi_2, \phi_1)$ (see Figure 3-2) whenever $X_1(t) < X_2(t)$ and assigns extreme point $\boldsymbol{r}_B = (\phi_1, \phi_2)$ whenever $X_1(t) \geq X_2(t)$. The LQHR policy therefore always operates on the extreme points of the dominant face. At time t, it chooses the extreme point based on the queue state $\boldsymbol{X}(t)$.

3.2.1 Stochastic Weak Majorization

We now proceed to prove the optimality of the LQHR policy in terms of (3.2). Central to our argument is a notion of ordering on vectors in $\mathbb{R}^M$. First, a bit of notation. For any $\boldsymbol{x} = (x_1, \ldots, x_M) \in \mathbb{R}^M$, let

$$x_{[1]} \geq \cdots \geq x_{[M]}$$

denote the components of $\boldsymbol{x}$ in decreasing order.

Definition 3.1 *For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^M$,*

$$\boldsymbol{x} \prec_w \boldsymbol{y} \quad \text{if} \;\; \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \; k = 1, \ldots, M. \tag{3.6}$$

The vector $\boldsymbol{x}$ is said to be weakly majorized *by $\boldsymbol{y}$. If, in addition, equality obtains in (3.6) for $k = M$, $\boldsymbol{x}$ is said to be* majorized *by $\boldsymbol{y}$, written $\boldsymbol{x} \prec \boldsymbol{y}$.*

That is, $\boldsymbol{x} \prec_w \boldsymbol{y}$ if the sum of the k largest components of $\boldsymbol{x}$ is less than or equal to the sum of the k largest components of $\boldsymbol{y}$, for every $k = 1, \ldots, M$. Vector $\boldsymbol{x}$ is majorized by $\boldsymbol{y}$ if, in addition, the sum of all M components is the same for both $\boldsymbol{x}$ and $\boldsymbol{y}$. In this case, $\boldsymbol{x}$ can be thought of as a more "evened-out" or "equalized" version of $\boldsymbol{y}$. For a thorough treatment of majorization, see [MO79].

We shall prove that property (3.5) leads to the result that the queue length vector under g_{LQHR} is *weakly majorized* by the queue length vector under any policy $g \in G_{\mathcal{D}}$, but only in a *stochastic* sense. Indeed, in view of the cost criterion in (3.2), the optimal policy need not be better than all other policies in a sample path sense, but only in a stochastic sense. This observation motivates the following definition.

Definition 3.2 *Let $\boldsymbol{X} = (X_1, \ldots, X_M)$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_M)$ be random vectors taking values in $\mathbb{R}^M$. $\boldsymbol{X}$ is said to be* stochastically weak-majorized *by $\boldsymbol{Y}$, written $\boldsymbol{X} \prec_w^{st} \boldsymbol{Y}$, if there exist random vectors $\tilde{\boldsymbol{X}}$ and $\tilde{\boldsymbol{Y}}$ taking values in $\mathbb{R}^M$ such that*

(a) *$\boldsymbol{X}$ and $\tilde{\boldsymbol{X}}$ are identically distributed.*

(b) *$\boldsymbol{Y}$ and $\tilde{\boldsymbol{Y}}$ are identically distributed.*

(c) *$\tilde{\boldsymbol{X}} \prec_w \tilde{\boldsymbol{Y}}$ a.s.*

Notice in Definition 3.2 that $\boldsymbol{X}(\boldsymbol{Y})$ and $\tilde{\boldsymbol{X}}(\tilde{\boldsymbol{Y}})$ have the same marginal distributions, but the joint distribution of $(\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{Y}})$ is in general different from that of $(\boldsymbol{X}, \boldsymbol{Y})$. This is the main point of the stochastic coupling method, which we will use for the proof of Theorem 3.1 below.

3.2.2 LQHR Minimizes Average Packet Delay

The following key result relies on the notion of stochastic weak majorization.

Theorem 3.1 *Let $\boldsymbol{x}_0$ be the vector of queue lengths in the system at time 0. Let $\boldsymbol{X}(t)$ be the vector of queue lengths under g_{LQHR} at time $t \geq 0$. Let $\boldsymbol{Y}(t)$ be the corresponding quantity under any policy $g \in G_{\mathcal{D}}$. Then*

$$\boldsymbol{X}(t) \prec_w^{st} \boldsymbol{Y}(t) \ \ \forall t \geq 0. \tag{3.7}$$

PROOF. We use a technique similar to that in [Wal88]. Define $r_{sum} \equiv W \log\left(1 + \frac{MP}{N_0 W}\right)$. Observe that the arrival times (over all queues) occur at jumps of a Poisson process with rate $M\lambda$. Now consider the set of *potential* service completion times. The potential service completion times are simply the actual service completion times when the queue is saturated (there is always a packet in the queue to be served). For a queue that is not always full, the potential service completion times are the times when departures would occur *if there were packets in the system at those times*. It is not hard to see that the potential service completion times occur at jumps of a Poisson process with rate μr_{sum} under both g and g_{LQHR}, since the two policies both belong to $G_{\mathcal{D}}$.

Consider the set of sample paths where $0 < t_1 < t_2 < \cdots < t_{n-1} < t_n < \cdots$ are the values of the arrival times and potential service completion times. These are the times at which the joint queue state can possibly change. Now the relation (3.7) holds at $t = 0$ since by assumption, $\boldsymbol{X}(0) = \boldsymbol{Y}(0) = \boldsymbol{x}_0$. Assume that (3.7) holds at some time t, where $t_{n-1} \leq t < t_n$ for some $n \geq 1$. It then suffices to show that $\boldsymbol{X}(t_n^+) \prec_w^{st} \boldsymbol{Y}(t_n^+)$.

Let $\tilde{\boldsymbol{X}}(t) = \boldsymbol{X}(t)$ and define $\tilde{\boldsymbol{Y}}(t)$ such that $\tilde{\boldsymbol{Y}}(t)$ and $\boldsymbol{Y}(t)$ are identically distributed. First, suppose t_n is an arrival time. Since the arrival processes are independent Poisson with the same rate λ, the probability that the arrival at t_n occurs on the jth longest queue of $\boldsymbol{X}(t)$, $1 \leq j \leq M$, is $1/M$. The same is true for $\boldsymbol{Y}(t)$. Let $\tilde{\boldsymbol{X}}(t_n^+) = \boldsymbol{X}(t_n^+)$ and define $\tilde{\boldsymbol{Y}}(t_n^+)$ by deciding that if the arrival occurs on the jth longest queue of $\tilde{\boldsymbol{X}}(t) = \boldsymbol{X}(t)$, then the same holds for $\tilde{\boldsymbol{Y}}(t)$. We have changed the joint distribution but not the marginals, so that the first two conditions in Definition 3.2 hold, with $\tilde{\boldsymbol{X}} = \tilde{\boldsymbol{X}}(t_n^+) = \boldsymbol{X}(t_n^+)$ and $\tilde{\boldsymbol{Y}} = \tilde{\boldsymbol{Y}}(t_n^+)$. So in the case of an arrival on the jth largest queue, $1 \leq j \leq M$,

$$\begin{aligned} \sum_{i=1}^{k} \tilde{X}_{[i]}(t_n^+) &= \left(\sum_{i=1}^{k} \tilde{X}_{[i]}(t) \right) + 1\{k \geq j\} \\ &\leq \left(\sum_{i=1}^{k} \tilde{Y}_{[i]}(t) \right) + 1\{k \geq j\} \\ &= \sum_{i=1}^{k} \tilde{Y}_{[i]}(t_n^+). \end{aligned}$$

for every $k = 1, \ldots, M$, Thus, (3.7) holds at time t_n^+.

Next suppose t_n is a potential service completion time. Let $\boldsymbol{r} \equiv (r_1, \ldots, r_M) \in \mathcal{D}$ be the rate vector assigned by policy g to the queues at time t. Let $r_{(i)}$ be the rate assigned to the ith longest queue of $\boldsymbol{Y}(t)$, $i = 1, \ldots, M$ ($r_{(i)}$ is *not* the same as $r_{[i]}$, which is the ith largest component of $(r_1, \ldots, r_M)$). Under policy g_{LQHR}, the ith longest queue of $\boldsymbol{X}(t)$

receives rate ϕ_i. By (3.5),

$$\sum_{i=1}^{j} r_{(i)} \leq \sum_{i=1}^{j} \phi_i \quad j = 1, \ldots, M, \text{ with equality for } j = M. \tag{3.8}$$

That is, $\boldsymbol{r} \prec \boldsymbol{\phi}$. Now since the process of potential service completions is Poisson with parameter μr_{sum} under both g and g_{LQHR}, the probability that the potential service completion at t_n occurs among the j longest queues of $\boldsymbol{Y}(t)$ is $\sum_{i=1}^{j} r_{(i)}/r_{sum}$, for each $j = 1, \ldots, M$. The corresponding probability for $\boldsymbol{X}(t)$ is $\sum_{i=1}^{j} \phi_i/r_{sum}$. Therefore, from (3.8), the probability that the potential service completion at t_n occurs among the j longest queues of $\boldsymbol{Y}(t)$ is less than or equal to the probability of the corresponding event for $\boldsymbol{X}(t)$, for each $j = 1, \ldots, M$, with equality for $j = M$. Then, let $\tilde{\boldsymbol{X}}(t_n^+) = \boldsymbol{X}(t_n^+)$ and define $\tilde{\boldsymbol{Y}}(t_n^+)$ by deciding that if the potential service completion occurs among the j longest queues of $\tilde{\boldsymbol{Y}}(t)$, then the same is true for $\tilde{\boldsymbol{X}}(t)$, for each $j = 1, \ldots, M$. This can be done precisely as follows. Let Q be a uniformly distributed random variable over $[0, 1]$. For each $j = 1, \ldots, M$, let $Q \in \left[\sum_{i=1}^{j-1} \phi_i/r_{sum}, \sum_{i=1}^{j} \phi_i/r_{sum}\right]$ if and only if the potential service completion at t_n occurs in the jth longest queue of $\tilde{\boldsymbol{X}}(t)$. Now define $\tilde{\boldsymbol{Y}}(t_n^+)$ such that the potential service completion at t_n occurs in the jth longest queue of $\tilde{\boldsymbol{Y}}(t)$ if and only if $Q \in \left[\sum_{i=1}^{j-1} r_{(i)}/r_{sum}, \sum_{i=1}^{j} r_{(i)}/r_{sum}\right]$. By (3.8), the above coupling implies that if the potential service completion occurs in the jth longest queue of $\tilde{\boldsymbol{Y}}(t)$, $1 \leq j \leq M$, then the potential service completion occurs in the lth largest queue of $\tilde{\boldsymbol{X}}(t)$, where $1 \leq l \leq j$. We now show that $\tilde{\boldsymbol{X}}(t_n^+) \prec_w \tilde{\boldsymbol{Y}}(t_n^+)$. For any fixed pair (l, j), $1 \leq l \leq j \leq M$, there are a number of cases.

Case 1: If $\tilde{Y}_{[j]}(t) = 0$, then $\tilde{Y}_{[i]}(t_n^+) = \tilde{Y}_{[i]}(t), \forall i$, and for each $k = 1, \ldots, M$,

$$\sum_{i=1}^{k} \tilde{X}_{[i]}(t_n^+) \leq \sum_{i=1}^{k} \tilde{X}_{[i]}(t) \leq \sum_{i=1}^{k} \tilde{Y}_{[i]}(t) = \sum_{i=1}^{k} \tilde{Y}_{[i]}(t_n^+).$$

Case 2: If $\tilde{Y}_{[j]}(t) > 0$, then $\sum_{i=1}^{k} \tilde{Y}_{[i]}(t_n^+) = \left(\sum_{i=1}^{k} \tilde{Y}_{[i]}(t)\right) - 1\{k \geq j\}$.

Case 2a: If $\tilde{X}_{[l]}(t) = 0$, then $\tilde{X}_{[i]}(t_n^+) = \tilde{X}_{[i]}(t), \forall i$, and $\tilde{X}_{[i]}(t_n^+) = \tilde{X}_{[i]}(t) = 0, \forall i \geq l$. For $1 \leq k < l \leq j$,

$$\sum_{i=1}^{k} \tilde{X}_{[i]}(t_n^+) = \sum_{i=1}^{k} \tilde{X}_{[i]}(t) \leq \sum_{i=1}^{k} \tilde{Y}_{[i]}(t) = \sum_{i=1}^{k} \tilde{Y}_{[i]}(t_n^+).$$

For $l \leq k \leq M$,

$$\begin{aligned}
\sum_{i=1}^{k} \tilde{X}_{[i]}(t_n^+) = \sum_{i=1}^{k} \tilde{X}_{[i]}(t) = \sum_{i=1}^{l-1} \tilde{X}_{[i]}(t) \quad &\leq \quad \sum_{i=1}^{l-1} \tilde{Y}_{[i]}(t) \\
&\stackrel{(a)}{=} \quad \left(\sum_{i=1}^{l-1} \tilde{Y}_{[i]}(t) \right) - 1\{l-1 \geq j\} \\
&= \quad \sum_{i=1}^{l-1} \tilde{Y}_{[i]}(t_n^+) \\
&\leq \quad \sum_{i=1}^{k} \tilde{Y}_{[i]}(t_n^+).
\end{aligned}$$

where (a) follows from the fact that $l \leq j$.

Case 2b: If $\tilde{X}_{[l]}(t) > 0$, then

$$\begin{aligned}
\sum_{i=1}^{k} \tilde{X}_{[i]}(t_n^+) \quad &= \quad \left(\sum_{i=1}^{k} \tilde{X}_{[i]}(t) \right) - 1\{k \geq l\} \\
&\leq \quad \left(\sum_{i=1}^{k} \tilde{Y}_{[i]}(t) \right) - 1\{k \geq l\} \\
&\stackrel{(b)}{\leq} \quad \left(\sum_{i=1}^{k} \tilde{Y}_{[i]}(t) \right) - 1\{k \geq j\} \\
&= \quad \sum_{i=1}^{k} \tilde{Y}_{[i]}(t_n^+).
\end{aligned}$$

where (b) follows from $l \leq j$. We have thus shown $\boldsymbol{X}(t_n^+) \prec_w^{st} \boldsymbol{Y}(t_n^+)$. $\square$

Theorem 3.1 is central to understanding the optimality of the LQHR policy. The essential idea is that a majorization order on the rates in $\mathcal{D}$ ($\boldsymbol{r} \prec \boldsymbol{\phi}$ for $\boldsymbol{r} \in \mathcal{D}$) translates into a stochastic weak majorization on the resulting queue vectors. Bringing out the theorem's full implications requires some discussion on the relations $\prec_w$ and $\prec_w^{st}$.

Definition 3.3 *Let $\mathcal{A} \subset \mathbb{R}^M$. A function $\varphi : \mathcal{A} \mapsto \mathbb{R}$ is said to be $\prec_w$-*preserving *if $\boldsymbol{x} \prec_w \boldsymbol{y} \Rightarrow \varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}$.*

The following lemma [MO79, p.483] draws the link between cost functions and stochastic weak majorization.

Lemma 3.1 *Let* $\boldsymbol{X} = (X_1, \ldots, X_M)$ *and* $\boldsymbol{Y} = (Y_1, \ldots, Y_M)$ *be random vectors taking values in* $\mathbb{R}^M$. *Then* $\boldsymbol{X} \prec_w^{st} \boldsymbol{Y}$ *if and only if* $\mathbb{E}[\varphi(\boldsymbol{X})] \leq \mathbb{E}[\varphi(\boldsymbol{Y})]$ *for all* $\prec_w$*-preserving functions* $\varphi : \mathbb{R}^M \mapsto \mathbb{R}$ *for which the expectations exist.*

PROOF. See [MO79].

We then immediately have a corollary to Theorem 3.1.

Corollary 3.1 *Let* $\boldsymbol{x}_0$ *be the vector of queue lengths in the system at time 0. Let* $\boldsymbol{X}(t)$ *be the vector of queue lengths under* g_{LQHR} *at time* $t \geq 0$. *Let* $\boldsymbol{Y}(t)$ *be the corresponding quantity under any policy* $g \in G_{\mathcal{D}}$. *Then*

$$\mathbb{E}\left[\varphi(\boldsymbol{X}(t))\right] \leq \mathbb{E}\left[\varphi(\boldsymbol{Y}(t))\right] \quad \forall t \geq 0$$

for all $\prec_w$*-preserving functions* $\varphi : \mathbb{R}^M \mapsto \mathbb{R}$ *for which the expectations exist.*

Which are the $\prec_w$-preserving functions? It turns out that a real-valued function φ defined on $\mathcal{A} \subset \mathbb{R}^M$ is $\prec_w$-preserving if and only if it is *increasing* and *Schur-convex* [MO79]. A real-valued function φ is *increasing* on $\mathcal{A}$ if for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}$, $x_i \leq y_i, \forall i \Rightarrow \varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. A real-valued function φ is *Schur-convex* on $\mathcal{A}$ if $\boldsymbol{x} \prec \boldsymbol{y} \Rightarrow \varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. That is, Schur-convex functions preserve majorization (as opposed to weak majorization).

The class of Schur-convex functions is well-studied [MO79]. Their characterization allows us to give some main examples of $\prec_w$-preserving functions:

- $\Phi_1 \equiv \{\varphi | \varphi : \mathbb{R}^M \mapsto \mathbb{R}$ is symmetric, convex and increasing$\}$. A function φ is *symmetric* on $\mathcal{A} \subset \mathbb{R}^M$ if $\varphi(\boldsymbol{x}) = \varphi(\boldsymbol{x}\mathsf{P})$ for any $\boldsymbol{x} \in \mathcal{A}$ and any M by M permutation matrix P.

- $\Phi_2 \equiv \{\varphi | \varphi(\boldsymbol{x}) = \sum_{i=1}^M \psi(x_i)$ where $\psi : \mathbb{R} \mapsto \mathbb{R}$ is a convex, increasing function$\}$. The class Φ_2 is clearly contained in Φ_1.

More specific examples of $\prec_w$-preserving functions include $\varphi(\boldsymbol{x}) = \max_{i=1,\ldots,M} |x_i|$, $\varphi(\boldsymbol{x}) = \max_{i_1 < i_2 < \cdots < i_k}(|x_{i_1}| + \cdots + |x_{i_k}|)$ for $1 \leq k \leq M$, $\varphi(\boldsymbol{x}) = \sum_{i=1}^M |x_i|^r$ for $r \geq 1$ or $r \leq 0$, and $\varphi(\boldsymbol{x}) = (\sum_{i=1}^M |x_i|^r)^{1/r}$ for $r \geq 1$. All this discussion shows that Corollary 3.1 is a significantly stronger result than an optimality result purely in terms of (3.2).

3.3 Average Bit Delay

The analysis in Section 3.2 has shown how the ideas of stochastic coupling and weak majorization can be combined to demonstrate the optimality of the LQHR policy in minimizing average packet delay for the symmetric Poisson/exponential case. In this section, we turn to the second version of the delay minimization problem. Here, we continue to consider independent Poisson arrival processes with a common parameter λ. However, we now allow the lengths of the packets from all sources to be i.i.d. according to some distribution function $F_Z(z)$ satisfying $\mathbb{E}[Z] < \infty$, independent of the arrival processes. As outlined in the introduction, our focus is on unfinished work and the goal is to find a policy $h^* \in H \equiv \{h : \mathcal{U} \mapsto \mathcal{C}\}$ to minimize (3.3). A natural idea is to apply the stochastic majorization techniques of the previous section to the present problem. Unfortunately, the sample path coupling arguments do not quite work here. Instead, dynamic programming techniques are used to prove that a modified version of the LQHR policy minimizes (3.3) within H for the *two-user* case.

3.3.1 The Modified LQHR Policy

To describe the modified version of the LQHR policy h_{LQHR}, we examine the generic form of the vector of unfinished work $\boldsymbol{U}(t) = (U_1(t), \ldots, U_M(t))$ at time t. Now $\boldsymbol{U}(t)$ satisfies $U_{[1]}(t) = \cdots = U_{[l_1]}(t) > U_{[l_1+1]}(t) = \cdots = U_{[l_2]}(t) > \cdots > U_{[l_{A-1}+1]}(t) = \cdots = U_{[l_A]}(t)$, where $l_A = M$. That is, at time t, there are in general a number of queues with the same amount of unfinished work. It is easy to see that if the LQHR policy given in Section 3.2 is applied without modification to $\boldsymbol{U}(t)$, infinitely rapid oscillations in rate allocation can arise (at least conceptually). This is because if one of two queues with equal unfinished work at time t receives the higher rate at time t, it immediately becomes the smaller of the two queues an infinitesimal amount of time after t, at which point the unmodified LQHR would give the other queue the higher rate, leading to an infinitely rapid oscillation in rate allocation. Since infinitely rapid oscillations are ill-defined both mathematically and physically, we give the *Modified LQHR policy* h_{LQHR} as follows. For $1 \leq i \leq M$, let $\mathcal{A}_{[i]}$ be the set of indices j such that $U_{[j]}(t) = U_{[i]}(t)$. For instance, $\mathcal{A}_{[1]} = \{1, \ldots, l_1\}$. At time t, let h_{LQHR} assign to $U_{[i]}(t)$ the rate

$$\phi_{\mathcal{A}_{[i]}} = \frac{\sum_{j \in \mathcal{A}_{[i]}} \phi_j}{|\mathcal{A}_{[i]}|} \tag{3.9}$$

where ϕ_j is defined in (3.4). For instance, $U_{[1]}(t)$ is assigned the rate $\phi_{\mathcal{A}_{[1]}} = \sum_{j=1}^{l_1} \phi_j / l_1$ under h_{LQHR} at time t. In other words, h_{LQHR} gives equal queues equal rates subject to the condition that longer queues receive higher rates.

The coding interpretation of the modified LQHR policy is slightly different from that of the LQHR policy in Section 3.2. Given a vector $\boldsymbol{U}(t)$ of unfinished work at time t, instead of successively decoding the users 1 to M, h_{LQHR} successively decodes *groups* of users $\{l_{A-1}+1, \ldots, M\}, \{l_{A-2}+1, \ldots, l_{A-1}\}, \ldots, \{l_1+1, \ldots, l_2\}, \{1, \ldots, l_1\}$ in that order. Within each group, h_{LQHR} in effect implements a coding strategy which gives equal rates to all users of that group. This may involve a rate-splitting scheme as in [GRUW01, RU96], or a time-sharing scheme as in [GR95, Rim97, Rim99] and Chapter 2 of this thesis. Geometrically, h_{LQHR} does not always operate on the extreme points of $\mathcal{D}$. In fact, whenever there are queues with equal amounts of unfinished work, h_{LQHR} operates at the "equal rate point" on the projection of $\mathcal{D}$ onto the coordinates corresponding to the equal queues. For instance, for $M = 2$, h_{LQHR} assigns the extreme point $\boldsymbol{r}_A = (\phi_2, \phi_1)$ (see Figure 3-2) whenever $U_1(t) < U_2(t)$ and assigns the extreme point $\boldsymbol{r}_B = (\phi_1, \phi_2)$ whenever $U_1(t) > U_2(t)$. When $U_1(t) = U_2(t)$, however, h_{LQHR} assigns the equal rate point $\boldsymbol{r}_C = ((\phi_1+\phi_2)/2, (\phi_1+\phi_2)/2)$.

To illustrate the workings of the modified LQHR policy, we examine the evolution of the unfinished vector as a function of time under h_{LQHR} for the two-user case. Since h_{LQHR} distinguishes between the two queues only in terms of their relative magnitudes, the queue state at time t can be taken as $(U_{[1]}(t), U_{[2]}(t))$ instead of $(U_1(t), U_2(t))$. Let the initial queue state a time 0 be $(U_{[1]}(0), U_{[2]}(0))$, and assume that no arrivals enter the system after time 0. There are two basic cases.

Case 1: $\frac{U_{[1]}(0)}{\phi_1} \leq \frac{U_{[2]}(0)}{\phi_2}$. In this case, $\Delta \equiv U_{[1]}(0) - U_{[2]}(0)$ is sufficiently small so that h_{LQHR} causes the two queues to become equal before they are depleted. The policy h_{LQHR} assigns rate ϕ_1 to queue [1] and rate ϕ_2 to queue [2] (where [1] and [2] are defined by $U_{[1]}(0) \geq U_{[2]}(0)$) from $t = 0$ to $t = \frac{\Delta}{\phi_1 - \phi_2}$, at which point the queues become equal. After the queues become equalized, h_{LQHR} allocates $(\phi_1 + \phi_2)/2$ to each queue until the queues are simultaneously depleted at $t = \frac{S}{\phi_1+\phi_2}$, where $S \equiv U_1(0) + U_2(0)$. Note that in

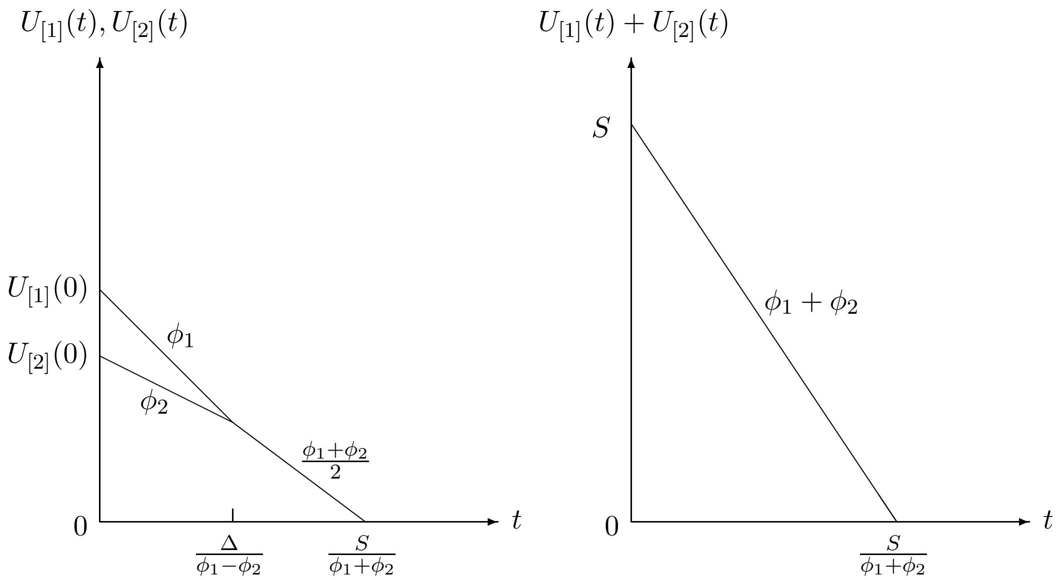

Figure 3-3: Plot of $U_{[1]}(t), U_{[2]}(t)$ and $U_{[1]}(t) + U_{[2]}(t)$ for the case of $\frac{U_{[1]}(0)}{\phi_1} \leq \frac{U_{[2]}(0)}{\phi_2}$. Here, $S \equiv U_{[1]}(0) + U_{[2]}(0), \Delta \equiv U_{[1]}(0) - U_{[2]}(0)$.

this case, h_{LQHR} processes the total unfinished work at the maximum rate $\phi_1 + \phi_2$ for all $t \in [0, \frac{S}{\phi_1+\phi_2}]$. See Figure 3-3.

Case 2: $\frac{U_{[1]}(0)}{\phi_1} \geq \frac{U_{[2]}(0)}{\phi_2}$. In this case, Δ is sufficiently large so that the queues cannot in general be equalized before they are depleted under h_{LQHR}. Here queue [1] always receives rate ϕ_1 and queue [2] always receives rate ϕ_2. Queue [2] is depleted at $t = \frac{U_{[2]}(0)}{\phi_2}$ before queue [1] is depleted at $t = \frac{U_{[1]}(0)}{\phi_1}$. Notice that in this case, h_{LQHR} processes total unfinished work at the maximum rate $\phi_1 + \phi_2$ from $t = 0$ to $t = \frac{U_{[2]}(0)}{\phi_2}$, then at rate ϕ_1 from $t = \frac{U_{[2]}(0)}{\phi_2}$ to $t = \frac{U_{[1]}(0)}{\phi_1}$. See Figure 3-4.

3.3.2 h_{LQHR} Minimizes Average Bit Delay for $M = 2$

We now demonstrate the optimality of h_{LQHR} in terms of (3.3) in the two-user case. Our approach is to first show that h_{LQHR} minimizes the expected integral of the total unfinished work for the set of sample paths corresponding to a fixed sequence of arrival epochs of the overall arrival process and lengths of arriving packets at those epochs. We will then appropriately interpret this "sample path" result in a more stochastic setting.

Consider the set of sample paths for which $0 < t_1 < t_2 < \cdots < t_{n-1} < t_n < \cdots$ are the arrival epochs of the overall arrival process and $z_1, z_2, \ldots, z_{n-1}, z_n \ldots$ are the lengths of the packets arriving at the corresponding times. At each arrival epoch, $t_k, k = 1, 2, \ldots,$ there

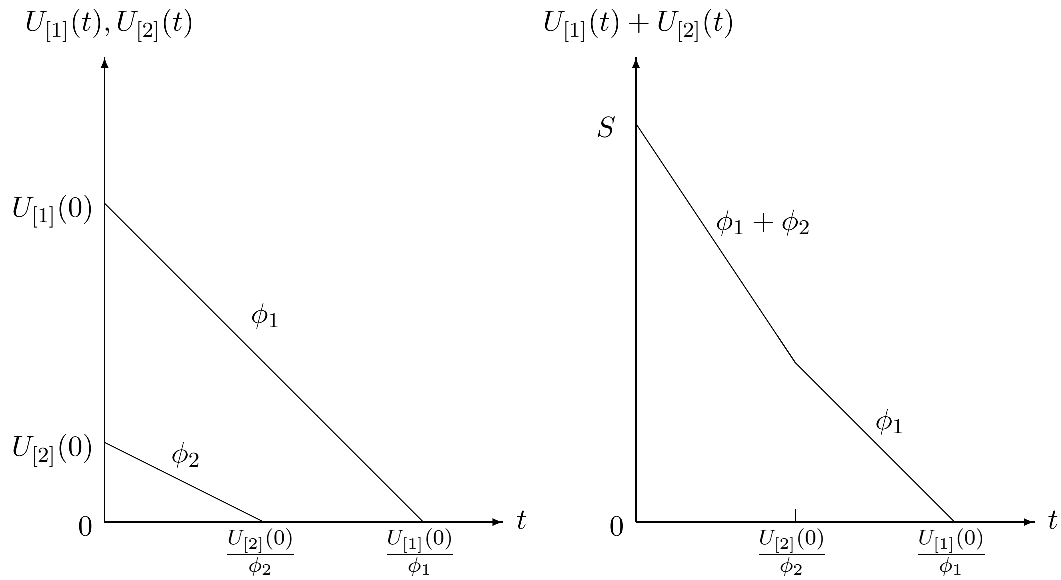

Figure 3-4: Plot of $U_{[1]}(t), U_{[2]}(t)$ and $U_{[1]}(t) + U_{[2]}(t)$ for the case of $\frac{U_{[1]}(0)}{\phi_1} \geq \frac{U_{[2]}(0)}{\phi_2}$. Here, $S \equiv U_{[1]}(0) + U_{[2]}(0)$.

is uncertainty as to which queue the arriving packet of length z_k enters (we will specify the details of this uncertainty shortly). Let $T > 0$ be fixed and let $N(T) = \max\{k | t_k < T\}$. For simplicity, we write N for $N(T)$. For a fixed admissible rate allocation policy $h \in H$, we are interested in studying the evolution of the queue state in terms of the unfinished work in the system as a function of time, from $t = 0$ to $t = T$. For $t \in [0, T]$, we choose to characterize the queue state at time t in terms of $(U_{[1]}(t), U_{[2]}(t))$ or equivalently, in terms of $(S(t), M(t))$, where $S(t) \equiv U_1(t) + U_2(t)$ is the sum and $M(t) \equiv U_{[1]}(t) = \max(U_1(t), U_2(t))$ is the maximum queue size. For all $t \in [0, T]$, the pair $(S(t), M(t))$ takes values in the set $\mathcal{V} \equiv \{(s, m) | 0 \leq s < \infty, s/2 \leq m \leq s\} \subset \mathbb{R}^2$. Figure 3-5 plots a sample path of the total unfinished work $S^h(t)$ under some policy $h \in H$. For each $k = 1, \ldots, N$, if we define $S_k^- \equiv S(t_k^-)$ and $M_k^- \equiv M(t_k^-)$, the queue state at t_k^- can be written as (S_k^-, M_k^-).

