A Robust Optimization Approach to Supply Chains and Revenue Management

by

Aurélie Thiele

Dipl. Ing., École des Mines de Paris (1999)
M.Sc., Electrical Engineering and Computer Science, MIT (2000)

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Author ....
Department of Electrical Engineering and Computer Science
May 21, 2004

Certified by ....
Dimitris J. Bertsimas
Boeing Professor of Operations Research
Thesis Supervisor

Accepted by ....
Arthur C. Smith
Chairman, Department Committee on Graduate Students

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Abstract

The contribution of this thesis is to provide a tractable framework for supply chains and revenue management problems subject to uncertainty that gives insight into the structure of the optimal policy and requires little knowledge of the underlying probability distributions. Instead, we build upon recent advances in the field of robust optimization to develop models of uncertainty that make few probabilistic assumptions and have an adjustable level of conservatism to ensure performance.

Specifically, we consider two classes of robust optimization approaches. First, we model the random variables as uncertain parameters belonging to a polyhedral uncertainty set, and optimize the system against the worst-case value of the uncertainty in the set. The polyhedron is affected by budgets of uncertainty, which reflect a trade-off between robustness and optimality. We apply this framework to supply chain management, show that the robust problem is equivalent to a deterministic problem with modified parameters, and derive the main properties of the optimal policy.

We also explore a second approach, which builds directly on the historical realizations of uncertainty, without requiring any estimation. In that model, a fraction of the best cases are removed to ensure robustness, and the system is optimized over the sample average of the remaining data. This leads to tractable mathematical programming problems. We apply this framework to revenue management problems, and show that in many cases, the optimal policy simply involves an appropriate ranking of the historical data.

Robust optimization emerges as a promising methodology to address a wide range of management problems subject to uncertainty, in particular in a dynamic setting, as it leads to representations of randomness that make few assumptions on the underlying probabilities, remain numerically tractable, incorporate the decision-maker's risk aversion, and provides theoretical insights into the structure of the optimal policy.

Thesis Supervisor: Dimitris J. Bertsimas
Title: Boeing Professor of Operations Research

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Chapter 1

Introduction

1.1 Background

Researchers have traditionally addressed the problem of optimally controlling stochastic systems by taking a probabilistic view of randomness, where the uncertain variables are assumed to obey known probability distributions and the goal is to minimize the expected cost. However, accurate probabilities are hard to obtain in practice, in particular for distributions varying over time, and even small specification errors might make the resulting policy very suboptimal. In a dynamic setting, this approach also suffers from tractability issues, as the size of the problem increases exponentially with the time horizon considered. This is known as "the curse of dimensionality" in dynamic programming. As a result, the optimal stochastic policy might be numerically intractable, and even when available, it might be of limited practical relevance if it was computed with the wrong distribution. In applications, this translates into significant revenue opportunities that are lost when companies misjudge demand for their product or are unable to solve the complex mathematical formulations that best represent their problem. Therefore, the following question emerges as a critical issue in the optimization of stochastic systems:

Can we develop a model of uncertainty that incorporates randomness in a tractable manner, makes few assumptions on the underlying probability distributions, gives theoretical insights into the structure of the optimal policy, and performs well in computational experiments?

The purpose of this thesis is to present such an approach, based on robust optimization. The focus of robust optimization is to protect the system against the worst-case value of the uncertainty in a prespecified set. It was originally developed by Ben-Tal and Nemirovski [4, 5, 6] and independently by El Ghaoui et al. [34, 35] to address the imperfect knowledge of parameters in mathematical programming problems with an ellipsoidal uncertainty structure, but can also be applied to uncertain probabilities in Markov decision problems as Nilim and El Ghaoui showed in [50]. Other robust techniques have been implemented to model trade-offs between performance and risk from a theoretical viewpoint as well as a practical one, using polyhedral uncertainty sets or risk measures (see for instance Bertsimas and Sim [15, 16], Bertsimas et al. [12, 17], Pachamanova [51]). In particular, appealing features of the framework described by Bertsimas and Sim in [15] and Bertsimas et. al. in [12] are that (a) such problems can be solved efficiently; and (b) their optimal solutions are not overly conservative. This motivates investigating further the applicability of robust optimization to the problem of optimally controlling systems subject to randomness. We use this technique in two broad classes of applications:

1. **Supply chain management:** The decision-maker wants to minimize the cost associated with operating a supply chain over time, but is faced with uncertain demand. Each station in the supply chain can only send goods it currently has in inventory to its successor. The decision-maker has a limited amount of information on the demands at his disposal. What should his optimal strategy be?

2. **Revenue management:** A newsvendor's goal is to maximize his revenue by selling perishable products. Uncertainty here can affect the demand for the items on the primary or secondary markets, or the number of goods ordered that are in good enough condition to be sold. Alternatively, an airline would like to allocate seats to customer classes or determine admission policies so that as many planes as possible are filled close to capacity, without denying requests by high-paying customers. What is the optimal policy?

1.2 Thesis Overview and Contributions

The main contribution of this thesis is to provide a tractable and insightful framework for stochastic systems that requires little knowledge of the underlying probability distributions, in the context of managerial problems with imperfect information. From a theoretical perspective, we develop a robust optimization approach with two main components, each adapted to a specific problem structure. The first one builds on uncertainty sets and the second one is data-driven. They yield numerically tractable formulations as well as key insights into the optimal policies. From a practical perspective, we apply these techniques to supply chains and revenue management, where we derive robust replenishment strategies for a wide range of inventory problems and robust seat allocations as well as admission policies to maximize airline revenue.

We conclude this introduction by giving a brief overview of each chapter and its specific contributions.

Robust Optimization with Uncertainty Sets

In this first robust optimization framework, we adopt a deterministic view of randomness, in the sense that we model random variables by uncertain parameters in (polyhedral) uncertainty sets and protect the system against the worst-case value in that set. We do not assume any specific distribution. This is a representation of stochasticity that, to the best of our knowledge, has not been considered before. To avoid overconservatism, rare events are excluded from the uncertainty sets through the use of budgets of uncertainty, which rule out large deviations. We study the implications of such a model for static and dynamic problems with convex cost functions, analyze the optimal policies, describe how to select the budgets of uncertainty and discuss some limitations of the approach. A key result is that the robust counterparts of convex problems remain convex problems, and hence are tractable.

Data-Driven Robust Optimization

We develop the second robust optimization framework to address some of the limitations of the approach with uncertainty sets. We consider the sample of available realizations, trim the best cases, and optimize the system over the remaining observations. This approach is related to risk aversion and avoids altogether the estimation of the underlying stochastic process. As in the first approach, it remains numerically tractable and provides insight into the structure of the optimal policy. For linear programming problems, we use duality arguments to show that the optimal solution corresponds to ranking quantities related to the observations in an appropriate manner.

Robustness with Uncertainty Sets and Data Samples: A Comparison

We compare the two robust optimization approaches, highlight their specific features, and summarize the problem structures where each is most appropriate. We then focus on linear programming problems, where both techniques are easily implemented, to give some deeper insight into the differences and connections between the methods. The purpose of this chapter is to provide a unifying framework to the robust optimization approach for management problems, with the methods presented so far emerging as two complementary sets of tools adapted to different situations but serving the same overarching goal.

Application to Supply Chains

We consider the problem of finding the optimal ordering strategies for supply chains subject to uncertain demand over a finite horizon. We assume linear dynamics, i.e., unmet demand is backlogged, a linear ordering cost with possibly an additional fee (fixed cost) incurred whenever an order is made, and piecewise linear holding/shortage costs. We show that the robust counterpart is a linear programming problem in situations with no fixed costs and is a mixed integer programming problem in situations with fixed costs. In particular, the class of the robust problem does not change as the network topology increases in complexity. Furthermore, the robust problem is equivalent to a deterministic problem with modified demands, and is qualitatively similar to the optimal stochastic policy obtained by dynamic programming when theoretical properties of the stochastic policy are known. We also obtain insights in situations where the optimal stochastic policy is not known, using Lagrangian duality arguments. Therefore, the robust optimization approach provides us with a better understanding of the way uncertainty affects the optimal policy.

Application to Revenue Management

We study two classes of revenue management problems: (a) the classical newsvendor problem and its extensions; and (b) seat allocation and admission policies for airlines. These problems can be formulated as linear programming problems, and their robust counterparts remain of the same class. The use of complementarity slackness allows us to derive the main properties of the optimal solution. We show that in many cases, the optimal policy can be characterized by the ranking of appropriate parameters related to the observations. The robust policy is also more intuitive than policies computed assuming only that the first two moments are known (in the newsvendor problem) and gives a more accurate representation of the real-world setting than the deterministic models commonly used as heuristics for tractability purposes (in airline revenue management). It also easily incorporates correlated demand, which is frequently encountered in practice but is not taken into account in traditional optimization models.

1.3 Thesis Structure

This thesis is structured as follows: in Chapters 2 and 3, we develop and analyze the theory of robust optimization applied to stochastic systems, first relying on a deterministic description of randomness based on uncertainty sets (Chapter 2), then building directly on the sample of available data (Chapter 3). We show that both frameworks yield tractable mathematical programming problems and study the corresponding optimal policies. In Chapter 4, we investigate further the connections and differences between the two approaches and describe the problem and uncertainty structures where each is most appropriate. In Chapters 5 and 6, we apply the robust optimization techniques to management problems. In Chapter 5, we consider supply chain management, to which the methodology developed in Chapter 2 is well suited. Chapter 6 focuses on revenue management, which is best addressed using the model presented in Chapter 3. Finally, in Chapter 7 we conclude the thesis with a summary of the results and suggest some future research directions.

Chapter 2

Robust Optimization with Uncertainty Sets

2.1 Background and Contributions

The traditional approach to stochastic optimization builds upon a probabilistic description of randomness, where the random variables are characterized by their probability distributions, which are assumed perfectly known, and minimizes the expected cost incurred. When the system evolves over time, the optimal policy is found by solving backwards recursive equations in the expected "cost-to-go". This method, known as dynamic programming, is described extensively by Bertsekas in [8] and [9], and has enjoyed much theoretical success over the years. However, it suffers from two major disadvantages: it assumes full knowledge of the underlying distributions, and the computational requirements increase exponentially with the size of the problem, which is commonly referred to as "the curse of dimensionality".

Researchers have long recognized that probability distributions are not always available in practice. For instance, in what may well have been the first attempt at implementing a robust optimization approach in the field, Scarf developed in 1958 a min-max approach to the classical newsvendor problem, assuming only that the mean and the variance of the demand were known [57]. His approach optimized the worst-case revenue over the class of probabilities with given mean and variance. Later, Gallego and his co-authors extended his results to static and dynamic problems in inventory and revenue management [36, 47, 48], without breaking the curse of dimensionality. In the 1960s and 1970s, min-max approaches began generating interest in a widely different research context, since they were used to optimally control dynamic systems subject to unknown-but-bounded disturbances. This set-membership description of uncertainty was pioneered by Witsenhausen [71, 72] in the late 1960s and Bertsekas [7, 10] in the early 1970s for common problems in control theory. For instance, Bertsekas used set membership in [10] to address the target set reachability problem, where one seeks to keep the state of the system close to a desired trajectory, i.e. in a target set, despite the effect of (unknown but bounded) perturbations. Much around the same time, Soyster applied in 1973 a similar technique to mathematical programming under uncertainty [62], where he guaranteed the feasibility of each constraint for any value of the parameters in a given uncertainty set. Specifically, his focus was on linear programming problems subject to column-wide uncertainty:

$$\begin{aligned} \max \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \sum_{j=1}^{n} \mathbf{A_j}\, x_j \leq \mathbf{b}, \quad \forall \mathbf{A_j} \in K_j,\ \forall j, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{2.1}$$

with each column $\mathbf{A_j}$ belonging to a convex set K_j. Soyster showed that this problem was equivalent to another linear programming problem:

$$\begin{aligned} \max \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \sum_{j=1}^{n} \widetilde{\mathbf{A}}_{\mathbf{j}}\, x_j \leq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{2.2}$$

where $\widetilde{A}_{ij} = \sup_{\mathbf{A_j} \in K_j} A_{ij}$ for all i and j. Unsurprisingly, this problem yields very conservative solutions, as it protects each constraint against its worst case. This issue of conservativeness prevented for many years the adoption of min-max approaches as a viable alternative to probabilistic descriptions of uncertainty.

Interest in such a framework was revived in the 1990s when Ben-Tal and Nemirovski developed robust optimization techniques to address parameter uncertainty in convex programming with ellipsoidal uncertainty sets [4, 5, 6]. Similar results were derived independently by El Ghaoui et al. [34, 35] using ideas from the field of robust control. The key idea was that the size of the ellipsoid can be chosen to guarantee feasibility with high probability (rather than with probability 1), and preserve an acceptable level of performance. In particular, if the uncertainty set is described by:

$$\mathcal{E} = \{\mathbf{a} = \overline{\mathbf{a}} + \mathbf{\Gamma^{1/2}}\,\mathbf{z},\ \|\mathbf{z}\|_2 \leq \theta\}, \tag{2.3}$$

with $\overline{\mathbf{a}}$ the mean and $\mathbf{\Gamma}$ the covariance matrix of the uncertain parameters, the size of the ellipsoid is controlled by the parameter θ, and increases as θ increases.

However, although this approach leads to robust counterparts that are tractable either exactly or approximately, it also increases the complexity of the problem considered. In particular, when applied to the linear programming problem (LP):

$$\begin{array}{ll} \min & \mathbf{c}'\mathbf{x} \\ \text{s.t.} & \mathbf{a_i'}\,\mathbf{x} \leq \mathbf{b_i}, \quad \forall i, \end{array} \tag{2.4}$$

where $\mathbf{a_i}$ belongs to the ellipsoid $\mathcal{E}_i = \{\mathbf{a_i} = \overline{\mathbf{a}}_\mathbf{i} + \mathbf{\Gamma_i^{1/2}}\,\mathbf{z},\ \|\mathbf{z}\|_2 \leq \theta_i\}$ for each i, it yields the second-order cone problem (SOCP):

$$\begin{array}{ll} \min & \mathbf{c}'\mathbf{x} \\ \text{s.t.} & \overline{\mathbf{a}}_\mathbf{i}'\,\mathbf{x} + \theta_i \|\mathbf{\Gamma_i^{1/2}}\,\mathbf{x}\|_2 \leq \mathbf{b_i}, \quad \forall i. \end{array} \tag{2.5}$$

While many LPs can be interpreted quite easily, SOCPs might not provide such insights. At the very least, this increase in complexity will change the nature of the underlying management problem. For instance, in a production planning problem, producing x_j units of item j will use resource i at a set rate of $\overline{a}_{ij}$ per unit in the nominal case, but at a variable rate depending on $\mathbf{x}$ (e.g., $\overline{a}_{ij} + \theta_i\,\sigma_{ij}/\|\Gamma_i^{1/2}\,\mathbf{x}\|_2$ if the uncertain parameters are independent) in the robust problem.

In contrast, Bertsimas and Sim propose in [15] an approach based on polyhedral uncertainty sets that yields linear robust counterparts of linear programming problems. They also quantify explicitly the relationship between the level of conservativeness of the solution and the probability of constraint violation, for which they coin the term “price of robustness”. Specifically, they model each a_{ij} as an uncertain parameter obeying a symmetric distribution and taking values in the interval $[\overline{a}_{ij} - \widehat{a}_{ij}, \overline{a}_{ij} + \widehat{a}_{ij}]$, with $\overline{a}_{ij}$ the mean of the distribution and $\widehat{a}_{ij}$ the half-length of its support. They further bound the total scaled deviation of the parameters in constraint i from their nominal value: $\sum_{j=1}^{n} \frac{|a_{ij} - \overline{a}_{ij}|}{\widehat{a}_{ij}} \leq \Gamma_i$, where Γ_i is called the budget of uncertainty for constraint i. This can be interpreted as requiring that at most Γ_i of the uncertain coefficients in constraint i take their worst-case value at the same time. If $\Gamma_i = 0$, one simply solves the nominal problem. Equivalently, this uncertainty set can be written in terms of scaled deviations:

$$\mathcal{A}_i = \{a_{ij} = \overline{a}_{ij} + \widehat{a}_{ij}\, z_{ij},\ |z_{ij}| \leq 1,\ \forall j,\ \sum_j |z_{ij}| \leq \Gamma_i\},\ \forall i, \tag{2.6}$$

where z_{ij} is the scaled deviation of a_{ij} from its nominal value $\overline{a}_{ij}$. It appears that the choice of ellipsoidal or polyhedral uncertainty sets is, in essence, a choice of norms: the $\mathcal{L}_2$-norm for ellipsoids and the $\mathcal{L}_1$-norm for polyhedra. The link between norms and uncertainty sets has been studied by Bertsimas and his co-authors in [17].

Bertsimas and Sim show that, in the framework described by (2.6), the robust counterpart of the linear programming problem subject to parameter uncertainty:

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{a_i'}\,\mathbf{x} \leq b_i, \quad \forall a_i \in \mathcal{A}_i,\ \forall i, \end{aligned} \tag{2.7}$$

is another LP:

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \sum_{j=1}^{n} \overline{a}_{ij}\, x_j + p_i\, \Gamma_i + \sum_{j=1}^{n} q_{ij} \leq b_i, && \forall i, \\ & p_i + q_{ij} \geq \widehat{a}_{ij} y_j, && \forall i, j, \\ & -y_j \leq x_j \leq y_j, && \forall j, \\ & p_i \geq 0,\ q_{ij} \geq 0, && \forall i, j. \end{aligned} \tag{2.8}$$

They derive the following probabilistic guarantee:

Theorem 2.1.1 (Probabilistic guarantee [15]) *The probability that the i-th constraint is violated satisfies:*

$$P\left(\sum_{j=1}^{n} a_{ij} x_j^* > b_i\right) \leq \frac{1}{2^{n_i}} \left\{ (1-\mu) \sum_{k=\lfloor \nu \rfloor}^{n_i} \begin{pmatrix} n_i \\ k \end{pmatrix} + \mu \sum_{k=\lfloor \nu \rfloor + 1}^{n_i} \begin{pmatrix} n_i \\ k \end{pmatrix} \right\}, \tag{2.9}$$

where n_i is the number of uncertain coefficients in row i, $\nu = \frac{\Gamma_i + n_i}{2}$ and $\mu = \nu - \lfloor \nu \rfloor$.

Bound (2.9) can be approximated using the probability function of the standard normal *distribution Φ to yield:*

$$P\left(\sum_{j=1}^{n} a_{ij}x_j^* > b_i\right) \leq 1 - \Phi\left(\frac{\Gamma_i - 1}{\sqrt{n_i}}\right). \tag{2.10}$$

An important consequence of this result is that Γ_i of the order of $\sqrt{n_i}$ will guarantee the feasibility of the solution with high probability. This is the model of uncertainty that we will use in the remainder of this thesis when we consider uncertainty sets, because of its appealing linear properties.

Since robust optimization techniques do not suffer from overconservatism, in contrast with the early min-max approaches, they might be of practical interest for the problem of optimally controlling stochastic systems, to address the imperfect knowledge of the distributions so frequently encountered in practice. Further motivation to use robust optimization with polyhedral uncertainty sets is provided by the dimensionality problems of dynamic programming, since the latter becomes intractable as the size of the problem increases. Methods proposed in the past to remedy this have their own drawbacks. Stochastic programming, described by Birge and Louveaux [20], optimizes the average cost over a set of scenarios generated in advance. However, it also suffers from the curse of dimensionality, as one needs to sample a large number of scenarios to obtain relevant results. (Stochastic programming is discussed further when we investigate robust data-driven methods in Chapter 3.) Approximate dynamic programming, originally developed by Schweitzer and Seidmann [59] and later studied by Bertsekas and Tsitsiklis [11], as well as Van Roy and de Farias [32], considers approximations of the value function to address the tractability issue, but its practical use remains limited because of the difficulty in computing good approximations. Myopic policies, often used in inventory management (see for instance Zipkin [74]), solve the optimal stochastic problem for the current time period, and therefore can be very sub-optimal.

Our goal in this chapter is to describe how robust optimization with polyhedral uncertainty sets can be used to develop a tractable and insightful framework to optimize static and dynamic systems subject to randomness. The remainder of this chapter is structured as follows. In Section 2.2, we introduce the main concepts using simple static problems. In Section 2.3, we extend these results to dynamic optimization with linear dynamics. We discuss how to select the budgets of uncertainty in Section 2.4 and consider the applicability of the method to some extensions in Section 2.5. Finally, Section 2.6 contains some concluding remarks.

2.2 Robust Static Optimization

In this section, we develop the robust optimization framework for static problems subject to randomness. This allows us to highlight the key ideas involved in the approach, before applying them to the dynamic case. We discuss the properties of the robust problem and of its optimal solution, and investigate probabilistic guarantees for the system.

2.2.1 Additive Uncertainty

We consider the unconstrained problem:

$$\min f\left(x - \sum_{i=1}^{n} w_i\right), \tag{2.11}$$

where $\sum_{i=1}^{n} w_i$ is random and f is a convex function such that $\lim_{|x|\to\infty} f(x) = \infty$. This problem is traditionally solved by assuming that $\sum_{i=1}^{n} w_i$ obeys a known probability distribution p and solving:

$$\min E_p\, f\left(x - \sum_{i=1}^{n} w_i\right). \tag{2.12}$$

The convexity of f implies that the optimal solution of Problem (2.12) is the unique solution of:

$$E_p\, f'\left(x - \sum_{i=1}^{n} w_i\right) = 0. \tag{2.13}$$

However, in practice the exact probability distribution is often not available, and solving (2.12) with the wrong distribution will lead to suboptimal performance. In contrast, the robust optimization approach we propose does not require any (specific) assumption on the distribution. It is robust in the sense that it explicitly protects the system against the worst-case realization of the random variable in a given uncertainty set. The counterpart of Problem (2.11) in the general robust framework is:

$$\min_{x} \left\{ \max_{\sum_{i=1}^{n} w_i \in \mathcal{W}} f\left(x - \sum_{i=1}^{n} w_i\right) \right\}, \tag{2.14}$$

where $\mathcal{W}$ is a convex set. As mentioned in Section 2.1, we will describe randomness by a polyhedral set of the type:

$$\mathcal{W} = \left\{ w_i = \overline{w}_i + \widehat{w}_i \, z_i, \ |z_i| \leq 1, \ \forall i, \ \sum_{i=1}^{n} |z_i| \leq \Gamma \right\}, \tag{2.15}$$

with Γ the budget of uncertainty. In this framework, Problem (2.14) becomes:

$$\begin{aligned} \min_{x} \quad \max_{z} \quad & f\left(x - \sum_{i=1}^{n} \overline{w}_i - \sum_{i=1}^{n} \widehat{w}_i \, z_i \right) \\ \text{s.t.} \quad & \sum_{i=1}^{n} |z_i| \leq \Gamma, \\ & |z_i| \leq 1, \ \forall i. \end{aligned} \tag{2.16}$$

Since f is convex, (2.16) is equivalent to:

$$\min_{x} \max \left\{ f\left(x - \sum_{i=1}^{n} \overline{w}_i - \sum_{i=1}^{n} \widehat{w}_i \, z_i^* \right), f\left(x - \sum_{i=1}^{n} \overline{w}_i + \sum_{i=1}^{n} \widehat{w}_i \, z_i^* \right) \right\}, \tag{2.17}$$

where:

$$\begin{aligned} \mathbf{z}^* = \arg \quad \max \quad & \sum_{i=1}^{n} \widehat{w}_i \, z_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} z_i \leq \Gamma, \\ & 0 \leq z_i \leq 1, \forall i. \end{aligned} \tag{2.18}$$

If Γ is given, we can first solve the auxiliary problem (2.18) and then reinject the corresponding $\mathbf{z}^*$ in (2.17), yielding a modified convex problem in x. Obviously, since the $\widehat{w}_i$ are nonnegative, we have $\sum_{i=1}^{n} \widehat{w}_i \, z_i^* = \sum_{i=1}^{\lfloor \Gamma \rfloor} \widehat{w}_{(i)} + \widehat{w}_{(\Gamma+1)} \cdot (\Gamma - \lfloor \Gamma \rfloor)$, where $\widehat{w}_{(1)} \geq \ldots \geq \widehat{w}_{(n)}$. We describe below how to solve the robust problem in a single step, without solving (2.18) beforehand or ranking the $\widehat{w}_i$. This is of practical interest for instance when the budgets of uncertainty are adjusted successively to meet specific performance criteria. We have the following theorem.

Theorem 2.2.1 (The robust problem) *The robust counterpart of Problem (2.11) is the convex problem:*

$$\begin{aligned} \min \quad & \max \left\{ f\left(x - \sum_{i=1}^{n} \overline{w}_i - \left[p\,\Gamma + \sum_{i=1}^{n} q_i \right] \right), \ f\left(x - \sum_{i=1}^{n} \overline{w}_i + \left[p\,\Gamma + \sum_{i=1}^{n} q_i \right] \right) \right\} \\ s.t. \quad & p + q_i \geq \widehat{w}_i, \ \forall i, \\ & p \geq 0, \ q_i \geq 0, \ \forall i. \end{aligned} \tag{2.19}$$

Proof: Since $\{\sum_{i=1}^n z_i \leq \Gamma,\ 0 \leq z_i \leq 1\}$ is bounded and nonempty, the optimal value of the maximization problem in (2.18) is equal to, by strong duality:

$$\begin{array}{lll} \min & p\,\Gamma + \sum_{i=1}^n q_i & \\ \text{s.t.} & p + q_i \geq \widehat{w}_i, & \forall i, \\ & p \geq 0,\ q_i \geq 0, & \forall i. \end{array} \tag{2.20}$$

Let $\mathcal{F}$ be the feasible set of (2.20), x any scalar and $X = x - \sum_{i=1}^n \overline{w}_i$. Let also $Y(p,q) = p\,\Gamma + \sum_{i=1}^n q_i$ for $(p,q) \in \mathcal{F}$. We want to show that:

$$\begin{aligned} &\max\left\{ f\left(X - \min_{(p,q)\in\mathcal{F}} Y(p,q) \right),\ f\left(X + \min_{(p,q)\in\mathcal{F}} Y(p,q) \right) \right\} = \\ &\min_{(p,q)\in\mathcal{F}} \max\left\{ f\left(X - Y(p,q) \right),\ f\left(X + Y(p,q) \right) \right\}. \end{aligned} \tag{2.21}$$

From the nonnegativity constraints on p and q and the unboundedness of the feasible set, $Y(p^*,q^*) \geq 0$ with $Y(p^*,q^*) = \min_{(p,q)\in\mathcal{F}} Y(p,q)$, and $Y(p,q)$ for $(p,q) \in \mathcal{F}$ can take any value in $[Y(p^*,q^*),\infty)$. Since f is convex and $\lim_{|x|\to\infty} f(x) = \infty$, there exists a unique x_f minimizing f, and f decreases, resp. increases, on $(-\infty, x_f]$, resp. $[x_f, \infty)$. Using these remarks, it is easy to show that (2.21) holds for each of the three (exhaustive and mutually exclusive) cases: (a) $X - Y(p^*,q^*) \leq X + Y(p^*,q^*) \leq x_f$, (b) $x_f \leq X - Y(p^*,q^*) \leq X + Y(p^*,q^*)$; and (c) $X - Y(p^*,q^*) \leq x_f \leq X + Y(p^*,q^*)$. (2.19) follows immediately. □

The robust counterpart (2.19) of the unconstrained convex problem (2.11) with n sources of uncertainty is a convex problem with linear constraints ($n+1$ nonnegativity constraints and n additional constraints). If the $\widehat{w}_i$ are ranked beforehand, the robust problem (2.19) becomes an unconstrained convex problem where, for each x, we evaluate the function $f\left(x - \sum_{i=1}^n w_i\right)$ in (2.11) for two (symmetric, diametrally opposed) realizations of the uncertainty and keep the greater one. Therefore, nominal and robust problems are closely connected.

We next study the optimal solution of the robust problem. For any $y > 0$, $f(x-y) = f(x+y)$ has exactly one solution because f is convex and $\lim_{|x|\to\infty} f(x) = \infty$. Let $F(y)$ be that solution, and let $F(0)$ be defined by continuity in 0, i.e., $F : \mathcal{R}^+ \to \mathcal{R}$ is defined by:

$$f(x-y) = f(x+y) \Leftrightarrow F(y) = x,\ \forall y > 0 \text{ and } F(0) = \lim_{y\to 0} F(y) = x_f, \tag{2.22}$$

where $x_f = \arg\min_x f(x)$ as before.

Examples:

- for $f(x) = x^2$, $F(y) = 0$ for all $y \geq 0$.
- for $f(x) = \max(h\,x, -p\,x)$, $F(y) = \dfrac{p-h}{p+h}\,y$ for all $y \geq 0$.

Theorem 2.2.2 (The robust solution) *Let x_Γ be the optimal solution of (2.17). We have:*

$$x_\Gamma = \sum_{i=1}^{n} \overline{w}_i + F\left(\sum_{i=1}^{n} \widehat{w}_i\, z_i^*\right), \tag{2.23}$$

where $\mathbf{z}^*$ is obtained by solving (2.18) and F is the function defined in (2.22).

Moreover, solving the robust problem is equivalent to solving the deterministic problem (2.11) when the uncertain parameter is equal to:

$$\sum_{i=1}^{n} w_i' = \sum_{i=1}^{n} \overline{w}_i + F\left(\sum_{i=1}^{n} \widehat{w}_i\, z_i^*\right) - x_f, \tag{2.24}$$

with $x_f = \arg\min_x f(x)$.

Proof: Let $X_\Gamma = x_\Gamma - \sum_{i=1}^n \overline{w}_i$ and $Y^* = \sum_{i=1}^n \widehat{w}_i z_i^*$. At optimality, $f(X_\Gamma - Y^*) = f(X_\Gamma + Y^*)$, otherwise it would be possible to strictly decrease the cost by increasing (resp. decreasing) X_Γ if $f(X_\Gamma - Y^*) > f(X_\Gamma + Y^*)$ (resp. $f(X_\Gamma - Y^*) < f(X_\Gamma + Y^*)$). Therefore, $X_\Gamma = F(Y^*)$. Since x_f is the unique optimal solution of $\min f(x)$, x_Γ will be the optimal solution of (2.11) for some modified parameter if and only if (2.11) can be written under the form $\min f(x - x_\Gamma + x_f)$. $\square$

This theorem yields insights into the way the uncertainty affects the optimal solution, and links the robust model to an equivalent nominal problem. The function F plays an important role in this analysis. For instance, if $f(x) = \max(h\,x,\, -p\,x)$,

$$x_\Gamma = \sum_{i=1}^{n} \overline{w}_i + \frac{p-h}{p+h} \sum_{i=1}^{n} \widehat{w}_i\, z_i^*. \tag{2.25}$$

In this case, the uncertainty has a greater impact on the robust solution x_Γ if p and h differ widely, as measured by the ratio $|(p-h)/(p+h)|$. On the other hand, if $p = h$, the uncertainty has no influence on x_Γ.

Finally, we interpret the optimal cost of the robust problem in terms of probabilistic guarantees for the cost in the stochastic setting.

Theorem 2.2.3 (Probabilistic guarantee) *Let C_Γ be the optimal cost in the robust problem, and $Y^* = \sum_{i=1}^n \widehat{w}_i z_i^*$ where $\mathbf{z}^*$ is obtained in (2.18). If the w_i are independent random variables obeying a symmetric distribution with mean $\overline{w}_i$ and support $[\overline{w}_i - \widehat{w}_i, \overline{w}_i + \widehat{w}_i]$ for each i, then:*

$$P\left(f\left(x_\Gamma - \sum_{i=1}^n w_i\right) > C_\Gamma\right) \leq 2\cdot\left(1 - \Phi\left(\frac{\Gamma-1}{\sqrt{n}}\right)\right). \tag{2.26}$$

Proof: We know from Theorem 2.2.2, its proof and Equation (2.23) that: $C_\Gamma = f\left(F(Y^*) - Y^*\right) = f\left(F(Y^*) + Y^*\right)$. Let $P_\Gamma = P\left(f\left(x_\Gamma - \sum_{i=1}^n w_i\right) > C_\Gamma\right)$. Then:

$$\begin{aligned}
P_\Gamma &= P\left(x_\Gamma - \sum_{i=1}^n w_i < F(Y^*) - Y^*\right) + P\left(x_\Gamma - \sum_{i=1}^n w_i > F(Y^*) + Y^*\right), && (2.27)\\
&= P\left(\sum_{i=1}^n w_i > \sum_{i=1}^n \overline{w}_i + \sum_{i=1}^n \widehat{w}_i z_i^*\right) + P\left(\sum_{i=1}^n w_i < \sum_{i=1}^n \overline{w}_i - \sum_{i=1}^n \widehat{w}_i z_i^*\right), && (2.28)\\
&= 2\cdot P\left(\sum_{i=1}^n w_i > \sum_{i=1}^n \overline{w}_i + \sum_{i=1}^n \widehat{w}_i z_i^*\right), && (2.29)\\
&\leq 2\cdot\left(1 - \Phi\left(\frac{\Gamma-1}{\sqrt{n}}\right)\right), && (2.30)
\end{aligned}$$

where we have used the convexity of f (and therefore monotonicity over $(-\infty, x_f]$ and $[x_f, \infty)$) in Eq. (2.27), the definition of x_Γ in Eq. (2.28), the symmetry of the random variables in Eq. (2.29) and Bound (2.10) in Eq. (2.30). □

Therefore, the probability that the cost in the real, stochastic world will exceed the optimal cost in the robust framework (2.19) is bounded from above by a function of the budgets of uncertainty and (the square root of) the number of random variables. This bound depends neither on the function f nor on the parameters $\overline{w}_i$, $\widehat{w}_i$. As an example, if $\Gamma = 2\sqrt{n} + 1$, the probability of exceeding the threshold C_Γ is guaranteed to be lower than 0.05, provided that the w_i are symmetric and independent.

2.2.2 Multiplicative Uncertainty

We now consider the convex problem subject to uncertainty:

$$\min \quad f\left(\sum_{i=1}^n w_i\, x_i\right) \quad \text{s.t.} \quad \mathbf{x} \in \mathcal{X}, \tag{2.31}$$

where the w_i are random, f is convex and $\mathcal{X}$ is a convex set. We use the robust optimization approach, where the uncertainty set is defined as in (2.15), to develop a model using little information on the random variables. The robust counterpart of Problem (2.31) can be formulated as:

$$
\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X}} \quad \max_{\mathbf{z}} \quad & f\left(\sum_{i=1}^{n} \overline{w}_i x_i + \sum_{i=1}^{n} \widehat{w}_i\, z_i\, x_i\right) \\
\text{s.t.} \quad & \sum_{i=1}^{n} |z_i| \le \Gamma, \\
& |z_i| \le 1,\ \forall i.
\end{aligned}
\tag{2.32}
$$

Theorem 2.2.4 (The robust problem) *Formulation (2.32) is equivalent to the convex problem:*

$$
\begin{aligned}
\min \quad & \max\left\{ f\left(\sum_{i=1}^{n} \overline{w}_i x_i - \left(p\,\Gamma + \sum_{i=1}^{n} q_i\right)\right),\ f\left(\sum_{i=1}^{n} \overline{w}_i x_i + \left(p\,\Gamma + \sum_{i=1}^{n} q_i\right)\right)\right\} \\
s.t. \quad & p + q_i \ge \widehat{w}_i\, y_i,\ \forall i, \\
& -y_i \le x_i \le y_i,\ \forall i, \\
& p \ge 0,\ q_i \ge 0,\ \forall i, \\
& \mathbf{x} \in \mathcal{X}.
\end{aligned}
\tag{2.33}
$$

Proof: From the convexity of f,

$$
\begin{aligned}
\max \quad & f\left(\sum_{i=1}^{n} \overline{w}_i\, x_i + \sum_{i=1}^{n} \widehat{w}_i\, z_i\, x_i\right) \\
\text{s.t.} \quad & \sum_{i=1}^{n} |z_i| \le \Gamma, \\
& |z_i| \le 1,\ \forall i,
\end{aligned}
\tag{2.34}
$$

is equivalent to $\max\left\{f\left(\sum_{i=1}^{n} \overline{w}_i\, x_i - \sum_{i=1}^{n} \widehat{w}_i\, |x_i|\, z_i^*(x)\right), f\left(\sum_{i=1}^{n} \overline{w}_i\, x_i + \sum_{i=1}^{n} \widehat{w}_i\, |x_i|\, z_i^*(x)\right)\right\}$ where $\mathbf{z}^*(\mathbf{x}) = \arg\max \sum_{i=1}^{n} \widehat{w}_i\, |x_i|\, z_i$ s.t. $\sum_{i=1}^{n} z_i \le \Gamma,\ 0 \le z_i \le 1$. Let $\mathbf{x}$ be a given feasible vector. By strong duality,

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} \widehat{w}_i |x_i| z_i & = \quad \min \quad & p\,\Gamma + \sum_{i=1}^{n} q_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} z_i \le \Gamma, & \text{s.t.} \quad & p + q_i \ge \widehat{w}_i\, y_i,\ \forall i, \\
& 0 \le z_i \le 1,\ \forall i, & & -y_i \le x_i \le y_i,\ \forall i, \\
& & & p \ge 0,\ q_i \ge 0,\ \forall i.
\end{aligned}
\tag{2.35}
$$

Eq. (2.21) remains valid, using the feasible set of the minimization problem in (2.35) instead of $\mathcal{F}$. The rest of the proof follows closely the proof of Theorem 2.2.1. □

The robust counterpart (2.33) of the constrained convex problem (2.32) with n sources of uncertainty is also a constrained convex problem, where $n+1$ nonnegativity constraints and $2 \cdot n$ other constraints, all linear, have been added to the original feasible set. In the robust formulation, the cost associated with a feasible $\mathbf{x} \in \mathcal{X}$ is evaluated by taking the greater value between two costs, realized for values of the uncertainty that are symmetric with respect to the mean.

Corollary 2.2.5 *Let $\mathbf{x}_{\boldsymbol{\Gamma}}$ be the optimal solution of Problem (2.33). If $\mathbf{x}_{\boldsymbol{\Gamma}}$ belongs to the interior of $\mathcal{X}$, then:*

$$\sum_{i=1}^{n} \overline{w}_i \, x_{\Gamma\, i} = F\left(\sum_{i=1}^{n} \widehat{w}_i \, |x_{\Gamma\, i}| \, z_{\Gamma\, i}\right), \tag{2.36}$$

where $\mathbf{z}_{\boldsymbol{\Gamma}} = \arg\max \sum_{i=1}^{n} \widehat{w}_i \, |x_{\Gamma\, i}| \, z_i$ *s.t.* $\sum_{i=1}^{n} z_i \leq \Gamma, \; 0 \leq z_i \leq 1.$

Proof: Is similar to Theorem 2.2.2. □

In the deterministic case, we have $\sum_{i=1}^{n} \overline{w}_i \, x_{\Gamma\, i} = x_f$ where $x_f = \arg\min_x f(x)$. Hence, $F\left(\sum_{i=1}^{n} \widehat{w}_i \, |x_{\Gamma\, i}| \, z_{\Gamma\, i}\right) - x_f$ quantifies the impact of the uncertainty on the robust solution. It also allows us to gain a better understanding of how the cost parameters in f (and F) and the volatility of the random variables, as measured by the $\widehat{w}_i$, affect the optimal solution.

We study next the probabilistic guarantees on $\sum_{i=1}^{n} \widehat{w}_i \, x_{\Gamma\, i}$ and the resulting cost.

Theorem 2.2.6 (Probabilistic guarantee) *Let C_Γ be the optimal cost in the robust problem (2.33), and $\mathbf{x}_{\boldsymbol{\Gamma}}$, resp. $\mathbf{z}_{\boldsymbol{\Gamma}}$ the corresponding optimal solution, resp. optimal scaled deviation. Let also $w_i^- = \overline{w}_i - \widehat{w}_i \, z_{\Gamma\, i}$ and $w_i^+ = \overline{w}_i + \widehat{w}_i \, z_{\Gamma\, i}$ for all i.*

We have:

$$P\left(f\left(\sum_{i=1}^{n} w_i \, x_{\Gamma\, i}\right) > C_\Gamma\right) \leq 2 \cdot \left(1 - \Phi\left(\frac{\Gamma - 1}{\sqrt{n}}\right)\right), \tag{2.37}$$

and:

$$P\left(\sum_{i=1}^{n} w_i \, x_{\Gamma\, i} \in \left[\sum_{i=1}^{n} w_i^- \, x_{\Gamma\, i}, \sum_{i=1}^{n} w_i^+ \, x_{\Gamma\, i}\right]\right) \geq 2 \cdot \Phi\left(\frac{\Gamma - 1}{\sqrt{n}}\right) - 1. \tag{2.38}$$

Proof: There are three cases:

(a) If $f\left(\sum_{i=1}^{n} w_i^- \, x_{\Gamma\, i}\right) = f\left(\sum_{i=1}^{n} w_i^+ \, x_{\Gamma\, i}\right)$, the proof is similar to the proof of Theorem 2.2.3.

(b) If $f\left(\sum_{i=1}^{n} w_i^- \, x_{\Gamma\, i}\right) > f\left(\sum_{i=1}^{n} w_i^+ \, x_{\Gamma\, i}\right)$, let A be such that:

$$f(A) = f\left(\sum_{i=1}^{n} w_i^- \, x_{\Gamma\, i}\right), \; A \neq \sum_{i=1}^{n} w_i^- \, x_{\Gamma\, i}. \tag{2.39}$$

A exists and is unique since f is convex and $\sum_{i=1}^n w_i^- \, x_{\Gamma\, i}$ is not its minimum. In particular, $A > \sum_{i=1}^n w_i^+ \, x_{\Gamma\, i}$. Let $P_\Gamma = P\left(f\left(\sum_{i=1}^n w_i \, x_{\Gamma\, i}\right) > C_\Gamma\right)$. We have:

$$\begin{aligned}
P_\Gamma &= P\left(\sum_{i=1}^n w_i \, x_i < \sum_{i=1}^n w_i^- \, x_{\Gamma\, i}\right) + P\left(\sum_{i=1}^n w_i \, x_i > A\right) && (2.40)\\
&\leq P\left(\sum_{i=1}^n w_i \, x_i < \sum_{i=1}^n w_i^- \, x_{\Gamma\, i}\right) + P\left(\sum_{i=1}^n w_i \, x_i > \sum_{i=1}^n w_i^+ \, x_{\Gamma\, i}\right) && (2.41)\\
&\leq 2 \cdot P\left(\sum_{i=1}^n w_i \, x_i < \sum_{i=1}^n w_i^- \, x_{\Gamma\, i}\right) \text{ by symmetry,} && (2.42)\\
&\leq 2 \cdot \left(1 - \Phi\left(\frac{\Gamma - 1}{\sqrt{n}}\right)\right) \text{ from (2.10).} && (2.43)
\end{aligned}$$

(c) If $f\left(\sum_{i=1}^n w_i^- \, x_{\Gamma\, i}\right) < f\left(\sum_{i=1}^n w_i^+ \, x_{\Gamma\, i}\right)$, the proof is similar to (b). □

Therefore, if the budgets of uncertainty are well chosen, the cost in the stochastic world will remain lower than a given threshold, and the actual state $\sum_{i=1}^n w_i \, x_i$ will remain within a prespecified interval, with high probabilities. The threshold and the limits of the interval are obtained by solving the robust problem (2.33).

2.2.3 The General Case

The approach developed in Sections 2.2.1 and 2.2.2 can be generalized to the problem:

$$\min \quad f\left(\sum_{i=1}^n \widetilde{v}_i \, \widetilde{x}_i + \sum_{j=1}^m \widetilde{w}_i\right) \quad \text{s.t.} \quad \widetilde{\mathbf{x}} \in \mathcal{X}, \tag{2.44}$$

where the $\widetilde{v}_i$, $\widetilde{w}_i$ are random, f is convex and $\mathcal{X}$ is a convex set.

Theorem 2.2.7 *All the results of Section 2.2.2 apply, with* $\mathbf{w}$ *defined such that* $(w_1, \ldots, w_n)' = \widetilde{\mathbf{v}}$, $(w_{n+1}, \ldots, w_{n+m})' = \widetilde{\mathbf{w}}$, *and* $\mathbf{x}$ *defined as* $(x_1, \ldots, x_n)' = \widetilde{\mathbf{x}}$, $(x_{n+1}, \ldots, x_{n+m})' = \mathbf{e}$.

Proof: Is immediate. □

As a result, the robust counterpart of (2.44) is a convex problem with the constraints of the deterministic model and new auxiliary variables belonging to a polyhedron. The value of the objective function for a feasible $\tilde{x} \in \mathcal{X}$ is evaluated by taking the greater cost associated with two possible realizations of the uncertainty. Hence, the robust problem is: (a) tractable, and (b) closely connected to the deterministic framework.

We will use this model further when we consider the dynamic case in Section 2.3 and discuss the selection of the budgets in Section 2.4.

2.3 Robust Dynamic Optimization with Linear Dynamics

In this section, we apply the robust optimization framework to the problem of optimally controlling, over a finite horizon of length T, a system subject to linear dynamics with random perturbations. We describe the methodology in discrete as well as continuous time, present the robust counterparts, and discuss their key features. In particular, we show that the robust approach yields numerically tractable formulations and analyze how uncertainty affects the optimal policy. We illustrate the approach on short examples of portfolio management and control of queueing networks. In Chapter 5, we model and analyze a supply chain management problem using the framework presented here.

2.3.1 The Discrete Case

The Model

For the sake of simplicity, we present the model for a scalar system, but it can be extended to vectors with no difficulty. We define, for $t = 0, \ldots, T$:

x_t : the state of the system at the beginning of the t-th period,

u_t : the control applied at the beginning of the t-th period,

$v_t,\ w_t$: the random perturbations during the t-th period.

The system evolves according to the linear dynamics:

$$x_{t+1} = a_t\, x_t + v_t'\, u_t + w_t, \qquad t = 0, \ldots, T-1, \tag{2.45}$$

where a_t is known and v_t and w_t are random for all t. u_t and v_t can be scalars or vectors and w_t is a scalar. This leads to the closed-form expression:

$$x_{t+1} = a_{0,t}\, x_0 + \sum_{s=0}^{t} a_{s+1,t}\, (v_s'\, u_s + w_s), \qquad t = 0, \ldots, T-1, \tag{2.46}$$

where:

$$a_{s,t} = \prod_{k=s}^{t} a_k, \quad \forall s \geq 0,\ \forall t. \tag{2.47}$$

We assume that the cost $C_t(x_t, u_t, v_t, w_t)$ incurred in period t is separable into two components, the cost $f_t(u_t)$ of applying the control u_t and the cost $g_t(x_{t+1})$ of being in state x_{t+1} at the end of the period, that is:

$$C_t(x_t, u_t, v_t, w_t) = f_t(u_t) + g_t(a_t\, x_t + v_t\, u_t + w_t), \tag{2.48}$$

where f_t and g_t are convex. (The assumption of convexity of the f_t can be somewhat relaxed, as we will illustrate in Chapter 5.) Let $\mathcal{U}_t$ be the feasible set for the control u_t at time t, which is assumed to be convex.

The deterministic problem, where the v_t and w_t are known, can therefore be formulated as:

$$
\begin{aligned}
\min \quad & \sum_{t=0}^{T-1} \left[f_t(u_t) + g_t \left(a_{0,t}\, x_0 + \sum_{s=0}^{t} a_{s+1,t} \left(v_s\, u_s + w_s \right) \right) \right] \\
\text{s.t.} \quad & u_t \in \mathcal{U}_t, \ \forall t,
\end{aligned}
\tag{2.49}
$$

where we have used the closed-form expression (2.46) of x_{t+1}. As a convex problem, (2.49) can be solved efficiently using standard optimization techniques.

In practice, $\mathbf{v}$ and $\mathbf{w}$ are random variables for which we have a limited amount of information. We model them as uncertain parameters belonging to a prespecified uncertainty set $\mathcal{S}$. A natural problem formulation for the robust approach would be:

$$
\begin{aligned}
\min \quad & \sum_{t=0}^{T-1} f_t(u_t) \quad + \max_{(\mathbf{v},\mathbf{w})\in\mathcal{S}} \ \sum_{t=0}^{T-1} g_t \left(a_{0,t}\, x_0 + \sum_{s=0}^{t} a_{s+1,t} \left(v_s'\, u_s + w_s \right) \right) \\
\text{s.t.} \quad & u_t \in \mathcal{U}_t, \forall t.
\end{aligned}
\tag{2.50}
$$

However, in (2.50) each uncertain v_s and w_s affects *all* convex functions g_t for $t \geq s$, making it difficult to maximize this sum of convex functions in a tractable manner. (An exception will be the case of linear functions g_t, where the inner maximization in (2.50) becomes a linear programming problem and can be solved with no difficulty.) For this reason, we focus instead on a slightly more conservative (hence, more robust) problem that is easier to solve:

$$
\begin{aligned}
\min \quad & \sum_{t=0}^{T-1} f_t(u_t) \quad + \sum_{t=0}^{T-1} \max_{(\mathbf{v},\mathbf{w})\in\mathcal{S}_t} g_t \left(a_{0,t}\, x_0 + \sum_{s=0}^{t} a_{s+1,t} \left(v_s'\, u_s + w_s \right) \right) \\
\text{s.t.} \quad & u_t \in \mathcal{U}_t, \forall t,
\end{aligned}
\tag{2.51}
$$

where we have for all t:

$$
\begin{aligned}
\mathcal{S}_t \quad = \quad & \Big\{ v_{sj} = \overline{v}_{sj} + \widehat{v}_{sj}\, y_{sj}, \quad w_s = \overline{w}_s + \widehat{w}_s\, z_s, \ |y_{sj}| \leq 1, \ |z_s| \leq 1, \ \forall s \leq t, \ \forall j, \\
& \left. \sum_{s=0}^{t} \left(\sum_{j} |y_{sj}| + |z_s| \right) \leq \Gamma_{(m+1)\cdot(t+1)} \right\},
\end{aligned}
\tag{2.52}
$$

Another possible choice would be to consider separate uncertainty sets for $\mathbf{v}$ and $\mathbf{w}$.

The budgets of uncertainty rule out large deviations in the cumulative perturbations.

They depend on the number of parameters that have been revealed so far. In particular, they are nondecreasing with the time horizon and increase by at most the number of new uncertain parameters at each time period:

$$0 \le \Gamma_{(m+1)\cdot(t+1)} - \Gamma_{(m+1)\cdot t} \le m+1, \quad \forall t. \tag{2.53}$$

Theorem 2.3.1 (The robust problem) *The robust counterpart of Problem (2.49) is the convex problem:*

$$\begin{array}{lll}
\min & \displaystyle\sum_{t=0}^{T-1} \left[f_t(u_t) + \max\left\{ g_t(\overline{x}_{t+1} - Y_t), g_t(\overline{x}_{t+1} + Y_t) \right\} \right] & \\
s.t. & Y_t = p_t \cdot \Gamma_{(m+1)\cdot(t+1)} + \displaystyle\sum_{s=0}^{t} \left(\sum_j q_{stj} + r_{st} \right), & \forall t, \\
 & \overline{x}_{t+1} = a_t\, \overline{x}_t + \overline{v}'_t\, u_t + \overline{w}_t, & \forall t, \\
 & p_t + q_{stj} \ge |a_{s+1,t}|\, \widehat{v}_{sj}\, u'_{sj}, & \forall t,\ \forall s \le t,\ \forall j, \\
 & p_t + r_{st} \ge |a_{s+1,t}|\, \widehat{w}_s, & \forall t,\ \forall s \le t, \\
 & -\mathbf{u}'_\mathbf{t} \le \mathbf{u}_\mathbf{t} \le \mathbf{u}'_\mathbf{t}, & \forall t, \\
 & \mathbf{u}_\mathbf{t} \in \mathcal{U}_t, & \forall t, \\
 & \mathbf{p},\ \mathbf{q},\ \mathbf{r} \ge \mathbf{0}. &
\end{array} \tag{2.54}$$

Proof: This is an immediate extension of the proof for Theorems 2.2.1 and 2.2.4. □

Remark: The robust problem (2.54) corresponds to a deterministic problem with the same dynamics as before, a modified convex cost function (not separable in the state and the control) and an extended control space $(\mathbf{u}, \mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{Y})$. The auxiliary variables belong to a polyhedral set, and the new state-related cost at the end of time period t is computed by taking the greater of two costs at the end of that period, corresponding to two possible states.

