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Abstract

This paper considers a connection between the deterministic and noisy behavior of nonlinear networks. Specifically, a particular bridge circuit is examined which has two possibly nonlinear energy storage elements. By proper choice of the constitutive relations for the network elements, the deterministic terminal behavior reduces to that of a single linear resistor. This reduction of the deterministic terminal behavior, in which a natural frequency of a linear circuit does not appear in the driving-point impedance, has been shown in classical circuit theory books (*e.g.* [1, 2]). The paper shows that, in addition to the reduction of the deterministic behavior, the thermal noise at the terminals of the network, arising from the usual Nyquist-Johnson noise model associated with each resistor in the network, is also exactly that of a single linear resistor. While this result for the linear time-invariant (LTI) case is a direct consequence of a well-known result for RLC circuits, the nonlinear result is novel. We show that the terminal noise current is precisely that predicted by the Nyquist-Johnson model for R if the driving voltage is zero or constant, but not if the driving voltage is time-dependent or the inductor and capacitor are time-varying.

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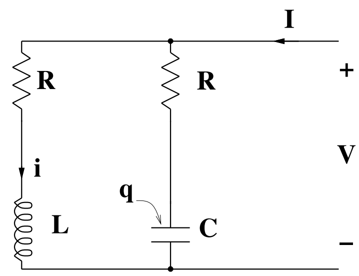

Figure 1: The linear noise-free bridge circuit is matched and has input impedance R is $L = R^2C$.

1 Introduction

Consider the bridge circuit of Figure 1. It is a standard result of linear circuit theory that under the matching condition $L = R^2\,C$ the natural frequency of the circuit does not appear as a pole in the driving-point impedance [1, 2]. One can intuitively see that – regardless of the values of the capacitor and inductor – for high frequencies, the capacitor is essentially a short circuit, whereas the inductor is essentially an open circuit; at low frequencies, the opposite occurs. The matching condition ensures that a balance is preserved for intermediate frequencies: the charging of the capacitor is matched by the fluxing of the inductor.

Example 2 on pp. 630-633 of Ref. [1] discusses the circuit shown in Figure 1. This circuit will be the main focus of this paper. However, the derivations can also be carried out for the dual circuit, which is in fact the version presented in Ref. [2]. Ref. [2] also discusses the circuit in terms of loss of observability and controllability: under the matching condition, the states of the system, namely the inductor flux and the capacitor charge, are neither observable nor controllable from the external terminals, *i.e.*, the state equations become *nonminimal*.

It is straightforward to verify directly in this linear case that if a Nyquist-Johnson [3, 4] noise model (as seen in Fig. 2) is associated with each resistor, then the spectrum of the short-circuit current is also that predicted by a Nyquist-Johnson noise model for a single resistor of value R. The verification can be done by standard frequency-domain techniques or by stochastic calculus. The highpass filtering of the RC branch is precisely balanced by the lowpass filtering of the RL branch, so that the terminal noise spectrum is flat. Of course, both resistors must be at the same temperature. As noted in [2], applying a d.c. voltage to the circuit would result in differential heating of the resistor in the RL branch. If the resistors were not properly connected to thermal reservoirs, one could heat up and become noisier than the other, and the noise spectrum would no longer be flat. This is a trivial nonequilibrium exception to the results of this paper, which assumes uniform, constant temperature.

The result above is a particular example of a general circuit theory result, namely, that a one-port network of (linear) passive elements with port impedance $Y(j\omega)$ presents a thermal noise voltage with power spectrum $2kT\ \mathrm{Re}\{Y(j\omega)\}$, where k is Boltzmann's constant and T is the absolute temperature [5]. Physicists regard such results as particular cases of the fluctuation-dissipation theorem [6].

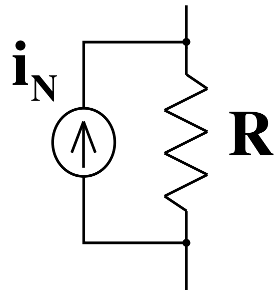

Figure 2: Nyquist-Johnson thermal noise model (Norton form) is a noiseless linear resistor in parallel with a Gaussian white noise current source i_N with power spectral density $2kT/R$.