We now specify the uncertainty in the arrivals at $t_k, k = 1, \ldots, N$. Let $W_k, k = 1, \ldots, N$, be a sequence of binary-valued random variables defined by

$$W_k = \begin{cases} +1 & \text{if the arrival at } t_k \text{ occurs on queue [1]}, \\ -1 & \text{if the arrival at } t_k \text{ occurs on queue [2]}, \end{cases} \tag{3.10}$$

where [1] and [2] are defined by $M_k^- = U_{[1]}(t_k^-) \geq U_{[2]}(t_k^-) = S_k^- - M_k^-$. The probability

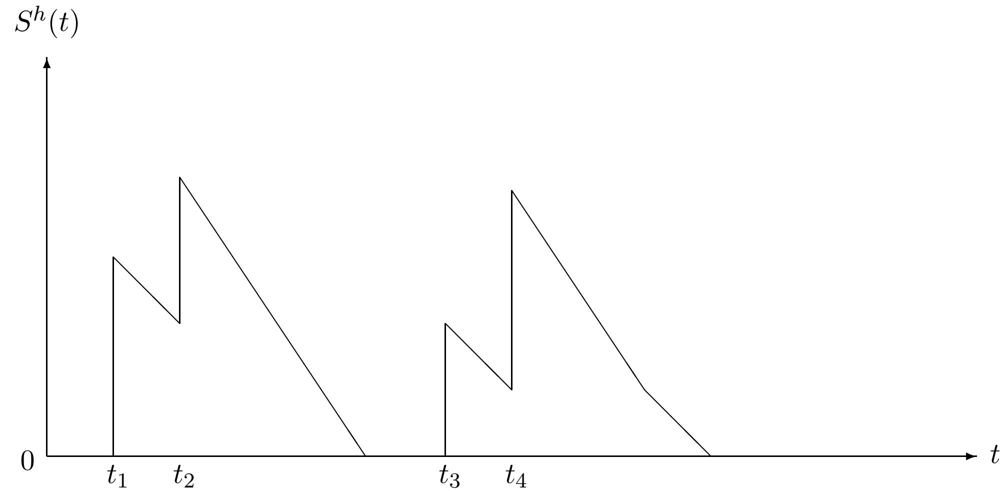

Figure 3-5: Plot of total unfinished work $S^h(t)$ as a function of time under the policy h. The arrival instants are t_1, t_2, t_3, t_4. $S^h(t)$ does not always decrease at the rate $\phi_1 + \phi_2$ since some queues empty before others. We have assumed $h \in H$.

distribution of W_k (conditioned on the fixed sequence of arrival epochs and packet lengths) is characterized by

$$p_k \equiv \Pr(W_k = +1). \tag{3.11}$$

For independent Poisson arrival processes with common rate λ and i.i.d. packet lengths, $\{W_k\}$ is a sequence of i.i.d. Bernoulli($\frac{1}{2}$) random variables, with $p_k = \frac{1}{2}, k = 1, \ldots, N$. Now given (S_k^-, M_k^-) and W_k, the queue state (S_k^+, M_k^+), where $S_k^+ \equiv S(t_k^+)$ and $M_k^+ \equiv M(t_k^+)$, is determined by

$$\begin{aligned} S_k^+ &= S_k^- + z_k, \\ M_k^+ &= \begin{cases} M_k^- + z_k & \text{if } W_k = +1, \\ \max(M_k^-, S_k^- - M_k^- + z_k) & \text{if } W_k = -1. \end{cases} \end{aligned}$$

For a fixed $h \in H$, the queue state (S_{k+1}^-, M_{k+1}^-) is determined by (S_k^+, M_k^+), and therefore by (S_k^-, M_k^-, W_k). We can therefore describe the evolution of the queue state at time instants $t_k^-, k = 1, \ldots, N$, (with t_{N+1} being defined as T) by

$$(S_{k+1}^-, M_{k+1}^-) = f_k^h(S_k^-, M_k^-, W_k). \tag{3.12}$$

For $h \in H$, let $(S^h(t), M^h(t))$ be the queue state under policy h at time t. Our main objective is to find $h^* \in H$ to minimize the integral of the total unfinished work from 0 to T, averaged over all realizations of the arrival variables

$$\mathbb{E}_{W_1,\dots,W_N}\left\{\int_0^T S^h(t)dt\right\}. \tag{3.13}$$

For each $k = 1, \dots, N$, we shall refer to the time interval $[t_k, t_{k+1})$ (where we define t_{N+1} to be T) as the kth period. For a fixed policy h, the queue evolution $(S^h(t), M^h(t))$ for $t \in (t_k, t_{k+1})$ is a function of $(S^h(t_k^+), M^h(t_k^+))$, and therefore of $(S^h(t_k^-), M^h(t_k^-), W_k)$, which we abbreviate as (S_k^-, M_k^-, W_k). We let

$$c_k^h(S_k^-, M_k^-, W_k) = \int_{t_k}^{t_{k+1}} S^h(t)dt$$

denote the integral of the total unfinished work, or cost, over the kth period under policy h.

As a function of the initial unfinished work (S_1^-, M_1^-), the expected integral in (3.13) can be expressed as

$$J_h(S_1^-, M_1^-) = \mathbb{E}_{W_1,\dots,W_N}\left\{\sum_{k=1}^{N} c_k^h(S_k^-, M_k^-, W_k)\right\}. \tag{3.14}$$

Our goal is to find $h^* \in H$ to minimize (3.14):

$$J_{h^*}(S_1^-, M_1^-) = \min_{h \in H} J_h(S_1^-, M_1^-). \tag{3.15}$$

To solve the minimization problem in (3.15), we resort to a slightly more general setting. Here, we allow the rate allocation policy to vary as a function of $k, k = 1, \dots, N$. That is, we allow different policies to be adopted over different periods. Let $\pi = (h_1, \dots, h_N)$ be a sequence of admissible policies, where $h_k \in H$ is the policy used over the kth period, $k = 1, \dots, N$. The corresponding total integral or cost for π given (S_1^-, M_1^-) is

$$J_\pi(S_1^-, M_1^-) = \mathbb{E}_{W_1,\dots,W_N}\left\{\sum_{k=1}^{N} c_k^{h_k}(S_k^-, M_k^-, W_k)\right\}. \tag{3.16}$$

The optimization is now over all admissible policy sequences:

$$J^*(S_1^-, M_1^-) = \min_{\pi \in \Pi} J_\pi(S_1^-, M_1^-). \tag{3.17}$$

where $\Pi \equiv \{(h_1, \ldots, h_N) | h_k \in H, k = 1, \ldots, N\}$ is the set of all admissible policy sequences. Clearly, $J_{h^*}(S_1^-, M_1^-) \geq J^*(S_1^-, M_1^-)$. We will show, however, that there exists $h^* \in H$ such that $J_{h^*}(S_1^-, M_1^-) = J^*(S_1^-, M_1^-)$, so that a *fixed* policy attains the minimum in (3.17). In fact, we will show that the optimal h^* is h_{LQHR}.

To solve (3.17), we use a dynamic programming approach. That is, we first choose a policy $h_N \in H$ to minimize the expected cost over the Nth period and then recurse backwards, choosing policy h_k to minimize the expected value of the current cost plus the optimal cost-to-go at the kth stage, for $k = 1, \ldots, N-1$. Consider the following recursion. Let

$$J_N(S_N^-, M_N^-) = \min_{h_N \in H} \mathbb{E}_{W_N} \left\{ c_N^{h_N}(S_N^-, M_N^-, W_N) \right\}, \tag{3.18}$$

and for $k = 1, \ldots, N-1$,

$$J_k(S_k^-, M_k^-) = \min_{h_k \in H} \mathbb{E}_{W_k} \left\{ c_k^{h_k}(S_k^-, M_k^-, W_k) + J_{k+1}\left(f_k^{h_k}(S_k^-, M_k^-, W_k) \right) \right\}. \tag{3.19}$$

Equations (3.18) and (3.19) specify the dynamic programming algorithm. The following lemma says that this algorithm gives the optimal solution to (3.17).

Lemma 3.2

$$J^*(S_1^-, M_1^-) = J_1(S_1^-, M_1^-).$$

where $J^*(S_1^-, M_1^-)$ is given by (3.17), and $J_1(S_1^-, M_1^-)$ is given by (3.18) and (3.19).

PROOF. The argument is similar in spirit to that in [Ber95]. We include it for completeness. For $k = 1, \ldots, N$, let $J_k^*(S_k^-, M_k^-)$ be the optimal cost for the $N-k+1$-stage version of the minimization problem in (3.16)-(3.17) which starts with (S_k^-, M_k^-) at time t_k^- and ends at time $t_{N+1} = T$. That is,

$$J_k^*(S_k^-, M_k^-) = \min_{(h_k,\dots,h_N)} \mathbb{E}_{W_k,\dots,W_N} \left\{ \sum_{i=k}^{N} c_i^{h_i}(S_i^-, M_i^-, W_i) \right\}.$$

For $k = N$,

$$J_N^*(S_N^-, M_N^-) = \min_{h_N \in H} \mathbb{E}_{W_N} \left\{ c_N^{h_N}(S_N^-, M_N^-, W_N) \right\} = J_N(S_N^-, M_N^-).$$

Assume that for some $k, 1 \le k \le N-2$, we have

$$J_{k+1}^*(S_{k+1}^-, M_{k+1}^-) = J_{k+1}(S_{k+1}^-, M_{k+1}^-).$$

Then we have

$$\begin{aligned}
J_k^*(S_k^-, M_k^-) &= \min_{(h_k,\dots,h_N)} \mathbb{E}_{W_k,\dots,W_N} \left\{ c_k^{h_k}(S_k^-, M_k^-, W_k) + \sum_{i=k+1}^{N} c_i^{h_i}(S_i^-, M_i^-, W_i) \right\} \\
&= \min_{h_k \in H} \mathbb{E}_{W_k} \Bigg\{ c_k^{h_k}(S_k^-, M_k^-, W_k) \\
&\quad + \min_{(h_{k+1},\dots,h_N)} \left[\mathbb{E}_{W_{k+1},\dots,W_N} \left\{ \sum_{i=k+1}^{N} c_i^{h_i}(S_i^-, M_i^-, W_i) \right\} \right] \Bigg\} \\
&= \min_{h_k} \mathbb{E}_{W_k} \left\{ c_k^{h_k}(S_k^-, M_k^-, W_k) + J_{k+1}^*(S_{k+1}^-, M_{k+1}^-) \right\} \\
&\overset{(a)}{=} \min_{h_k} \mathbb{E}_{W_k} \left\{ c_k^{h_k}(S_k^-, M_k^-, W_k) + J_{k+1}(S_{k+1}^-, M_{k+1}^-) \right\} \\
&\overset{(b)}{=} \min_{h_k \in H} \mathbb{E}_{W_k} \left\{ c_k^{h_k}(S_k^-, M_k^-, W_k) + J_{k+1}(f_k^{h_k}(S_k^-, M_k^-, W_k) \right\} \\
&\overset{(c)}{=} J_k(S_k^-, M_k^-).
\end{aligned}$$

where (a) follows from the inductive assumption, (b) follows from (3.12) and (c) follows from (3.19). □

We now proceed to show that the sequence of policies $(h_{LQHR}, \dots, h_{LQHR})$ solves the dynamic programming recursion in (3.18)-(3.19) whenever $p_k(s_k^-, m_k^-) \ge \frac{1}{2}$ for all $(s_k^-, m_k^-) \in \mathcal{V}$. We first prove a key lemma which states that h_{LQHR} simultaneously minimizes S_{k+1}^-, M_{k+1}^-, and $c_k^{h_k}$ as functions of S_k^-, M_k^-, and W_k. Since S_{k+1}^-, M_{k+1}^-, and $c_k^{h_k}$ are functions of S_k^+ and M_k^+, we will often write $c_k^{h_k}(S_k^-, M_k^-, W_k)$ as $\tilde{c}_k^{h_k}(S_k^+, M_k^+)$,

$S^-_{k+1}(S^-_k, M^-_k, W_k)$ as $\tilde{S}^-_{k+1}(S^+_k, M^+_k)$, and $M^-_{k+1}(S^-_k, M^-_k, W_k)$ as $\tilde{M}^-_{k+1}(S^+_k, M^+_k)$.

Lemma 3.3 *Let $h \in H$ be given. For $k = 1, \ldots, N$, let $(\tilde{S}^-_{k+1}(S^+_k, M^+_k), \tilde{M}^-_{k+1}(S^+_k, M^+_k))$ and $(\tilde{S}^{-'}_{k+1}(S^+_k, M^+_k), \tilde{M}^{-'}_{k+1}(S^+_k, M^+_k))$ be the queue states at t^-_{k+1} as functions of (S^+_k, M^+_k) under h_{LQHR} and h, respectively. Let $\tilde{c}_k(S^+_k, M^+_k)$ and $\tilde{c}'_k(S^+_k, M^+_k)$ be the costs over the kth period as functions of (S^+_k, M^+_k) under h_{LQHR} and h, respectively. Then for all $(s^+_k, m^+_k) \in \mathcal{V} \equiv \{(s,m)|0 \leq s < \infty, s/2 \leq m \leq s\}$, and for all $k = 1, \ldots, N$,*

$$\tilde{S}^-_{k+1}(s^+_k, m^+_k) \leq \tilde{S}^{-'}_{k+1}(s^+_k, m^+_k), \quad \tilde{M}^-_{k+1}(s^+_k, m^+_k) \leq \tilde{M}^{-'}_{k+1}(s^+_k, m^+_k),$$
$$\tilde{c}_k(s^+_k, m^+_k) \leq \tilde{c}'_k(s^+_k, m^+_k).$$

PROOF. Let (s^+_k, m^+_k) be any element of $\mathcal{V}$ and suppose $(S^+_k, M^+_k) = (s^+_k, m^+_k)$. Then the unfinished work vector $(u^h_1(t), u^h_2(t))$ under any policy $h \in H$ is determined (up to a permutation on the queue indices) for $t \in (t_k, t_{k+1})$.

We first show that for any admissible policy $h \in H$, there exists some $h' \in H_{\mathcal{D}} \equiv \{h : \mathcal{U} \mapsto \mathcal{D}\}$, *i.e.* a policy operating strictly on the dominant face, such that $u^{h'}_1(t) \leq u^h_1(t)$ and $u^{h'}_2(t) \leq u^h_2(t)$ $\forall t \in (t_k, t_{k+1})$. Let $r^h_i(u_1, u_2)$ be the rate allocated to queue i under h for the state $(u_1, u_2), i = 1, 2$. By the definition of $\mathcal{D}$, for all (u_1, u_2) and for $i = 1, 2$, there exists $r'_i(u_1, u_2)$ such that $r^h_i(u_1, u_2) \leq r'_i(u_1, u_2)$ (See discussion in Chapter 2). Define h' by $h'(u_1, u_2) = (r'_1(u_1, u_2), r'_2(u_1, u_2))$. Since h' allocates higher rates than h to both queues 1 and 2 in all states (u_1, u_2), we conclude that $u^{h'}_1(t) \leq u^h_1(t)$ and $u^{h'}_2(t) \leq u^h_2(t)$ $\forall t \in (t_k, t_{k+1})$. From this it follows that $u^{h'}_1(t^-_{k+1}) + u^{h'}_2(t^-_{k+1}) \leq u^h_1(t^-_{k+1}) + u^h_2(t^-_{k+1})$, $\max(u^{h'}_1(t^-_{k+1}), u^{h'}_2(t^-_{k+1})) \leq \max(u^h_1(t^-_{k+1}), u^h_2(t^-_{k+1}))$, and $\int_{t_k}^{t_{k+1}} [u^{h'}_1(t^-_{k+1}) + u^{h'}_2(t^-_{k+1})]dt \leq \int_{t_k}^{t_{k+1}} [u^h_1(t^-_{k+1}) + u^h_2(t^-_{k+1})]dt$. Thus, we need only show that h_{LQHR} performs no worse than all $h \in H_{\mathcal{D}}$ in proving the lemma.

Let $h \in H_{\mathcal{D}}$ be given. The key observation is that queues start to empty out under policy h no later than they do under h_{LQHR}. For $t \in (t_k, t_{k+1})$, let $(u_{[1]}(t), u_{[2]}(t))$ and $(u'_{[1]}(t), u'_{[2]}(t))$ be the evolution of unfinished work under h_{LQHR} and h, respectively. Then, $u_{[1]}(t^+_k) = u'_{[1]}(t^+_k) = m^+_k$ and $u_{[2]}(t^+_k) = u'_{[2]}(t^+_k) = s^+_k - m^+_k$. Let $t_e \equiv \sup\{t_k \leq t \leq t_{k+1}|u_{[1]}(t) > 0, u_{[2]}(t) > 0\}$ and $t'_e \equiv \sup\{t_k \leq t \leq t_{k+1}|u'_{[1]}(t) > 0, u'_{[2]}(t) > 0\}$. We show that $t'_e \leq t_e$.

There are two basic cases to consider, corresponding to Figures 3-3 and 3-4. If $\frac{u_{[1]}(t^+_k)}{\phi_1} \geq$

$\frac{u_{[2]}(t_k^+)}{\phi_2}$, $t_e = t_k + \min\left(\tau_k, \frac{u_{[2]}(t_k^+)}{\phi_2}\right)$, where $\tau_k = t_{k+1} - t_k$. Now for any $h \in H_{\mathcal{D}}$, the rate allocated to the larger queue can never exceed ϕ_1 and the rate allocated to the smaller queue can never be less than ϕ_2. Then, since $\frac{u_{[1]}(t_k^+)}{\phi_1} \geq \frac{u_{[2]}(t_k^+)}{\phi_2}$, the smaller queue must empty out before the larger queue does, for any $h \in H_{\mathcal{D}}$. Therefore, $t'_e = \sup\{t_k \leq t \leq t_{k+1} | u'_{[2]}(t) > 0\} \leq t_k + \min\left(\tau_k, \frac{u_{[2]}(t_k^+)}{\phi_2}\right) = t_e$ (once again, since the rate allocated to the smaller queue can never be less than ϕ_2).

If $\frac{u_{[1]}(t_k^+)}{\phi_1} \leq \frac{u_{[2]}(t_k^+)}{\phi_2}$, then $t_e = t_k + \min\left(\tau_k, \frac{s_k^+}{\phi_1+\phi_2}\right)$. If $\tau_k \leq \frac{s_k^+}{\phi_1+\phi_2}$, then $t'_e \leq t_{k+1} = t_k + \tau_k = t_e$. Otherwise, $t_e = t_k + \frac{s_k^+}{\phi_1+\phi_2}$. Suppose $t'_e > t_e$. Then $u'_{[1]}(t) > 0$ and $u'_{[2]}(t) > 0$ for all $t \in [t_k, t'_e)$. In particular, $u'_{[1]}\left(t_k + \frac{s_k^+}{\phi_1+\phi_2}\right) > 0$ and $u'_{[2]}\left(t_k + \frac{s_k^+}{\phi_1+\phi_2}\right) > 0$. Since $h \in H_{\mathcal{D}}$, h reduces total unfinished work at a rate of $\phi_1 + \phi_2$ at all $t, t_k \leq t \leq t_k + \frac{s_k^+}{\phi_1+\phi_2}$. Then $u'_{[1]}\left(t_k + \frac{s_k^+}{\phi_1+\phi_2}\right) + u'_{[2]}\left(t_k + \frac{s_k^+}{\phi_1+\phi_2}\right) = s_k^+ - (\phi_1 + \phi_2)\left(\frac{s_k^+}{\phi_1+\phi_2}\right) = 0$, which gives a contradiction. Thus, $t'_e \leq t_e$.

We now show $\tilde{s}_{k+1}^- \equiv u_{[1]}(t_{k+1}^-) + u_{[2]}(t_{k+1}^-) \leq u'_{[1]}(t_{k+1}^-) + u'_{[2]}(t_{k+1}^-) \equiv \tilde{s}_{k+1}^{-'}$ and $\tilde{c}_k \equiv \int_{t_k}^{t_{k+1}} [u_{[1]}(t) + u_{[2]}(t)]dt \leq \int_{t_k}^{t_{k+1}} [u'_{[1]}(t) + u'_{[2]}(t)]dt \equiv \tilde{c}'_k$. Observe that h_{LQHR} reduces the total unfinished work at a rate of $\phi_1 + \phi_2$ from t_k to t_e, and then at a rate of ϕ_1 from t_e until t_{k+1} or until the system empties. Policy h, on the other hand, reduces the total unfinished work at a rate of $\phi_1 + \phi_2$ from t_k to t'_e, and then at a rate of ϕ_1 from t'_e until t_{k+1} or until the system empties. Thus, from t_k to t'_e, h_{LQHR} and h both reduce the total unfinished work at the same maximum rate $\phi_1 + \phi_2$, implying $u_{[1]}(t'_e) + u_{[2]}(t'_e) = u'_{[1]}(t'_e) + u'_{[2]}(t'_e)$ and $\int_{t_k}^{t'_e} [u_{[1]}(t) + u_{[2]}(t)]dt = \int_{t_k}^{t'_e} [u'_{[1]}(t) + u'_{[2]}(t)]dt$. Since $t_e \geq t'_e$, h_{LQHR} reduces total unfinished work at a higher rate than h at all times from t'_e to t_{k+1}. This implies $\tilde{s}_{k+1}^- \leq \tilde{s}_{k+1}^{-'}$ and $\int_{t'_e}^{t_{k+1}} [u_{[1]}(t) + u_{[2]}(t)]dt \leq \int_{t'_e}^{t_{k+1}} [u'_{[1]}(t) + u'_{[2]}(t)]dt$, with equality in both inequalities if $t_e = t'_e$. Therefore, we also have $\tilde{c}_k \leq \tilde{c}'_k$, with equality if $t_e = t'_e$.

Finally, we show $\tilde{m}_{k+1}^- \equiv u_{[1]}(t_{k+1}^-) \leq u'_{[1]}(t_{k+1}^-) \equiv \tilde{m}_{k+1}^{-'}$. We show this by demonstrating that $\Delta_{k+1}^- \equiv u_{[1]}(t_{k+1}^-) - u_{[2]}(t_{k+1}^-) \leq u'_{[1]}(t_{k+1}^-) - u'_{[2]}(t_{k+1}^-) \equiv \Delta_{k+1}^{-'}$. Since $\tilde{m}_{k+1}^- = (\tilde{s}_{k+1}^- + \Delta_{k+1}^-)/2$, $\tilde{m}_{k+1}^{-'} = (\tilde{s}_{k+1}^{-'} + \Delta_{k+1}^{-'})/2$, and $\tilde{s}_{k+1}^- \leq \tilde{s}_{k+1}^{-'}$, the claim follows. Here, once again, there are two cases. First assume $t'_e < t_{k+1}$. Then, $\Delta_{k+1}^{-'} = \tilde{s}_{k+1}^{-'}$ and $\Delta_{k+1}^- \leq \tilde{s}_{k+1}^- \leq \tilde{s}_{k+1}^{-'} = \Delta_{k+1}^{-'}$. Next, assume $t'_e = t_{k+1}$. Then, $t_e = t_{k+1}$ also. For $t \in (t_k, t_{k+1})$, let $\Delta(t) = u_{[1]}(t) - u_{[2]}(t)$ and $\Delta'(t) = u'_{[1]}(t) - u'_{[2]}(t)$. Now $\Delta(t) = \left(\Delta(t_k^+) - (\phi_1 - \phi_2)(t - t_k)\right)^+$, where $(x)^+ \equiv \max(x, 0)$. Under any policy $h \in H_{\mathcal{D}}$, the larger queue can receive a rate which is at most ϕ_1 and the smaller queue can receive a rate which is at least ϕ_2. Thus, the magnitude of the difference of the queues (as long as it is positive) can decrease at a rate no larger than $\phi_1 - \phi_2$. Thus, $\Delta(t) \le \Delta'(t)$ at all $t \in (t_k, t_{k+1})$ for which $\Delta(t) > 0$. And of course $\Delta(t) \le \Delta'(t)$ when $\Delta(t) = 0$. Thus, $\Delta(t) \le \Delta'(t)\ \forall t \in (t_k, t_{k+1})$. In particular, $\Delta^-_{k+1} \le \Delta^{-'}_{k+1}$. □

Next, we develop some properties of the functions $\tilde{S}^-_{k+1}$, $\tilde{M}^-_{k+1}$, and $\tilde{c}_k$, $k = 1, \ldots, N$, under h_{LQHR}. First we examine the cost over the kth period $\tilde{c}_k(S^+_k, M^+_k)$ as a function of the queue state at (S^+_k, M^+_k) at t^+_k. Define $\tau_k \equiv t_{k+1} - t_k$. There are two main cases, corresponding to Figures 3-3 and 3-4.

Case 1: $\frac{M^+_k}{\phi_1} \le \frac{S^+_k - M^+_k}{\phi_2}$ (See Figure 3-3). The integral over the kth period for this case is given by

$$\tilde{c}_k(S^+_k, M^+_k) = \begin{cases} S^+_k \tau_k - \frac{1}{2}(\phi_1 + \phi_2)\tau_k^2 & \text{for } \tau_k \le \frac{S^+_k}{\phi_1 + \phi_2}, \\ \frac{{S^+_k}^2}{2(\phi_1 + \phi_2)} & \text{for } \tau_k \ge \frac{S^+_k}{\phi_1 + \phi_2}. \end{cases}$$

More compactly,

$$\tilde{c}_k(S^+_k, M^+_k) = S^+_k \tau_k - \frac{1}{2}(\phi_1 + \phi_2)\tau_k^2 + \frac{1}{2}(\phi_1 + \phi_2)\left[\left(\tau_k - \frac{S^+_k}{\phi_1 + \phi_2}\right)^+\right]^2,$$

where $(x)^+ \equiv \max(x, 0)$.

Case 2: $\frac{M^+_k}{\phi_1} \ge \frac{S^+_k - M^+_k}{\phi_2}$ (See Figure 3-4). The integral in this case is

$$\tilde{c}_k(S^+_k, M^+_k) = \begin{cases} S^+_k \tau_k - \frac{1}{2}(\phi_1 + \phi_2)\tau_k^2 & \text{for } \tau_k \le \frac{S^+_k - M^+_k}{\phi_2}, \\ \frac{(S^+_k - M^+_k)^2}{2\phi_2} + M^+_k \tau_k - \frac{1}{2}\tau_k^2 \phi_1 & \text{for } \frac{S^+_k - M^+_k}{\phi_2} \le \tau_k \le \frac{M^+_k}{\phi_1}, \\ \frac{(S^+_k - M^+_k)^2}{2\phi_2} + \frac{{M^+_k}^2}{2\phi_1} & \text{for } \tau_k \ge \frac{M^+_k}{\phi_1}. \end{cases}$$

More compactly,

$$\tilde{c}_k(S^+_k, M^+_k) = S^+_k \tau_k - \frac{1}{2}(\phi_1 + \phi_2)\tau_k^2 + \frac{\phi_2}{2}\left[\left(\tau_k - \frac{S^+_k - M^+_k}{\phi_2}\right)^+\right]^2 + \frac{\phi_1}{2}\left[\left(\tau_k - \frac{M^+_k}{\phi_1}\right)^+\right]^2.$$

From the above expressions, it is easy to verify the following lemma. We shall say that a function $c : \mathbb{R}^2 \mapsto \mathbb{R}$ is *increasing* in (x, y) if $c(x_1, y_1) \leq c(x_2, y_2)$ whenever $x_1 \leq x_2$ and $y_1 \leq y_2$.

Lemma 3.4 *For each $k = 1, \ldots, N$, let $\tilde{c}_k(S_k^+, M_k^+)$ be the integral of the total unfinished work (or cost) over the kth period under h_{LQHR} as a function of the queue state (S_k^+, M_k^+) at t_k^+. Then $\tilde{c}_k$ is increasing in (S_k^+, M_k^+).*

It is also possible to verify (by calculating first and second partial derivatives) that $\tilde{c}_k$ is convex in the pair (S_k^+, M_k^+). We shall prove this in Lemma 3.6, however, using a more physically motivated argument. For now, we move on to examine $\tilde{S}_{k+1}^-$ and $\tilde{M}_{k+1}^-$ as functions of (S_k^+, M_k^+), $k = 1, \ldots, N$, under h_{LQHR}. Again, there are two main cases.

Case 1: $\frac{M_k^+}{\phi_1} \leq \frac{S_k^+ - M_k^+}{\phi_2}$ (See Figure 3-3). For $\tau_k \leq \frac{2M_k^+ - S_k^+}{\phi_1 - \phi_2}$, $\tilde{M}_{k+1}^- = M_k^+ - \phi_1 \tau_k$ and $\tilde{S}_{k+1}^- = S_k^+ - (\phi_1 + \phi_2)\tau_k$. For $\frac{2M_k^+ - S_k^+}{\phi_1 - \phi_2} \leq \tau_k \leq \frac{S_k^+}{\phi_1 + \phi_2}$, $\tilde{M}_{k+1}^- = \frac{1}{2}(S_k^+ - (\phi_1 + \phi_2)\tau_k)$ and $\tilde{S}_{k+1}^- = S_k^+ - (\phi_1 + \phi_2)\tau_k$. For $\tau_k \geq \frac{S_k^+}{\phi_1 + \phi_2}$, $\tilde{M}_{k+1}^- = \tilde{S}_{k+1}^- = 0$. More compactly,

$$\begin{aligned}
\tilde{M}_{k+1}^- &= \max\left(\frac{1}{2}\left(S_k^+ - (\phi_1 + \phi_2)\tau_k\right)^+, M_k^+ - \phi_1 \tau_k\right), \\
\tilde{S}_{k+1}^- &= \left(S_k^+ - (\phi_1 + \phi_2)\tau_k\right)^+.
\end{aligned}$$

Case 2: $\frac{M_k^+}{\phi_1} \geq \frac{S_k^+ - M_k^+}{\phi_2}$ (See Figure 3-4). For $\tau_k \leq \frac{S_k^+ - M_k^+}{\phi_2}$, $\tilde{M}_{k+1}^- = M_k^+ - \phi_1 \tau_k$ and $\tilde{S}_{k+1}^- = S_k^+ - (\phi_1 + \phi_2)\tau_k$. For $\frac{S_k^+ - M_k^+}{\phi_2} \leq \tau_k \leq \frac{M_k^+}{\phi_1}$, $\tilde{M}_{k+1}^- = \tilde{S}_{k+1}^- = M_k^+ - \phi_1 \tau_k$. For $\tau_k \geq \frac{M_k^+}{\phi_1}$, $\tilde{M}_{k+1}^- = \tilde{S}_{k+1}^- = 0$. More compactly,

$$\begin{aligned}
\tilde{M}_{k+1}^- &= \left(M_k^+ - \phi_1 \tau_k\right)^+, \\
\tilde{S}_{k+1}^- &= \max\left(\left(M_k^+ - \phi_1 \tau_k\right)^+, S_k^+ - (\phi_1 + \phi_2)\tau_k\right).
\end{aligned}$$

From the above expressions, the following lemma is immediate.

Lemma 3.5 *For each $k = 1, \ldots, N$, let $(\tilde{S}_{k+1}^-(S_k^+, M_k^+), \tilde{M}_{k+1}^-(S_k^+, M_k^+))$ be the queue state at t_{k+1}^- under h_{LQHR} as a function of the queue state (S_k^+, M_k^+) at t_k^+. Then $\tilde{S}_{k+1}^-$ and $\tilde{M}_{k+1}^-$ are both increasing in (S_k^+, M_k^+).*

We now prove an important convexity result.

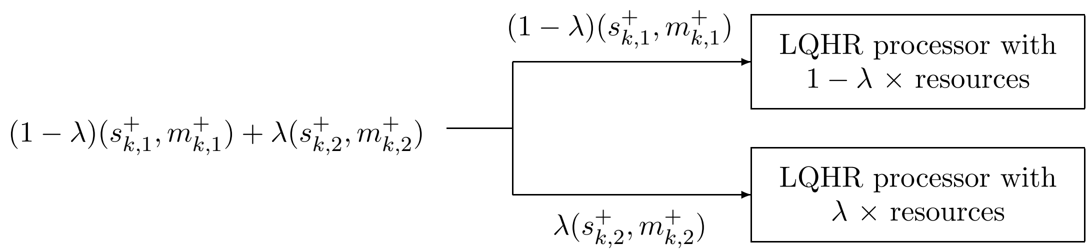

Figure 3-6: Illustration of the h_{divide} policy.

Lemma 3.6 *For $k = 1, \ldots, N$, let $\tilde{c}_k(S_k^+, M_k^+)$ be the integral of the total unfinished work (or cost) over the kth period under h_{LQHR} as a function of the queue state (S_k^+, M_k^+) at t_k^+, and let $(\tilde{S}_{k+1}^-(S_k^+, M_k^+), \tilde{M}_{k+1}^-(S_k^+, M_k^+))$ be queue state at t_{k+1}^- under h_{LQHR} as a function of the queue state (S_k^+, M_k^+) at t_k^+. Then, $\tilde{c}_k$, $\tilde{S}_{k+1}^-$ and $\tilde{M}_{k+1}^-$ are all convex in (S_k^+, M_k^+).*

PROOF. Let $(s_{k,1}^+, m_{k,1}^+)$ and $(s_{k,2}^+, m_{k,2}^+)$ be any two elements of $\mathcal{V}$. For any $\lambda \in [0, 1]$, consider the convex combination $(1-\lambda)(s_{k,1}^+, m_{k,1}^+) + \lambda(s_{k,2}^+, m_{k,2}^+)$, which lies in $\mathcal{V}$. Suppose the queue state at t_k^+ takes on this convex combination as its value. Now consider the following rate allocation policy for the kth period. Divide the unfinished work vector at t_k^+ into two parts, on corresponding to $(1-\lambda)(s_{k,1}^+, m_{k,1}^+)$ and the other corresponding to $\lambda(s_{k,2}^+, m_{k,2}^+)$. Process the first part of the unfinished work using h_{LQHR}, but with $1-\lambda$ times the resources available from $\mathcal{C}$. That is, h_{LQHR} is allowed to operate with rates from the region $\{(r_1, r_2) | r_1 \leq (1-\lambda)\phi_1, r_2 \leq (1-\lambda)\phi_2, r_1 + r_2 \leq (1-\lambda)(\phi_1 + \phi_2)\}$. In parallel, process the second part of the unfinished work using h_{LQHR}, but with λ times the resources available from $\mathcal{C}$. That is, h_{LQHR} is allowed to operate with rates from the region $\{(r_1, r_2) | r_1 \leq \lambda\phi_1, r_2 \leq \lambda\phi_2, r_1 + r_2 \leq \lambda(\phi_1 + \phi_2)\}$. Figure 3-6 illustrates the situation.