Theoretical properties of (2.54) depend on the choice of the cost and the dynamics. Below, we illustrate the methodology on a simple example in portfolio management. Chapter 5 describes in detail the linear case (with or without a fixed ordering cost), in the context of supply chain management.

An Example: Portfolio Optimization

We follow Bertsekas [8] to formulate the problem. An investor wants to maximize his wealth at the end of a finite horizon T by investing in a collecting of risky and riskless assets over time. Let n be the number of risky assets available at each time period. We define, for $t = 0, \ldots, T$:

x_t : the wealth of the investor at the beginning of the t-th period,

u_t^i : the amount invested at the start of the t-th period in the i-th risky asset,

r_t^i : the rate of return of the i-th risky asset during the t-th period,

s_t : the rate of return of the riskless asset during the t-th period.

The investor wealth evolves according to the dynamics:

$$x_{t+1} = \sum_{i=1}^{n} r_t^i \, u_t^i + s_t \left(x_t - \sum_{i=1}^{n} u_t^i \right), \quad t = 0, \ldots, T-1, \tag{2.55}$$

or equivalently:

$$x_{t+1} = s_t \, x_t + \sum_{i=1}^{n} (r_t^i - s_t) \, u_t^i, \quad t = 0, \ldots, T-1. \tag{2.56}$$

In closed form, this yields:

$$x_{t+1} = S_{0,t} \, x_0 + \sum_{i=1}^{n} \sum_{\tau=0}^{t} S_{\tau+1,t} \, (r_t^i - s_t) \, u_t^i, \quad t = 0, \ldots, T-1, \tag{2.57}$$

with

$$S_{\tau,t} = \prod_{k=\tau}^{t} s_k, \quad \forall t, \tau \leq t. \tag{2.58}$$

The goal is to maximize the utility of the investor wealth at time T, $U(x_T)$, where the utility is a concave and nondecreasing function. The traditional stochastic setting assumes probability distributions for the risky assets and solves:

$$\max E_{\mathbf{r_0}, \ldots, \mathbf{r_{T-1}}} [U(x_T)], \tag{2.59}$$

where negative amounts invested indicate short sales. Bertsekas [8] describes the optimal policy for utility functions with linear risk tolerance.

In the robust framework, each return r_i is modelled as an uncertain parameter in a symmetric interval $[\overline{r}_i - \widehat{r}_i, \overline{r}_i + \widehat{r}_i]$ and the problem becomes:

$$\max_{\mathbf{u}} \min_{\mathbf{r} \in \mathcal{R}} U(x_T), \tag{2.60}$$

where (a first try for) the uncertainty set is defined by:

$$\mathcal{R} = \left\{ r_\tau^i = \overline{r}_\tau^i + \widehat{r}_\tau^i \, z_\tau^i, \ |z_\tau^i| \leq 1, \ \forall i, \ \forall \tau, \ \sum_{i=1}^{n} \sum_{\tau=0}^{T-1} |z_\tau^i| \leq \Gamma_{nT} \right\}. \tag{2.61}$$

An important point in the robust approach is that since U is nondecreasing and the model takes a deterministic viewpoint of uncertainty, Problem (2.60) is equivalent to:

$$U\left(\max_{\mathbf{u}} \min_{\mathbf{r}\in\mathcal{R}} x_T\right), \tag{2.62}$$

that is, the optimal allocation will *not* depend on the utility function, but only on the final wealth itself. However, this raises the possibility of arbitrage, as the amounts invested have not been constrained. Therefore, we have to impose "no short sales" constraints which were not required in the stochastic setting:

$$u_t^i \geq 0, \forall i,\ t,\ \sum_{i=1}^{n} u_t^i \leq x_t,\ \forall t. \tag{2.63}$$

In turn, enforcing these constraints at all times requires uncertainty sets that depend on the current time period:

$$\mathcal{R}_t = \left\{ r_\tau^i = \overline{r}_\tau^i + \widehat{r}_\tau^i\, z_\tau^i,\ |z_\tau^i| \leq 1,\ \forall i,\ \forall \tau \leq t,\ \sum_{i=1}^{n}\sum_{\tau=0}^{t-1} |z_\tau^i| \leq \Gamma_{n\,t} \right\}. \tag{2.64}$$

This issue only arises because the investor wealth in the intermediate time periods is not included in his objective. Applying Theorem 2.3.1 to this setting, we derive the following linear programming problem for robust portfolio optimization:

$$\begin{array}{lll}
\max\limits_{\mathbf{u}} & \overline{x}_T\ - \left(p_T\,\Gamma_{nT} + \displaystyle\sum_{i=1}^{n}\sum_{t=0}^{T-1} q_{\tau,t}^i\right) & \\
\text{s.t.} & \overline{x}_{t+1} = S_{0,t}\, x_0 + \displaystyle\sum_{i=1}^{n}\sum_{\tau=0}^{t} S_{\tau+1,t}\,(\overline{r}_t^i - s_t)\, u_t^i, & \forall t, \\
& \displaystyle\sum_{i=1}^{n} u_t^i \leq \overline{x}_t - \left(p_t\,\Gamma_{nt} + \displaystyle\sum_{i=1}^{n}\sum_{\tau=0}^{t-1} q_{\tau,t}^i\right), & \forall t, \\
& p_t + q_{\tau,t}^i \geq S_{\tau+1,t-1}\,\widehat{r}_\tau^i\, u_\tau^i, & \forall i,\ t,\ \tau \leq t, \\
& p_t,\ q_{\tau,t}^i,\ u_t^i \geq 0, & \forall i,\ t,\ \tau \leq t.
\end{array} \tag{2.65}$$

Interpretations:

1. This corresponds to a problem with an extended control space $(\mathbf{u}, \mathbf{p}, \mathbf{q})$, where $\mathbf{u}$ is the main control driving the nominal dynamics $\overline{x}_{t+1}$, and $\mathbf{p}, \mathbf{q}$ are auxiliary control variables that do not affect $\overline{x}_{t+1}$ but affect the terminal cost as well as the resources available to u_t.

2. Alternatively, if we set the variables $\mathbf{p}$ and $\mathbf{q}$ to their optimal values, (2.65) can be interpreted as maximizing the nominal wealth subject to (a) "buffer" constraints, where the investor puts aside $p_t \, \Gamma_{nt} + \sum_{i=1}^{n} \sum_{\tau=0}^{t-1} q^i_{\tau,t}$ from his nominal wealth at each time period (instead of reinvesting it); and (b) upper bounds on the amount invested in each asset u^i_τ, which are of the order of $1/\widehat{r}^i_\tau$. In particular, these bounds decrease as the lengths of the confidence intervals for the r_i increase.

Bound (2.10) indicates that in this case, protecting $\sqrt{n\,t}$ risky assets for the horizon from 0 to t will provide good probabilistic guarantees. This number can be much smaller than nt, even for the early time periods, if n is sufficiently large. Therefore, an appealing feature of the approach is that it will not be necessary to protect the portfolio against many fluctuations to guarantee a good performance with high probability.

2.3.2 The Continuous Case

Generalities

In this section, we develop the robust model in the case of continuous time. We keep the same notations as in Section 2.3.1, with time in parenthesis (e.g., $x(t)$) rather than as an index (x_t). We consider infinite-dimensional linear programming problems (see Anderson and Nash [1]):

$$\begin{array}{lll} \min & \displaystyle\int_0^T \mathbf{c}'\mathbf{x}(\mathbf{t})\, dt & \\ \text{s.t.} & \dot{\mathbf{x}}(t) = \mathbf{F}\,\mathbf{u}(\mathbf{t}) + \mathbf{g}, & \forall t, \\ & \mathbf{A}\,\mathbf{u}(\mathbf{t}) \leq \mathbf{b}, & \forall t, \\ & \mathbf{x}(\mathbf{t}),\ \mathbf{u}(\mathbf{t}) \geq \mathbf{0}, & \forall t, \end{array} \tag{2.66}$$

with uncertainty on $\mathbf{F}$ and $\mathbf{g}$. In Section 2.3.1, we presented an example where the uncertainty did not depend on the control: in portfolio management, the investor can observe the fluctuations of the stock prices whether the stocks belong to his portfolio or not. Here, we will apply the robust methodology to (the fluid approximation of) queueing networks, where service times can only be observed when the decision-maker chooses to process the corresponding classes. In this sense, uncertainty depends on the control. This will be addressed by choosing the uncertainty sets appropriately. To make our point clearer, we describe the model in the context of the proposed application.

Queueing Networks

Stochastic queueing systems are often hard to analyze because of the large number of possible transitions in the state space, and most mathematical models assume exponential (that is, memoryless) inter-arrival and service distributions for tractability. Fluid approximations have been developed by Newell [49] and Chen and Mandelbaum [26] to yield some insights into the behavior of the network. They take a deterministic view of the system and replace the flow of discrete jobs by the flow of a continuous fluid. Surprisingly, there are some strong connections between the stability of the fluid model and its stochastic counterpart, as established by Dai in [30]. Fluid approximations have also been implemented to develop optimal policies for queueing networks (see Ricard [54]). However, because they do not incorporate any information on the variability of the arrival and service rates, they might lead in practice to very suboptimal policies. The robust optimization approach applied to fluid networks seems well suited to model stochasticity in a tractable manner.

We define:

$\mathbf{x}(\mathbf{t})$: the fluid content at time t,

$\mathbf{u}(\mathbf{t})$: the policy at time t,

$\mathbf{c}$: the cost vector,

λ: the vector of external arrival rates,

μ: the vector of service rates,

$\mathbf{M}$: the diagonal matrix of service rates ($\mathbf{M} = diag(\mu)$),

$\mathbf{R}$: the (deterministic) routing matrix of the different classes,

$\mathbf{A}$: the topology matrix of the network,

$\mathbf{e}$: the vector $(1, \dots, 1)'$.

The general control problem of a fluid network can be formulated as:

$$
\begin{array}{lll}
\min & \displaystyle\int_0^T \mathbf{c}'\mathbf{x}(\mathbf{t})\, dt & \\
\text{s.t.} & \dot{\mathbf{x}}(\mathbf{t}) = \lambda + \mathbf{R}\,\mathbf{M}\,\mathbf{u}(\mathbf{t}), & \forall t, \\
& \mathbf{A}\,\mathbf{u}(\mathbf{t}) \leq \mathbf{e}, & \forall t, \\
& \mathbf{x}(\mathbf{t}),\ \mathbf{u}(\mathbf{t}) \geq \mathbf{0}, & \forall t,
\end{array}
\tag{2.67}
$$

where $\mathbf{x}(\mathbf{0})$ is given. Using the closed-form expression of $\mathbf{x}(\mathbf{t})$, and removing the terms independent of the control, we obtain:

$$
\begin{aligned}
\min \quad & \mathbf{c}' \mathbf{R}\mathbf{M} \int_0^T \left[\int_0^t \mathbf{u}(\mathbf{s})\, ds \right] dt & \\
\text{s.t.} \quad & \mathbf{x}(\mathbf{0}) + \lambda\, t + \mathbf{R}\mathbf{M} \int_0^t \mathbf{u}(\mathbf{s})\, ds \geq \mathbf{0}, & \forall t, \\
& \mathbf{A}\, \mathbf{u}(\mathbf{t}) \leq \mathbf{e}, & \forall t, \\
& \mathbf{u}(\mathbf{t}) \geq \mathbf{0}, & \forall t.
\end{aligned}
\tag{2.68}
$$

In practice, λ and μ are subject to uncertainty, and randomness affects the cost as well as the nonnegativity constraints in $\mathbf{x}(\mathbf{t})$. Let $\overline{\mathbf{x}}(t)$ be, as before, the state of the system if the uncertain parameters are equal to their nominal value. To preserve the same stability region for the robust model as for the original fluid approximation, we will consider uncertainty on the cost only, and apply the nonnegativity constraints to $\overline{\mathbf{x}}(t)$. This is further motivated by the fact that the system evolves in a continuous manner, so that when $x_i(t)$ reaches 0 for some i, the control can always be adjusted in an ad-hoc manner to keep the state nonnegative. If the confidence intervals for λ and μ are sufficiently small so that the robust fluid network remains stable in the worst case, it is also possible to use the robust approach on $\mathbf{x}(\mathbf{t}) \geq \mathbf{0}$. Therefore, we focus on:

$$
\begin{aligned}
\min \quad & \mathbf{c}' \mathbf{R}\mathbf{M} \int_0^T \left[\int_0^t \mathbf{u}(\mathbf{s})\, ds \right] dt & \\
\text{s.t.} \quad & \mathbf{x}(\mathbf{0}) + \overline{\lambda}\, t + \mathbf{R}\overline{\mathbf{M}} \int_0^t \mathbf{u}(\mathbf{s})\, ds \geq \mathbf{0}, & \forall t, \\
& \mathbf{A}\, \mathbf{u}(\mathbf{t}) \leq \mathbf{e}, & \forall t, \\
& \mathbf{u}(\mathbf{t}) \geq \mathbf{0}, & \forall t,
\end{aligned}
\tag{2.69}
$$

where $\mathbf{M} = diag(\mu)$ is uncertain.

Although the arrival rates might be uncertain as well, (2.69) only depends on their nominal value. For this reason, we only describe the uncertainty set for the service rates μ_i. They are modelled as uncertain parameters in $[\overline{\mu}_i - \widehat{\mu}_i, \overline{\mu}_i + \widehat{\mu}_i]$, subject to an additional constraint (involving budgets of uncertainty) that we describe below. The cost function can be rewritten as:

$$
\min \sum_{i,j} c_i\, R_{ij} \left[\overline{\mu}_j \int_0^T (T-s)\, u_j(s)\, ds + \widehat{\mu}_j \int_0^T (T-s)\, u_j(s)\, z_j(s)\, ds \right], \tag{2.70}
$$

where the $z_j(s)$ are the scaled deviations of the service rates at time s. In particular, it is unnecessary to use budgets of uncertainty at each time period as described in Section 2.3.1. Instead, we focus on maximizing $\sum_{ij} c_i\, R_{ij}\, \widehat{\mu}_j \int_0^T (T-s)\, u_j(s)\, z_j(s)\, ds$ and consider the constraint:

$$\int_0^T |z_j(s)|\, ds \leq \Gamma, \tag{2.71}$$

for some Γ. The performance of the method hinges on the choice of the budget of uncertainty. As we have noted above, the "averaging effect" takes place across jobs being processed, i.e., at the station level rather than at the class level. This justifies indexing the budgets of uncertainty by $K\,\tau$, where K is the number of stations and τ is approximately the emptying time for the network. In particular, Bound (2.10) suggests that $\Gamma \approx \sqrt{K\,\tau}$ will perform well in practice.

We now derive the equivalent of the robust problem (2.54) in the continuous case.

Theorem 2.3.2 (The robust problem) *The robust counterpart of (2.69) is the infinite-dimensional linear programming problem:*

$$\begin{array}{lll}
\min & \displaystyle\sum_{i,j} c_i\, R_{ij}\, \overline{\mu}_j \int_0^T (T-s)\, u_j(s)\, ds + p\,\Gamma + \sum_j \int_0^T q_j(s)\, ds & \\
s.t. & p + q_j(s) \geq \left|\displaystyle\sum_i c_i\, R_{ij}\right| \widehat{\mu}_j\, (T-s)\, u_j(s), & \forall s,\ \forall j, \\
& \mathbf{x(0)} + \overline{\lambda}\, t + \mathbf{R}\,\overline{\mathbf{M}} \displaystyle\int_0^t \mathbf{u(s)}\, ds \geq \mathbf{0}, & \forall t, \\
& \mathbf{A}\,\mathbf{u(t)} \leq \mathbf{e}, & \forall t, \\
& \mathbf{u(t)},\ p,\ \mathbf{q(t)} \geq \mathbf{0}, & \forall t.
\end{array} \tag{2.72}$$

Proof: Let $A_j(s) = |\sum_i c_i\, R_{ij}|\ \widehat{\mu}_j\, (T-s)\, u_j(s)$ for all j and s. ($A_j(s) \geq 0$.) We want to show that strong duality holds for the (infinite-dimensional) auxiliary problem:

$$\begin{array}{rcll}
Z_P & = & \max & \displaystyle\sum_j \int_0^T A_j(s)\, z_j(s)\, ds \\
& & \text{s.t.} & \displaystyle\sum_j \int_0^T z_j(s) \leq \Gamma, \\
& & & 0 \leq z_j(s) \leq 1, \qquad \forall j,\ \forall s.
\end{array} \tag{2.73}$$

The dual of Problem (2.73) is:

$$\begin{array}{rcll}
Z_D & = & \min & p\,\Gamma + \displaystyle\sum_j \int_0^T q_j(s)\, ds \\
& & \text{s.t.} & p + q_j(s) \geq A_j(s), \qquad \forall j,\ \forall s, \\
& & & p,\ q_j(s) \geq 0, \qquad \forall j,\ \forall s.
\end{array} \tag{2.74}$$

Weak duality holds:

$$Z_P \leq Z_D, \tag{2.75}$$

since for any feasible $\mathbf{z}$, p, $\mathbf{q}$, we have:

$$\sum_j \int_0^T A_j(s)\, z_j(s)\, ds \le \sum_j \int_0^T A_j(s)\, z_j(s)\, ds + p\left(\Gamma - \sum_j \int_0^T z_j(s)\, ds\right) + \sum_j \int_0^T q_j(s)(1 - z_j(s))\, ds, \quad (2.76)$$

$$\sum_j \int_0^T A_j(s)\, z_j(s)\, ds \le p\,\Gamma + \sum_j \int_0^T q_j(s)\, ds + \underbrace{\sum_j \int_0^T z_j(s)\,[A_j(s) - p - q_j(s)]\, ds}_{\le 0}, \quad (2.77)$$

Furthermore, let $p = A$, $S_j = \{s | A_j(s) \ge A\}$ for all j, where A is chosen such that $\sum_j \int_0^T 1_{\{s \in S_j\}} d\, s = \Gamma$, and $q_j(s) = \max(0, A_j(s) - A)$. Let also $z_j(s) = 1$ if $s \in S_j$, 0 otherwise. Then it is easy to check that both primal and dual solutions are feasible and achieve the same cost. Therefore, strong duality holds. □

Interpretations:

1. With p, $\mathbf{q}$ set to their optimal values p^*, $\mathbf{q}^*$, Problem (2.72) is equivalent to a nominal problem with the additional upper bound on each $u_j(t)$:

$$u_j(t) \le \frac{p^* + q_j^*(t)}{|\sum_i c_i\, R_{ij}|\ \widehat{\mu}_j\, (T - t)}. \quad (2.78)$$

   This bound increases as either of the following quantities decreases: (a) the length of the remaining time horizon, (b) the length of the confidence interval for the service rates; or (c) the difference (in absolute value) between the cost of the class and that of the next class in the network.

2. With (only) p set to its optimal value p^*, Problem (2.72) becomes:

$$\begin{aligned} p^*\,\Gamma + \min \sum_{i,j} c_i\, R_{ij}\, \overline{\mu}_j \int_0^T (T - s)\, u_j(s)\, ds + \sum_j \int_0^T \max\left(0, \left|\sum_i c_i\, R_{ij}\right| \widehat{\mu}_j\, (T - s)\, u_j(s) - p^*\right) ds \\ \text{s.t. } \mathbf{A}\, \mathbf{u}(\mathbf{t}) \le \mathbf{e}, \\ \mathbf{u}(\mathbf{t}),\ \overline{\mathbf{x}}(\mathbf{t}) \ge \mathbf{0},\ \forall t. \end{aligned} \quad (2.79)$$

   In particular, the effective service rate $\mu_j(s)$ of class j at time s is given by:

$$\mu'_j = \begin{cases} \overline{\mu}_j + sign\left(\sum_i c_i R_{ij}\right) \widehat{\mu}_j, & \text{if } |\sum_i c_i\, R_{ij}|\ \widehat{\mu}_j\, (T - s)\, u_j(s) \ge p^*, \\ \overline{\mu}_j, & \text{otherwise.} \end{cases} \quad (2.80)$$

As a special case, it is easy to see that, for nontrivial values of Γ, the first classes to be processed (resp. the last ones) will have an effective service rate equal to $\overline{\mu}_j + sign\left(\sum_i c_i R_{ij}\right)\widehat{\mu}_j$ (resp. $\overline{\mu}_j$).

2.4 Selecting the Budgets of Uncertainty

In this section, we show how to select the budgets of uncertainty to guarantee performance. The performance of the system can be evaluated in two ways:

(a) Using probabilistic guarantees for the cost or the state at any time period,

(b) Using bounds on the expected cost, for a worst-case distribution.

2.4.1 Probabilistic Guarantees

A possible approach is to focus on probabilistic guarantees. We consider the dynamic optimization problem described in Section 2.3.1:

$$\begin{aligned}
\min \quad & \sum_{t=0}^{T-1} f_t(u_t) + \sum_{t=0}^{T-1} \max_{(\mathbf{y},\mathbf{z})\in\mathcal{S}_t} g_t\left(\overline{x}_{t+1} + \sum_{s=0}^{t}\left[\sum_{j=1}^{m} \widehat{v}_{sj}\, u_{sj}\, y_{sj} + \widehat{w}_s\, z_s\right]\right) \\
\text{s.t.} \quad & \overline{x}_{t+1} = a_t\,\overline{x}_t + \overline{v}_t\, u_t + \overline{w}_t,\ \forall t, \\
& \mathbf{u_t} \in \mathcal{U}_t,\ \forall t,
\end{aligned} \tag{2.81}$$

where $\mathbf{y}$ and $\mathbf{z}$ are the scaled deviations of the uncertain parameters. Let x_{t+1} be the state of the system at time $t+1$ in the stochastic setting. We also define $C_{\Gamma\, t}$ the state-related cost incurred at time t in the robust setting, Y_t the value of $\sum_{s=0}^{t}\left[\sum_{j=1}^{m}\widehat{v}_{sj}\,|u_{sj}\,y_{sj}| + \widehat{w}_s\,|z_s|\right]$ at optimality, and $x^{+}_{\Gamma,t+1} = \overline{x}_{t+1} + Y_t$, $x^{-}_{\Gamma,t+1} = \overline{x}_{t+1} - Y_t$.

Theorem 2.4.1 (Probabilistic guarantees) *We have:*

$$P\left(g_t\left(x_{t+1}\right) > C_\Gamma\right) \le 2\cdot\left(1 - \Phi\left(\frac{\Gamma_{(m+1)\cdot(t+1)} - 1}{\sqrt{(m+1)\cdot(t+1)}}\right)\right), \tag{2.82}$$

and:

$$P\left(x_{t+1} \in \left[x^{-}_{\Gamma,t+1}, x^{+}_{\Gamma,t+1}\right]\right) \ge 2\cdot\Phi\left(\frac{\Gamma_{(m+1)\cdot(t+1)} - 1}{\sqrt{(m+1)\cdot(t+1)}}\right) - 1. \tag{2.83}$$

Proof: Follows from Theorem 2.2.7. □

In other words, it is possible to select the budgets of uncertainty so that the state-related cost at each time period remains below a threshold, and the state at each time period remains within a prespecified interval, with high probabilities. The threshold and the bounds of the interval are given by the robust model.

Example 1: For $\Gamma_{(m+1)\cdot(t+1)} = 2\sqrt{(m+1)\cdot(t+1)} + 1$, $P\left(g_t\left(x_{t+1}\right) > C_\Gamma\right) \leq 0.05$ and $P\left(x_{t+1} \in \left[x^-_{\Gamma,t+1}, x^+_{\Gamma,t+1}\right]\right) \geq 0.95$. For instance, if $m = 3$ and $t = 100$, this yields $\Gamma_{(m+1)\cdot(t+1)} \approx 41.2$ for 404 uncertain parameters. In other words, we only need here to protect the system against the fluctuations of one tenth of its random variables to guarantee performance with very high probability.

Example 2: Figure 2-1 shows how (2.83) evolves as a function of Γ for $(m+1)\cdot(t+1) = 50$. Obviously, this bound is not tight for small values of Γ, as it is negative for $\Gamma = 0$. However, it increases quite rapidly towards 1. At $\Gamma = 10$, resp. $\Gamma = 15$, the probability that the state will actually be in the bounds given by the robust problem is 0.80, resp. 0.95.

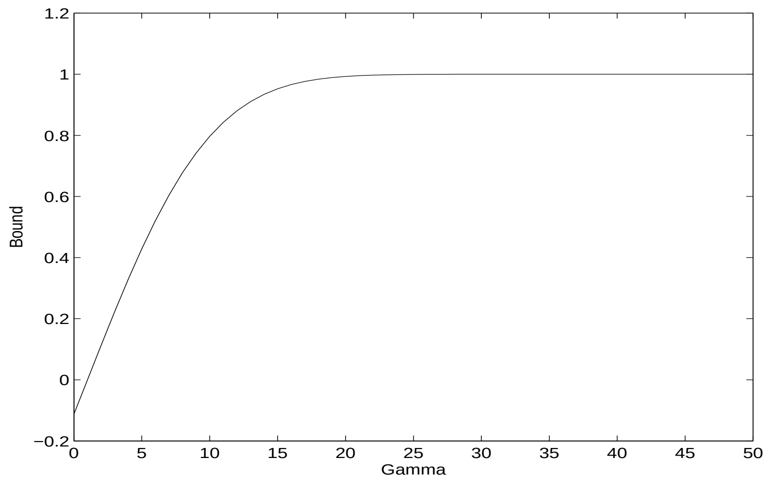

Figure 2-1: Lower bound on probability.

2.4.2 Bounds on the Expected Value

Another criterion to evaluate performance is the worst-case expected value of the cost, where the worst case is computed over the set of distributions with given properties such as their mean. Let $\mathbf{w}$ be the vector of random variables, $\mathcal{P}$ the set of feasible distributions, $\mathbf{x_\Gamma}$ the optimal solution given by the robust problem. Let also $C(\mathbf{x_\Gamma}, \mathbf{w})$ be the cost incurred by implementing $\mathbf{x_\Gamma}$ in the stochastic setting, when the realization of the randomness is $\mathbf{w}$. We want to solve in Γ:

$$\min_{\Gamma} \max_{p \in \mathcal{P}} E_p[C(\mathbf{x_\Gamma}, \mathbf{w})]. \tag{2.84}$$

This is conceptually different from the problem of minimizing in $\mathbf{x}$ the worst-case expectation:

$$\min_{\mathbf{x}} \max_{p \in \mathcal{P}} E_p[C(\mathbf{x}, \mathbf{w})]. \tag{2.85}$$

Specifically, the approach we have proposed leads to robust problems that are tightly connected to the original deterministic formulation, while the problems that arise from Problem (2.85) are stochastic problems for the worst-case distribution, and hence have a very different cost structure. To give an example, the robust model in Section 2.2.1 with a piecewise linear cost function is equivalent to a deterministic problem of the same class (i.e., a linear programming problem) with a modified parameter, but in the stochastic setting where the mean and the variance of the cumulative uncertainty are given, the problem becomes a quadratic programming problem (see Bound (2.94)).

To solve Problem (2.84) in an efficient manner, we follow the approach developed by Bertsimas and Popescu in [13] and Popescu in [52], and use strong duality to transform the inner maximization problem into an equivalent minimization problem, which can be solved in closed form. We will consider two possible choices for the set $\mathcal{P}$, depending on the information at our disposal:

1. the set of symmetric, independent random variables with given mean and support, and verifying the probabilistic guarantee (2.10),

2. the set of random variables with given mean and variance (possibly with a nonnegativity assumption).

We can also combine these two cases, or incorporate information on other moments, as described by Popescu in [52].

When the optimal robust solution $\mathbf{x}_{\mathbf{\Gamma}}$ is given, the problem of selecting the budgets of uncertainty amounts to finding tight upper bounds to $E\left[f\left(a - \sum_{i=1}^{n} w_i\right)\right]$, where f is a convex function, a is a constant and the w_i are random variables, all possibly depending on $\mathbf{x}_{\mathbf{\Gamma}}$. In the formulations we are studying, $\sum_{i=1}^{n} w_i$ is a linear combination of $\mathbf{x}_{\mathbf{\Gamma}}$ and the "real" sources of uncertainty, say $\mathbf{v}$. Therefore, the information we have on $\mathbf{v}$, such as its mean, its support or its variance, can easily be expressed in terms of $\sum_{i=1}^{n} w_i$.

Case 1: Symmetric, independent random variables with given mean, support and a probabilistic guarantee

We assume that the random variables are symmetric with given mean and support. The assumption of independence across the v_i is necessary to use the probabilistic guarantee (2.10).

Theorem 2.4.2 (Upper bound, Case 1) *Let $\mathcal{W}$ be the set of symmetric random variables W with mean $\overline{W} = \sum_{i=1}^n \overline{w}_i$, support of half-length $\widehat{W} = \sum_{i=1}^n \widehat{w}_i$, for which:*

$$P\left(\overline{W} \leq W \leq \overline{W} + Y\right) \geq \Phi\left(\frac{\Gamma - 1}{\sqrt{n}}\right) - \frac{1}{2}, \tag{2.86}$$

where $Y = \sum_{i=1}^n \widehat{w}_i z_i^*$ and $\mathbf{z}^*$ is the optimal scaled deviation for $\mathbf{w}$, obtained by solving the robust problem.

We have, with $b = a - \overline{W}$ and $\phi = \Phi\left(\frac{\Gamma-1}{\sqrt{n}}\right)$:

$$\max_{w \in \mathcal{W}} E\left[f(a-w)\right] = \left[f\left(b - Y\right) + f\left(b + Y\right)\right]\left(\phi - \frac{1}{2}\right) + \left[f\left(b - \widehat{W}\right) + f\left(b + \widehat{W}\right)\right](1 - \phi)\,. \tag{2.87}$$

Proof: We want to solve, using the symmetry of the density function around the mean:

$$\begin{aligned} \max \quad & \int_{\overline{W}}^{\overline{W}+\widehat{W}} \left[f(a - 2\,\overline{W} + w) + f(a-w)\right] g(w)\,dw \\ \text{s.t.} \quad & \int_{\overline{W}}^{\overline{W}+\widehat{W}} g(w)\,dw = \frac{1}{2}, \\ & \int_{\overline{W}}^{\overline{W}+Y} g(w)\,dw \geq \phi - \frac{1}{2}, \\ & g(w) \geq 0,\ \forall w \in \left[\overline{W}, \overline{W} + \widehat{W}\right]. \end{aligned} \tag{2.88}$$

Following Popescu [52], we consider the dual of (2.88):

$$\begin{aligned} \min \quad & \frac{\alpha}{2} + \beta\left(\phi - \frac{1}{2}\right) \\ \text{s.t.} \quad & \alpha + \beta\, 1_{\{\overline{W} \leq w \leq \overline{W}+Y\}} \geq f(a - 2\,\overline{W} + w) + f(a - w),\ \forall w \in \left[\overline{W}, \overline{W} + \widehat{W}\right], \\ & \beta \leq 0. \end{aligned} \tag{2.89}$$

The left-hand side is a piecewise constant, nondecreasing function, and the right-hand side is a convex nondecreasing function (convex because f is convex, nondecreasing from the convexity and the fact that the derivative at $\overline{W}$ is 0.) Therefore, at optimality, left-hand side and right-hand side are equal for $w = \overline{W} + Y$ and $w = \overline{W} + \widehat{W}$. These two equations allow us to determine α and β. Reinjecting into the cost function of Problem (2.89) yields Equation (2.87). $\square$

An attractive feature of this upper bound on the expected cost is that it is available in closed form for *any* cost function f.

Algorithm 2.4.3 (Selection of the budgets of uncertainty) *The budgets of uncertainty are obtained by minimizing the right-hand side of Bound (2.87) in* Γ.

If the dependence of $\mathbf{x}_{\mathbf{\Gamma}}$ in Γ is not explicit, we will need to find the optimal budgets of uncertainty iteratively, by a gradient-descent method. In the proposed robust approach, such iterations can be performed with ease, as they are approximately of the same difficulty as the nominal problem.

Example: We plot Bound (2.87) as a function of Γ for the simple case given in Section 2.2.1, with $f(x) = \max(h\,x, -p\,x)$ for h, $p > 0$ and i.i.d. random variables. Eq. (2.23) yields the robust optimal solution in closed form:

$$x_\Gamma = n\,\overline{w} + \frac{p-h}{p+h}\,\widehat{w}\,\Gamma. \tag{2.90}$$

In turn, we obtain the following expression for Bound (2.87):

$$B_1(\Gamma) = \widehat{w}\,\left[\frac{2\,p\,h}{p+h}\,\Gamma\cdot\left(2\Phi\left(\frac{\Gamma-1}{\sqrt{n}}\right)-1\right)+(p+h)\left(1-\Phi\left(\frac{\Gamma-1}{\sqrt{n}}\right)\right)\left\{n-\Gamma\left(\frac{p-h}{p+h}\right)^2\right\}\right]. \tag{2.91}$$

We omit the term in $\widehat{w}$ and take $n = 50$, $p = 5$ and h between 2 and 8. There is a clear trade-off between performance and robustness. As we start protecting the system against uncertainty, the performance of the system (in terms of worst-case bound on expected cost) improves markedly. It reaches its optimum for $\Gamma \approx 11$, and deteriorates rapidly as Γ increases past this value. Another interesting point is that the optimal value of Γ does not seem to depend significantly of the cost parameters for this choice of cost function.

Case 2: Random variables with given mean and variance

Here, we describe how to obtain a tight upper bound on the expectation when mean $\overline{W}$ and variance Σ (for the cumulative uncertainty) are known. The other notations are similar to

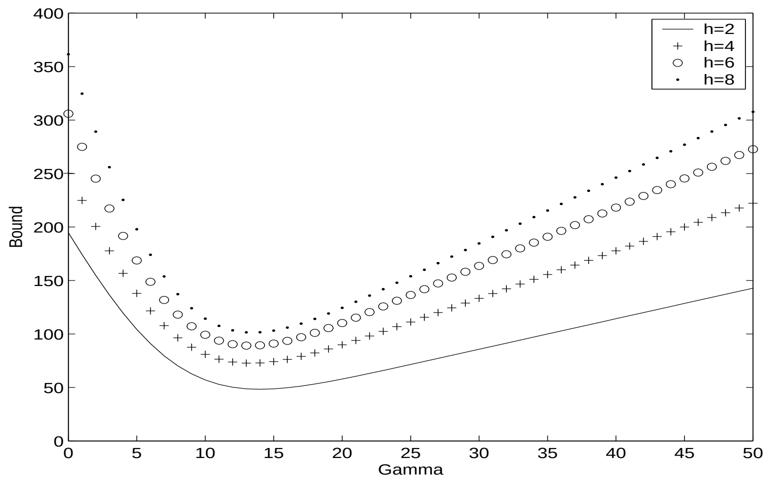

Figure 2-2: Bound on expected cost, case 1.

the previous cases. We follow closely Popescu [52].

Theorem 2.4.4 (Upper bound, Case 2) $\max_{p \in \mathcal{P}} E_p[f(a-w)]$ *is equivalent to:*

$$\begin{aligned} \min \quad & \alpha + \beta\,\overline{W} + \gamma\,\left(\overline{W}^2 + \Sigma^2\right) \\ s.t. \quad & \alpha + \beta\,w + \gamma\,w^2 \geq f(a-w), \ \forall w. \end{aligned} \tag{2.92}$$

Proof: This follows from taking the dual of:

$$\begin{aligned} \max \quad & \int_{-\infty}^{\infty} f(a-w)\,g(w)\,d\,w \\ \text{s.t.} \quad & \int_{-\infty}^{\infty} g(w)\,dw = 1, \\ & \int_{-\infty}^{\infty} w\,g(w)\,dw = \overline{W}, \\ & \int_{-\infty}^{\infty} w^2\,g(w)\,dw = \overline{W}^2 + \Sigma^2, \\ & g(w) \geq 0, \end{aligned} \tag{2.93}$$

in a manner similar as in the proof of Theorem 2.4.2. □

The closed-form expression of the bound, if available, is specific to the function f. It is obtained by computing the tangent points between the quadratic function $\alpha + \beta\,w + \gamma\,w^2$ and the convex function $f(a-w)$ to derive α, β and γ, and reinjecting into the cost of Problem (2.92).

Algorithm 2.4.5 (Selection of the budgets of uncertainty) *The budgets of uncertainty are obtained by minimizing the right-hand side of Bound (2.92) in* Γ.

Example: We consider the same example as above. Here, the bound will depend on $\widehat{w}$ (through x_Γ) as well as Σ. To implement Bound (2.92), we need the following lemma.

Lemma 2.4.6 (Optimal upper bound, Lo [45] and Bertsimas and Popescu [13]) *Let X be a random variable with mean $\overline{W}$ and variance Σ^2, which we denote as $X \sim (\overline{W}, \Sigma^2)$. For any k, we have:*

$$\max_{X\sim(\overline{W},\Sigma^2)} E[\max(0, X-k)] = \frac{1}{2}\left[-(k-\overline{W}) + \sqrt{\Sigma^2 + (k-\overline{W})^2}\right]. \tag{2.94}$$

Proof: See Bertsimas and Popescu [13]. This result was also derived by Moon and Gallego in [47], and an equivalent tight lower bound for $E\min(X,k)$ is due to Scarf [57]. Although the bound in the references above is derived in the case of nonnegative random variables, it is straightforward to adapt the proof by Bertsimas and Popescu in [13] to obtain the (simpler) formula above. □

This yields the closed-form expression for Bound (2.92):

$$B_2(\Gamma) = -\frac{(p-h)^2}{2(p+h)}\,\widehat{w}\,\Gamma + \frac{(h+p)}{2}\sqrt{\sigma^2\,n + \left(\frac{p-h}{p+h}\,\widehat{w}\,\Gamma\right)^2}, \tag{2.95}$$

where σ is the standard deviation of any w_i. We take $n = 50$, $p = 5$, and h between 2 and 8 as before. We also take $\widehat{w} = 50$ and $\sigma = 20$ for each of the random variables. Figure 2-3 shows the worst-case bound on the expected cost as a function of Γ. Here, $\sum_{i=1}^{n} w_i$ (from the formulation in Section 2.2.1) is modelled as a single random variable with known mean and variance. In contrast, the previous bound incorporated the fact that it was a sum of random variables, and as a result exhibited a sharper trade-off between performance and robustness. That trade-off was approximately independent of h and p, with an optimum for $\Gamma \approx 11$. In Figure 2-3, the shape of the curves depends strongly on the values of p and h. In particular, the curves are almost flat when $p \approx h$. For the other curves, the optimum is reached for $\Gamma \approx 5$, and there does not seem to be a significant benefit in taking $\Gamma > 0$ rather than $\Gamma = 0$ for these numerical values. This preliminary evidence suggests that incorporating the averaging effect (through probabilistic guarantee) into the bound leads to deeper insights and is better aligned with the goals of the robust optimization approach. The implementation of this approach is illustrated in further detail in Chapter 5, in the context of supply chain management.

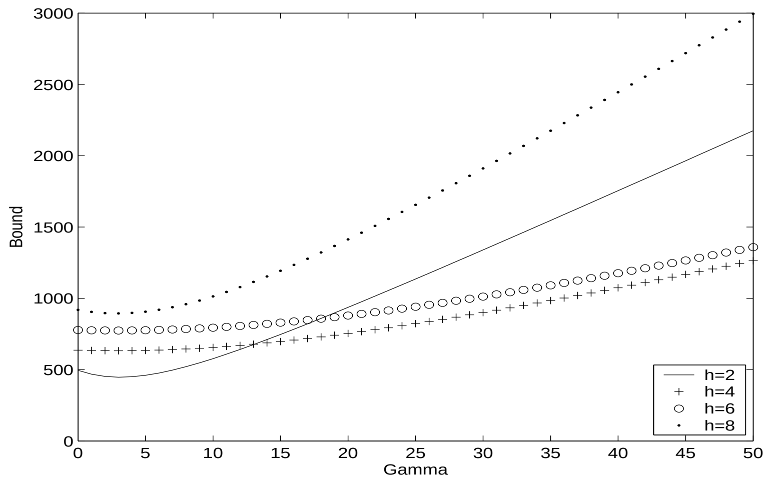

Figure 2-3: Bound on expected cost, case 2.

2.5 Extensions and Limitations

The problems studied in Sections 2.2 and 2.3 have a particular structure that plays a crucial role in the performance of the robust optimization approach. Specifically, when formulated in terms of the deviations of the uncertain parameters from their nominal value, these problems involve (weighted) sums of the scaled deviations and an averaging effect in the cost incurred from the real-world randomness. In other words, realizations that are worse than average can be counterbalanced by others that are better than average. Although this is often the case in practice, it also leaves out important classes of problems. We illustrate this point and highlight the limitations of the proposed methodology on the following example. We consider the class of linear programming problems with uncertain bounds on the decision variables, which can be used to model revenue management problems with random demand. The deterministic problem is:

$$\begin{aligned} \max \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\,\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{0} \leq \mathbf{x} \leq \mathbf{d}, \end{aligned} \tag{2.96}$$

where only $\mathbf{d}$ is subject to randomness, for instance the demand in a revenue-maximizing problem with capacity constraints. For simplicity we will assume that $a_{ij} \geq 0$ for all i, j. In the robust optimization approach, each d_i is modelled by an uncertain parameter in $[\overline{d}_i - \widehat{d}_i, \overline{d}_i + \widehat{d}_i]$ with $\sum_{i=1}^{n} |d_i - \overline{d}_i|/\widehat{d}_i \leq \Gamma$. However, if $\Gamma \geq 1$, protecting the constraints $x_i \leq \overline{d}_i + \widehat{d}_i\, z_i$, $|z_i| \leq 1$ always yields $x_i \leq \overline{d}_i - \widehat{d}_i$, i.e., is very conservative. To address this issue raised by column-wise uncertainty, a possibility is to consider an alternative formulation, which is equivalent to Eq. (2.96) when the problem is deterministic:

$$
\begin{aligned}
\max \quad & \mathbf{c}' \min(\mathbf{x}, \mathbf{d}) \\
\text{s.t.} \quad & \mathbf{A}\,\mathbf{x} \le \mathbf{b}, \\
& \mathbf{x} \ge \mathbf{0}.
\end{aligned}
\tag{2.97}
$$

Since all the uncertainty is expressed in the cost, we can hope that the robust optimization approach applied to Problem (2.97) will not be so conservative. Unfortunately, this is not the case, because $\min(\mathbf{x}, \mathbf{d})$ limits the revenue upward when demands are higher than expected. Alternatively, in this model the resource is wasted if it is not used by the class it was originally allocated to. Therefore, there is no averaging effect. We detail below from a more technical perspective what happens if we implement the robust optimization approach in this context.

Applying the framework of Sections 2.2 and 2.3, we define the robust counterpart of Problem (2.97) as:

$$
\begin{aligned}
\max \quad & \left[\min_{\mathbf{z} \in \mathcal{Z}} \quad \sum_{i=1}^{n} c_i \min(x_i, \overline{d}_i + \widehat{d}_i\, z_i) \right] \\
\text{s.t.} \quad & \mathbf{A}\,\mathbf{x} \le \mathbf{b}, \\
& \mathbf{x} \ge \mathbf{0},
\end{aligned}
\tag{2.98}
$$

where $\mathcal{Z} = \left\{ |z_i| \le 1, \ \forall i, \ \sum_{i=1}^{n} |z_i| \le \Gamma \right\}$, or equivalently:

$$
\begin{aligned}
\max \quad & \min \quad && \sum_{i=1}^{n} c_i \min(x_i, \overline{d}_i - \widehat{d}_i\, z_i) \\
& \text{s.t.} && \sum_{i=1}^{n} z_i \le \Gamma, \\
& && 0 \le z_i \le 1, \ \forall i, \\
\text{s.t.} \quad & \mathbf{A}\,\mathbf{x} \le \mathbf{b}, && \\
& \mathbf{x} \ge \mathbf{0}, &&
\end{aligned}
\tag{2.99}
$$

as the worst case is reached when the demand is less than expected. If Γ is integer, the minimum over $\mathcal{Z}$ will be reached for $z_i \in \{0, 1\}$ $\forall i$ (the minimum of a concave function over a convex set is reached on the boundary of the feasible set, and it is easy to show that if there are i, j such that $0 < z_i$, $z_j < 1$, the revenue can be increased by bringing z_i and z_j to 0 or 1.) Therefore, the robust problem (2.99) can be rewritten as a linear combination of the z_i.

For any feasible $\mathbf{x}$, we have:

$$\sum_{i=1}^{n} c_i \, \min(x_i, \overline{d}_i - \widehat{d}_i \, z_i) = \sum_{i=1}^{n} c_i \left[\min(x_i, \overline{d}_i) \, (1 - z_i) + \min(x_i, \overline{d}_i - \widehat{d}_i) \, z_i \right]. \tag{2.100}$$

If at optimality $\mathbf{x}^*$ is such that $x_i^* > \overline{d}_i$ for some i, $\min(\overline{x}^*, \overline{d})$ is also feasible (since we have assumed $a_{ij} \geq 0 \; \forall i, j$) and yields the same revenue. So we can limit ourselves to considering:

$$\begin{array}{lll} \max & \displaystyle\sum_{i=1}^{n} c_i \, x_i + & \min \quad \displaystyle\sum_{i=1}^{n} c_i \, z_i \, \min(0, \overline{d}_i - \widehat{d}_i - x_i) \\ & & \text{s.t.} \quad \displaystyle\sum_{i=1}^{n} z_i \leq \Gamma, \\ & & \qquad\;\; 0 \leq z_i \leq 1, \;\; \forall i, \\ \text{s.t.} & \mathbf{A}\,\mathbf{x} \leq \mathbf{b}, & \\ & \mathbf{0} \leq \mathbf{x} \leq \overline{\mathbf{d}}. & \end{array} \tag{2.101}$$

As a result, we have found a linear robust counterpart to the original problem:

$$\begin{array}{lll} \max & \displaystyle\sum_{i=1}^{n} c_i \, x_i - \left(p \, \Gamma + \sum_{i=1}^{n} q_i \right) & \\ \text{s.t.} & p + q_i - c_i \, y_i \geq 0, & \forall i, \\ & x_i - y_i \leq \overline{d}_i - \widehat{d}_i, & \forall i, \\ & \mathbf{A}\,\mathbf{x} \leq \mathbf{b}, & \\ & \mathbf{0} \leq \mathbf{x} \leq \overline{\mathbf{d}}, & \\ & p, \, \mathbf{q}, \, \mathbf{y} \geq 0. & \end{array} \tag{2.102}$$

The issue with this formulation is that it is possible, and sometimes optimal, to select $x_i = \overline{d}_i - \widehat{d}_i$ more than Γ times and yet have $\sum_{i=1}^{n} z_i \leq \Gamma$. In that case the robust framework yields very conservative results, and the meaning of the budget of uncertainty is lost.

It follows from this analysis that, although the robust approach with uncertainty sets performs well for a wide range of problems, alternative approaches need to be developed for problems that do not allow for an averaging effect across the sources of uncertainty. This is the purpose of the next chapter.

2.6 Concluding Remarks

In this chapter, we have proposed an approach, based on robust optimization with uncertainty sets, that addresses the problem of optimally controlling stochastic systems without assuming specific probability distributions. Instead, we have modelled random variables as uncertain parameters belonging to polyhedral sets, and optimized the system against the worst-case value of the uncertainty in that set. The robust framework builds upon the features of the deterministic convex problem to yield tractable formulations of a similar class. This innovative way to model randomness also provides key insights into the optimal solution, in the static as well as in the dynamic case. It is particularly well suited to problems with an averaging effect in the uncertainty. Furthermore, we have described how to select the parameters to ensure practical performance. The robust optimization approach with uncertainty sets emerges as a powerful methodology, which will be further investigated in its application to supply chain management in Chapter 5.

Chapter 3

Data-Driven Robust Optimization

3.1 Background and Contributions

In the traditional approach to stochastic optimization, the decision-maker has at his disposal the past realizations of the random variables, and uses them to estimate the distribution in the next time period. This of course is nothing more than an educated guess, without any guarantee that the future will indeed behave like the past. The historical data can also be used to extract some properties of the random variables, such as the mean or the confidence interval, and build the uncertainty sets described in Chapter 2. On the other hand, we have seen that the performance of that approach is affected by the problem structure, and the data sample might contain more information than what is captured by the mean and support width. Therefore, it seems legitimate to develop a robust optimization approach based directly on the data, which eliminates the need for an estimation procedure.

The data-driven approach should also reflect the decision-maker's attitude towards risk. We assume that he is risk-averse, that is, he prefers a sure reward to a stochastic reward with the same mean, despite the possibility of a higher gain in the latter case. This question of preferences is central to any realistic framework of optimization under uncertainty, and has been addressed in the past by considering the expected value of utility functions. Unfortunately, in practice it is very difficult to articulate a particular investor's utility. We have also noted in Chapter 2 that accurate probabilities are rarely available in real-life applications. This raises the following point:

Can we develop a tractable framework, built directly on the available data, that will incorporate risk aversion and protect the system against uncertainty without assuming a specific utility and/or probability distribution?

In this chapter, we present such an approach, based on robust optimization techniques. The idea is to remove the best realizations of the random data and optimize the system over the remaining sample. As a result, the solution will be robust against downside risk. In mathematical terms, we replace the expected utility $E[U(X)]$ by the conditional expectation $E[X|X \leq q_\alpha(X)]$, where $q_\alpha(X)$ is the α-quantile of the random variable X:

$$q_\alpha(X) = \inf\{x|P(X \leq x) \geq \alpha\}, \ \alpha \in (0,1). \tag{3.1}$$

Note that if we consider a minimization problem with cost X, we will remove the cases yielding the *lowest* costs, and hence, focus on the tail expectation $E[X|X \geq q_\alpha(X)]$. A nonparametric estimator of $E[X|X \leq q_\alpha(X)]$ is provided by:

$$\widehat{R}_\alpha = \frac{1}{N_\alpha}\sum_{k=1}^{N_\alpha} X_{(k)}, \tag{3.2}$$

where N is the total number of observations available, N_α is the number of cases remaining after trimming to the level α ($N_\alpha = \lfloor N \cdot (1-\alpha) + \alpha \rfloor \approx N \cdot (1-\alpha)$) and $X_{(k)}$ is the k-th smallest component of $(X_1, \ldots. X_N)$, yielding $X_{(1)} \leq \ldots \leq X_{(N)}$. In minimization problems, $X_{(k)}$ will be defined as the k-th greatest component. The exact definition of N_α, and specifically the addition of α to $N \cdot (1-\alpha)$, is motivated by the extreme value $\alpha = 1$, where we want to keep the worst-case realization only.

