Model Order Reduction for Determining Bubble Parameters to Attain a Desired Fluid Surface Shape

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***Abstract*—In this paper, a new methodology for predicting fluid free surface shape using Model Order Reduction (MOR) is presented. Proper Orthogonal Decomposition (POD) combined with a linear interpolation procedure for its coefficient is applied to a problem involving bubble dynamics near to a free surface. A model is developed to accurately and efficiently capture the variation of the free surface shape with different bubble parameters. In addition, a systematic approach is developed within the MOR framework to find the best initial locations and pressures for a set of bubbles beneath the quiescent free surface such that the resultant free surface attained is close to a desired shape. Predictions of the free surface in two-dimensions and three-dimensions are presented.**

***Index Terms*— Bubble Dynamics, Model Order Reduction, Proper Orthogonal Decomposition**

I. INTRODUCTION

A set of underwater gas bubbles created by explosions can lead to the formation of a water plume in the sea surface. It is often desired that the water plume attain a particular shape to serve certain functionality, such as obstructing objects skimming above the water surface. This shape can be achieved by solving an optimization problem which minimizes the deviation of the simulated free surface from the desired shape with the decision variable set being the bubble parameters such as lateral positions, depths and strengths. Since it is not efficient and cost effective to couple a full model of bubble and free surface in an optimization problem, the Proper Orthogonal Decomposition (POD) is used to build a low-cost, low-order approximation model for the current problem. A series of linear POD basis functions are obtained from a set of solution snapshots of the bubble and free surface interaction problem. These basis functions and their POD coefficients are then used to simplify the optimization of bubble parameters to attain the desired water plume.

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II. PROPER ORTHOGONAL DECOMPOSITION

Given a set of snapshots $\{u(x)\}$ which are solutions of the system collected at different instants in time, the optimal POD basis functions $\{\phi_j(x)\}_{j=1}^{\infty}$ are chosen to minimize the truncation error due to the construction of the snapshots $u(x)$ using M basis functions

$$\max_{\psi} \frac{\left\langle \left|(u,\psi)\right|^2 \right\rangle}{(\psi,\psi)} = \frac{\left\langle \left|(u,\phi)\right|^2 \right\rangle}{(\phi,\phi)} \quad (1)$$

The POD basis functions $\{\phi_j(x)\}_{j=1}^{\infty}$ satisfy

$$\int_{\Omega} K(x,x')\phi(x')dx' = \lambda\phi(x) \quad (2)$$

where the kernel $K(x,x')$ is the autocorrelation function with the dimension of $N \times N$ (N is dimension of u, usually in order of 10^4), (Newman, 1996).

In the Method of Snapshots (Sirovich, 1987), the POD basis vectors are calculated as the linear combination of the snapshots

$$\phi_j(x) = \sum_{i=1}^{M} b_i^j u_i(x),\ j = 1,\dots,M \quad (3)$$

where b^j satisfies

$$Rb = \lambda b \quad (4)$$

Here R is the modified correlation matrix,

$$R_{ij} = \frac{1}{M}(u_i,u_j) \quad (5)$$

The approximation is given by the linear combination of the basis functions

$$u(x) \approx \sum_{j=1}^{q} a_j\phi_j(x) \quad (6)$$

where q is chosen to satisfy a desired level of accuracy ($q \le M$ and $q << N$); the POD coefficient a_j is determined as a function of time.

The POD procedure can also be applied to parameter-dependent problems whose snapshots are collected at a series of varying parameters (Bui-Thanh et al., 2003). The coefficient a_j is then a function of these varied parameters.

Let $\{u^{\delta_i}\}$ be the snapshot taken corresponding to the parameter value $\eta_i, i = 1,...,M$. Basic POD is applied on the set of snapshots $\{u^{\delta_i}\}_{i=1}^{M}$ to obtain the orthonormal basis $\{\phi_j\}_{j=1}^{M}$. The coefficient $a_j^{\delta_i}$ is given by the projection

$$a_j^{\delta_i} = \left(\phi_j, u^{\delta_i}\right) \tag{7}$$

The POD coefficients a_j^{δ} for intermediate values of η which is in the range of interest but not included in the ensemble can be found by interpolation among the $a_j^{\delta_i}$. The prediction of u^{δ} is given by

$$u^{\delta} = \sum_{j=1}^{q} a_j^{\delta}\phi_j \tag{8}$$

In the next section, POD in parametric space coupled with bubble dynamics will be used in an optimization context in the problem of water barrier construction.

III. Water Barrier Simulation

The free surface formed by N_0 number of bubbles can be approximated by the linear superposition of the individual free surfaces, Our objective is to minimize the difference between the constructed surface $S = \sum_{k=1}^{N_o} S^k$ and the desired surface S_0.