Refer to the above "divide and conquer" policy as h_{divide}. For $t \in (t_k, t_{k+1})$, let $\boldsymbol{r}_i(t) \equiv (r_{i1}(t), r_{i2}(t))$ be the rates assigned under h_{divide} to the ith part of the unfinished work, $i = 1, 2$. It is then clear that $\boldsymbol{r}_1(t) + \boldsymbol{r}_2(t) \in \mathcal{C}$ for all $t \in (t_k, t_{k+1})$. Thus, h_{divide} is an admissible policy. Now let $\tilde{S}_{k+1}^-((1-\lambda)(s_{k,1}^+, m_{k,1}^+) + \lambda(s_{k,2}^+, m_{k,2}^+))$ and $\tilde{S}_{k+1}^{-'}((1-\lambda)(s_{k,1}^+, m_{k,1}^+) + \lambda(s_{k,2}^+, m_{k,2}^+))$ be the sum of queue sizes at t_{k+1}^- under h_{LQHR} and h_{divide}, respectively, given the queue state $(1-\lambda)(s_{k,1}^+, m_{k,1}^+) + \lambda(s_{k,2}^+, m_{k,2}^+)$ at t_k^+. Let $\tilde{M}_{k+1}^-((1-\lambda)(s_{k,1}^+, m_{k,1}^+) + \lambda(s_{k,2}^+, m_{k,2}^+))$ and $\tilde{M}_{k+1}^{-'}((1-\lambda)(s_{k,1}^+, m_{k,1}^+) + \lambda(s_{k,2}^+, m_{k,2}^+))$ be the maximum queue sizes at t_{k+1}^- under h_{LQHR} and h_{divide}, respectively. Finally, let $\tilde{c}_k((1-\lambda)(s_{k,1}^+, m_{k,1}^+) + \lambda(s_{k,2}^+, m_{k,2}^+))$

and $\tilde{c}'_k((1-\lambda)(s^+_{k,1}, m^+_{k,1}) + \lambda(s^+_{k,2}, m^+_{k,2}))$ be the integral of the total unfinished work over the kth period under h_{LQHR} and h_{divide}, respectively. Then by Lemma 3.3,

$$\begin{aligned}
\tilde{S}^-_{k+1}\left((1-\lambda)(s^+_{k,1}, m^+_{k,1}) + \lambda(s^+_{k,2}, m^+_{k,2})\right) &\leq \tilde{S}^{-'}_{k+1}\left((1-\lambda)(s^+_{k,1}, m^+_{k,1}) + \lambda(s^+_{k,2}, m^+_{k,2})\right), \\
\tilde{M}^-_{k+1}\left((1-\lambda)(s^+_{k,1}, m^+_{k,1}) + \lambda(s^+_{k,2}, m^+_{k,2})\right) &\leq \tilde{M}^{-'}_{k+1}\left((1-\lambda)(s^+_{k,1}, m^+_{k,1}) + \lambda(s^+_{k,2}, m^+_{k,2})\right), \\
\tilde{c}_k((1-\lambda)\left(s^+_{k,1}, m^+_{k,1}) + \lambda(s^+_{k,2}, m^+_{k,2})\right) &\leq \tilde{c}'_k\left((1-\lambda)(s^+_{k,1}, m^+_{k,1}) + \lambda(s^+_{k,2}, m^+_{k,2})\right).
\end{aligned}$$

Now notice that under policy h_{divide}, subsystem 1 uses the LQHR policy with $1-\lambda$ times the resources of $\mathcal{C}$, and needs to process $1-\lambda$ times the unfinished work corresponding to the state $(s^+_{k,1}, m^+_{k,1})$. Thus, the resulting unfinished work as a function of time is just a $(1-\lambda)$-scaled version of the unfinished work function when h_{LQHR} uses all the resources of $\mathcal{C}$ to process the unfinished work corresponding to the state $(s^+_{k,1}, m^+_{k,1})$. The same argument applies to subsystem 2. Thus, we see that the RHS of the above three inequalities are simply given by

$$\begin{aligned}
\tilde{S}^{-'}_{k+1}\left((1-\lambda)(s^+_{k,1}, m^+_{k,1}) + \lambda(s^+_{k,2}, m^+_{k,2})\right) &= (1-\lambda)\tilde{S}^-_{k+1}(s^+_{k,1}, m^+_{k,1}) + \lambda\tilde{S}^-_{k+1}(s^+_{k,2}, m^+_{k,2}), \\
\tilde{M}^{-'}_{k+1}\left((1-\lambda)(s^+_{k,1}, m^+_{k,1}) + \lambda(s^+_{k,2}, m^+_{k,2})\right) &= (1-\lambda)\tilde{M}^-_{k+1}(s^+_{k,1}, m^+_{k,1}) + \lambda\tilde{M}^-_{k+1}(s^+_{k,2}, m^+_{k,2}), \\
\tilde{c}'_k\left((1-\lambda)(s^+_{k,1}, m^+_{k,1}) + \lambda(s^+_{k,2}, m^+_{k,2})\right) &= (1-\lambda)\tilde{c}_k(s^+_{k,1}, m^+_{k,1}) + \lambda\tilde{c}_k(s^+_{k,2}, m^+_{k,2}).
\end{aligned}$$

This establishes the lemma. □

To solve a dynamic programming recursion such as (3.18)-(3.19), it is usually helpful to have some special structure (such as monotonicity and convexity) in the cost-to-go functions $J_k(S^-_k, M^-_k)$. In Lemmas 3.4, 3.5, and 3.6, we have revealed some structural properties of the functions $\tilde{S}^-_{k+1}(S^+_k, M^+_k)$, $\tilde{M}^-_{k+1}(S^+_k, M^+_k)$, and $\tilde{c}_k(S^+_k, M^+_k)$ under h_{LQHR}. We will prove that these properties also hold for $J_k(S^-_k, M^-_k), k = 1, \ldots, N$, whenever $p_k \equiv \Pr(W_k = +1) \geq \frac{1}{2}$ for all $k = 1, \ldots, N$. In doing so, we will need the following key lemma.

Lemma 3.7 *Let $c : \mathbb{R}^2 \mapsto \mathbb{R}$ be a real functions of two variables s and m. Suppose c is increasing in (s, m) and convex in (s, m) over the region $\mathcal{V} \equiv \{(s, m) | 0 \leq s < \infty, s/2 \leq$*

$m \leq s\}$. Then, for any $z > 0$ and any $\frac{1}{2} \leq p \leq 1$, the function

$$f_z(s,m) = pc(s+z, m+z) + (1-p)c(s+z, \max(m, s-m+z))$$

is also increasing in (s,m) and convex in (s,m) over $\mathcal{V}$.

PROOF. We first check convexity. Note that $s+z, m+z, m$ and $s-m+z$ are all convex in (s,m). Since $\max(x,y)$ is convex and increasing in (x,y), $\max(m, s-m+z)$ is convex in (s,m). Since $(s+z, m+z)$ and $(s+z, \max(m, s-m+z))$ are both in $\mathcal{V}$ and since c is increasing and convex in its arguments, f_z is convex in (s,m).

Next, we check that f_z is increasing in (s,m). For a fixed m, f_z is increasing in s by inspection. Now fix s. If $m \geq s-m+z$ or $m \geq (s+z)/2$, f_z is clearly increasing in m. For $m \leq (s+z)/2$, we show

$$f_z(s,m) = pc(s+z, m+z) + (1-p)c(s+z, s-m+z)$$

is increasing in m. For simplicity, write $c(x)$ for $c(s+z, x)$. Let m_1 and m_2 be any two numbers satisfying $s/2 \leq m_1 < m_2 \leq (s+z)/2$. Note that since $m_1 \geq s/2$ by definition, we have $m_2 + z > m_1 + z \geq s - m_1 + z > s - m_2 + z$. Now since c is convex in its second argument fixing the first, and since convex functions must have increasing slope,

$$\frac{c(m_2+z) - c(m_1+z)}{m_2 - m_1} \geq \frac{c(s-m_1+z) - c(s-m+z)}{m_2 - m_1}.$$

Thus, $p[c(m_2+z) - c(m_1+z)] \geq (1-p)[c(s-m_1+z) - c(s-m_2+z)]$ for all $\frac{1}{2} \leq p \leq 1$. That is, $pc(s+z, m+z) + (1-p)c(s+z, s-m+z)$ is increasing in m. Since $f_z(s,m)$ is continuous at $m = (s+z)/2$, the lemma follows. $\square$

Lemma 3.7 holds for all $\frac{1}{2} \leq p \leq 1$. As we shall see below, this implies that h_{LQHR} is optimal whenever (conditioned on the fixed sequence of arrival epochs and packet lengths) each arrival occurs on the currently larger queue with probability greater than $\frac{1}{2}$. That is, for each $k = 1, \ldots, N$, $p_k \equiv \Pr(W_k = +1) \geq \frac{1}{2}$. We are now ready to prove the main theorem.

Theorem 3.2 *For each $k = 1, \ldots, N$, h_{LQHR} attains the minimum in the dynamic programming recursion of (3.18)-(3.19), whenever $p_k \equiv \Pr(W_k = +1) \geq \frac{1}{2}$ for all $k = 1, \ldots, N$.*

PROOF. We use backwards induction on k. Let $k = N$. By Lemma 3.3, h_{LQHR} minimizes $\tilde{c}_N^{h_N}(s_N^+, m_N^+)$ among all admissible policies, for all possible queue state values $(s_N^+, m_N^+) \in \mathcal{V}$ at time t_N^+. Thus, h_{LQHR} minimizes $c_N^{h_N}(s_N^-, m_N^-, w_N)$ among all admissible policies, for all $(s_N^-, m_N^-, w_N) \in \mathcal{V} \times \{+1, -1\}$ at time t_N^-. So h_{LQHR} also minimizes $\mathbb{E}_{W_N}\left\{c_N^{h_N}(s_N^-, m_N^-, W_N)\right\}$ for all $(s_N^-, m_N^-) \in \mathcal{V}$. That is, $J_N(s_N^-, m_N^-) = \mathbb{E}_{W_N}\left\{c_N^{h_{LQHR}}(s_N^-, m_N^-, W_N)\right\}$ for all $(s_N^-, m_N^-) \in \mathcal{V}$.

Next, we show that J_N is increasing and convex in (S_N^-, M_N^-). By Lemma 3.4 and 3.6, $\tilde{c}_N^{h_{LQHR}}$ is increasing and convex in (S_N^+, M_N^+). Now

$$\begin{aligned}
J_N(S_N^-, M_N^-) &= p_N \cdot c_N^{h_{LQHR}}(S_N^-, M_N^-, +1) + (1 - p_N) \cdot c_N^{h_{LQHR}}(S_N^-, M_N^-, -1) \\
&= p_N \cdot \tilde{c}_N^{h_{LQHR}}(S_N^- + z_N, M_N^- + z_N) \\
&\quad + (1 - p_N) \cdot \tilde{c}_N^{h_{LQHR}}(S_N^- + z_N, \max(M_N^-, S_N^- - M_N^- + z_N)).
\end{aligned}$$

Since $\frac{1}{2} \leq p_N \leq 1$, J_N is increasing and convex in (S_N^-, M_N^-) by Lemma 3.7.

Now assume for some $k, 1 \leq k \leq N - 2$, $J_{k+1}(S_{k+1}^-, M_{k+1}^-)$ is increasing and convex in (S_{k+1}^-, M_{k+1}^-). By Lemma 3.3, h_{LQHR} minimizes $\mathbb{E}_{W_k}\left\{c_k^{h_k}(s_k^-, m_k^-, W_k)\right\}$ among all admissible policies, for all $(s_k^-, m_k^-) \in \mathcal{V}$. Also by Lemma 3.3, h_{LQHR} minimizes $\tilde{S}_{k+1}^-$ and $\tilde{M}_{k+1}^-$ among all admissible policies, for all $(s_k^+, m_k^+) \in \mathcal{V}$. Thus, h_{LQHR} minimizes S_{k+1}^- and M_{k+1}^- among all admissible policies, for all $(s_k^-, m_k^-, w_k) \in \mathcal{V} \times \{+1, -1\}$, where $(S_{k+1}^-, M_{k+1}^-) = f_k^{h_k}(S_k^-, M_k^-, W_k)$. Since J_{k+1} is increasing in (S_{k+1}^-, M_{k+1}^-), we have for all $(s_k^-, m_k^-, w_k) \in \mathcal{V} \times \{+1, -1\}$,

$$J_{k+1}\left(f_k^{h_{LQHR}}(s_k^-, m_k^-, w_k)\right) = \min_{h_k \in H} J_{k+1}\left(f_k^{h_k}(s_k^-, m_k^-, w_k)\right).$$

Hence, for all $(s_k^-, m_k^-) \in \mathcal{V}$, h_{LQHR} minimizes $\mathbb{E}_{W_k}\left\{J_{k+1}\left(f_k^{h_k}(s_k^-, m_k^-, w_k)\right)\right\}$. We have therefore shown that h_{LQHR} attains the minimum in (3.19) at the kth step. That is, for all $(s_k^-, m_k^-) \in \mathcal{V}$,

$$J_k(s_k^-, m_k^-) = \mathbb{E}_{W_k}\left\{c_k^{h_{LQHR}}(s_k^-, m_k^-, W_k) + J_{k+1}\left(f_k^{h_{LQHR}}(s_k^-, m_k^-, W_k)\right)\right\}.$$

It remains to show that J_k is increasing and convex in (S_k^-, M_k^-). By repeating the argument above for J_N (replacing N by k), we can show that $\mathbb{E}_{W_k}\left\{c_k^{h_{LQHR}}(S_k^-, M_k^-, W_k)\right\}$ is increasing and convex in (S_k^-, M_k^-). Now

$$
\begin{aligned}
&\mathbb{E}_{W_k}\left\{J_{k+1}\left(f_k^{h_{LQHR}}(S_k^-, M_k^-, W_k)\right)\right\} \\
&= p_k \cdot J_{k+1}\left(f_k^{h_{LQHR}}(S_k^-, M_k^-, +1)\right) + (1-p_k)\cdot J_{k+1}\left(f_k^{h_{LQHR}}(S_k^-, M_k^-, -1)\right) \\
&= p_k \cdot J_{k+1}\left(\tilde{f}_k^{h_{LQHR}}(S_k^- + z_k, M_k^- + z_k)\right) \\
&\quad +(1-p_k)\cdot J_{k+1}\left(\tilde{f}_k^{h_{LQHR}}(S_k^- + z_k, \max(M_k^-, S_k^- - M_k^- + z_k))\right)
\end{aligned}
$$

where $\tilde{f}_k^{h_{LQHR}}(S_k^+, M_k^+) \equiv (\tilde{S}_{k+1}^-(S_k^+, M_k^+), \tilde{M}_{k+1}^-(S_k^+, M_k^+))$. By Lemmas 3.5 and 3.6, under h_{LQHR}, both $\tilde{S}_{k+1}^-$ and $\tilde{M}_{k+1}^-$ are increasing and convex functions of (S_k^+, M_k^+). Since J_{k+1} is increasing and convex in its arguments, it can be verified that $J_{k+1} \circ \tilde{f}_k^{h_{LQHR}}$ is increasing and convex in (S_k^+, M_k^+). Now since $\frac{1}{2} \le p_k \le 1$, we may apply Lemma 3.7 and conclude that $\mathbb{E}_{W_k}\left\{J_{k+1}\left(f_k^{h_{LQHR}}(S_k^-, M_k^-, W_k)\right)\right\}$ is increasing and convex in (S_k^-, M_k^-). Thus, we have shown that $J_k(S_k^-, M_k^-)$ is increasing and convex in (S_k^-, M_k^-). □

By Lemma 3.2 and Theorem 3.2, we have the following corollary.

Corollary 3.2 *Consider the set of sample paths for which $0 < t_1 < t_2 < \cdots < t_{n-1} < t_n < \cdots$ are the arrival epochs and $z_1, z_2, \ldots, z_{n-1}, z_n \ldots$ are the lengths of the packets arriving at the corresponding times. Let $T > 0$ be given and let $N(T) = \max\{k | t_k < T\}$. Let $p_k, k = 1, \ldots, N(T)$, be as defined in (3.11). Then, whenever $p_k \ge \frac{1}{2}$ for all $k = 1, \ldots, N(T)$, then*

$$J_{h_{LQHR}}(S_1^-, M_1^-) = J_{h^*}(S_1^-, M_1^-) = J^*(S_1^-, M_1^-) = J_1(S_1^-, M_1^-).$$

where $J_{h^*}(S_1^-, M_1^-)$, $J^*(S_1^-, M_1^-)$, and $J_1(S_1^-, M_1^-)$ are given by (3.15), (3.17), and (3.18)-(3.19), respectively. That is, h_{LQHR} minimizes (3.13) among all $h \in H$.

So far, we have considered our problem within the finite time horizon $[0, T]$ and we have fixed the arrival epochs and the lengths of the packets arriving at those epochs. The only randomness we have allowed lies in the W_k's. We would now like to place our problem in a more stochastic setting. We continue to assume that sources 1 and 2 both generate packets according to independent Poisson processes with a common parameter λ, and that all packets of both sources are i.i.d. according to distribution $F_Z(z)$ satisfying $\mathbb{E}[Z] < \infty$.

For a fixed policy $h \in H$, let $(U^h_{[1]}(t), U^h_{[2]}(t))$ or equivalently $(S^h(t), M^h(t))$ be the resulting joint queue state at time $t, t \geq 0$.

For any given sequence of arrival epochs $0 < t_1 < t_2 < \cdots < t_{n-1} < t_n < \cdots$, define W_k and $p_k, k = 1, 2, \ldots$ as in (3.10) and (3.11). Under our assumptions regarding the arrival process and packet lengths, $p_k = \frac{1}{2}$ for all $k = 1, 2, \ldots$. Then by Corollary 3.2, for any $T > 0$, h_{LQHR} minimizes

$$\frac{1}{T} \int_0^T \mathbb{E}_{W_1, \ldots, W_{N(T)}} \left\{ S^h(t) dt \right\}. \tag{3.20}$$

among all $h \in H$, for every realization of the arrival processes and packet lengths. Thus, h_{LQHR} minimizes (3.20) where $S^h(t)$ is now regarded as a random function of the arrivals and the packet lengths of the arrivals.

Now for fixed λ and fixed packet length distributions $F_Z(z)$, consider the set H_{stable} of policies $h \in H$ for which the multiaccess queueing system is stable, *i.e.* $S^h(t)$ hits 0 with probability 1 and the expected length of the busy period under h is finite. Then, we may view the starting points of the busy periods of the system under h as a non-arithmetic (since the arrival processes are non-arithmetic)[11] renewal process, and $\mathbb{E}_{W_1, \ldots, W_{N(T)}} \left\{ S^h(t) dt \right\}$ as a renewal reward function. In this case,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}_{W_1, \ldots, W_{N(T)}} \left\{ S^h(t) dt \right\} \quad \text{and} \quad \lim_{t \to \infty} \mathbb{E} \left\{ S^h(t) \right\} \tag{3.21}$$

both exist and are equal [Gal96]. The expectation in the second limit of (3.21) is with respect to the joint arrival process and the packet lengths. Notice that whenever $H_{stable} \neq \emptyset$, $h_{LQHR} \in H_{stable}$. We then have the following corollary.

Corollary 3.3 *Consider a two-user multiaccess queueing system where the arrival processes are independent Poisson with parameter λ, and all packets of both sources are i.i.d. according to distribution $F_Z(z)$ satisfying $\mathbb{E}[Z] < \infty$. Then if $H_{stable} \neq \emptyset$, h_{LQHR} minimizes $\lim_{t \to \infty} \mathbb{E} \left\{ S^h(t) \right\}$.*

We now note that the results in this section would continue to hold if the arrival processes are not necessarily Poisson, but nevertheless satisfy the condition that $p_k \geq \frac{1}{2}$ for all $k =$

[11] An inter-arrival distribution is said to *arithmetic* if arrivals occur only at integer multiples of some real number d.

$1, 2, \ldots$. For instance, consider a situation where the arrivals to the two queues are generated by a single renewal process $\{A(t); t \geq 0\}$ and the packet lengths corresponding to the arrivals are i.i.d. according to distribution $F_Z(z)$ satisfying $\mathbb{E}[Z] < \infty$. For each arrival, a switch uses the outcome of a fair coin flip to decide whether the arrival enters queue 1 or queue 2. Then if the multiaccess queueing system is stable, h_{LQHR} still minimizes $\lim_{t\to\infty} \mathbb{E}\left\{S^h(t)\right\}$. We can even go a step further and consider the situation where p_k is allowed to depend on the state (S_k^-, M_k^-). If for every realization of the overall arrival epochs and packet lengths, and for all $(s_k^-, m_k^-) \in \mathcal{V}$, $p_k(s_k^-, m_k^-) \geq \frac{1}{2}, k = 1, 2, \ldots$, then the conclusion of this section still holds. This rather strange situation roughly corresponds to a scenario where the arrival processes "looks at" the queue state and somehow forces more arrivals to enter queues which are already more backed up than other queues. In this case, it is intuitively clear that h_{LQHR} is the optimal policy. Since these scenarios do not seem very physically motivated, however, we do not place great emphasis on them.

3.4 Summary and Discussion

The fundamental intuition underlying this work is that the optimal queue control strategy in the given non-work-conserving setting should be based on "load-balancing" or "equalization" of queue sizes. We have shown that for the symmetric case where transmission powers, arrival rates and packet length distributions are the same for all users, versions of the LQHR policy can implement load-balancing and minimize the steady-state average packet delay and average bit delay. In general, however, LQHR may not be sufficient for load-balancing and thus optimality. We now elaborate on this point.

First, the optimality of LQHR is quite dependent on the symmetry conditions. In Section 3.2, we crucial use of symmetry in the stochastic majorization argument. In Section 3.3, the symmetry assumptions are reflected in the probability distribution of the W_k's. If the Poisson processes had different arrival rates, or if the packets from different sources had different distributions, then the conditional distribution of the W_k's (given the fixed set of arrival epochs and packet lengths) would not be Bernoulli($\frac{1}{2}$). In many cases of asymmetry, it is easy to see why LQHR would not be optimal. Consider, for example, the case of two users where the arrival rate of the first user is much larger than the arrival rate of the second user: $\lambda_1 \gg \lambda_2$. Assume that the packets are still i.i.d. and transmission powers are the same for both users. Assume that at time t, the length of the first queue is slightly smaller than the length of the second queue. In an effort to keep the queues equalized on average, the optimal controller may still allocate a higher rate to the first queue even though it has a shorter length, in expectation of a large number of arrivals on queue 1 in the near future due to a large λ_1. Thus, while there is some reason to believe that the optimal policy for the asymmetric case may still be based on thresholds, it seems that the thresholds will not be given strictly in terms of the queue state.

Second, the optimality of LQHR seems to depend on a certain "homogeneity" in the arrival processes. For the Poisson case, this homogeneity certainly holds, since given there are N Poisson arrivals in some interval $[0, T]$, the arrival epochs of these N arrivals are uniformly distributed over $[0, T]$. If arrivals were not Poisson, the control policy may need to observe not only the queue state, but also the pattern of arrivals in order to perform load-balancing, even when the symmetry conditions hold. For instance, if packet arrivals were "bursty," then a single arrival on a queue may signal the imminent arrivals of many more packets. The control policy would then need to equalize the queues by observing the sequence of arrivals as well.

Next, we give an operational interpretation to the results obtained in this work. We have assumed that the controller can assign rates at any time from anywhere in the capacity region. Thus, the objective value in (3.2) and (3.3) associated with the optimal policies g_{LQHR} and h_{LQHR} (for $M = 2$) provide lower bounds to the corresponding objective values for all coding schemes which seek to meet any given level of decoding error probability. This is because the multiaccess converse theorem puts a lower bound on error probability (via Fano's Inequality) of any coding scheme operating at rates outside the capacity region $\mathcal{C}$. This is the "converse" side of the story. On the "achievability" side, to implement an approximation of the LQHR policy requires operating at rates arbitrarily close to the dominant face $\mathcal{D}$. We know from information theory that this requires codes (block or convolutional) which operate over a large number of information bits at a time. This is not a problem when the typical arriving information packet is very long or if the arrival rate is sufficiently high that the probability of having a very small number of packets (or bits) in queue is negligible. In this case, the transmitter always has enough information bits to operate at the appropriate rate and block length. However, if the arrival rate is relatively low and the packets are short, then one may need to pad the packet with dummy information bits in order to have the code operate at rates close to capacity limits.

Finally, we discuss two other issues concerning the relationship between our queue control policies and multiaccess coding at the physical layer. First, our assumption that the controller can, at any time, allocate any rate from $\mathcal{C}$ implicitly assumes the existence of codes which can change rate at very high speeds. Whether such codes can be designed in practice is an open question. Fortunately, at least in the second version of our delay minimization problem (which is probably the more practically significant of the two problems considered), the optimal queue control policy tends to operate one of three rate regimes $((\phi_1, \phi_2), (\phi_2, \phi_1),$ or $((\phi_1 + \phi_2)/2, (\phi_1 + \phi_2)/2))$ for a significant amount of time before switching to another regime, implying that the code rate does not need to change as rapidly as one would think. Second, our identification of the LQHR policy as an adaptive successive decoding method seems to imply that the delays associated with the LQHR policies are also encountered in successive decoding schemes. This need not, however, be the case in practice. In our model, packets can exit the queue immediately after receiving enough mutual information service rate. In a practical successive decoding scheme, however, packets belonging to a user at the bottom of a particular successive decoding order cannot be decoded before the packets belonging to a user at the top of the decoding order are decoded, even if enough service rate has been offered. This implies extra packet and bit delays in practice. In any case, we can conclude that the delays associated with the LQHR policy in our model can serve as a lower bound to those achievable in practice.

In this chapter, we have tried to take a more cohesive view of networks by combining fundamental communication limits at the physical layer with QOS issues at the higher layers in a multiaccess setting. In our analysis, the rate allocation policy which minimizes average packet delay in the higher layers turn out to have significant interpretations as coding schemes at the physical layer. We believe that this "inter-layer" view may reap more dividends in many other communication contexts.

Chapter 4

Broadcasting Over Fading Channels

4.1 Introduction

In the previous two chapters, we focused on the problem of multiaccess interference among users in a communication network. We examined the multiaccess problem in the context of coding for the physical layer of the network, and with respect to quality-of-service issues such as packet delay for higher layers of the network. We now turn to the second fundamental issue in the design of wireless networks: the time-varying nature of the wireless channel. As we discussed in Chapter 1, in analyzing communication over time-varying channels, it is necessary to consider several key issues: the nature of the traffic carried by the system, the time scale of the channel fading relative to the decoding delay requirement, and the amount of information available to the transmitter and receiver regarding the state of the channel. In this work, we are primarily interested in the performance of single-user wireless systems carrying data traffic. Unlike wireless voice systems, data systems are allowed to operate at variable rates, as long as the average rate over time meets user requirements. As in voice systems, however, there is a finite decoding delay requirement placed on each transmission. The delay requirement may or may not be long compared to the time scale of the fading. As we show, this has implications for the appropriateness of various capacity metrics. Finally, we are interested in studying systems the channel state may be measured and made available to the receiver and transmitter. Out of practical considerations, we allow the channel state to be known immediately at the receiver, but only with a certain delay at the transmitter. We show that this asymmetry of channel state information (CSI) leads to an interesting transmission strategy.

We focus on a single transmitter/receiver pair[1] communicating over a slowly varying (symbol duration $T_s \ll$ channel coherence time T_{coh}) flat fading (signal bandwidth $W \ll$ channel coherence bandwidth B_{coh}) additive white Gaussian noise (AWGN) channel where the time varying channel gain stays constant over each block of T seconds, where $T \leq T_{coh}$. Discretizing this continuous time channel over the kth block ($k \in \mathbb{Z}$) by projecting on roughly $2WT$ orthonormal basis functions [Gal68], we have the discrete-time channel model

$$\boldsymbol{Y}_k = H_k \boldsymbol{X}_k + \boldsymbol{Z}_k \tag{4.1}$$

where $N = 2WT$, $\boldsymbol{X}_k = (X_{k1}, \ldots, X_{kN})$ and $\boldsymbol{Y}_k = (Y_{k1}, \ldots, Y_{kN})$ take values in $\mathbb{R}^N$ and represent the inputs and outputs of the channel over the kth block. $\boldsymbol{Z}_k$ is a Gaussian random vector with mean zero and covariance matrix $\sigma^2 \mathsf{I}_N$ with the process $\{\boldsymbol{Z}_k\}$ being i.i.d. Here, I_N is the N by N identity matrix. Assume $\mathcal{H}$, the space of the channel state process $\{H_k\}$, is a finite subset of $\mathbb{R}$ and that $\min \mathcal{H} > 0$. Moreover, assume that $\{H_k\}$ is *stationary* and *ergodic*, and that the channel state for block k is independent of channel inputs up to block k when conditioned on channel states for previous blocks. Specifically, let $H_{-\infty}^{k-1}$ denote the sequence $(\ldots, H_{k-n}, \ldots, H_{k-2}, H_{k-1})$. Then for any $h_k \in \mathcal{H}$, $h_{-\infty}^{k-1} \in \mathcal{H}_{-\infty}^{k-1}$, $\boldsymbol{x}_{-\infty}^{k} \in \mathbb{R}_{-\infty}^{k}$,

$$\Pr\left(H_k = h_k | H_{-\infty}^{k-1} = h_{-\infty}^{k-1}, \boldsymbol{X}_{-\infty}^{k} = \boldsymbol{x}_{-\infty}^{k}\right) = \Pr\left(H_k = h_k | H_{-\infty}^{k-1} = h_{-\infty}^{k-1}\right).$$

This is a particular example of the *block fading channel* [CTB99, OSW94]. Even though we present the block fading channel mainly in the time domain, it has equally useful representations in frequency (as in a multi-carrier system) and in time-frequency (as in a slow frequency hopping system) [CTB99].

Assume that during the kth block, the receiver has perfect knowledge of the channel gains $H_{-\infty}^{k}$ for all the blocks up to and including the current one.[2] The transmitter, on

[1] The transmitter and receiver are each assumed to have only one antenna.

[2] A slowly-varying flat-fading channel is underspread in the sense that the product of the multipath spread T_m and the doppler spread B_d is much less than one. Under such conditions, the channel is easily measured [BPS98] when the signal-to-noise ratio (SNR) is large enough. Thus, our assumption on CSI is reasonable when the SNR is sufficiently large.

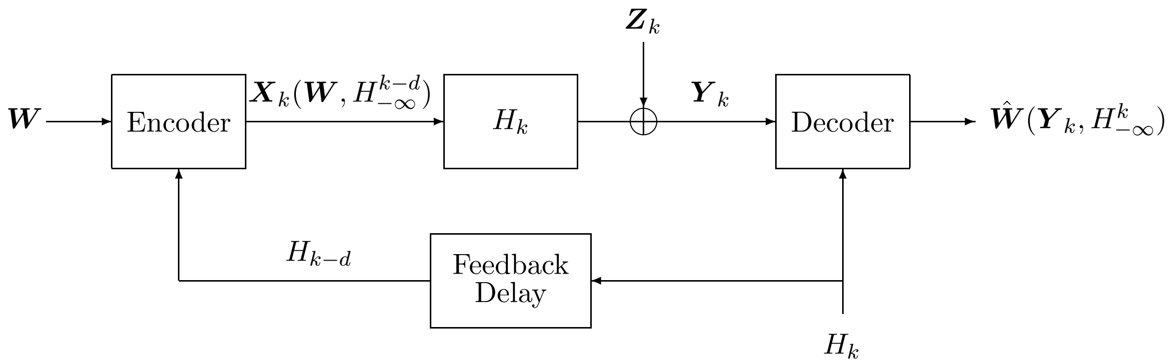

Figure 4-1: Diagram of the block fading AWGN channel with delayed channel state feedback. The decoder is allowed to decode at different rates according to the channel state. The decoding delay constraint is KN symbols.

the other hand, has perfect knowledge only of the channel gains which are at least d blocks previous to the current block, where $d \geq 1$, via *delayed* noiseless feedback. That is, it knows $H_{-\infty}^{k-d}$ at the kth block. Next, suppose the system has a maximum allowable decoding delay of Δ seconds. We assume $\Delta = KT$ where K is an integer. In terms of channel symbols, the delay constraint is KN. A codeword of length KN symbols is referred to as a *frame* [KH00]. Since a frame is encoded across K channel realizations, the parameter K measures the amount of time diversity present in the system. For the multi-carrier and frequency hopping examples, K measures frequency and time-frequency diversity. There, K is given by the bandwidth constraint. Assume that after receiving a whole frame of symbols, the decoder is forced to decode as much of the frame as possible, and declare "erasure" on those parts of the frame that it is unable to decode.[3]

Note that if there is no decoding delay constraint Δ, that is, $N \leq \infty$ and K can be arbitrarily large, the (ergodic) capacity of the block fading channel is well-defined under an average transmitted power constraint. The capacity with perfect CSI at the receiver and no CSI at the transmitter is achievable by a "single-codebook, constant-power" scheme. The capacity with perfect instantaneous CSI at both the transmitter and the receiver is achievable via a "waterfilling" power control strategy (over the channel states) with either a "single-codebook, variable-power" or a "variable-rate, variable-power" scheme [CTB99, GV97]. Finally, the capacity in the case of ideal CSI available to the receiver with noiseless delayed feedback to the transmitter has been found in [Vis99] assuming the process $\{H_k\}$

[3]The undecoded information is regarded as lost and is not retransmitted.

is Markov.

The presence of a decoding delay constraint (finite Δ) forces K to be finite and fixed. In this case, arbitrarily long code lengths are no longer possible and the ergodic capacity may not be defined. It is here that the notions of capacity versus outage and delay-limited capacity become important [Ber00, BPS98, CTB99, KH00, HT98] These concepts are natural for applications such as real-time speech, where the system seeks to guarantee both a constant rate and a maximum decoding delay per transmission.

This work takes a different perspective on delay-limited communications. The main application of interest here is time-sensitive data transmission (*e.g.* stock quotes, weather reports) where the system often has no need to guarantee a constant rate per transmission, but must observe strict decoding delay constraints and ensure low error probability. In such a setting, the system may allow the decoder to decode at different rates according to the observed channel state so that the "reliably received rate" is a random variable. A reasonable goal is then to maximize the *expected reliably received rate* [EG98] (where the expectation is over the fading process $\{H_k\}$) over the block fading channel, subject to a decoding delay constraint of KN channel symbols, and an average power constraint over KN symbols:

$$\frac{1}{KN}\sum_{k=1}^{K}\sum_{n=1}^{N}\mathbb{E}[X_{kn}^2] \leq P \tag{4.2}$$

and low error probability.

To make the expected rate problem statement more precise, one would like to resort to an information theoretic setting and obtain a capacity-like quantity with a coding theorem and a converse. It is possible to rigorously obtain converses via Fano's inequality which place upper bounds on achievable expected rates when arbitrarily small error probability is required, for a given *fixed and finite block length.* Converses are, in fact, the main focus of this section. Achievability, on the other hand, is more problematic since one cannot obtain arbitrarily small probability of error with a finite block length. Indeed, for a fixed finite N, the best bound on error probability is given by an exponential dependence on the error exponent. Unfortunately, for rates very close to capacity, the exponent is near zero and the required code length for a given error probability requirement may be very large [Gal68].