Although two-sided trimming has been extensively studied in statistics, for instance by Rousseeuw and Leroy [55], Ryan [56], Wilcox [69], to develop robust estimators, one-sided trimming has received little attention outside the field of portfolio optimization, where it has been shown to have some attractive properties, in particular a linear programming formulation (see Bertsimas et al. [12], Krokhmal et. al. [41], Pachamanova [51], Uryasev and Rockafellar [64]). It has also been shown to be a coherent risk measure (see Artzner et. al. [2]). Levy and Kroll have characterized in [43] investor preferences in terms of the quantile functions of their investments.

Theorem 3.1.1 (Levy and Kroll [43]) *$E[U(X)] \geq E[U(Y)]$ for all U increasing and concave if and only if $E[X|X \leq q_\alpha(X)] \geq E[Y|Y \leq q_\alpha(Y)]$ for any α in $(0,1)$, and we have strict inequality for some α.*

This theorem establishes that a portfolio chosen to maximize $E[X|X \leq q_\alpha(X)]$ is non-dominated, i.e., no other portfolio would be preferred by all investors with increasing and concave utilities.

The conditional expectation at the quantile level is closely related to the expected shortfall studied by Bertsimas et. al. [12], which is defined by:

$$s_\alpha(X) = E[X] - E[X|X \leq q_\alpha(X)]. \tag{3.3}$$

In the context of portfolio management, it measures the expected loss if the return of the portfolio drops below its α-quantile. In what follows, we will refer to $E[X|X \leq q_\alpha(X)]$ (or $E[X|X \geq q_\alpha(X)]$ in minimization problems) as the Conditional Value-at-Risk (CVaR).

Before studying CVaR further, we point out two major differences with the methodology developed in Chapter 2: (a) the robust approach with uncertainty sets rules out large deviations, good and bad, while CVaR only removes the best cases; and (b) the base value of the budgets of uncertainty ($\Gamma = 0$) corresponds to the nominal, deterministic problem, while CVaR with $\alpha = 0$ corresponds to the sample stochastic problem.

The contributions of this chapter can be summarized as follows:

1. We develop a data-driven approach that is based on Conditional Value-at-Risk, and is tightly connected to the risk aversion of the decision-maker,

2. We derive robust formulations that are computationally tractable, as the robust counterpart of a convex problem remains convex, and illustrate the applicability of the method on a wide range of problem structures, in the static and dynamic cases,

3. We gain important insights into the structure of the optimal solution for linear programming problems with uncertain right-hand side, in particular we show that the optimal solution can be found by ranking quantities related to the historical data and the dual variables.

In Section 3.2, we build the theoretical framework for the data-driven approach. We apply the method to static problems in Section 3.3 and extend the results to the dynamic case in Section 3.4. We discuss how to select the trimming factor in Section 3.5. Finally, Section 3.6 presents some concluding remarks.

3.2 The Data-Driven Framework

3.2.1 The Model

We consider problems of the type:

$$\begin{aligned} \min \quad & f_0\left(\mathbf{A_0}\,\mathbf{x} + \mathbf{b_0}\right) \\ \text{s.t.} \quad & f_i\left(\mathbf{A_i}\,\mathbf{x} + \mathbf{b_i}\right) \leq 0, \quad \forall i \geq 1, \\ & \mathbf{x} \in \mathcal{X}, \end{aligned} \tag{3.4}$$

where $\mathbf{x}$ is the decision vector, $(\mathbf{A_i}, \mathbf{b_i})$, $i \geq 0$, are the parameters subject to randomness, the f_i, $i \geq 0$, are convex functions and $\mathcal{X}$ is a convex set. There are two ways to formulate (3.4) in the stochastic case using Conditional Value-at-Risk:

1. We can incorporate $f_i\left(\mathbf{A_i}\,\mathbf{x} + \mathbf{b_i}\right) \leq 0$, $i \geq 1$, to the cost using convex penalty functions, and minimize the CVaR of the modified cost.

2. We can replace each random quantity $f_i\left(\mathbf{A_i}\,\mathbf{x} + \mathbf{b_i}\right)$, $i \geq 0$, by its CVaR, in the cost and the constraints.

In this chapter, we will only consider Case 1. We study Case 2 in Chapter 4 when we compare the robust approaches in the context of linear programming. (Other methods, such as random sampling presented by Calafiore and Campi in [24], are also available to address the stochasticity of the constraints, but are outside the scope of this thesis.)

Eq. (3.4) can be rewritten as:

$$\begin{aligned} \min \quad & f(\mathbf{A}\,\mathbf{x} + \mathbf{b}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X}, \end{aligned} \tag{3.5}$$

where f incorporates the penalty function. We assume that we have at our disposal N realizations of the random variables, $(\mathbf{A}^1, \mathbf{b}^1), \ldots, (\mathbf{A}^N, \mathbf{b}^N)$. The traditional framework assumes specific distributions for $\mathbf{A}$ and $\mathbf{b}$, and solves:

$$\begin{aligned} \min \quad & E_{\mathbf{A},\mathbf{b}}\left[f(\mathbf{A}\mathbf{x} + \mathbf{b})\right] \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X}, \end{aligned} \tag{3.6}$$

or its sample-based equivalent:

$$\begin{aligned} \min \quad & \frac{1}{N} \sum_{k=1}^{N} f(\mathbf{A^k}\,\mathbf{x} + \mathbf{b^k}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X}. \end{aligned} \tag{3.7}$$

Let α be the fraction of data removed, also called the trimming factor, and $N_\alpha = \lfloor N \cdot (1-\alpha) + \alpha \rfloor$ the number of observations remaining after trimming. In particular, if $\alpha = 0$, $N_\alpha = N$ and we simply compute the sample mean. If $\alpha = 1$, $N_\alpha = 1$ and we only keep the worst case. Selecting α in $(0,1)$ allows us to adjust the level of conservatism of the solution. The data-driven robust counterpart can be written as:

$$\begin{aligned} \min \quad & \frac{1}{N_\alpha} \sum_{k=1}^{N_\alpha} f\left(\mathbf{A}^{\cdot}\,\mathbf{x} + \mathbf{b}^{\cdot}\right)_{(k)} \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X}, \end{aligned} \tag{3.8}$$

where $s_{(k)}$ is the k-th greatest element of $\mathbf{s}$ $(s_{(1)} \geq \ldots \geq s_{(N)})$.

Theorem 3.2.1 (The robust problem)

(a) The robust counterpart of (3.5) is the convex problem:

$$\begin{aligned} \min \quad & \phi + \frac{1}{N_\alpha} \sum_{k=1}^{N} \psi_k \\ s.t. \quad & \phi + \psi_k \geq f\left(\mathbf{A}^{\mathbf{k}}\,\mathbf{x} + \mathbf{b}^{\mathbf{k}}\right), \quad && \forall k, \\ & \psi_k \geq 0, && \forall k, \\ & \mathbf{x} \in \mathcal{X}. \end{aligned} \tag{3.9}$$

Furthermore, this is a linear programming problem if f is (convex) piecewise linear.

(b) Problem (3.9) is equivalent to:

$$\begin{aligned} \min \quad & \phi + \frac{1}{N_\alpha} \sum_{k=1}^{N} \max\left(0, f\left(\mathbf{A}^{\mathbf{k}}\,\mathbf{x} + \mathbf{b}^{\mathbf{k}}\right) - \phi\right) \\ s.t. \quad & \mathbf{x} \in \mathcal{X}. \end{aligned} \tag{3.10}$$

At ϕ given, this is a stochastic optimization problem similar to Problem (3.7), with the same feasible set $\mathcal{X}$, the same realizations $(\mathbf{A}^{\mathbf{k}}, \mathbf{b}^{\mathbf{k}})$ for $k = 1, \ldots, N$, and a truncated cost function $\max\left(f(\mathbf{A}\,\mathbf{x} + \mathbf{b}), \phi\right)$, where the cost values that fall below a threshold ϕ are replaced by ϕ.

(c) Problem (3.9) can be interpreted a posteriori as a stochastic problem of the type of Problem (3.7), solved for a specific distribution P^* over the set of past realizations. We have:

$$P^*\left((\mathbf{A}, \mathbf{b}) = (\mathbf{A}^{\mathbf{k}}, \mathbf{b}^{\mathbf{k}})\right) = \begin{cases} \dfrac{1}{N_\alpha}, & \textit{if } k \in \mathcal{S}_\alpha, \\ 0, & \textit{otherwise}, \end{cases} \tag{3.11}$$

where $\mathcal{S}_\alpha$ is defined as follows. Let $\phi^* = f(\mathbf{A}^{\cdot}\,\mathbf{x}^* + \mathbf{b}^{\cdot})_{(k)}$, $\psi_k^* = \max\left(0, f(\mathbf{A}^{\mathbf{k}}\,\mathbf{x}^* + \mathbf{b}^{\mathbf{k}})\right)$

for all k, $S_\alpha^+ = \{k | \psi_k^* > 0\}$ and $S_\alpha^0 = \{k | \phi^* = f(\mathbf{A^k x^*} + \mathbf{b^k})\}$. If S_α^0 has more than $N_\alpha - |S_\alpha^+|$ elements, we only keep $N_\alpha - |S_\alpha^+|$ of them (chosen arbitrarily) in the set. We take $\mathcal{S}_\alpha = \mathcal{S}_\alpha^+ \cup \mathcal{S}_\alpha^0$.

Proof: (a) For any vector $\mathbf{s}$ with ranked components $s_{(1)} \geq \dots s_{(N)}$, $\sum_{k=1}^{N_\alpha} s_{(k)}$ is the optimal solution of:

$$\begin{aligned} \max \quad & \sum_{k=1}^{N} s_k \, y_k \\ \text{s.t.} \quad & \sum_{k=1}^{N} y_k = N_\alpha, \\ & 0 \leq y_k \leq 1, \qquad \forall k. \end{aligned} \tag{3.12}$$

The feasible set of Problem (3.12) is nonempty and bounded, therefore by strong duality Problem (3.12) is equivalent to:

$$\begin{aligned} \min \quad & N_\alpha \cdot \phi + \sum_{k=1}^{N} \psi_k \\ \text{s.t.} \quad & \phi + \psi_k \geq s_k, \qquad \forall k, \\ & \psi_k \geq 0, \qquad\qquad \forall k. \end{aligned} \tag{3.13}$$

Reinjecting Eq. (3.13) into Eq. (3.8) with $s_k = f\left(\mathbf{A^k x} + \mathbf{b^k}\right)$ for all k yields Eq. (3.9). This is a convex problem because f is convex. Moreover, if f is piecewise linear, that is $f(\mathbf{y}) = \max_l [\mathbf{c_l}'\mathbf{y} + d_l]$ for all $\mathbf{y}$, then Problem (3.9) is a linear programming problem.
(b) We note that at optimality,

$$\psi_k = \max\left(0, f\left(\mathbf{A^k x} + \mathbf{b^k}\right) - \phi\right), \quad \text{for all } k. \tag{3.14}$$

Eq. (3.10) is obtained by injecting Eq. (3.14) into Eq. (3.9).
(c) Finally, Eq. (3.9) can be interpreted a posteriori as solving the stochastic problem:

$$\begin{aligned} \min \quad & \frac{1}{N_\alpha} \sum_{k=1}^{N} f\left(\mathbf{A^k x} + \mathbf{b^k}\right) y_k^* \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X}, \end{aligned} \tag{3.15}$$

where $y_k^* = 1$ if scenario k is among the N_α worst cases and 0 otherwise. The y_k^* are obtained from Problem (3.9) once it has been solved to optimality. In particular, it is easy to see that $\phi^* + \psi_k^* = s_k$ if k is among the N_α greatest scenarios, with $\phi^* = s_{(N_\alpha)}$ and $\psi_k^* = \max(0, s_k - \phi^*)$. However, if several scenarios achieve the N_α-th worst case, the set

$S = \{k|\phi^* + \psi_k^* = s_k\}$ has more than N_α elements. We distinguish between the scenarios k in S whose cost is strictly worse than the N_α-th worst case ($\psi_k^* > 0$) and the ones that make the tie (k such that $\phi^* = f(\mathbf{A^k}\,\mathbf{x} + \mathbf{b^k})$), to obtain a set of N_α worst cases. □

Theorem 3.2.1 establishes that the data-driven robust framework leads to tractable convex problems (point (a)), which can be reformulated with only one new variable, the same feasible set as in the deterministic problem (3.5) and a similar, truncated cost function (point (b)), and have a strong connection with the traditional stochastic problem (3.7) (point (c)). In particular, once the set of worst-case scenarios has been identified, we can use all the insights available for the optimal stochastic policy to characterize its robust counterpart.

3.2.2 Incorporating New Information

The robust problem (3.8) adds $2 \cdot N$ constraints to the nominal problem: N constraints involving ϕ and ψ, which we will call the main constraints, and N nonnegativity constraints on ψ. This can be a large number if many scenarios are available. However, the number of scenarios considered after trimming, N_α, might be much smaller than N. In this section, we describe how we can solve the robust problem by considering convex problems of smaller size as new information is revealed over time.

Let K be the number of realizations (out of N) we have observed so far, and let $N_\alpha = \lfloor N \cdot (1-\alpha) + \alpha \rfloor$ be the number of worst cases considered in the robust approach. Let also S_K be the optimal set of worst-case scenarios for the robust problem solved at time K (with the K historical observations).

Theorem 3.2.2 (Reduced robust problem)

If $K \leq N_\alpha$, the robust problem is equivalent to solving the stochastic problem (3.7) for those realizations. In particular, $|S_K| = K$ and $S_{N_\alpha} = \{1, \ldots, N_\alpha\}$.

If $K > N_\alpha$, $|S_K| = N_\alpha$ and the robust problem at time K is equivalent to solving a reduced problem with $2\cdot(N_\alpha+1)$ (rather than $2\cdot N$) constraints added to the deterministic formulation (3.4):

$$
\begin{aligned}
\min \quad & \phi + \frac{1}{N_\alpha}\sum_{k=1}^{K}\psi_k & & \\
s.t. \quad & \phi + \psi_k \geq f(\mathbf{A^k}\,\mathbf{x} + \mathbf{b^k}), & & k \in S_{K-1} \cup \{K\}, \\
& \psi_k \geq 0, & & k \in S_{K-1} \cup \{K\}, \\
& \mathbf{x} \in \mathcal{X}. & &
\end{aligned}
\tag{3.16}
$$

Once (3.16) has been solved to optimality at time K, we define S_K similarly to $\mathcal{S}_\alpha$ in

Theorem 3.2.1. Moreover, it is unnecessary to resolve the problem at time K if the optimal solution of the problem at time $K-1$ is still feasible with $\psi_K = 0$, in which case $S_K = S_{K-1}$.

Proof: The result for $K \leq N_\alpha$ is straightforward. At time K for $K > N_\alpha$, we use that $S_K \subset S_{K-1} \cup \{K\}$, since either K will be among the N_α worst cases or this new observation will not change the optimal set. S_K is updated in the same manner as $\mathcal{S}_\alpha$ in Theorem 3.2.1. Moreover, the optimal solution at time $K-1$ will still be optimal if and only if it is still feasible with $\psi_K = 0$. □

Example: if the decision-maker only wants to keep the 10 worst cases among the realizations he has observed so far, and has observed a number $N \geq 10$, which can possibly be very large, he only needs to consider 11 scenarios at each time period.

3.3 Robust Static Optimization

In this section, we revisit the static problems presented in Section 2.2 from a data-driven perspective. Then we apply the robust methodology to the linear programming problem described in Section 2.5, which could not be satisfactorily addressed with uncertainty sets, and characterize the optimal policy. A last example further illustrates the applicability of the robust approach in linear programming. The data-driven methodology allows us to derive key insights on the structure of the optimal solution.

3.3.1 Additive and Multiplicative Uncertainty

Additive Uncertainty

We consider the unconstrained problem of minimizing $f\left(x - \sum_{i=1}^n w_i\right)$ where $\sum_{i=1}^n w_i$ is random and f is a convex function such that $\lim_{|x|\to\infty} f(x) = \infty$. This problem was addressed in Section 2.2.1 using uncertainty sets. Here, we assume that we have at our disposal a set of N scenarios, $\sum_{i=1}^n w_i^1, \ldots, \sum_{i=1}^n w_i^N$. The data-driven model (3.8) becomes:

$$\min \frac{1}{N_\alpha} \sum_{k=1}^{N_\alpha} \left[f\left(x - \sum_{i=1}^n w_i^{\cdot}\right)\right]_{(k)}, \tag{3.17}$$

where $\left[f\left(x - \sum_{i=1}^n w_i^{\cdot}\right)\right]_{(k)}$ denotes the k-th greatest $f\left(x - \sum_{i=1}^n w_i^k\right)$.

Theorem 3.3.1 (The robust problem)

(a) The robust problem is:

$$
\begin{aligned}
\min \quad & \phi + \frac{1}{N_\alpha}\sum_{k=1}^{N}\psi_k \\
s.t. \quad & \phi + \psi_k \geq f\left(x - \sum_{i=1}^{n} w_i^k\right), \quad \forall k, \\
& \psi_k \geq 0, \qquad\qquad\qquad\qquad \forall k.
\end{aligned}
\tag{3.18}
$$

(b) Let $(\sum_{i=1}^{n} w_i^{\cdot})_{(k)}$ be the k-th greatest element of $\left(\sum_{i=1}^{n} w_i^j\right)$, $j = 1, \ldots, N$. The optimal solution x_α to (3.18) can be found by solving $(N_\alpha + 1)$ stochastic problems, where the K-th subproblem, $K = 0, \ldots, N_\alpha$, is defined by:

$$
\min \frac{1}{N_\alpha}\left\{\sum_{k=1}^{K} f\left(x - \left(\sum_{i=1}^{n} w_i^{\cdot}\right)_{(k)}\right) + \sum_{k=M_K+1}^{N} f\left(x - \left(\sum_{i=1}^{n} w_i^{\cdot}\right)_{(k)}\right)\right\}, \tag{3.19}
$$

with $M_K = K + N - N_\alpha$. x_α is the solution of the subproblem with the greatest cost.

Proof: (a) follows from Theorem 3.2.1. Since f is convex and $\lim_{|x|\to\infty} f(x) = \infty$, f has a unique, finite minimum x_f and is decreasing, resp. increasing, over $(-\infty, x_f]$, resp. $[x_f, \infty)$. Therefore, if $x - (\sum_{i=1}^{n} w_i^{\cdot})_{(k)} \leq x_f$, then $f\left(x - (\sum_{i=1}^{n} w_i^{\cdot})_{(j)}\right)$ decreases in j for $j \leq k$. The case where $x - (\sum_{i=1}^{n} w_i^{\cdot})_{(k)} \geq x_f$ is similar. As a result, the set of worst-case scenarios is of the type $S_\alpha = \{1, \ldots, K\} \cup \{M_K, \ldots, N\}$, where $M_K = K + N - N_\alpha$ since $|S_\alpha| = N_\alpha$. (b) follows by enumerating the possible values of K. □

Theorem 3.3.1 presents a tractable formulation for the data-driven problem (point (a)) and gives some insight into the worst-case scenarios (point (b)). In particular, it establishes that the worst cases are a combination of the higher and the smaller values of the uncertainty.

Multiplicative uncertainty

Here, we consider the problem of minimizing $f(\sum_{i=1}^{n} w_i x_i)$ subject to $\mathbf{x} \in \mathcal{X}$, where f is a convex function and $\mathcal{X}$ a convex set. Its data-driven robust counterpart is:

$$
\min \frac{1}{N_\alpha}\sum_{k=1}^{N_\alpha} f\left(\sum_{i=1}^{n} w_i^{\cdot} x_i\right)_{(k)} \quad \text{s.t. } \mathbf{x} \in \mathcal{X}. \tag{3.20}
$$

Theorem 3.3.2 (The robust problem) *The robust problem is:*

$$
\begin{aligned}
\min \quad & \phi + \frac{1}{N_\alpha}\sum_{k=1}^{N}\psi_k \\
s.t. \quad & \phi + \psi_k \geq f\left(\sum_{i=1}^{n} w_i^k x_i\right), \quad \psi_k \geq 0, \quad \forall k, \\
& \mathbf{x} \in \mathcal{X}.
\end{aligned}
\tag{3.21}
$$

Proof: Follows from Theorem 3.2.1. □

Therefore, the data-driven formulation allows us to optimize the system and determine the worst-case scenarios in a single step, by solving a convex problem with a linear cost function and $2 \cdot N$ additional (linear) constraints.

3.3.2 Application to Linear Programming: Uncertain Bounds

We consider the linear programming problem described in Section 2.5:

$$\begin{aligned} \max \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\,\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{0} \leq \mathbf{x} \leq \mathbf{d}, \end{aligned} \tag{3.22}$$

where $\mathbf{d}$ is subject to uncertainty. In practical applications, Problem (3.22) arises from:

$$\begin{aligned} \max \quad & \mathbf{c}' \min(\mathbf{x}, \mathbf{d}) \\ \text{s.t.} \quad & \mathbf{A}\,\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{3.23}$$

which we will use to build the data-driven model. An example for such a setting is seat allocation in airline revenue management, which is described in further detail in Section 6.3.1. The worst cases here are those bringing in the *smallest* revenues. The robust counterpart of Problem (3.23) can be written as:

$$\begin{aligned} \max \quad & \frac{1}{N_\alpha} \sum_{k=1}^{N_\alpha} \left(\sum_{i=1}^{n} c_i \min(x_i, d_i^{\cdot}) \right)_{(k)} \\ \text{s.t.} \quad & \mathbf{A}\,\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{3.24}$$

where $y_{(k)}$ denotes the k-th *smallest* component of a vector $\mathbf{y}$.

Theorem 3.3.3 (The robust problem) *The robust model is:*

$$\begin{aligned} \max \quad & \phi + \frac{1}{N_\alpha} \sum_{k=1}^{N} \psi_k \\ s.t. \quad & \phi + \psi_k - \sum_{i=1}^{n} c_i\, y_i^k \leq 0, \qquad \forall k, \\ & y_i^k - x_i \leq 0,\ y_i^k \leq d_i^k, \qquad \forall i,\ k, \\ & \mathbf{A}\,\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0},\ \psi \leq \mathbf{0}. \end{aligned} \tag{3.25}$$

Proof: Is a direct application of Theorem 3.2.1. □

In particular, it is a linear programming problem as its deterministic counterpart, and thus can be solved efficiently with standard optimization packages.

We now characterize the optimal solution. For that purpose we will need the dual of Problem (3.25), where we have indicated the corresponding primal variables on the right:

$$\begin{array}{lllll}
\min & \pi'\mathbf{b} + \sum_{k=1}^{N}\sum_{i=1}^{n} d_i^k\, \delta_i^k & & & \\
\text{s.t.} & c_i\, \rho_k - (\gamma_i^k + \delta_i^k) = 0, & \forall i,\ k, & : y_i^k & \\
& (\mathbf{A}'\pi)_i - \sum_{k=1}^{N} \gamma_i^k \geq 0, & \forall i, & : x_i & \\
& 0 \leq \rho_k \leq \dfrac{1}{N_\alpha}, & \forall k, & : \psi_k & \\
& \sum_{k=1}^{N} \rho_k = 1, & & : \phi & \\
& \rho, \gamma, \delta, \pi \geq \mathbf{0}. & & &
\end{array} \tag{3.26}$$

We assume that the set of the N_α worst cases S_α is "nondegenerate", in the sense that at optimality, the N_α-th worst-case scenario yields a revenue strictly lower than the $(N_\alpha+1)$-st one. This assumption is necessary to take full advantage of complementary slackness. Let π^* be the optimal π in Problem (3.26). Also, let $[d_i]_{S_\alpha}^{(k)}$ be the k-th *greatest* d_i among the N_α worst cases.

Theorem 3.3.4 (The robust solution)
(a) If $x_i^* > 0$ and $\left(\dfrac{(\mathbf{A}'\pi^*)_i}{c_i} \cdot N_\alpha\right)$ is not an integer, then:

$$x_i^* = [d_i]_{S_\alpha}^{(k)} \quad \text{where } k = \left\lceil \frac{(\mathbf{A}'\pi^*)_i}{c_i} \cdot N_\alpha \right\rceil. \tag{3.27}$$

(b) If $(\mathbf{A}'\pi^*)_i > c_i$, then $x_i^* = 0$.

Proof: By complementarity slackness, the following hold at optimality:

1. Scenario k will be among the N_α worst cases if and only if $\rho_k^* = 1/N_\alpha$, otherwise $\rho_k^* = 0$. Specifically, for any scenario that is not in S_α, $\phi + \psi_k - \sum_{i=1}^{n} c_i\, y_i^k < 0$ because of our assumption of "nondegeneracy" of S_α, so that $\rho_k^* = 0$. For any of the $(N_\alpha - 1)$ worst-case scenarios, $\psi_k^* > 0$, forcing $\rho_k^* = 1/N_\alpha$. Finally, we use that $\sum_{k=1}^{N} \rho_k^* = 1$ to show that $\rho_k^* = 1/N_\alpha$ for the N_α-th worst case as well.

2. For any scenario k that does not belong to S_α, $\gamma_i^{*k} = \delta_i^{*k} = 0$ for all i. This is a direct consequence of $\rho_k^* = 0$ in conjunction with $c_i\, \rho_k^* - (\gamma_i^{*k} + \delta_i^{*k}) = 0$ and $\delta, \gamma \geq 0$.

3. For any scenario k in S_α,

   (i) if $x_i^* < d_i^k$, we have $\delta_i^{*k} = 0$ and $\gamma_i^{*k} = \frac{c_i}{N_\alpha}$.

   (ii) if $x_i^* > d_i^k$, we have $\delta_i^{*k} = \frac{c_i}{N_\alpha}$ and $\gamma_i^{*k} = 0$.

   (iii) if $x_i^* = d_i^k$, we have $\gamma_i^{*k} + \delta_i^{*k} = \frac{c_i}{N_\alpha}$ with $\gamma_i^{*k}, \delta_i^{*k} \geq 0$.

4. If $x_i^* > 0$, then $(\mathbf{A}'\pi^*)_i = \sum_{k=1}^N \gamma_i^{*k}$. Using the expressions of γ_i^{*k} derived above, this yields:

   (i) If there is no k such that $x_i^* = d_i^k$ (instead, x_i^* would exhaust a resource, for instance it would be the last fare class admitted on a flight of given capacity), then:
$$(\mathbf{A}'\pi^*)_i = \frac{c_i}{N_\alpha} N_i^0, \tag{3.28}$$
   where N_i^0 the number of scenarios k in S_α such that $x_i^* < d_i^k$, which implies that $\left(\frac{(\mathbf{A}'\pi^*)_i}{c_i} \cdot N_\alpha\right)$ is an integer.

   (ii) If $\left(\frac{(\mathbf{A}'\pi^*)_i}{c_i} \cdot N_\alpha\right)$ is not an integer, then there exists a scenario k^* such that $x_i^* = d_i^{k^*}$ and:
$$(\mathbf{A}'\pi^*)_i = \frac{c_i}{N_\alpha} \cdot (N_i - \beta_i), \tag{3.29}$$
   with N_i the number of scenarios k in S_α such that $x_i \leq d_i^k$ and $\beta_i \in [0,1]$ such that $\delta_i^{*k^*} = (c_i/N_\alpha)\beta_i$. This leads to:
$$N_i = \left\lceil \frac{(\mathbf{A}'\pi^*)_i}{c_i} \cdot N_\alpha \right\rceil. \tag{3.30}$$
   Eq. (3.27) follows immediately. This proves (a).

5. If $(\mathbf{A}'\pi^*)_i > \sum_{k=1}^N \gamma_i^{*k}$, then $x_i^* = 0$. In that case, $N_i = N_\alpha$ and $\beta_i = 0$. But we have seen that $\sum_{k=1}^N \gamma_i^{*k} = \frac{c_i}{N_\alpha} \cdot (N_i - \beta_i)$. This yields (b).

□

Remark: The index k for variable i in Eq. (3.27) is proportional (neglecting the rounding-up effects) to $(\mathbf{A}'\pi^*)_i/c_i$. It increases if the unit profit c_i decreases or the opportunity cost $(\mathbf{A}'\pi^*)_i$ increases. Unsurprisingly, as $[d_i]_{S_\alpha}^{(k)}$ has been defined as the k-th greatest d_i among the N_α worst cases, x_i^* (when $x_i^* > 0$ and the nonintegrality condition is verified) decreases when $(\mathbf{A}'\pi^*)_i/c_i$ increases. Moreover, the probability that the demand for product i is not fully met is approximately proportional to $(\mathbf{A}'\pi^*)_i/c_i$, in the risk-neutral as well as the risk-averse cases. This suggests that the ratio $(\mathbf{A}'\pi^*)_i/c_i$ (rather than, for instance,

$(\mathbf{A}'\pi^*)_i - c_i)$ plays a key role in optimally allocating scarce resources under uncertainty. Since $N_\alpha \approx N \cdot (1-\alpha)$, the probability that demand for any product in stock is not fully met is also approximately proportional to $1-\alpha$, and in particular, this probability decreases as the trimming factor α increases.

Example: Limited shelf space at a retailer

A retailer can choose among n products to display on a shelf, where the total space available is b. Each item of type i uses a_i units of space and sells for a unit price c_i. Product i is subject to demand d_i. Let π^* be the optimal dual variable associated with the shelf space. We know from Theorem 3.3.4 that, if it is optimal to stock product i and $(a_i/c_i)\,\pi^*\,N_\alpha$ is not an integer, $x_i^* = [d_i]_{S_\alpha}^{(k)}$ with $k = \lceil (a_i/c_i)\,\pi^*\,N_\alpha \rceil$. As a result, the probability of shortage for such a product i is approximately proportional to $(1-\alpha)\,a_i/c_i$, which does not depend on π^*. Moreover, if we consider two such products i and j, at optimality the ratio of their shortage probabilities is approximately equal to $(a_i/c_i)/(a_j/c_j)$, independently of the trimming factor. As an example, if products 1 and 2 are of the same size, but product 1 sells for twice as much as product 2, at optimality product 1 is half as likely to be out of stock than product 2, for any α. The risk aversion of the decision-maker will however affect the choice of the products which are indeed stocked at the beginning of the period.

The numerical experiment below illustrates the impact of the decision-maker's risk aversion on the optimal allocation. The retailer has the choice between 5 products of equal size ($a_i = 1$ for all i), shelf space is 100 units, unit price is $c_i = 10 \cdot (6-i)$ for all i), and the demands for all products are i.i.d. with mean 30 units and standard deviation 10 units. We consider (symmetric) Bernoulli and Gaussian distributions. For each of 100 iterations,

1. we generate 21 historical scenarios (therefore the number N_α of scenarios remaining after trimming between 1 and 20),
2. use them to determine the optimal allocation at N_α given,
3. and test them on 100 new realizations of the demand.

Figure 3-1 shows how mean and standard deviation of the actual revenue vary with the number of discarded scenarios $N - N_\alpha$, when the optimal quantities of products in stock are chosen as above. Trimming the data sample by up to 30% has only a small impact on the mean, which decreases by at most 1.5%, but significantly affects the standard deviation, which decreases by up to 7%, resp. 25%, in the Gaussian, resp. Bernoulli (Binomial) case. Figure 3-2 shows the optimal allocation for the 5 products as a function of the number of discarded scenarios. The average demand for each product is 30, and we note that the risk-neutral retailer orders more of product 1 than average because of the potential for high profits. On the other hand, he orders less of products 4 and 5 than average. As his risk aversion increases, he orders less of product 1 and more of products 4 and 5, i.e., he trades off high profit opportunities for the likelihood that he will indeed sell the products on the shelf. Finally, Figure 3-3 shows the probability ratios P_i/P_1 for $i = 2, \ldots, 5$ for both distributions, and confirms that for $x_i^* > 0$, and assuming the nonintegrality condition is verified, P_i/P_1 is of the order of $i \cdot (1-\alpha)$, although this prediction seems more accurate when the distribution is Bernoulli than Gaussian.

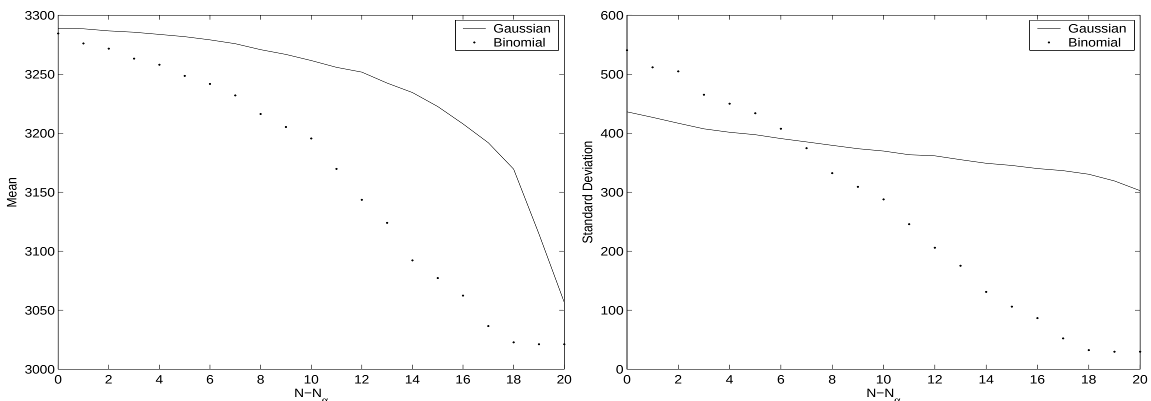

Figure 3-1: Mean and standard deviation of actual revenue.

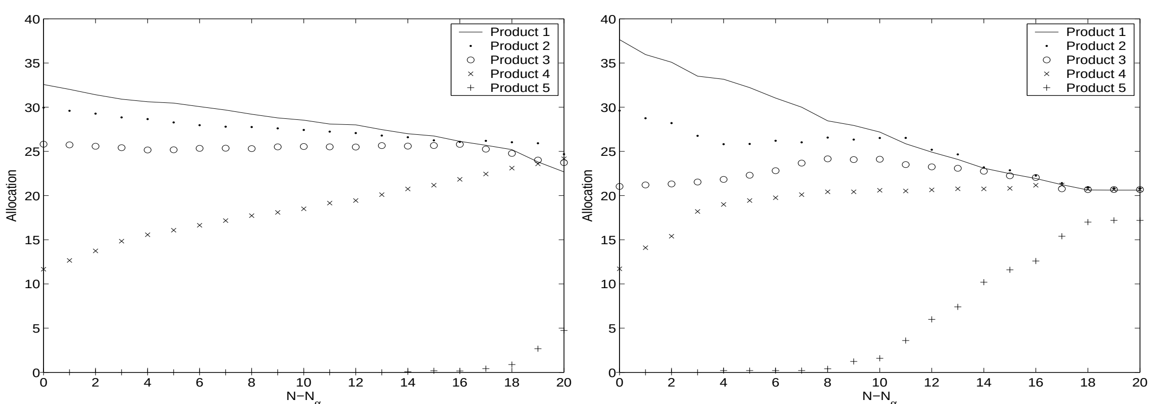

Figure 3-2: Optimal allocation with Gaussian (left) and Bernoulli (right) distributions.

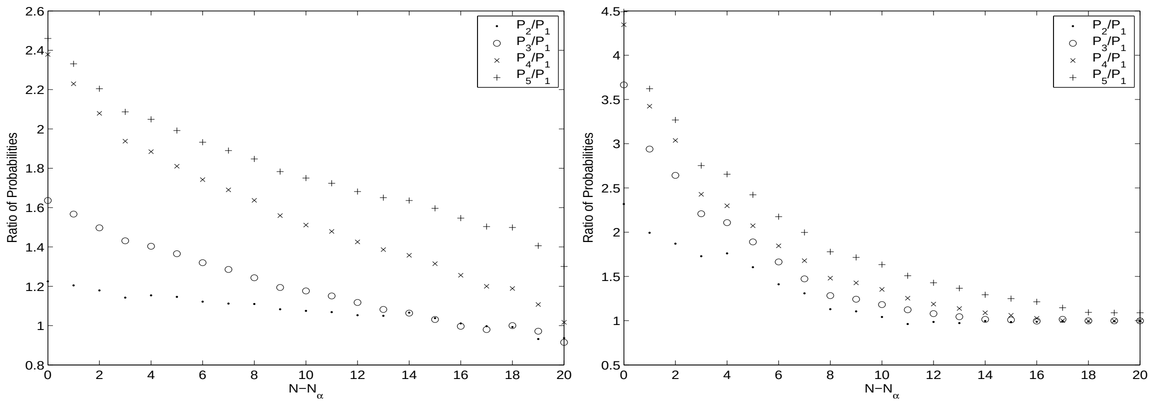

Figure 3-3: Probability ratios with Gaussian (left) and Bernoulli (right) distributions.

This example illustrates that the data-driven approach provides a tractable framework to incorporate randomness and risk aversion in revenue management problems. It also allows us to gain valuable insights into the properties of the optimal solution when the decision-maker is risk-averse, in terms of product allocations and probabilities of unmet demand. In particular, we have shown that, under some conditions, if it is optimal to have a positive inventory of product i, then (a) x_i^* is equal to a historical realization of the demand for i; and (b) the rank of this observation depends on the ratio between unit price and opportunity costs for the product, as well as the number of scenarios N_α that the decision-maker keeps after trimming.

3.3.3 Application to Linear Programming: Robust Bid Prices

A common way to analyze the deterministic problem (3.22) is to consider its dual. It is well known from duality theory (see Bertsimas and Tsitsiklis [19]) that:

$$\begin{aligned} \max \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\,\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{0} \leq \mathbf{x} \leq \mathbf{d} \end{aligned} \tag{3.31}$$

has the same optimal solution as:

$$\min_{\pi \geq \mathbf{0}} \quad \pi'\mathbf{b} + \quad \begin{aligned} \max \quad & (\mathbf{c} - \mathbf{A}'\pi)'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{0} \leq \mathbf{x} \leq \mathbf{d}. \end{aligned} \tag{3.32}$$

Therefore, in the deterministic setting it is optimal to take $x_i = d_i$ if $c_i \geq (A'\pi^*)_i$ and $x_i = 0$ otherwise, where π^* is the dual vector corresponding to the capacity constraints at optimality. The quantity $c_i - (A'\pi^*)_i$ is the "net profitability" of class i (revenue minus opportunity cost, per unit), also called "bid price" in applications such as airline revenue management, for instance by Talluri and van Ryzin in [63]. The example of airline revenue management is investigated further in Section 6.3.2. Problem (3.32) is equivalent to:

$$\begin{aligned} \min_{\pi \geq \mathbf{0}} \quad \pi'\mathbf{b} + \quad & \max \quad (\mathbf{c} - \mathbf{A}'\pi)'\mathbf{D}\,\mathbf{x} \\ & \text{s.t.} \quad \mathbf{0} \leq \mathbf{x} \leq \mathbf{e}, \end{aligned} \tag{3.33}$$

with $\mathbf{D} = diag(\mathbf{d})$.

In the data-driven framework, we have at our disposal N past observations of the uncertain vector $\mathbf{d}$. Let π^k be the optimal dual vector associated with the capacity constraints in scenario k. The revenue generated in scenario k by a feasible $\mathbf{x}$ is $(\pi^k)'\mathbf{b} + (\mathbf{c} - \mathbf{A}'\pi^{\mathbf{k}})'\mathbf{D}^{\mathbf{k}}\,\mathbf{x}$. The robust counterpart of Problem (3.33) can be written as:

$$\begin{aligned} \max \quad & \frac{1}{N_\alpha} \sum_{k=1}^{N_\alpha} \left[(\pi^\cdot)'\mathbf{b} + (\mathbf{c} - \mathbf{A}'\pi^\cdot)'\mathbf{D}^\cdot\,\mathbf{x}\right]_{(k)} \\ \text{s.t.} \quad & \mathbf{0} \leq \mathbf{x} \leq \mathbf{e}. \end{aligned} \tag{3.34}$$

As explained in Section 3.2, Problem (3.34) is equivalent to:

$$\begin{aligned} \max \quad & \phi + \frac{1}{N_\alpha} \sum_{k=1}^{N_\alpha} \psi_k \\ \text{s.t.} \quad & \phi + \psi_k \leq (\pi^k)'\mathbf{b} + (\mathbf{c} - \mathbf{A}'\pi^{\mathbf{k}})'\mathbf{D}^{\mathbf{k}}\,\mathbf{x}, \quad \forall k, \\ & \mathbf{0} \leq \mathbf{x} \leq \mathbf{e}, \ \psi_k \leq 0, \qquad \forall k. \end{aligned} \tag{3.35}$$

Let $\mathcal{S}_\alpha$ be the set of N_α worst-case scenarios, which can be obtained from (3.35) by complementarity slackness, and for any vector $\mathbf{y}$, let $< \mathbf{y} >= \frac{1}{N_\alpha} \sum_{k \in \mathcal{S}_\alpha} y_k$ be the average of the components of $\mathbf{y}$ in S_α. We assume that $\mathcal{S}_\alpha$ is "nondegenerate", i.e., it is uniquely defined.

Theorem 3.3.5 (Robust bid prices) *The optimal solution of Problem (3.34) verifies:*

$$x_i^* = \begin{cases} 1 & \textit{if } c_i \geq \dfrac{< \mathbf{D}\,\mathbf{A}'\,\pi >_i}{< \mathbf{D} >_i}, \\ 0 & \textit{otherwise.} \end{cases} \tag{3.36}$$

Proof: Follows from using the actual scenarios in $\mathcal{S}_\alpha$ to evaluate (3.34), and studying the sign of the coefficient in front of x_j for all j. □

For variable i to be selected ($x_i^* = 1$), the unit profit has to be greater than a weighted average of the opportunity costs, the weights being the actual demands in each scenario.

This can be contrasted with the deterministic model, where variable i is selected when the unit profit is greater than the opportunity cost of the nominal problem.

Example: Supply chain contract

A supplier is considering contracts with $n = 5$ possible retailers. If he enters into a contract with retailer i, he will fill any order made by i. Each such order is processed by the supplier at a unit profit $c_i = 10 \cdot (6 - i)$, uses $a_i = 1$ units of his production capacity, and the total number of orders d_i is a random variable with mean 30 units and standard deviation 10 units. Orders across retailers are independent. We consider Gaussian and (symmetric) Bernoulli distributions. The total production capacity at the supplier is equal to $b = 100$. The supplier has to decide which retailers he should sign a contract with. He has at his disposal $N = 21$ historical realizations of the demands and has computed the opportunity cost of his production capacity for each scenario $k = 1, \ldots, N$. In this framework, an admission policy $\mathbf{x} \in \{0, 1\}^n$ yields the profit $\sum_{i=1}^{n}(c_i - \pi^k)\, d_i^k\, x_i + b\, \pi^k$ in scenario $k = 1, \ldots, N$. The supplier maximizes the trimmed mean over the N_α lowest profits. To evaluate the policies obtained by this approach, we proceed as follows, for each of 100 vectors of historical realizations (of size $21 \cdot 5 = 105$) drawn from either a Gaussian or a Bernoulli distribution:

1. we determine the optimal admission policies when the number of discarded scenarios $N - N_\alpha$ varies from 0 to $N - 1 = 20$ (Figure 3-4 represents the policies averaged over the historical scenarios),

2. then we evaluate the policy by computing the mean profit on a sample of 100 vectors of realizations, and comparing it to the mean profit realized by implementing the traditional approach, where the bid prices are derived from the nominal model (Figure 3-5). Here, the robust shadow price associated with the resource varies between 0 and 30 depending on the scenarios, and the nominal shadow price is 20. The profit in each approach is defined as:

   either the total revenue $\sum_{i=1}^{n} c_i\, d_i\, x_i$ (Figure 3-5, left),

   or the total revenue minus the loss incurred by paying a unit penalty of 10 whenever the supplier oversells his production capacity (Figure 3-5, right)

Figure 3-4 shows that, as the risk aversion of the supplier increases, he is more likely to accept contracts with the lowest-paying retailers 4 and 5, as he values the extra income more than risking the unit penalty (shadow price) π^k associated with overcommitting himself in scenario k. Figure 3-5 shows that the robust approach yields higher profits than the traditional method, even after factoring in the overselling penalty. Moreover, trimming the available data by up to 25% increases the ratio by up to 2%. In the case without penalty, the data-driven approach outperforms the nominal approach by at least 12%, for both distributions and any trimming factor. With penalty, the ratio is about 0.8%, resp. 2.2%, in the Gaussian, resp. Bernoulli, case.

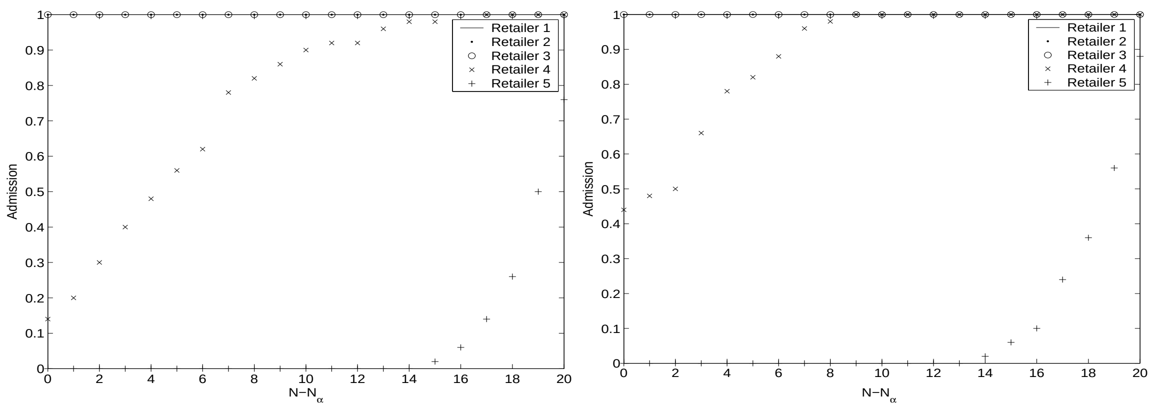

Figure 3-4: Admission policies.

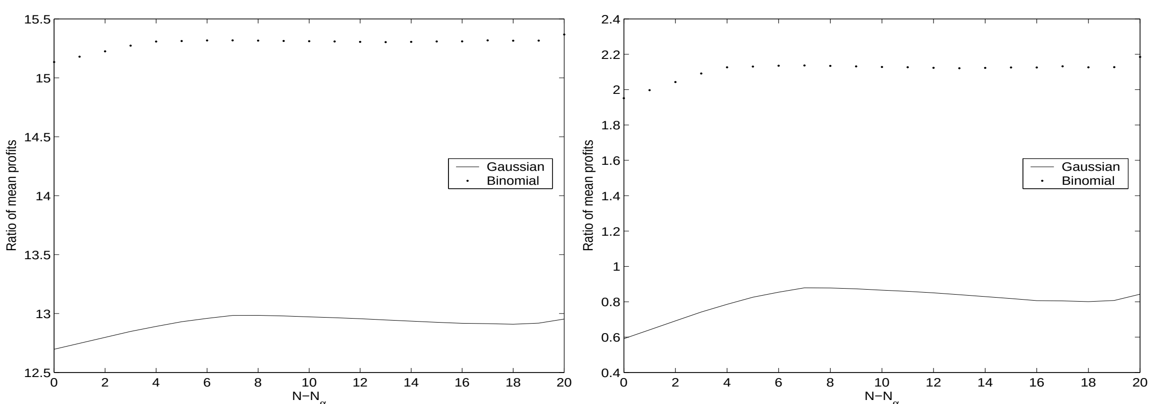

Figure 3-5: Ratio of mean profits without (left) and with (right) penalty.

This example suggests that incorporating the historical data in a robust approach to determine the optimal admission policies can have a significant impact on revenue, in particular when compared to the traditional bid price approach. It also illustrates the supplier's trade-off between the opportunity for high profit and the possibility of overcommitting his production capacity as his aversion to risk increases. An appealing feature of this framework is that its interpretation remains similar to the traditional method, where the unit profit of a class needs to exceed a certain threshold depending on the opportunity costs of the resources used, for the class to be admitted.

3.4 Robust Dynamic Optimization

In this section, we apply the methodology developed in Section 3.2 to dynamic systems.

3.4.1 Two-Stage Stochastic Programming

Stochastic programming, studied by Birge and Louveaux in [20] and Kall and Wallace in [40], is concerned with sequential decision-making. In its simplest form, it involves decisions at two time periods in an uncertain environment modelled by scenarios. The first-stage decisions must be made before the true scenario is known, but the second-stage decisions can (and should) take into account which scenario has been realized. Such a setting was originally presented by Dantzig in [31]. Let $\mathbf{x}$ be the first-stage decision variables, which must satisfy the constraints:

$$\mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}, \tag{3.37}$$

with $\mathbf{A}$ and $\mathbf{b}$ of appropriate dimensions. We have previously observed N realizations of the uncertainty. (It is also possible to generate random scenarios from simulations, as Shapiro and Homem-de-Mello explain in [60].) The true scenario is revealed after $\mathbf{x}$ is chosen and before the second-stage decisions are made. Let $\mathbf{y_k}$ be the vector of decisions made if scenario k ($k = 1, \ldots, N$) is realized. It must satisfy the constraints:

$$\mathbf{B_k}\,\mathbf{x} + \mathbf{D}\,\mathbf{y_k} = \mathbf{d_k},\ \mathbf{y_k} \geq \mathbf{0}, \tag{3.38}$$

with $\mathbf{B_k}$, $\mathbf{D}$, $\mathbf{d_k}$ of appropriate dimensions. We are also given cost vectors $\mathbf{c}$ and $\mathbf{f}$ associated with the first and second-stage decisions.

The traditional stochastic programming framework considers the problem:

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} + \frac{1}{N}\sum_{k=1}^{N} \mathbf{f}'\mathbf{y_k} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{B_k}\,\mathbf{x} + \mathbf{D}\,\mathbf{y_k} = \mathbf{d_k}, \quad \forall k, \\ & \mathbf{x} \geq \mathbf{0},\ \mathbf{y_k} \geq \mathbf{0}, \qquad \forall k. \end{aligned} \tag{3.39}$$

As mentioned in Section 3.2, this framework assumes that the risk of second-stage infeasibility induced by the uncertainty can be modelled by an appropriate penalty function. Here, we will use piecewise linear penalty functions to preserve the linear structure of the problem. Alternatively, we can follow Dantzig's assumption in [31] that for any feasible first-stage decision, there is at least one feasible second-stage decision available. The Conditional Value-at-Risk counterpart of Problem (3.39) is:

$$\begin{aligned} \min \quad & \mathbf{c'x} + \frac{1}{N_\alpha}\sum_{k=1}^{N_\alpha}(\mathbf{f'y}_\cdot)_{(k)} \\ \text{s.t.} \quad & \mathbf{Ax} = \mathbf{b}, \\ & \mathbf{B_k\,x} + \mathbf{D\,y_k} = \mathbf{d_k}, \qquad \forall k, \\ & \mathbf{x},\ \mathbf{y_k} \geq \mathbf{0}, \qquad \forall k. \end{aligned} \tag{3.40}$$

Theorem 3.4.1 (The robust problem)

(a) The robust formulation (3.40) is equivalent to:

$$\begin{aligned} \min \quad & \mathbf{c'x} + \phi + \frac{1}{N_\alpha}\sum_{k=1}^{N}\psi_k \\ s.t. \quad & \phi + \psi_k \geq \mathbf{f'y_k}, \qquad \forall k, \\ & \mathbf{Ax} = \mathbf{b}, \\ & \mathbf{B_k\,x} + \mathbf{D\,y_k} = \mathbf{d_k}, \qquad \forall k, \\ & \mathbf{x},\ \psi,\ \mathbf{y_k} \geq \mathbf{0}, \qquad \forall k. \end{aligned} \tag{3.41}$$

(b) Let S_K be the optimal set of worst-case scenarios at time K (with K observations). Assume we know beforehand that we will observe the realizations of the random variables until we have N observations, and that we will keep N_α of these. At time K, (3.41) is equivalent to a smaller problem involving $\min(K, N_\alpha + 1)$ scenarios. Specifically:

(i) *if $K \leq N_\alpha$, we solve the stochastic problem averaging (without trimming) over all the realizations so far.*

(ii) *if $K > N_\alpha$, we solve (3.41) by considering only the scenarios in S_{K-1} and the new observation vector.*

Proof: Theorem 3.2.1 applied to Eq. (3.40) yields (a). (b) follows from Theorem 3.2.2. □

Theorem 3.4.1 establishes that risk aversion can be incorporated to two-stage stochastic programming without any difficulty, and without overly increasing the size of the problem.