It is difficult to obtain directly the bubble parameters that give the desired shape, so we divided the problem into two stages.

In the first stage, the lateral position $\{r_0^k\}_{k=1}^{N_o}$ and the POD coefficient $\{a_j^k\}_{k=1,..N_0;j=1,..,q}$ for the bubbles are solved:

BPS1: $$\min_{r_0^k, a_j^k, k=1...N_o, j=1...q} \sum_{i=1}^{M_o}\left[\left(S - S^0\right)_i\right]^2$$

s.t. $$S = \sum_{k=1}^{N_o}\left(\bar{\phi}^k + \sum_{j=1}^{q} a_j^k \phi_j^k\right)$$

$$\bar{\phi}^k = \bar{\phi}\left(r_0^k\right) \qquad k = 1...N_o$$

$$\phi_j^k = \phi_j\left(r_0^k\right) \qquad k = 1...N_o,\ j = 1...q$$

$$\left|r_0^{k_1} - r_0^{k_2}\right| \geq dr \qquad k_1 \neq k_2; k_1, k_2 = 1..N_o$$

$$a_j^{\min} \leq a_j^k \leq a_j^{\max} \qquad k = 1...N_o,\ j = 1...q$$

Here, $\bar{\phi}^k$ and ϕ_j^k are the snapshot mean and the j^{th} POD mode, respectively, for the k^{th} bubble. The minimum lateral distance between centers of two arbitrary bubbles is $dr = 2.0$ to ensure the validity of the superposition. The range a_j is determined by the range of interest of the bubble strength and depth, $[\varepsilon_{\min}, \varepsilon_{\max}]$ and $[\gamma_{\min}, \gamma_{\max}]$.

The first-stage problem can be solve approximately using the Approximation Function (AF) algorithm which uses exponential functions to approximate the first POD mode, ϕ_1^k, and the mean, $\bar{\phi}^k$. The functions have the form of and

$$f(r) = C_1 e^{-C_2 r^2} \tag{9}$$

The coefficients C_1 and C_2 are specified by solving the optimization problem

AF: $$\min_{C_1, C_2} \left\|\left(f - f^0\right)_i\right\|^2$$

s.t. $$f(r) = C_1 e^{-C_2 r^2}$$

where f^0 is the mean or the first POD mode. Using these functions, the free surface corresponding to the k^{th} bubble can be represented by

$$S^k(r) = C_{m1} e^{-C_{m2}\left(r - r_0^k\right)^2} + a_1^k C_{11} e^{-C_{12}\left(r - r_0^k\right)^2} \tag{10}$$

BPS1b: $$\min_{r_0^k, a_1^k, k=1...N_o} \sum_{i=1}^{M_o}\left[\left(S - S^0\right)_i\right]^2$$

s.t. $$S(r) = \sum_{k=1}^{N_o}\left(C_{m1} e^{-C_{m2}\left(r - r_0^k\right)^2} + a_1^k C_{11} e^{-C_{12}\left(r - r_0^k\right)^2}\right)$$

$$\left|r_0^{k_1} - r_0^{k_2}\right| \geq dr \qquad k_1 \neq k_2, k_1, k_2 = 1...N_o$$

$$a_1^{\min} \leq a_1^k \leq a_1^{\max} \qquad k = 1...N_o$$

The lateral positions of the bubbles, $\{r_0^k\}_{k=1}^{N_o}$, are obtained by solving this problem. Then, we substitute these lateral positions back to the BPS1 to obtain the complete set of POD coefficients

$$\left\{\left(a^*\right)_j^k,\ k = 1,..., N_o,\ j = 1,..., q\right\} \tag{11}$$

In the second-stage, the strength ε and depth γ of each bubble are obtained from the POD coefficients by interpolation of a piecewise linear function

$$a_j(\varepsilon, \gamma) = \left\{a_j^{s,t}(\varepsilon, \gamma),\ \varepsilon_t \leq \varepsilon \leq \varepsilon_{t+1}, \gamma_s \leq \gamma \leq \gamma_{s+1}, s = 1,..., n-1, t = 1,..., m-1\right\} \tag{12}$$

The second-stage is written as

BPS2: $$\min_{\varepsilon^k, \gamma^k} \sum_{j=1}^{q}\left[\left(a^*\right)_j^k - a_j\right]^2$$

s.t. $$a_j\left(\varepsilon^k, \gamma^k\right) = a_j^{st}\left(\varepsilon^k, \gamma^k\right)$$

$$\varepsilon_t \leq \varepsilon^k \leq \varepsilon_{t+1}$$

$$\gamma_s \leq \gamma^k \leq \gamma_{s+1}$$

$$s = 1,..., n-1; t = 1,..., m-1;\ j = 1...q$$

The problem BPS2 involves minimizing of a piecewise function and this is given by the global minimum of the solutions of its piecewise function elements. The bubble problem is fully solved when the lateral position, the depth and the strength of the bubbles are determined.