One way to deal with this difficulty is to study the *expected capacity* of a *compound* *channel* (with prior probabilities) corresponding to the block fading channel. Here, we follow the approach of [Ber00]. Consider a compound channel consisting of a family of channels $\{\Gamma(\theta) : \theta \in \Theta_K\}$ indexed by the set $\Theta_K \subset \mathbb{R}^K$, where Θ_K is the set of all length K sequences of channel gains $\theta = \{h_1, h_2, \ldots, h_K\}$ which occur with positive probability π_θ under the (stationary) joint distribution of $(H_1, H_2, \ldots, H_K)$. Let π_θ be the *a priori* probability associated with $\theta \in \Theta_K$. Now let $N \geq 1$ be fixed. Consider the *memoryless* vector Gaussian channel

$$\hat{\boldsymbol{Y}}_n = \mathsf{H}\hat{\boldsymbol{X}}_n + \hat{\boldsymbol{Z}}_n \tag{4.3}$$

where $n = 1, \ldots, N$, $\hat{\boldsymbol{X}}_n = (X_{1n}, \ldots, X_{Kn})$ and $\hat{\boldsymbol{Y}}_n = (Y_{1n}, \ldots, Y_{Kn})$ are vectors made up of the nth components of the vectors $\boldsymbol{X}_k$ and $\boldsymbol{Y}_k, k = 1, \ldots, K$, respectively, in (4.1). The vector $\hat{\boldsymbol{Z}}_n$ is a K-dimensional Gaussian random vector with zero mean and covariance matrix $\sigma^2 \mathsf{I}_K$. The random matrix $\mathsf{H} \equiv \mathrm{diag}(H_1, \ldots, H_K)$. It is clear that for each $N \geq 1$, there is a one-to-one map which takes an instance of the block-fading channel in (4.1) under a particular realization of the fading process $(H_1, \ldots, H_K) = (h_1, \ldots, h_K) = \theta$ to a corresponding instance $\Gamma(\theta)$ of the vector Gaussian channel in (4.3). With this correspondence, we can prove a coding theorem for the sequence of block fading channels in (4.1) indexed by the block length $N = 1, 2, \ldots$ by proving a corresponding coding theorem for the compound channel $\{\Gamma(\theta) : \theta \in \Theta_K\}$. Thus, by permitting N to get arbitrarily large, one can find the expected capacity of the block fading channel by finding the expected capacity of the underlying compound channel with prior probabilities.

The above process, however, presents a physical problem. For taking $N = 2WT$ arbitrarily large in the block fading channel implies having arbitrarily large coherence times (for a fixed bandwidth W). Since the coherence time of the actual channel is fixed, the process of taking N to infinity is physically meaningless. On the other hand, for a typical practical system, N is fairly large. For instance, $N = 114$ in GSM systems [CKH98] and $N = 320$ in the IS-54 standard [OSW94]. Thus we shall assume that N is large enough for reliable communication but still small compared to the coherence time in channel symbols, and expect the theoretical performance limits obtained from the compound channel model by letting $N \to \infty$ to give a reasonable indication of what is achievable in practical systems. In the course of the section, we point out strategies to approach expected capacity in this sense. We will also analyze the minimal achievable error probability performance of these capacity-approaching strategies for each finite N.

An intuitive and reasonable approach to the problem of maximizing expected rates over the block fading channel with delayed channel state feedback is the broadcast strategy first proposed by Cover [BC74, Cov72]. In this approach, states in a fading channel are associated with corresponding receivers in a broadcast channel, and the idea of superposition coding is used to "successfully refine" decoded information according to channel state.[4] The broadcast approach has recently been applied to the case of a flat fading Gaussian channel with no dynamics ($N \to \infty$ and $K = 1$) [BPS98, Sha97]. In that work, an attempt was made to combine the concepts of broadcast and outage probability and thereby generalize the notion of capacity versus outage. The approach in [Sha97] is to use superposition coding where the power allocation among the subcodes is determined by maximizing the expected capacity conditioned on the channel gain being larger than some threshold level. In this way, expected capacity is traded off against a notion of outage [Sha97].

We adopt the broadcast viewpoint and present a precise analysis of the broadcast strategy as applied to the block fading channel. Our work differs from [Sha97] in that we are interested only in maximizing expected rates. Moreover, we shall utilize tools and known results from the literature of degraded broadcast channels to deal with the case where the block fading channel does experience some dynamics ($K > 1$). To adopt notation more consistent with the traditional literature in broadcast channels, we use the following equivalent model (in the sense that the maximum achievable expected rate of the two channels are the same) of the block fading AWGN channel in (4.1). The equivalent channel is obtained by simply dividing both sides of (4.1) by H_k, yielding

$$\boldsymbol{Y}'_k = \boldsymbol{X}_k + \boldsymbol{Z}'_k \tag{4.4}$$

where $\boldsymbol{X}_k, \boldsymbol{Y}'_k = \boldsymbol{Y}_k/H_k$ are the inputs and outputs of the channel over the kth block, $\boldsymbol{Z}'_k = \boldsymbol{Z}_k/H_k$ is a Gaussian random vector with mean zero and covariance matrix $S_k I$. Here the noise variance process $\{S_k\}$ is assumed to be stationary and ergodic, and the $\boldsymbol{Z}'_k$'s are independent. All assumptions regarding CSI and feedback carry over directly from the

[4]Note that a system designer can dovetail this approach to channel coding with the corresponding "successive refinement" source coding technique, where the rate-distortion trade-off is determined by the channel realization [BPS98, Rim94].

fading states H_k in (4.1) to the noise variances S_k in (4.4). We refer to the channel in (4.4) as the *block Gaussian channel*. For simplicity, we will drop the primes in (4.4) for the analysis in the rest of the section.

In Section 4.2, we state results for the case of $K = 1$ (decoding delay constraint of one N-block), which show that a broadcast strategy based on channel history (available via delayed feedback) maximizes the expected rate for the case where the noise variance changes according to a stationary and ergodic process from one N-block to the next. This is a natural generalization of Bergman's results for degraded broadcast channels [Ber74] (which is applicable to the case where the noise variance process is i.i.d.). In proving this result, we demonstrate a useful technique, first introduced by El Gamal [Gam80], which turns out to also yield the main result for the case of $K = 2$. In addition, we give an analysis of the error probability performance of successive decoders used in superposition codes implementing the broadcast strategy. Finally, we assess the performance of a practical broadcast strategy utilizing only a finite segment of the fed-back channel history.

In Section 4.3, we turn to our main result and show that when $K = 2$ (decoding delay constraint of two N-blocks), the broadcast strategy with superposition coding again maximizes the expected rate for the case where the noise variance changes according to a two-state i.i.d. process. This is somewhat surprising since the underlying parallel Gaussian broadcast channels in this case are not degraded in the same direction. Our result requires new analysis of two-parallel Gaussian broadcast channels investigated by El Gamal [Gam80] and Poltyrev [Pol77], and is not a direct extension of Bergmans's results [Ber74]. Finally, we demonstrate that a technique developed in [Tse99] can be used to obtain the optimal power splitting parameters for the superposition coding strategy in the two-block case.

4.2 Decoding Delay of One Block ($K = 1$)

Cover first advocated the use of a broadcast strategy for communication over composite channels [BC74, Cov72]. Under the broadcast strategy the transmitter regards the block Gaussian channel as a degraded broadcast channel (DBC) with common information where each noise variance level is associated with a receiver in the DBC. The transmission strategy is easily illustrated for the two-state i.i.d. case with noise variances $\eta_1 > \eta_2$. The transmitter sends only common information (coarse bits) to the receiver with high noise variance η_1 and additional independent information (fine bits) to the good receiver with low noise variance η_2. The (actual) receiver observes the state of the channel for the current block. If the noise variance is high, it decodes only the coarse bits and marks the packet of fine bits as "erased." If the noise variance is low, it decodes both the coarse and fine bits. This strategy can be implemented using superposition coding.

4.2.1 A Converse Theorem for Stationary Channels

For the case of $K = 1$, the above strategy is clearly achievable, but is it the best possible given the channel model? In the i.i.d. case, the answer is in fact implied by the converse to the DBC given by Bergmans [Ber74]. The maximum expected rate per block is attained by a broadcast strategy which associates noise variance levels in the block Gaussian channel with corresponding receivers in a degraded broadcast channel. The optimal power splitting parameters of the superposition code are chosen according to the probabilities of the current channel state assuming the various possible values. In what follows, we consider a more general situation where the noise power of the block Gaussian channel varies according to a stationary and ergodic process $\{S_k, k \in \mathbb{N}\}$. We find that within a class of coding systems where the decoded rate depends on the channel state, the broadcast strategy again attains the maximum expected rate. The optimizing power splitting parameters of the superposition code are now chosen, quite intuitively, according to the *conditional* probabilities of the current channel state given all previous channel states available via feedback to the transmitter.

Consider a block Gaussian channel with noiseless state feedback delayed by d blocks. Let the noise variance process $\{S_k\}$ have finite state space $\mathcal{S} = \{\eta_1, \ldots, \eta_L\}$, $\eta_1 > \cdots > \eta_L > 0$. An $(N, R_1, \ldots, R_L)$ code for the kth block of this channel consists of the following.

(a) Index sets $\mathcal{W}_1, \ldots, \mathcal{W}_L$, where $\mathcal{W}_l = \{1, \ldots, M_l\}, M_l \equiv \lceil e^{NR_l} \rceil, 1 \leq l \leq L$.

(b) An encoding function that maps L sets of messages to channel input words of length N given the channel states up to block $k - d$

$$f : \mathcal{W}_1 \times \cdots \times \mathcal{W}_L \times \mathcal{S}_{-\infty}^{k-d} \mapsto \mathbb{R}^N, \quad f(W_1, \ldots, W_L, s_{-\infty}^{k-d}) = \boldsymbol{X}_k \equiv (X_{k1}, \ldots, X_{kN}).$$

(c) A set of decoding functions $g_1, \ldots, g_L$, where g_l is the decoding function when $S_k = \eta_l$.

The function $g_l, 1 \leq l \leq L$, maps a received sequence of N channel outputs over the kth block and channel states up to block $k-1$ to the first l sets of messages

$$g_l : \mathbb{R}^N \times \mathcal{S}_{-\infty}^{k-1} \mapsto \mathcal{W}_1 \times \cdots \mathcal{W}_l, \;\; g_l(\boldsymbol{Y}_k, s_{-\infty}^{k-1}) = (\hat{W}_1, \ldots, \hat{W}_l)$$

where $\boldsymbol{Y}_k \equiv (Y_{k1}, \ldots, Y_{kN})$.

For each $l = 1, \ldots, L$, define the probability of decoding error in state l averaged over all messages

$$P_{el} = \frac{1}{M_1 \cdots M_L} \sum_{(w_1, \ldots, w_L) \in \mathcal{W}_1 \times \cdots \times \mathcal{W}_L} \int_{Y_k^c} p_{\boldsymbol{Y}|\boldsymbol{X},S} \left(\boldsymbol{y}_k | f(s_{-\infty}^{k-d}, w_1, \ldots, w_L), \eta_l \right) d\boldsymbol{y}_k \quad (4.5)$$

where $Y_k^c \equiv \{\boldsymbol{y}_k : g_l(\boldsymbol{y}_k, s_{-\infty}^{k-1}) \neq (w_1, \ldots, w_l)\}$ and

$$p_{\boldsymbol{Y}|\boldsymbol{X},S}(\boldsymbol{y}|\boldsymbol{x}, \eta) = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi\eta}} \exp\left\{ \frac{-(y_n - x_n)^2}{2\eta} \right\}.$$

For any $k \in \mathbb{Z}$, we shall require an $(N, R_1, \ldots, R_L)$ code for the kth block of this channel to satisfy the following average power constraint.[5]

$$\frac{1}{N} \sum_{n=1}^{N} \mathbb{E}\left[X_{kn}^2 | S_{-\infty}^{k-d} = s_{-\infty}^{k-d} \right] \leq P \quad (4.6)$$

for *any sample history* $s_{-\infty}^{k-d} \in \mathcal{S}_{-\infty}^{k-d}$, where the expectation is with respect to the probability measure on the codewords. Note that (4.6) is somewhat stringent in the sense that we do not allow the encoder/transmitter to vary the average energy for $\boldsymbol{X}_k$ based on the observed history $s_{-\infty}^{k-d}$. A more relaxed power constraint would allow for a power allocation function $P_k : \mathcal{S}_{-\infty}^{k-d} \mapsto \mathbb{R}^+$ where $P_k(s_{-\infty}^{k-d})$ is the average energy for $\boldsymbol{X}_k$ given observed channel history $s_{-\infty}^{k-d}$. In this case, $P_k(s_{-\infty}^{k-d})$ replaces P in the RHS of (4.6). Any admissible power function P_k, of course, must still satisfy the constraint $\mathbb{E}_{S_{-\infty}^{k-d}}[P_k(S_{-\infty}^{k-d})] \leq P$.[6] Finding the

[5]In practical systems, one uses amplifiers with a certain degree of backoff known as the peak-to-average ratio. Excessive excursions in the peak signal lead to deterioration of the amplifier which is unable to dissipate the heat produced quickly enough. This seems to justify the use of a per-codeword power constraint of the type $\frac{1}{N}\sum_{n=1}^{N} x_{kn}^2 \leq P$ for every codeword $\boldsymbol{x}_k = (x_{k1}, \ldots, x_{kN})$. It turns out the expected capacity is the same under the expected constraint in (4.6) or under a per-codeword constraint.

[6]This is the distinction made in [CTB99] between long-term and short-term power constraints.

maximum expected rate with this general power constraint seems to be a difficult problem; we do not treat it here.

We state the following converse concerning the maximum expected rate per block of an $(N, R_1, \ldots, R_L)$ code. For convenience, define $C(x) \equiv \frac{1}{2}\ln(1+x)$.

Theorem 4.1 *Consider a block Gaussian channel with an average transmit power constraint P according to* (4.6) *and noise power varying according to a stationary process $\{S_k, k \in \mathbb{Z}\}$ with state space $\mathcal{S} = \{\eta_1, \ldots, \eta_L\}$, $\eta_1 > \eta_2 > \ldots > \eta_L$. Suppose the decoding delay constraint is one block of N symbols and noiseless channel state feedback to the transmitter is delayed by d blocks. For any $(N, R_1, \ldots, R_L)$ code, if P_{el} is required to be arbitrarily small for every $l = 1, \ldots, L$, the expected rate per block must satisfy*

$$\mathbb{E}[R] \leq \mathbb{E}_{S_{-\infty}^{k-d}} \left[\sum_{l=1}^{L} Q_l \left(S_{-\infty}^{k-d} \right) C \left(\frac{\alpha_l^*(S_{-\infty}^{k-d})P}{\sum_{j>l} \alpha_j^*(S_{-\infty}^{k-d})P + \eta_l} \right) \right] \tag{4.7}$$

where $Q_l\left(s_{-\infty}^{k-d}\right) \equiv \sum_{j=l}^{L} \Pr\left(S_k = \eta_j | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}\right)$, *and* $\boldsymbol{\alpha}^*(s_{-\infty}^{k-d}) \equiv \left(\alpha_1^*\left(s_{-\infty}^{k-d}\right), \ldots, \alpha_L^*\left(s_{-\infty}^{k-d}\right)\right)$ *maximizes*

$$\sum_{l=1}^{L} Q_l\left(s_{-\infty}^{k-d}\right) C\left(\frac{\alpha_l P}{\sum_{j>l}\alpha_j P + \eta_l}\right) \tag{4.8}$$

subject to $\alpha_l \geq 0, \sum_{l=1}^{L} \alpha_l = 1$.

It is important to note that Theorem 4.1 and other converses to follow apply for any block length N and are *not asymptotic* results as $N \to \infty$. Also, notice that the converse applies for any stationary block Gaussian channel, and does not require ergodicity. When we consider coding schemes to approach the upper bounds stated in Theorem 4.1, however, we shall need ergodicity.

A number of auxiliary results are needed to prove the theorem. First, we show that that the quantity $\Pr\left(S_k = \eta | S_{-\infty}^{k-d}\right)$ in Theorem 4.1 is almost surely well-defined. To this end, we state the following lemma due to Khinchin.

Lemma 4.1 (*Khinchin*) *Let $\{S_k,\ k \in \mathbb{Z}\}$ be a stationary process over a finite alphabet $\mathcal{S}$. Define $P_{d,n}(\eta) \equiv \Pr\left(S_0 = \eta | S_{-d}, \cdots, S_{-(d+n)}\right)$, where $\eta \in \mathcal{S}$. Then the sequence $\{P_{d,n}(\eta), n \geq 1\}$ is a martingale.*

PROOF. A proof in [Kin57] shows $\{P_{1,n}(\eta), n \geq 1\}$ is a martingale. For a simpler proof, let $(\Omega, \mathcal{F}, P)$ be our probability triple. Let $\mathcal{F}_n \equiv \sigma(S_{-d}, \ldots, S_{-d-n})$ be the σ-algebra generated by random variables $S_{-d}, \ldots, S_{-d-n}$. Then $\{\mathcal{F}_n : n \geq 1\}$ is a filtration. Now for every $X \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$, $\{\mathbb{E}[X|\mathcal{F}_n], n \geq 1\}$ is a martingale [Wil91]. Letting $X = 1_{\{S_0=\eta\}}$ completes the proof. □

Since $\{P_{d,n}(\eta), n \geq 1\}$ is a bounded martingale, it converges almost everywhere to a random variable by Doob's theorem (see for instance [Wil91]). Thus, $P(S_k = \eta | S_{-\infty}^{k-d})$ in Theorem 4.1 is well-defined, almost surely. In fact, since we have a sequence of probabilities, $\{P_{d,n}(\eta), n \geq 1\}$ is a martingale bounded in $\mathcal{L}^2$, and we may further deduce that the sequence converges in $\mathcal{L}^2$ [Wil91].

Lemma 4.2 *For $\eta \in \mathcal{S}$, $P_{d,\infty}(\eta) \equiv \lim_{n\to\infty} P_{d,n}(\eta)$ exists almost surely and $P_{d,n}(\eta) \to P_{d,\infty}(\eta)$ in $\mathcal{L}^2$. That is,* $\lim_{n\to\infty} \mathbb{E}\left[(P_{d,\infty}(\eta) - P_{d,n}(\eta))^2\right] = 0$.

Next, we need a lemma due to Bergmans [Ber74] which is sometimes referred to as the conditional entropy power inequality. Define $g(\eta) \equiv \frac{1}{2} \ln 2\pi e \eta$ as in [Ber74].

Lemma 4.3 (*Bergmans*) *Consider the ensemble $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{W})$, where $\boldsymbol{X}$ is a continuous random vector in $\mathbb{R}^N$ and $\boldsymbol{W}$ is a set of conditioning random variables. Let $\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{Z}$ where $\boldsymbol{Z}$ is a N-vector of i.i.d. Gaussian random variables with variance η. If $H(\boldsymbol{X}|\boldsymbol{W}) \geq Nv$, then*

$$H(\boldsymbol{Y}|\boldsymbol{W}) \geq N g(g^{-1}(v) + \eta).$$

If $H(\boldsymbol{Y}|\boldsymbol{W}) \leq Nv$ with $v > g(\eta)$, then

$$H(\boldsymbol{X}|\boldsymbol{W}) \leq N g(g^{-1}(v) - \eta).$$

PROOF. The proof follows from the usual form of the entropy power inequality [CT91] and application of Jensen's inequality to the function $g(g^{-1}(v) + \eta)$, which is increasing and convex in v, and the function $g(g^{-1}(v) - \eta)$, which is increasing and concave in v. □

The following proof of Theorem 4.1 uses Fano's inequality and the conditional entropy power inequality given in Lemma 4.3. The argument illustrates a technique, first used by El Gamal [Gam80], which "pegs" certain conditional entropies via scalar parameters and links upper bounds on various rates using the entropy power inequality. We shall see that the same technique yields the desired results in the two-block case to be discussed later.

PROOF of Theorem 4.1. Let $k \in \mathbb{Z}$. Suppose an $(N, R_1, \ldots, R_L)$ code for the kth of the time-varying Gaussian channel satisfies the power constraint in (4.6). Let $\boldsymbol{X}_k \equiv (X_{k1}, \ldots, X_{kN}) = f(W_1, \ldots, W_L, S_{-\infty}^{k-d})$ be the codeword input into the channel and let $\boldsymbol{Y}_k = (Y_{k1}, \ldots, Y_{kN})$ be the output of the channel over N uses. We shall obtain upper bounds on rates $R_l, l = 1, \ldots, L$, which are "linked" via the parameters $\{\alpha_l\}$:

$$\begin{aligned}
NR_1 &= H(W_1) \\
&= H(W_1|S_{-\infty}^{k-1} = s_{-\infty}^{k-1}, S_k = \eta_1) && (4.9) \\
&= H(W_1|\boldsymbol{Y}_k, S_{-\infty}^{k-1} = s_{-\infty}^{k-1}, S_k = \eta_1) + I(W_1; \boldsymbol{Y}_k|S_{-\infty}^{k-1} = s_{-\infty}^{k-1}, S_k = \eta_1) \\
&\leq h(P_{e1}) + P_{e1}\ln(M_1 - 1) + H(\boldsymbol{Y}_k|S_{-\infty}^{k-1} = s_{-\infty}^{k-1}, S_k = \eta_1) \\
&\quad -H(\boldsymbol{Y}_k|S_{-\infty}^{k-1} = s_{-\infty}^{k-1}, S_k = \eta_1, W_1) && (4.10) \\
&= \epsilon_1(P_{e1}) + H(\boldsymbol{Y}_k|S_{-\infty}^{k-d} = s_{-\infty}^{k-d}, S_k = \eta_1) \\
&\quad -H(\boldsymbol{Y}_k|S_{-\infty}^{k-d} = s_{-\infty}^{k-d}, S_k = \eta_1, W_1) && (4.11)
\end{aligned}$$

The quantity $H(W_1|S_{-\infty}^{k})$ is defined to be the almost-sure limit $\lim_{n\to\infty} H(W_1|S_{k-n}^{k})$. Similarly, $H(\boldsymbol{Y}_k|S_{-\infty}^{k})$ is the almost-sure limit $\lim_{n\to\infty} H(\boldsymbol{Y}_k|S_{k-n}^{k})$. The existence of these limits follows from Lemma 4.2 and a dominated convergence argument. In this proof, it is understood that quantities involving conditioning on the infinite past are all evaluated on a set of sample paths with probability one. The measure-zero set of sample paths on which these quantities do not exist will not affect the end result, as we will be taking expectations. With the above caveat, equation (4.9) follows from the fact that the messages are independent of the channel states. Fano's inequality gives (4.10). Since the transmitter has access only to the side information $S_{-\infty}^{k-d} = s_{-\infty}^{k-d}$,

$$\begin{aligned}
p_{\boldsymbol{Y}_k|S_{-\infty}^{k-1}, S_k, W_1}(\boldsymbol{y}_k|s_{-\infty}^{k-1}, \eta_1, w_1) &= p_{\boldsymbol{Y}_k|S_{-\infty}^{k-d}, S_k, W_1}(\boldsymbol{y}_k|s_{-\infty}^{k-d}, \eta_1, w_1), \\
p_{\boldsymbol{Y}_k|S_{-\infty}^{k-1}, S_k}(\boldsymbol{y}_k|s_{-\infty}^{k-1}, \eta_1) &= p_{\boldsymbol{Y}_k|S_{-\infty}^{k-d}, S_k}(\boldsymbol{y}_k|s_{-\infty}^{k-d}, \eta_1).
\end{aligned}$$

Thus, (4.11) follows, with $\epsilon_1(P_{e1}) \equiv h(P_{e1}) + P_{e1} \ln(M_1 - 1)$.

Next, observe that

$$\begin{aligned}
H(\boldsymbol{Y}_k | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}, S_k = \eta_1) &\leq \sum_{n=1}^{N} H(Y_{kn} | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}, S_k = \eta_1) \\
&= \sum_{n=1}^{N} H(X_{kn} + Z_{kn} | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}, S_k = \eta_1) \\
&\leq \sum_{n=1}^{N} g\left(\mathbb{E}\left[X_{kn}^2 | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}\right] + \eta_1\right) \qquad (4.12) \\
&\leq Ng\left(\frac{1}{N}\sum_{n=1}^{N} \mathbb{E}\left[X_{kn}^2 | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}\right] + \eta_1\right) \qquad (4.13) \\
&\leq Ng(P + \eta_1). \qquad (4.14)
\end{aligned}$$

Inequality (4.12) follows from

$$\mathbb{E}\left[(X_{kn} + Z_{kn})^2 | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}, S_k = \eta_1\right] = \mathbb{E}\left[X_{kn}^2 | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}\right] + \mathbb{E}\left[Z_{kn}^2 | S_k = \eta_1\right]$$

which holds by independence. Inequality (4.13) is a consequence of the concavity of g. Finally, the power constraint (4.6) enforces (4.14).

Now note that since $Ng(\eta_1) = H(\boldsymbol{Y}_k | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}, S_k = \eta_1, W_1, \ldots, W_L) \leq H(\boldsymbol{Y}_k | S_{-\infty}^{k-d} = S_{-\infty}^{k-d}, S_k = \eta_1, W_1) \leq H(\boldsymbol{Y}_k | S_{-\infty}^{k-d} = S_{-\infty}^{k-d}, S_k = \eta_1) \leq Ng(P + \eta_1)$, there exists $\gamma_1 \in [0,1]$ such that

$$H(\boldsymbol{Y}_k | S_{-\infty}^{k-d} = S_{-\infty}^{k-d}, S_k = \eta_1, W_1) = Ng(\overline{\gamma_1} P + \eta_1)$$

where $\overline{\gamma_1} \equiv 1 - \gamma_1$. Combining this with (4.11) and (4.14), we have

$$R_1 \leq C\left(\frac{\gamma_1 P}{\overline{\gamma_1} P + \eta_1}\right) + \frac{\epsilon_1(P_{e1})}{N}$$

where $\epsilon_1(P_{e1}) \to 0$ as $P_{e1} \to 0$.

Now suppose that for $i = 1, \ldots, l-1$,

$$H(\boldsymbol{Y}_k | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}, S_k = \eta_i, W_1, \ldots, W_i) = Ng(\overline{\gamma_i} P + \eta_i) \qquad (4.15)$$

have been so defined such that $0 \equiv \gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_{i-1} \leq 1$, and

$$R_i \leq C\left(\frac{(\gamma_i - \gamma_{i-1})P}{\overline{\gamma_i}P + \eta_i}\right) + \frac{\epsilon_i(P_{ei})}{N}$$

where $\epsilon_i(P_{ei}) \to 0$ as $P_{ei} \to 0$. Then,

$$\begin{aligned}
NR_l = H(W_l) &= H(W_l | S_{-\infty}^{k-1} = s_{-\infty}^{k-1}, S_k = \eta_l) \\
&= H(W_l | \boldsymbol{Y}_k, S_{-\infty}^{k-1} = s_{-\infty}^{k-1}, S_k = \eta_l) + I(W_l; \boldsymbol{Y}_k | S_{-\infty}^{k-1} = s_{-\infty}^{k-1}, S_k = \eta_l) \\
&\leq H(W_1, \ldots, W_l | \boldsymbol{Y}_k, S_{-\infty}^{k-1} = s_{-\infty}^{k-1}, S_k = \eta_l) \\
&\quad + I(W_l; \boldsymbol{Y}_k | S_{-\infty}^{k-1} = s_{-\infty}^{k-1}, S_k = \eta_l, W_1, \ldots, W_{l-1}) \\
&\leq H(W_1, \ldots, W_l | \boldsymbol{Y}_k, S_{-\infty}^{k-1} = S_{-\infty}^{k-1}, S_k = \eta_l) \\
&\quad + H(\boldsymbol{Y}_k | S_{-\infty}^{k-1} = s_{-\infty}^{k-1}, S_k = \eta_l, W_1, \ldots, W_{l-1}) \\
&\quad - H(\boldsymbol{Y}_k | S_{-\infty}^{k-1} = s_{-\infty}^{k-1}, S_k = \eta_l, W_1, \ldots, W_l). \\
&\leq \epsilon_l(P_{el}) + H(\boldsymbol{Y}_k | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}, S_k = \eta_l, W_1, \ldots, W_{l-1}) \\
&\quad - H(\boldsymbol{Y}_k | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}, S_k = \eta_l, W_1, \ldots, W_l).
\end{aligned} \tag{4.16}$$

where $\epsilon_l(P_{el}) \to 0$ as $P_{el} \to 0$ in (4.16).

By (4.15) and Lemma 4.3, $H(\boldsymbol{Y}_k | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}, S_k = \eta_l, W_1, \ldots, W_{l-1}) \leq Ng(\overline{\gamma_{l-1}}P + \eta_l)$. Then, since $Ng(\eta_l) = H(\boldsymbol{Y}_k | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}, S_k = \eta_l, W_1, \ldots, W_L) \leq H(\boldsymbol{Y}_k | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}, S_k = \eta_l, W_1, \ldots, W_l) \leq H(\boldsymbol{Y}_k | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}, S_k = \eta_l, W_1, \ldots, W_{l-1}) \leq Ng(\overline{\gamma_{l-1}}P + n_l)$, there exists $\gamma_l \in [\gamma_{l-1}, 1]$ such that

$$H(\boldsymbol{Y}_k | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}, S_k = \eta_l, W_1, \ldots, W_l) = Ng(\overline{\gamma_l}P + \eta_l).$$

It follows that

$$R_l \leq C\left(\frac{(\gamma_l - \gamma_{l-1})P}{\overline{\gamma_l}P + \eta_l}\right) + \frac{\epsilon_l(P_{el})}{N}.$$

Finally, for $l = L$, we have

$$\begin{aligned}
NR_L &\leq \epsilon_L(P_{eL}) + H(\boldsymbol{Y}_k | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}, S_k = \eta_L, W_1, \ldots, W_{L-1}) \\
&\quad - H(\boldsymbol{Y}_k | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}, S_k = \eta_L, W_1, \ldots, W_L)
\end{aligned}$$

$$\leq \quad \epsilon_L(P_{eL}) + Ng(\overline{\gamma_{L-1}}P + \eta_L) - Ng(\eta_L)$$

where $\epsilon_L(P_{eL}) \to 0$ as $P_{eL} \to 0$. Thus, we have

$$R_L \leq C\left(\frac{(1-\gamma_{L-1})P}{\eta_L}\right) + \frac{\epsilon_L(P_{eL})}{N}.$$

Setting $\alpha_l \equiv \gamma_l - \gamma_{l-1}, l = 1, \ldots, L$, with $\gamma_L \equiv 1$, we find that for $l = 1, \ldots, L$,

$$R_l \leq C\left(\frac{\alpha_l P}{(\sum_{j>l}\alpha_j)P + \eta_l}\right) + \frac{\epsilon_l(P_{el})}{N}$$

where $\alpha_l \geq 0, l = 1, \ldots, L$ and $\sum_{l=1}^{L}\alpha_l = 1$. Note that the α parameters are in fact functions of the sample history $s_{-\infty}^{k-d}$.

If we now require P_{el} to be arbitrarily small for every $l = 1, \ldots, L$ and every sample history $s_{-\infty}^{k-d}$, the expected rate per block must satisfy

$$\mathbb{E}[R] \leq \mathbb{E}_{S_{-\infty}^{k-d}}\left[\sum_{l=1}^{L} Q_l\left(S_{-\infty}^{k-d}\right) C\left(\frac{\alpha_l(S_{-\infty}^{k-d})P}{\sum_{j>l}\alpha_j(S_{-\infty}^{k-d})P + \eta_l}\right)\right]$$

where $Q_l\left(s_{-\infty}^{k-d}\right) \equiv \sum_{j=l}^{L} \Pr\left(S_k = \eta_j | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}\right)$. From this, the theorem follows. $\square$

A widely-used model for time-varying channels is the finite-state Markov channel (FSMC) [GV97, Vis99]. In this model, it is assumed that for any $\eta \in \mathcal{S}$, $s_{-\infty}^{k-1} \in \mathcal{S}_{-\infty}^{k-1}$, $\boldsymbol{x}_{-\infty}^{k} \in \mathbb{R}_{-\infty}^{k}$,

$$\begin{aligned}
\Pr\left(S_k = \eta | S_{-\infty}^{k-1} = s_{-\infty}^{k-1}, \boldsymbol{X}_{-\infty}^{k} = \boldsymbol{x}_{-\infty}^{k}\right) &= \Pr\left(S_k = \eta | S_{-\infty}^{k-1} = s_{-\infty}^{k-1}\right) \\
&= \Pr\left(S_k = \eta | S_{k-1} = s_{k-1}\right).
\end{aligned}$$

It is easy to show from this that for any $d \geq 1$,

$$\begin{aligned}
\Pr\left(S_k = \eta | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}, \boldsymbol{X}_{-\infty}^{k} = \boldsymbol{x}_{-\infty}^{k}\right) &= \Pr\left(S_k = \eta | S_{-\infty}^{k-d} = s_{-\infty}^{k-d}\right) \\
&= \Pr\left(S_k = \eta | S_{k-d} = s_{k-d}\right).
\end{aligned}$$

The following corollary is a specialization of Theorem 4.1.

Corollary 4.1 *Consider a block Gaussian channel with an average transmit power constraint P according to (4.6) and noise power varying according to an irreducible, aperiodic, homogeneous FSMC $\{S_k, k \in \mathbb{Z}\}$ with state space $\mathcal{S} = \{\eta_1, \ldots, \eta_L\}$, $\eta_1 > \eta_2 > \ldots > \eta_L$. Let $\boldsymbol{\pi} = (\pi(\eta_1), \ldots, \pi(\eta_L))$ be the (unique) steady-state probability distribution and let A be the one-step state transition probability matrix of the Markov chain. Suppose the decoding delay constraint is N symbols and noiseless channel state feedback to the transmitter is delayed by d blocks. For any $(N, R_1, \ldots, R_L)$ code, if P_{el} is required to be arbitrarily small for every $l = 1, \ldots, L$, the expected rate per block must satisfy*

$$\mathbb{E}[R] \leq \sum_{i=1}^{L} \pi(\eta_i) \left[\sum_{l=1}^{L} \left(\sum_{j=l}^{L} A^d(\eta_j, \eta_i) \right) C \left(\frac{\alpha_l^*(\eta_i) P}{\sum_{j>l} \alpha_j^*(\eta_i) P + \eta_l} \right) \right]$$

where $A^d(\eta_j, \eta_i)$ is the (j,i)th element the d-step transition probability matrix A^d and $\boldsymbol{\alpha}^*(\eta_i) = (\alpha_1^*(\eta_i), \ldots, \alpha_L^*(\eta_i))$ maximizes

$$\sum_{l=1}^{L} \left(\sum_{j=l}^{L} A^d(\eta_j, \eta_i) \right) C \left(\frac{\alpha_l(\eta_i) P}{\sum_{j>l} \alpha_j(\eta_i) P + \eta_l} \right)$$

subject to $\alpha_l \geq 0, \sum_{l=1}^{L} \alpha_l = 1$.