3.4.2 The General Dynamic Model

We now extend the framework developed in Section 3.4.1 to the multi-period case, where N scenarios of length T are available. Our goal is to determine the optimal control at time 0. We define, for $t = 0, \ldots, T$ and $k = 1, \ldots, N$:

x_t^k : the state of the system at time t in scenario k,

u_t : the control implemented at time $t = 0, \ldots, T-1$,

w_t^k : the random perturbation at time t in scenario k.

The trajectories (x_t^k) for each scenario k start from the same initial state x_0. The state of the system in scenario k evolves according to the dynamics:

$$x_{t+1}^k = h_t(x_t^k, u_t, w_t^k), \quad \forall t, \ \forall k. \tag{3.42}$$

We assume that the cost incurred at time t in scenario k is separable into a control-related and a state-related component:

$$C_t(x_t^k, u_t, w_t^k) = f_t(u_t) + g_t(x_{t+1}^k), \tag{3.43}$$

where f_t and g_t are convex. Let $\mathcal{U}_t$ be the feasible (convex) set for the control u_t at time t.

The data-driven robust problem takes the form:

$$\begin{aligned}
\min \quad & \sum_{t=0}^{T-1} f_t(u_t) + \frac{1}{N_\alpha} \sum_{k=1}^{N_\alpha} \left[\sum_{t=0}^{T-1} g_t(x_{t+1}^{\cdot}) \right]_{(k)} & \\
\text{s.t.} \quad & x_{t+1}^k = h_t(x_t^k, u_t, w_t^k), & \forall t, \ \forall k, \\
& u_t \in \mathcal{U}_t, & \forall t.
\end{aligned} \tag{3.44}$$

Theorem 3.4.2 (The robust problem)

(a) The robust problem (3.44) is equivalent to:

$$\begin{aligned}
\min \quad & \sum_{t=0}^{T-1} f_t(u_t) + \phi + \frac{1}{N_\alpha} \sum_{k=1}^{N} \psi_k & \\
s.t. \quad & \phi + \psi_k \geq \sum_{t=0}^{T-1} g_t(x_{t+1}^k), & \forall k, \\
& x_{t+1}^k = h_t(x_t^k, u_t, w_t^k), & \forall t, \ \forall k, \\
& \psi_k \geq 0, & \forall k \\
& u_t \in \mathcal{U}_t, & \forall t.
\end{aligned} \tag{3.45}$$

Therefore, the robust formulation adds $N+1$ variables and $2 \cdot (N+1)$ constraints to the *original problem.*

(b) Problem (3.45) can also be written as:

$$\begin{aligned}
\min \quad & \sum_{t=0}^{T-1} f_t(u_t) + \phi + \frac{1}{N_\alpha}\sum_{k=1}^{N} \max\left(0, \sum_{t=0}^{T-1} g_t(x_{t+1}^k) - \phi\right) \\
s.t. \quad & x_{t+1}^k = h_t(x_t^k, u_t, w_t^k), && \forall t,\ \forall k, \\
& u_t \in \mathcal{U}_t, && \forall t.
\end{aligned} \tag{3.46}$$

In (3.46), there is only one new variable ϕ and no new constraint, and the state-related cost for each scenario is truncated so that, if it falls below the threshold ϕ, it is replaced by that value. The control-related cost and the feasible set for the control remain identical.

Proof: This is a direct application of Theorem 3.2.1. □

An important consequence of Theorem 3.4.2 is that the robust data-driven model is tightly connected to the original model, increases the problem considered by only one variable and no constraint, and has an intuitive interpretation.

3.4.3 The Piecewise Linear Case with Additive Perturbations

We now illustrate the data-driven methodology on the case of linear dynamics with additive perturbations:

$$x_{t+1}^k = x_t^k + u_t - w_t^k, \quad \forall k,\ \forall t, \tag{3.47}$$

a piecewise linear cost:

$$C_t(x_t^k, u_t, w_t^k) = c\, u_t + \max(h\, x_{t+1}^k, -p\, x_{t+1}^k), \quad \forall k,\ \forall t, \tag{3.48}$$

and nonnegativity constraints on the control: $u_t \geq 0$ for all t.

Theorem 3.4.3 (The robust problem) *The robust problem can be formulated as a linear programming problem:*

$$\begin{aligned}
\min \quad & c\sum_{t=0}^{T-1} u_t + \phi + \frac{1}{N_\alpha}\sum_{k=1}^{N}\psi_k \\
s.t. \quad & \phi + \psi_k - \sum_{t=0}^{T-1} z_{t+1}^k \geq 0, && \forall k, \\
& z_{t+1}^k - h\sum_{s=0}^{t} u_s \geq h\left(x_0 - \sum_{s=0}^{t} w_s^k\right), && \forall k,\ \forall t, \\
& z_{t+1}^k + p\sum_{s=0}^{t} u_s \geq -p\left(x_0 - \sum_{s=0}^{t} w_s^k\right), && \forall k,\ \forall t, \\
& \psi_k \geq 0,\ u_t \geq 0, && \forall k,\ \forall t.
\end{aligned} \tag{3.49}$$

Proof: Theorem 3.4.2 yields the following robust counterpart:

$$\begin{aligned}
\min \quad & c\sum_{t=0}^{T-1} u_t + \phi + \frac{1}{N_\alpha}\sum_{k=1}^{N}\psi_k \\
\text{s.t.} \quad & \phi + \psi_k \geq \sum_{t=0}^{T-1} \max\left(h\, x_{t+1}^k, -p\, x_{t+1}^k\right), \quad \forall k, \\
& x_{t+1}^k = x_0 + \sum_{s=0}^{t}(u_s - w_s^k), \qquad \forall k,\ \forall t, \\
& \psi_k \geq 0,\ u_t \geq 0, \qquad \forall k,\ \forall t.
\end{aligned} \tag{3.50}$$

We then use the closed-form expression of x_{t+1}^k and introduce auxiliary variables z_{t+1}^k with $z_{t+1}^k \geq h\, x_{t+1}^k$ and $z_{t+1}^k \geq -p\, x_{t+1}^k$ to obtain (3.49). □

We next provide some insight into the structure of the optimal solution at time 0, under the assumption that the set of worst cases is "nondegenerate", i.e., at optimality the cost of the $(N_\alpha + 1)$-st worst scenario is strictly lower than the cost of the (N_α)-th worst one. We assume $u_0 > 0$. Let $\tau = \arg\min\{t \geq 1 | u_t > 0\}$ be the next time period a control is implemented. Let also $\left[\sum_{s=0}^{\tau-1} w_s^{\cdot}\right]^{(k)}$ be the k-th greatest $\sum_{s=0}^{\tau-1} w_s^k$.

Theorem 3.4.4 (The robust solution) *We have:*
(a) If $\tau < \infty$ *and* $\dfrac{h\,N_\alpha}{h+p}$ *is not an integer,*

$$u_0 = -x_0 + \left[\sum_{s=0}^{\tau-1} w_s^{\cdot}\right]^{(k)} \quad \text{with } k = \left\lceil \frac{h}{h+p}\, N_\alpha \right\rceil. \tag{3.51}$$

(b) If $\tau = \infty$ *and* $\dfrac{(h+c)\,N_\alpha}{h+p}$ *is not an integer,*

$$u_0 = -x_0 + \left[\sum_{s=0}^{T-1} w_s^{\cdot}\right]^{(k)} \quad \text{with } k = \left\lceil \frac{h+c}{h+p}\, N_\alpha \right\rceil. \tag{3.52}$$

Proof: The dual of (3.49) is:

$$\begin{aligned}
\max \quad & \sum_{t=0}^{T-1}\sum_{k=1}^{N}(h\, H_{t+1}^k - p\, P_{t+1}^k)\left(x_0 - \sum_{s=0}^{t} w_s^k\right) \\
\text{s.t.} \quad & H_{t+1}^k + P_{t+1}^k - S_k = 0, \qquad \forall k, t \quad : z_{t+1}^k \\
& \sum_{k=1}^{N} S_k = 1, \qquad : \phi \\
& S_k \leq \frac{1}{N_\alpha}, \qquad \forall k \quad : \psi_k \\
& \sum_{t \geq s}\sum_{k=1}^{N}\left(p\, P_{t+1}^k - h\, H_{t+1}^k\right) \leq c, \qquad \forall s \quad : u_s \\
& \mathbf{S}, \mathbf{H}, \mathbf{P} \geq \mathbf{0}.
\end{aligned} \tag{3.53}$$

where we have indicated the corresponding primal variables on the right, or equivalently:

$$
\begin{aligned}
\max \quad & \sum_{t=0}^{T-1}\sum_{k=1}^{N}(h\, H^k_{t+1} - p\, P^k_{t+1})\left(x_0 - \sum_{s=0}^{t} w^k_s\right) & \\
\text{s.t.} \quad & \sum_{k=1}^{N}\left(H^k_{t+1} + P^k_{t+1}\right) = 1, & \\
& H^k_{t+1} + P^k_{t+1} \leq \frac{1}{N_\alpha}, & \forall k, \\
& \sum_{t\geq s}\sum_{k=1}^{N}\left(p\, P^k_{t+1} - h\, H^k_{t+1}\right) \leq c, & \forall s, \\
& \mathbf{H}, \mathbf{P} \geq \mathbf{0}. &
\end{aligned}
\tag{3.54}
$$

Following the same line of argument as in the proof of Theorem 3.3.4, we have at optimality by complementarity slackness:

1. if k is not among the N_α worst-case scenarios, $H^k_{t+1} = P^k_{t+1} = 0$ for all t, because of the assumption of "nondegeneracy" of S_α.

2. if k is among the N_α worst-case scenarios,

   (i) if $x^k_{t+1} < 0$, $H^k_{t+1} = 0$ and $P^k_{t+1} = \frac{1}{N_\alpha}$,

   (ii) if $x^k_{t+1} > 0$, $H^k_{t+1} = \frac{1}{N_\alpha}$ and $P^k_{t+1} = 0$,

   (iii) if $x^k_{t+1} = 0$, $H^k_{t+1} + P^k_{t+1} = \frac{1}{N_\alpha}$ for $H^k_{t+1}, P^k_{t+1} \geq 0$.

3. if $u_s > 0$ at time s,

$$\sum_{t\geq s}\sum_{k=1}^{N}\left(p\, P^k_{t+1} - h\, H^k_{t+1}\right) = c. \tag{3.55}$$

4. Let assume that $u_0 > 0$, $u_1 > 0$ and $\frac{h\, N_\alpha}{h+p}$ is not an integer, so that we are guaranteed to have at least one scenario k for which $x^k_1 = 0$. Let S^-_α, resp. S^0_α, be the set of scenarios in S_α for which $x^k_1 < 0$, resp. $x^k_1 = 0$. Eq. (3.55) becomes, using the expressions of H^k_1 and P^k_1 derived above:

$$|S^-_\alpha| + \sum_{k\in S^0_\alpha}(1 - h_k) = \frac{h}{p+h}\, N_\alpha, \tag{3.56}$$

   where the h_k are numbers in $[0, 1]$. This yields:

$$|S^-_\alpha| \leq \left\lfloor \frac{h}{h+p}\, N_\alpha \right\rfloor \text{ and } |S^-_\alpha \cup S^0_\alpha| \geq \left\lceil \frac{h}{h+p}\, N_\alpha \right\rceil, \tag{3.57}$$

In particular, the k-th greatest demand $w_0^{(k)}$ with $k = \left\lceil \frac{h}{h+p} N_\alpha \right\rceil$ corresponds to a scenario in S_α^0, i.e., $x_1^k = 0$. Equivalently,

$$u_0 = -x_0 + w_0^{(k)} \text{ with } k = \left\lceil \frac{h}{h+p} N_\alpha \right\rceil, \tag{3.58}$$

where the $w_0^{(k)}$ are ranked so that $w_0^{(1)} \geq \ldots \geq w_0^{(N)}$.

5. If $u_0 > 0$, $\frac{h\,N_\alpha}{h+p}$ is not an integer and no control is implemented before time τ ($\tau = \arg\min\{t \geq 1 | u_t > 0\}$) with $\tau < \infty$, the situation is equivalent to the case where $u_1 > 0$ if we aggregate the demand from time 0 to time $\tau - 1$. Therefore, we have:

$$u_0 = -x_0 + \left[\sum_{s=0}^{\tau-1} w_s\right]^{(k)} \text{ with } k = \left\lceil \frac{h}{h+p} N_\alpha \right\rceil, \tag{3.59}$$

6. If $u_0 > 0$, time 0 is the last time period where a control is implemented and $\frac{(h+c)\,N_\alpha}{h+p}$ is not an integer, a similar analysis yields:

$$u_0 = -x_0 + \left[\sum_{s=0}^{T-1} w_s\right]^{(k)} \text{ with } k = \left\lceil \frac{h+c}{h+p} N_\alpha \right\rceil, \tag{3.60}$$

□

An important consequence of this result is that the optimal policy in the data-driven, risk-averse framework is basestock[1], as it brings the total amount of stock on hand and on order up to a certain level. In particular, we have expressed the threshold in terms of the historical demand for a specific scenario.

3.5 Selecting the Trimming Factor

We now discuss how to select the trimming factor in a minimization problem.

3.5.1 Expected Value over Discarded and Remaining Data

Let $Y = f(\mathbf{A}\,\mathbf{x} + \mathbf{b})$. We focus on the mean $E[Y]$, the mean of the discarded tail distribution $E[Y|Y \leq q_\alpha(Y)]$, and the mean of the remaining tail distribution after trimming, $E[Y|Y \geq q_\alpha(Y)]$.

Theorem 3.5.1 (The trimming factor) *The trimming factor α quantifies how much of an outlier the discarded part of the distribution will be on average. Specifically, the mean*

[1]See Chapter 5 for a definition.

will be $\left(\frac{1}{\alpha}-1\right)$ *closer to the one-sided trimmed mean than to the mean of the discarded tail distribution:*

$$\frac{E[Y]-E[Y|Y\le q_\alpha(Y)]}{E[Y|Y\ge q_\alpha(Y)]-E[Y]}=\frac{1}{\alpha}-1. \tag{3.61}$$

Proof: Follows from $E[Y]=E[Y|Y\le q_\alpha(Y)]\cdot\alpha+E[Y|Y\ge q_\alpha(Y)]\cdot(1-\alpha)$. Equivalently,

$$\frac{E[Y|Y\ge q_\alpha(Y)]-E[Y]}{E[Y|Y\ge q_\alpha(Y)]-E[Y|Y\le q_\alpha(Y)]}=\alpha. \tag{3.62}$$

□

Example: If we trim 10%, resp. 20%, of the data, the mean will be nine, resp. four, times closer to the one-sided trimmed mean than to the mean of the discarded tail distribution. In Figure 3-6, we plot Bound (3.61) for $\alpha\in[0.1,0.9]$.

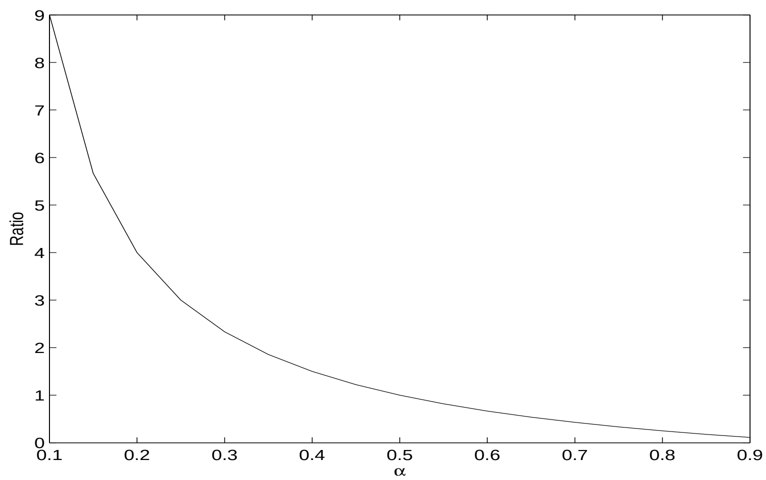

Figure 3-6: Ratio between distances of tail expectations from the mean.

3.5.2 Gaussian and Worst-Case Distributions for Shortfall

We have:

Theorem 3.5.2 (Bounds on CVaR)

(a) If Y *follows a Gaussian distribution with mean* $E[Y]$ *and variance* $Var(Y)$,

$$E[Y|Y\ge q_\alpha(Y)]=E[Y]+\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}\sqrt{Var(Y)}, \tag{3.63}$$

with ϕ *the density of the standard Gaussian distribution, and* Φ *its cumulative function.*

(b) For any distribution with mean $E[Y]$ *and variance* $Var(Y)$,

$$E[Y|Y \geq q_\alpha(Y)] \geq E[Y] + \sqrt{\frac{\alpha}{1-\alpha}}\sqrt{Var(Y)} \quad (3.64)$$

Proof: (a) We have that $E[Z|Z \leq q_\beta(Z)] = E[Z] - \frac{\phi(\Phi^{-1}(1-\beta))}{\beta}\sqrt{Var(Z)}$ for a Gaussian random variable Z and any trimming factor $\beta \in (0,1)$ (see Bertsimas et al. [12]). We apply this result to $Z = -Y$, $\beta = 1-\alpha$, and use that $q_{1-\alpha}(Y) = -q_\alpha(-Y)$. The proof for (b) is similar, using another result of Bertsimas et al. [12]. □

Theorem 3.5.2 gives some insights into the role of the distribution on the relationship between CVaR and standard deviation.

Figure 3-7 shows the impact of the trimming factor on the CVaR when the distribution of Y is Gaussian (left panel) or unknown with mean and variance given (right panel). In both cases, we take $E[Y] = 100$ and $Var(Y) = 400$. In practice, the trimming factor is

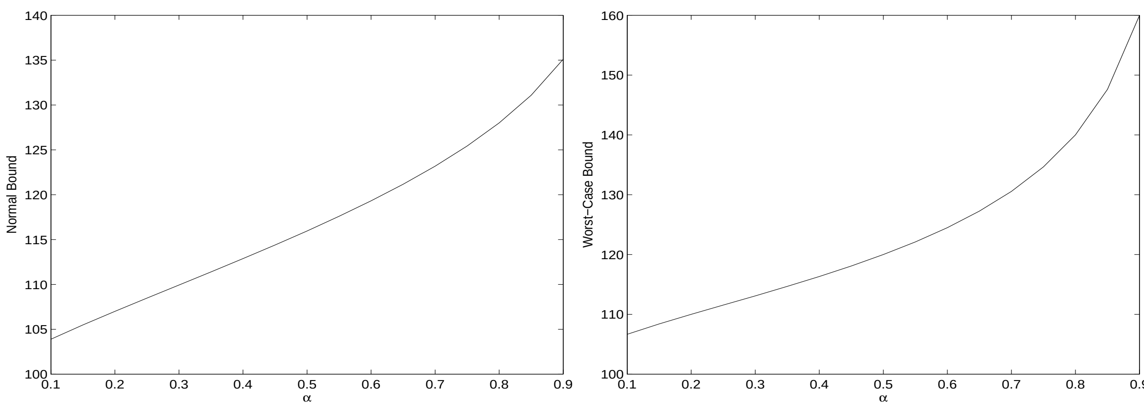

Figure 3-7: CVaR with Gaussian (left) and Worst-Case (right) Distributions.

often chosen between 0.1 and 0.2.

3.6 Concluding Remarks

We have presented here a robust approach, based on Conditional Value-at-Risk, that directly uses the historical data and does not require any estimation procedure. Its appealing features include its connection to risk aversion, its numerical tractability and its theoretical insights. It also addresses the limitations encountered by the approach described in Chapter 2. Specifically,

(a) risk aversion is captured by a single parameter, the trimming factor, rather than the whole utility, which makes the approach easy to implement in real-life applications,

(b) the robust counterparts are convex formulations with close ties to the stochastic model, and the methodology can be successfully applied to any convex problem,

(c) the data-driven framework allows for deeper insights into the optimal solution, notably for linear programming problems with uncertain right-hand side.

Therefore, CVaR holds a significant potential as a robust technique applied to management problems. It is further analyzed in later chapters of the thesis: in Chapter 4, we compare it to the robust approach with uncertainty sets, and in Chapter 6, we apply the framework to common problems in revenue management.

Chapter 4

Robustness with Uncertainty Sets and Data Samples: A Comparison

4.1 Background and Contributions

In Chapters 2 and 3, we have taken a robust optimization approach to model randomness and optimize stochastic systems in a tractable manner. This has involved two techniques: in Chapter 2, we have represented random variables as unknown parameters belonging to polyhedral uncertainty sets, and in Chapter 3, we have applied one-sided trimming to the historical data. While each framework has appealing features, these ideas build on very different models of uncertainty. The purpose of this chapter is to offer a unified perspective on robust optimization of stochastic systems, by:

- contrasting the main features of the approaches, and describing the problem structures where each is most appropriate (Section 4.2),

- studying in depth a case where both methods can be implemented successfully: linear programming, with an emphasis on how to choose the uncertainty sets and the parameters so that the frameworks become equivalent (Section 4.3).

Finally, Section 4.4 contains some concluding remarks.

4.2 Overview

4.2.1 Summary of Features

Table 4.1 summarizes the key features of each approach. The reader is referred to Chapters 2 and 3 for more details on these points.

	Approach with Uncertainty Sets **Chapter 2**	**Approach with Data Samples** **Chapter 3**
Framework	Worst-case approach over an uncertainty set of adjustable size Removes best and worst cases	Worst-case approach over the set of past realizations Removes the best cases only
Parameter	Budgets of uncertainty	Trimming factor
Required data	Nominal value of parameters Half-length of confidence interval	All past realizations (can be somewhat reduced)
Insights	Link with deterministic model	Link with stochastic model
Features	Probabilistic guarantee of constraint violation in some cases Models "averaging effect" between good and bad realizations	Link with risk aversion and robust optimization on proba- -bilities, incorporates learning No need for estimation
Correlated uncertainty	Need to adapt uncertainty sets Use covariance matrix	Well suited to correlated random variables - no change
Good for	Row-wise uncertainty Problems with linear combinations of random variables	Column-wise uncertainty Problems with sums of convex functions of one random variable
Example	Supply Chain Management Chapter 5	Revenue Management Chapter 6

Table 4.1: Summary of key features.

4.2.2 The Averaging Effect

The main difference between the approaches presented in Chapters 2 and 3 can be described as follows. In the data-driven framework, we remove a number of realizations that we consider too optimistic. The other good scenarios are counterbalanced by the bad ones when we average them to compute the Conditional Value-at-Risk over the remaining data. In the method with uncertainty sets, the good or bad cases are not evaluated at the global level (i.e., in terms of cost or revenue for the whole system), but rather at the local level (i.e., in terms of the random variables being greater or smaller than their mean, or the uncertain parameters being greater or smaller than their estimate). In that setting, the good and bad cases are not necessarily counterbalanced by each other. Such a phenomenon will indeed only occur when the problem structure allows for an "averaging effect". Loosely speaking, this involves linear combinations of (a large number of) symmetric uncertain parameters, although the assumption of symmetry can be somewhat relaxed. The fundamental insight here is that, for the approach with uncertainty sets to perform well[1] in practice, upside and downside risk of each random variable need to cancel each other out when the system is considered as a whole. This is the basis for diversification in portfolio management: keeping many different stocks in one's portfolio decreases the risk of major losses, as it is more likely that some stocks will appreciate and others will depreciate during the time period. Table 4.2 gives a few examples that are well suited for the approach with uncertainty sets. The term "linear in uncertainty" below refers to the problem structure and is not related to the actual choice of the uncertainty sets, which can be polyhedral or ellipsoidal. Table 4.3 lists problems that do not exhibit the averaging effect. The references on the right provide detailed descriptions of the exact problems considered.

Examples	**Why**	**References**
Inventory management	Linear dynamics Same	Bertsimas and Thiele [18] Thesis, Chapter 5
Knapsack problems	Linear in uncertainty	Bertsimas and Sim [15]
Network flows	Linear in uncertainty	Bertsimas and Sim [16]
Portfolio management	Linear in uncertainty Same Same	Goldfarb and Iyengar [38] Bertsimas and Sim [15] Pachamanova [51]

Table 4.2: Some problems with an averaging effect.

Examples	**Why**	**References**
LP with column-wise uncertainty	Need to protect each row	Soyster [62]
Newsvendor problem	Profit in $\mathbf{f}' \min(\mathbf{x}, \mathbf{d})$ Higher demand is lost	Thesis, Chapter 6
Optimal Stopping Time (Asset Selling)	Highest offer only is retained	Bertsekas [8]
Seat allocation in airline revenue management	Profit in $\mathbf{f}' \min(\mathbf{x}, \mathbf{d})$ Unused seats are lost	Thesis, Chapter 6

Table 4.3: Some problems without averaging effect.

[1]By "perform well", we mean that the approach should protect the system against a reasonable amount of uncertainty without being overly conservative.

4.3 Linear Programming Problems

4.3.1 Generalities

In this section, we apply the robust approaches to linear programming problems with row-wise uncertainty. It follows from the discussion in Section 4.2 that the method with uncertainty sets will perform well in this setting, and the data-driven framework can be applied to any convex problem with ease. This allows us to compare in greater depth the features of each approach. We are particularly interested in studying constraint protection from both perspectives, and investigating whether the uncertainty sets can be chosen so that the frameworks become equivalent.

We consider a linear programming problem of the type:

$$\begin{aligned} \max \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\,\mathbf{x} \leq \mathbf{b}, \end{aligned} \tag{4.1}$$

where we assume w.l.o.g. that only $\mathbf{A}$ is random, and focus on any row of that formulation:

$$\mathbf{a}'\mathbf{x} \leq b, \tag{4.2}$$

where we have dropped the row index for simplicity. We have at our disposal N past realizations of the random vector, $\mathbf{a^1}, \ldots, \mathbf{a^N}$ and use this data sample to either build uncertainty sets or to implement the data-driven framework.

Uncertainty set:

Following the approach developed in Chapter 2, we first model each a_j as an uncertain parameter in the interval $[\overline{a}_j - \widehat{a}_j, \overline{a}_j + \widehat{a}_j]$, and bound the total deviation $\sum_{j=1}^{n} |a_j - \overline{a}_j| / \widehat{a}_j$ by Γ. The choice of $\overline{a}_j$ and $\widehat{a}_j$ as a function of the observations is made more precise below. We assume that Γ is integer for our analysis. (4.2) becomes:

$$\begin{aligned} \overline{\mathbf{a}}'\mathbf{x} + \quad \max \quad & \sum_{j=1}^{n} \widehat{a}_j \, |x_j| \, z_j \qquad \leq b, \\ \text{s.t.} \quad & \sum_{j=1}^{n} z_j \leq \Gamma, \\ & 0 \leq z_j \leq 1, \ \forall j. \end{aligned} \tag{4.3}$$

Therefore, (4.3) is equivalent to:

$$\overline{\mathbf{a}}'\mathbf{x} + \sum_{j=1}^{\Gamma} \left(\widehat{a}_{\cdot}\,|x_{\cdot}|\right)_{(j)} \leq b, \tag{4.4}$$

where $y_{(k)}$ is the k-th greatest component of a vector $\mathbf{y}$. In particular, the level of constraint protection is:

$$\beta_u(\mathbf{x}) = \sum_{j=1}^{\Gamma} \left(\widehat{a}_{\cdot}\,|x_{\cdot}|\right)_{(j)}. \tag{4.5}$$

Data samples:

In the robust data-driven approach, we are given the trimming factor α and the number N_α of realizations left after trimming. (4.2) is rewritten as:

$$E[\mathbf{a}'\mathbf{x}|\mathbf{a}'\mathbf{x} \geq q_\alpha(\mathbf{a}'\mathbf{x})] \leq b, \tag{4.6}$$

which in turn is estimated by:

$$\frac{1}{N_\alpha} \sum_{k=1}^{N_\alpha} (\mathbf{a}_{\cdot}{}'\mathbf{x})_{(k)} \leq b. \tag{4.7}$$

With $\overline{\mathbf{a}} = (1/N) \cdot \sum_{k=1}^{N} \mathbf{a_k}$, (4.7) is equivalent to:

$$\overline{\mathbf{a}}'\mathbf{x} + \frac{1}{N_\alpha} \sum_{k=1}^{N_\alpha} \left((\mathbf{a}_{\cdot} - \overline{\mathbf{a}})'\mathbf{x}\right)_{(k)} \leq b, \tag{4.8}$$

yielding the level of constraint protection:

$$\beta_v(\mathbf{x}) = \frac{1}{N_\alpha} \sum_{k=1}^{N_\alpha} \left((\mathbf{a}_{\cdot} - \overline{\mathbf{a}})'\mathbf{x}\right)_{(k)}. \tag{4.9}$$

4.3.2 Models of Randomness

Theorem 4.3.1 (Preliminary results) *The levels of constraint protection $\beta_u(\mathbf{x})$ and $\beta_v(\mathbf{x})$ are nondecreasing in Γ and α, respectively. Furthermore, we have:*

$$[\beta_u(\mathbf{x})]_{\Gamma=0} = [\beta_v(\mathbf{x})]_{\alpha=0} = 0. \tag{4.10}$$

Proof: Follows immediately from (4.5) and (4.9). □

In the general case, let S_Γ be the set of Γ greatest $\widehat{a}_j\,|x_j|$ and let S_α be the set of N_α

worst-case scenarios. We define the half-length $\widehat{a}_j$ as:

$$\widehat{a}_j = \max_k |a_{kj} - \overline{a}_j|,\ \forall j. \tag{4.11}$$

Therefore, for any scenario k, there exists $z_{kj} \in [-1, 1]$ such that $a_{kj} = \overline{a}_j + \widehat{a}_j \cdot z_{kj}$. Let $< z_j >$ be the average of z_{kj} over the set of worst-case scenarios S_α. We want to compare $\beta_u(\mathbf{x}) = \sum_{j \in S_\Gamma} \widehat{a}_j\, |x_j|$ with $\beta_v(\mathbf{x}) = \sum_{j=1}^n \widehat{a}_j\, x_j < z_j >$.

First we consider the extreme cases $\Gamma = n$ and $\alpha = 1$ (i.e., $N_\alpha = 1$).

Theorem 4.3.2 (Extreme cases) *We have:*

$$[\beta_v(\mathbf{x})]_{\alpha=1} \leq [\beta_u(\mathbf{x})]_{\Gamma=n}\, . \tag{4.12}$$

(4.12) is an equality:

- *for a given* $\mathbf{x}$ *if and only if the vector* $\widetilde{\mathbf{a}}$ *where* $\widetilde{a}_j = \overline{a}_j + \widehat{a}_j\, sign(x_j)$ *for all* j *is among the scenarios,*

- *for any* $\mathbf{x}$ *if and only if the* 2^n *vectors* $\widetilde{\mathbf{a}}$ *such that, for all* j, $\widetilde{a}_j = \overline{a}_j + \widehat{a}_j$ *or* $\widetilde{a}_j = \overline{a}_j - \widehat{a}_j$ *are among the scenarios.*

In other words, there must be a scenario that realizes the worst cases for each *component. Therefore, the probability, when the scenarios are generated randomly, that (4.12) is an equality decreases exponentially fast with* n.

Proof: (4.5) becomes:

$$[\beta_u(\mathbf{x})]_{\Gamma=n} = \sum_{j=1}^n \widehat{a}_j\, |x_j|. \tag{4.13}$$

On the other hand, (4.9) is now:

$$[\beta_v(\mathbf{x})]_{\alpha=1} = \sum_{j=1}^n \widehat{a}_j\, x_j\, z_{(1)j}, \tag{4.14}$$

where $z_{(1)j}$ denotes the components z_{kj} of the worst-case scenario. Since $z_{kj} \in [-1, 1]$ for all k, j, we have $x_j\, z_{kj} \leq |x_j|$ for all j. This proves (4.12). We analyze the case where (4.12) is an equality by studying when $x_j\, z_{(1)j} = |x_j|$ for all j and for a given $\mathbf{x}$ or any $\mathbf{x}$. $\square$

Theorem 4.3.3 (Interdependence of the models) *We have:*

$$\forall \alpha \in [0,1],\ \exists\, \Gamma \in [0,n],\ s.t.\ [\beta_u(\mathbf{x})]_\Gamma \leq [\beta_v(\mathbf{x})]_\alpha < [\beta_u(\mathbf{x})]_{\Gamma+1}\,. \tag{4.15}$$

Therefore, the data-driven approach applied to linear constraints is equivalent to the approach with uncertainty sets for well-chosen parameters.

Proof: Follows from $[\beta_u(\mathbf{x})]_{\Gamma=0} = [\beta_v(\mathbf{x})]_{\alpha=0} = 0$,$[\beta_v(\mathbf{x})]_{\alpha=1} \leq [\beta_u(\mathbf{x})]_{\Gamma=n}$, and the monotonicity of $\beta_u(\mathbf{x})$ in Γ, resp. $\beta_v(\mathbf{x})$ in α. $\square$

Figure 4-1 shows the resulting constraints protections when $\mathbf{x} = \mathbf{e}$ and the a_j, $j = 1, \ldots, n$, are i.i.d. random variables in $\{-1, 1\}$ with equal probability, for $n = 10$ or 50 random variables and $N = 50$ or $5,000$ scenarios generated. (For clarity, the x-axis for Γ is not shown, but Γ varies from 0 to n.) As expected from Theorem 4.3.2, for $n = 10$ and $N = 5,000$, enough scenarios are generated for the two approaches to be equivalent. In the other cases however, the constraint can be much more protected when the method with uncertainty sets is used. For $n = 10$ and $N = 50$, all $\beta_v(\mathbf{x})$ are smaller than $\beta_u(\mathbf{x})$ computed with $\Gamma = 5$. For $n = 50$, all $\beta_v(\mathbf{x})$ are smaller than $\beta_u(\mathbf{x})$ computed with $\Gamma = 15$ for $N = 50$ and $\Gamma = 25$ for $N = 5,000$.

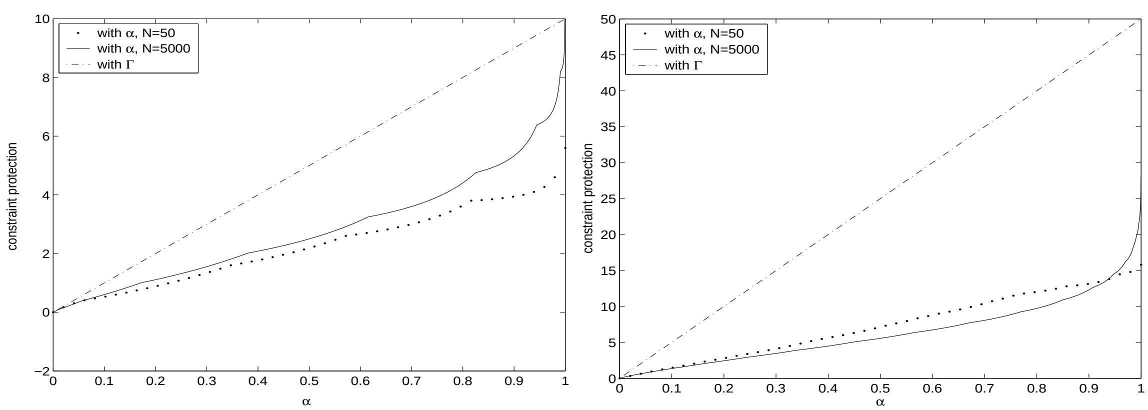

Figure 4-1: Constraint protections with $n = 10$ (left) and $n = 50$ (right).

4.3.3 Extensions

It is legitimate to ask if the results derived in Section 4.3.2 depend on the assumption of symmetry and independence of the random variables. We address this question next by considering two extensions: (a) asymmetric and (b) correlated sources of uncertainty.

Asymmetric random variables

In some cases, the historical data might strongly indicate that the random variables are asymmetric. As a result, we might want to model the a_j by uncertain parameters in asymmetric intervals $[\overline{a}_j - \widehat{a}_j^-, \overline{a}_j + \widehat{a}_j^+]$, with:

$$\widehat{a}_j^+ = \max_k \left(a_{jk} - \overline{a}_j\right), \ \widehat{a}_j^- = \max_k \left(\overline{a}_j - a_{jk}\right). \tag{4.16}$$

The approach with uncertainty sets will consider:

$$\begin{aligned} \overline{\mathbf{a}}'\mathbf{x} + \quad \max \quad & \sum_{j=1}^{n} \left(\widehat{a}_j^+ \, x_j \, z_j^+ - \widehat{a}_j^- \, x_j \, z_j^-\right) \quad \leq b, \\ \text{s.t.} \quad & \sum_{j=1}^{n} \left(z_j^+ + z_j^-\right) \leq \Gamma, \\ & z_j^+ + z_j^- \leq 1, \ \forall j, \\ & z_j^+, z_j^- \geq 0, \ \forall j. \end{aligned} \tag{4.17}$$

(4.17) is equivalent to:

$$\overline{\mathbf{a}}'\mathbf{x} \ + \underbrace{\sum_{j=1}^{\Gamma} \left(\max\left(\widehat{a}_j^+ \, x_j \, , -\widehat{a}_j^- \, x_j\right)\right)_{(j)}}_{=\beta_u(\mathbf{x})} \leq b. \tag{4.18}$$

We also have:

$$\beta_v(\mathbf{x}) = \frac{1}{N_\alpha} \sum_{k=1}^{N_\alpha} \left(\sum_{j=1}^{n} \left(\widehat{a}_j^+ \, x_j \, z_j^+ - \widehat{a}_j^- \, x_j \, z_j^-\right)\right)_{(k)} = \sum_{j=1}^{n} \left(\widehat{a}_j^+ \, x_j < z_j^+ > -\widehat{a}_j^- \, x_j < z_j^- >\right), \tag{4.19}$$

where $<z_j^->$ and $<z_j^+>$ are the average of z_{jk} over the worst-case scenarios.

Theorem 4.3.4 (Asymmetric random variables) *$\beta_u(\mathbf{x})$, resp. $\beta_v(\mathbf{x})$, is nondecreasing in Γ, resp. α, and we have:*

$$[\beta_u(\mathbf{x})]_{\Gamma=0} = [\beta_v(\mathbf{x})]_{\alpha=0} = 0, \ [\beta_u(\mathbf{x})]_{\Gamma=n} \geq [\beta_v(\mathbf{x})]_{\alpha=1.} \tag{4.20}$$

Consequently,

$$\forall \alpha \in [0,1], \ \exists \ \Gamma \in [0,n], \ s.t. \ [\beta_u(\mathbf{x})]_\Gamma \leq [\beta_v(\mathbf{x})]_\alpha < [\beta_u(\mathbf{x})]_{\Gamma+1}. \tag{4.21}$$

Therefore, the data-driven approach applied to linear constraints is equivalent to the approach with uncertainty sets.

Proof: Follows immediately from above. □

Correlated random variables

In practice, the random variables a_j might be correlated. The data-driven approach does not require this information and therefore can be implemented without any change. We now discuss how correlation affects uncertainty sets.

For simplicity, we assume that the random variables are symmetric. If we define $\widehat{a}_j = \max_k |a_{j\,k} - \overline{a}_j|$ for all j as before, the realizations of the random variables still fall within the box $[\overline{a}_1 - \widehat{a}_1, \overline{a}_1 + \widehat{a}_1] \times \ldots \times [\overline{a}_n - \widehat{a}_n, \overline{a}_n + \widehat{a}_n]$. However, the assumption that the total scaled deviation from the mean remains relatively small ($\sum_{j=1}^{n} |z_i| \leq \Gamma$) is not justified any more, as for instance $(1, \ldots, 1)$ and $(-1, \ldots, -1)$ might be the only possible values for $\mathbf{z}$. Hence, we need to consider another polyhedron to model correlated uncertainty. Let $\overline{\mathbf{a}}$ be the sample mean and $\mathbf{A}$ be the covariance matrix of the realizations $(\mathbf{a_1}, \ldots, \mathbf{a_N})$. We define $\widehat{a}_i = \max_k \left| \mathbf{A}^{-1/2}(\mathbf{a_k} - \overline{\mathbf{a}}) \right|_i$ and use the polyhedral set:

$$\mathcal{P} = \left\{ \mathbf{z},\; a_j = \overline{a}_j + \sum_i A_{ji}^{1/2}\, \widehat{a}_i\, z_i,\; |z_i| \leq 1, \forall i,\; \sum_{i=1}^{n} |z_i| \leq \Gamma \right\}. \tag{4.22}$$

Let $\mathbf{z_k} = diag(\mathbf{1}/\widehat{\mathbf{a}})\, \mathbf{A}^{-1/2}(\mathbf{a_k} - \overline{\mathbf{a}})$ for any scenario k. Straightforward calculations lead to, at $\mathbf{x}$ given:

$$\beta_u(\mathbf{x}) = \sum_{i=1}^{\Gamma} \left(\widehat{a}_{.} \left| \sum_{j=1}^{n} A_{j.}^{1/2}\, x_j \right| \right)_{(i)}, \tag{4.23}$$

where we note $y_{(1)} \geq \ldots y_{(n)}$, and:

$$\beta_v(\mathbf{x}) = \frac{1}{N_\alpha} \sum_{k=1}^{N_\alpha} \left((\mathbf{a} - \overline{\mathbf{a}})'\mathbf{x} \right)_{(k)} = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} A_{ji}^{1/2}\, x_j \right) \widehat{a}_i <z_i>, \tag{4.24}$$

where $< z_i >$ is the average of the z_{ik} over the set of worst-case scenarios. It follows immediately that:

Theorem 4.3.5 (Correlated random variables) *$\beta_u(\mathbf{x})$, resp. $\beta_v(\mathbf{x})$, is nondecreasing in Γ, resp. α, and $[\beta_u(\mathbf{x})]_{\Gamma=0} = [\beta_v(\mathbf{x})]_{\alpha=0} = 0$, $[\beta_u(\mathbf{x})]_{\Gamma=n} \geq [\beta_v(\mathbf{x})]_{\alpha=1}$. Therefore,*

$\forall\alpha \in [0,1],\ \exists\, \Gamma \in [0,n],\ s.t.\ [\beta_u(\mathbf{x})]_\Gamma \le [\beta_v(\mathbf{x})]_\alpha < [\beta_u(\mathbf{x})]_{\Gamma+1}$ *and the data-driven approach is equivalent to the approach with uncertainty sets.*

4.3.4 An Example

Here, we consider the example of a knapsack linear programming problem with random cost coefficients, which are independent and symmetric:

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} c_j\, x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} w_j\, x_j \ge W, \\ & 0 \le x_j \le 1,\ \forall j, \end{aligned} \tag{4.25}$$

where $w_j = 20 + j$ for all j. We study two distributions to generate the historical data for the c_j: Gaussian with mean $\overline{c}_j = 10 + 2\,j$, standard deviation $0.067\,\overline{c}_j$, and Bernoulli such that $c_j = 0.8\,\overline{c}_j$ w.p. $1/2$, and $c_j = 1.2\,\overline{c}_j$ w.p. $1/2$. This corresponds to $\widehat{c}_j = 0.2\,\overline{c}_j$ for all j. The standard deviation of the Gaussian distribution is chosen so that the realizations for both distributions fall approximately within $[\overline{c}_j - \widehat{c}_j, \overline{c}_j + \widehat{c}_j]$ for all j. We generate 100 scenarios to serve as historical data. Then we implement the robust approaches, and test the solutions they give us on a sample of $1,000$ new realizations of the same distribution. Let C_{rob} be the optimal cost given by the robust approach considered. We evaluate performance by computing $P\left(\sum_{j=1}^{n} c_j\, x_j > C_{rob}\right)$ and $E[\max(0, \sum_{j=1}^{n} c_j\, x_j - C_{rob})]$. They represent the probability that the actual cost will be greater than the threshold C_{rob}, and the expected value of the cost in excess. In the figures, the results for the approach with uncertainty sets, resp. the data-driven approach, are shown on the left, resp. on the right. To compare the trends more easily, the plots for the data-driven approach are function of $N - N_\alpha$ rather than α.

First set of experiments: $n = 10$

We take $W = 150$. Figure 4-2 shows the evolution of the robust cost, i.e., the optimal value of the robust formulations, as Γ varies from 0 to n and α varies from 0 to 1. We observe that the cost increases much more in the approach with uncertainty sets than in the one with trimming. This is related to the fact, noted above, that the approach with uncertainty sets allows for more constraint protection. In Figures 4-3 and 4-4, we observe that the probability and expected value of constraint violation decrease more sharply when we use uncertainty sets.

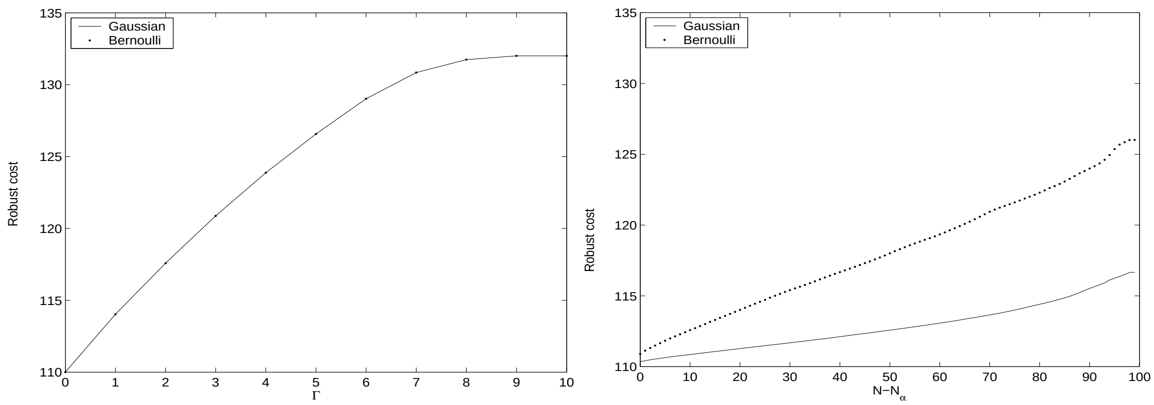

Figure 4-2: Robust cost, Case 1.

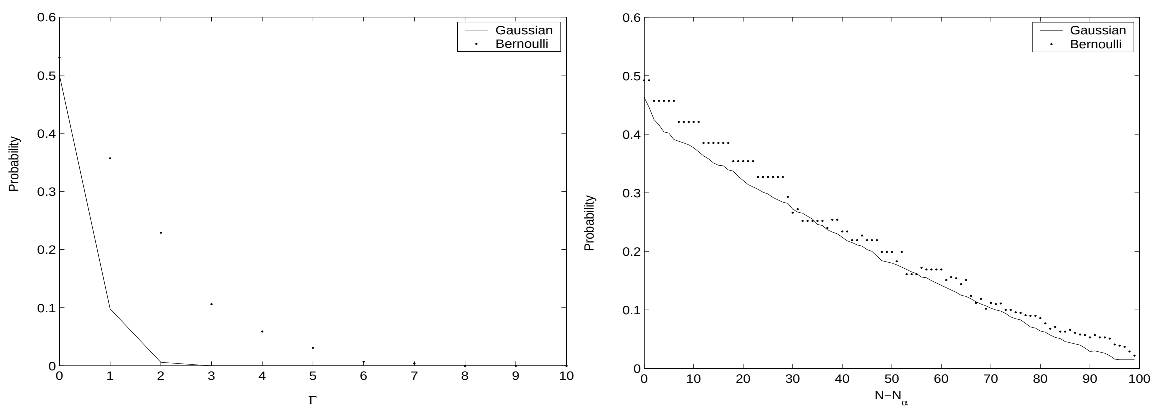

Figure 4-3: Probability of constraint violation, Case 1.

Second set of experiments: $n = 50$

We take $W = 1,500$. When $n = 50$, the approach with uncertainty sets has the potential to be much more conservative (Figure 4-5, left), but the probability of constraint violation (Figure 4-6, left) decreases very fast (for instance, it is less than 0.005 for both distributions when $\Gamma = 15$). This is also true of the expected value of the constraint violation (Figure 4-7, left), which is less than 0.1 for both distributions at $\Gamma = 15$, although it was of the order of 10 to 20 at $\Gamma = 0$. On the other hand, the results for the data-driven approach are somewhat disappointing, as the probability (Figure 4-6, right) and expected value (Figure

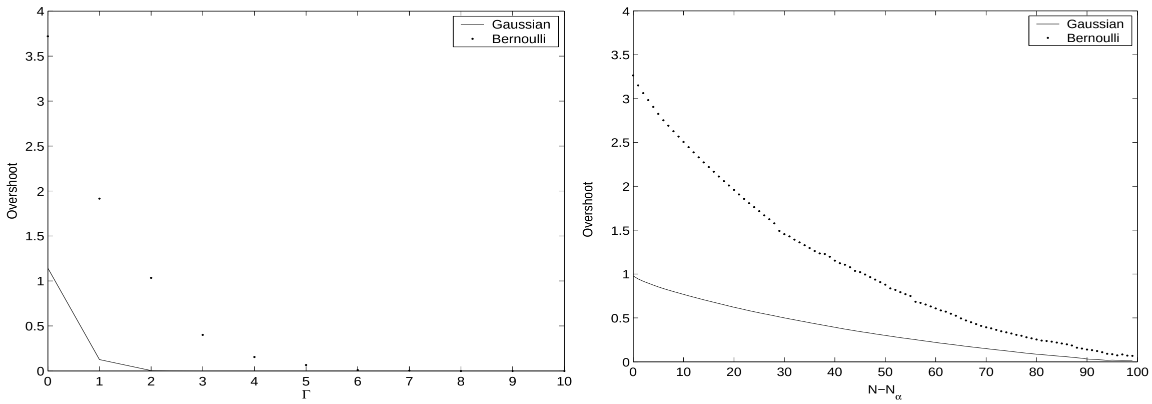

Figure 4-4: Expected value of constraint violation, Case 1.