IV. Results

The parametric POD is applied to the ensemble that contains 312 snapshots corresponding to 39 values of initial depth in the range [-1.25, -5.05] with interval step of 0.1, and 8 values of strength in the range [100, 800] with interval step of 100. The following figures and tables show the results for the water barrier problems using the AF algorithm implemented in LOQO. We can see that the AF algorithm is very efficient for solving 2D and reasonable-size 3D problems.

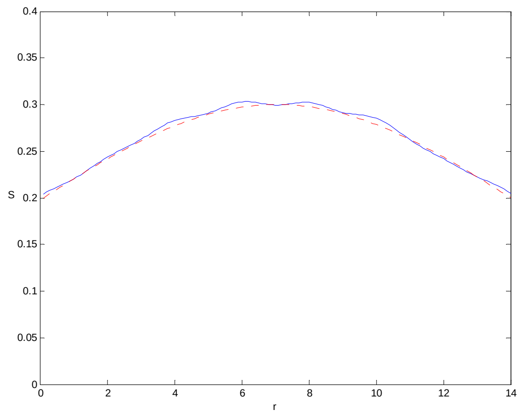

Fig. 1. Solution to 2D convex sine-shape surface using 8 bubbles (dash line is desired surface, solid line is predicted surface)

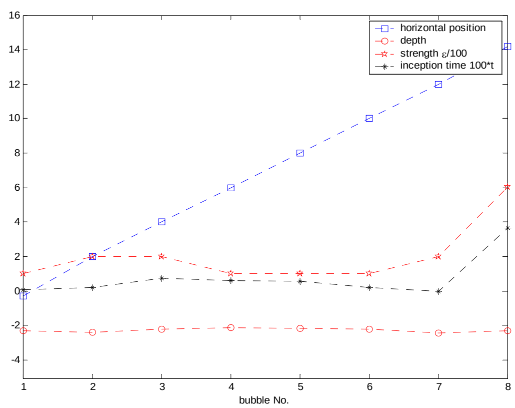

Fig. 2. Optimal parameter values of the bubbles (depth, strength, and inception time and position).

TABLE I
COMPUTATION TIME FOR 2D PROBLEMS

NO. OF BUBBLES	COMPUTATION TIME
5 bubbles	16.0s
6 bubbles	19.7s
7 bubbles	23.2s
8 bubbles	28.6s
Full simulation of 1 bubble	45.0s

Computation time for solving two-dimensional optimization problem in the domain of [0,14], 141 grid points (grid size 0.1) using AF algorithm. Solution is obtained by running a LOQO program on a Pentium 4 1.6MHz processor, RAM 256Mb

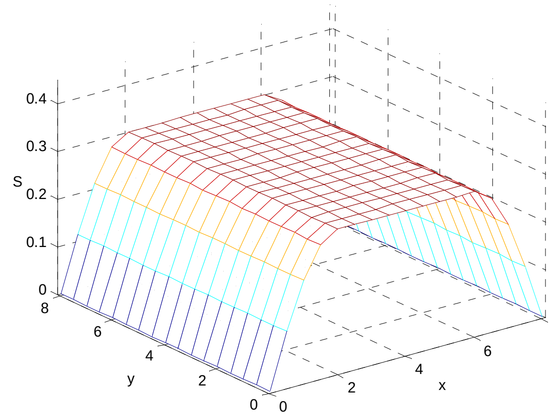

Fig. 3. Desired surface shape for 3D problem.

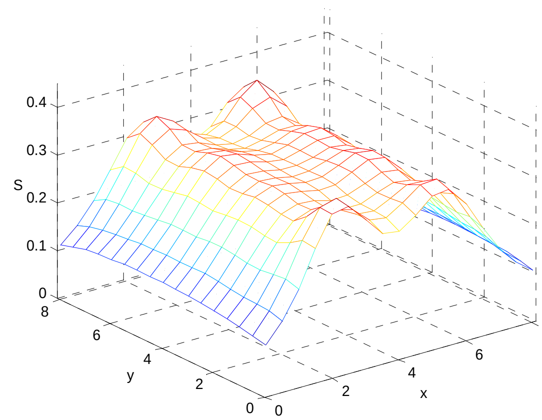

Fig. 4. Constructed surface using 10 bubbles.