We shall discuss methods to solve the maximization of (4.8) in Section 4.2.4. But first, we need to gain some understanding for coding schemes which approach the upper bound on expected rate per block in (4.7).

4.2.2 Achievability Using Superposition Codes

We assume for the following that the fading process $\{S_k, k \in \mathbb{Z}\}$ is both stationary and ergodic. The main implication of Theorem 4.1 is that the upper bound on expected rate per block when $K = 1$ can be approached by a broadcast strategy implemented via superposition coding with successive decoding, where the optimal power splitting parameters are chosen according to the conditional probabilities of the current channel state given all previous channel states available via feedback. Indeed, under the broadcast strategy, it is possible to attain expected rates arbitrarily close to the upper bound in (4.7) if we allow the block length N to be arbitrarily large. This is equivalent to saying that the *expected capacity* of the corresponding compound channel (recall the correspondence established in the introduction) is the RHS of (4.7).

The broadcast strategy is briefly outlined as follows. Suppose at block k, the receiver observes $S_{-\infty}^{k-1} = s_{-\infty}^{k-1}, S_k = \eta_l$ $(1 \le l \le L)$ and the transmitter observes $S_{-\infty}^{k-d} = s_{-\infty}^{k-d}$. Let $\boldsymbol{\alpha}^*(s_{-\infty}^{k-d}) = \left(\alpha_1^*(s_{-\infty}^{k-d}), \ldots, \alpha_L^*(s_{-\infty}^{k-d})\right)$ maximize (4.8) subject to $\alpha_i \ge 0, \sum_{i=1}^{L} \alpha_i = 1$. For convenience, we drop the argument and refer to $\boldsymbol{\alpha}^*(s_{-\infty}^{k-d})$ simply as $\boldsymbol{\alpha}^*$. For a given block length N, construct an $(N, R_1, \ldots, R_L)$ superposition code with L codebooks where the ith codebook $(1 \le i \le L)$ has average power $\alpha_i^* P$ and rate

$$R_i < C\left(\frac{\alpha_i^* P}{(\sum_{j>i} \alpha_j^*)P + \eta_i}\right). \tag{4.17}$$

The ith codebook may be generated by selecting its $\lceil e^{NR_i} \rceil$ codewords by choosing each codeword independently as an i.i.d. sequence of Gaussian random variables with zero mean and variance $\alpha_i^* P$. In this way, the codewords of other codebooks appear as Gaussian noise to a given codebook.

Let f_i be the encoding function of the ith codebook. To send the message $(W_1, \ldots, W_L)$, the transmitter sends the codeword $f(W_1, \ldots, W_L) = f_1(W_1) + \cdots + f_L(W_L)$. Having observed $S_k = \eta_l$, the receiver uses a successive decoding strategy to generate message estimates $(\hat{W}_1, \ldots, \hat{W}_l)$ by first decoding $f_1(W_1)$ regarding $f_2(W_2) + \cdots + f_L(W_L)$ as noise, then subtracting $f_1(\hat{W}_1)$ from the received signal and decoding $f_2(W_2)$ regarding $f_3(W_3) + \cdots + f_L(W_L)$ as noise, and so on.

It is important to comment that since the superposition coding strategy (the selection of $\boldsymbol{\alpha}^*$ in particular) depends on the entire past history of the channel, the encoder-decoder pair may potentially use a countably infinite number of different codebooks in the general case of Theorem 4.1 (although clearly only L codebooks are needed in the FSMC case of Corollary 4.1). This is obviously infeasible in practice. In Section 4.2.5, we study the effect on expected capacity of choosing $\boldsymbol{\alpha}^*$ based on only a finite segment of the past, subject to certain assumptions about the channel state process.

4.2.3 Error Probability Performance of Successive Decoders

The minimum achievable error probability performance of the superposition coding process discussed above can be upper-bounded for any fixed N as follows. We first condition on the event $S_{-\infty}^{k-1} = s_{-\infty}^{k-1}, S_k = \eta_l$. Let an $(N, R_1, \ldots, R_L)$ superposition code be constructed as in the previous section. Assume the message tuple $(W_1, \dots, W_L)$ is uniformly chosen over its alphabet $\mathcal{W}_1 \times \cdots \times \mathcal{W}_L$. We shall analyze the probability that the successive decoder decodes $(W_1, \dots, W_l)$ incorrectly under these conditions.

Let the decoding order be $1, \dots, l$. Refer to the decoder responsible for generating the estimate $\hat{W}_i$ in the ith stage $(1 \le i \le l)$ of the decoding process as the ith decoder. For each $i = 1, \dots, l$, let E_i be the event that the ith decoder makes an error in decoding when provided with message estimates $\hat{W}_1, \dots, \hat{W}_{i-1}$. Let E_i' be the event that the ith decoder makes an error in decoding when provided with the true messages $W_1, \dots, W_{i-1}$.

Let $P_{el}(s_{-\infty}^{k-1})$ be the probability of decoding error in state l averaged all messages, conditioned on the event $S_{-\infty}^{k-1} = s_{-\infty}^{k-1}$, so that $P_{el} = \mathbb{E}_{S_{-\infty}^{k-1}}[P_{el}(S_{-\infty}^{k-1})]$ in (4.5). Then $P_{el}(s_{-\infty}^{k-1}) = \Pr\left(\bigcup_{i=1}^{l} E_i\right)$. A crucial observation is that, in addition, $\Pr\left(\bigcup_{i=1}^{l} E_i\right) = \Pr\left(\bigcup_{i=1}^{l} E_i'\right)$. This "genie-aided" argument, shown in [RU96], allows us to analyze the error probability performance of the superposition code in terms of the error performances of a number of single-user codes.

Following the usual random coding approach to studying error probabilities, we now construct an ensemble of $(N, R_1, \dots, R_L)$ superposition codes as follows. Let $\boldsymbol{\alpha}^*$ be given as in the previous section. For each $i = 1, \dots, L$, let

$$q_i(x) = \frac{1}{\sqrt{2\pi\alpha_i^* P}} \exp\left\{\frac{-x^2}{2\alpha_i^* P}\right\}$$

be the Gaussian density with zero mean and variance $\alpha_i^* P$. Let the joint density $q(x_1, \dots, x_L)$ satisfy $q(x_1, \dots, x_L) = \prod_{i=1}^{L} q_i(x_i)$. Let $R_i, i = 1, \dots, L$, be given as in (4.17). For each $i = 1, \dots, L$, generate $M_i = \lceil e^{NR_i} \rceil$ codewords by choosing each codeword $\boldsymbol{x}_m^i, 1 \le m \le M_i$, independently according to the product distribution $q_i(\boldsymbol{x}^i) = \prod_{n=1}^{N} q_i(x_n^i)$, where $\boldsymbol{x}^i = (x_1^i, \dots, x_N^i)$.

For each code in the ensemble, assume the ith decoder employs maximum likelihood (ML) decoding, for each $i = 1, \dots, l$. Note that the ML decoder minimizes error probability since we assume the messages are equally likely. Furthermore, for the ML decoder, $g_l(\boldsymbol{y}_k, s_{-\infty}^{k-1}) = g_l(\boldsymbol{y}_k, s_{-\infty}^{k-d})$ since the likelihood function depends only on $s_{-\infty}^{k-d}$. Thus, $P_{el}(s_{-\infty}^{k-1}) = P_{el}(s_{-\infty}^{k-d})$. Let $\overline{P_{el}(s_{-\infty}^{k-d})}$ be $P_{el}(s_{-\infty}^{k-d})$ averaged over the ensemble. Then

$$\overline{P_{el}(s_{-\infty}^{k-d})} = \overline{\Pr\left(\cup_{i=1}^{l} E_i'\right)} \le \overline{\Sigma_{i=1}^{l} \Pr\left(E_i'\right)} = \Sigma_{i=1}^{l} \overline{\Pr\left(E_i'\right)} \tag{4.18}$$

where $\overline{\Pr(E_i')}$ is $\Pr(E_i')$ averaged over the ensemble.

It follows from the ensemble construction that given $W_1, \ldots, W_{i-1}$, the ith decoder must estimate W_i facing i.i.d. Gaussian noise of zero mean and variance $(\sum_{j>i} \alpha_j^*)P + \eta_l$. Hence, we have can apply Gallager's bound [Gal85]:

$$\overline{\Pr(E_i')} \leq \exp[-NE_{ri}(R_i)] \tag{4.19}$$

where

$$E_{ri}(R_i) = \max_{0 \leq \rho_i \leq 1} \left[\rho_i \cdot C\left(\frac{A_i}{1+\rho_i}\right) - \rho_i R_i \right], \quad A_i = \frac{\alpha_i^* P}{(\sum_{j>i} \alpha_j^*)P + \eta_l}.$$

The function $E_{ri}(R_i)$ behaves as follows. For

$$R_i \geq C\left(\frac{A_i}{2}\right) - \frac{A_i}{4(2+A_i)} \equiv R_{cr,i},$$

$E_{ri}(R_i)$ and R_i are parametrically related in terms of ρ_i for $0 \leq \rho_i \leq 1$.

$$R_i = C\left(\frac{A_i}{1+\rho_i}\right) - \frac{A_i \rho_i}{2(1+\rho_i)(1+\rho_i+A_i)},$$
$$E_{ri}(R_i) = \frac{\rho_i^2 A_i}{2(1+\rho_i)(1+\rho_i+A_i)}.$$

For $R_i < R_{cr,i}$,

$$E_{ri}(R_i) = C\left(\frac{A_i}{2}\right) - R_i.$$

We note that by (4.17), $R_i < C(A_i)$ for all $1 \leq i \leq l$, and thus $E_{ri}(R_i) > 0$ for all $1 \leq i \leq l$. Then, combining (4.18) and (4.19), we have

$$\begin{aligned} \overline{P_{el}(s_{-\infty}^{k-d})} &\leq \sum_{i=1}^{l} \exp[-NE_{ri}(R_i)] \\ &\leq l \cdot \exp[-NE_r(R_1, \ldots, R_l)] \end{aligned}$$

where

$$E_r(R_1, \ldots, R_l) \equiv \min_{i=1,\ldots,l} E_{ri}(R_i) > 0.$$

To reflect the dependence of the error exponents on the observed channel history, let $E_{r,s_{-\infty}^{k-d}}(R_1,\ldots,R_l)$ be the minimum exponent for the exponential bound on $\overline{P_{el}(s_{-\infty}^{k-d})}$. Then, the ensemble probability of error averaged over the channel states and the channel history is bounded as

$$\mathbb{E}_{S_{-\infty}^{k-d}}\left[\sum_{l=1}^{L}\Pr\left(S_k=\eta_l|S_{-\infty}^{k-d}\right)\overline{P_{el}(S_{-\infty}^{k-d})}\right] \\ \leq \mathbb{E}_{S_{-\infty}^{k-d}}\left[\sum_{l=1}^{L}\Pr\left(S_k=\eta_l|S_{-\infty}^{k-d}\right)\ l\cdot\exp\left\{-NE_{r,S_{-\infty}^{k-d}}(R_1,\ldots,R_l)\right\}\right]. \quad (4.20)$$

Finally, we note that since the ensemble probability error is bounded as in (4.20), there must exist at least one $(N,R_1,\ldots,R_L)$ superposition code in the ensemble for which the error probability averaged over the channel states and the channel history is upper bounded by the RHS of (4.20).

4.2.4 Optimization of Expected Rate

We now return to the optimization problem in (4.8). An equivalent formulation is the following. For given probability coefficients $\boldsymbol{Q}=(Q_1,\ldots,Q_L), 1=Q_1\geq Q_2\geq\cdots\geq Q_L\geq 0$, noise variances $\eta_1>\cdots>\eta_L$, and total transmission power P, consider the problem

$$\text{maximize}\ \ \boldsymbol{Q}^T\boldsymbol{R} \quad (4.21)$$

$$\text{subject to}\ \ R_l\leq\frac{1}{2}\ln\left(1+\frac{\alpha_l P}{\sum_{j>l}\alpha_j P+\eta_l}\right),\ l=1,\ldots,L, \quad (4.22)$$

$$\alpha_l\geq 0,\forall l=1,\ldots,L,\ \sum_{l=1}^{L}\alpha_l=1. \quad (4.23)$$

Note that the constraints in (4.22) and (4.23) define the capacity region of a Gaussian broadcast channel. It follows that the feasible set $\mathcal{R}$ of rates $\boldsymbol{R}=(R_1,\ldots,R_L)$ is convex (it contains rates achieved using time-sharing). Since the objective function is linear in $\boldsymbol{R}$, any maxima are attained on the boundary $\partial\mathcal{R}$ of $\mathcal{R}$, where the $\partial\mathcal{R}$ is found by setting all inequalities in (4.22) to equalities. We can therefore recast the maximization directly in terms of the power parameters:

$$\begin{aligned} &\text{maximize} \;\; \sum_{l=1}^{L} \frac{Q_l}{2} \left[\ln\left(\sum_{j \geq l} \alpha_j P + \eta_l \right) - \ln\left(\sum_{j > l} \alpha_j P + \eta_l \right) \right] \\ &\text{subject to} \;\; \alpha_l \geq 0, \forall l = 1, \ldots, L, \; \sum_{l=1}^{L} \alpha_l = 1. \end{aligned} \tag{4.24}$$

Write the objective function as $f(\boldsymbol{\alpha})$. Then the Kuhn-Tucker necessary conditions for a global maximum $\boldsymbol{\alpha}^*$ in (4.24) are

$$\frac{\partial f(\boldsymbol{\alpha}^*)}{\partial \alpha_j} \begin{cases} = \lambda^*, & \text{for } \alpha_j^* > 0 \\ \leq \lambda^*, & \text{for } \alpha_j^* = 0. \end{cases} \tag{4.25}$$

for $j = 1, \ldots, L$, where λ^* is a scalar (whose existence is assured since the constraints are linear) and

$$\frac{\partial f(\boldsymbol{\alpha})}{\partial \alpha_j} = \sum_{l<j} \frac{Q_l}{2} \left[\frac{P}{\sum_{i \geq l} \alpha_i P + \eta_l} - \frac{P}{\sum_{i > l} \alpha_i P + \eta_l} \right] + \frac{Q_j P}{2\left(\sum_{i \geq j} \alpha_i P + \eta_j \right)}.$$

To further spell out the conditions in (4.25), let $B(\boldsymbol{\alpha}^*) = \{l | \alpha_l^* = 0\}$. For convenience, write B for $B(\boldsymbol{\alpha}^*)$ and let $b = |B|$. Note that B can be any *proper* subset of $\{1, \ldots, L\}$ and $0 \leq b < L$. If $|B^c| = L - b = 1$, then $\alpha_j^* = 1$ for some j and $\alpha_k = 0, \forall k \neq j$. Otherwise, for each $j \in B^c$ such that $j < \max B^c$, let $k = \min B^c \cap \{j+1, \ldots, L\}$. Since both α_j^* and α_k^* are positive, the first part of (4.25) says that $\partial f(\boldsymbol{\alpha}^*)/\partial \alpha_j = \partial f(\boldsymbol{\alpha}^*)/\partial \alpha_k$. It can be verified that this translates into the following condition:

$$\frac{Q_j}{2\left(\sum_{l>j} \alpha_l^* P + \eta_j \right)} = \frac{Q_k}{2\left(\sum_{l>j} \alpha_l^* P + \eta_k \right)}. \tag{4.26}$$

Necessary conditions for these $L-b-1$ equations to hold are $Q_j \neq Q_k$ and $Q_k/\eta_k > Q_j/\eta_j$ for each pair of j and k in (4.26). For the second part of (4.25), let $j \in B$. If $B^c \cap \{j+1, \ldots, L\} \neq \emptyset$, let $k = \min B^c \cap \{j+1, \ldots, L\}$. Otherwise, let $k = \max B^c \cap \{1, \ldots, j-1\}$. Since $\alpha_j^* = 0$

and $\alpha_k^* > 0$, we must have $\partial f(\boldsymbol{\alpha}^*)/\partial \alpha_j \leq \partial f(\boldsymbol{\alpha}^*)/\partial \alpha_k$. This leads to the condition

$$\frac{Q_j}{2\left(\sum_{l>j} \alpha_l^* P + \eta_j\right)} \leq \frac{Q_k}{2\left(\sum_{l>j} \alpha_l^* P + \eta_k\right)}. \tag{4.27}$$

There are b inequalities of this type.

The Kuhn-Tucker conditions (4.26)-(4.27) have revealed the significance of the decreasing and convex functions $\frac{Q_l}{2(z+\eta_l)}, l = 1, \ldots, L$. It turns out that for given parameters, these functions completely determine the optimal solution to the maximization in (4.21). This is shown in [Tse99] within the more general setting of parallel Gaussian broadcast channels. As described in Theorem 3.2 of [Tse99], the unique optimal solution to (4.21) can be explicitly obtained via a greedy procedure. We state a version of the theorem appropriate for our context.

Theorem 4.2 *(Tse) Consider the optimization problem in (4.21). Define for $l = 1, \ldots, L$ the marginal utility functions*

$$\begin{aligned} u_l(z) &\equiv \frac{Q_l}{2(z+\eta_l)} \\ u^*(z) &\equiv \max_l u_l(z) \end{aligned}$$

and sets

$$\mathcal{A}_l \equiv \{z \in [0, P] : u_l(z) = u^*(z)\}.$$

Then the optimal solution to (4.21) is

$$\int_0^P u^*(z) dz$$

attained at the unique point

$$R_l^* = \int_{\mathcal{A}_l} \frac{1}{2(z+\eta_l)} dz = \int_{\sum_{j>l} \alpha_j^* P}^{\sum_{j\geq l} \alpha_j^* P} \frac{1}{2(z+\eta_l)} dz = C\left(\frac{\alpha_l^* P}{\sum_{j>l} \alpha_j^* P + \eta_l}\right) \tag{4.28}$$

where

$$\begin{aligned}\mathcal{A}_l &= \left[\sum_{j>l} \alpha_j^* P, \sum_{j\geq l} \alpha_j^* P\right], \\ \alpha_l^* P &= |\mathcal{A}_l|\end{aligned}$$

for $l = 1, \dots, L$.

PROOF. See [Tse99]. The uniqueness of the optimal solution follows from the fact that $\eta_i \neq \eta_j$ for $i \neq j$. For this implies that u_i and u_j, $i \neq j$, can intersect at most once. So except for a finite set of intersection points, each $z \in [0, P]$ has a unique $l, 1 \leq l \leq L$, such that $z \in \mathcal{A}_l$. Thus, the solution is unique. The argument does not require assuming $Q_i \neq Q_j$ for $i \neq j$ as in [Tse99]. If $Q_i = Q_j$ for $i < j$ so that $\eta_i > \eta_j$, u_i never intersects u_j. In particular, $u_i(z) < u_j(z)$ for all $z > 0$ and $\mathcal{A}_i = \emptyset$. □

The interpretation of functions $u_l(z)$ as marginal utility functions is intimately connected to the idea of superposition coding with successive decoding. Following the association between states in a block Gaussian channel and users in a degraded broadcast channel, we refer to a single-user code within the superposition code as a "virtual user." As pointed out in [Tse99], functions $u_l(z)$ are marginal utility functions in the sense that $u_l(z)\delta P$ can be interpreted as the marginal increase $Q_l \delta R_l$ in the objective $\boldsymbol{Q}^T \boldsymbol{R}$ due to an amount δP of power given to virtual user l at interference level $z + \eta_l$. The value z can be interpreted as the amount of interference caused by virtual users with better channels (larger indices) plus the power already allocated to virtual user l. Theorem 4.2 says that the optimal solution is found in a greedy manner whereby at each power level z, the transmitter allocates power to the virtual user with the largest marginal utility function. In the following example, we demonstrate the greedy optimization procedure, and show how the Kuhn-Tucker necessary conditions (4.26)-(4.27) directly follow.

Figure 4-2 illustrates the optimization procedure for a four-state channel. Marginal utilities $u_l(z) \equiv Q_l/2(z+\eta_l), l = 1, 2, 3, 4$, corresponding to the four virtual users are plotted as functions of the interference level z. Here, we have the ordering $Q_4/\eta_4 > Q_3/\eta_3 > Q_2/\eta_2 > Q_1/\eta_1$. The figure shows that any two utility functions u_i and $u_j, i \neq j$, can intersect at most once and that sets $\mathcal{A}_l$'s are contiguous intervals. Moreover, the intervals

Figure 4-2: Optimization procedure for a four-state channel. At each power level z, the transmitter allocates power to the virtual user with the largest marginal utility function $u(z)$.

$\mathcal{A}_l$'s (which can be empty) are ordered on the positive reals in increasing η_l's (decreasing indices). These properties are found to hold in general [Tse99].

Theorem 4.2 dictates that the transmitter allocates power to the virtual user with the highest marginal utility at each interference level z. Thus, we have $\mathcal{A}_4 = [0, \alpha_4^* P], \mathcal{A}_3 = [\alpha_4^* P, (\alpha_3^* + \alpha_4^*)P], \mathcal{A}_2 = [(\alpha_3^* + \alpha_4^*)P, P], \mathcal{A}_1 = \emptyset$, and $P_l^* = |\mathcal{A}_l|, l = 1, 2, 3, 4$. The optimal rate R_l^* for virtual user l is given by (4.28). Note that virtual user 1 is given zero power and thus zero rate, since its utility function never dominates in the interval $[0, P]$. The rates R_l^* are achieved using superposition coding with successive decoding, where the order of decoding is 2, 3, 4. That is, virtual user 2 decodes its message regarding virtual users 3 and 4 as noise. Virtual user 3 decodes virtual user 2's message, subtracts it off, and then decodes its own message regarding virtual user 4 as noise. Finally, virtual user 4 decodes both 2 and 3, subtracts them off, and then decodes its own message facing only background noise.

The simple structure of Figure 4-2 also readily gives the Kuhn-Tucker necessary conditions discussed above. We have from the graph

$$\frac{Q_2}{2\left((\alpha_3^*+\alpha_4^*)P+\eta_2\right)} = \frac{Q_3}{2\left((\alpha_3^*+\alpha_4^*)+\eta_3\right)} \tag{4.29}$$

$$\frac{Q_3}{2\left(\alpha_4^*P+\eta_3\right)} = \frac{Q_4}{2\left(\alpha_4^*P+\eta_4\right)} \tag{4.30}$$

$$\frac{Q_1}{2(P+\eta_1)} \leq \frac{Q_2}{2(P+\eta_2)}. \tag{4.31}$$

But (4.29)-(4.31) are precisely the conditions (4.26)-(4.27) for this problem.

4.2.5 A Coding Strategy using Finite History of the Fading Process

A difficulty with Theorem 4.1 is that the bound is stated in terms of probabilities conditioned on the infinite past. As we showed, these probabilities are almost surely well defined. In practice, however, it is impossible to use such quantities for the purposes of encoding and decoding signals. As we now show, the bound given in Theorem 4.1 can be approached by using probabilities conditioned on the *finite* past. We shall continue to assume that $\{S_k, k \in \mathbb{Z}\}$ is both stationary and ergodic.

The first key observation, already suggested by Lemma 4.1 and 4.2, is that the probability of seeing a noise level η_l in the kth block conditioned on channel history n steps into the past is a good estimate of the probability of η_l conditioned on the infinite past, as n gets large. Indeed, by Lemma 4.2, $\{P_{d,n}(\eta), n \geq 1\}$ converges in $\mathcal{L}^2$. Thus by Chebychev's inequality, for a given noise (fading) level η_l, fixed k and given $\epsilon, \delta > 0, \exists N_0$ such that $\forall\ n \geq N_0$,

$$\Pr\left(\left|\Pr\left(S_k = \eta_l | S_{-\infty}^{k-d}\right) - \Pr\left(S_k = \eta_l | S_{k-d-n}^{k-d}\right)\right| \geq \epsilon\right) < \delta.$$

Since there are only a finite number of fading levels, we may conclude:

Lemma 4.4 *Given $\epsilon, \delta > 0$ and fixed k, $\exists N_0$ such that $\forall n \geq N_0$*

$$\Pr\left(\sum_{l=1}^{L}\left|Q_l(S_{-\infty}^{k-d}) - Q_l(S_{k-d-n}^{k-d})\right| \geq \epsilon\right) < \delta$$

where the Q_l's are defined as in Theorem 4.1.

Now recall the maximization problem in (4.8), or equivalently, in (4.21). Consider a given

$s_{-\infty}^{k-d}$ for which the conditional probability vector $\boldsymbol{Q}(s_{-\infty}^{k-d}) = \Big(Q_1(s_{-\infty}^{k-d}), \ldots, Q_L(s_{-\infty}^{k-d})\Big)$, where $1 = Q_1(s_{-\infty}^{k-d}) \geq Q_2(s_{-\infty}^{k-d}) \geq \cdots \geq Q_L(s_{-\infty}^{k-d}) \geq 0$, is well-defined. If we now replace conditional probabilities based on the infinite past with approximations based on finite history (replacing $\boldsymbol{Q}(s_{-\infty}^{k-d})$ with $\boldsymbol{Q}(s_{k-d-n}^{k-d})$ for some n), the objective function in (4.8) or (4.21) is changed slightly. Such a slight change in the objective should result in a corresponding small change in the optimum. This amounts to saying that the optimal value of (4.8) or (4.21) is a continuous function of $\boldsymbol{Q}$.

Continuous Dependence of the Optimum on $\boldsymbol{Q}$

For a given conditional probability vector $\boldsymbol{Q}$, let $R_l^*(\boldsymbol{Q})$ be the optimal rate allocation for the lth single-user code, $l = 1, \ldots, L$, according to Theorem 4.2. We show that $\boldsymbol{R}^* \equiv (R_1^*, \ldots, R_L^*)$ is a continuous function of $\boldsymbol{Q}$. To prove this fact, it is convenient to express the intervals $\mathcal{A}_l$ in Theorem 4.2 in another form.

Lemma 4.5 *For $l = 1, \ldots, L$, the intervals $\mathcal{A}_l$ in Theorem 4.2 are given by*

$$\mathcal{A}_l = [\max\{0, \{r(k,l) : k > l\}\}. \min\{P, \{r(k,l) : k < l\}\}].$$

where

$$r(k,l) \equiv \begin{cases} \textit{root of equation } u_k(z) = u_l(z) & \textit{if } Q_k \neq Q_l, \\ +\infty & \textit{if } Q_k = Q_l. \end{cases}$$

for $k, l \in \{1, \ldots, L\}$.

PROOF. Note first that, for $k \neq l$, a solution to the equation $\frac{Q_k}{z+\eta_k} = \frac{Q_l}{z+\eta_l}$ exists and is unique whenever $Q_k \neq Q_l$. Thus, $r(k,l)$ is well-defined. Since the functions $u_l(z)$ are strictly decreasing, we can observe the following. For $k > l$, if $Q_k \neq Q_l$, then $z < r(k,l)$ if and only if $u_l(z) < u_k(z)$ while $z > r(k,l)$ if and only if $u_l(z) > u_k(z)$. For $k > l$ and $Q_k = Q_l$, $u_k(z) > u_l(z)\ \forall z$.

Let $z \in [\max\{0, \{r(k,l) : k > l\}\}, \min\{P, \{r(k,l) : k < l\}\}]$. Since the set is nonempty, $r(k,l) \leq P, \forall k > l$. Since $z \geq r(k,l), \forall k > l$, $u_l(z) \geq u_k(z), \forall k > l$. Also, $z \leq r(k,l), \forall k < l$, implying $u_l(z) \geq u_k(z), \forall k < l$. Therefore $z \in \mathcal{A}_l$.

Suppose $z \in \mathcal{A}_l$, then $u_l(z) \geq u_k(z), \forall k \neq l$. Note that for $k > l$, we cannot have $Q_k = Q_j$, since that would imply $u_l(x) < u_k(x), \forall x$. Therefore, we must have $z \geq r(k,l), \forall k > l$. On the other hand, for all $k < l$, we also must have $z \leq r(k,l)$. Thus, $z \in [0, P] \cap [\max\{r(k,l) : k > l\}, \min\{r(k,l) : k < l\}]$. $\square$

From Lemma 4.5, the continuous dependence of the optimum on $\boldsymbol{Q}$ follows easily.

Lemma 4.6 *There exists a continuous function $\psi : \mathcal{Q} \mapsto \mathcal{R}$, where $\mathcal{Q} \equiv \{\boldsymbol{Q} \equiv (Q_1, \ldots, Q_L) : 1 = Q_1 \geq Q_2 \geq \cdots \geq Q_L \geq 0\}$ and $\mathcal{R}$ is the feasible region defined by (4.22)-(4.23), such that $\psi(\boldsymbol{Q}) = \boldsymbol{R}^*$ is the optimal rate vector corresponding to $\boldsymbol{Q}$ in the maximization (4.8) or (4.21).*

PROOF. The existence of a function from $\mathcal{Q}$ to $\mathcal{R}$ follows directly from Theorem 4.2. To prove continuity, we show that for each pair $(k,l), k \neq l$, $r(k,l)$ is a continuous function of $\boldsymbol{Q}$. Assume $k > l$. When $Q_k \neq Q_l$ ($Q_k < Q_l$), the root $r(k,l) = (Q_k\eta_l - Q_l\eta_k)/(Q_l - Q_k)$ is clearly continuous in $\boldsymbol{Q}$. As $Q_k \uparrow Q_l$, $Q_l/\eta_l < Q_k/\eta_k$ or $Q_k\eta_l - Q_l\eta_k > 0$, and $\lim_{Q_k \uparrow Q_l}(Q_k\eta_l - Q_l\eta_k)/(Q_l - Q_k) = +\infty$. Thus, $r(k,l)$ is a continuous function of $\boldsymbol{Q}$. A similar argument applies for the case of $k < l$. It then follows that $\max\{0, \{r(k,l) : k > l\}\}$ and $\min\{P, \{r(k,l) : k < l\}\}$ are continuous functions of $\boldsymbol{Q}$. This implies by Lemma 4.5 and Theorem 4.2 that the optimal power allocations $\alpha_l^* P$ and optimal rates R_l^* are continuous functions of $\boldsymbol{Q}$. $\square$

The Coding Theorem

We now use our previous results and the assumption of ergodicity to obtain a coding theorem. Let N be the number of channel symbols corresponding to a channel block (over which the gain is constant). For convenience, consider the system at time 0. We construct a code using history of the process from $-d$ to $-d - N_H$, where d is given and fixed and N_H will be set subsequently. Since there are L fading levels there can be at most L^{N_H} sequences $s_{-d-N_H}^{-d}$ of length N_H. For each such sequence, calculate the conditional probabilities and thus $\boldsymbol{Q}(s_{-d-N_H}^{-d})$. Now obtain the optimal choice of power parameters $\{\alpha_l^*(\boldsymbol{Q}(s_{-d-N_H}^{-d}))\}_{l=1}^{L}$ corresponding to $\boldsymbol{Q}(s_{-d-N_H}^{-d})$ as in Theorem 4.2. Let $\{R_l^*(\boldsymbol{Q}(s_{-d-N_H}^{-d}))\}_{l=1}^{L}$ be the resulting optimal rates under the power allocation $\{\alpha_l^*(\boldsymbol{Q}(s_{-d-N_H}^{-d}))P\}_{l=1}^{L}$ according to (4.28). For each $l = 1, \ldots, L$, we may then design the lth single-user code within the superposition code to have rate

$$R_l(\boldsymbol{Q}(s_{-d-N_H}^{-d})) = R_l^*(\boldsymbol{Q}(s_{-d-N_H}^{-d})) - \nu_l(P_{el}, N)$$

where ν_l is a function (via the error exponent) of some given target error probability P_{el} (averaged over messages) of the lth single-user code and block length N.

Given a sequence $s_{-\infty}^{-d}$, the capacity bound in Theorem 4.1 is defined almost surely. Define the event

$$A_\delta(n) \equiv \left\{ s_{-\infty}^{-d} : \lim_{n\to\infty} Q_l(s_{-d-n}^{-d}) = Q_l(s_{-\infty}^{-d}) \ \text{(exists)}, \sum_l |Q_l(s_{-\infty}^{-d}) - Q_l(s_{-d-n}^{-d})| < \delta \right\}.$$

For all $s_{-\infty}^{-d} \in A_\delta(n)$,

$$\begin{aligned}
\sum_l R_l^*(\boldsymbol{Q}(s_{-\infty}^{-d}))Q_l(s_{-\infty}^{-d}) &\le \sum_l \left(R_l^*(\boldsymbol{Q}(s_{-d-n}^{-d}) + \epsilon(\delta) \right) Q_l(s_{-\infty}^{-d}) && (4.32)\\
&\le \sum_l \left(R_l(\boldsymbol{Q}(s_{-d-n}^{-d})) + \nu_l + \epsilon(\delta) \right) Q_l(s_{-\infty}^{-d}) && (4.33)\\
&\le \sum_l \left(R_l(\boldsymbol{Q}(s_{-d-n}^{-d})) + \nu_l \right) Q_l(s_{-\infty}^{-d}) + M\epsilon(\delta) && (4.34)
\end{aligned}$$

where (4.32) follows from Lemma 4.6 and (4.33) follows from our codebook construction. The last inequality (4.34) holds since $\sum_l Q_l \le \sum_l 1 \le M$.