4-7, right) of the constraint violation decrease very slowly. This can be explained as follows: the 100 historical realizations represent only a tiny fraction of the possible values taken by the uncertainty. For instance, if the random variables are binomial, we can have at most 100 different scenarios, while the total number of possible cases is $2^50 \approx 10^{15}$. This exacerbates the trends mentioned in the first set of experiments, as the data-driven approach does not have a range of possible scenarios large enough to adequately protect the system.

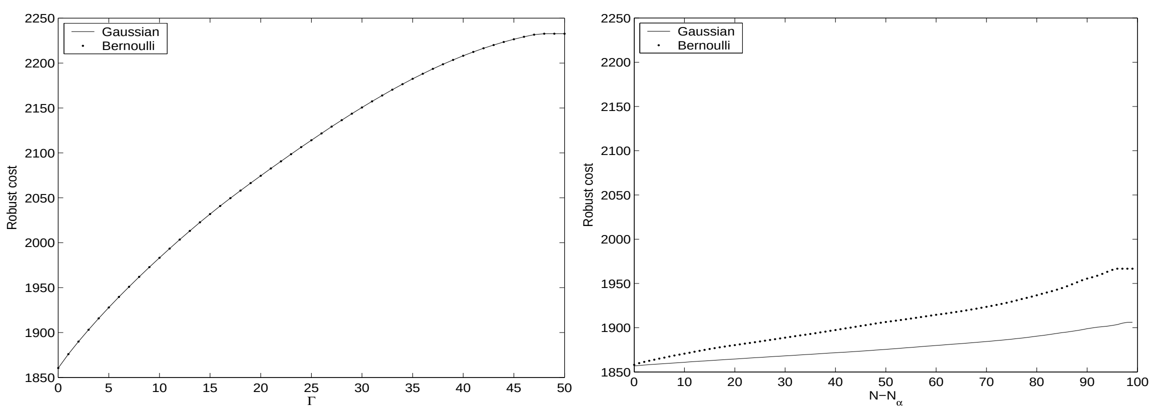

Figure 4-5: Robust cost, Case 2.

Conclusions:

This numerical experiment suggests that the approach with uncertainty sets performs better than the data-driven approach for problems with row-wise uncertainty.

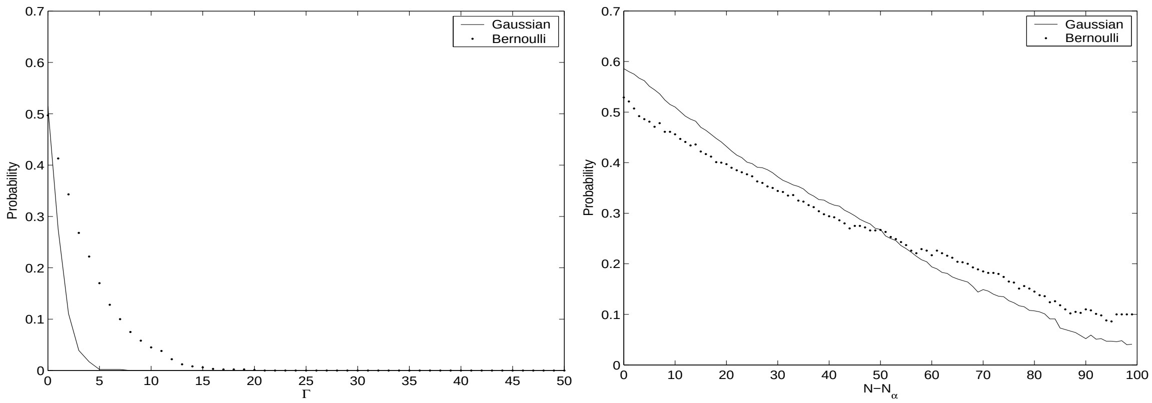

Figure 4-6: Probability of constraint violation, Case 2.

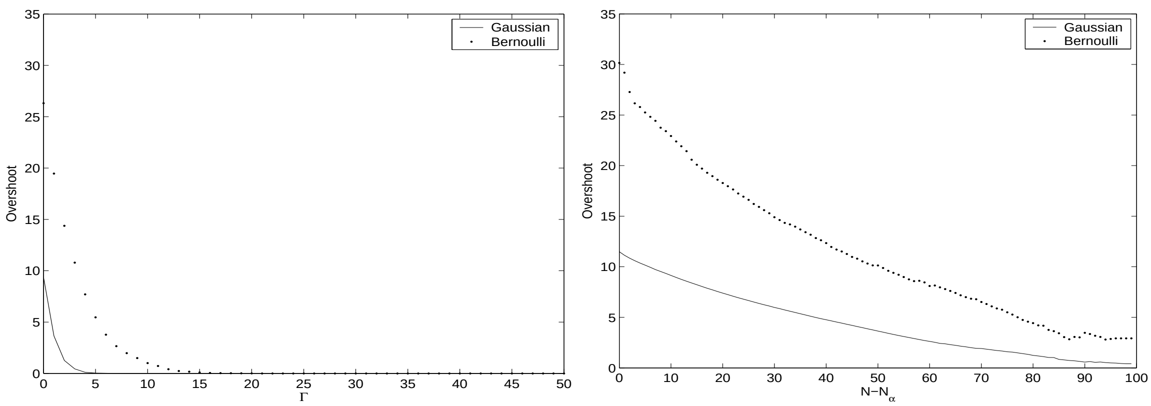

Figure 4-7: Expected value of constraint violation, Case 2.

4.4 Concluding Remarks

In this chapter, we have unified the techniques developed in Chapters 2 and 3 in a single robust optimization framework, by comparing the key features of the approaches and discussing when each is most appropriate. We have also analyzed the methods in detail in the case of linear programming problems with row-wise uncertainty. It emerges from this work that the approach with uncertainty sets should be preferred whenever possible, i.e., whenever the uncertainty structure allows for an averaging effect.

Chapter 5

Application to Supply Chains

5.1 Background and Contributions

Optimal supply chain management has been extensively studied in the past with much theoretical success. Dynamic programming has long emerged as the standard tool for this purpose, and has lead to significant breakthroughs as early as 1960, when Clark and Scarf proved the optimality of basestock policies for series systems in their landmark paper [28]. It has generated a considerable amount of research interest, with contributions by Iglehart [39], Veinott [66] and Veinott and Wagner [67], to name only a few. Although dynamic programming is a powerful technique as to the theoretical characterization of the optimal ordering policy for simple systems, the complexity of the underlying recursive equations over a growing number of state variables makes it ill-suited for the computation of the actual policy parameters, which is crucial for real-life applications. As a result, preference for implementation purposes is given to more intuitive policies that are much easier to compute, but also suboptimal.

Bramel and Simchi-Levi in [21] and Zipkin in [74] describe policies widely used in practice, such as the Economic-Order-Quantity model, where the demand is constant over time, and the Dynamic-Economic-Lotsize model, which incorporates time-varying demands, for single installations. In both cases, the demand is considered to be without any uncertainty. Myopic policies, which minimize the cost solely at the current time period, are also often used as a substitute for the optimal policy obtained by dynamic programming. For supply chains more complex than series systems, "the curse of dimensionality" plagues even the theoretical use of dynamic programming to find the structure of the optimal policy, thus making it necessary to resort to approximations.

Dynamic programming also assumes full knowledge of the underlying distributions, which further limits its practical usefulness. The first attempt to address the issue of imperfect information in inventory control is due to Scarf [57], who studied the optimal policy for the most adverse distribution in a one-period one-stage inventory model where only the mean and the variance of the demand are known. Moon and Gallego later extended this approach to single-period newsboy problems [47] and to one-stage inventory models with a fixed reorder quantity and under periodic review [48]. Similar ideas were applied to finite-horizon inventory models by Gallego et. al. [36], under the assumption that demand is a discrete random variable taking values in a known countable set. However, their approach relies on dynamic programming and, as a result, suffers from similar practical limitations.

Hence, the need arises to develop a new optimization approach that incorporates the stochastic character of the demand in the supply chain without making any assumptions on its distribution, and combines computational tractability with the structural properties of the optimal policy. In the light of the results derived in the previous chapters, robust optimization appears as a promising technique to develop such an approach.

Specifically, the contributions of this chapter are:

1. We develop an approach that incorporates demand randomness in a deterministic manner, remains numerically tractable as the dimension of the problem increases and leads to high-quality solutions without assuming a specific demand distribution. In particular, preliminary computational results are quite promising.

2. The robust problem is of the same class as the nominal problem, that is, a linear programming problem if there are no fixed costs or a mixed integer programming problem if fixed costs are present, independently of the topology of the network. Moreover, the optimal robust policy is identical to the optimal nominal policy for a modified demand sequence.

3. The optimal robust policy is qualitatively similar to the optimal policy obtained by dynamic programming when known. In particular, it remains basestock when the optimal stochastic policy is basestock, as well as in some other cases where the optimal stochastic policy is not known.

4. We derive closed-form expressions of key parameters defining the optimal policy.

These expressions provide a deeper insight into the way uncertainty affects the optimal policy in supply chain problems.

The remainder of this chapter is structured as follows: Section 5.2 introduces the framework in the single station case, and Section 5.3 considers general networks. Extensions are discussed in Section 5.4. Section 5.5 presents computational results. The last section summarizes our findings.

5.2 Single Station

5.2.1 Problem Formulation

In this section we apply the robust optimization framework to the problem of ordering, at a single installation, a single type of item subject to stochastic demand over a finite discrete horizon of T periods, so as to minimize a given cost function. We follow closely Bertsekas [8] in our setting. We define, for $t = 0, \ldots, T$:

x_t : the stock available at the beginning of the t-th period,

u_t : the stock ordered at the beginning of the t-th period,

w_t : the demand during the t-th period.

The stock ordered at the beginning of the t-th period is delivered before the beginning of the $(t+1)$-st period, that is, all orders have a constant leadtime equal to 0. Excess demand is backlogged. Therefore, the evolution of the stock over time is described by the following linear equation:

$$x_{t+1} = x_t + u_t - w_t, \qquad t = 0, \ldots, T-1, \tag{5.1}$$

leading to the closed-form expression:

$$x_{t+1} = x_0 + \sum_{\tau=0}^{t} (u_\tau - w_\tau), \qquad t = 0, \ldots, T-1. \tag{5.2}$$

Neither the stock available nor the quantity ordered at each period are subject to upper bounds. Section 5.4.1 deals with the capacitated case.

The demands w_t are random variables. Because the dynamics of the system are linear, the discussion in Chapter 4 motivates using the robust optimization approach with uncertainty sets developed in Chapter 2. In other words, the fact that at time $t+1$ the uncertainty affects the state x_{t+1} through the sum of random variables $\sum_{\tau=0}^{t} w_\tau$ (rather than, say, w_t only) creates an averaging effect for the demands across the different time periods. As we have explained in Section 4.2, this makes a description of randomness based on uncertainty sets well suited for this type of problems. Therefore, we model w_t for each t as an uncertain parameter that takes values in $[\overline{w}_t - \widehat{w}_t, \overline{w}_t + \widehat{w}_t]$, and impose budgets of uncertainty at each time period t for the scaled deviations up to that time: $\sum_{\tau=0}^{t} |z_\tau| \leq \Gamma_t$. As in Section 2.3, the main assumption we make on the Γ_t is that they are increasing in t, i.e., we feel that uncertainty increases with the number of time periods considered. We also constrain the Γ_t to be increasing by at most 1 at each time period, i.e., the increase of the budgets of uncertainty should not exceed the number of new parameters added at each time period.

Finally, we specify the cost function. The cost incurred at period t consists of two parts: a purchasing cost $C(u_t)$ and a holding/shortage cost resulting from this order $R(x_t+u_t-w_t)$, which is computed at the end of the end of the period, after the shipment u_t has been received and the demand w_t has been realized. Here, we consider a purchasing cost of the form:

$$C(u) = \begin{cases} K + c \cdot u, & \text{if } u > 0, \\ 0, & \text{if } u = 0, \end{cases} \tag{5.3}$$

with $c > 0$ the unit variable cost and $K \geq 0$ the fixed cost. If $K > 0$, a fixed positive cost is incurred whenever an order is made. The holding/shortage cost represents the cost associated with having either excess inventory (positive stock) or unfilled demand (negative stock). We consider a convex, piecewise linear holding/shortage cost:

$$R(x) = \max(hx, -px), \tag{5.4}$$

where h and p are nonnegative. We assume $p > c$, so that ordering stock remains a possibility up to the last period. In mathematical terms, the inventory problem at demand given is:

$$\begin{aligned} \min \quad & \sum_{t=0}^{T-1} \Big(c\, u_t + K\, 1_{\{u_t > 0\}} + \max(h\, x_{t+1}, -p\, x_{t+1}) \Big) \\ \text{s.t.} \quad & x_{t+1} = x_t + u_t - w_t, && \forall t, \\ & u_t \geq 0, && \forall t. \end{aligned} \tag{5.5}$$

If $K = 0$, the cost function is convex. If $K > 0$, it is K-convex, where K-convexity is defined by Scarf in [58] and Bertsekas in [8] as:

Definition 5.2.1 (K-convexity) *A real-valued function f is K-convex with $K \geq 0$ if, for any y, any $z \geq y$ and $b > 0$, we have:*

$$K + f(z) \geq f(y) + \frac{z-y}{b}(f(y) - f(y-b)). \tag{5.6}$$

Chen and Simchi-Levi propose in [27] the equivalent definition, which might be more insightful to contrast K-convexity with convexity:

Definition 5.2.2 (Equivalent definition of K-convexity) *A real-valued function f is K-convex with $K \geq 0$ if, for any y, any $z \geq y$ and $\lambda \in [0,1]$, we have:*

$$f\left((1-\lambda)\, y + \lambda\, z\right) \leq (1-\lambda)\, f(y) + \lambda\, f(z) + \lambda\, K. \tag{5.7}$$

Using the piecewise linearity and convexity of the holding/shortage cost function, and modelling the fixed ordering cost with binary variables, (5.5) can be written as a mixed integer programming (MIP) problem:

$$\begin{array}{lll} \min & \displaystyle\sum_{t=0}^{T-1} (c\, u_t + K\, v_t + y_t) & \\ \text{s.t.} & y_t \geq h\left(x_0 + \displaystyle\sum_{\tau=0}^{t}(u_\tau - w_\tau)\right), & t = 0, \ldots, T-1, \\ & y_t \geq -p\left(x_0 + \displaystyle\sum_{\tau=0}^{t}(u_\tau - w_\tau)\right), & t = 0, \ldots, T-1, \\ & 0 \leq u_t \leq M v_t,\ v_t \in \{0,1\}, & t = 0, \ldots, T-1, \end{array} \tag{5.8}$$

where $w_\tau = \overline{w}_\tau + \widehat{w}_\tau \cdot z_\tau$ such that $\mathbf{z} \in \mathcal{P} = \{|z_\tau| \leq 1\ \forall \tau \geq 0,\ \sum_{\tau=0}^{t} |z_\tau| \leq \Gamma_t\ \forall t \geq 0\}$. The following theorem presents the robust counterpart of this formulation.

Theorem 5.2.1 (The robust problem) *The robust inventory problem is:*

$$\begin{array}{lll} \mathit{min} & \displaystyle\sum_{t=0}^{T-1} (c\, u_t + K\, v_t + y_t) & \\ \mathit{s.t.} & y_t \geq h\left(x_0 + \displaystyle\sum_{\tau=0}^{t}(u_\tau - \overline{w}_\tau) + q_t \Gamma_t + \sum_{\tau=0}^{t} r_{\tau t}\right), & \forall t, \\ & y_t \geq p\left(-x_0 - \displaystyle\sum_{\tau=0}^{t}(u_\tau - \overline{w}_\tau) + q_t \Gamma_t + \sum_{\tau=0}^{t} r_{\tau t}\right), & \forall t, \\ & q_t + r_{\tau t} \geq \widehat{w}_\tau, & \forall t,\ \forall \tau \leq t, \\ & q_t \geq 0,\ r_{\tau t} \geq 0, & \forall t,\ \forall \tau \leq t, \\ & 0 \leq u_t \leq M\, v_t,\ v_t \in \{0,1\}, & \forall t, \end{array} \tag{5.9}$$

where M is a large positive number. The robust problem is of the same class as its deterministic counterpart: a LP if there are no fixed costs ($K = 0$) and a MIP if fixed costs are present ($K > 0$).

Proof: Follows from Theorem 2.3.1. □

Interpretation: The variables q_t and $r_{\tau t}$ quantify the sensitivity of the cost to infinitesimal changes in the key parameters of the robust approach, namely the level of conservativeness and the bounds of the uncertain variables. At each time period t, $q_t\,\Gamma_t + \sum_{\tau=0}^{t} r_{\tau t}$ represents the worst-case deviation of the cumulative demand from its nominal value, subject to the budgets of uncertainty.

Therefore, the robust model can readily be solved numerically through standard optimization tools, which is of course very appealing. It is also desirable to have some theoretical understanding of the optimal policy, in particular with respect to the optimal nominal policy and, if known, the optimal stochastic policy. We address these questions next.

5.2.2 Theoretical Properties

First we define basestock policies, which play a critical role in the analysis of inventory systems.

Definition 5.2.3 ((S,S) and (s,S) policies) *The optimal policy of a discrete-horizon inventory problem is said to be (s, S), or basestock, if there exists a threshold sequence (s_t, S_t) such that, at each time period t, it is optimal to order $S_t - x_t$ if $x_t < s_t$ and 0 otherwise, with $s_t \leq S_t$. If there is no fixed ordering cost ($K = 0$), $s_t = S_t$.*

In order to analyze the optimal robust policy, we need the following lemma:

Lemma 5.2.2 (Optimal nominal and stochastic policy)
(a) The optimal policy in the stochastic case, where the cost to minimize is the expected value of the cost function over the random variables w_k, is (s, S). As a result, the optimal policy for the nominal problem is also (s, S).
(b) For the nominal problem without fixed cost, the optimal policy for the nominal case is (S,S) with the threshold at time t being $S_t = \overline{w}_t$.
(c) For the nominal problem with fixed cost, if we denote by t_j ($j = 1, \ldots, J$) the times where stock is ordered and s_j, S_j the corresponding thresholds at time t_j, we have:

$$S_j = \sum_{\tau=0}^{I_j} \overline{w}_{t_j+\tau}, \tag{5.10}$$

and

$$s_1 = x_0 - \sum_{\tau=0}^{t_1-1} \overline{w}_\tau, \qquad s_j = -\sum_{\tau=I_{j-1}+1}^{L_{j-1}-1} \overline{w}_{t_{j-1}+\tau},\ j \geq 2, \tag{5.11}$$

where $L_j = t_{j+1} - t_j$ *and* $I_j = \left\lfloor \dfrac{pL_j - c\,1_{\{j=J\}}}{h+p} \right\rfloor$.

Proof: (a) See [8] for the optimality of basestock policies in the stochastic case. The nominal problem is a special case where the random variables are equal to their nominal value with probability 1.

(b) For the nominal case without fixed cost, the policy $\mathbf{u}$ defined by:

$$u_t = \begin{cases} \overline{w}_t - x_t, & \text{if } x_t < \overline{w}_t, \\ 0, & \text{otherwise,} \end{cases} \tag{5.12}$$

is feasible and incurs the cost $COST = c\left(\sum_{t=0}^{T-1} \overline{w}_t - x_0\right) + h\sum_{t=0}^{I}\left(x_0 - \sum_{\tau=0}^{t} \overline{w}_\tau\right)$, where I is the largest integer t such that $x_0 - \sum_{\tau=0}^{t} \overline{w}_\tau \geq 0$. We assume $I < T-1$, otherwise the problem is trivial. We consider the dual of this linear programming problem:

$$\begin{aligned} \max \quad & \sum_{t=1}^{T}\left(x_0 - \sum_{\tau=0}^{t-1} \overline{w}_\tau\right)(h\,\alpha_t - p\,\beta_t) & \\ \text{s.t.} \quad & -h\sum_{\tau=t}^{T}\alpha_\tau + p\sum_{\tau=t}^{T}\beta_\tau \leq c, & \forall t, \\ & \alpha_t + \beta_t = 1, & \forall t, \\ & \alpha_t \geq 0,\ \beta_t \geq 0, & \forall t. \end{aligned} \tag{5.13}$$

The following solution is dual feasible with cost equal to $COST$, proving (b) by strong duality:

$$\alpha_t = \begin{cases} 1, & \text{if } x_0 - \sum_{\tau=0}^{t-1} \overline{w}_\tau \geq 0, \\ \dfrac{p - c\,1_{\{t=T\}}}{h+p}, & \text{otherwise} \end{cases}, \quad \beta_t = 1 - \alpha_t. \tag{5.14}$$

(c) In the case with fixed cost, we consider the optimal ordering times as given (that is, $\mathbf{v}^*$ is given). The problem becomes a linear programming problem. Let t_j, $j = 1, \ldots, J$, be the times when an amount of stock u_j is ordered. The cost function can be decomposed in $J+1$ pieces, the jth piece ($j=0,\ldots,J$) representing the cost incurred from time t_j up to t_{j+1} (non included), with the conventions $t_0=-1$ and $t_{J+1}=T$. The minimization problem is solved recursively backwards for $j=1,\ldots,J$, for the cumulative cost from step j onward. Let I_j be the greatest integer i in $[1,L_j]$ such that $x_{t_j+i}>0$, where $L_j=t_{j+1}-t_j$. (If $x_{t_j+i}\leq 0$ for all $i\in[1,L_j]$, we take $I_j=0$.) The cost function for the J-th piece can be rewritten as:

$$cu_J + h\sum_{t=0}^{I_J-1}\left(x_{t_J}+u_J-\sum_{\tau=0}^{t}\overline{w}_{t_J+\tau}\right)+p\sum_{t=I_J}^{L_J-1}\left(-x_{t_J}-u_J+\sum_{\tau=0}^{t}\overline{w}_{t_J+\tau}\right), \tag{5.15}$$

and is therefore linear in u_J with slope $(h+p)I_J+c-pL_J$, with u_J subject to the constraint: $\sum_{\tau=0}^{I_J-1}\overline{w}_{t_J+\tau}<x_{t_J}+u_J\leq\sum_{\tau=0}^{I_J}\overline{w}_{t_J+\tau}$ from the definition of I_J. This function is minimized for:

$$I_J^*=\left\lfloor\frac{pL_J-c}{h+p}\right\rfloor,\quad x_{t_J}+u_J^*=\sum_{\tau=0}^{I_J^*}\overline{w}_{t_J+\tau}. \tag{5.16}$$

Moreover, we have $x_{t_J+L_J}=x_{t_{J+1}}=x_{t_J}+u_J^*-\sum_{\tau=0}^{L_J-1}\overline{w}_{t_J+\tau}=-\sum_{\tau=I_J^*+1}^{L_J-1}\overline{w}_{t_J+\tau}$. At optimality, the cost function at the last time period is equal to $c\left(\sum_{\tau=0}^{I_J^*}\overline{w}_{t_J+\tau}-x_{t_J}\right)+h\sum_{t=0}^{I_J-1}\left(\sum_{\tau=t+1}^{I_J^*}\overline{w}_{t_J+\tau}\right)+p\sum_{t=I_J+1}^{L_J-1}\left(\sum_{\tau=I_J^*+1}^{t}\overline{w}_{t_J+\tau}\right)$.

After step $j+1$, we see that at optimality the cumulative cost at step $j+1$ affects the cumulative cost at step j only through $-c\,x_{t_{j+1}}$, which depends on I_j and the data of the problem. The cumulative cost function at step j is minimized for:

$$I_j^*=\left\lfloor\frac{pL_j}{h+p}\right\rfloor,\quad x_{t_j}+u_j^*=\sum_{\tau=0}^{I_j^*}\overline{w}_{t_j+\tau}. \tag{5.17}$$

Moreover, we have $x_{t_j+L_j}=x_{t_{j+1}}=x_{t_j}+u_j^*-\sum_{\tau=0}^{L_j-1}\overline{w}_{t_j+\tau}=-\sum_{\tau=I_j^*+1}^{L_j-1}\overline{w}_{t_j+\tau}$. Using the definition of s and S in a (s,S) policy, it follows immediately that:

$$s_j=-\sum_{\tau=I_{j-1}^*+1}^{L_{j-1}-1}\overline{w}_{t_{j-1}+\tau},\ \forall j\geq 2,\qquad S_j=\sum_{\tau=0}^{I_j^*}\overline{w}_{t_j+\tau},\ \forall j\geq 1, \tag{5.18}$$

with $I_j^*=\left\lfloor\dfrac{pL_j-c\,1_{\{j=J\}}}{h+p}\right\rfloor$. s_1 is obtained by the dynamics equation, using $s_1=x_{t_1}$. □

We next present the main result regarding the structure of the optimal robust policy.

Theorem 5.2.3 (Optimal robust policy)

(a) The optimal policy in the robust formulation (5.9)*, evaluated at time* 0 *for the rest of*

the horizon, is the optimal policy for the nominal problem with the modified demand:

$$w'_t = \overline{w}_t + \frac{p-h}{p+h}\left(A_t - A_{t-1}\right), \tag{5.19}$$

where $A_t = q^*_t \Gamma_t + \sum_{\tau=0}^{t} r^*_{\tau t}$ *is the deviation of the cumulative demand from its mean at time* t*,* $\mathbf{q}^*$ *and* $\mathbf{r}^*$ *being the optimal* $\mathbf{q}$ *and* $\mathbf{r}$ *variables in* (5.9)*. (By convention* $q_{-1} = r_{\cdot,-1} = 0$*.) In particular it is* (S,S) *if there is no fixed cost and* (s,S) *if there is a fixed cost.*

(b) If there is no fixed cost, the optimal robust policy is (S,S) with $S_t = w'_t$ *for all* t*.*

(c) If there is a fixed cost, the corresponding thresholds S_j*,* s_j*, where* $j = 1, \ldots, J$ *indexes the ordering times, are given by Equations (5.10) and (5.11) applied to the modified demand* w'_t*.*

(d) The optimal cost of the robust problem (5.9) is equal to the optimal cost for the nominal problem with the modified demand plus a term representing the extra cost incurred by the robust policy, $\frac{2ph}{p+h}\sum_{t=0}^{T-1} A_t$*.*

Proof: Let $\overline{x}_{t+1} = x_0 + \sum_{\tau=0}^{t}(u_\tau - \overline{w}_\tau)$ be the inventory that we would have at time t if there was no uncertainty on the demand. We use that:

$$\max\left(h(\overline{x}_{t+1} + A_t), p(-\overline{x}_{t+1} + A_t)\right) = \max\left(h\, x'_{t+1}, -p\, x'_{t+1}\right) + \frac{2ph}{p+h}\sum_{t=0}^{T-1} A_t, \tag{5.20}$$

with:

$$x'_{t+1} = \overline{x}_{t+1} - \frac{p-h}{p+h} A_t, \ \forall t. \tag{5.21}$$

$\mathbf{x}'$ can be interpreted as a modified stock variable with the following dynamics:

$$x'_{t+1} = x'_t + u_t - \underbrace{\left(\overline{w}_t + \frac{p-h}{p+h}(A_t - A_{t-1})\right)}_{=w'_t}, \tag{5.22}$$

with $x'_0 = x_0$. Note that, at $\mathbf{q}$ and $\mathbf{r}$ given, the modified demand w'_t is not subject to uncertainty. The reformulation of the robust model as a nominal inventory problem in the modified stock variable x'_k (plus the fixed cost $\frac{2ph}{p+h}\sum_{k=0}^{T-1} A_k$) follows immediately. This proves (a) and (d). We conclude that (b) and (c) hold by invoking Lemma 5.2.2. □

Remarks:

1. Since $\Gamma_{t-1} \leq \Gamma_t$ for all t, we have $A_{t-1} \leq A_t$ for all t. ($A_t = \max \sum_{\tau=0}^{t} \widehat{w}_\tau z_\tau$ s.t. $\sum_{\tau=0}^{t} z_\tau \leq \Gamma_t,\ 0 \leq z_\tau \leq 1$, so the feasible domain increases in t.) Therefore, w'_t will

be greater than $\overline{w}_t$ if $p > h$ (that is, if shortage costs are more expensive than holding costs, we increase the "safety stock"), smaller than $\overline{w}_t$ if $p < h$ (if holding costs are more expensive, we want to make sure that we will not be left with extra items), and equal to $\overline{w}_t$ if $p = h$.

2. Using a similar argument as above, and since $\Gamma_t \leq \Gamma_{t-1} + 1$ for all t, we have $A_t \leq A_{t-1} + \widehat{w}_t$ for all t. Therefore, at time t, w'_t belongs to $\left[\overline{w}_t, \overline{w}_t + \frac{p-h}{p+h}\widehat{w}_t\right]$ if $p \geq h$ and $\left[\overline{w}_t + \frac{p-h}{p+h}\widehat{w}_t, \overline{w}_t\right]$ if $p < h$. The extreme case is $p = h$, where $w'_t = \overline{w}_t$ for all t and Γ_t.

3. For the case without fixed cost, and for the case with fixed cost when the optimal ordering times are given, the robust approach leads to the thresholds in closed form. For instance, if the demand is i.i.d. ($\overline{w}_t = \overline{w}$, $\widehat{w}_t = \widehat{w}$ for all t), we have $A_t = \widehat{w}\,\Gamma_t$ and, if there is no fixed cost, $S_t = w'_t = \overline{w} + \frac{p-h}{p+h}\,\widehat{w}\ (\Gamma_t - \Gamma_{t-1})$ for all t.

Hence, the robust approach protects against the uncertainty of the demand while maintaining striking similarities with the nominal problem, remains computationally tractable and is easy to understand intuitively.

5.3 Series Systems and Trees

5.3.1 Problem Formulation

We now extend the results of Section 5.2 to the network case. We focus on tree networks, which are well suited to describe supply chains because of their hierarchical structure: the main storage hubs (the sources of the network) receive their supplies from outside manufacturing plants and send items throughout the network, each time bringing them closer to their final destination, until they reach the stores (the sinks of the network). Let S be the number of sink nodes. When there is only one sink node, the tree network is called a series system.

We define echelon k, for $k = 1, \ldots, N$ with N the total number of nodes in the network, to be the union of all the installations, including k itself, that can receive stock from installation k, and the links between them. This is the definition used by Clark and Scarf in [28] as well as Zipkin in [74] when they consider tree networks. In the special case of series systems, we number the installations such that for $k = 1, \ldots, N$, the items transit from installation $k+1$ to k, with installation N receiving its supply from the plant and installation 1 being the only sink node, as in [28]. In that case, the demand at installation $k+1$ at time t is the amount of stock ordered at installation k at the same time t.

We also define, for $k = 1, \dots, N$:

$I_k(t)$: the stock available at the beginning of period t at installation k,

$X_k(t)$: the stock available at the beginning of period t at echelon k,

$U_{i_k k}(t)$: the stock ordered at the beginning of period t at echelon k to its supplier i_k,

$W_s(t)$: the demand at sink node s during period t, $s = 1, \dots, S$.

Let $N(k)$ be the set of installations supplied by installation k and $O(k)$ the set of sink nodes in echelon k. We assume constant leadtimes equal to 0, backlog of excess demand, and linear dynamics for the stock at installation k over time ($k = 1, \dots, N$):

$$I_k(t+1) = I_k(t) + U_{i_k k}(t) - \sum_{j \in N(k)} U_{kj}(t), \qquad t = 0, \dots, T-1, \tag{5.23}$$

By convention, if k is a sink node s, $\sum_{j \in N(k)} U_{kj}(t) = W_s(t)$. This leads to the following dynamics for the stock at echelon k:

$$X_k(t+1) = X_k(t) + U_{i_k k}(t) - \sum_{s \in O(k)} W_s(t), \qquad t = 0, \dots, T-1. \tag{5.24}$$

Furthermore, the stock ordered by echelon k at time t is subject to the coupling constraint:

$$\sum_{i \in N(k)} U_{ki}(t) \leq \max(I_k(t), 0), \ \forall k, \ \forall t, \tag{5.25}$$

that is, the total order made to a supplier cannot exceed what the supplier has currently in stock, or, equivalently, the supplier can only send through the network items that it really has. Since the network was empty when it started operating at time $t_0 = -\infty$, it follows by induction on t that $I_k(t) \geq 0$ for all $k \geq 2$. Therefore the coupling constraint between echelons is linear and can be rewritten as:

$$\sum_{i \in N(k)} U_{ki}(t) \leq \overline{X}_k(t) - \sum_{i \in N(k)} \overline{X}_i(t), \ \ \forall k, \ \forall t. \tag{5.26}$$

Neither the echelon inventories nor the orders are capacitated, although the approach can be extended to incorporate this case (see Section 5.4).

Finally, we specify the cost function. We assume that each echelon k has the same cost structure as the single installation modelled in Section 5.2 with specific parameters (c_k, K_k, h_k, p_k). We also keep here the same notations and assumptions as in Section 5.2 regarding the uncertainty structure at each sink node. In particular, each sink node s has its own threshold sequence $\Gamma_s(t)$ evolving over time that represents the total budget of uncertainty allowed up to time t for sink s. We have $W_s(t) = \overline{W}_s(t) + \widehat{W}_s(t) \cdot Z_s(t)$ such that the $Z_s(t)$ belong to the set $\mathcal{P}_s = \{|Z_s(t)| \leq 1 \ \forall t,\ \sum_{\tau=0}^{t} Z_s(\tau) \leq \Gamma_s(t),\ \forall t\}$. We assume $0 \leq \Gamma_s(t) - \Gamma_s(t-1) \leq 1$ for all s and t, that is, the budgets of uncertainty are increasing in t at each sink node, but cannot increase by more than 1 at each time period.

Theorem 5.3.1 (The robust problem) *The robust problem is:*

$$
\begin{aligned}
\min \quad & \sum_{t=0}^{T-1} \sum_{k=1}^{N} \sum_{i \in N(k)} \{c_{ki} U_{ki}(t) + K_{ki} V_{ki}(t) + Y_i(t)\} \\
s.t. \quad & Y_i(t) \geq h_i \left\{ \overline{X}_i(t+1) + \sum_{s \in O(i)} \left(q_s(t)\Gamma_s(t) + \sum_{\tau=0}^{t} r_s(\tau, t) \right) \right\}, && \forall i,\ \forall t, \\
& Y_i(t) \geq p_i \left\{ -\overline{X}_i(t+1) + \sum_{s \in O(i)} \left(q_s(t)\Gamma_s(t) + \sum_{\tau=0}^{t} r_s(\tau, t) \right) \right\}, && \forall i,\ \forall t, \\
& \sum_{i \in N(k)} U_{ki}(t) \leq \overline{X}_k(t) - \sum_{i \in N(k)} \overline{X}_i(t), && \forall k,\ \forall t, \\
& q_s(t) + r_s(\tau, t) \geq \widehat{W}_s(\tau), && \forall s,\ \forall t,\ \forall \tau \leq t, \\
& q_s(t) \geq 0,\ r_s(\tau, t) \geq 0, && \forall s,\ \forall t,\ \forall \tau \leq t, \\
& 0 \leq U_{ki}(t) \leq M V_{ki}(t),\ V_{ki}(t) \in \{0, 1\}, && \forall k,\ \forall i \in N(k),\ \forall t,
\end{aligned}
\tag{5.27}
$$

with $\overline{X}_i(t+1) = X_i(0) + \sum_{\tau=0}^{t} \left\{ U_{ki}(\tau) - \sum_{s \in O(i)} \overline{W}_s(\tau) \right\}$ *for all* i, t, *where* k *supplies* i.

This is a LP if there are no fixed costs and a MIP if fixed costs are present.

Proof: Follows from Theorem 2.3.1. □

As in the single-station case, an attractive feature of this approach is that the robust model of a supply chain remains of the same class as the nominal model, that is, a linear programming problem if there are no fixed costs and a mixed integer programming problem if fixed costs are present. Therefore, the proposed method is numerically tractable for very general topologies.

5.3.2 Theoretical Properties

We now investigate the structural properties of the optimal solution in the robust framework. First, we need to study the optimal policy in the nominal case.

Lemma 5.3.2 (Optimal nominal policy)
(a) For the problem without uncertainty, the optimal policy for each echelon k is the optimal policy obtained for a single installation with time-varying capacity on the orders, subject to the demand $\sum_{s\in O(k)} \overline{W}_s(t)$ at time t.
(b) In the case without fixed costs, it is also the optimal policy obtained for a single installation with new, time-varying cost coefficients, without capacity, subject to the demand $\sum_{s\in O(k)} \overline{W}_s(t)$ at time t.
(c) In the case without fixed costs, the optimal policy at each echelon is basestock, for the new parameters of the system.

Proof: Let first analyze the series system case. The coupling constraint for echelon k at time t is then $U_k(t) \leq I_{k+1}(t)$. We analyze the optimal orders by setting the right-hand sides of the coupling constraints to their optimal values. The coupling constraint for echelon k at time t becomes $U_k(t) \leq C_k(t)$ for some given $C_k(t)$. Hence, the problem is now decoupled in the echelons and is equivalent to solving a capacitated single-station inventory problem with or without fixed cost at each echelon, subject to the nominal demand at the sink node and with the original cost parameters. This proves (a) for series systems.

In the general network case, since the coupling constraints (5.26) bound the total order made at an installation by its customers, it cannot be directly interpreted as a capacity on the orders made by each customer. We analyze the nominal problem by duplicating the coupling constraint and writing it as:

$$U_{ki}(t) \leq \overline{X}_k(t) - \sum_{i\in N(k)} \overline{X}_j(t) - \sum_{j\in N(k), j\neq i} U_{kj}(t),\ \forall k,\ \forall t \tag{5.28}$$

for each echelon i supplied by installation k, and setting the right-hand side of this new constraint to its optimal value, to obtain a time-varying capacity on the orders made by each echelon. (a) follows immediately.

For any network, the inventory problem in the case without fixed costs is a linear programming problem. We dualize the coupling constraints (5.26) through a Lagrangian multiplier approach. The feasible set of the relaxation is now separable in the echelons, and the cost function of the relaxation can be rewritten as the sum of separable single-installation problems, with new cost parameters that incorporate the Lagrangian multipliers. It follows from the theory of Lagrangian relaxation for linear programming problems that the cost of the relaxation of the problem is equal to the cost of the original problem. This proves (b).

(c) follows from applying Lemma 5.2.2 to (b). □

We now give the main theorem regarding the optimal policy in the robust approach:

Theorem 5.3.3 (Optimal robust policy)
(a) The optimal policy in the robust formulation (5.27) for echelon k is the optimal policy obtained for the supply chain subject to the modified, deterministic demand at sink node s (for $s \in O(k)$):

$$W'_{s,k}(t) = \overline{W}_s(t) + \frac{p_k - h_k}{p_k + h_k}\left(A_s(t) - A_s(t-1)\right), \tag{5.29}$$

where $A_s(t) = q_s^*(t)\Gamma_s(t) + \sum_{\tau=0}^{t} r_s^*(\tau, t)$, $\mathbf{q_s^*}$ and $\mathbf{r_s^*}$ being the optimal $\mathbf{q}$ and $\mathbf{r}$ variables associated with sink node s in (5.27).
(b) The optimal cost in the robust case for the tree network is equal to the optimal cost of the nominal problem for the modified demands, plus a term representing the extra cost incurred by the robust policy, $\sum_{k=1}^{N} \frac{2p_k h_k}{p_k + h_k} \sum_{t=0}^{T-1} \sum_{s \in O(k)} A_s(t)$.

Proof: We reformulate the problem as a nominal problem in the same way as in the proof of Theorem 5.2.3 and invoke Lemma 5.3.2. □

5.4 Extensions

5.4.1 Capacity

So far, we have assumed that there is no upper bound on the amount of stock that can be ordered, nor on the amount of stock that can be held in the facility. Here, we consider the more realistic case where such bounds exist. For simplicity, we present this extension on single stations, but the results apply to more complex networks as well. The other assumptions remain the same as in Section 5.2.

Capacitated orders

The extension of the model to capacitated orders of maximal size d is immediate, by adding the constraint $u_t \leq d, \ \forall t$ to (5.9).

Theorem 5.4.1 (Optimal robust policy) *The optimal robust policy is the optimal policy for the nominal problem with capacity d on the links and with the modified demand defined in (5.19).*

Proof: The reformulation of the robust model as a problem without uncertainty does not affect constraints on the orders u_t. Adding $u_t \leq d$ for all t to this new problem, we obtain a deterministic model with capacity d on the links. □

Capacitated inventory

We now consider the case where stock can only be stored up to an amount C. This adds the following constraint to (5.9):

$$x_0 + \sum_{\tau=0}^{t} (u_\tau - w_\tau) \leq C, \tag{5.30}$$

where $w_\tau = \overline{w}_\tau + \widehat{w}_\tau \cdot z_\tau$ such that $\mathbf{z} \in \{|z_\tau| \leq 1 \ \forall \tau, \ \sum_{\tau=0}^{t} |z_\tau| \leq \Gamma_t \ \forall t\}$. This constraint depends on the uncertain parameters w_τ. Applying the technique developed in Chapter 2, we rewrite the constraint in the robust framework as:

$$\overline{x}_{t+1} + q_t \Gamma_t + \sum_{\tau=0}^{t} r_{\tau t} \leq C, \ \forall t, \tag{5.31}$$

where q_t and $r_{\tau t}$ are defined in (5.9). Therefore, capacitated stock can be incorporated to the robust formulation without changing the class, and therefore the numerical tractability, of the problems considered.

We now analyze the optimal robust policy. We define the modified stock variables x'_t by $x'_{t+1} = x'_t + u_t - w'_t$ and $x'_0 = x_0$, with w'_t given by (5.19) for all t. The inventory capacity constraint (5.31) becomes:

$$x'_{t+1} \leq C - \frac{2p}{p+h} A_t, \ \forall t. \tag{5.32}$$

This deterministic problem in x'_t is *not* equivalent to a nominal problem with inventory capacity, since the right-hand side in the new capacity constraint depends on the time period t, and worse, decreases with t. However, it never threatens the feasibility of the problem, in the following sense:

Lemma 5.4.2 *For all t, if $x'_t \leq C_t$, then $x'_t - w'_t \leq C_{t+1}$, where $C_t = C - \dfrac{2p}{p+h} A_{t-1}$.*

Therefore, if x'_t is feasible, it is always possible to satisfy the inventory capacity at time $t+1$ by not ordering.

Proof: We need to show that $x'_t \leq C - \dfrac{2p}{p+h}A_{t-1}$ implies $x'_t \leq C - \dfrac{2p}{p+h}A_t + w'_t$ for all t. Since $w'_t = \overline{w}_t + \dfrac{p-h}{p+h}(A_t - A_{t-1})$, it suffices to prove that: $A_t - A_{t-1} \leq \overline{w}_t$. But we have seen that $A_t - A_{t-1} \leq \widehat{w}_t$ from $\Gamma_t - \Gamma_{t-1} \leq 1$, and $\widehat{w}_t \leq \overline{w}_t$ since the demand is always nonnegative. □

We then have the following theorem:

Theorem 5.4.3 (Optimal robust policy) *The optimal robust policy is the optimal policy for the nominal problem subject to the modified demand defined in (5.19), and with inventory capacity at time* 0 *equal to* C*, and inventory capacity at time* $t+1$*,* $t \geq 0$*, equal to* $C - \dfrac{2p}{p+h}A_t$.

Proof: Follows from incorporating (5.32) to (5.9). □

5.4.2 Lead Times, Cost Structure and Network Topology

Lead times

The robust methodology does not depend on the actual $\overline{X}_k(t)$, $t \geq 0$, but only uses that the uncertainty is additive. Therefore, it can readily be extended to arbitrary constant leadtimes L, using $\overline{X}_k(t+1) = X_k(0) + \sum_{\tau=0}^{t-L} U_{i_k\, k}(\tau) - \sum_{\tau=0}^{t}\sum_{s\in O(k)} \overline{W}_s(\tau)$ for all k, where i_k is the supplier of echelon k and $O(k)$ the sink nodes of that echelon. In particular, the robust problem remains a linear programming problem if there is no fixed cost and a mixed integer programming problem if fixed costs are present.

Cost structure

The cost structure can be extended in several different ways, without changing the class of problems considered.

Station versus echelon: Although we have considered cost at the echelon level in networks, we obviously can apply the methodology to costs computed at the station level, where the uncertainty will only affect the sink nodes of the network.

Other order- and state-related costs: The framework we have presented here can also be used as an approximation for more complex cost functions, where order- and state-related costs will be modelled as piecewise linear.

Time-varying cost parameters: The methodology can also be applied when the unit cost parameters vary in time.

Network topology

Finally, it is important to acknowledge that the numerical tractability of the proposed approach is not dependent on the network topology. The robust method will still apply if we consider a problem with, for instance, inward trees, e.g., several warehouses supplying by the same shop.

Example

As an example, we give below the robust formulation of a supply chain management problem for a station ($i = 0$) with n suppliers ($i = 1, \ldots, n$). Each supplier has a capacity constraint d_i on the order size $u_i(t)$ it can send to the customer, and sends goods with a lead time L_i. The suppliers face neither capacity nor lead times on their own orders $U_i(t)$ for all i and t, and costs are computed at the installation level. Capital letters denote costs and variables associated with the suppliers. Cost parameters vary in time.

$$
\begin{array}{lll}
\min & \displaystyle\sum_{t=0}^{T-1} \left\{ \sum_{i=1}^{n} [c_i(t)\, u_i(t) + k_i(t)\, v_i(t)] + y(t) + \sum_{i=1}^{n} [C_i(t)\, U_i(t) + K_i(t)\, V_i(t) + Y_i(t)] \right\} & \\
\text{s.t.} & \displaystyle y_0(t) \geq h_0(t) \left(x_0(0) + \sum_{i=1}^{n} \sum_{\tau=0}^{t-L_i} u_i(\tau) - \sum_{\tau=0}^{t} \overline{w}_\tau + q_t \Gamma_t + \sum_{\tau=0}^{t} r_{\tau t} \right), & \forall t, \\
& \displaystyle y_0(t) \geq p_0(t) \left(-x_0(0) - \sum_{i=1}^{n} \sum_{\tau=0}^{t-L_i} u_i(\tau) - \sum_{\tau=0}^{t} \overline{w}_\tau + q_t \Gamma_t + \sum_{\tau=0}^{t} r_{\tau t} \right), & \forall t, \\
& \displaystyle Y_i(t) \geq h_i(t) \left(X_i(0) + \sum_{\tau=0}^{t} [U_i(\tau) - u_i(\tau)] \right), & \forall i,\ \forall t, \\
& \displaystyle Y_i(t) \geq p_i(t) \left(-X_i(0) - \sum_{\tau=0}^{t} [U_i(\tau) - u_i(\tau)] \right), & \forall i,\ \forall t, \\
& \displaystyle \sum_{\tau=0}^{t} u_i(\tau) - \sum_{\tau=0}^{t-1} U_i(\tau) \leq X_i(0), & \forall i,\ \forall t, \\
& q_t + r_{\tau t} \geq \widehat{w}_\tau, & \forall t,\ \forall \tau \leq t, \\
& q_t \geq 0,\ r_{\tau t} \geq 0, & \forall t,\ \forall \tau \leq t, \\
& 0 \leq u_i(t) \leq d_i\, v_i(t),\ v_i(t) \in \{0,1\}, & \forall i,\ \forall t, \\
& 0 \leq U_i(t) \leq M\, V_i(t),\ V_i(t) \in \{0,1\}, & \forall i,\ \forall t,
\end{array}
\tag{5.33}
$$

where M is a large number and by convention $\sum_{\tau=0}^{t-L} u_\tau$ is 0 if $t < L$.

5.5 Computational Experiments

5.5.1 The Budgets of Uncertainty

We now present some computational results. We assume that the random variables are uncorrelated and that we know the mean $\overline{W}_s(t)$ and the variance $\sigma_s^2(t)$ of the demand at each sink and each time period. Therefore, the budgets of uncertainty are selected using the mean and variance of the random variables as described in Chapter 2, where we also incorporate the nonnegativity in a bound similar to (2.4.4). Specifically, the budgets are computed as in [18], using the following algorithm:

Algorithm 5.5.1 (Selection of the budgets of uncertainty) *If the demands are i.i.d., we solve:*

$$
\begin{aligned}
\min \quad & \sum_{k=1}^{N} \sum_{i \in N(k)} c_{ki} \frac{p_i - h_i}{p_i + h_i} \sum_{s \in O(i)} \widehat{W}_s \Gamma_s(T-1) \\
& + \sum_{t=0}^{T-1} \sum_{k=1}^{N} \sum_{i \in N(k)} \left\{ h_i \overline{X}_i(t+1) + (h_i + p_i)\, f\left(\overline{X}_i(t+1), M_i(t+1), S_i^2(t+1)\right)\right\} \\
s.t. \quad & \overline{X}_i(t+1) = \frac{p_i - h_i}{p_i + h_i} \sum_{s \in O(i)} \widehat{W}_s \Gamma_s(t), && \forall i,\ t, \\
& 0 \le \Gamma_s(t) - \Gamma_s(t-1) \le 1, && \forall s,\ t,
\end{aligned}
\tag{5.34}
$$

where f is defined by:

$$
f(x, \mu, \sigma^2) = \begin{cases} \frac{1}{2}\left[-x + \sqrt{\sigma^2 + x^2}\right], & \text{if } x \ge \dfrac{\sigma^2 - \mu^2}{2\mu}, \\ -x \dfrac{\mu^2}{\mu^2 + \sigma^2} + \mu \dfrac{\sigma^2}{\mu^2 + \sigma^2}, & \text{if } x < \dfrac{\sigma^2 - \mu^2}{2\mu}, \end{cases} \tag{5.35}
$$

and with $M_i(t+1) = \sum_{\tau=0}^{t} \sum_{s \in O(i)} \overline{W}_s(\tau)$ and $S_i(t+1) = \sqrt{\sum_{\tau=0}^{t} \sum_{s \in O(i)} \sigma_s^2(\tau)}$ for all i, t.

If the demands are not i.i.d., we replace $\widehat{W}_s$ by $\sum_{\tau=0}^{T-1} \widehat{W}_s(\tau)/T$ in the cost function of (5.34) and by $\sum_{\tau=0}^{t} \widehat{W}_s(\tau)/(t+1)$ in the definition of $X_i(t+1)$, for all t and $s \in O(i)$.

5.5.2 Example of a Single Station

We first apply the proposed methodology to the example of minimizing the cost at a single station. The horizon is $T = 20$ time periods, the station has zero initial inventory, with an ordering cost per unit $c = 1$, a holding cost $h = 4$ and a shortage cost $p = 6$, in appropriate measurement units. There is no fixed ordering cost. The stochastic demand is i.i.d. with mean $\overline{w} = 100$ and standard deviation $\sigma = 20$ (unless specified otherwise). In the robust framework, we take $\widehat{w} = 2 \cdot \sigma$, that is, the demand belongs to the interval $[\overline{w} - 2 \cdot \sigma, \overline{w} + 2 \cdot \sigma]$.

In the first set of experiments, the stochastic policy is computed using a binomial distribution. In the second set of experiments, the stochastic policy is computed using an approximation of the gaussian distribution on five points $(\overline{w} - 2\sigma, \overline{w} - \sigma, \overline{w}, \overline{w} + \sigma, \overline{w} + 2\sigma)$. In both cases, the actual distribution is Gamma, Lognormal or Gaussian, with the same mean $\overline{w}$ and standard deviation σ.