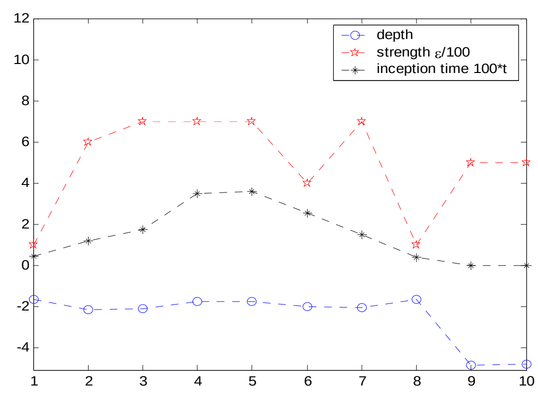

Fig. 5. Optimal parameter values of the bubbles (depth, strength, and inception time).

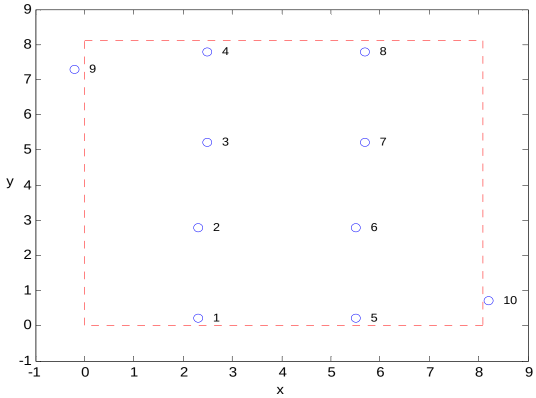

Fig. 6. Lateral positions of the bubbles

TABLE II
COMPUTATION TIME FOR 3D PROBLEMS

No. bubbles	8x8 grid	15x15 grid	29x29 grid
6 bubbles	19.9s	24.7s	44.2s
8 bubbles	27.7s	39.3s	84.7s
10 bubbles	43.7s	63.0s	136.0s
12 bubbles	64.2s	255.0s	598.0s
16 bubbles	340.1s	1310.6s	2216.6s
Full simulation of 1 bubble (1717 nodes)			542.0s

Computation time for solving three-dimensional optimization problems with different numbers of bubbles and grid sizes using AF algorithm. Solution is obtained by running a LOQO program on a Pentium 4 1.6MHz processor, RAM 256Mb

V. Conclusion

We presented an efficient method to determine the bubble parameters that will give a desired fluid surface shape for both 2D and 3D problems. The POD method using linear interpolation in parametric space, together with the AF algorithm, provides a good approximate solution to this optimization problem

References

[1] Wang Q.X., Yeo K.S., Khoo B.C. and Lam K.Y. Strong Interaction Between a Buoyancy Bubble and a Free Surface. Theoretical Computational Fluid Dynamics, Vol. 8, 73-88, 1996.

[2] Rungsiyaphornrat S., Klaseboer E., Khoo B.C. and Yeo K.S. The Merging of Two Gaseous Bubbles with an Application to Underwater Explosion. Computer & Fluid, Vol. 32, 1049-1074, 2003.

[3] Holmes P., Lumley J.L. and Berkooz G. Turbulence Coherent structures, Dynamical Systems and Symmetry. Cambridge University Press, 1996.

[4] Newman A.J. Model Reduction via the Karhunen-Lo `eve Expansion. Part I and II. T.R. 96-32, T.R. 96-33, 1996.

[5] Sirovich L. Turbulence and the Dynamics of Coherent Structures, Part 1: Coherent Structures. Quarterly of Applied Mathematics, Vol. 45, No. 3, 561-571, October 1987.

[6] Dowell E.H., Hall K.C., Thomas J.P., Florea R. and Epureanu B.I. Reduced-Order Models in Unsteady Aerodynamic. AIAA Paper, 99-0655, 1999.

[7] Epureanu B.I., Dowell E.H. and Hall K.C. A Parametric Analysis of Reduced-Order Models of Potential Flows in Turbomachinery using Proper Orthogonal Decomposition. 2001-GT-0434, Proceedings of ASME TURBO EXPO 2001, New Orleans, Louisiana, June 2001.

[8] Willcox K. and Peraire J. Balanced Model Reduction via the Proper Orthogonal Decomposition. AIAA paper, vol. 40, no. 11, 2323-2330, November 2002.

[9] Hung V. Ly and Hien T. Tran. Modeling and Control of Physical Processes using Proper Orthogonal Decomposition. Journal of Mathematical and Computer Modeling.

[10] Everson R. and Sirovich L. The Karhunen-Loeve for Gappy Data. J.Opt.Soc.Am.,12: 1657-1664, 1995.

[11] Bui-Thanh T., Damodaran M. and Willcox K. Aerodynamic Data Reconstruction and Inverse Design using Proper Orthogonal Decomposition. AIAA paper, 2003-4231.