We now take expectations over the sequences $s_{-\infty}^{-d}$:

$$\begin{aligned}
&\mathbb{E}\left[\sum_l R_l^*(\boldsymbol{Q}(S_{-\infty}^{-d}))Q_l(S_{-\infty}^{-d}) \right] \\
=\ &\mathbb{E}\left[\sum_l R_l^*(\boldsymbol{Q}(S_{-\infty}^{-d}))Q_l(S_{-\infty}^{-d}) | S_{-\infty}^{-d} \in A_\delta(n) \right] \Pr\left(S_{-\infty}^{-d} \in A_\delta(n) \right) \\
&+ \mathbb{E}\left[\sum_l R_l^*(\boldsymbol{Q}(S_{-\infty}^{-d}))Q_l(S_{-\infty}^{-d}) | S_{-\infty}^{-d} \notin A_\delta(n) \right] \Pr\left(S_{-\infty}^{-d} \notin A_\delta(n) \right) \\
\le\ &\mathbb{E}\left[\sum_l \left(R_l(\boldsymbol{Q}(S_{-d-n}^{-d})) + \nu_l \right) Q_l(S_{-\infty}^{-d}) \right] \Pr\left(S_{-\infty}^{-d} \in A_\delta(n) \right) \\
&+ M\epsilon(\delta) + D\Pr\left(S_{-\infty}^{-d} \notin A_\delta(n) \right)
\end{aligned}$$

where D is an upper bound on $\mathbb{E}\left[\sum_l R_l^*(\boldsymbol{Q}(S_{-\infty}^{-d}))Q_l(S_{-\infty}^{-d})|S_{-\infty}^{-d} \notin A_\delta(n)\right]$. Then,

$$\begin{aligned}\mathbb{E}\left[\sum_l R_l^*(\boldsymbol{Q}(S_{-\infty}^{-d}))Q_l(S_{-\infty}^{-d})\right] \;\; &\leq \;\; \mathbb{E}\left[\sum_l \left(R_l(\boldsymbol{Q}(S_{-d-n}^{-d}))) + \nu_l\right) Q_l(S_{-\infty}^{-d})\right] \\ &\quad +M\epsilon(\delta) + D\Pr\left(S_{-\infty}^{-d} \notin A_\delta(n)\right).\end{aligned}$$

Note that D can be defined independently of $A_\delta(n)$. For a given $\varepsilon > 0$, we may choose δ so that $M\epsilon(\delta) \leq \varepsilon/2$. Also choose N_H large enough so that

$$D\Pr\left(S_{-\infty}^{-d} \notin A_\delta(N_H)\right) \leq \frac{\varepsilon}{2}$$

since for $s_{-\infty}^{-d}$ such that $\lim_{n\to\infty} Q_l(s_{-d-n}^{-d}) = Q_l(s_{-\infty}^{-d})$,

$$\Pr\left(\sum_l |Q_l(S_{-\infty}^{-d}) - Q_l(S_{-d-n}^{-d})| \geq \delta\right) \to 0 \text{ as } n \to \infty$$

from Lemma 4.4. Then,

$$\mathbb{E}\left[\sum_l R_l^*(\boldsymbol{Q}(S_{-\infty}^{-d}))Q_l(S_{-\infty}^{-d})\right] \leq \mathbb{E}\left[\sum_l R_l(\boldsymbol{Q}(S_{-d-N_H}^{-d}))Q_l(S_{-\infty}^{-d})\right] + \overline{\nu} + \varepsilon, \tag{4.35}$$

where $\overline{\nu} = \mathbb{E}\left[\sum_l Q_l(S_{-\infty}^{-d})\nu_l\right]$. We have thus shown that the expected rate at time 0 in our coding scheme based on the finite history $S_{-d-N_H}^{-d}$ is within $\varepsilon + \overline{\nu}$ of the upper bound stated in Theorem 4.1. By construction, the probability of error averaged over the messages for the lth single-user code is less than or equal to $P_{el}, l = 1, \ldots, L$.

We are now ready to use the ergodicity of the channel state process. Consider the following mapping,

$$f : \Omega \mapsto \mathbb{R}^+, \quad f(s) = \sum_{l=1}^{L} R_l(\boldsymbol{Q}(s_{-d-N_H}^{-d}))1_{l,0}(s)$$

where $s \equiv s_{-\infty}^{+\infty} \in \Omega$ is a given sample path of channel states, $1_{l,0}(s)$ is the indicator of the event that the noise variance at time 0 is less than or equal to η_l (index of noise variance is greater than or equal to l), and $f(s)$ is the information rate obtained with our coding scheme at the receiver at time 0 for the sample path s. Since f is integrable, we have by the ergodic theorem [Bil95],

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1} f(T^k s) &= \mathbb{E}\left[\sum_l R_l(\boldsymbol{Q}(S_{-d-N_H}^{-d}))Q_l(S_{-\infty}^{-d})\right] \quad \text{a.s.}\\ &\geq \mathbb{E}\left[\sum_l R_l^*(\boldsymbol{Q}(S_{-\infty}^{-d}))Q_l(S_{-\infty}^{-d})\right] - \overline{\nu} - \varepsilon\end{aligned}$$

where $T^k s$ denotes the sequence s shifted in time by k positions. We have thus obtained the following theorem.

Theorem 4.3 *Consider a block Gaussian channel with an average transmit power constraint P according to* (4.6) *and noise power varying according to a stationary ergodic process $\{S_k,\ k \in \mathbb{Z}\}$ with state space $\mathcal{S} = \{\eta_1, \ldots, \eta_L\}$, $\eta_1 > \eta_2 > \ldots > \eta_L$. Suppose the decoding delay constraint is N symbols and noiseless channel state feedback to the transmitter is delayed by d blocks.*

Let $\varepsilon > 0$, and desired average (over messages) error probabilities $P_{el}, l = 1, \ldots, L$, be given. Then there exists $N_H \in \mathbb{Z}^+$ such that for each finite sequence $s_{-d-N_H}^{-d}$, a corresponding set of codebooks $\mathcal{C}_1, \cdots, \mathcal{C}_L$ exist so that for almost every realization of the block Gaussian channel the limiting time average reliably received rate is within $\varepsilon + \overline{\nu}$ of the LHS of (4.35), where $\overline{\nu} = \mathbb{E}\left[\sum_l Q_l(S_{-\infty}^{-d})\nu_l\right]$.

Theorem 4.3 says that we can approach the performance limit promised by Theorem 4.1 with codebooks designed according to a sufficiently long finite segment of the channel history. Given $\varepsilon > 0$ and the desired error probabilities, however, the theorem does not specify the required N_H. For this, we would essentially need convergence rates for the conditional probabilities in Lemma 4.2. This, in turn, would require more specific assumptions on the fading process itself.

4.3 Decoding Delay of Two Blocks ($K = 2$)

In view of the results obtained for the one-block case, it is interesting to examine what happens when K is gradually increased. Does the broadcast strategy continue to be optimal? In the following, we examine the case of a two-state i.i.d. block Gaussian channel with a decoding delay constraint of *two* N-blocks ($K = 2$). By that we mean that at the end of $2N$ symbols the decoder must decode as much of the frame as possible, and declare "erasure" on those parts of the frame that it is unable to decode. In addition, since the state process is i.i.d., delayed state feedback does not aid the encoder-decoder pair in exploiting the channel state.

A moment of reflection will show that the generalization of Theorem 4.1 to the two-block case is far from obvious. In the two-state i.i.d. case, suppose the noise variance $S = \eta_1$ (High noise) w.p. q and $S = \eta_2$ (Low noise) w.p. $1 - q$, where $\eta_1 > \eta_2$.[7] Over the course of two blocks, four types of channels are possible: Low-Low (LL), Low-High (LH), High-Low (HL), and High-High (HH). Notice that unlike the one-block case, these channels are not in general degraded with respect to each other, and thus a direct extension of the DBC results is not possible. Instead, one needs some general results concerning optimal encoding strategies over parallel Gaussian broadcast channels. Here the results are incomplete. Hughes-Hartog [HH75] derived the capacity region of parallel Gaussian broadcast channels in which all channels are degraded in the same direction, but only an achievable region for the general case. El Gamal [Gam80] derived the capacity region for the two-receiver two-parallel Gaussian broadcast channel (BC) in the general case, allowing for the presence of common information. In our setting, El Gamal's analysis would deal with only the Low-High (LH) and High-Low (HL) channels. It has been shown that with only independent information, El Gamal's capacity region coincides with Hughes-Hartog's achievable region in the two-by-two case. Results concerning the capacity of the general parallel Gaussian BC with only independent information are reported in [Tse99] and [LG99].

4.3.1 Converse Theorem for a Two-state I.I.D. Channel

As in the one-block case, we make the correspondence between the four channel states in the i.i.d. two-block Gaussian channel and the four receivers in the two-parallel Gaussian BC. We are thus interested in achievable rates for a four-receiver (HH, HL, LH, LL) two-parallel Gaussian BC, seen in Figure 4-3.

Fortunately, we do not require the entire capacity region, but merely want an upper bound on a linear combination of rates for the four receivers (corresponding to the possible channel states). Notice that the HL, LH, and HH channels are all degraded versions of the LL channel, and the HH channel is also degraded with respect to the HL and LH channels.

[7] A physical scenario for the i.i.d. case may be a GSM slow frequency hopping system where the interleaving depth is two blocks, the carrier separations are greater than B_{coh}, and $T_s \ll T_{coh}$ [CKH98].

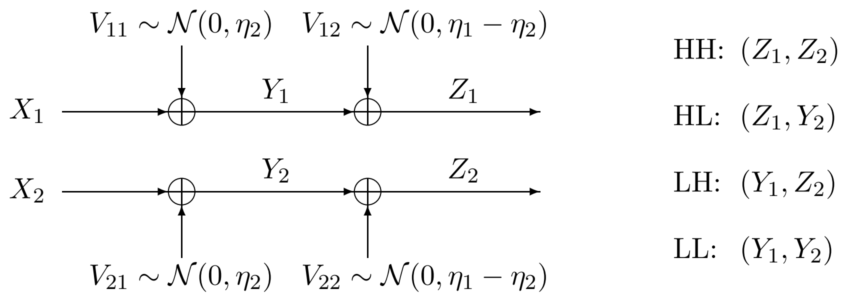

Figure 4-3: A two-parallel Gaussian broadcast channel with four receivers Low-Low (LL), Low-High (LH), High-Low (HL) and High-High (HH). The noise variables $V_{11}, V_{12}, V_{21}, V_{22}$ are assumed to be mutually independent.

The HL and LH channels, however, are not degraded with respect to each other, and so we need to allow for common information between those two receivers. In the following, we first obtain upper bounds on the rates for each of the four receivers. These upper bounds translate directly to upper bounds on rates for the four channel states in the two-block Gaussian channel. We then show that the optimized bound on the expected rate over the channel states is in fact obtained by a broadcast strategy which sends independent information for the HH and LL states and only common information for the HL and LH states.

Consider the four-receiver two-parallel Gaussian broadcast channel (GBC) in Figure 4-3. This consists of two Gaussian broadcast channels described by

$$Y_i = X_i + V_{i1}, \ \ Z_i = Y_i + V_{i2}, \ i = 1, 2.$$

where $V_{i1} \sim \mathcal{N}(0, \eta_2)$ and $V_{i2} \sim \mathcal{N}(0, \eta_1 - \eta_2)$ for $i = 1, 2$. The two broadcast channels are assumed to be *independent* in the sense that $V_{11}, V_{12}, V_{21}, V_{22}$ are mutually independent. The LL receiver of the two-parallel GBC observes outputs (Y_1, Y_2). The LH and HL receivers observe (Y_1, Z_2) and (Z_1, Y_2), respectively, and the HH receiver observes (Z_1, Z_2). For each $i = 1, 2$, let $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{iN})$ be a sequence of N inputs into the ith component channel of the two-parallel GBC and let $\boldsymbol{y}_i = (y_{i1}, \ldots, y_{iN})$ and $\boldsymbol{z}_i = (z_{i1}, \ldots, z_{iN})$ be output sequences from the ith component channel. Suppose the two-parallel GBC is *memoryless* in that the $\boldsymbol{X}_i$'s, $\boldsymbol{Y}_i$'s and $\boldsymbol{Z}_i$'s have joint conditional probability density

$$p_N(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{z}_1, \boldsymbol{z}_2 | \boldsymbol{x}_1, \boldsymbol{x}_2) = \prod_{i=1}^{2} \prod_{n=1}^{N} p_{Y|X}(y_{in}|x_{in}) p_{Z|Y}(z_{in}|y_{in})$$

where

$$p_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi\eta_2}} \exp\left\{ \frac{-(y-x)^2}{2\eta_2} \right\}, \quad p_{Z|Y}(z|y) = \frac{1}{\sqrt{2\pi(\eta_1 - \eta_2)}} \exp\left\{ \frac{-(z-y)^2}{2(\eta_1 - \eta_2)} \right\}.$$

Consider the four-receiver two-parallel Gaussian broadcast channel in Figure 4-3. A $(2N, R_{LL}, R_0, R_{LH}, R_{HL}, R_{HH})$ code for this channel consists of the following.

(a) Index sets $\mathcal{W}_{LL} = \{1, \dots, M_{LL}\}$, $\mathcal{W}_0 = \{1, \dots, M_0\}$, $\mathcal{W}_{LH} = \{1, \dots, M_{LH}\}$, $\mathcal{W}_{HL} = \{1, \dots, M_{HL}\}$, $\mathcal{W}_{HH} = \{1, \dots, M_{HH}\}$, where $M_{LL} = \lceil e^{NR_{LL}} \rceil$, $M_0 = \lceil e^{NR_0} \rceil$, $M_{LH} = \lceil e^{NR_{LH}} \rceil$, $M_{HL} = \lceil e^{NR_{HL}} \rceil$, $M_{HH} = \lceil e^{NR_{HH}} \rceil$.

(b) Encoder $f : \mathcal{W}_{LL} \times \mathcal{W}_0 \times \mathcal{W}_{LH} \times \mathcal{W}_{HL} \times \mathcal{W}_{HH} \mapsto \mathbb{R}^{2N}$, $f(W_{LL}, W_0, W_{LH}, W_{HL}, W_{HH}) = (\boldsymbol{X}_1, \boldsymbol{X}_2)$, where $\boldsymbol{X}_i = (X_{i1}, \dots, X_{iN}), i = 1, 2$.

(c) Decoding functions

$$\begin{aligned}
g_{LL} \quad &: \quad \mathbb{R}^{2N} \mapsto \mathcal{W}_{LL} \times \mathcal{W}_0 \times \mathcal{W}_{LH} \times \mathcal{W}_{HL} \times \mathcal{W}_{HH}, \\
& \quad\quad g_{LL}(\boldsymbol{Y}_1, \boldsymbol{Y}_2) = (\hat{W}_{LL}, \hat{W}_0, \hat{W}_{LH}, \hat{W}_{HL}, \hat{W}_{HH}), \\
g_{LH} \quad &: \quad \mathbb{R}^{2N} \mapsto \mathcal{W}_0 \times \mathcal{W}_{LH} \times \mathcal{W}_{HH}, \;\; g_{LH}(\boldsymbol{Y}_1, \boldsymbol{Z}_2) = (\hat{W}_0, \hat{W}_{LH}, \hat{W}_{HH}), \\
g_{HL} \quad &: \quad \mathbb{R}^{2N} \mapsto \mathcal{W}_0 \times \mathcal{W}_{HL} \times \mathcal{W}_{HH}, \;\; g_{HL}(\boldsymbol{Z}_1, \boldsymbol{Y}_2) = (\hat{W}_0, \hat{W}_{HL}, \hat{W}_{HH}), \\
g_{HH} \quad &: \quad \mathbb{R}^{2N} \mapsto \mathcal{W}_{HH}, \;\; g_{HH}(\boldsymbol{Z}_1, \boldsymbol{Z}_2) = \hat{W}_{HH}, \qquad (4.36)
\end{aligned}$$

where channel outputs $\boldsymbol{Y}_i = (Y_{i1}, \dots, Y_{iN})$, $\boldsymbol{Z}_i = (Z_{i1}, \dots, Z_{iN})$, i=1,2.

Note that R_{LH} and R_{HL} denote the rates of independent information for the LH and HL receivers and R_0 denote the rate of common information between the LH and HL receivers. R_{LL} and R_{HH} are the independent information rates for the LL and HH receivers.

Assuming $(W_{LL}, W_0, W_{LH}, W_{HL}, W_{HH})$ is uniformly drawn over $\mathcal{W}_{LL} \times \mathcal{W}_0 \times \mathcal{W}_{LH} \times$

$\mathcal{W}_{HL} \times \mathcal{W}_{HH}$, we define error probabilities

$$\begin{aligned}
P_{e1} &= \Pr\left(g_{LL}(\boldsymbol{Y}_1, \boldsymbol{Y}_2) \neq (W_{LL}, W_0, W_{LH}, W_{HL}, W_{HH})\right), \\
P_{e2} &= \Pr\left(g_{LH}(\boldsymbol{Y}_1, \boldsymbol{Z}_2) \neq (W_0, W_{LH})\right), \\
P_{e3} &= \Pr\left(g_{HL}(\boldsymbol{Z}_1, \boldsymbol{Y}_2) \neq (W_0, W_{HL})\right), \\
P_{e4} &= \Pr\left(g_{HH}(\boldsymbol{Z}_1, \boldsymbol{Z}_2) \neq W_{HH}\right).
\end{aligned}$$

Specifically,

$$P_{e1} = \frac{1}{M_{LL}M_0M_{LH}M_{HL}M_{HH}} \cdot \sum_{(w_{LL},w_0,w_{LH},w_{HL},w_{HH})} \int_{Y^c} p_{\boldsymbol{Y}_1,\boldsymbol{Y}_2|\boldsymbol{X}_1,\boldsymbol{X}_2}(\boldsymbol{y}_1, \boldsymbol{y}_2 | f(w_{LL}, w_0, w_{LH}, w_{HL}, w_{HH}))d\boldsymbol{y}_1 d\boldsymbol{y}_2$$

where $Y^c = \{(\boldsymbol{y}_1, \boldsymbol{y}_2) : g_{LL}(\boldsymbol{y}_1, \boldsymbol{y}_2) \neq (w_{LL}, w_0, w_{LH}, w_{HL}, w_{HH})\}$ and

$$p_{\boldsymbol{Y}_1,\boldsymbol{Y}_2|\boldsymbol{X}_1,\boldsymbol{X}_2}(\boldsymbol{y}_1, \boldsymbol{y}_2 | f(w_{LL}, w_0, w_{LH}, w_{HL}, w_{HH})) = \prod_{i=1}^{2}\prod_{n=1}^{N} p_{Y|X}(y_{in}|x_{in}).$$

The quantities P_{e2}, P_{e3}, P_{e4} are similarly specified.

Lemma 4.7 *Consider the four-receiver two-parallel Gaussian broadcast channel with an average transmit power constraint P over $2N$ channel symbols according to (4.2) (for $K = 2$). Let $\eta_1 > \eta_2$ be the variances for the High and Low channels and let $q > 0$ be the probability of encountering a High channel. Then for any $(2N, R_{LL}, R_0, R_{LH}, R_{HL}, R_{HH})$ code and any $\beta \in [0,1]$, there exist $\theta_i \in [0,1]$, $\gamma_i \in [\theta_i, 1]$, $i = 1, 2$, such that*

$$R_{LL} \leq \frac{1}{2}\left[C\left(\frac{2\beta P\overline{\gamma_1}}{\eta_2}\right) + C\left(\frac{2\overline{\beta}P\overline{\gamma_2}}{\eta_2}\right)\right] + \frac{\epsilon_1(P_{e1})}{N} \tag{4.37}$$

$$R_0 + R_{LH} \leq \frac{1}{2}\left[C\left(\frac{2\beta P(\gamma_1 - \theta_1)}{2\beta P\overline{\gamma_1} + \eta_2}\right) + C\left(\frac{2\overline{\beta}P(\gamma_2 - \theta_2)}{2\overline{\beta}P\overline{\gamma_2} + \eta_1}\right)\right] + \frac{\epsilon_2(P_{e2})}{N} \tag{4.38}$$

$$R_0 + R_{HL} \leq \frac{1}{2}\left[C\left(\frac{2\beta P(\gamma_1 - \theta_1)}{2\beta P\overline{\gamma_1} + \eta_1}\right) + C\left(\frac{2\overline{\beta}P(\gamma_2 - \theta_2)}{2\overline{\beta}P\overline{\gamma_2} + \eta_2}\right)\right] + \frac{\epsilon_3(P_{e3})}{N} \tag{4.39}$$

$$R_{HH} \leq \frac{1}{2}\left[C\left(\frac{2\beta P\theta_1}{2\beta P\overline{\theta_1} + \eta_1}\right) + C\left(\frac{2\overline{\beta}P\theta_2}{2\overline{\beta}P\overline{\theta_2} + \eta_1}\right)\right] + \frac{\epsilon_4(P_{e4})}{N} \tag{4.40}$$

where $\epsilon_i(P_{ei}) \to 0$ as $P_{ei} \to 0$, $i = 1, 2, 3, 4$ and $\overline{\theta}$ denotes $1 - \theta$.

PROOF. The proof uses essentially the same techniques seen in the proof of Theorem 4.1.

Conditional entropies are "pegged" via the parameters θ_i and $\gamma_i, i = 1, 2$. This leads to parameterized upper bounds on the quantities R_{LL}, $R_0 + R_{LH}$, $R_0 + R_{HL}$, and R_{HH}.

Suppose a $(2N, R_{LL}, R_0, R_{LH}, R_{HL}, R_{HH})$ code satisfies the overall power constraint

$$\frac{1}{2N}\sum_{n=1}^{N}\mathbb{E}\left[X_{1n}^2 + X_{2n}^2\right] \leq P \tag{4.41}$$

where $\boldsymbol{X}_1 = (X_{11}, \ldots, X_{1N})$ are the channel input symbols for the first N-block and $\boldsymbol{X}_2 = (X_{21}, \ldots, X_{2N})$ are the symbols for the second N-block. The parameter β is defined via the power constraint on the symbols for the second N-block,

$$\overline{\beta}P \equiv \frac{1}{2N}\sum_{n=1}^{N}\mathbb{E}\left[X_{2n}^2\right].$$

Hence it follows from (4.41) that

$$\beta P \geq \frac{1}{2N}\sum_{n=1}^{N}\mathbb{E}\left[X_{1n}^2\right].$$

Suppose messages $W_{LL} \in \mathcal{W}_{LL}$, $W_0 \in \mathcal{W}_0$, $W_{LH} \in \mathcal{W}_{LH}$, $W_{HL} \in \mathcal{W}_{HL}$ and $W_{HH} \in \mathcal{W}_{HH}$ are to be sent. For given $\beta \in [0,1]$, we may define $\theta_i \in [0,1], \gamma_i \in [0,1], i = 1, 2$, as follows:

$$\begin{aligned} Ng(2\beta P\overline{\theta_1} + \eta_1) &\equiv H(\boldsymbol{Z}_1|W_{HH}), \qquad (4.42)\\ Ng(2\overline{\beta}P\overline{\theta_2} + \eta_1) &\equiv H(\boldsymbol{Z}_2|\boldsymbol{Z}_1, W_{HH}),\\ Ng(2\beta P\overline{\gamma_1} + \eta_2) &\equiv H(\boldsymbol{Y}_1|W_0, W_{LH}, W_{HL}, W_{HH}),\\ Ng(2\overline{\beta}P\overline{\gamma_2} + \eta_2) &\equiv H(\boldsymbol{Y}_2|\boldsymbol{Y}_1, W_0, W_{LH}, W_{HL}, W_{HH}). \end{aligned}$$

We have $Ng(\eta_1) \leq H(\boldsymbol{Z}_1|W_{HH}) \leq H(\boldsymbol{Z}_1) \leq Ng(2\beta P + \eta_1)$. Thus, there exists $\theta_1 \in [0,1]$ such that $H(\boldsymbol{Z}_1|W_{HH}) = Ng(2\beta P\overline{\theta_1} + \eta_1)$. The other definitions in (4.42) are similarly justified. We will also show below that in fact, $\theta_i \leq \gamma_i, i = 1, 2$.

We have

$$\begin{aligned} 2NR_{LL} &= H(W_{LL})\\ &= I(W_{LL}; \boldsymbol{Y}_1, \boldsymbol{Y}_2) + H(W_{LL}|\boldsymbol{Y}_1, \boldsymbol{Y}_2)\\ &\leq I(W_{LL}; \boldsymbol{Y}_1, \boldsymbol{Y}_2|W_0, W_{LH}, W_{HL}, W_{HH}) + H(W_{LL}, W_0, W_{LH}, W_{HL}, W_{HH}|\boldsymbol{Y}_1, \boldsymbol{Y}_2) \end{aligned}$$

$$
\begin{aligned}
&\leq I(W_{LL}; \boldsymbol{Y}_1, \boldsymbol{Y}_2 | W_0, W_{LH}, W_{HL}, W_{HH}) \\
&\quad + h(P_{e1}) + P_{e1} \ln(M_{LL} M_0 M_{LH} M_{HL} M_{HH} - 1) && (4.43) \\
&= H(\boldsymbol{Y}_1, \boldsymbol{Y}_2 | W_0, W_{LH}, W_{HL}, W_{HH}) + \epsilon_1(P_{e1}) && (4.44) \\
&\quad - H(\boldsymbol{Y}_1, \boldsymbol{Y}_2 | W_0, W_{LL}, W_{LH}, W_{HL}, W_{HH}) \\
&= H(\boldsymbol{Y}_1 | W_0, W_{LH}, W_{HL}, W_{HH}) + H(\boldsymbol{Y}_2 | \boldsymbol{Y}_1, W_0, W_{LH}, W_{HL}, W_{HH}) \\
&\quad - H(\boldsymbol{Y}_1, \boldsymbol{Y}_2 | W_0, W_{LL}, W_{LH}, W_{HL}, W_{HH}) + \epsilon_1(P_{e1}) \\
&= Ng(2\beta P \overline{\gamma_1} + \eta_2) + Ng(2\overline{\beta} P \overline{\gamma_2} + \eta_2) - 2Ng(\eta_2) + \epsilon_1(P_{e1}). && (4.45)
\end{aligned}
$$

Fano's inequality gives (4.43). In (4.44), $\epsilon_1(P_{e1}) \equiv h(P_{e1}) + P_{e1} \ln(M_{LL} M_0 M_{LH} M_{HL} M_{HH} - 1)$. Equation (4.45) follows from the definitions in (4.42). We therefore have (4.37). Notice that as the error probability requirement $P_{e1} \to 0$, $\epsilon_1(P_{e1}) \to 0$ *for every* N. This converse result is *not* asymptotic in the block length.

For $R_0 + R_{LH}$,

$$
\begin{aligned}
2N(R_0 + R_{LH}) &\leq I(W_0, W_{LH}; \boldsymbol{Y}_1, \boldsymbol{Z}_2 | W_{HH}) + \epsilon_2(P_{e2}) \\
&= I(W_0, W_{LH}; \boldsymbol{Y}_1 | W_{HH}) + I(W_0, W_{LH}; \boldsymbol{Z}_2 | \boldsymbol{Y}_1, W_{HH}) + \epsilon_2(P_{e2}) \\
&= H(\boldsymbol{Y}_1 | W_{HH}) - H(\boldsymbol{Y}_1 | W_0, W_{LH}, W_{HH}) + H(\boldsymbol{Z}_2 | \boldsymbol{Y}_1, W_{HH}) \\
&\quad - H(\boldsymbol{Z}_2 | W_0, W_{LH}, W_{HH}, \boldsymbol{Y}_1) + \epsilon_2(P_{e2})
\end{aligned}
$$

where $\epsilon_2(P_{e2}) \to 0$ as $P_{e2} \to 0$. By equation (4.42) and Lemma 4.3, $H(\boldsymbol{Y}_1 | W_{HH}) \leq Ng(2\beta P \overline{\theta_1} + \eta_2)$ and $H(\boldsymbol{Z}_2 | W_0, W_{LH}, W_{HH}, \boldsymbol{Y}_1) \geq Ng(2\overline{\beta} P \overline{\gamma_2} + \eta_1)$. By the data processing inequality, $I(\boldsymbol{Y}_1; \boldsymbol{Z}_2 | W_{HH}) \geq I(\boldsymbol{Z}_1; \boldsymbol{Z}_2 | W_{HH})$, so $H(\boldsymbol{Z}_2 | \boldsymbol{Y}_1, W_{HH}) \leq H(\boldsymbol{Z}_2 | \boldsymbol{Z}_1, W_{HH}) \equiv Ng(2\overline{\beta} P \overline{\theta_2} + \eta_1)$. Finally, since conditioning reduces entropy, we have

$$
\begin{aligned}
2N(R_0 + R_{LH}) &\leq Ng(2\beta P \overline{\theta_1} + \eta_2) - Ng(2\beta P \overline{\gamma_1} + \eta_2) + Ng(2\overline{\beta} P \overline{\theta_2} + \eta_1) \\
&\quad - Ng(2\overline{\beta} P \overline{\gamma_2} + \eta_1) + \epsilon_2(P_{e2}).
\end{aligned}
$$

Since $Ng(2\beta P \overline{\theta_1} + \eta_2) \geq H(\boldsymbol{Y}_1 | W_{HH}) \geq H(\boldsymbol{Y}_1 | W_0, W_{LH}, W_{HH}) \geq Ng(2\beta P \overline{\gamma_1} + \eta_2)$, we must have $\theta_1 \leq \gamma_1$. Also, $Ng(2\overline{\beta} P \overline{\theta_2} + \eta_1) \geq H(\boldsymbol{Z}_2 | \boldsymbol{Y}_1, W_{HH}) \geq H(\boldsymbol{Z}_2 | W_0, W_{LH}, W_{HH}, \boldsymbol{Y}_1)$ $\geq Ng(2\overline{\beta} P \overline{\gamma_2} + \eta_1)$ implies $\theta_2 \leq \gamma_2$. Thus (4.38) holds.

Similarly,

$$\begin{aligned}
2N(R_0 + R_{HL}) &\leq I(W_0, W_{HL}; \boldsymbol{Z}_1, \boldsymbol{Y}_2 | W_{HH}) + \epsilon_3(P_{e3}) \\
&= I(W_0, W_{HL}; \boldsymbol{Y}_2 | W_{HH}, \boldsymbol{Z}_1) + I(W_0, W_{HL}; \boldsymbol{Z}_1 | W_{HH}) + \epsilon_3(P_{e3}) \\
&= H(\boldsymbol{Y}_2 | W_{HH}, \boldsymbol{Z}_1) - H(\boldsymbol{Y}_2 | W_0, W_{HL}, W_{HH}, \boldsymbol{Z}_1) \\
&\quad + H(\boldsymbol{Z}_1 | W_{HH}) - H(\boldsymbol{Z}_1 | W_0, W_{HL}, W_{HH}) + \epsilon_3(P_{e3})
\end{aligned}$$

where $\epsilon_3(P_{e3}) \to 0$ as $P_{e3} \to 0$. By (4.42), Lemma 4.3, and the fact that conditioning reduces entropy, $H(\boldsymbol{Y}_2|W_{HH}, \boldsymbol{Z}_1) \leq Ng(2\overline{\beta}P\overline{\theta_2} + \eta_2)$, $H(\boldsymbol{Z}_1|W_0, W_{HL}, W_{HH}) \geq Ng(2\beta P\overline{\gamma_1} + \eta_1)$. Now since $I(\boldsymbol{Z}_1; \boldsymbol{Y}_2|W_0, W_{HL}, W_{HL}, W_{HH}) \leq I(\boldsymbol{Y}_2; \boldsymbol{Y}_1|W_0, W_{HL}, W_{HL}, W_{HH})$, $H(\boldsymbol{Y}_2|W_0, W_{HL}, W_{HH}, \boldsymbol{Z}_1) \geq H(\boldsymbol{Y}_2|W_0, W_{LH}, W_{HL}, W_{HH}, \boldsymbol{Y}_1) \equiv Ng(2\overline{\beta}P\overline{\gamma_2}+\eta_2)$. Hence,

$$\begin{aligned}
2N(R_0 + R_{HL}) &\leq Ng(2\overline{\beta}P\overline{\theta_2} + \eta_2) - Ng(2\overline{\beta}P\overline{\gamma_2} + \eta_2) + Ng(2\beta P\overline{\theta_1} + \eta_1) \\
&\quad - Ng(2\beta P\overline{\gamma_1} + \eta_1) + \epsilon_3(P_{e3}),
\end{aligned}$$

from which (4.39) follows. As before, $\theta_i \leq \gamma_i, i = 1, 2$.

Finally,

$$\begin{aligned}
2NR_{HH} &\leq I(W_{HH}; \boldsymbol{Z}_1, \boldsymbol{Z}_2) + \epsilon_4(P_{e4}) \\
&\leq H(\boldsymbol{Z}_1) + H(\boldsymbol{Z}_2) - H(\boldsymbol{Z}_1|W_{HH}) - H(\boldsymbol{Z}_2|\boldsymbol{Z}_1, W_{HH}) + \epsilon_4(P_{e4}) \\
&\leq Ng(2\beta P + \eta_1) - Ng(2\overline{\beta}P + \eta_1) - Ng(2\beta P\overline{\theta_1} + \eta_1) \\
&\quad - Ng(2\overline{\beta}P\overline{\theta_2} + \eta_1) + \epsilon_4(P_{e4})
\end{aligned}$$

where $\epsilon_4(P_{e4}) \to 0$ as $P_{e4} \to 0$. The last inequality follows directly from (4.42) and the fact that $H(\boldsymbol{Z}_1) \leq Ng(2\beta P + \eta_1)$ and $H(\boldsymbol{Z}_2) \leq Ng(2\overline{\beta}P + \eta_1)$. This results in (4.40). □

Although we have stated the converse in terms of the error probability of the entire frame, Fano's inequality can just as easily be used to obtain a converse in terms of the bit error probability [Gal68]. The point is, however, that it is often the frame error probability that is important in most practical systems [KH00].

As suggested before, we now make the correspondence between the four channel states in the i.i.d. two-block Gaussian channel and the four receivers in the two-parallel Gaussian broadcast channel. Thus, R_{LH} and R_{HL} denote the rates of independent information for the LH and HL states and R_0 is the rate of common information between the LH and HL states. Let R_{LL} and R_{HH} denote the independent information rates for the LL and HH states. We have the following corollary.