The key metric we consider is the relative performance of the robust policy compared to the stochastic policy obtained by dynamic programming, as measured by the ratio $R = 100 \cdot (E(DP) - E(ROB))/E(DP)$, in percent. The expectations are computed with respect to the actual probability distribution, on a sample of size $1,000$. In particular, when $R > 0$ the robust policy leads to lower costs on average than the stochastic policy. We are also interested in the sample probability distribution of the costs DP and ROB.

The numerical experiments aim to provide some insight into the relationship between the performance of the robust policy and:

- the actual and assumed distributions,
- the standard deviation of the demand,
- the cost parameters c, h, p.

Budgets of uncertainty:

The budgets of uncertainty are computed as in Algorithm 5.5.1. Figure 5-1 (resp. Figure 5-2) shows Γ_k as a function of k, for p varying and $h = 4$ (resp. h varying and $p = 4$), with $c = 0$ on the left panel and $c = 1$ on the right panel. The general trend is that Γ_k evolves as $\sqrt{k+1}$, although there are differences for the last few time periods. In particular:

- From Algorithm 5.5.1, the budgets of uncertainty for $c = 0$ verify for all k:

$$\Gamma_k = \frac{\sigma}{\widehat{w}} \sqrt{\frac{k+1}{1-\alpha^2}}, \tag{5.36}$$

  with $\alpha = \dfrac{p-h}{p+h}$. (The constraint $0 \leq \Gamma_k - \Gamma_{k-1} \leq 1$ is verified for all the numerical values of interest here.)

- The budgets of uncertainty for $c = 0$ and $c = 1$ are identical for the early time periods, suggesting that c is not a factor there, but differ in the last few periods.

- If p varies at h given, the budgets of uncertainty increase in p, for $c = 0$ as well as $c = 1$. If h varies at p given, the budgets of uncertainty increase in h for $c = 0$, and for $c = 1$ except for the last few time periods, where they decrease in h.

- If p varies at h given (here $h = 4$), having $c > 0$ rather than $c = 0$ results in less conservative budgets of uncertainty. In particular, if $c = 1$, the budgets stop increasing once a certain threshold is reached. This threshold decreases with p.

- If h varies at p given (here $p = 4$), having $c > 0$ rather than $c = 0$ results in much more conservative budgets of uncertainty in the last time periods, in particular as h is small (with $h > p$).

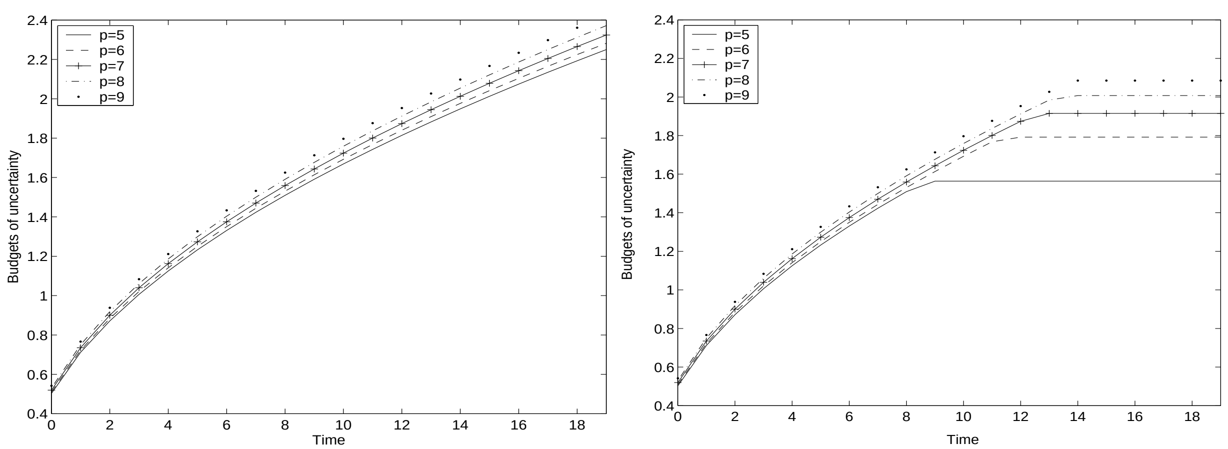

Figure 5-1: Budgets of uncertainty for $c = 0$ (left) and $c = 1$ (right).

In the rest of this example (Figures 5-3 to 5-9), results obtained when dynamic programming assumes a binomial (resp. approximate gaussian) distribution are shown in the left (resp. right) panel of all figures.

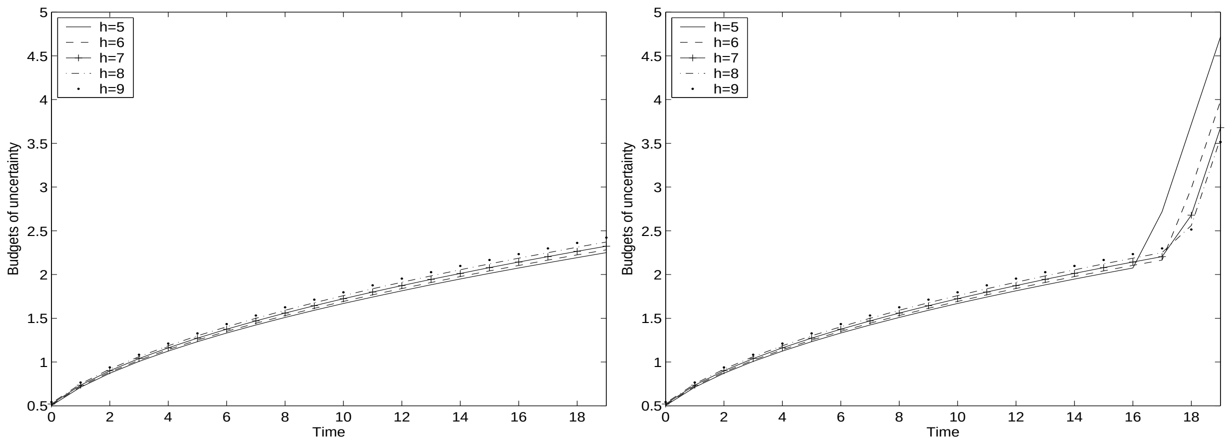

Figure 5-2: Budgets of uncertainty for $c = 0$ (left) and $c = 1$ (right).

Impact of the standard deviation:

Figure 5-3 shows how the ratio $R = 100 \cdot (E(DP) - E(ROB))/E(DP)$, in percent, evolves as the ratio $\sigma/\overline{w}$ increases (i.e. as the standard deviation increases, since we keep $\overline{w}$ constant). When the assumed and actual distributions are very different beyond their first two moments, the ratio R increases as the standard deviation increases and the robust policy outperforms dynamic programming by up to 10 to 13%, depending on the actual distribution. When the assumed and actual distributions are similar, the two methods are equivalent, since the robust policy outperforms dynamic programming by at most 0.4%. In both cases, R shows the same qualitative trend as σ increases for the three actual distributions implemented here, although spreads in numerical values increase as σ increases.

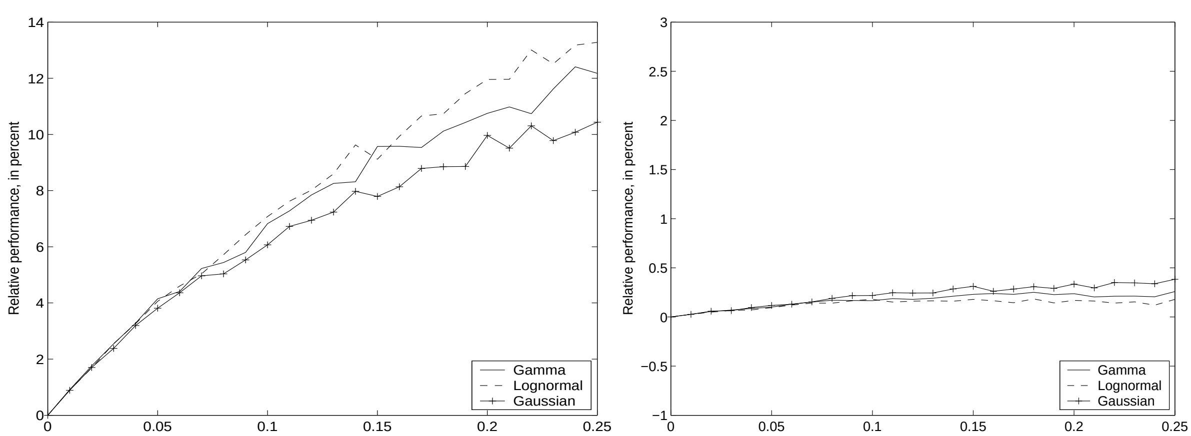

Figure 5-3: Impact of the standard deviation on performance.

A probabilistic view of performance:

Figure 5-4 shows the sample probability distribution of the costs ROB and DP. The sample distributions have the same shape, and although DP has slightly less variance, ROB has the potential of yielding much lower costs.

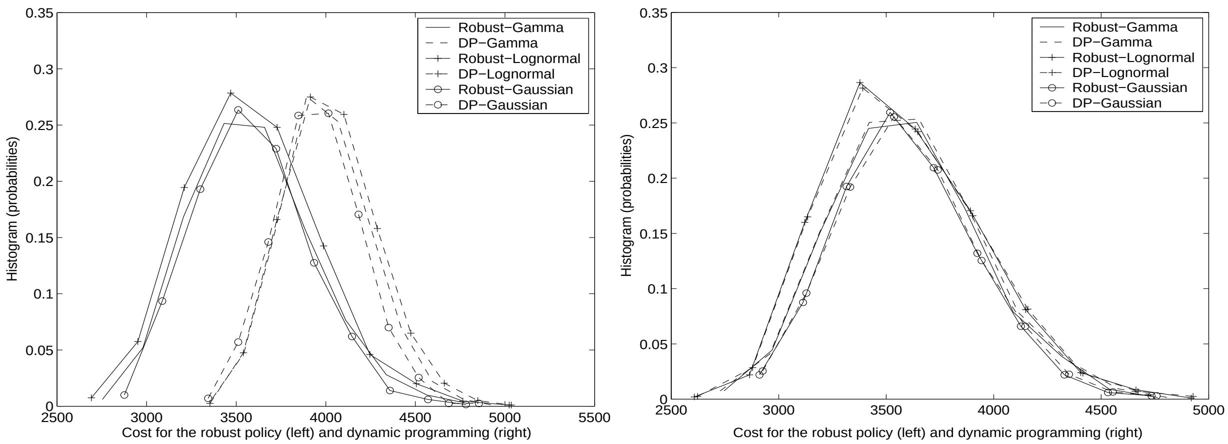

Figure 5-4: Sample probability distributions.

Impact of the cost parameters:

In Figures 5-5 to 5-9, we study the impact on performance of the cost parameters c, h and p, for $\sigma = 20$.

In Figures 5-5 to 5-7, we change one parameter (c, h or p) and show the results for the three actual distributions implemented. The numerical evidence suggests that:

- The choice of the actual distribution does not affect the qualitative trends in performance, although it slightly changes the numerical values obtained.

- When assumed and actual distributions vary widely, the relative cost benefit of using the robust approach rather than dynamic programming decreases as c increases (with $c < p$), down to 4%. When assumed and actual distributions are similar, the difference between the robust and stochastic policies is not statistically significant.

- The stochastic policy leads to better results when h is small. The exact numbers depend on the distribution used to compute the stochastic policy.

- The robust approach performs better than dynamic programming for a wide range of parameter values. In particular here, when p and h are of the same order of magnitude and assumed and actual distributions differ widely, R can exceed 15%.

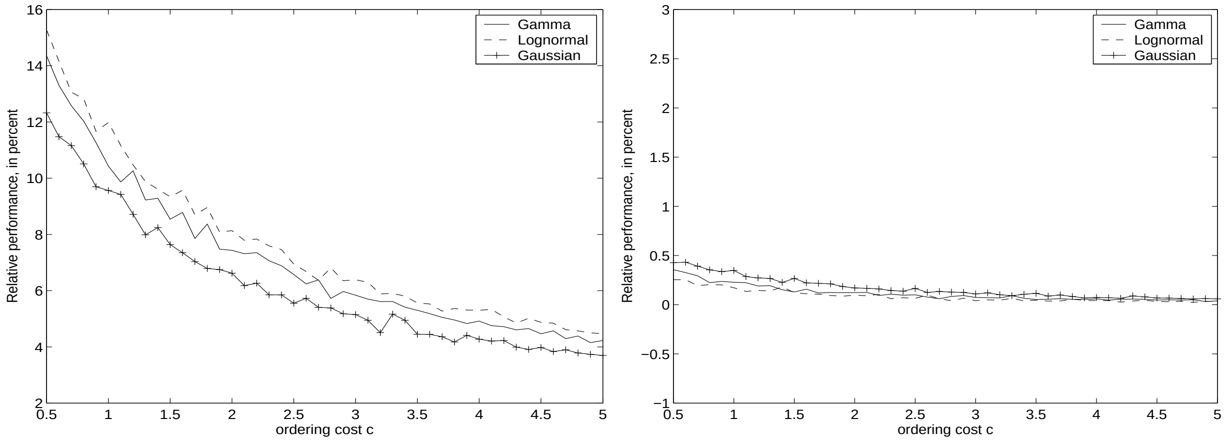

Figure 5-5: Impact of the ordering cost.

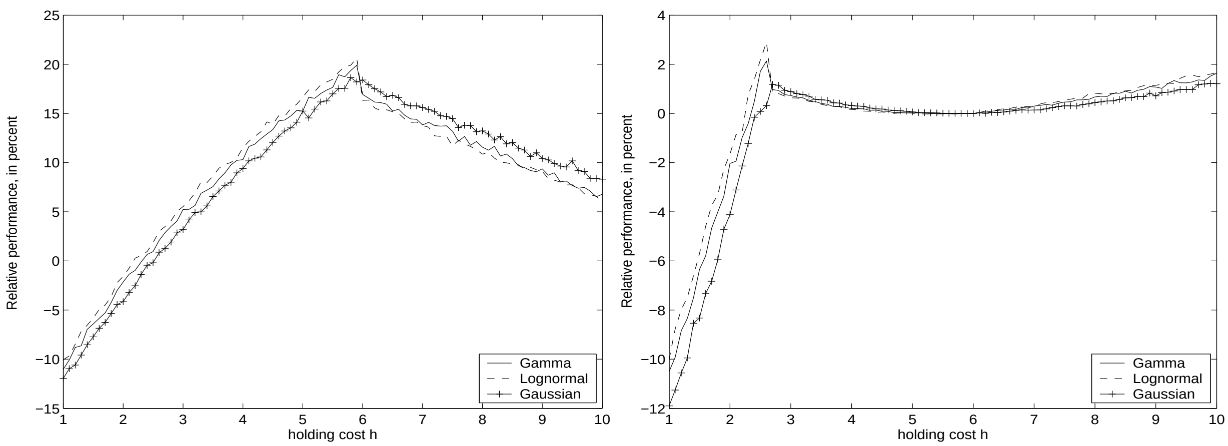

Figure 5-6: Impact of the holding cost.

In Figures 5-8 and 5-9, we change two parameters and, for clarity, only show the results for the Gamma distribution, since the choice of the actual distribution does not appear to have a large impact on performance. The numerical results lead to the following additional insights:

- The relative advantage of the robust policy over dynamic programming decreases as the unit ordering cost c increases. When actual and assumed distributions vary widely, the relative advantage increases in h until $h = p$, then decreases. When they are similar, it increases in h for $h < p$ and $h > p$, despite a slight discontinuity around $h = p$.

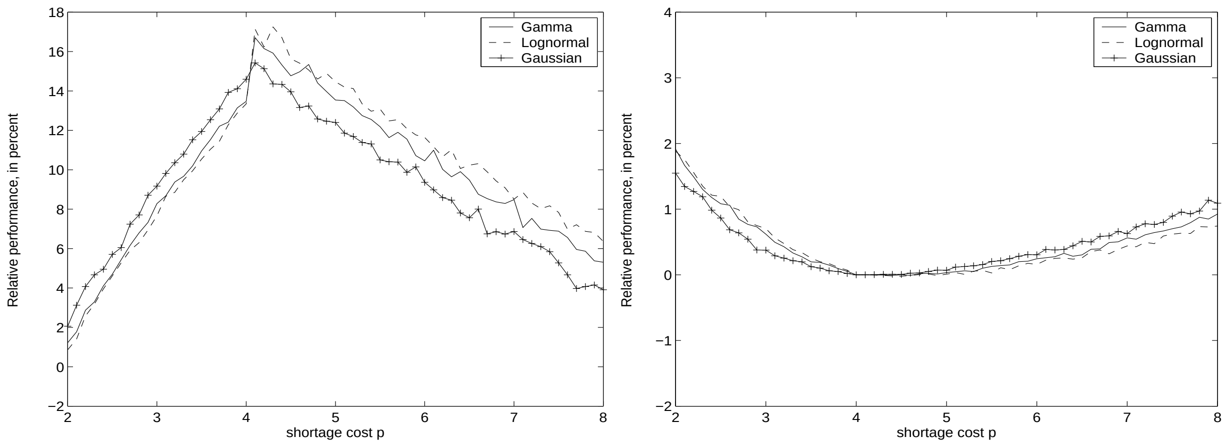

Figure 5-7: Impact of the shortage cost for various distributions.

- Dynamic programming outperforms the robust policy for very low values of the holding cost h, especially for large values of p, and in some cases where p is very small and h is large.

- When actual and assumed demand distributions vary widely, the relative performance as a function of the shortage cost p is shifted to the right and upward as h increases. When actual and assumed demand distributions are similar, the relative performance as a function of the shortage cost p is shifted to the right for $h \geq 4$.

- The robust policy outperforms dynamic programming for a wide range of parameters, although not always, and its relative performance can reach 20% in some cases.

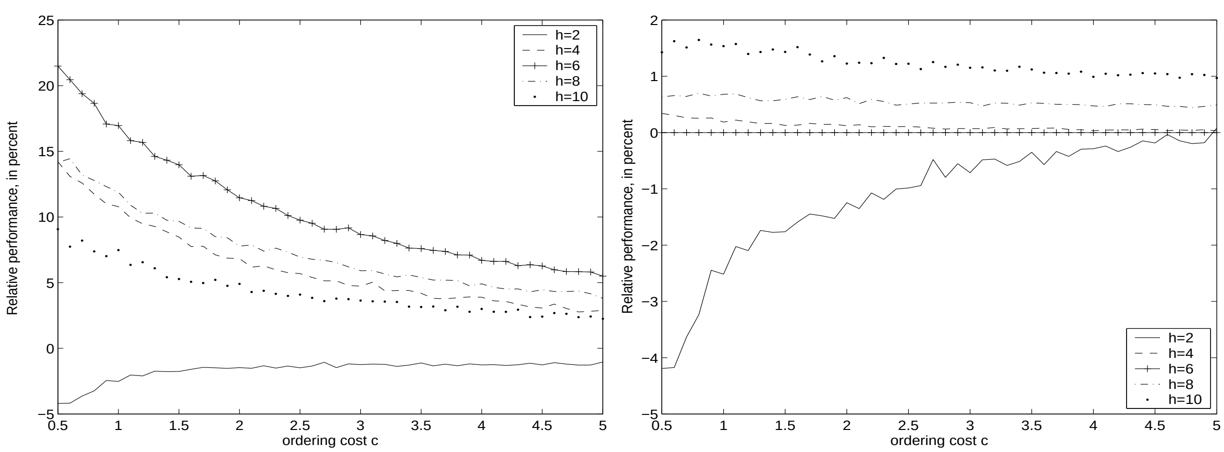

Figure 5-8: Impact of the ordering cost for various holding costs.

This numerical evidence suggests that the robust policy performs significantly better

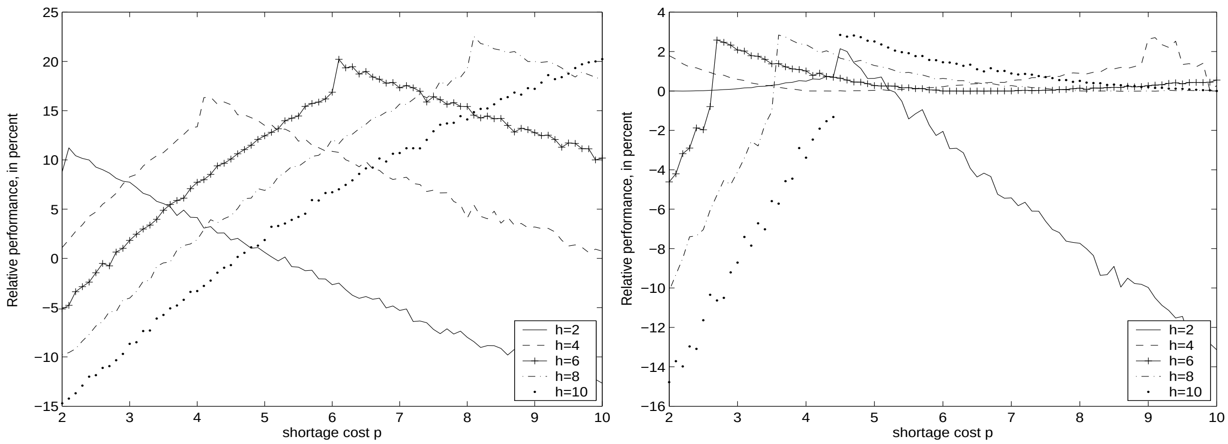

Figure 5-9: Impact of the shortage cost for various holding costs.

than dynamic programming when assumed and actual distributions differ widely beyond their first two moments, and performs similarly to dynamic programming when assumed and actual distributions are close. The results are thus quite promising.

5.5.3 Examples of Networks

A series system

In this section, we apply the proposed approach to the series system in Figure 5-10 over $T = 10$ time periods. Each order made at echelon 1 or 2 incurs a fixed cost $K_1 = K_2 = 10$

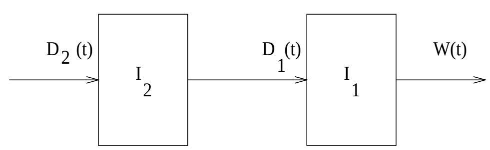

Figure 5-10: A series system.

and a cost per unit ordered $c_1 = c_2 = 1$. The holding and shortage costs at echelon 1 are $h_1 = 4$, $p_1 = 12$, while holding and shortage are penalized equally at echelon 2: $h_2 = p_2 = 4$. The stochastic demand is i.i.d. with mean $\overline{W} = 100$ and $\sigma = 20$. In the robust model, we take $\widehat{W} = 2 \cdot \sigma$. Echelons 1 and 2 hold zero inventory at time 0.

Here we are interested in comparing the performance of the robust policy and the myopic policy obtained for the same two assumed distributions as in Section 5.5.2 (binomial and approximate gaussian on five points). The actual distributions remain Gamma, Lognormal or Gaussian with the same first two moments.

The budgets of uncertainty at time 0 for all remaining time periods are computed using Algorithm 5.5.1. We reoptimize the problem at each time period, for both the robust and the myopic approach. For the robust approach, we update the budgets of uncertainty at time t, $t = 1, \ldots, T-1$, as follows:

$$\Gamma^{(t)}(\tau) = \Gamma^{(t-1)}(\tau+1) - \Gamma^{(t-1)}(1), \tag{5.37}$$

for $\tau = 0, \ldots, T-t-1$. (If we reoptimized the budgets of uncertainty, we would consistently overprotect the first time period in the horizon.) This updating rule does not change $\Gamma^{(t)}(\tau) - \Gamma^{(t)}(\tau-1)$, which we use to define the modified demand, and therefore is consistent with the implementation of the basestock levels in Section 5.5.2.

The performance is measured by the sample probability distributions of the costs MYO and ROB, and the ratio $r = 100 \cdot (MYO - ROB)/MYO$, in percent. As before, the left (resp. right) panels of the figures show the results obtained assuming a binomial distribution (resp. approximate gaussian).

Costs of the robust and the myopic policies:

The actual distribution of the demand and, more surprisingly, the distribution assumed to compute the myopic policy, do not appear to play a significant role in the sample probability distribution of the costs of the two policies. The robust policy clearly outperforms the myopic policy, since it leads to costs with a lower mean and variance.

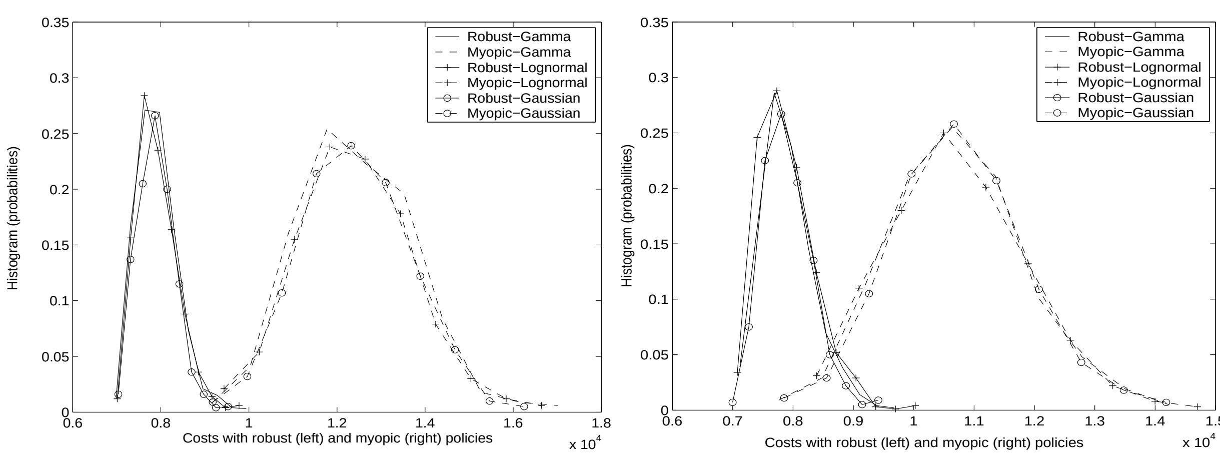

Figure 5-11: Costs of robust and myopic policies .

Sample distribution of performance ratio:

When actual and assumed distributions vary widely, all values of r are positive and the sample probability distribution of r is mostly concentrated on the range $[20\%, 45\%]$. When actual and assumed distributions are similar, $r < 0$ has a very low probability and the sample probability distribution of r is mostly concentrated on the range $[15\%, 35\%]$. Therefore, it appears that having similar actual and assumed distributions does reduce the relative difference between the two policies, but the very use of a myopic policy leads to a significant cost disadvantage. This further demonstrates the high potential of the robust policy when applied to supply chains.

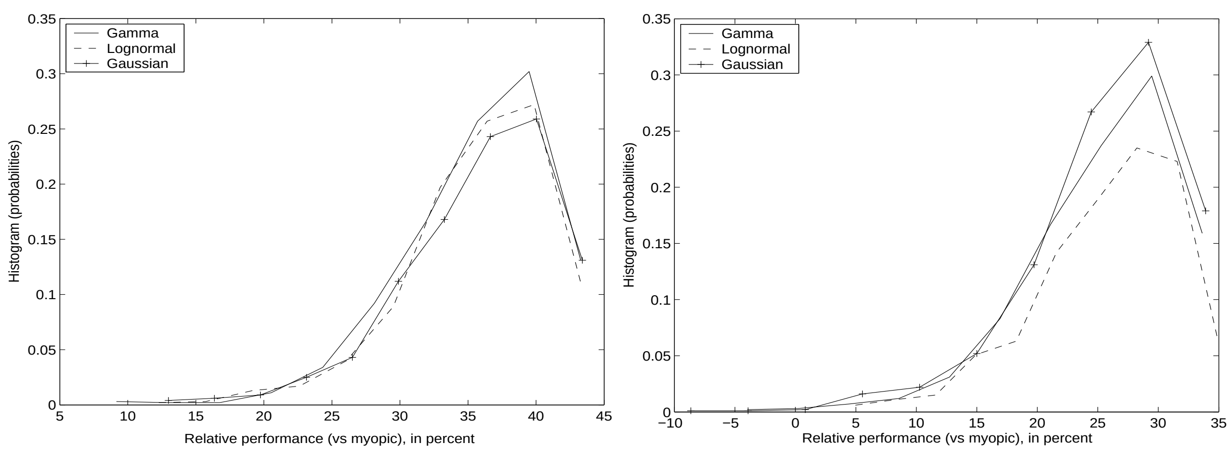

Figure 5-12: Sample distribution of relative performance.

A tree network

In this section, we apply the proposed approach to the supply chain in Figure 5-13 over $T = 5$ time periods. Echelons 1 and 2 consist of installations 1 and 2, respectively. Echelon

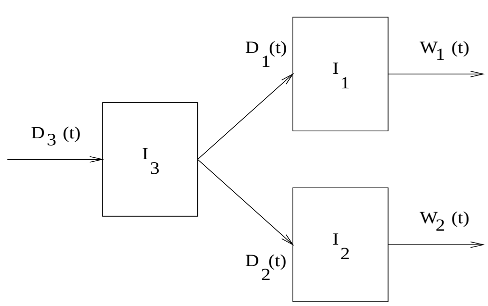

Figure 5-13: A supply chain.

3 consists of installations 1, 2 and 3, and the links in-between. The unit ordering costs are $c_1 = c_2 = c_3 = 1$. There are no fixed costs. Holding and shortage are penalized equally at echelons 1 and 2: $h_1 = p_1 = 8$, $h_2 = p_2 = 8$, while the holding and shortage costs at echelon 3 are $h_3 = 5$, $p_3 = 7$. Demands at installations 1 and 2 are i.i.d., with the same nominal demand $\overline{W} = 100$ and the same standard deviation σ. What distinguishes echelons 1 and 2 are their initial inventory: 150 items at echelon 1, 50 at echelon 2, and the actual demand distributions, which are either Gamma, Lognormal or Gaussian, with the same mean and standard deviation. Echelon 3 holds initially 300 items. Expected costs are computed using a sample of size $1,000$.

As in Section 5.5.3, we are interested in comparing the performance of the robust policy and the myopic policy obtained for the same two assumed distributions as in Section 5.5.2 (binomial and approximate gaussian on five points). Here, we assume the same distribution at stations 1 and 2 when computing the myopic policy. The actual distributions remain Gamma, Lognormal or Gaussian with the same first two moments, and can differ between stations. In the robust framework, $\widehat{W} = 2 \cdot \sigma$ and the budgets of uncertainty are computed at time 0 for all remaining time periods using Algorithm 5.5.1. We reoptimize the problem at each time period, for both the robust and the myopic approach, and update the budgets of uncertainty as in Section 5.5.3.

The performance is measured by the sample probability distributions of the costs MYO and ROB, and the ratio $r = 100 \cdot (MYO - ROB)/MYO$, in percent. As before, the left (resp. right) panels of the figures show the results obtained assuming a binomial distribution (resp. approximate gaussian).

Sample distribution of performance ratio:

As shown in Figure 5-14, the robust policy performs significantly better than the myopic policy, independently of the actual probability distributions, although there is a small positive probability that the myopic policy will perform better. This last point is not surprising given the short time horizon we are considering.

Costs of the robust and the myopic policies:

For clarity, and because the actual distribution does not seem to play a major role, we only show in Figure 5-15 the costs for Gamma distributions at both sink nodes. The robust

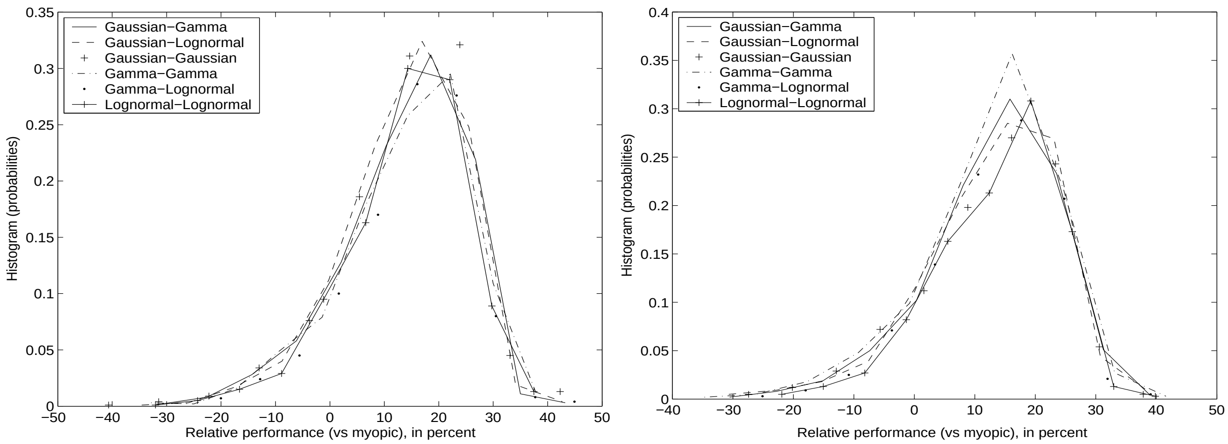

Figure 5-14: Relative performance.

policy has lower mean and variance than the myopic policy, when actual and assumed distributions vary widely or are similar.

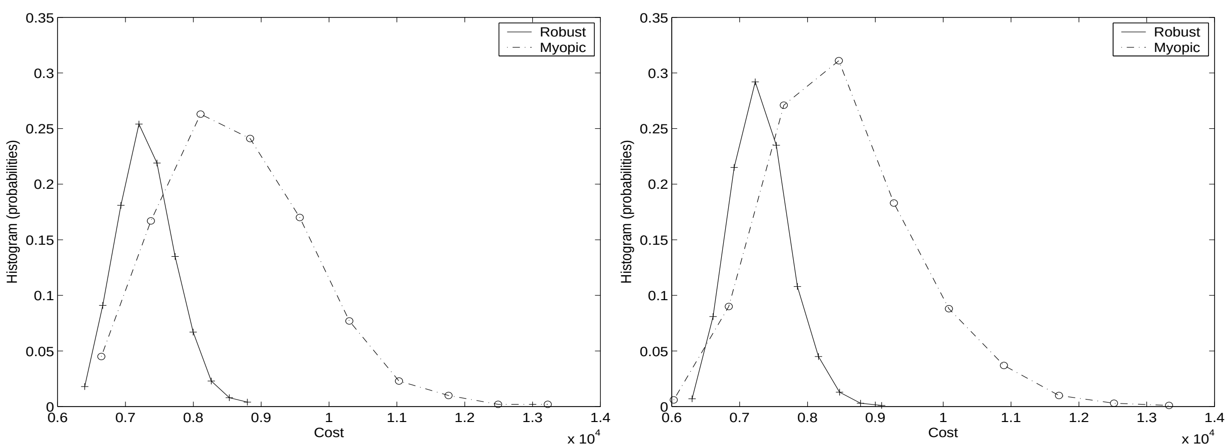

Figure 5-15: Sample probability distributions of costs.

Impact of the horizon:

We study here the impact of the time horizon T on the relative performance of the robust and myopic policies as measured by the ratio r, for Gamma demand distributions at both stations 1 and 2. The results presented in Figure 5-16 confirm the intuition that the robust policy tends to perform better as the horizon T increases. Although the peak of the relative performance r stays around 20% for a sample probability of about 0.3, the spread of the sample distribution seems to be reduced as T increases, thus making it more and more likely that the robust policy will outperform the myopic policy as the horizon increases.

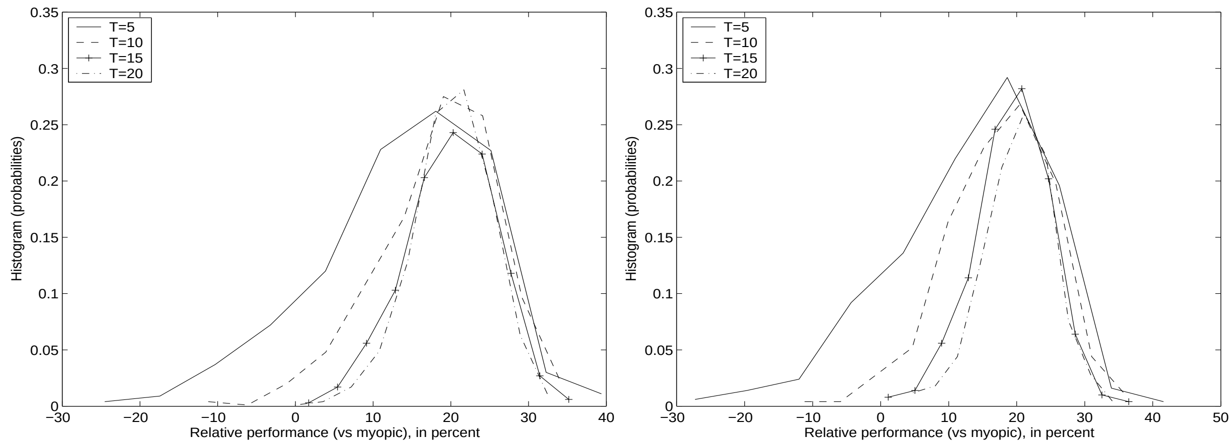

Figure 5-16: Impact of the horizon.

5.5.4 Summary of Results

The numerical evidence that we have presented suggests that:

- The robust approach leads to high-quality solutions and often outperforms other commonly used approaches, such as dynamic programming in single stations or myopic policies in more complex supply chains.

- For single stations, the robust approach outperforms dynamic programming for a wide range of parameters when assumed and actual distributions vary widely, and performs similarly otherwise.

- For more complex supply chains, the robust policy performs significantly better than a myopic policy, in particular over many time periods, even when actual and assumed demand distributions are close.

- The exact actual distribution of the demand does not play a significant role in the qualitative trends.

5.6 Concluding Remarks

In this chapter, we have proposed a deterministic, numerically tractable methodology to optimally control supply chains subject to random demand. Using robust optimization ideas, we have built an equivalent model without uncertainty of the same class as the nominal problem, with a modified demand sequence. Specifically, the proposed model is a linear programming problem if there are no fixed costs throughout the supply chain and a mixed integer programming problem if fixed costs are present.

This model incorporates a wide variety of phenomena, including demands that are not identically distributed over time and capacity on the echelons and links. When the parameters are chosen appropriately, the proposed approach preserves performance while protecting against uncertainty. One of its most appealing features is that it uses very little information on the demand distributions, and therefore is widely applicable. In particular, if we only know the mean and the variance of the distributions, the robust policy often outperforms the nominal policy, as well as policies computed assuming full but erroneous knowledge of the distributions for the correct mean and variance.

This approach also provides valuable theoretical insights. We have derived the expression of key parameters of the robust policy, and shown optimality of basestock policies in the proposed framework when the optimal stochastic policy is basestock, but also in other instances where the optimal stochastic policy is not known. Hence, the methodology is not only validated on benchmark problems where the optimal stochastic policy is already known (yet in general hard to compute numerically), but also provides a framework to analyze complex supply chains for which the traditional tools of stochastic optimization face serious dimensionality problems.

Chapter 6

Application to Revenue Management

6.1 Background and Contributions

Revenue management is concerned with the theory and practice of maximizing profits in industries facing uncertain customer demand. This encompasses a broad range of problems. In this work, we focus our attention on those involving perishable assets, for which it is particularly difficult to obtain accurate probabilistic information. As an example, the demand for a newspaper varies greatly depending on what is making the headlines that day. In Scarf's words [57], "we may have reason to suspect that the future demand will come from a distribution which differs from that governing past history in an unpredictable way".

We consider two classes of applications. In the first part of this chapter (Section 6.2), we apply robust techniques to management problems maximizing net revenue (profits minus costs) in presence of uncertain demand for a perishable product. A classical example of such a setting is the newsvendor problem, which Porteus summarizes in [53]. As one of the building blocks of inventory theory, it has received much attention in the literature, often under the assumption that the demand distribution is known exactly. However, in practice, the volatility of the demand for perishable products makes it difficult to obtain an accurate forecast. The issue of imperfect information has been addressed in the past by assuming that only the first two moments are available. In 1958, Scarf [57] derived the optimal ordering quantity for the classical newsboy problem at mean and variance given, and his work was later extended by Gallego and his co-authors [36, 47, 48]. Here, we apply the robust optimization framework to address uncertainty in the demand distribution. We assume that demand for a product is lost if the product is out-of-stock. Such a profit structure rules out the possibility of an averaging effect across different items, which motivates using the data-driven technique developed in Chapter 3 rather than the method with uncertainty sets. It also allows us to incorporate risk aversion in a very intuitive manner. The first attempt to model the attitude of the decision-maker towards risk in the newsboy problem is due to Lau [42], who considers two alternative criteria to the expected revenue: the expected utility and the probability of reaching a prespecified profit target. More recently, Eeckhoudt et al. [33] have revisited the framework based on the expected utility of the newsvendor, and Chen and Federgruen [25] have analyzed some simple inventory models from a mean-variance perspective. The Conditional Value-at-Risk approach that we propose here incorporates risk without requiring the exact knowledge of the utility function, and yields more tractable formulations that models using the variance of the revenue. For the newsvendor problem with a single source of uncertainty, we show in Section 6.2.1 that the optimal order can be found by ranking the historical data appropriately. When several sources of uncertainty are present, we derive linear programming formulations, which we analyze in detail in Section 6.2.2. Section 6.2.3 presents numerical results.

In the second part of this chapter (Section 6.3), we focus on capacity-constrained industries. This describes many services such as hotels or car rentals, but most of the research efforts have focused on airlines, as they have the longest history of work in revenue management. For instance, as early as 1972, Littlewood proposed a simple rule to optimally control discount and full fare bookings on a single-leg flight [44]. Belobaba developed in [3] an approach to compute protection levels for multiple classes, which became famously known as the Expected Marginal Seat Revenue (EMSR) approach, however it is only optimal for two classes. Brumelle and McGill extended Littlewood's results to multiple classes in [22]. Interest in yield management techniques was fuelled by the airline industry deregulation in the 1970s, and the potential of generating high profits by efficiently allocating scarce resources. For instance, Cook notes in [29] that American Airlines increased its revenue by one billion dollars in 1997 following the implementation of such methods. A comprehensive survey by McGill and van Ryzin [46] discusses the research developments in the field over the last forty years, and points out that demand dependencies between booking classes, among other factors, dramatically increase the complexity of the forecasting process. The same authors have proposed in an earlier paper [65] an algorithm for determining airline seat protection levels without forecasting, but their technique is limited to independent demands on a single leg. Brumelle et al. have considered correlated demand with perfect knowledge of the distribution in [23], again on a single leg. The seat allocation problem is far more complex at the network level than on individual routes. Glover et al. [37] have studied network flow models when the demand is deterministic, and the case of random demands has been investigated by Wang [68] and Wollmer [73]. However, the introduction of bidprice control, which was proposed by Simpson in [61] and further analyzed by Williamson in [70], is widely regarded as the most important conceptual breakthrough in the theory of network revenue management. In that model, booking requests are accepted if the corresponding fares are higher than the opportunity cost of the itineraries. A drawback of the approach is that opportunity costs are computed in a static manner, using the nominal value of the demands. Taking into account dynamic effects across the network adds another layer of complexity to the formulation. This is indeed a crucial component of yield management in the airline industry, as cost-conscious travellers tend to buy their tickets before those willing to pay the full fare. Talluri and van Ryzin [63] suggested a mechanism relying on adaptive bidprices to control the system, while Bertsimas and Popescu proposed and analyzed in [14] an algorithm based on approximate dynamic programming. Moreover, McGill and van Ryzin [46] mention implementable dynamic programming approaches as an important area to investigate further in the context of airline revenue management. Because the data-driven approach developed in Chapter 3 does not require any forecasting and yields tractable formulations, it seems particularly well suited to such a dynamic network environment with a complex demand structure. Specifically, we consider robust seat allocations (Section 6.3.1) and admission policies (Section 6.3.2), and analyze the optimal solution in terms of the historical data and the shadow prices on each leg of the network. We report encouraging computational results in Section 6.3.3.

Finally, we conclude this chapter in Section 6.4.

6.2 The Newsvendor Problem

In this section, we apply the data-driven approach to the newsvendor problem and its extensions. Section 6.2.1 presents three possible cost structures when only the demand in the primary market is random: (a) the classical setting, (b) the model with holding and shortage cost; and (c) the problem with a set ordering fee. We describe how the optimal policy can be found by ranking the historical data in an appropriate manner. Our results also quantify explicitly the impact of risk aversion on the optimal order. Section 6.2.2 considers the case of multiple sources of uncertainty: (a) the demand and the yield, (b) multiple items; and (c) the demand in the primary as well as secondary markets. We show that the robust counterparts are linear or mixed integer programming problems, and give insights into the structure of the optimal solution. We implement the data-driven approach on some examples in Section 6.2.3.

6.2.1 With a Single Source of Uncertainty

The purpose of the newsboy problem, in its most elementary version, is to find the optimal order for a perishable item in a single-period horizon, in order to maximize revenue. We follow closely Gallego ([47] and [48]) in our notations and assumptions.

The classical problem

We define:

$c > 0$:	the unit cost,
$p = c\,(1+m)$:	the unit selling price,
$s = (1-t)\,c$:	the unit salvage price,
Q:	the order quantity,
D:	the random demand.

m (resp. t) is commonly referred to as the markup (resp. the discount) factor. The random profit is given by:

$$\pi(Q, D) = p\,\min(Q, D) + s\,\max(0, Q-D) - c\,Q, \tag{6.1}$$

or, equivalently:

$$\pi(Q, D) = (p-c)\,Q + (p-s)\,\min(0, D-Q). \tag{6.2}$$

Let $d_1, \ldots, d_N$ be previous observations of the demand, and $d_{(1)}, \ldots, d_{(N)}$ the same observations ranked in increasing order, so that $d_{(1)} \leq \ldots \leq d_{(N)}$. We assume $d_k \geq 0$ for all k and integers. The robust approach described in Chapter 3 maximizes the sample profit over the $N_\alpha = \lfloor N(1-\alpha) + \alpha \rfloor$ worst cases, that is:

$$\max_{Q \geq 0} \ (p-c)Q + (p-s) \cdot \frac{1}{N_\alpha} \sum_{k=1}^{N_\alpha} (\min(0, d_{\cdot} - Q))_{(k)}. \tag{6.3}$$

(There is no need to impose an integrality constraint on Q since at optimality it will be one of the breakpoints of the objective function, that is: one of the d_j, which are all integers.) Since $\min(0, D-Q)$ is nondecreasing in D, the k-th smallest $\min(0, D-Q)$ at Q given is equal to $\min(0, d_{(k)} - Q)$. Therefore, in this case the robust formulation follows immediately, without requiring the use of strong duality:

$$\max_{Q \geq 0} \ (p-c)\, Q + (p-s) \cdot \frac{1}{N_\alpha} \sum_{k=1}^{N_\alpha} \min(0, d_{(k)} - Q). \tag{6.4}$$

Theorem 6.2.1 (Optimal policy) *The optimal order Q^* solution of Problem (6.4) verifies:*

$$Q^* = d_{(j)} \ \textit{with} \ j = \left\lceil \frac{m}{m+t} N_\alpha \right\rceil. \tag{6.5}$$

Proof: For $d_{(k)} \leq Q \leq d_{(k+1)}$, the optimum is reached at $Q = d_{(k)}$ if $p - c \leq (p-s)k/N_\alpha$ and at $Q = d_{(k+1)}$ otherwise. Similarly, for $Q \leq d_{(1)}$ (resp. $Q \geq d_{(N)}$), the optimum is reached at $d_{(1)}$ (resp. $d_{(N)}$). Hence, the optimum over $Q \geq 0$ is reached at $Q = d_{(j)}$ where $(j-1)/N_\alpha < (p-c)/(p-s) \leq j/N_\alpha$. The result follows from $(p-c)/(p-s) = m/(m+t)$.
□

Remarks:

- An appealing feature of this framework is that the optimal order is very easy to compute, by ranking the historical data.

- Unsurprisingly, the risk-averse newsvendor orders less than his risk-neutral counterpart, as $N_\alpha \leq N$. The impact of his aversion to risk on Q^* depends on $m/(m+t)$.

- Alternatively, for $Q^* = d_{(j)}$ where $j = \left\lceil \frac{m}{m+t} N_\alpha \right\rceil$ is given, N_α can take any value such that:

  $$\left(1 + \frac{t}{m}\right)(j-1) \leq N_\alpha < \left(1 + \frac{t}{m}\right) j, \tag{6.6}$$

  which corresponds to an interval that has $\lfloor 1 + t/m \rfloor$ integer points (i.e. possible values for N_α) in it. As a result, if $t \gg m$, the problem is less sensitive in the choice of N_α than if $m \ll t$. This also appears in the numerical experiments in Section 6.2.3.

Comparison with the optimal order for the worst-case distribution:
Scarf gives in [57] the following ordering rule for the newsboy problem when only the mean μ and the standard deviation σ of the demand distribution are known:

$$Q^0 = \mu + \frac{\sigma}{2}\left(\sqrt{\frac{m}{t}} - \sqrt{\frac{t}{m}}\right) \tag{6.7}$$

(Q^0 is obtained by optimizing the worst-case expected profit over all nonnegative distributions with those first two moments.) The optimal policies in Scarf's model and in the data-driven approach have in common that they do not depend on the unit cost c, but rather on the markup and discount factors m and t.

We note that if $m = t$, the optimal order is equal to the mean in Scarf's formulation, and to $d_{(\lceil N_\alpha/2 \rceil)}$ in the data-driven framework. In particular, if $\alpha = 0$, the robust optimal order is equal to the *median*, rather than the mean. Therefore, using the historical data or assuming only that the first two moments are known can result in very different ordering strategies, in particular for skewed distributions.

We next show that trimming can improve the average a-posteriori ratio to optimality, computed with the historical observations. If $D = d_{(k)}$, the a-posteriori optimal order for the classical newsvendor problem is $Q = d_{(k)}$, yielding the profit $(p-c)\, d_{(k)}$. We are interested in measuring the average ratio between the random profit $\pi(d_{(j)}, d_{(k)})$ obtained for $Q = d_{(j)}$ and $D = d_{(k)}$ (all $d_{(k)}$, $k = 1, \ldots, N$, occurring with equal probability), and the a-posteriori optimal profit $(p-c)\, d_{(k)}$.

Theorem 6.2.2 (Average ratio to optimality) *The average ratio to optimality is minimized for $Q = d_{(j)}$ where j is the smallest integer such that:*

$$\sum_{k=1}^{j} \frac{1}{d_{(k)}} \geq \frac{m}{m+t} \sum_{k=1}^{N} \frac{1}{d_k}. \tag{6.8}$$

The optimal j is always less than or equal to $\left\lceil \frac{m}{m+t} N \right\rceil$.

Proof: For $D = d_{(k)}$ and $Q = d_{(j)}$, $j, k = 1, \ldots, N$, the ratio to optimality is given by:

$$\rho_{kj} = 1 - \frac{d_{(j)}}{d_{(k)}} - \left(1 + \frac{t}{m}\right) \min\left(0, 1 - \frac{d_{(j)}}{d_{(k)}}\right). \tag{6.9}$$

Averaging over all d_k yields a piecewise linear, convex function in $d_{(j)}$. (6.8) follows by studying the slope of this function. To prove that $j \leq \left\lceil \frac{m}{m+t} N \right\rceil$, where j is the smallest integer verifying (6.8), it is sufficient to prove that $\left\lceil \frac{m}{m+t} N \right\rceil$ itself verifies (6.8). Since $d_{(1)} \leq \ldots \leq d_{(N)}$, the function $j \to \sum_{k=1}^{j} \frac{1}{d_{(k)}}$ is concave in j, where j is an integer between 0 and N, and the function takes the value 0 for $j = 0$. Therefore,

$$\sum_{k=1}^{\left\lceil \frac{m\,N}{m+t} \right\rceil} \frac{1}{d_{(k)}} \geq \frac{\left\lceil \frac{m\,N}{m+t} \right\rceil}{N} \sum_{k=1}^{n} \frac{1}{d_k} \geq \frac{m}{m+t} \sum_{k=1}^{N} \frac{1}{d_k}. \tag{6.10}$$

□

Since $j = \left\lceil \frac{m}{m+t} N \right\rceil$, there exists $\alpha^* \in [0, 1]$ such that $j = \left\lceil \frac{m}{m+t} N_{\alpha^*} \right\rceil$. Therefore, trimming a fraction $\alpha \in [0, \alpha^*]$ will improve the average ratio of optimality, as compared to the case without trimming.

Remark: The property that the order $\widetilde{Q}$ minimizing the average ratio to optimality is less than or equal to the order $\overline{Q}$ maximizing the (non-trimmed) sample profit is quite general in practice and arises under the assumption that the a-posteriori optimal profit (here, $\pi^*(D) = (p - c)\,D$) increases with the demand.

This is because the average ratio to optimality $1 - (1/N) \sum_k \frac{\pi(Q, d_{(k)})}{\pi^*(d_{(k)})}$ gives more relative weight to the cases with small demand than they have in the sample profit $(1/N) \sum_k \pi(Q, d_{(k)})$ (since $\pi^*(d_{(1)}) \leq \ldots \leq \pi^*(d_{(N)})$), and the slope of the profit for a specific demand is negative as soon as Q is greater than the demand. As a result, the slope in Q of $\sum_k \frac{\pi(Q, d_{(k)})}{\pi^*(d_{(k)})}$ (which we want to maximize) decreases and changes signs before the slope of the sample profit $\sum_i \pi(Q, d_{(i)})$. In particular, since the order Q^*_α minimizing the trimmed sample profit is always less than or equal to $\overline{Q}$ (and can be equal to any $d_{(j)}$ in $[d_{(1)}, \overline{Q}]$), it is possible to find a trimming factor α such that $Q^*_\alpha = \widetilde{Q} \leq \overline{Q}$.

The model with holding and shortage cost

Let h^+ (resp. h^-) be the unit holding cost (resp. the unit shortage cost), incurred when the newsvendor has items left (resp. does not have enough items) at the end of the time period. As before, the unit cost is c and the unit price is p. The random profit is given by:

$$\pi(Q, D) = (p - c + h^-)\,Q + (p + h^+ + h^-) \min(0, D - Q) - h^- D. \tag{6.11}$$

Our goal is to maximize the trimmed mean of the profit:

$$\max_{Q \geq 0} \, (p - c + h^-) \, Q + \frac{1}{N_\alpha} \sum_{k=1}^{N_\alpha} \left[(p + h^+ + h^-) \min(0, d_\cdot - Q) - h^- d_\cdot \right]_{(k)}, \tag{6.12}$$

where for any $\mathbf{y} \in \mathcal{R}^n$, $y_{(k)}$ is the k-th smallest component of $\mathbf{y}$.

Theorem 6.2.3 (Optimal policy)

(a) The optimal order Q^* in (6.12) is solution of the linear programming problem:

$$\begin{array}{lll} \max & (p - c + h^-)Q + \phi + \frac{1}{N_\alpha} \sum_{k=1}^{N} \psi_k & \\ s.t. & \phi + \psi_k - (p + h^+ + h^-) \, Z_k \leq -h^- d_k, & \forall k, \\ & Z_k + Q \leq d_k, & \forall k, \\ & Z_k \leq 0, \; \psi_k \leq 0, & \forall k, \\ & Q \geq 0. & \end{array} \tag{6.13}$$

Moreover, $Q^* = d_{(j)}$ for some j.
(b) Let $M_\alpha = \left\lceil \frac{p - c + h^-}{p + h^+ + h^-} N_\alpha \right\rceil$. Q^* verifies:

$$Q^* = \min \left\{ d_{(j)} | d_{(j)} \geq \frac{p + h^+}{p + h^+ + h^-} \, d_{(M_\alpha)} + \frac{h^-}{p + h^+ + h^-} \, d_{(N - N_\alpha + M_\alpha)} \right\}. \tag{6.14}$$

(c) Let S_α be the set of the N_α worst-case scenarios at optimality, and $d_{(j)}^{S_\alpha}$ the j-th lowest demand within that set. We have:

$$Q^* = d_{(M_\alpha)}^{S_\alpha}, \tag{6.15}$$

where M_α is defined in (b).

Proof: (a) follows from applying Theorem 3.2.1 to (6.12). At optimality, $Q = d_{(j)}$ for some j because the function to maximize in (6.12) is concave piecewise linear with breakpoints in the set (d_i).

(b) This uses ideas similar to the proof of Theorem 3.3.1. The slope of the profit function is: $(p - c + h^-) - \frac{1}{N_\alpha}(p + h^+ + h^-) \cdot |\{i \in S(Q), d_i \leq Q\}|$, where $S(Q)$ is the set of indices of the N_α smallest $(p + h^+ + h^-) \min(0, d_i - Q) - h^- d_i$ at Q given. It is easy to show that for any $i \in S(Q)$ and any k such that $d_k \leq d_i \leq Q$, $k \in S(Q)$ as well. Similarly, for any $i \in S(Q)$ and any k such that $d_k \geq d_i \geq Q$, $k \in S(Q)$. Hence, $S(Q)$ consists of the indices of $d_{(1)}, \ldots, d_{(M_\alpha)}$ and $d_{(N - N_\alpha + M_\alpha + 1)}, \ldots, d_{(N)}$, for some $0 \leq M_\alpha \leq N$, with $d_{(m_\alpha)} \leq Q \leq d_{(N - N_\alpha + M_\alpha + 1)}$. The slope of the trimmed profit function is then proportional to $\frac{p - c + h^-}{p + h^+ + h^-} N_\alpha - M_\alpha$, and at optimality M_α is equal to $\left\lceil \frac{p - c + h^-}{p + h^+ + h^-} N_\alpha \right\rceil$. We now have to determine the optimal value of Q.

Let $f_i^j = (p + h^+ + h^-)\min(0, d_{(i)} - d_{(j)}) - h^- d_{(i)}$ be the profit realized when $Q = d_{(j)}$ and $D = d_{(i)}$, for all i and j. The optimal M_α is the greatest integer less than or equal to N_α such that $f^j_{M_\alpha} \leq f^j_{N-N_\alpha+M_\alpha}$. (Otherwise, we would remove M_α from $S(Q)$ and add $N - N_\alpha + M_\alpha$ instead.) Plugging the expressions of $f^j_{M_\alpha}$ and $f^j_{N-N_\alpha+M_\alpha}$ yields:

$$(p + h^+)d_{(M_\alpha)} - (p + h^+ + h^-)d_{(j)} \leq -h^- d_{(N-N_\alpha+M_\alpha)}. \tag{6.16}$$

Combining the previous results, (6.14) follows immediately.

(c) Considering only the scenarios in S_α, we inject $N = N_\alpha$ into (6.14). □

Remarks:

1. For $\alpha = 0$, we obtain the data-driven version of the optimal policy obtained when the exact distribution of the demand is known [36]:

$$Q = \arg\min \left\{ y | P(D \leq y) \geq \frac{p - c + h^-}{p + h^+ + h^-} \right\}. \tag{6.17}$$

2. This framework can also be used to model the newsvendor problem with recourse. In this case, we consider the problem described in Section 6.2.1, and have the additional assumption that, once the demand has been revealed, we place a second order at a unit cost of $c' = c(1 + e)$ with $0 < e < m$, whenever the first order is not enough to satisfy the demand. The profit becomes:

$$\pi(Q, D) = p\,D + s\ \max(0, Q - D) - c\,Q - c'\ \max(0, D - Q), \tag{6.18}$$

or equivalently:

$$\pi(Q, D) = (c' - c)\,Q + (c' - s)\min(0, D - Q) + (p - c')D. \tag{6.19}$$

The profit function is identical to the one for the model with holding and shortage cost, with $c' = p + h^-$ and $s = -h^+$. Therefore, all the results presented in Section 6.2.1 apply here. In particular, it is straightforward to show that the optimal policy with recourse is the same as the optimal policy without recourse where the markup factor m has been replaced by the cost premium e. This property is also verified by the optimal policy when only mean and variance of the distribution are known [47].

Theorem 6.2.4 (Average ratio to optimality) *The average ratio to optimality is min-*

imized for $Q = d_{(j)}$ when j is the smallest integer such that:

$$\sum_{k=1}^{j} \frac{1}{d_{(k)}} \geq \frac{p-c+h^-}{p+h^+ + h^-} \sum_{k=1}^{N} \frac{1}{d_k}. \tag{6.20}$$

In particular, there exists $\alpha^* \in [0,1]$ such that trimming by a fraction $\alpha \in [0, \alpha^*]$ will improve the average ratio to optimality.

Proof: Is similar to the proof of Theorem 6.2.2. □

The model with fixed ordering cost

In this section, we consider the case where a fixed cost A is incurred whenever an order is made. Let $I \geq 0$ be the initial inventory. All the other assumptions remain the same as in Section 6.2.1. The random profit is here:

$$\pi(Q, D) = -A\, 1_{\{Q>0\}} + p\, \min(Q + I, D) + s\, \max(Q + I - D, 0) - c\, Q, \tag{6.21}$$

or, defining $S = Q + I$:

$$\tilde{\pi}(S, D) = -A\, 1_{\{S>I\}} + c\, I + (p-c)S + (p-s)\, \min(0, D - S). \tag{6.22}$$

Since $c\, I$ is a constant and $\min(0, D-S)$ is increasing in D, the data-driven approach solves:

$$\max_{S \geq I}\ -A\, 1_{\{S>I\}} + (p-c)S + \frac{(p-s)}{N_\alpha} \sum_{k=1}^{N_\alpha} \min(0, d_{(k)} - S), \tag{6.23}$$

with $d_{(1)} \leq \ldots \leq d_{(N)}$.

Theorem 6.2.5 (Optimal policy) *It is optimal to order $S^* - I$ if $I \leq s^*$ and 0 otherwise, with:*

$$S^* = d_{(j)} \ \text{ where } j = \left\lceil \frac{m}{m+t}\, N_\alpha \right\rceil, \tag{6.24}$$

and:

$$s^* = S^* - \frac{\dfrac{A}{c} + \dfrac{m+t}{N_\alpha} \displaystyle\sum_{i=k+1}^{j} (S^* - d_{(i)})}{m - (m+t)\, \dfrac{k}{N_\alpha}}, \tag{6.25}$$

where k is such that:

$$m\, d_{(k)} + \frac{m+t}{N_\alpha} \sum_{i=1}^{k-1} (d_{(i)} - d_{(k)}) \leq -\frac{A}{c} + m\, S^* + \frac{m+t}{N_\alpha} \sum_{i=1}^{j} (d_{(i)} - S^*) < m\, d_{(k+1)} + \frac{m+t}{N_\alpha} \sum_{i=1}^{k} (d_{(i)} - d_{(k+1)}). \tag{6.26}$$

Proof: We have seen in Section 6.2.1 that the optimal solution for $A = 0$ is $S^* = d_{(j)}$ with $j = \left\lceil \frac{m}{m+t}\, N_\alpha \right\rceil$, yielding a Conditional Value-at-Risk of:

$$K_\alpha(S^*) = c\left[m\,S^* + \frac{(m+t)}{N_\alpha}\sum_{i=1}^{j-1}(d_{(i)} - S^*)\right]. \tag{6.27}$$

If we choose not to order, the trimmed average profit becomes:

$$K_\alpha(I) = c\left[m\,I + \frac{(m+t)}{N_\alpha}\sum_{i=1}^{N}\min(0, d_{(i)} - I)\right]. \tag{6.28}$$

We want to find $s = \arg\min\{I|K_\alpha(I) \geq -A + K_\alpha(S^*)\}$, the threshold value for the inventory that makes ordering optimal (note that s here is not the salvage value. The only parameter related to the salvage value in this section is the discount factor t.) It is easy to see that $s \leq S$. Since $K_\alpha(x)$ is piecewise linear, nondecreasing for $x \leq S$, we first identify which interval $[d_{(k)}, d_{(k+1)})$ s belongs to by solving in k: $K_\alpha(d_{(k)}) \leq -A + K_\alpha(S) < K_\alpha(d_{(k+1)})$. Then $K_\alpha(s) = c\left[m\,s + \frac{m+t}{N_\alpha}\sum_{i=1}^{k}(d_{(i)} - s)\right]$ is linear in s and the value of s for which $K_\alpha(s) = -A + K_\alpha(S^*)$ follows from simple algebraic manipulations. □

Comparison with the optimal policy for the worst-case distribution: For $\alpha = 0$, we can compare the optimal policy with the one obtained in [48] when only the mean and the standard deviation of the distribution are known, namely:

$$S^0 = \mu + \frac{\sigma}{2}\left(\sqrt{\frac{m}{t}} - \sqrt{\frac{t}{m}}\right), \quad s^0 = \mu + \frac{(m-t)\widehat{A} - (m+t)\sqrt{\widehat{A}^2 - m\,t\,\sigma^2}}{2mt}, \tag{6.29}$$

where $\widehat{A} = \sigma\sqrt{m\,t} + \frac{A}{c}$. The differences between S^* and S^0 have been described in Section 6.2.1. Here we focus on $S^* - s^*$ and $S^0 - s^0$.

- For A/c large, $k = 0$ and:

$$S^* - s^* \approx S^0 - s^0 \approx \frac{A}{m\,c}. \tag{6.30}$$

  In this case, both approaches leads to a similar value of $S - s$.

- For A/c small, $k = j - 1$ and:

$$S^* - s^* \approx \frac{A}{c\left(m - (m+t)\frac{j-1}{N}\right)}. \tag{6.31}$$

  In the case where only mean and variance are known:

$$S^0 - s^0 \approx \frac{m+t}{(mt)^{3/4}} \cdot \sqrt{\frac{A\sigma}{2c}}. \tag{6.32}$$

  Here, the data-driven approach yields very different results from the case where only the first two moments are known.

The average ratio to optimality can be computed by distinguishing first between the cases where $I \leq s^*$ and $I > s^*$ (order or no order in the robust model) and then, for $D = d_{(i)}$, between $d_{(i)} \geq I + \dfrac{A}{p-c}$ and $d_{(i)} < I + \dfrac{A}{p-c}$ (order or no order in the a-posteriori optimal policy). However, since the formulas are not particularly insightful, we omit them here.

6.2.2 With Multiple Sources of Uncertainty

The model with random yield

We consider the newsboy problem described in Section 6.2.1, with the additional assumption that an order for Q units results in the delivery of Q units, only $G(Q)$ of which are good. The random profit can be written as:

$$\pi(Q, D) = (s-c)\, Q + (p-s) \min(G(Q), D). \tag{6.33}$$

We define the yield r as $G(Q) = r\, Q$. The data observed in realization i is (d_i, r_i). (We assume that the yield is independent of the quantity ordered.) Relaxing the integrality constraint on $G(Q)$, we consider the problem:

$$\max_{Q \in \mathcal{Z}^+} \ (s-c)\, Q + \frac{p-s}{N_\alpha} \sum_{k=1}^{N_\alpha} \left[\min(r.Q, d.)\right]_{(k)}. \tag{6.34}$$

Let S_α be the set of the N_α worst-case scenarios at optimality. At $\mathbf{x}$ given, let $x_{(j)}^{S_\alpha}$ be the j-th smallest component of $\mathbf{x}$ among the components in the set S_α. Here, we consider $\mathbf{x}$ such that $x_i = d_i/r_i$ for all i. We also define $S_\alpha^j = \left\{ i \in S_\alpha,\, \frac{d_i}{r_i} \geq \left(\frac{d}{r}\right)_{(j)}^{S_\alpha} \right\}$.

Theorem 6.2.6 (Optimal policy)

(a) The optimal order Q^* is solution of the mixed integer programming problem:

$$\begin{array}{lll} \max & (s-c)\, Q + (p-s) \left(\phi + \dfrac{1}{N_\alpha} \displaystyle\sum_{k=1}^{N} \psi_k \right) & \\ s.t. & \phi + \psi_k \leq d_k, & \forall k, \\ & \phi + \psi_k - r_k\, Q \leq 0, & \forall k, \\ & \psi_k \leq 0, & \forall k, \\ & Q \in \mathcal{Z}^+. & \end{array} \tag{6.35}$$

(b) If the set of worst-case scenarios S_α is given, if we relax the integrality constraint on

Q^*, Q^* verifies:

$$Q^* = \left(\frac{d}{r}\right)^{S_\alpha}_{(j)} \quad \textit{with } j \textit{ s.t.} \sum_{i \in S_\alpha^{j+1}} r_i \leq \frac{t}{m+t} N_\alpha < \sum_{i \in S_\alpha^j} r_i. \tag{6.36}$$

Proof: (a) follows from Theorem 3.2.1. The proof of (b) is similar to the proof of Theorem 6.2.1, where we consider the subintervals $[(d_i/r_i)^{S_\alpha}_{(j)}, (d_i/r_i)^{S_\alpha}_{(j+1)})$ and study the slope in Q of the sample profit. □

Note that if $r_i = 1$ for all i, S_α is the set of scenarios corresponding to the N_α lowest demands, and the condition in (b) yields again $j = \left\lceil \frac{m}{m+t} N_\alpha \right\rceil$, as in Theorem 6.2.1.

Remark: The formula in (b) is much more intuitive than the optimal order obtained when only the first two moments of the demand and the exact probability ρ of each unit being good are known [48]:

$$Q^0 = \frac{1}{\rho} \left\{ \mu - \frac{\overline{\rho}}{2} + \frac{1}{2} \left(\sqrt{\frac{m}{t}} - \sqrt{\frac{t}{m}} \right) \sqrt{\mu^2 + \sigma^2 - \left(\frac{\overline{\rho}}{2} - \mu \right)^2} \right\}, \tag{6.37}$$

where $\overline{\rho} = 1 - \rho$.

The model with multiple products

This case is also referred to in the literature as "stochastic product mix". It is an extension of the newsboy problem to multiple items in presence of a budget constraint. We define for item $i = 1, \ldots, n$:

$c_i > 0$:	the unit cost,
$p_i = c_i\,(1 + m_i)$:	the unit selling price,
$s_i = (1 - t_i)\,c_i$:	the unit salvage price,
Q_i:	the order quantity,
D_i:	the random demand.

The budget constraint is:

$$\sum_{i=1}^{n} c_i\, Q_i \leq B, \tag{6.38}$$

where B is the total budget. The random profit can then be written as:

$$\pi(\mathbf{Q}, \mathbf{D}) = \sum_{i=1}^{n} c_i \left[m_i\, Q_i + (m_i + t_i)\, \min(0, D_i - Q_i) \right]. \tag{6.39}$$

The data observed in realization k, $k = 1, \ldots, N$, is $(d_1^k, \ldots, d_n^k)$, the demand for each item. We consider the following problem:

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} c_i\, m_i\, Q_i + \frac{1}{N_\alpha} \sum_{k=1}^{N_\alpha} \left(\sum_{i=1}^{n} c_i\, (m_i + t_i) \min(0, d_i^{\cdot} - Q_i) \right)_{(k)} \\
\text{s.t.} \quad & \sum_{i=1}^{n} c_i\, Q_i \leq B, \\
& Q_i \in \mathcal{Z}^+, \ \forall i.
\end{aligned}
\tag{6.40}
$$

Theorem 6.2.7 (Optimal policy)

(a) The optimal orders $\mathbf{Q}^*$ *are solutions of the mixed integer programming problem:*

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} c_i\, m_i\, Q_i + \phi + \frac{1}{N_\alpha} \sum_{k=1}^{N} \psi_k \\
s.t. \quad & \sum_{i=1}^{n} c_i\, Q_i \leq B, \\
& \phi + \psi_k - \sum_{i=1}^{n} c_i\, (m_i + t_i) Z_i^k \leq 0, \quad && \forall k, \\
& Z_i^k + Q_i \leq d_i^k, && \forall i, k, \\
& Z_i^k \leq 0, \ \psi_k \leq 0, \ Q_i \in \mathcal{Z}^+, && \forall i, k.
\end{aligned}
\tag{6.41}
$$

(b) If we relax the integrality constraint on $\mathbf{Q}$*, we can separate the multi-item problem into single-item subproblems with a new markup factor* $m_i' = m_i - \lambda$ *and a new discount factor* $t_i' = t_i + \lambda$ *for any item* $i = 1, \ldots, n$*, where* $\lambda \geq 0$ *is the optimal Lagrangian multiplier for the budget constraint. For any* i*:*

$$
Q_i^* = (d_i^{\cdot})_{(j_i)} \ \textit{with}\ j_i = \left\lceil \frac{m_i - \lambda}{m_i + t_i}\, N_\alpha \right\rceil, \tag{6.42}
$$

i.e., the optimal order for item i *is equal to its* $\left\lceil \frac{m_i - \lambda}{m_i + t_i}\, N_\alpha \right\rceil$*-th smallest historical demand.*

Proof: (a) follows from Theorem 3.2.1. Moreover, when the integrality constraint on $\mathbf{Q}$ is relaxed, the problem becomes a linear programming problem and we can dualize the budget constraint by assigning a Lagrangian multiplier λ to it without changing the optimal cost of the problem (when λ is selected optimally). The stochastic mix problem becomes then separable in the items, and we can use the results of Section 6.2.1. This proves (b). □

The model with random demand at the salvage value

Finally, we consider the case where the demand at the salvage value V is random. All the other assumptions remain as in Section 6.2.1. The random profit is now:

$$
\pi(Q, D, V) = (p - c)\, Q + \min[p \min(0, D - Q) + s\, V, (p - s) \min(0, D - Q)]. \tag{6.43}
$$

The data in realization k is here (d_k, v_k), where d_k is the demand at the beginning of the period and v_k is the demand at the end. These demands might be correlated. An appealing feature of the proposed approach is that, by building directly upon the data sample, it incorporates such dependency without requiring the estimation of key parameters such as the covariance, since the realizations of the data will capture the correlation.

Since the trimmed profit is piecewise linear with breakpoints at the (d_j), which are integer, we can relax the integrality constraint on Q^* in the problem formulation.

Theorem 6.2.8 (Optimal policy) *The optimal order Q^* is solution of:*

$$\begin{array}{lll} \max & (p-c)Q + \phi + \dfrac{1}{N_\alpha}\displaystyle\sum_{k=1}^{N} \psi_k & \\ s.t. & \phi + \psi_k - p\, Z_k \leq s\, v_k, & \forall k, \\ & \phi + \psi_k - (p-s)\, Z_k \leq 0, & \forall k, \\ & Z_k + Q \leq d_k, & \forall k, \\ & \psi_k \leq 0,\ Z_k \leq 0, & \forall k, \\ & Q \geq 0. & \end{array} \tag{6.44}$$

Proof: This is a straightforward application of Theorem 3.2.1. □

The optimal policy when the first few moments are known can also be obtained by solving a linear programming problem on the worst-case probabilities [36], when the demands belong to a known countable set. However, the worst-case approach is quite conservative since the probabilities it yields can be very different from the actual distribution, and unrealistic given the historical data at our disposal. (Another drawback is that its extension to a multi-period setting relies on recursive equations as in dynamic programming, and therefore suffers from the same dimensionality problems.) An appealing feature of the methodology proposed here is that the optimal policy incorporates the information revealed by the data sample.

6.2.3 Computational Experiments

Example of the classical newsvendor problem

In this section, we apply the proposed methodology to the classical newsvendor problem described in Section 6.2.1 and seek to gain some insight into the following questions:

- does the data-driven approach lead to high-quality results, as compared to the optimal policy computed with the accurate demand distribution?

- does the data-driven approach outperform Scarf's approach (i.e. the optimal policy computed when only the first two moments are known)?
- how sensitive is the data-driven approach to the choice of the trimming factor?
- what is the influence of the markup and discount factors?
- how does the standard deviation of the demand affect the results?
- what happens if the demand is not i.i.d.?

We will consider two types of distributions: gamma and lognormal, both with mean and standard deviation equal to 200 items unless specified otherwise. (Throughout the simulations, we will round down the realizations of the random variables, to have integer demands.) We study these distributions because we feel that positively skewed distributions depict accurately the situation in practice, where the "bulk" of the demand (repeat buyers, who for instance buy the newspaper most of the time) is followed by a long tail (one-time buyers who, say, will buy the newspaper only if there is something that interests them in the news). We generate $N+1$ values of the demand with $N = 50$ over $1,000$ iterations. Each time, the first N values are taken to be the historical data and are used to compute the optimal order quantity for the $(N+1)$st scenario. At mean, standard deviation and type of distribution given, this data sample of size $1,000 \cdot (N+1)$ is generated only once at the beginning of the simulation, so that the effects of the parameters on the expected revenue are measured over exactly the same data.

Influence of N_α on performance

Figures 6-1 to 6-3 show how the sample average revenue obtained in the data-driven, robust approach evolves as a function of N_α for different values of the markup and discount factors m and t. We consider 3 cases: in Case 1, we take $m = 0.4$, $t = 0.4$, in Case 2, $m = 0.2$, $t = 0.6$, and in Case 3, $m = 0.6$, $t = 0.2$. The sample average is taken over the $1,000$ $(N+1)$st scenarios. Since we compute the expected revenue over the same data sample for all N_α, and since $Q = d_{(j)}$ for some j, plateaux appear in the expected revenue when changing N_α does not change the index j determined by Theorem 6.2.1. Throughout this section, the results obtained with the gamma (resp. lognormal) distribution are shown on the left (resp. right) panel. We take $c = 10$.

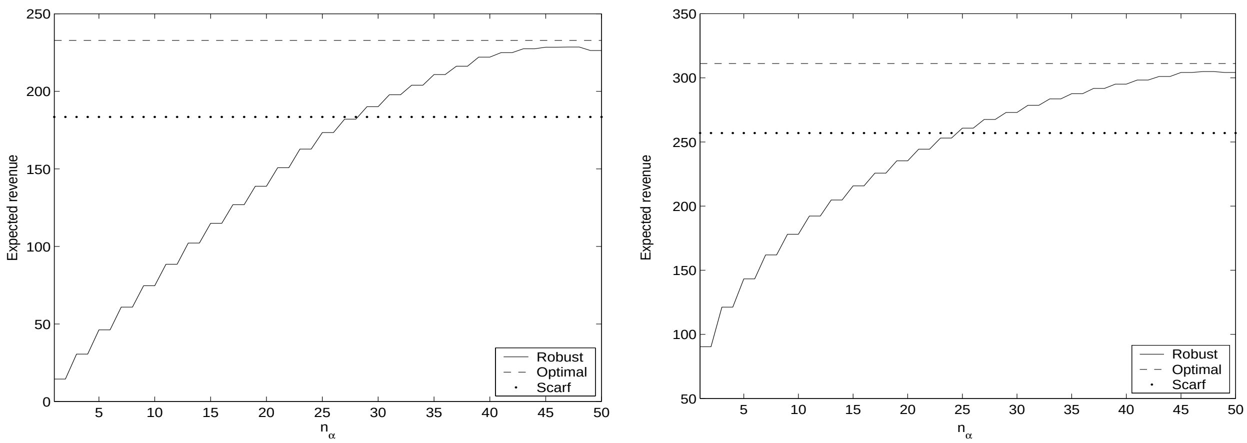

Figure 6-1: Influence of N_α, Case 1.

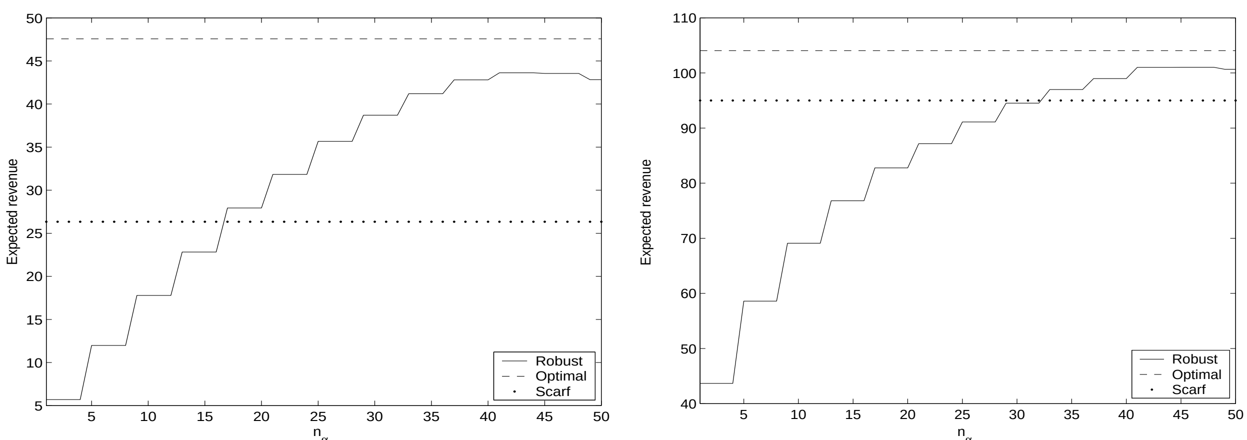

Figure 6-2: Influence of N_α, Case 2.

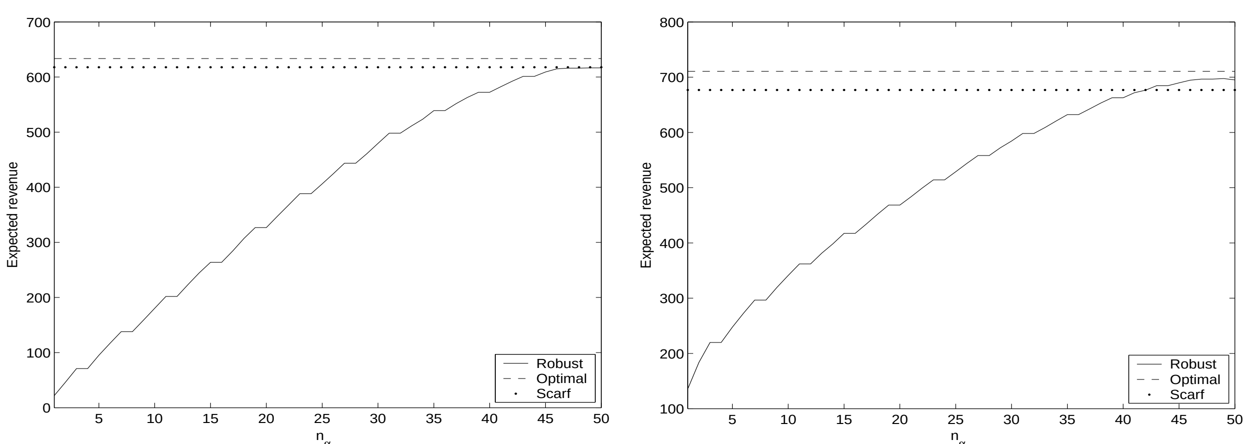

Figure 6-3: Influence of N_α, Case 3.

We notice that the curve of the expected revenue obtained in the robust approach (i.e. where the optimal order is computed by optimizing the trimmed revenue) levels off as N_α tends towards N, suggesting a very small performance loss, as measured by the difference between the expected revenues. The curve labeled "optimal" is obtained by optimizing the expected revenue for the accurate demand distribution (gamma or lognormal). The robust and optimal approaches yield similar results for trimming factors up to $10-20\%$, depending on the values of the markup and discount factors. The curve labeled "Scarf" shows the expected revenue when Scarf's policy - using only the first two moments - is implemented. The mean and standard deviation that are necessary to implement Scarf's ordering policy are computed from the data sample available at each iteration. The data-driven approach leads to a substantive performance gain as compared to Scarf's approach when $t \geq m$, which corresponds to a perishable product whose value deteriorates fast (the unit profit incurred by a sale at the beginning of the period is smaller than the unit loss incurred by a sale at the salvage value at the end of the period). For $m > t$, Scarf's approach and the data-driven methodology yields similar results when the trimming factor is chosen appropriately. The figures with gamma and lognormal distributions exhibit the same qualitative trends, suggesting that the actual type of the distribution is less important for the performance of the robust approach than the markup and discount factors.

Influence of the markup and discount factors

Figures 6-4 to 6-6 show in further detail how the markup and discount factors affect the performance of the robust, data-driven approach, when we take $\alpha = 0.15$ (i.e., $N_\alpha = 42$, since $N = 50$.) In Figure 6-5, resp. 6-6, we consider the ratio $t/m = 2$, resp. $t/m = 0.5$.

Again, the figures show the same qualitative trends both for the gamma and the lognormal distributions, and high-quality results from the robust approach as compared to the optimal approach. This is very encouraging since in practice we would not know the actual distribution and therefore we would not be able to compute the optimal expected revenue. The data-driven approach seems to have a significant advantage over Scarf's approach when $t \geq m$. At t/m given, the expected revenues (in the robust, optimal and worst-case approaches) appear to be linear in the markup cost m, although the actual slopes vary.

Influence of the standard deviation

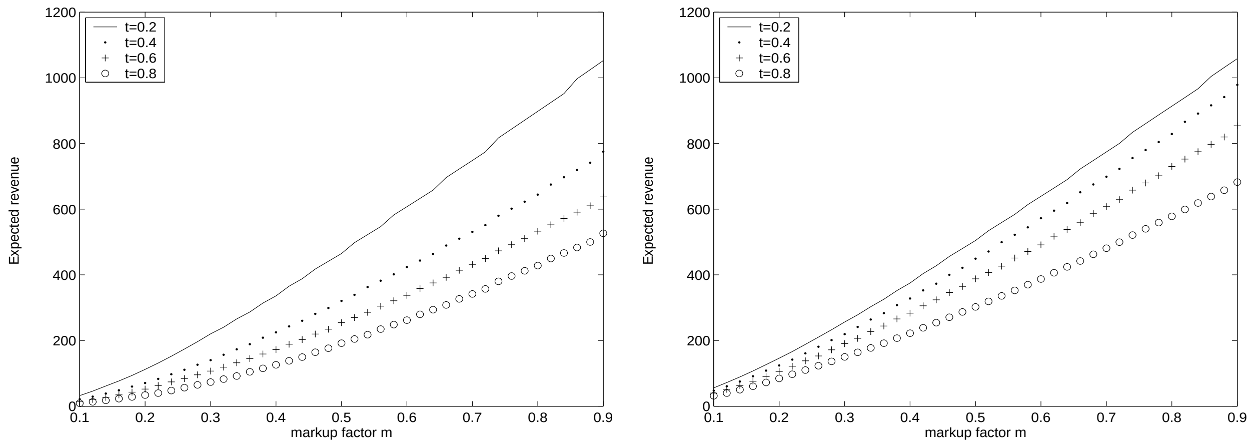

Figure 6-4: Influence of the markup and discount factors.

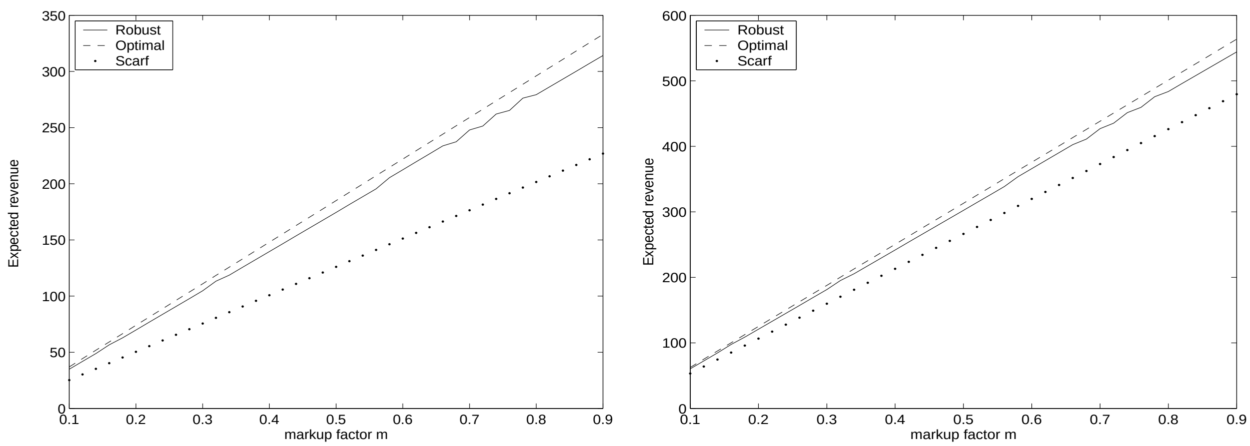

Figure 6-5: Influence of the markup and discount factors, Ratio 1.

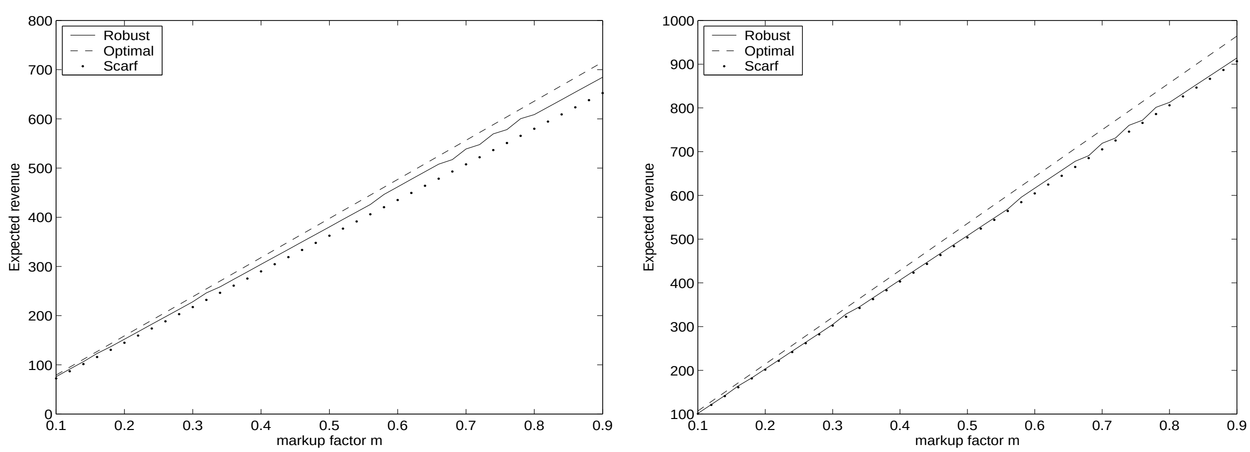

Figure 6-6: Influence of the markup and discount factors, Ratio 2.

Here we study the impact of the standard deviation on the performance of the system, at mean given. We take $m = 0.4$, $t = 0.6$. (As before, the standard deviation to be used in Scarf's ordering policy is estimated from the past realizations of the data.) The robust and optimal approaches behave similarly, and the performance of Scarf's policy is adversely affected by the increasing standard deviation.

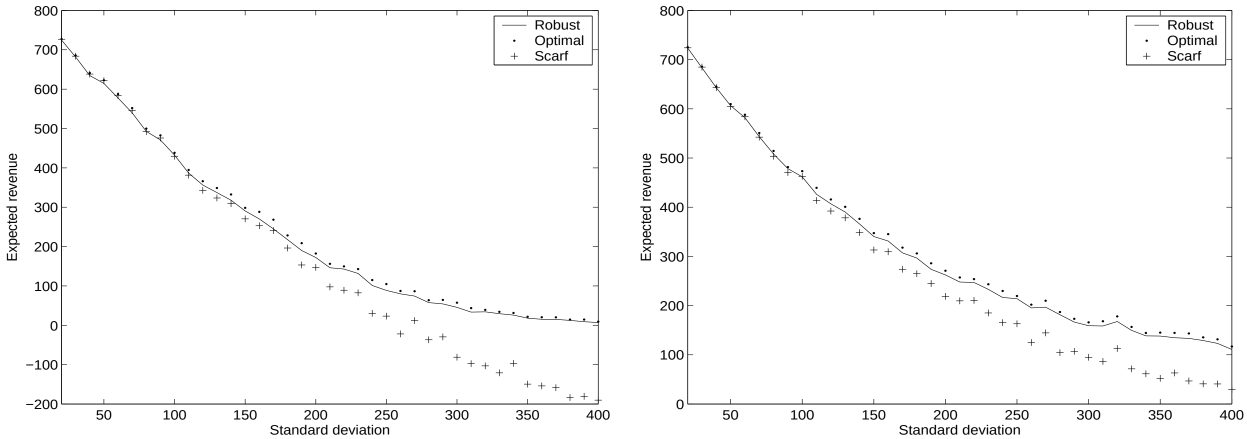

Figure 6-7: Influence of the standard deviation.

Influence of demand distributions that are not i.i.d.

Here the distributions are still gamma or lognormal, but this time the mean is random, and the standard deviation remains equal to the mean. In this example the mean follows a gaussian distribution with mean 200 and standard deviation 40. The ratio t/m is equal to 2.

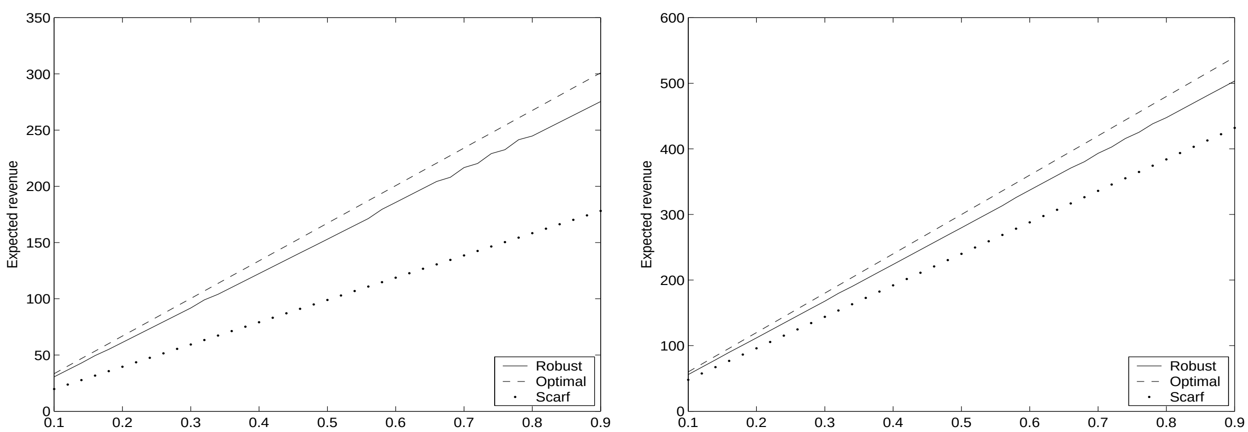

Figure 6-8: Influence of demand distributions that are not i.i.d.

It appears that the data-driven approach is robust to changes in the parameters of the distribution and performs well despite this additional randomness. Scarf's policy is not as robust to such changes, although the performance loss is more striking for the gamma distribution than for the lognormal one.

Example of the newsvendor problem with random demand at the salvage value

In this section, we apply the data-driven approach to the newsvendor problem with random demand at the salvage value, as described in Section 6.2.2. Of particular interest is the impact on performance of a possible correlation between the demands. We will also study the influence of the standard deviation of the demand at the full price.

We will consider three policies:

- the robust policy, obtained in the data-driven approach,
- the optimal policy computed under the (wrong) assumption that both demands are uncorrelated and Gaussian (those assumptions are often encountered in practice), with mean and standard deviation estimated from the available data,
- Scarf's policy obtained under the (wrong) assumption of infinite demand at the salvage value, when only the first two moments of the demand at the full price are known. (In other words, the newsvendor assumes, perhaps erroneously, that the salvage price is so low that the remaining items will always be sold.)

For each of 50 test cases, we generate 120 observations that will be used as follows: the first 20 observations represent the historical data ($N = 20$), used to find the optimal policy. Then the expected revenue for that order will be computed over the 100 remaining observations, each representing a possible value of the demand in the coming time period. This represents a total of $5,000$ test cases. The distributions we consider are (a) gamma for the demand at the full price, and a linear combination of the demand at the full price and another gamma distribution for the demand at the salvage value, representing different motivations of the buyers (left panel of the figures below) (b) lognormal for the demand at the full price, and a linear combination of the demand at the full price and another lognormal distribution for the demand at the salvage value (right panel). The coefficients of the linear combination depend on the desired correlation.

To select N_α, we start at $N_\alpha = N$ ($\alpha = 0$) for each of the 50 test cases, solve the data-driven problem and reiterate if necessary, decreasing N_α until a prespecified number of scenarios (βN, with $\beta = 0.9$) have a revenue greater than the trimmed mean given by the robust approach. This corresponds to a Value-at-Risk requirement. We take $c = 10$, $m = 0.2$, $t = 0.7$, means and standard deviations equal to 200.

Influence of the correlation

Figure 6-9 shows the influence of the correlation on the expected revenue for the three policies. The correlation varies from -0.9 to 0.9. The gamma and the lognormal cases present the same qualitative features:

1. when the correlation is negative, the expected revenue for the three approaches increases with the correlation, and the three approaches perform somewhat similarly,

2. when the correlation is positive, the expected revenue in the robust approach shows little dependence in the correlation. The robust approach performs significantly better than the other two approaches.

Figure 6-10 shows the influence of the correlation on the mean and median of N_α, when N_α is selected to satisfy the Value-at-Risk requirement. It appears that the sign of the correlation between the two demands plays a key role in the mean and median value of N_α: if the demands are negatively correlated, the newsvendor will be more conservative in order to guarantee performance, while if the demands are positively correlated, the newsvendor does not have to trim as much. The type of distribution does not significantly affect the choice of N_α.

Influence of the standard deviation

Figure 6-11 shows the influence of the standard deviation on the expected revenue in the three approaches. The data-driven policy appears to perform better than the other two strategies as the standard deviation increases.

6.2.4 Summary of Results

In this section, we have applied the data-driven approach to the classical newsvendor problem and its extensions when the probability distributions of the underlying sources of un-

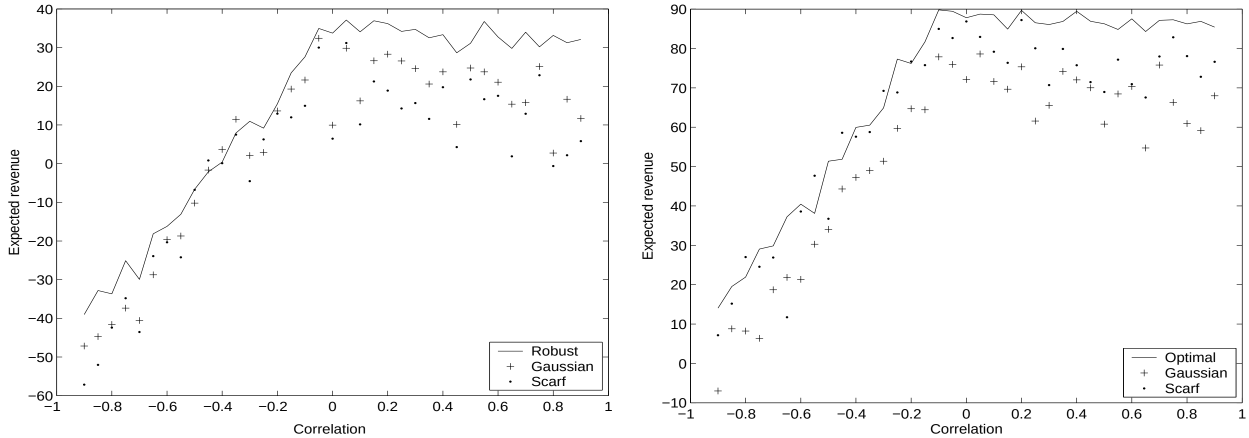

Figure 6-9: Influence of the correlation on the expected revenue.

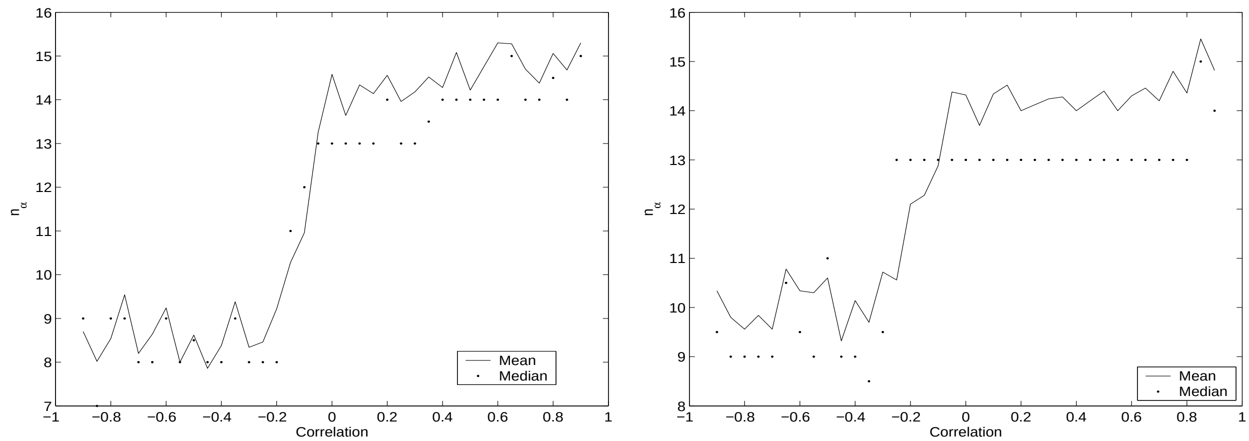

Figure 6-10: Influence of the correlation on N_α.

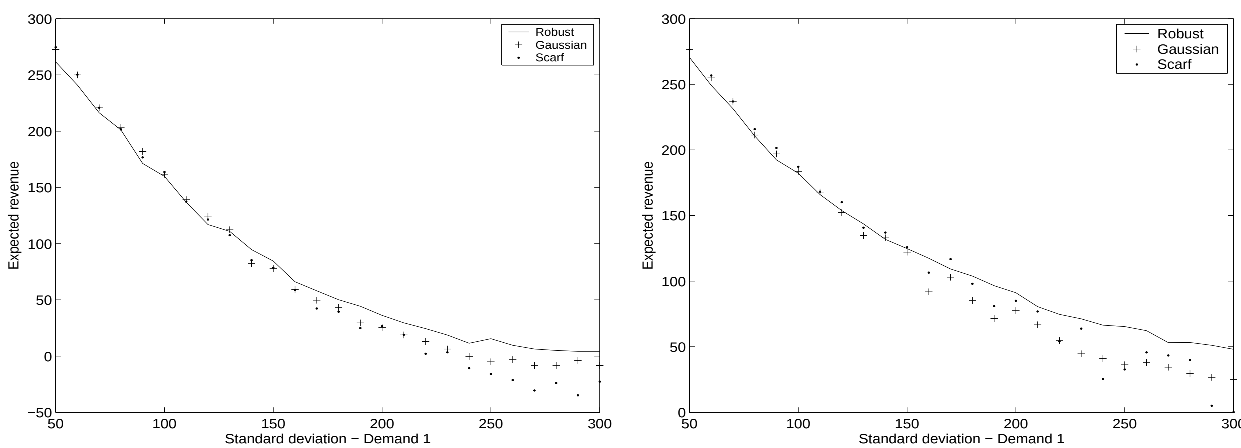

Figure 6-11: Influence of the standard deviation on the expected revenue.

certainty are not accurately known. The corresponding robust formulations are linear or mixed integer programming problems, and therefore can be solved efficiently. This model also allows us to gain a deeper insight into the impact of risk aversion on the optimal orders. We have derived closed-form expressions for several models, which highlight the role of the trimming factor and of the profit parameters on the optimal solution. They generally correspond to a well-chosen historical realization of the demand. Finally, we have presented encouraging computational results. In particular, the numerical evidence suggests that:

1. a small trimming factor (up to about 20%) does not significantly affect the expected revenue,
2. the data-driven approach performs better than Scarf's policy, which uses only the first two moments,
3. the benefit of using the data-driven methodology rather than Scarf's formula increases when the demand is more volatile, i.e., the standard deviation increases,
4. the data-driven approach is robust to non-i.i.d. distributions,
5. using such a framework that incorporates demand correlation increases profits, sometimes substantially.