Corollary 4.2 *Consider a two-state i.i.d. block Gaussian channel with an average transmit power constraint P over $2N$ channel symbols according to (4.2) ($K = 2$) and a decoding delay constraint of two N-blocks. Let $\eta_1 > \eta_2$ be the variances for the High and Low channels. For any $(2N, R_{LL}, R_0, R_{LH}, R_{HL}, R_{HH})$ code, if P_{el} is required to be arbitrarily small for every $l = 1, 2, 3, 4$, then for any $\beta \in [0, 1]$, there exist $\theta_i \in [0, 1]$, $\gamma_i \in [\theta_i, 1]$, $i = 1, 2$, such that the expected rate satisfies*

$$\begin{aligned} R^* = \mathbb{E}[R] &= R_{HH} + (1-q^2)\left[\frac{1}{2}(R_0 + R_{LH}) + \frac{1}{2}(R_0 + R_{HL})\right] + (1-q)^2 R_{LL} \\ &\leq \frac{1}{2}\left[C\left(\frac{2\beta P\theta_1}{2\beta P\overline{\theta_1} + \eta_1}\right) + C\left(\frac{2\overline{\beta}P\theta_2}{2\overline{\beta}P\overline{\theta_2} + \eta_1}\right)\right] \\ &\quad + \frac{1}{4}(1-q^2)\left[C\left(\frac{2\beta P(\gamma_1 - \theta_1)}{2\beta P\overline{\gamma_1} + \eta_2}\right) + C\left(\frac{2\overline{\beta}P(\gamma_2 - \theta_2)}{2\overline{\beta}P\overline{\gamma_2} + \eta_1}\right)\right. \\ &\quad \left. + C\left(\frac{2\beta P(\gamma_1 - \theta_1)}{2\beta P\overline{\gamma_1} + \eta_1}\right) + C\left(\frac{2\overline{\beta}P(\gamma_2 - \theta_2)}{2\overline{\beta}P\overline{\gamma_2} + \eta_2}\right)\right] \\ &\quad + \frac{1}{2}(1-q)^2\left[C\left(\frac{2\beta P\overline{\gamma_1}}{\eta_2}\right) + C\left(\frac{2\overline{\beta}P\overline{\gamma_2}}{\eta_2}\right)\right]. \end{aligned} \tag{4.46}$$

The next lemma shows that for every coding scheme which achieves a given expected rate R^* with power split $(\beta, \overline{\beta})$ over the two blocks of N symbols, there is a second scheme which achieves the same expected rate but uses uniform power, i.e. $\beta = 1/2$. From this it follows that schemes which approach the optimal expected rate can be taken to have uniform power.

Lemma 4.8 *Consider a coding scheme meeting an average power constraint of P over $2N$ channel symbols according to (4.2) ($K = 2$) which allocates a fraction β ($\overline{\beta}$) of the power to the symbols for the first (second) block of N symbols. Suppose the scheme achieves an expected rate R^* over a two-state i.i.d. two-block Gaussian channel with block length N. Then there is another coding scheme meeting an average power constraint of P over $4N$*

channel symbols which achieves the same expected rate over the corresponding two-block Gaussian channel with block length $2N$, but uses uniform power: $\beta = \overline{\beta} = \frac{1}{2}$.

PROOF. Consider a two-state i.i.d. two-block Gaussian channel with block length N. For a given $\beta \in [0, 1]$, suppose we have a $(2N, R_{LL}, R_0, R_{LH}, R_{HL}, R_{HH})$ code $\mathcal{C}$ with encoding function $f : \mathcal{W}_{LL} \times \mathcal{W}_0 \times \mathcal{W}_{LH} \times \mathcal{W}_{HL} \times \mathcal{W}_{HH} \mapsto \mathbb{R}^{2N}$ where $f(W_{LL}, W_0, W_{LH}, W_{HL}, W_{HH}) = (\boldsymbol{X}_1, \boldsymbol{X}_2)$ such that

$$\beta P = \frac{1}{2N} \sum_{n=1}^{N} \mathbb{E}\left[X_{1n}^2\right], \quad \overline{\beta} P = \frac{1}{2N} \sum_{n=1}^{N} \mathbb{E}\left[X_{2n}^2\right]. \tag{4.47}$$

Transmission of the codewords is depicted in part (a) of Figure 4-4, where $\boldsymbol{X}_1$ is sent over N uses of the upper channel in Figure 4-3 and $\boldsymbol{X}_2$ over N uses of the lower channel.

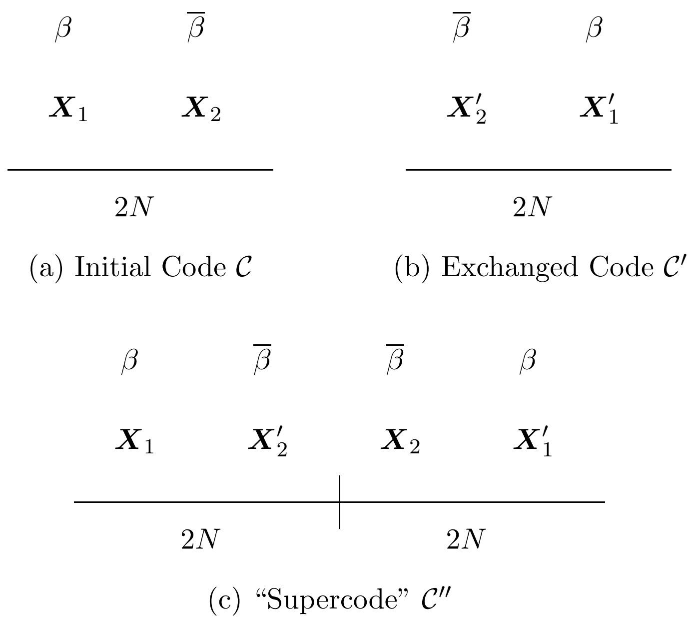

Figure 4-4: Construction of code with uniform power allocation by code exchange.

Let the decoding functions g_{LL}, g_{LH}, g_{HL} and g_{HH} be as in (4.36). We shall construct a new coding scheme $\mathcal{C}'$ with code length $2N$ and rates $(R'_{LL}, R'_0, R'_{LH}, R'_{HL}, R'_{HH})$ by "exchanging" the power allocation over the first N symbols and that over the second N symbols, as well as the rates of the HL state with that of the LH state. Define the new encoding function $f' : \mathcal{W}_{LL} \times \mathcal{W}_0 \times \mathcal{W}_{HL} \times \mathcal{W}_{LH} \times \mathcal{W}_{HH} \mapsto \mathbb{R}^{2N}$ by

$$\begin{aligned} f'(W'_{LL}, W'_0, W'_{LH}, W'_{HL}, W'_{HH}) &= \phi\left(f(W'_{LL}, W'_0, W'_{HL}, W'_{LH}, W'_{HH})\right) \\ &= \phi(\boldsymbol{X}'_1, \boldsymbol{X}'_2) \\ &= (\boldsymbol{X}'_2, \boldsymbol{X}'_1) \end{aligned}$$

where the function ϕ exchanges the first N components of its argument with the second N components. Note carefully that the component W'_{LH} is selected from $\mathcal{W}_{HL}$ and is thus transmitted at the rate $R'_{LH} = R_{HL}$ while W'_{HL} is transmitted at the rate $R'_{HL} = R_{LH}$. The rates $R'_0 = R_0$, $R'_{LL} = R_{LL}$ and $R'_{HH} = R_{HH}$ remain the same. Correspondingly, the decoding functions $g'_{LL} : \mathbb{R}^{2N} \mapsto \mathcal{W}_{LL} \times \mathcal{W}_0 \times \mathcal{W}_{LH} \times \mathcal{W}_{HL} \times \mathcal{W}_{HH}$, $g'_{LH} : \mathbb{R}^{2N} \mapsto \mathcal{W}_0 \times \mathcal{W}_{HL} \times \mathcal{W}_{HH}$, $g'_{HL} : \mathbb{R}^{2N} \mapsto \mathcal{W}_0 \times \mathcal{W}_{LH} \times \mathcal{W}_{HH}$, and $g_{HH} : \mathbb{R}^{2N} \mapsto \mathcal{W}_{HH}$ are defined by $g'_{LL}(\boldsymbol{Y}'_1, \boldsymbol{Y}'_2) = g_{LL}(\boldsymbol{Y}'_2, \boldsymbol{Y}'_1)$, $g'_{LH}(\boldsymbol{Y}'_1, \boldsymbol{Z}'_2) = g_{HL}(\boldsymbol{Z}'_2, \boldsymbol{Y}'_1)$, $g'_{HL}(\boldsymbol{Z}'_1, \boldsymbol{Y}'_2) = g_{LH}(\boldsymbol{Y}'_2, \boldsymbol{Z}'_1)$, and $g'_{HH}(\boldsymbol{Z}'_1, \boldsymbol{Z}'_2) = g_{HH}(\boldsymbol{Z}'_2, \boldsymbol{Z}'_1)$. Since the HL and LH states are *equally likely*, the new code $\mathcal{C}'$ achieves the *same expected rate* $R^* = R_{HH} + (1-q^2)[\frac{1}{2}(R_0 + R_{LH}) + \frac{1}{2}(R_0 + R_{HL})] + (1-q)^2 R_{LL}$ as the old code $\mathcal{C}$ over the block Gaussian channel with block length N. Meanwhile, the power allocation has been switched by the new code:

$$\overline{\beta} P = \frac{1}{2N} \sum_{n=1}^{N} \mathbb{E}\left[{X'_{2n}}^2\right], \quad \beta P = \frac{1}{2N} \sum_{n=1}^{N} \mathbb{E}\left[{X'_{1n}}^2\right]. \tag{4.48}$$

As shown in part (b) of Figure 4-4, $\boldsymbol{X}'_2$ is transmitted over N uses of the *upper* channel of Figure 4-3 using a fraction $\overline{\beta}$ of the power and $\boldsymbol{X}'_1$ over N uses of the *lower* channel using a fraction β of the power.

We now combine the codes $\mathcal{C}$ and $\mathcal{C}'$ to create a "supercode" $\mathcal{C}''$ with a block length of $4N$ as follows. Define the encoding function $f'' : \mathcal{W}_{LL} \times \mathcal{W}_0 \times \mathcal{W}_{LH} \times \mathcal{W}_{HL} \times \mathcal{W}_{HH} \times \mathcal{W}_{LL} \times \mathcal{W}_0 \times \mathcal{W}_{HL} \times \mathcal{W}_{LH} \times \mathcal{W}_{HH} \mapsto \mathbb{R}^{4N}$ by

$$f''(W_{LL}, W_0, W_{LH}, W_{HL}, W_{HH}, W'_{LL}, W'_0, W'_{LH}, W'_{HL}, W'_{HH}) = (\boldsymbol{X}_1, \boldsymbol{X}'_2, \boldsymbol{X}_2, \boldsymbol{X}'_1)$$

where $f(W_{LL}, W_0, W_{LH}, W_{HL}, W_{HH}) = (\boldsymbol{X}_1, \boldsymbol{X}_2)$ and $f'(W'_{LL}, W'_0, W'_{LH}, W'_{HL}, W'_{HH}) = (\boldsymbol{X}'_2, \boldsymbol{X}'_1)$. As shown in part (c) of Figure 4-4, $(\boldsymbol{X}_1, \boldsymbol{X}'_2)$ is transmitted over $2N$ uses of the upper channel of Figure 4-3 and $(\boldsymbol{X}_2, \boldsymbol{X}'_1)$ over $2N$ uses of the lower channel. Here,

$(\boldsymbol{X}_1, \boldsymbol{X}_2)$ satisfies (4.47) and $(\boldsymbol{X}_1, \boldsymbol{X}_2)$ satisfies (4.48) and so the new code clearly meets an average power constraint of P over $4N$ symbols.

The decoding functions $g''_{LL} : \mathbb{R}^{4N} \mapsto \mathcal{W}_{LL} \times \mathcal{W}_0 \times \mathcal{W}_{LH} \times \mathcal{W}_{HL} \times \mathcal{W}_{HH} \times \mathcal{W}_{LL} \times \mathcal{W}_0 \times \mathcal{W}_{LH} \times \mathcal{W}_{HL} \times \mathcal{W}_{HH}$, $g''_{HH} : \mathbb{R}^{4N} \mapsto \mathcal{W}_{HH} \times \mathcal{W}_{HH}$, $g''_{LH} : \mathbb{R}^{4N} \mapsto \mathcal{W}_0 \times \mathcal{W}_{LH} \times \mathcal{W}_{HH} \times \mathcal{W}_0 \times \mathcal{W}_{HL} \times \mathcal{W}_{HH}$ and $g''_{HL} : \mathbb{R}^{4N} \mapsto \mathcal{W}_0 \times \mathcal{W}_{HL} \times \mathcal{W}_{HH} \times \mathcal{W}_0 \times \mathcal{W}_{LH} \times \mathcal{W}_{HH}$ are given by

$$
\begin{aligned}
g''_{LL}(\boldsymbol{Y}_1, \boldsymbol{Y}'_2, \boldsymbol{Y}_2, \boldsymbol{Y}'_1) &= (g_{LL}(\boldsymbol{Y}_1, \boldsymbol{Y}_2), g'_{LL}(\boldsymbol{Y}'_2, \boldsymbol{Y}'_1)), \\
g''_{HH}(\boldsymbol{Z}_1, \boldsymbol{Z}'_2, \boldsymbol{Z}_2, \boldsymbol{Z}'_1) &= (g_{HH}(\boldsymbol{Z}_1, \boldsymbol{Z}_2), g'_{HH}(\boldsymbol{Z}'_2, \boldsymbol{Z}'_1)), \\
g''_{LH}(\boldsymbol{Y}_1, \boldsymbol{Y}'_2, \boldsymbol{Z}_2, \boldsymbol{Z}'_1) &= (g_{LH}(\boldsymbol{Y}_1, \boldsymbol{Z}_2), g'_{LH}(\boldsymbol{Y}'_2, \boldsymbol{Z}'_1)), \\
g''_{HL}(\boldsymbol{Z}_1, \boldsymbol{Z}'_2, \boldsymbol{Y}_2, \boldsymbol{Y}'_1) &= (g_{HL}(\boldsymbol{Z}_1, \boldsymbol{Y}_2), g'_{HL}(\boldsymbol{Z}'_2, \boldsymbol{Y}'_1)).
\end{aligned}
$$

If we now consider the i.i.d. block Gaussian channel with block length $2N$ (that is, each "high" and "low" state applies over $2N$ channel uses), we easily see that the rates of $\mathcal{C}''$ satisfy $R''_{LL} = R_{LL}$, $R''_{HH} = R_{HH}$, $R''_0 = R_0$, $R''_{LH} = R''_{HL} = \frac{1}{2}(R_{LH} + R_{HL})$ so that the expected rate over the $4N$-symbol block Gaussian channel is again R^*. Moreover, by construction, the fraction of power allocated to each half of the block Gaussian channel is $\frac{1}{2}$. Part (b) of Figure 4-4 illustrates the transmission of codewords from the new code over the block Gaussian channel. □

Given any coding scheme (with some power allocation) achieving expected rate R^* over a two-state i.i.d. two-block Gaussian channel with block length N, Lemma 4.8 gives a code with uniform power allocation and twice the code length which achieves the same expected rate over a corresponding two-block Gaussian channel with block length $2N$. Since Corollary 4.2 applies to two-state i.i.d block Gaussian channels with *any block length*, we see that the expected rate of any code over the block Gaussian channel is bounded above by (4.46) for $\beta = \frac{1}{2}$. Thus, in optimizing the upper bounds, we need only consider coding schemes which use uniform power allocation.

Lemma 4.9 *For $\beta = \overline{\beta} = \frac{1}{2}$, the upper bound in* (4.46) *holds with $\theta_1 = \theta_2$ and $\gamma_1 = \gamma_2$.*

PROOF. For $\beta = \frac{1}{2}$, there exist θ_i, γ_i, $i = 1, 2$ such that (4.37)-(4.40) hold. Now *fixing* the values of the bounds in (4.37) and (4.40), we vary the parameters θ_i and γ_i to maximize the sum of the bounds in (4.38)-(4.39). Removing the logarithms in (4.46), we suppose (equivalently) that the bound is achieved with

$$\begin{aligned}(P\overline{\gamma_1} + \eta_2)(P\overline{\gamma_2} + \eta_2) &= A \\ (P\overline{\theta_1} + \eta_1)(P\overline{\theta_2} + \eta_1) &= B,\end{aligned}$$

using $\beta = 1/2$. The first constraint corresponds to fixing the bound for R_{LL} and the second to fixing the bound for R_{HH}. Subject to these constraints we wish to maximize the constraint on the sum rate $R_0 + 1/2\,(R_{HL} + R_{LH})$ or equivalently,

$$\frac{(P\overline{\theta_1} + \eta_1)(P\overline{\theta_2} + \eta_1)(P\overline{\theta_1} + \eta_2)(P\overline{\theta_2} + \eta_2)}{(P\overline{\gamma_1} + \eta_2)(P\overline{\gamma_2} + \eta_2)(P\overline{\gamma_1} + \eta_1)(P\overline{\gamma_2} + \eta_1)}$$

which reduces immediately to maximizing

$$\frac{(P\overline{\theta_1} + \eta_2)(P\overline{\theta_2} + \eta_2)}{(P\overline{\gamma_1} + \eta_1)(P\overline{\gamma_2} + \eta_1)}.$$

This latter maximization is easily seen to be two separate maximizations. Set

$$x := P\overline{\theta_1} + \eta_1$$

$$y := P\overline{\theta_2} + \eta_1$$

and we wish then to maximize $xy - (\eta_1 - \eta_2)(x + y) + (\eta_1 - \eta_2)^2$ subject to $xy = B$. This occurs when $x = y$ or $\overline{\theta_1} = \overline{\theta_2}$. A similar argument applies to the denominator. Thus the optimization has a solution with $\theta_1 = \theta_2$ and $\gamma_1 = \gamma_2$. Hence, (4.46) holds with $\beta = \frac{1}{2}$ and $\theta_1 = \theta_2$, $\gamma_1 = \gamma_2$. □

The previous lemmas lead immediately to the following theorem.

Theorem 4.4 *Consider the two-state i.i.d block Gaussian channel with an average transmit power constraint P as in (4.2) ($K = 2$) and decoding delay constraint of two N-blocks. Let $\eta_1 > \eta_2$ be the variances for the High and Low channels and let $q > 0$ be the probability of encountering a High channel. For any $(2N, R_{LL}, R_0, R_{LH}, R_{HL}, R_{HH})$ code, if P_{el} is* *required to be arbitrarily small for every $l = 1,2,3,4$, then the expected rate satisfies*

$$\begin{aligned}\mathbb{E}[R] \le\; & C\left(\frac{\alpha_1^* P}{\eta_1 + \overline{\alpha_1^*} P}\right) + \frac{1}{2}(1-q^2)\left[C\left(\frac{\alpha_2^* P}{\eta_1 + \alpha_3^* P}\right) + C\left(\frac{\alpha_2^* P}{\eta_2 + \alpha_3^* P}\right)\right] \\ & + (1-q)^2 C\left(\frac{\alpha_3^* P}{\eta_2}\right)\end{aligned} \tag{4.49}$$

where $\boldsymbol{\alpha}^* = (\alpha_1^*, \alpha_2^*, \alpha_3^*)$ maximizes

$$C\left(\frac{\alpha_1 P}{\eta_1 + \overline{\alpha_1} P}\right) + \frac{1}{2}(1-q^2)\left[C\left(\frac{\alpha_2 P}{\eta_1 + \alpha_3 P}\right) + C\left(\frac{\alpha_2 P}{\eta_2 + \alpha_3 P}\right)\right] + (1-q)^2 C\left(\frac{\alpha_3 P}{\eta_2}\right) \tag{4.50}$$

subject to $\alpha_l \ge 0, l = 1,2,3$ and $\sum \alpha_l = 1$.

We now observe that a broadcast strategy attains the upper bound in (4.49). Such a strategy would allocate equal power P across the $2N$ uses of the two-block Gaussian channel. A fraction $\alpha_1^* P$ of the power would be used to send independent information to the HH state (also common to the HL, LH, and LL states); a fraction $\alpha_2^* P$ is used to send information common to the HL and LH states (as well as LL). Finally, a fraction $\alpha_3^* P$ is used to send independent information to the LL state. The receiver decodes as much information as it can depending on the channel, using successive decoding. The broadcast strategy, then, remains optimal in terms of expected rate for the two-state i.i.d. block Gaussian channel with a decoding delay constraint of two N-blocks.

4.3.2 Optimization of Expected Rate

Since the two-block channel considered above retains the broadcast structure, we would expect that the technique used in Theorem 4.2 can be used to solve the maximization problem in (4.50). For convenience, we write the optimization as

$$\begin{aligned} \text{maximize} \;\; & R_1 + (1-q^2)R_2 + (1-q)^2 R_3 \\ \text{subject to} \;\; & R_1 \le C\left(\frac{\alpha_1 P}{(\alpha_2+\alpha_3)P+\eta_1}\right), \\ & R_2 \le \frac{1}{2}C\left(\frac{\alpha_2 P}{\alpha_3 P+\eta_1}\right) + \frac{1}{2}C\left(\frac{\alpha_2 P}{\alpha_3 P+\eta_2}\right), \\ & R_3 \le C\left(\frac{\alpha_3 P}{\eta_2}\right), \\ & \alpha_l \ge 0, l = 1,2,3; \; \sum_{l=1}^{3} \alpha_l = 1. \end{aligned} \tag{4.51}$$

Theorem 4.5 *Consider the optimization problem in (4.51). Define the marginal utility functions*

$$\begin{aligned} u_1(z) &\equiv \frac{1}{2(z+\eta_1)}, \\ u_2(z) &\equiv \frac{1}{4}(1-q^2)\left[\frac{1}{z+\eta_1} + \frac{1}{z+\eta_2}\right], \\ u_3(z) &\equiv (1-q^2)\frac{1}{2(z+\eta_2)}, \\ u^*(z) &\equiv \max_{l=1,2,3} u_l(z), \end{aligned}$$

and sets

$$\mathcal{A}_l \equiv \{z \in [0,P] : u_l(z) = u^*(z)\}.$$

Then the optimal solution to (4.51) is

$$\int_0^P u^*(z)dz$$

attained at the unique point

$$R_1^* = \int_{\mathcal{A}_1} \frac{1}{2(z+\eta_1)} dz, \tag{4.52}$$

$$R_2^* = \int_{\mathcal{A}_2} \frac{1}{4}\left[\frac{1}{z+\eta_1} + \frac{1}{z+\eta_2}\right] dz, \tag{4.53}$$

$$R_3^* = \int_{\mathcal{A}_3} \frac{1}{2(z+\eta_2)} dz, \tag{4.54}$$

where

$$\begin{aligned}\mathcal{A}_l &= \left[\sum_{j>l}\alpha_j^* P, \sum_{j\geq l}\alpha_j^* P\right],\\ \alpha_l^* P &= |\mathcal{A}_l|\end{aligned}$$

for $l = 1, 2, 3.$

PROOF. The proof is similar in spirit to that for Theorem 3.2 of [Tse99]. We provide the argument for completeness. First, note that since the objective function is linear in $\boldsymbol{R}$, any maxima are attained on the boundary on the feasible region. Let the optimal solution to (4.51) be achieved at

$$\begin{aligned}R_1^* &= C\left(\frac{\alpha_1^* P}{(\alpha_2^*+\alpha_3^*)P+\eta_1}\right),\\ R_2^* &= \frac{1}{2}C\left(\frac{\alpha_2^* P}{\alpha_3^* P+\eta_1}\right)+\frac{1}{2}C\left(\frac{\alpha_2^* P}{\alpha_3^* P+\eta_2}\right),\\ R_3^* &= C\left(\frac{\alpha_3^* P}{\eta_2}\right),\end{aligned}$$

where $\alpha_l^* \geq 0, l = 1, 2, 3, \sum_{l=1}^{3}\alpha_l^* = 1$. Then the optimal value J^* satisfies

$$\begin{aligned}J^* &= R_1^* + (1-q^2)R_2^* + (1-q)^2 R_3^*\\ &= \frac{1}{2}\left[\ln(P+\eta_1) - \ln((\alpha_2^*+\alpha_3^*)P+\eta_1)\right]\\ &\quad +(1-q^2)\frac{1}{4}\left[\ln((\alpha_2^*+\alpha_3^*)P+\eta_1) - \ln(\alpha_3^* P+\eta_1)\right]\\ &\quad +(1-q^2)\frac{1}{4}\left[\ln((\alpha_2^*+\alpha_3^*)P+\eta_2) - \ln(\alpha_3^* P+\eta_2)\right]\\ &\quad +(1-q)^2\frac{1}{2}\left[\ln(\alpha_3^* P+\eta_2) - \ln(\eta_2)\right]\\ &= \int_{(\alpha_2^*+\alpha_3^*)P}^{P}\frac{1}{2(z+\eta_1)}dz + \int_{\alpha_3^* P}^{(\alpha_2^*+\alpha_3^*)P}\frac{1}{4}(1-q^2)\left[\frac{1}{z+\eta_1}+\frac{1}{z+\eta_2}\right]dz\\ &\quad + \int_0^{\alpha_3^* P}(1-q)^2\frac{1}{2(z+\eta_2)}dz\\ &= \sum_{l=1}^{3}\int_{\sum_{j>l}\alpha_j^* P}^{\sum_{j\geq l}\alpha_j^* P} u_l(z)dz\\ &\leq \int_0^P u^*(z)dz. \end{aligned} \tag{4.55}$$

We now produce a power allocation to achieve this upper bound. Define $\mathcal{A}_l, l = 1, 2, 3$, as in the statement of the theorem. Note that the $\mathcal{A}_l$'s form a partition of $[0, P]$. Now each $u_l(z)$ is strictly decreasing as a function of z for all $z \geq 0$. It is not hard to see that for $q > 0$, the equation $u_i(z) = u_j(z), i \neq j$, has a unique finite solution. This implies that sets $\mathcal{A}_l$ are single intervals.

Next we show that the $\mathcal{A}_l$'s are ordered on the real line from left to right as $\mathcal{A}_3, \mathcal{A}_3, \mathcal{A}_1$. The equation $u_1(z) = u_2(z)$ has a unique positive solution $\overline{z} > 0$ if and only if $u_2(0) > u_1(0)$. In this case, $u_2(z) > u_1(z), \forall z < \overline{z}$ and $u_2(z) < u_1(z), \forall z > \overline{z}$. Thus $\mathcal{A}_2$ lies to the left of $\mathcal{A}_1$. The equation $u_1(z) = u_2(z)$ has a unique negative solution $\overline{z} < 0$ if and only if $u_2(0) < u_1(0)$. In this case, $u_2(z) < u_1(z), \forall z \geq 0$, so that $\mathcal{A}_2 = \emptyset$. The above argument may be repeated for the pair $u_2(z), u_3(z)$ to show that $\mathcal{A}_3$ lies to the left of $\mathcal{A}_2$.

Given the above ordering, we may choose $\alpha_l^* \geq 0, l = 1, 2, 3, \sum_{l=1}^{3} \alpha_l^* = 1$ such that

$$\mathcal{A}_1 = [0, \alpha_3^* P],\ \mathcal{A}_2 = [\alpha_3^* P, (\alpha_2^* + \alpha_3^*)P],\ \mathcal{A}_3 = [(\alpha_2^* + \alpha_3^*)P, P].$$

where $\alpha_l^* P = |\mathcal{A}_l|, l = 1, 2, 3$. Let $R_l^*, l = 1, 2, 3$, be as in (4.52)-(4.54). Then

$$\begin{aligned}
R_1^* + (1-q^2)R_2^* + (1-q)^2 R_3^* &= \int_{\mathcal{A}_1} \frac{1}{2(z+\eta_1)} dz + \int_{\mathcal{A}_2} \frac{1}{4}(1-q^2)\left[\frac{1}{z+\eta_1} + \frac{1}{z+\eta_2}\right] dz \\
&\quad + \int_{\mathcal{A}_3} (1-q)^2 \frac{1}{2(z+\eta_2)} dz \\
&= \sum_{l=1}^{3} \int_{\mathcal{A}_l} u_l(z) dz \\
&= \int_0^P u^*(z) dz.
\end{aligned}$$

So we attain the upper bound in (4.55). As mentioned before, superposition coding combined with successive decoding can be used to achieve the expected rate $R_1^* + (1-q^2)R_2^* + (1-q)^2 R_3^*$. Finally, the uniqueness of our solution follows from the fact that for all $z \geq 0$ (except for a finite set of intersection points), there exists a unique $l, 1 \leq l \leq 3$, such that $z \in \mathcal{A}_l$. □

Let us associate states in the two-block Gaussian channel (HH, {LH,HL},LL) with "virtual users" in a superposition code. Just as in the one-block case, the optimal solution to (4.51) is found in a greedy manner where at each interference level z, the transmitter allocates power to the virtual user with the largest marginal utility function.

Figure 4-5: Optimization procedure for a two-block Gaussian channel.

Figure 4-5 demonstrates the optimization procedure of Theorem 4.5. Marginal utilities $u_l(z), l = 1, 2, 3$, are plotted as functions of interference level z. Here, we have $\frac{1}{4}(1-q^2)(\frac{1}{\eta_1} + \frac{1}{\eta_2}) > \frac{1}{2\eta_1} > \frac{(1-q)^2}{2\eta_2}$. The intervals are $\mathcal{A}_3 = \emptyset$, $\mathcal{A}_2 = [0, \alpha_2^* P]$, $\mathcal{A}_1 = [\alpha_2^* P, P]$. The optimal rates R_l^* are given by (4.52)-(4.54). Virtual user 3 (LL state) receives zero power and therefore zero rate, since $u_3(z)$ never dominates in $[0, P]$. Rates R_1^*, R_2^* are achieved by superposition coding with successive decoding, in the order 1,2.

4.4 Summary and Discussion

In this work, we showed how a certain set of assumptions regarding the fading process, channel state information, and the nature of the traffic carried, leads to an optimal transmission strategy over a single-user wireless channel. In particular, we considered transmission of time-sensitive data over a slowly-varying flat-fading additive white Gaussian channels with finite decoding delay constraints and delayed channel state feedback to the transmitter. This formulation led to a view of delay-limited communications emphasizing the expected rate reliably decoded at the receiver. Using Fano's inequality and the entropy power inequality, we have shown that the broadcast strategy maximizes the expected rate when the decoding delay constraint is one N-block and in certain cases when the decoding delay constraint is two N-blocks. It is clearly desirable to extend our results to the general case of L noise levels and K delay blocks. This requires a more advanced understanding of the underlying general non-degraded parallel Gaussian broadcast channels. Finally, a deeper understanding of delayed feedback in the context of delay-limited communications is needed to fully assess the interaction between the delay parameter d and the coding depth KN.

Chapter 5

Conclusions

We close with a few comments on the central role of successive decoding in our work, the "inter-layer" view of multi-user communications, and performance measures for communication over fading channels.

A common theme that runs through all parts of our study is the concept of successive decoding. In Chapter 2, successive decoding was combined with time-sharing to produce a low-complexity coding scheme for multiaccess channels. In Chapter 3, the optimal multiaccess queue control strategy turns out to be a form of adaptive successive decoding, where the decoding order at any particular time depends on the queue state. Finally, in Chapter 4, successive decoding plays a central role in the broadcast or successive refinement transmission strategy. Given its clear importance of a communication technique for multi-user and fading channels, it seems that more extensive studies on the subject, both theoretical and experimental, are warranted. In Chapters 2 and 4, we presented some simple bounds on overall error probability of successive decoders using a genie-aided argument. It is desirable to investigate whether these bounds are tight, and exactly how much is lost in error probability performance when successive decoding is used in place of more complex joint decoding. Further studies are also needed on how successive decoding schemes are to be implemented in practice. Recall that in analyzing the overall probability of error, we assumed that at each stage of the successive decoding algorithm, a genie provided the correct estimate of the codeword at the previous level to the current decoder. In practice, however, there is always a positive probability that the estimate at the previous level is incorrect, no matter how powerful the error correcting code used. Thus, it seems that some sort of iteration is needed among the various levels of successive decoding. Exactly how a low-complexity iterative scheme should be designed is an open problem. Finally, there is the question of how successive decoding can be used over fading channels, particularly when the channel is not known with perfect precision at the receiver. Here, it is necessary for the decoder to simultaneously decode the codeword and estimate the channel. At every stage of successive decoding, one would like to subtract the true value of the channel times the correct codeword. However, in practice, one can only subtract an estimate of the channel times an estimate of the codeword. It is easy to see how these two sources of error can lead to a catastrophic propagation of errors in successive decoding. This problem can be particularly acute in multiaccess fading settings where there are many different channels to estimate. The problem may be less severe in single-user cases as considered in Chapter 4, since there is only one channel to estimate. Although there have been some fine studies of successive decoding on fading channels (for instance [Gal94]), this remains an open area.

In this dissertation, we attempted to give an "inter-layer" analysis of multiaccess which treats issues of noise, interference, randomness of arrivals, delay, and error probability in a more cohesive framework. Our analysis combined elements of multiaccess information theory, which adequately models the noise and interference aspects of the problem but ignores the random arrival of messages, and queueing theory, which focuses on issues of source burstiness and delay but ignores physical-layer modeling. Our approach was to assume optimal coding at the physical layer so that all rates in the multiaccess capacity region are achievable, and then seeking the optimal rate allocation policy to minimize the overall packet delay (or bit delay) in the system. One can interpret our result as a "converse," in the sense that the average packet (bit) delay associated with our optimal policy gives a lower bound to the corresponding packet (bit) delays of all coding schemes which seek to meet any given level of decoding error probability. There are undoubtedly many other approaches that one can take within this "inter-layer" framework. Our primary aim is to show that such problems can and should be analyzed. Our belief is that as multi-user communication systems become increasingly complex, the need for joint network design over many layers will become more acute, and therefore the pursuit of a more comprehensive theory of networks is of paramount importance. Such a theory should have the same kind of explanatory and predictive power possessed by information theory for classical point-to-point channels. While this is surely a very long-term goal, there is no inherent reason why more concrete steps cannot be taken towards that end. Our "inter-layer" analysis, then, can be seen as a small step in this process.

In the latter part of the thesis, we examined single-user communications over wireless fading channels. We showed that the appropriate performance metric and the optimal transmission strategy depend on the interplay of fading parameters, decoding delay constraint, the ability of the transmitter and receiver to track channel variations, and the nature of the traffic carried by the system. While measures such as capacity *vs.* outage and delay-limited capacity are appropriate for delay-constrained constant-rate voice traffic, the expected reliably received rate seems to be a more fitting measure for delay-constrained variable-rate data traffic (such as those to be carried by Third Generation wireless networks). Under our particular assumptions of delayed channel state feedback to the transmitter, a broadcast or successive refinement strategy was shown to maximize the expected reliably received rate when the decoding delay is relatively short compared to the time scale of the channel fading process. An interesting question is whether some version of the broadcast strategy remains optimal when the decoding delay is gradually increased. The answer to this will depend on a deeper understanding of parallel broadcast channel models, and is left to future work.