Therefore, the data-driven robust approach applied to the newsvendor problem and its extensions opens a promising area of research.

6.3 Airline Revenue Management

In this section, we apply the robust data-driven framework developed in Chapter 3 to airline revenue management. Specifically, we consider the problems of finding optimal seat allocations (Section 6.3.1) and admission policies (Section 6.3.2) for a network subject to random demand over time.

6.3.1 Robust Seat Allocation

Notations

A booking request is identified by its origin o, destination d and fare class f, which is summarized in the notation odf. We have N past realizations of the random demand at our disposal, where each scenario k consists of the vector $\mathbf{d^k}$ of past demands over time in that scenario, where demand for booking type *odf* at time t in scenario k noted d_{odf}^{kt}. Let N_α be the number of scenarios we keep after trimming. We define:

T: the time horizon,

N_{odf}: the total number of possible *odf*,

S_l: the set of *odf* using leg l in the network,

d_{odf}^{kt}: the demand for booking of type *odf* at time $t = 1, \ldots, T$ in scenario k,

f_{odf}^{t}: the fare for booking of type *odf* at time $t = 1, \ldots, T$,

x_{odf}^{t}: the number of seats reserved for booking requests of type *odf* at time t.

The Model

The nominal problem can be formulated as:

$$\begin{aligned} \max \quad & \sum_{t=1}^{T} \sum_{odf=1}^{N_{odf}} f_{odf}^{t}\, x_{odf}^{t} \\ \text{s.t.} \quad & \sum_{t=1}^{T} \sum_{odf \in S_l} x_{odf}^{t} \le C_l, \quad \forall l, \\ & 0 \le x_{odf}^{t} \le d_{odf}^{t}, \qquad \forall odf,\ \forall t. \end{aligned} \tag{6.45}$$

We have the following theorem.

Theorem 6.3.1 (The robust problem) *The robust counterpart to Problem (6.45) is:*

$$\begin{aligned} \max \quad & \phi + \frac{1}{N_\alpha} \sum_{k=1}^{N} \psi_k \\ s.t. \quad & \phi + \psi_k - \sum_{t=1}^{T} \sum_{odf=1}^{N_{odf}} f_{odf}^{t}\, y_{odf}^{kt} \le 0, \quad \forall k, \\ & y_{odf}^{kt} - x_{odf}^{t} \le 0,\ y_{odf}^{k} \le d_{odf}^{kt}, \qquad \forall odf,\ t,\ k, \\ & \sum_{t=1}^{T} \sum_{odf \in S_l} x_{odf}^{t} \le C_l, \qquad \forall l, \\ & \mathbf{x} \ge \mathbf{0},\ \psi \le \mathbf{0}. \end{aligned} \tag{6.46}$$

Proof: Is an immediate application of Theorem 3.3.3. □

Let π^* be the optimal dual vector corresponding to the capacity constraints of (6.46) and for any vector $\mathbf{y}$, let $[\mathbf{y}]_{S_\alpha}^{(k)}$ be the k-th greatest y^k among the N_α worst cases.

Theorem 6.3.2 (The robust seat allocation)

(a) If $x^{*t}_{odf} > 0$ *and* $\left(\sum_{l\in odf} \pi^*_l \cdot \frac{N_\alpha}{f^t_{odf}}\right)$ *is not an integer, then:*

$$x^{*t}_{odf} = \left[d^{\cdot t}_{odf}\right]^{(k)}_{S_\alpha} \;\; \textit{where } k = \left\lceil \sum_{l\in odf} \pi^*_l \cdot \frac{N_\alpha}{f^t_{odf}} \right\rceil. \tag{6.47}$$

(b) If $\sum_{l\in odf} \pi^*_l > f^t_{odf}$*, then* $x^{*t}_{odf} = 0$.

Proof: Follows from Theorem 3.3.4. □

Remarks:

1. In the nominal model, we have (a) $x^{*t}_{odf} = 0$ if $\sum_{l\in odf} \overline{\pi}^*_l > f^{*t}_{odf}$; and (b) if $x^{*t}_{odf} > 0$ and $\sum_{l\in odf} \overline{\pi}^*_l < f^{*t}_{odf}$ then $x^{*t}_{odf} = \overline{d}^t_{odf}$. Therefore, it appears that, while in the nominal case the sign of $f^{*t}_{odf} - \sum_{l\in odf} \overline{\pi}^*_l$ determines the optimal allocation, in the robust case what matters is the ratio between unit revenue and opportunity costs $\frac{f^t_{odf}}{\sum_{l\in odf} \pi^*_l}$, rather than their difference.

2. Everything else being equal, if the fare for a ticket of type *odf* increases with time, the index k in Eq. (6.47) decreases, and as a result $x^{\cdot t}_{odf}$ increases with time (assuming that $x^{*t}_{odf} > 0$ and $\sum_{l\in odf} \pi^*_l \cdot \frac{N_\alpha}{f^t_{odf}}$ is not an integer). Unsurprisingly, the decision-maker allocates more seats to the *odf* when it generates more revenue. This is also true if the trimming factor increases, i.e., if the decision-maker becomes more conservative he will tend to allocate more seats to *odf* that clearly generate revenue.

3. If two fare classes for the same origin and destination (possibly at different time periods) are allocated seats and verify the nonintegrality condition, the ratio of the probabilities that demands for those classes is not fully met is equal to the inverse of the fare ratio. For instance, under these conditions, if a ticket for a specific *odf* is twice as expensive at time t_2 than t_1, the probability that demand at time t_2 is not fully met is approximately half the probability at time t_1.

6.3.2 Robust Admission Policies

We keep the same notations as in Section 6.3.1, with the difference that x^t_{odf} now represents the decision to admit *odf* at time t. We consider the linear relaxation of the admission problem, so that $x^t_{odf} \in [0, 1]$. For practical purposes, we admit *odf* at time t whenever $x^t_{odf} > 0$. The nominal problem is therefore:

$$
\begin{aligned}
\max \quad & \sum_{t=1}^{T} \sum_{odf=1}^{N_{odf}} f_{odf}^{t}\, d_{odf}^{t}\, x_{odf}^{t} \\
\text{s.t.} \quad & \sum_{t=1}^{T} \sum_{odf \in S_l} d_{odf}^{t}\, x_{odf}^{t} \le C_l, \quad \forall l, \\
& 0 \le x_{odf}^{t} \le 1, \qquad\qquad\qquad \forall odf,\ \forall t.
\end{aligned}
\tag{6.48}
$$

We follow the approach outlined in Section 3.3.3 to build the data-driven problem. Let π^k be the optimal dual vector associated with the capacity constraints in Problem (6.48) in scenario k.

Theorem 6.3.3 (The data-driven approach in airline revenue management)
(a) The optimal admission policy is the solution of:

$$
\begin{aligned}
\max \quad & \phi + \frac{1}{N_\alpha} \sum_{k=1}^{n} \psi_k \\
s.t. \quad & \phi + \psi_k - \sum_{t=1}^{T} \sum_{odf=1}^{N_{odf}} \left(f_{odf}^{t} - \sum_{l \in odf} \pi_l^k \right) d_{odf}^{kt} x_{odf}^{t} \le \sum_{l} \pi_l^k\, C_l, \quad \forall k, \\
& \psi_k \le 0, \qquad \forall k, \\
& 0 \le x_{odf}^{t} \le 1, \qquad \forall\ odf,\ \forall\ t.
\end{aligned}
\tag{6.49}
$$

(b) Let S_α the optimal set of the N_α worst cases. The optimal admission policy is to accept all requests of type odf at time t if:

$$
f_{odf}^{t} \ge \frac{\displaystyle\sum_{k \in S_\alpha} \sum_{l \in odf} \pi_l^k\, d_{odf}^{kt}}{\displaystyle\sum_{k \in S_\alpha} d_{odf}^{kt}},
\tag{6.50}
$$

and reject otherwise.

Proof: Follows from Theorem 3.3.5. □

In particular, the optimal policy is to accept requests of type *odf* at time t if the unit fare at that time is greater than a weighted average of the opportunity costs, where the weights are the historical demands across scenarios.

Example: A single-leg static network

Here, we consider the case of a single-leg static network, where either the capacity is not reached or it is always reached for the same class of travellers, i.e., if $\pi^k > 0$, then $\pi^k = \pi$, where π corresponds to the fare of the last class admitted. Let S_α^+ be the set of scenarios k in S_α for which $\pi^k > 0$. Then, from Eq. (6.50), class i is admitted if:

$$f_i \geq \underbrace{\left(\sum_{k \in S_\alpha^+} d_i^k / \sum_{k \in S_\alpha} d_i^k \right)}_{\leq 1} \pi. \tag{6.51}$$

An important consequence of this result is that a class i with a fare *lower* than the opportunity cost might well be admitted, if the demand for that class is negatively correlated with the saturation of the available capacity. In other words, if demand for that class happens to be low in many of the worst-case scenarios where capacity is reached, we will have $\sum_{k \in S_\alpha^+} d_i^k / \sum_{k \in S_\alpha} d_i^k \ll 1$, and therefore $f_i \geq \left(\sum_{k \in S_\alpha^+} d_i^k / \sum_{k \in S_\alpha} d_i^k \right) \pi$ although $f_i < \pi$. Intuitively, the decision-maker admits this class because if the capacity is not reached, it makes sense to admit as many classes as possible, and if it is, the demand for this class will likely be lower than ordinary, and therefore will not play an important role in meeting the capacity of the aircraft.

Remark: Because both sides of Eq. (6.50) vary with time, some requests in this approach might be accepted at a time period, rejected later and finally accepted again. For instance, consider the case with two scenarios in $\mathcal{S}$, a single leg, two classes and an horizon of three time periods. The fare and the demand of the first class are constant over time and scenarios, equal to 50 and 20, respectively. The fare for the second class is $(10, 15, 30)$ and its demand in scenario 1, resp. 2, is $(1, 20, 50)$, resp. $(10, 10, 10)$. Capacity is 100. It follows that $\pi^1 = 30$ and $\pi^2 = 0$. As a result, the optimal admission policy in the data-driven approach is to always admit class 1 ($50 \geq 20$) and admit class 2 at times 1 and 3, and reject it at time 2 (Theorem 6.3.3 (b) yields $10 \geq 30/11$, $15 < 20$ and $30 \geq 25$). This is counter-intuitive as one would expect that any booking request will be accepted once the price is high enough. In practice, this might indicate that the prices have not been set optimally.

6.3.3 Computational Experiments

In this section, we implement the robust approach on the huh-and-spoke network shown in Figure 6-12. This network has one hub, noted 1, and 8 auxiliary airports, noted $2, \ldots, 9$. Directions of travel on the legs are from left to right on Figure 6-12, and legs are numbered from left to right and from top to bottom (for instance the leg from airport 2 to the hub is leg 1, the leg from airport 3 to the hub is leg 2, etc.) There is only one fare class. The fares

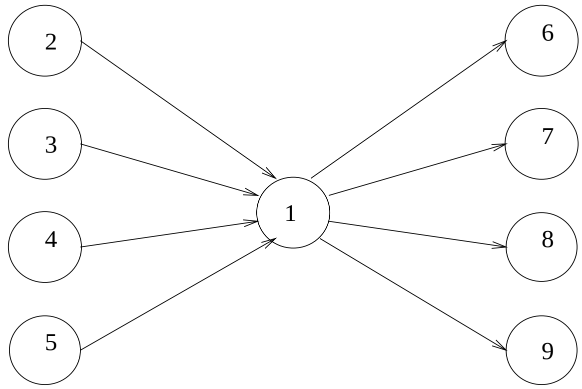

Figure 6-12: The airline network.

for each origin and destination are summarized in Table 6.1.

To From	1	6	7	8	9
1	x	25	30	35	40
2	37	30	35	40	45
3	45	40	50	55	60
4	50	30	40	50	60
5	25	15	20	25	30

Table 6.1: The fares.

Demands are assumed to be i.i.d. with mean 100 and standard deviation 30. We consider two types of distributions: Gaussian and symmetric Bernoulli (binomial). Unless specified otherwise, the left, resp. right, panel of the figures shows the results for the Gaussian, resp. Bernoulli distribution, and capacity on each leg is 350.

Our goal is to find optimal seat allocations in a static environment. In each of 50 iterations,

1. we start with $N = 11$ previous realizations of the demand across the network,
2. we determine the optimal seat allocations,
3. we evaluate the corresponding solution on 50 new realizations of the demand and compute various statistics.

Influence of the trimming factor:

First we analyze the influence of the trimming factor on the optimal solution. The figures below show the mean and standard deviation of the actual revenue (Figure 6-13) and the optimal seat allocations (Figures 6-14 to 6-21) as a function of the number of discarded scenarios $N - N_\alpha$.

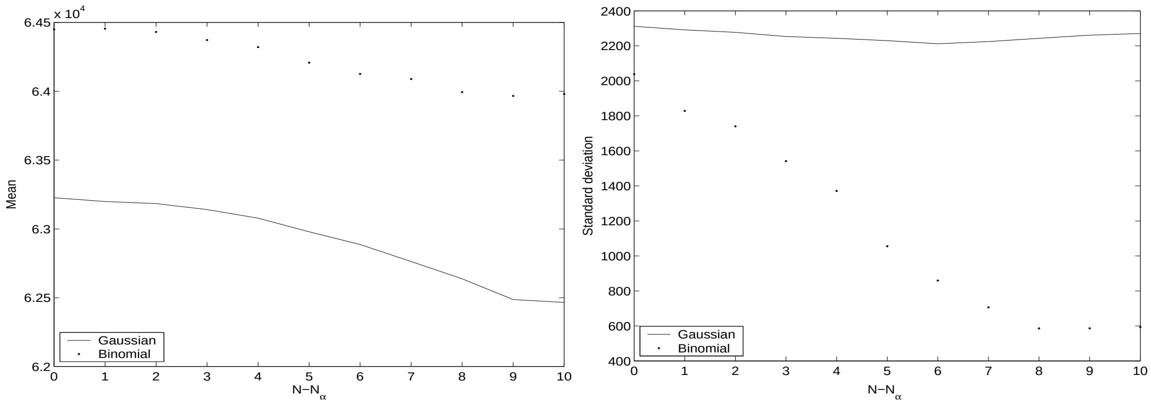

Figure 6-13: Influence of the trimming factor on mean and standard deviation of actual revenue.

In this example, increasing the trimming factor decreases the mean revenue for both distributions. It also significantly decreases the standard deviation of the revenue when the distribution is Bernoulli (binomial). When the distribution is Gaussian, it decreases the standard deviation if the trimming factor is less than 0.6, but increases it otherwise.

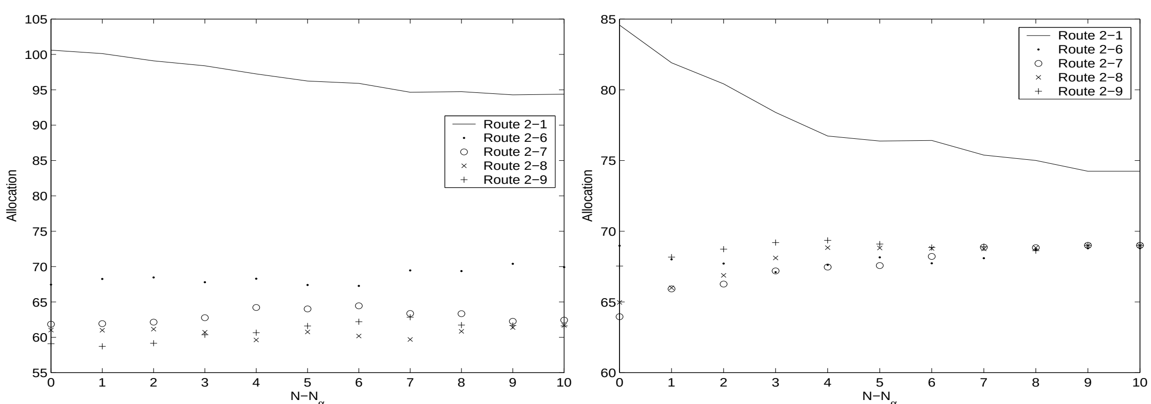

Figure 6-14: Influence of the trimming factor on allocations on leg 1.

The effect of the trimming factor on the actual allocations is qualitatively the same for both distributions, although the actual allocations differ, sometimes significantly. (As an example, Route $2-1$ is allocated between 94 and 101 seats when the distribution is Gaussian, but only between 74 and 85 when it is Bernoulli, see Figure 6-14.) In particular, the number of seats given to each route generally follows the same ranking in both cases. In Figure 6-15, for both distributions, Route $3-1$ receives the highest number of seats, Routes $3-7$, $3-8$ and $3-9$ are assigned comparable quantities, and Route $3-6$ has the lowest allocation.

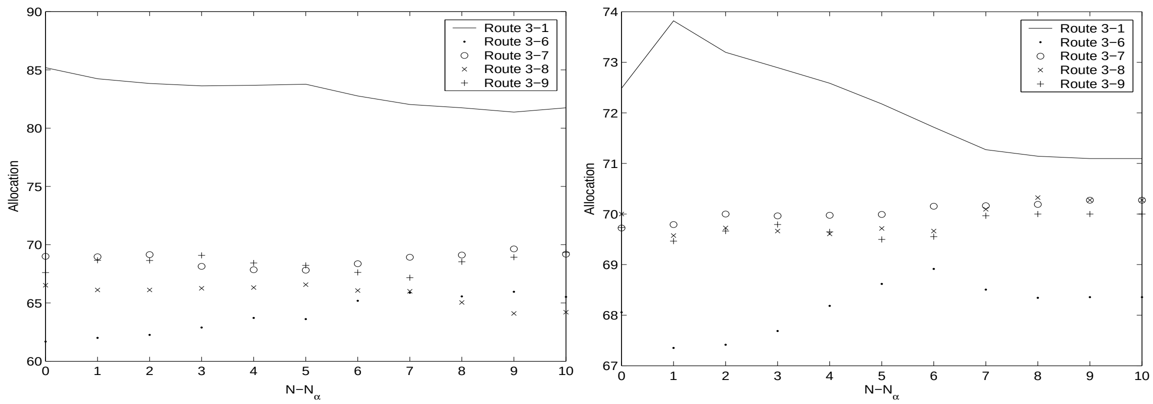

Figure 6-15: Influence of the trimming factor on allocations on leg 2.

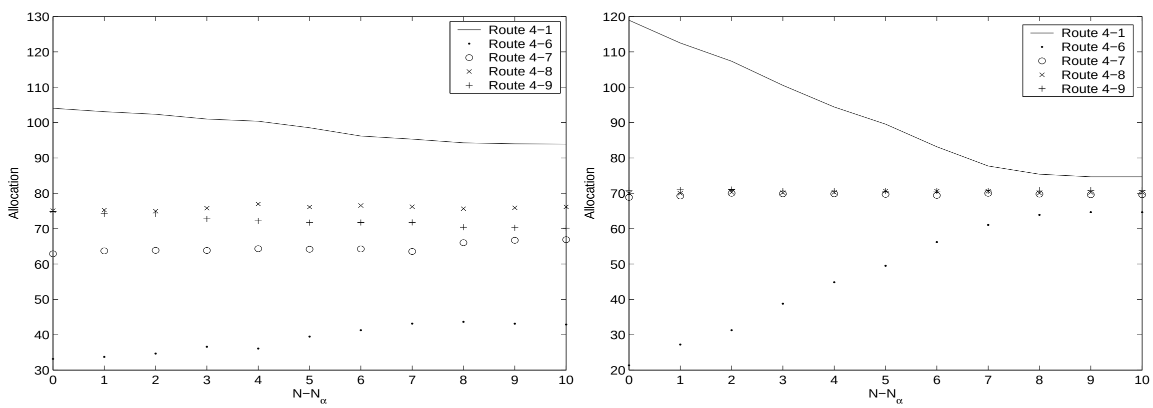

Figure 6-16: Influence of the trimming factor on allocations on leg 3.

We note that, for this fare structure, the number of seats allocated to one-leg routes, i.e., routes that have the hub as their origin or destination, is always nonincreasing with the number of scenarios discarded. Moreover, there is often a substitution effect, where the seats taken from the one-leg routes are re-assigned to a specific two-leg route using that same leg, while the allocations of the remaining origin-destination pairs show little change. This effect is somewhat stronger when the distribution is Bernoulli. Intuitively, as his risk aversion increases, the decision-maker prefers a higher likelihood to sell the seats rather than the opportunity for larger profits.

For instance, on leg 3, Route $4-1$ from Airport 4 to the hub loses seats to Route $4-6$ from Airport 4 to Airport 6 (Figure 6-16). On leg 5, Route $1-6$ loses seats to Route $4-6$

as well (Figure 6-18). Specifically, Routes $4-1$ and $1-6$ combined bring in a total unit revenue of $50+25=75$ units, which is much larger than the unit revenue of 30 units for Route $4-6$, but they are also allocated a number of seats that exceed the mean demand by the risk-neutral decision-maker (about 120 each in the Bernoulli case), while Route $4-6$ receives a far smaller number of seats (about 20). In contrast, the most risk-averse decision-maker in the Bernoulli case assigns about 75, resp. 78, 65, seats to Route $4-1$, resp. $1-6$, $4-6$.

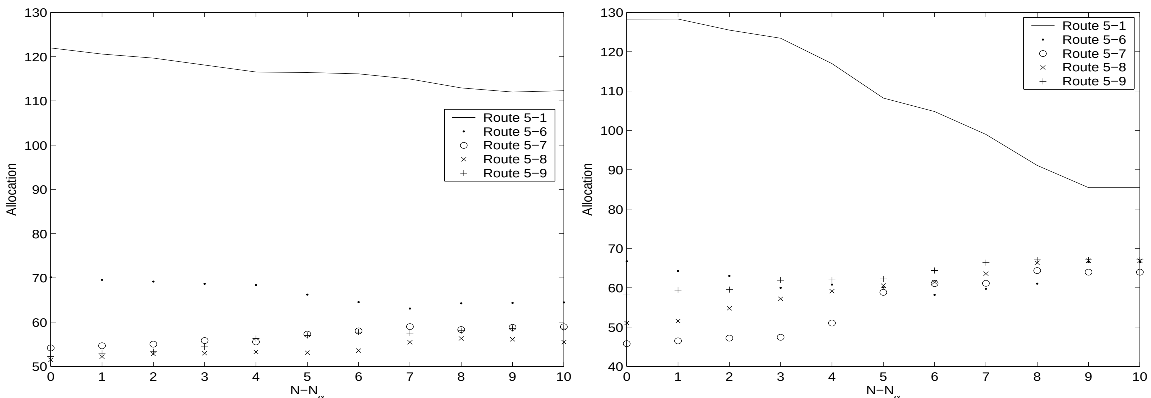

Figure 6-17: Influence of the trimming factor on allocations on leg 4.

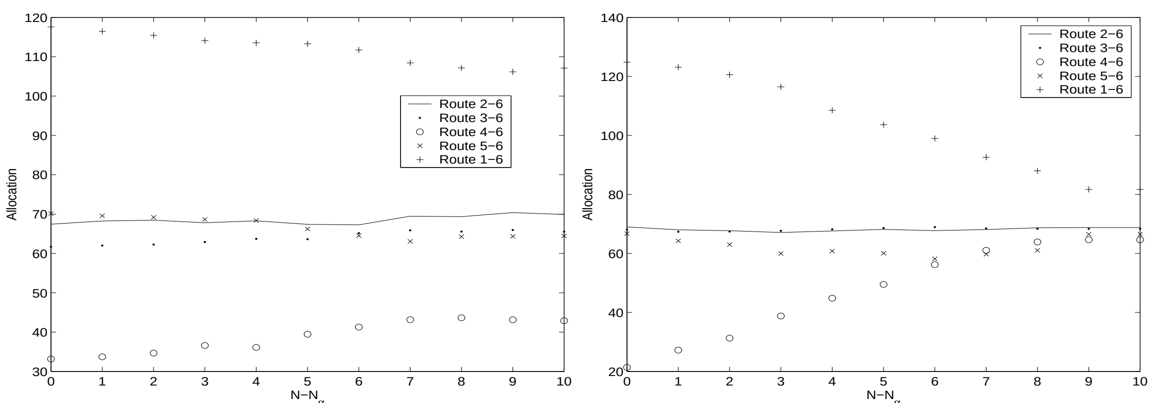

Figure 6-18: Influence of the trimming factor on allocations on leg 5.

Other examples of the same phenomenon can be observed on Routes $5-1$ and $1-7$ compared to Route $5-7$ (Figures 6-17 and 6-19), Routes $5-1$ and $1-8$ compared to $5-8$ (Figures 6-17 and 6-20) and Routes $5-1$ and $1-9$ compared to $5-9$ (Figures 6-17 and 6-21).

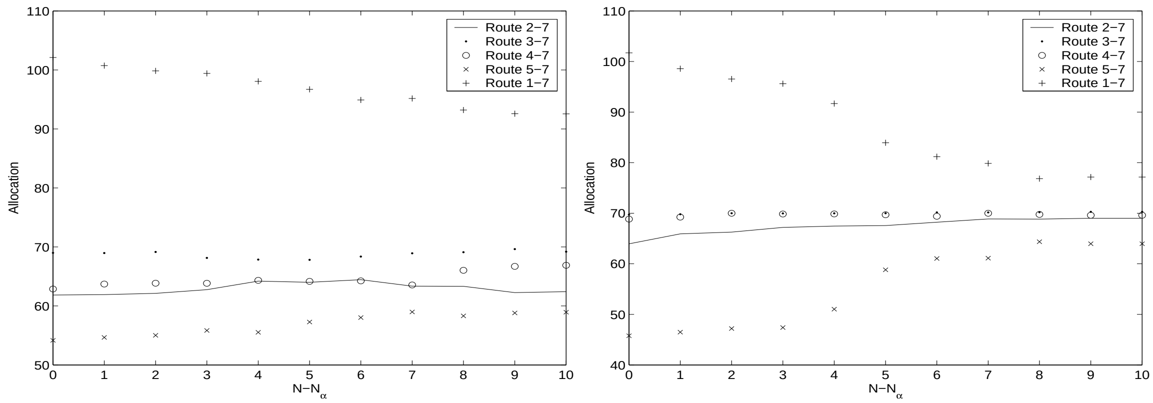

Figure 6-19: Influence of the trimming factor on allocations on leg 6.

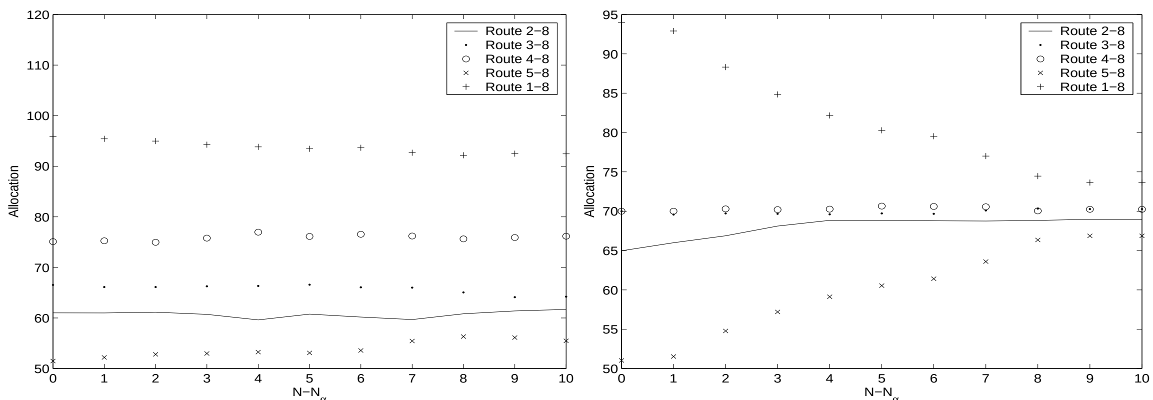

Figure 6-20: Influence of the trimming factor on allocations on leg 7.

Influence of leg capacity:

We also study how leg capacity affects the optimal solution. The figures below show the mean and standard deviation of the actual revenue (Figure 6-22) and the optimal seat allocations (Figures 6-23 to 6-30) as a function of the capacity of leg 1, which varies between 0 and 700. In this case, N_α is taken equal to 9, which corresponds to a trimming factor of about 20%.

Mean and standard deviation exhibit the same qualitative behavior for both distributions (Figure 6-22). Unsurprisingly, increasing the capacity increases the mean revenue, as it allows more demand to be satisfied. We note that the mean revenue plateaus when the capacity of leg 1 reaches a value of about 500 seats, which corresponds to the sum of the mean demands over all routes using that leg. As noted before, the allocations for the different routes approximately follow the same ranking in the Gaussian and Bernoulli cases. We make the following additional observations:

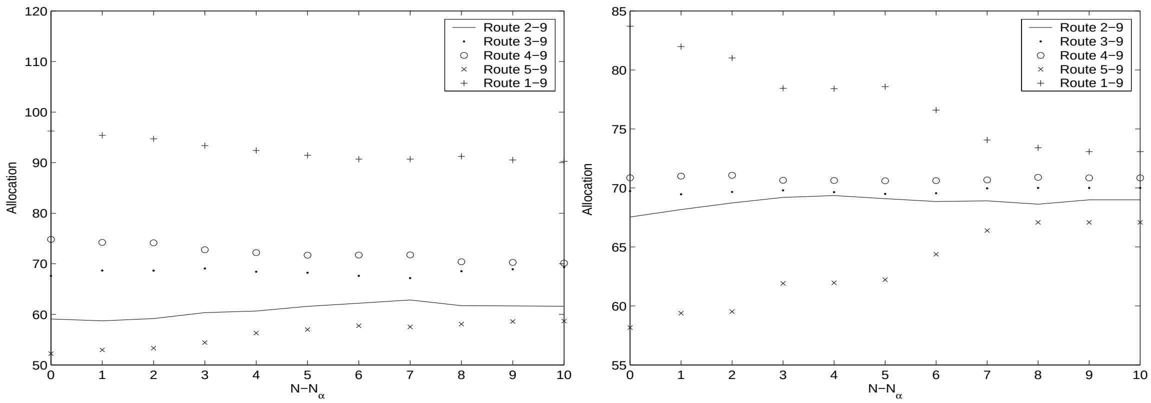

Figure 6-21: Influence of the trimming factor on allocations on leg 8.

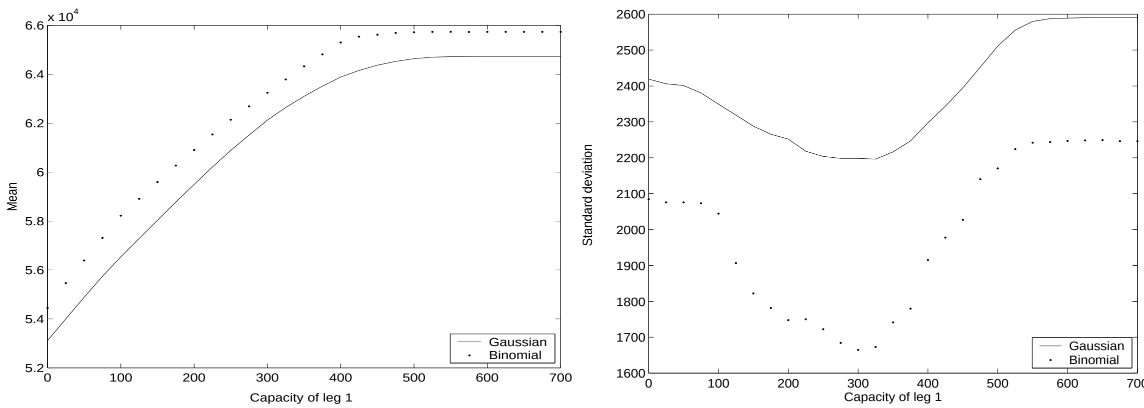

Figure 6-22: Influence of capacity of leg 1 on mean and standard deviation of actual revenue.

- On leg 1 (Figure 6-23), the number of seats assigned to each origin-destination pair is nondecreasing in the capacity, and the highest number of seats is consistently allocated to the single-leg route $2-1$. When the capacity exceeds 500, the number of seats for the two-leg routes $2-6$, $2-7$, $2-8$ and $2-9$ reaches a plateau, as these quantities are also constrained by the capacity of the other leg. Route $2-1$ has no such constraint and therefore the corresponding number of seats continues to grow.

- Legs 2 to 4 (Figures 6-24 to 6-26) are affected by the capacity of leg 1 in an indirect manner, as the passengers on these legs compete for a seat towards the destinations 6 to 9. As the capacity of leg 1 increases, it becomes more lucrative for the decision-maker to let the travellers coming from City 2 go to 6, 7, 8 or 9. This results in a shift in the seat allocations on legs 2 to 4 from two-leg routes to the single-leg routes $3-1$, $4-1$ and $5-1$.

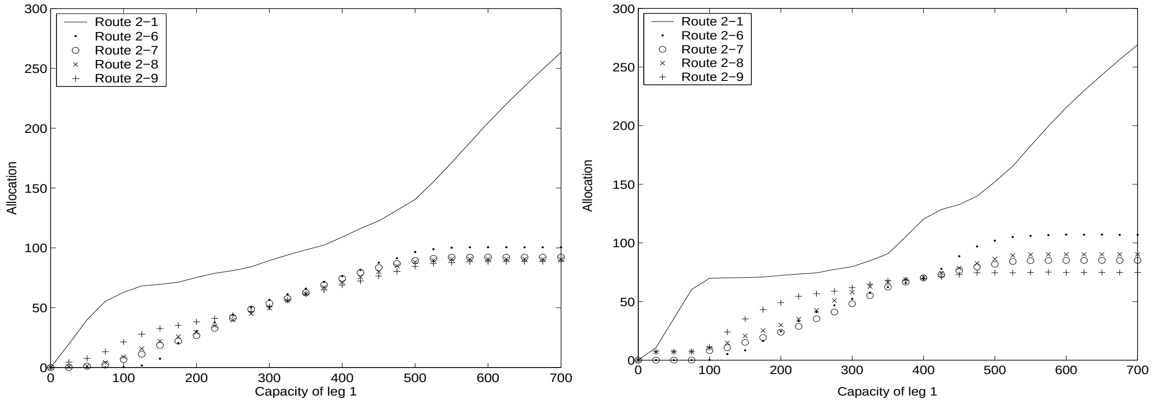

Figure 6-23: Influence of capacity of leg 1 on allocations on leg 1.

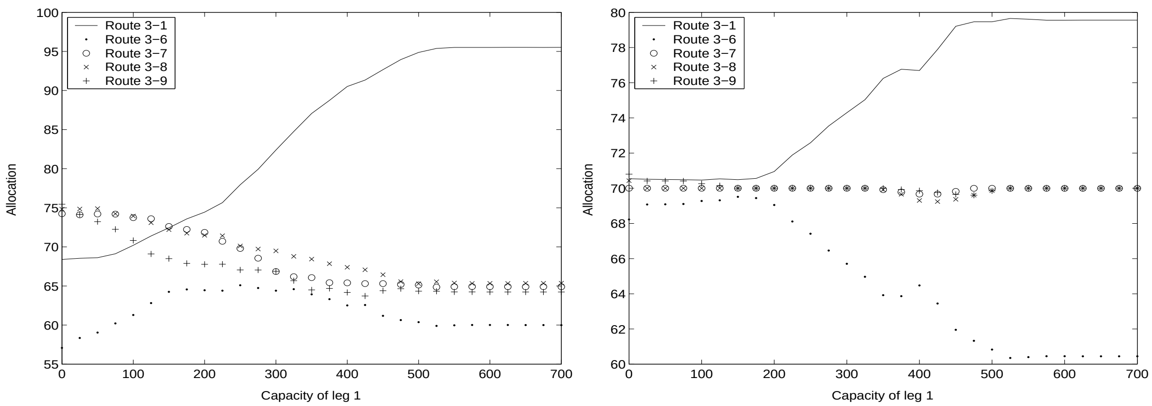

Figure 6-24: Influence of capacity of leg 1 on allocations on leg 2.

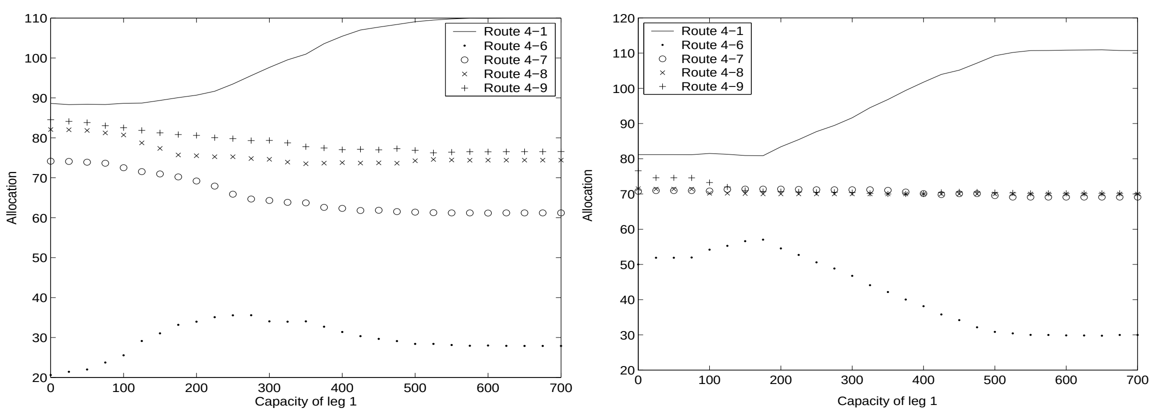

Figure 6-25: Influence of capacity of leg 1 on allocations on leg 3.

- On legs 5 to 8 (Figures 6-27 to 6-30), the routes coming from City 2 are assigned an increasing number of seats, which were previously attributed for the most part to the single-leg routes $1-6$ to $1-9$. After the capacity reaches 500, the allocations remain constant.

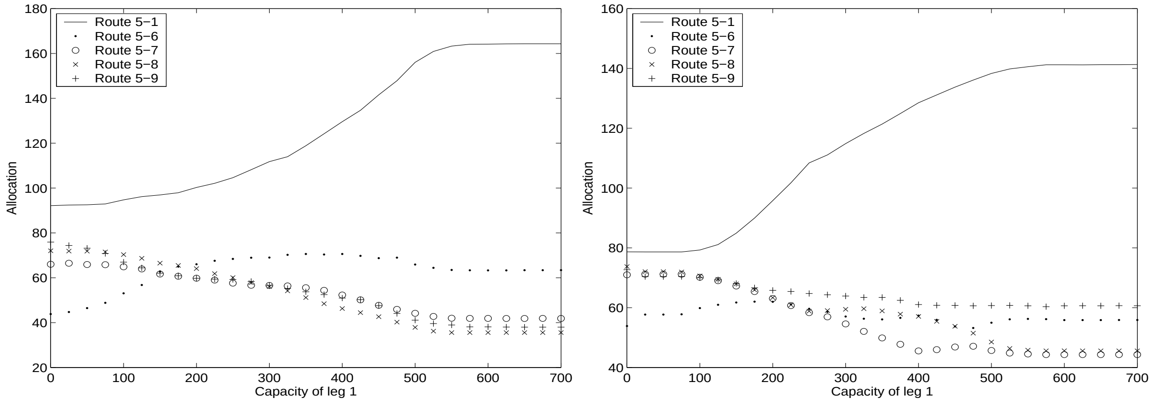

Figure 6-26: Influence of capacity of leg 1 on allocations on leg 4.

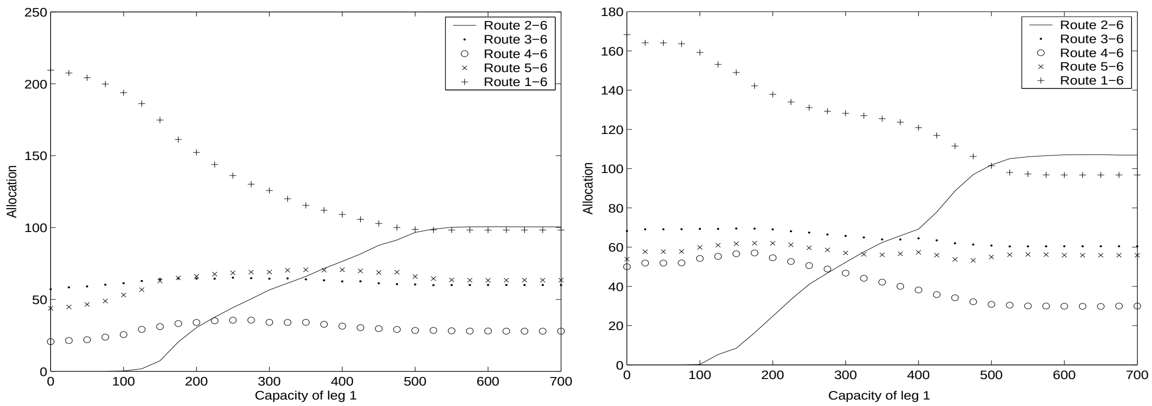

Figure 6-27: Influence of capacity of leg 1 on allocations on leg 5.

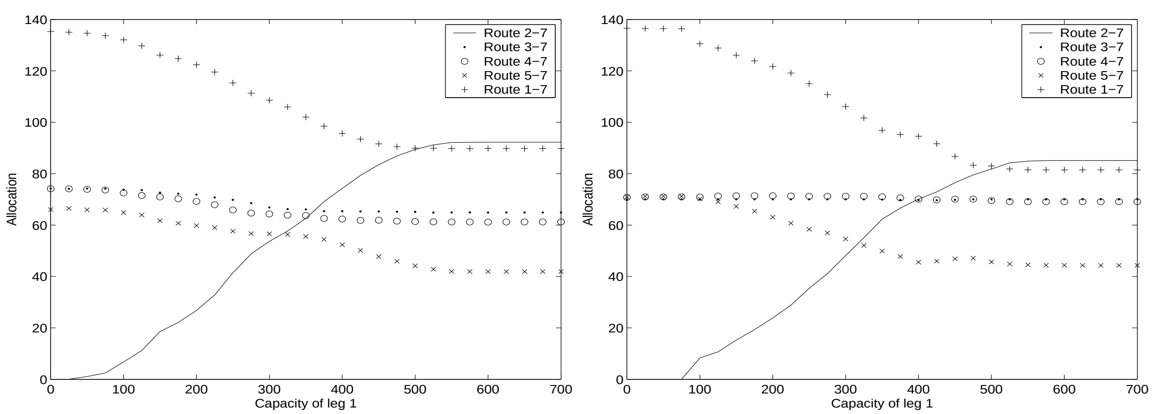

Figure 6-28: Influence of capacity of leg 1 on allocations on leg 6.

Therefore, the data-driven approach appears as a promising tool to incorporate risk aversion in a tractable and insightful manner.

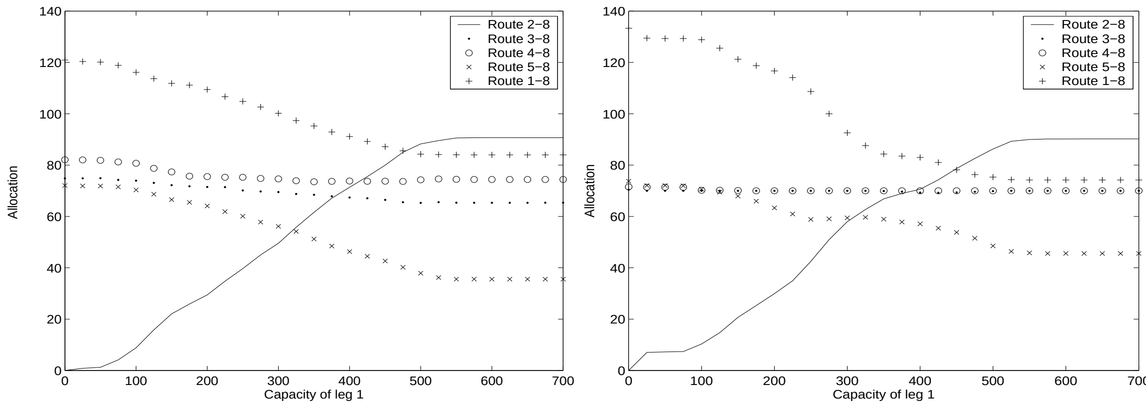

Figure 6-29: Influence of capacity of leg 1 on allocations on leg 7.

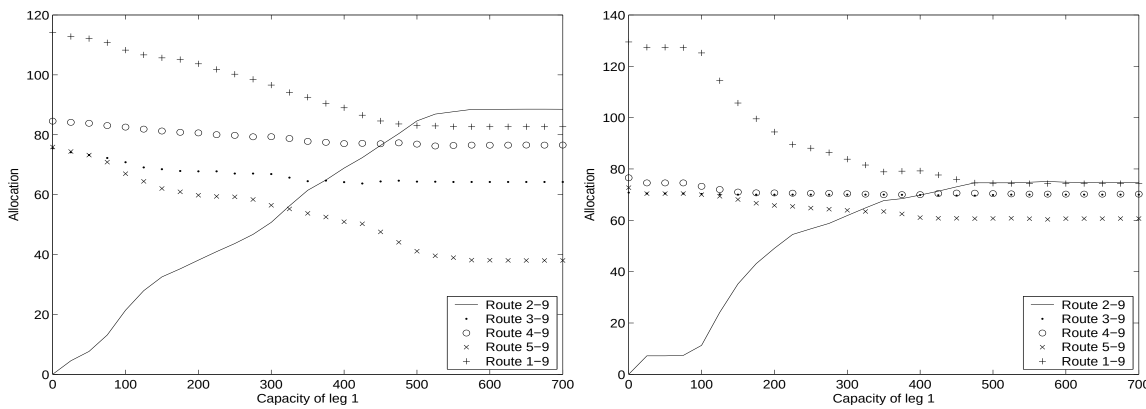

Figure 6-30: Influence of capacity of leg 1 on allocations on leg 8.

6.3.4 Summary of Results

In this section, we have applied data-driven techniques to airline revenue management. Our goal was to obtain robust seat allocations and admission policies that (a) would make full use of the available historical data without assuming specific distribution; and (b) would incorporate the decision-maker's risk aversion. We have formulated these problems in a linear programming framework, and derived closed-form expressions for the optimal policy. The proposed methodology highlights the role of each parameters (risk aversion, demand, fare price) in the choice of the allocation or admission strategy. Furthermore, we have implemented this approach on a network example. The insights we derived from this numerical experiment can be summarized as follows:

1. the robust framework is well-suited for network topologies, as it incorporates uncertainty while preserving the linear structure of the deterministic problem, and can

therefore be solved with standard optimization packages, using if necessary large-scale methods available commercially.

2. the optimal solution can be affected substantially by the decision-maker's risk aversion, which makes it all the more important to incorporate this factor in the decision process,

3. the data-driven approach also helps to identify which legs and routes play the most important role in the optimal strategy.

One of the major challenges in airline revenue management is to integrate different levels of the decision-making process, such as crew scheduling, pricing, aircraft assignment, admission policies, which are currently addressed sequentially, in one single model. The robust framework holds much potential in that respect as it incorporates uncertainty in a tractable and efficiently solvable manner.

6.4 Concluding Remarks

In this chapter, we have applied a data-driven methodology to common revenue management problems. Using the sample of historical realizations of the demand, we have built a tractable framework that does not require any estimation of the future demand process, and instead have used the ideas from the field of robust statistics and portfolio management developed in Chapter 3 to obtain linear programming problems for all the applications considered. These formulations allow us to gain valuable insights into the impact of uncertainty and risk aversion on the optimal solutions of the newsvendor problem, as well as seat allocation and admission policies in airline revenue management. In particular, we show that the optimal solution can be formulated in terms of auxiliary variables, and sometimes historical demands, appropriately ranked. Therefore, the data-driven approach opens a promising area of research in the field of revenue management.

Chapter 7

Conclusions and Future Research Directions

The unifying theme of this thesis is that robust optimization provides a powerful framework to model stochastic systems in a tractable and insightful manner. It is well suited to problems with volatile uncertainty, where the randomness is hard to describe in exact probabilistic terms. Supply chains and revenue management represent an important class of such problems, as customer demand is notoriously difficult to estimate accurately. Robust optimization also offers an elegant technique to address dynamic environments, since in contrast with other methods available to the decision-maker, e.g., dynamic programming, the robust formulations can be solved efficiently for large problem sizes.

In this thesis, we first presented a deterministic approach to randomness, where we modelled the random variables as uncertain parameters in polyhedral uncertainty sets. We derived tractable robust counterparts of the original convex problems, studied the static as well as the dynamic cases, analyzed the optimal solutions in detail, and described how to select the parameters to achieve a trade-off between robustness and optimality.

We then considered a second technique, which builds directly on the sample of historical data and incorporates risk aversion through a single parameter. An appealing feature of this approach is that it does not require any estimation process. We showed that this data-driven framework leads to convex problems, and hence is tractable. Furthermore, we provided insights into the structure of the optimal policy. In linear programming examples, we characterized the optimal solution in terms of auxiliary variables ranked appropriately.

In a third part, we compared the two methods and argued that the approach with uncertainty sets was best suited to problems exhibiting an averaging effect in the random variables. We also studied the implications of each framework in settings where both could be applied successfully, and established some conditions for the models to be equivalent.

Following these theoretical results, we applied the robust optimization techniques to examples from the fields of supply chains and revenue management. In the context of supply chains with backlogged demand and piecewise linear costs, we showed that the robust problem was equivalent to a nominal problem with modified demand. We derived the optimality of basestock policies in cases where the optimal stochastic policy is basestock as well, and in cases where the optimal stochastic policy is not known.

We also proposed robust strategies for the newsvendor problem and airline revenue management. We formulated the data-driven counterparts as linear programming problems and characterized the optimal policy as a function of the different demand scenarios and the dual variables. Our findings provide a deeper insight into the impact of risk aversion and demand stochasticity on the system.

In conclusion, we believe that robust optimization holds great promise as a modelling tool for management problems. It opens many research directions, as virtually any problem in supply chains or revenue management can be revisited using the techniques that we have developed. This is particularly attractive in a dynamic setting, where traditional methods quickly become intractable. In future work, we intend to address management problems with endogenous uncertainty. The randomness that we have considered so far was exogenous to the system, that is, could not be controlled by the decision-maker. In many applications however, pricing schemes allow for a much greater flexibility in regulating the demand process and therefore optimizing revenue. We hope that integrating these various decision levels in a single tractable formulation through a robust optimization approach will lead to a better understanding of the problem at hand, and a more efficient allocation of the resources available.

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