Appendix A

Proofs for Multiaccess Time-sharing

A.1 All faces $\mathcal{F}_S$ in an M-user ($M \geq 2$) Gaussian MAC have dimension $M-2$

The proof will be by induction on the number of users $M, M \geq 2$. We use the notation $C(P, \sigma^2) \equiv \frac{1}{2}\log(1 + \frac{P}{\sigma^2})$. We start with $M = 2$. Let $S = \{1\}$, the hyperplane $\mathcal{H}_{\{1\}} = \{(R_1, R_2) : R_1 = C(P_1, P_2 + \sigma^2)\}$, then $\mathcal{F}_{\{1\}} = \mathcal{H}_{\{1\}} \cap \mathcal{D} = (C(P_1, P_2 + \sigma^2), C(P_2, \sigma^2))$. Thus, $\dim(\mathcal{F}_{\{1\}}) = 0$. Similarly, $\dim(\mathcal{F}_{\{2\}}) = 0$.

Now assume that for any M-user Gaussian MAC, $M \geq 2$, $\dim(\mathcal{F}_S) = M-2$ for all $\emptyset \subset S \subset \{1, \dots, M\}$. For the $M+1$-user Gaussian MAC, let $\emptyset \subset S_1 \subset \{1, \dots, M+1\}$. By its definition, $\mathcal{F}_{S_1}$ is the intersection of $\mathcal{D}$ and the hyperplane $\mathcal{H}_{S_1} = \{(R_1, \dots, R_{M+1}) : \sum_{i \in S_1} R_i = C(\sum_{i \in S_1} P_i, \sigma^2 + \sum_{j \in S_1^c} P_j)$. Clearly, $\dim(\mathcal{F}_{S_1}) \leq M-1$.

Let $k \in \{1, \dots, M+1\} \backslash S_1$. Fix $R_k = C(P_k, \sigma^2)$. If $S_1 = \{1, \dots, M+1\} \backslash \{k\}$, then $\mathcal{F}_{S_1}$ projected onto the S_1 coordinates is the dominant face of an M-user Gaussian MAC with users $X_i, i \in S_1$, and noise variance $P_k + \sigma^2$. From Lemma 2.1, $\dim(\mathcal{F}_{S_1}) = M-1$. If $S_2 \equiv \{1, \dots, M+1\} \backslash (\{k\} \cup S_1) \neq \emptyset$, let $\mathcal{F}_1$ be the set of rate tuples $(R_1, \dots, R_{M+1})$ in $\mathcal{F}_{S_1}$ such that $R_k = C(P_k, \sigma^2)$, $\sum_{i \in S_1} R_i = C\left(\sum_{i \in S_1} P_i, \sum_{j \in S_1^c} P_j + \sigma^2\right)$, $\sum_{i \in S_2} R_i = C\left(\sum_{i \in S_2} P_i, P_k + \sigma^2\right)$. It is easy to see that the projection of $\mathcal{F}_1$ onto the $\{1, \dots, M+1\} \backslash \{k\}$ coordinates is a face $\mathcal{F}'_{S_1}$, $\emptyset \subset S_1 \subset \{1, \dots, M+1\} \backslash \{k\}$, of an M-user Gaussian MAC with users $X_i, i \in \{1, \dots, M+1\} \backslash \{k\}$, and noise variance $P_k + \sigma^2$. By the inductive assumption,

$\dim(\mathcal{F}_1) = M-2$. Now choose the rate tuple $\mathbf{\Phi}$ such that $\Phi_k = C(P_k, \sigma^2 + \sum_{j\in S_2} P_j)$, $\sum_{i\in S_1} \Phi_i = C\left(\sum_{i\in S_1} P_i, \sum_{j\in S_1^c} P_j + \sigma^2\right)$, $\sum_{i\in S_2} \Phi_i = C\left(\sum_{i\in S_2} P_i, \sigma^2\right)$. Notice that $\mathbf{\Phi} \in \mathcal{F}_{S_1}$, but since $\sum_{i\in S_2} \Phi_i \neq C(\sum_{i\in S_2} P_i, P_k + \sigma^2)$, $\mathbf{\Phi}$ does not lie in the affine hull of $\mathcal{F}_1$. Since $\mathcal{F}_{S_1}$ is convex, the convex hull $conv(\mathcal{F}_1, \mathbf{\Phi}) \subseteq \mathcal{F}_{S_1}$. Hence, $\dim(\mathcal{F}_{S_1}) \geq M-1$. Thus, $\dim(\mathcal{F}_{S_1}) = M-1$. $\square$

A.2 Proof of Lemma 2.2

Let $\emptyset \subset S \subset \{1,\dots,M\}$. We first prove $\boldsymbol{R} \in \mathcal{F}_S \Rightarrow \boldsymbol{R}_S \in \mathcal{D}_{P_S}$ and $\boldsymbol{R}_{S^c} \in \mathcal{D}_{P_{S^c|S}}$. Since $\boldsymbol{R} \in \mathcal{F}_S$, $\sum_{i\in S} R_i = I(X_S;Y)$. Let $V \subseteq S$. Then, $S\backslash V \subseteq S$. Since $\boldsymbol{R} \in \mathcal{F}_S \subset \mathcal{D}$, $\sum_{i\in V} R_i \geq I(X_V;Y)$ and $\sum_{i\in S\backslash V} R_i \geq I(X_{S\backslash V};Y)$. Then,

$$\sum_{i\in V} R_i = \sum_{i\in S} R_i - \sum_{i\in S\backslash V} R_i \leq I(X_S;Y) - I(X_{S\backslash V};Y) = I(X_V;Y|X_{S\backslash V}).$$

Thus, $\boldsymbol{R}_S \in \mathcal{D}_{P_S}$. Now, $\boldsymbol{R} \in \mathcal{F}_S$ also implies $\sum_{i\in S^c} R_i = I(X_{S^c};Y|X_S)$. Let $V \subseteq S^c$. $\boldsymbol{R} \in \mathcal{D} \Rightarrow \sum_{i\in V} R_i \leq I(X_V;Y|X_{V^c})$. Also, $\sum_{i\in S^c\backslash V} R_i \leq I(X_{S^c\backslash V};Y|X_S,X_V)$. Then,

$$\begin{aligned}
\sum_{i\in V} R_i &= \sum_{i\in S^c} R_i - \sum_{i\in S^c\backslash V} R_i \\
&\geq I(X_S;Y|X_S) - I(X_{S^c\backslash V};Y|X_S,X_V) \\
&= I(X_V;Y|X_S).
\end{aligned}$$

Thus, $\boldsymbol{R}_{S^c} \in \mathcal{D}_{P_{S^c|S}}$.

Next, we show $\boldsymbol{R}_S \in \mathcal{D}_{P_S}, \boldsymbol{R}_{S^c} \in \mathcal{D}_{P_{S^c|S}} \Rightarrow (\boldsymbol{R}_S, \boldsymbol{R}_{S^c}) \in \mathcal{F}_S \subset \mathcal{D}$. Let $V \subseteq \{1,\dots,M\}$. Then, $V = V_1 \cup V_2$, where $V_1 = V \cap S, V_2 = V \cap S^c$. Note that $V^c = (S\backslash V_1) \cup (S^c\backslash V_2)$. By definition of $\mathcal{D}_{P_S}$ and $\mathcal{D}_{P_{S^c|S}}$ via (2.9)-(2.10),

$$\begin{aligned}
\sum_{i\in V_1} R_i &\leq I(X_{V_1};Y|X_{S\backslash V_1}) \leq I(X_{V_1};Y|X_{S\backslash V_1}, X_{S^c\backslash V_2}), \\
\sum_{i\in V_2} R_i &\leq I(X_{V_2};Y|X_{V_2^c}) = I(X_{V_2};Y|X_{S\backslash V_1}, X_{S^c\backslash V_2}, X_{V_1}).
\end{aligned}$$

Therefore, by the chain rule,

$$\sum_{i\in V} R_i = \sum_{i\in V_1} R_i + \sum_{i\in V_2} R_i \le I(X_V;Y|X_{S\setminus V_1}, X_{S^c\setminus V_2}) = I(X_V;Y|X_{V^c}).$$

Now we have $\sum_{i\in V_1} R_i \ge I(X_{V_1};Y)$ and $\sum_{i\in V_2} R_i \ge I(X_{V_2};Y|X_S) \ge I(X_{V_2};Y|X_{V_1})$. So $\sum_{i\in V} R_i \ge I(X_V;Y)$ by the chain rule. Thus, we have $\boldsymbol{R} = (\boldsymbol{R}_S, \boldsymbol{R}_{S^c}) \in \mathcal{D}$. But $\boldsymbol{R}_S \in \mathcal{D}_{P_S} \Rightarrow \sum_{i\in S} R_i = I(X_S;Y)$ and $\boldsymbol{R}_{S^c} \in \mathcal{D}_{P_{S^c|S}} \Rightarrow \sum_{i\in S^c} R_i = I(X_{S^c};Y|X_S)$. Therefore, $\boldsymbol{R} = (\boldsymbol{R}_S, \boldsymbol{R}_{S^c}) \in \mathcal{F}_S \subset \mathcal{D}$. □

A.3 Proof of Theorem 2.1

For $\rho \in [0,1]$, define

$$E_n(\rho, Q) \equiv -\ln \sum_y \left\{ \sum_x Q(x) P_{s_n}(y|x)^{1/(1+\rho)} \right\}^{1+\rho}; \quad n = 1, \ldots, N \tag{A.1}$$

$$E_o(\rho, Q, N) \equiv \frac{1}{N} \sum_{n=1}^{N} E_n(\rho, Q), \tag{A.2}$$

$$E_r\left(R, Q, N\right) \equiv \max_{0\le\rho\le1} \left[E_o(\rho, Q, N) - \rho R\right]. \tag{A.3}$$

Let N be any positive integer and R be any positive number. Let $P_{\boldsymbol{s}}^N$ be a (discrete memoryless) deterministically-varying channel. Consider an (N, R, Q) random code ensemble. Let $\overline{P}_{e,l}$ be the ensemble average probability of decoding error for message $l, 1 \le l \le \lceil e^{NR} \rceil = L$, using maximum likelihood (ML) decoding. Let $\overline{P}_e$ be the average error probability of the ensemble over all messages under ML decoding. We first show

$$\begin{aligned} \overline{P}_{e,l} &\le \exp\left[-NE_r\left(R, Q, N\right)\right]; \qquad 1 \le l \le L \\ \overline{P}_e &\equiv \sum_{l=1}^{L} \Pr\left(l\right) \overline{P}_{e,l} \le \exp\left[-NE_r\left(R, Q, N\right)\right]. \end{aligned} \tag{A.4}$$

This argument is a consequence of the Parallel Channels Result of [Gal68, pp. 149-150]. Regard the DVC $P_{\boldsymbol{s}}^N = \{P_{s_n}(y|x) : s_n \in \mathcal{S}, n = 1, \ldots, N\}$ as a composite vector channel made up of N independent parallel discrete channels. The composite channel takes as input N-tuples $\boldsymbol{x} = (x_1, \ldots, x_N) \in \mathcal{X}^N$ and produces as its output N-tuples $\boldsymbol{y} = (y_1, \ldots, y_N) \in$

$\mathcal{Y}^N$. The channel transition probability is given by (2.11). Construct an (N, R, Q) random code ensemble, and regard each codeword $\boldsymbol{x} = (x_1, \ldots, x_N)$ of a code in the (N, R, Q) ensemble as an N-tuple input letter of an (N', R') code into the vector channel, where the new block length $N' = 1$, and the new rate $R' = NR$. This gives an ensemble of (N', R') codes for the vector channel. The error exponent for this new code ensemble, for $\rho \in [0, 1]$, is

$$\begin{aligned} E'_o(\rho, Q, N) &\equiv -\ln \sum_{\boldsymbol{y}} \left\{ \sum_{\boldsymbol{x}} Q^N(\boldsymbol{x}) P^N_{\boldsymbol{s}}(\boldsymbol{y}|\boldsymbol{x})^{1/(1+\rho)} \right\}^{1+\rho} \\ &= -\ln \sum_{y_1} \cdots \sum_{y_N} \left\{ \sum_{x_1} \cdots \sum_{x_N} \prod_{n=1}^{N} Q(x_n) \prod_{n=1}^{N} P_{s_n}(y_n|x_n)^{1/(1+\rho)} \right\}^{1+\rho} \\ &= \sum_{n=1}^{N} E_n(\rho, Q) \end{aligned}$$

where $E_n(\rho, Q)$ is given by (A.1). Applying Theorem 5.6.2 of [Gal68] to the composite channel with the ensemble of (N', R') codes, we have for each $l, 1 \leq l \leq \lceil e^{NR} \rceil = L$,

$$\begin{aligned} \overline{P}_{e,l} &\leq \exp\left\{ -\left[E'_o(\rho, Q, N) - \rho R' \right] \right\} \\ &= \exp\left\{ -\left[\sum_{n=1}^{N} E_n(\rho, Q) - \rho R' \right] \right\} \\ &= \exp\left\{ -N \left[\frac{1}{N} \sum_{n=1}^{N} E_n(\rho, Q) - \rho R \right] \right\}. \end{aligned} \tag{A.5}$$

Since (A.5) holds for each message in the (N, R, Q) code ensemble, the average error probability of the ensemble over all messages satisfies

$$\overline{P}_e = \sum_{l=1}^{L} \Pr(l)\, \overline{P}_{e,l} \leq \exp\left\{ -N \left[E_o(\rho, Q, N) - \rho R \right] \right\}.$$

Since maximizing the exponent over $\rho \in [0, 1]$ can only improve the bound, we have established (A.4). The theorem now follows immediately from (A.4) and Definition 2.1. □

A.4 Proof of Theorem 2.2

Let N be any positive integer. We first show that for any (discrete memoryless) deterministically varying channel $P_{\boldsymbol{s}}^N$, $E_r(R,Q,N)$, as defined in Equations (A.2) and (A.3), is a convex, decreasing, and positive function of R for $0 \leq R < \frac{1}{N}\sum_{n=1}^N I(Q;P_{s_n})$. For this, it is again convenient to consider N uses of the DVC as a single use of a composite channel made up of N independent parallel discrete channels $P_{s_n}, n = 1, \ldots, N$. Construct an ensemble of (N', R') codes as before, with $N' = 1$. Notice that $R' = \sum_{n=1}^N R_n$, where R_n is the rate over the nth channel P_{s_n}. Thus, the rate of the corresponding ensemble of (N,R) code over the DVC is $R = \frac{1}{N}\sum_{n=1}^N R_n$.

Since the composite channel is discrete, we may apply the proof of Theorem 5.6.4 in [Gal68] to optimize $E_o(\rho,Q,N) - \rho R$ over $0 \leq \rho \leq 1$. It can be verified that the resulting $E_r(R,Q,N)$ behaves as follows. For

$$\left.\frac{\partial E_o(\rho,Q,N)}{\partial \rho}\right|_{\rho=1} = \frac{1}{N}\sum_{n=1}^N \left.\frac{\partial E_n(\rho,Q)}{\partial\rho}\right|_{\rho=1} \leq R \leq \left.\frac{\partial E_o(\rho,Q,N)}{\partial \rho}\right|_{\rho=0}$$
$$= \frac{1}{N}\sum_{n=1}^N \left.\frac{\partial E_n(\rho,Q)}{\partial\rho}\right|_{\rho=0} = \frac{1}{N}\sum_{n=1}^N I(Q;P_{s_n}),$$

$E_r(R,Q,N)$ and R are given parametrically in terms of ρ:

$$R = \frac{\partial E_o(\rho,Q,N)}{\partial\rho} = \frac{1}{N}\sum_{n=1}^N \frac{\partial E_n(\rho,Q)}{\partial\rho}; \qquad 0 \leq \rho \leq 1$$
$$E_r(R,Q,N) = E_o(\rho,Q,N) - \rho\frac{\partial E_o(\rho,Q,N)}{\partial\rho} = \frac{1}{N}\sum_{n=1}^N \left(E_n(\rho,Q) - \rho\frac{\partial E_n(\rho,Q)}{\partial\rho}\right).$$

The point $\left.\frac{\partial E_o(\rho,Q,N)}{\partial\rho}\right|_{\rho=1}$ is the critical rate R_{cr} for the given Q,N. For $R < R_{cr}$,

$$E_r(R,Q,N) = E_o(1,Q,N) - R = \frac{1}{N}\sum_{n=1}^N \left(E_n(1,Q) - R_n\right).$$

For $R > \frac{1}{N}\sum_{n=1}^N I(Q;P_{s_n})$, $E_r(R,Q,N) = 0$. The parameter ρ is the magnitude of the slope of the $E_r(R,Q,N)$ versus R curve. Finally, as a function of R, $E_r(R,Q,N)$ is convex, decreasing and positive for all $R < \frac{1}{N}\sum_{n=1}^N I(Q;P_{s_n})$. The theorem now follows immediately from Definition 2.1. $\square$

A.5 Proof of Theorem 2.4

Assume that the U^K bits are Bernoulli($\frac{1}{2}$) due to prior source coding. Then

$$\begin{aligned} NR &= K \\ &= H(U^K) \\ &= H(U^K|\hat{U}^K(Y^N)) + I(U^K;\hat{U}^K(Y^N)) \\ &\leq H(U^K|\hat{U}^K(Y^N)) + I(X^N;Y^N) \\ &\leq KP_b^K \ln(|\mathcal{U}|-1) + KH(P_b^K) + I(X^N;Y^N) \\ &= NR\cdot P_b^K \ln(|\mathcal{U}|-1) + NR\cdot H(P_b^K) + \sum_{n=1}^{N} I(Q;P_{s_n}) \end{aligned} \tag{A.6}$$

The first inequality follows from the data processing inequality. The second inequality is a consequence of Fano's inequality for sequences (Theorem 4.3.2 in [Gal68]). Rewriting (A.6) and applying the definition for fixed fraction DVC's gives the theorem. □

A.6 Proof of Lemma 2.4

1): The proof will be by induction on the argument m of the function $h(m)$. The claim trivially holds for $m = 1, 2$. Assume that the statement holds for all $m \leq M$. We shall prove for it $m = M + 1$. Assume without loss of generality that $k \leq M + 1 - k$. Let $g(k, M) = h(k) + h(M + 1 - k) + \min(k, M + 1 - k)$.

$$\begin{aligned} g(k,M) &= h(k) + h(M+1-k) + k \\ &= h\left(\left\lfloor \frac{k}{2} \right\rfloor\right) + h\left(\left\lceil \frac{k}{2} \right\rceil\right) + \left\lfloor \frac{k}{2} \right\rfloor + h\left(\left\lfloor \frac{M+1-k}{2} \right\rfloor\right) \\ &\quad + h\left(\left\lceil \frac{M+1-k}{2} \right\rceil\right) + \left\lfloor \frac{M+1-k}{2} \right\rfloor + k \end{aligned} \tag{A.7}$$

where equation (A.7) follow from the induction hypothesis. Assuming M is odd and k even, we have $\lfloor \frac{k}{2} \rfloor = \lceil \frac{k}{2} \rceil = \frac{k}{2}$, $\lceil \frac{M+1-k}{2} \rceil = \lfloor \frac{M+1-k}{2} \rfloor = \frac{M+1-k}{2}$. Continuing with the chain of equalities, we have

$$\begin{aligned}
g(k,M) &= h\left(\frac{k}{2}\right)+h\left(\frac{k}{2}\right)+\frac{k}{2} \\
&\quad +h\left(\frac{M+1-k}{2}\right)+h\left(\frac{M+1-k}{2}\right)+\frac{M+1-k}{2}+k \\
&= h\left(\frac{k}{2}\right)+h\left(\frac{M+1-k}{2}\right)+\frac{k}{2} \\
&\quad +h\left(\frac{k}{2}\right)+h\left(\frac{M+1-k}{2}\right)+\frac{k}{2}+\frac{M+1}{2} \\
&\leq h\left(\frac{M+1}{2}\right)+h\left(\frac{M+1}{2}\right)+\frac{M+1}{2} \qquad \text{(A.8)} \\
&= h\left(\left\lceil\frac{M+1}{2}\right\rceil\right)+h\left(\left\lfloor\frac{M+1}{2}\right\rfloor\right)+\left\lfloor\frac{M+1}{2}\right\rfloor \qquad \text{(A.9)}
\end{aligned}$$

The inequality in equation (A.8) follows from the induction hypothesis (for $m = \frac{M+1}{2}$). Equation (A.9) is the result of our assumption on M and k. Thus, $h(k) + h(M+1-k) + \min(k, M+1-k)$ is bounded above by $h(\lceil\frac{M+1}{2}\rceil) + h(\lfloor\frac{M+1}{2}\rfloor) + \lfloor\frac{M+1}{2}\rfloor$. The other cases ($M$ odd and k odd, M even and k odd, M even and k even) are treated in a similar way, and (A.9) holds for each case. Also, in each case, $k = \lfloor\frac{M+1}{2}\rfloor$ achieves the bound in (A.9). Hence, we have $h(M+1) \equiv \max_{1\leq k\leq M} h(k) + h(M+1-k) + \min(k, M+1-k) = h(\lceil\frac{M+1}{2}\rceil) + h(\lfloor\frac{M+1}{2}\rfloor) + \lfloor\frac{M+1}{2}\rfloor$.

2): Again, the argument is by induction on m. The claim is obviously true for $m = 1, 2$. Now assume $h(m) \leq \frac{1}{2}m\log_2 m + m$ for all even $m < M$. If M is even,

$$\begin{aligned}
h(M) &= 2h\left(\frac{M}{2}\right)+\frac{M}{2} \\
&\leq 2\left(\frac{1}{2}\frac{M}{2}\log_2\frac{M}{2}+\frac{M}{2}\right)+\frac{M}{2} \\
&= \frac{M}{2}(\log_2 M - 1) + M + \frac{M}{2} \\
&= \frac{1}{2}M\log_2 M + M,
\end{aligned}$$

where the first equality follows from part 1 of the lemma and the first inequality results from the inductive assumption. If M is odd,

$$\begin{aligned} h(M) &= h\left(\frac{M+1}{2}\right) + h\left(\frac{M-1}{2}\right) + \frac{M-1}{2} && \text{(A.10)} \\ &\leq \frac{1}{2}\left(\frac{M+1}{2}\right)\log_2\left(\frac{M+1}{2}\right) + \frac{M+1}{2} \\ &\quad + \frac{1}{2}\left(\frac{M-1}{2}\right)\log_2\left(\frac{M-1}{2}\right) + M - 1. && \text{(A.11)} \end{aligned}$$

Equation (A.10) follows from part 1 of the lemma and (A.11) results from the inductive assumption. Subtracting the RHS of (A.11) from $\frac{1}{2}M\log_2 M + M = \frac{1}{2}\left(\frac{M+1}{2}\right)\log_2 M + \frac{1}{2}\left(\frac{M-1}{2}\right)\log_2 M + M$, we have

$$\begin{aligned} & \frac{1}{2}\left(\frac{M+1}{2}\right)\log_2\left(\frac{M+1}{2M}\right) + \frac{1}{2}\left(\frac{M-1}{2}\right)\log_2\left(\frac{M-1}{2M}\right) + \frac{M-1}{2} \\ = & \frac{1}{2}\left(\frac{M-1}{2}\right)\log_2\left(\frac{M+1}{2M}\right) + \frac{1}{2}\log_2\left(\frac{M+1}{2M}\right) \\ & + \frac{1}{2}\left(\frac{M-1}{2}\right)\log_2\left(\frac{M-1}{2M}\right) + \frac{M-1}{2} \\ = & \frac{1}{2}\left(\frac{M-1}{2}\right)\left[\log_2\left(\frac{M+1}{M}\right) + \log_2\left(\frac{M-1}{M}\right)\right] + \frac{1}{2}\log_2\left(\frac{M+1}{2M}\right) \\ = & \frac{M-1}{4}\log_2\left(1+\frac{1}{M}\right) + \frac{M-1}{4}\log_2\left(1-\frac{1}{M}\right) \\ & + \frac{1}{2}\log_2\left(1+\frac{1}{M}\right) - \frac{1}{2} \\ \leq & \frac{1}{\ln 2}\left[\left(\frac{M-1}{4}\right)\left(\frac{-1}{M^2}\right) + \frac{1}{2}\left(\frac{1}{M} - \frac{1}{2M^2} + \frac{1}{3M^3}\right)\right] - \frac{1}{2} && \text{(A.12)} \\ = & \frac{1}{\ln 2}\left[\frac{1}{4M} + \frac{1}{6M^3}\right] - \frac{1}{2} && \text{(A.13)} \end{aligned}$$

where (A.12) follows from the inequality $\ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$ for $|x| < 1$. Now the RHS of (A.13) is negative for $M = 2$ and is strictly decreasing as a function of M. We have therefore shown $h(M) \leq \frac{1}{2}M\log_2 M + M$ for all $M \geq 1$.

Now for $M = 2^r, r \in \mathbb{Z}^+$, we have by part 1 of the lemma,

$$\begin{aligned} h(M) = h(2^r) &= 2h(2^{r-1}) + 2^{r-1} \\ &= 2[2h(2^{r-2}) + 2^{r-2}] + 2^{r-1} \\ &= 2^2 \cdot h(2^{r-2}) + 2 \cdot 2^{r-1}. \end{aligned}$$

Continuing to iterate downwards,

$$
\begin{aligned}
h(2^r) &= 2^k h(2^{r-k}) + k \cdot 2^{r-1}, \quad k = 1, \dots, r-1 \\
&= 2^{r-1} h(2) + (r-1)2^{r-1} \\
&= 3 \cdot 2^{r-1} + (r-1)2^{r-1} \\
&= (r+2)\frac{M}{2} \\
&= (\log_2 M + 2)\frac{M}{2} \\
&= \frac{1}{2} M \log_2 M + M.
\end{aligned}
$$

□

Bibliography

[Ahl71] R. Ahlswede. Multi-way communication channels. In *Proceedings 2nd International Symposium on Information Theory*, pages 23–52, 1971.

[BC74] P. Bergmans and T. Cover. Cooperative broadcasting. *IEEE Trans. Information Theory*, 20(3):253–259, 1974.

[Ber74] P. Bergmans. A simple converse for broadcast channels with additive white gaussian noise. *IEEE Trans. Information Theory*, 20(2):279–280, 1974.

[Ber95] D. Bertsekas. *Dynamic Programming and Optimal Control*. Athena Scientific, Belmont, Massachusetts, 1995.

[Ber00] R. Berry. *Power and Delay Trade-offs in Fading Channels*. PhD thesis, Massachusetts Institute of Technology, 2000.

[Bil95] P. Billingsley. *Probability and Measure*. Wiley, New York, 1995.

[BPS98] E. Bigleri, J. Proakis, and S. Shamai. Fading channels: Information theoretic and communication aspects. *IEEE Transactions on Information Theory*, 44(6):2619–2692, 1998.

[CFRU01] S-Y. Chung, G.D. Forney, T. Richardson, and R. Urbanke. On the design of low-density parity-check codes within 0.0045 db of the shannon limit. *IEEE Communications Letters*, 5(2):58–60, 2001.

[CKH98] G. Caire, R. Knopp, and P. Humblet. System capacity of f-tdma cellular systems. *IEEE Transactions on Communications*, 46(12):1649–1661, 1998.

[Cov72] T. Cover. Broadcast channels. *IEEE Trans. Information Theory*, 18(1):1–14, 1972.

[CT91] T. Cover and J. Thomas. *Elements of Information Theory*. Wiley, New York, 1991.

[CTB99] G. Caire, G. Taricco, and E. Biglieri. Optimum power control over fading channels. *IEEE Transactions on Information Theory*, 45(5):1468–1489, 1999.

[Edm70] J. Edmonds. Submodular functions, matroids, and certain polyhedra. In *Proceedings of the International Conference on Combinatorial Structures and Their Applications*, pages 69–87, New York, 1970. Gordon and Beach.

[EG98] M. Effros and A. Goldsmith. Capacity definitions and coding strategies for general channels with receiver side information. In *1998 IEEE Int. Symp. Information Theory*, page 39, Cambridge, MA, 1998.

[EH98] A. Ephremides and B. Hajek. Information theory and communication networks: An unconsummated union. *IEEE Trans. Information Theory*, 44(6):2416–2434, 1998.

[Gal68] R. Gallager. *Information Theory and Reliable Communication*. Wiley, New York, 1968.

[Gal85] R. Gallager. A perspective on multiaccess channels. *IEEE Trans. Information Theory*, 31(2):124–142, 1985.

[Gal94] R. Gallager. Residual noise after stripping on fading multipath channels. Technical report, MIT, Laboratory for Information and Decision Systems, 1994.

[Gal96] R. Gallager. *Discrete Stochastic Processes*. Kluwer, Boston, 1996.

[Gam80] A. El Gamal. Capacity of the product and sum of two unmatched broadcast channels. *Problems of Information Transmission*, 16(1):1–16, 1980.

[GLS85] M. Grotschel, L. Lovasz, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer-Verlag, New York, 1985.

[GM80] E. Gelenbe and I. Mitrani. *Analysis and Synthesis of Computer Systems*. Academic Press, New York, 1980.

[GR95] A. Grant and L. Rasmussen. A new coding method for the asynchronous multiple access channel. In *Proceedings of the 33rd Allerton Conference on Communication, Control, and Computing*, Monticello, IL, 1995.

[GRUW01] A. Grant, B. Rimoldi, R. Urbanke, and P. Whiting. Rate-splitting multiple access for discrete memoryless channels. *IEEE Transactions on Information Theory*, 47(3):873–890, 2001.

[GV97] A. Goldsmith and P. Varaiya. Capacity of fading channels with channel side information. *IEEE Transactions on Information Theory*, 43:1986–1992, 1997.

[HH75] D. Hughes-Hartog. The capacity of the degraded spectral gaussian broadcast channel. Technical report, Stanford University, Information Systems Laboratory, Center for Systems Research, 1975.

[HH85] J. Hui and P. Humblet. The capacity region of totally asynchronous multiple-access channel. *IEEE Transactions on Information Theory*, 31(2):207–216, 1985.

[HT98] S. Hanly and D. Tse. Multi-access fading channels: Part ii: Delay-limited capacitie. *IEEE Transactions on Information Theory*, 44(7):2816–2831, 1998.

[HW94] S. Hanly and P. Whiting. Constraints on the capacity in a multi-user channel. In *Proceedings 1994 IEEE International Symposium on Information Theory*, page 54, Trondheim, Norway, 1994.

[KH00] R. Knopp and P. Humblet. On coding for block fading channels. *IEEE Transactions on Information Theory*, 46(1):189–205, 2000.

[Kin57] A. Kinchin. *Mathematical Foundations of Information Theory*. Dover, New York, 1957.

[LG99] L. Li and A. Goldsmith. 'capacity and optimal resource allocation for fading broadcast channels: Part i: Ergodic capacity. Submitted to *IEEE Transactions on Information Theory*, 1999.

[Lia72] H. Liao. *Multiple Access Channels*. PhD thesis, University of Hawaii, Department of Electrical Engineering, 1972.

[MO79] A. Marshall and I. Olkin. *Inequalities: Theory of Majorization and Its Applications*. Academic Press, New York, 1979.

[OSW94] L. Ozarow, S. Shamai, and A. Wyner. Information theoretic considerations for mobile radio. *IEEE Trans. Vehicular Technology*, 43(2):359–378, May 1994.

[Pol77] G. Poltyrev. Carrying capacity for parallel broadcast channels with degraded components. *Probl. Pered. Inform*, 13(2):23–35, 1977.

[Pol83] G. Poltyrev. Coding for channel with asynchronous multiple access. *Problemy Peredachi Informatsii*, 19:12–21, 1983.

[Rim94] B. Rimoldi. Successive refinement of information: Characterization of the achievable rates. *IEEE Trans. Information Theory*, 40(1):253–259, 1994.

[Rim97] B. Rimoldi. Generalized time-sharing for multiple access channels. In *Proceedings 1997 IEEE International Symposium on Information Theory*, page 26, Ulm, Germany, 1997.

[Rim99] B. Rimoldi. Time-splitting multiple-access. Submitted to *IEEE Transactions on Information Theory*, 1999.

[RU96] B. Rimoldi and R. Urbanke. A rate-splitting approach to the gaussian multiple-access channel. *IEEE Trans. Information Theory*, 42:364–375, 1996.

[Sha97] S. Shamai. A broadcast strategy for the gaussian slowly fading channel. In *Proceedings of the 1997 International Symposium on Information Theory*, page 150, Ulm, Germany, 1997.

[TE93] L. Tassiulas and A. Ephremides. Dynamic server allocation to parallel queues with randomly varying connectivity. *IEEE Transactions on Information Theory*, 39(2):466–478, 1993.

[Tel95] İ. E. Telatar. Multiple access information theory and job scheduling. 1995.

[TG95] İ. E. Telatar and R. Gallager. Combining queueing theory with information theory for multiaccess. *IEEE Journal on Selected Areas in Communications*, 13(6):963–969, 1995.

[TH98] D. Tse and S. Hanly. Multi-access fading channels: Part i: Polymatroid structure, optimal resource allocation and throughput capacities. *IEEE Transactions on Information Theory*, 44(7):2796–2815, 1998.

[Tse99] D. Tse. Optimal power allocation over parallel gaussian broadcast channels. Submitted to *IEEE Transactions on Information Theory*, 1999.

[Vis99] H. Viswanathan. Capacity of markov channels with receiver csi and delayed feedback. *IEEE Transactions on Information Theory*, 45:761–771, 1999.

[Wal88] J. Walrand. *Queueing Networks.* Prentice Hall, Englewood Cliffs, NJ, 1988.

[Wel76] D. Welsh. *Matroid Theory.* Academic Press, London, 1976.

[Wil91] D. Williams. *Probability with Martingales.* Cambridge University Press, Cambridge, 1991.

[Zie95] G. Ziegler. *Lectures on Polytopes.* Springer, New York, 1995.