This paper extends these results to the nonlinear case and to some nonequilibrium situations. In layman's terms, the question is as follows: suppose one has two black boxes, one with a matched bridge inside, and one with a single equivalent linear resistor. Is it possible to distinguish the two using the noise behavior?

This paper studies one carefully-chosen example, motivated by the question of whether some form of fluctuation-dissipation theorem holds for some class of nonlinear circuits. Our initial formulation appears below as a conjecture for any pair of two-terminal networks, each comprising an interconnection of LTI resistors at a uniform, constant temperature, described by the Nyquist-Johnson model, and possibly also capacitors and inductors that may be nonlinear or time-varying. Two such networks are said to be *zero-state deterministically equivalent* if every applied terminal voltage waveform $v(t), t \geq 0$, produces the same current response $i(t)$ from both networks, provided all capacitor voltages and inductor currents are initially zero and all noise sources in the resistor models are set to zero. (In the LTI case this just means the two input admittances are identical.)

> *Preliminary Fluctuation-Dissipation Conjecture for Networks*:
> No two zero-state deterministically equivalent networks can be distinguished by their terminal noise current responses to any applied voltage waveform.

The conjecture just hypothesizes that the deterministic terminal behavior uniquely determines the noise current response for all voltage drives, independent of the details of the network. The conjecture is true in the LTI case. (Closely related formulations for the current-driven and multiport cases [5] also hold true for LTI networks, but we ignore them here for simplicity.) However, an examination of the bridge circuit will show that this conjecture is wrong in other cases and must be revised.

The linear results are first derived in Sections 2-5 to motivate the nonlinear case and to familiarize the reader with the notation. In Section 6 we derive the matching condition corresponding to $L = R^2C$ for the bridge circuit with nonlinear, time-invariant inductor and capacitor, under which it becomes deterministically equivalent at the terminals to a single linear resistor R. In Section 7, we show that such a matched nonlinear bridge gives a short-circuit port current noise statistically identical to that of the Nyquist-Johnson model for R at thermal equilibrium. Nonequilibrium situations are considered in Section 8.

2 Linear, Noise-Free Case

Kirchoff's Laws for the bridge circuit of Figure 1 yield differential equations for the capacitor charge

$$\frac{dq}{dt} = \frac{V}{R} - \frac{1}{RC}\,q, \tag{1}$$

and for the inductor current

$$L\frac{di}{dt} = V - R\,i. \tag{2}$$

Great insight can be extracted from the quantity $i - q/RC$. Using the two previous equations, the time evolution of this quantity is

$$\begin{aligned} \frac{d}{dt}\left(i - \frac{q}{RC}\right) &= \frac{1}{L}(V - R\,i) - \frac{1}{RC}\left(\frac{V}{R} - \frac{1}{RC}\,q\right) \\ &= V\left(\frac{1}{L} - \frac{1}{R^2C}\right) - \frac{R}{L}\left(i - \frac{L}{R^2C}\frac{1}{RC}\,q\right). \end{aligned} \tag{3}$$

Suppose now that the element values are balanced with the matching condition

$$\frac{L}{R^2C} = 1 \quad \text{or} \quad L = R^2\,C. \tag{4}$$

Then, Eq. (3) reduces to

$$\frac{d}{dt}\left(i - \frac{q}{RC}\right) = -\frac{R}{L}\left(i - \frac{q}{RC}\right). \tag{5}$$

With zero initial conditions, or for non-zero initial conditions satisfying $i(0) = q(0)/RC$, this guarantees that $i(t) = q(t)/RC$ for all time. More significantly, all other initial conditions are exponentially attracted to the line $i = q/RC$. Further, by Kirchoff's Current Law,

$$\begin{aligned} I &= i + \frac{dq}{dt} = i + \frac{V}{R} - \frac{q}{RC} \\ &= \frac{V}{R}, \end{aligned} \tag{6}$$

so that the entire circuit appears as simply a single resistor of value R. The mode of the circuit corresponding to $i - q/RC$ is neither observable (it does not appear in the output equation for I) nor controllable (its dynamics are unaffected by the input V).

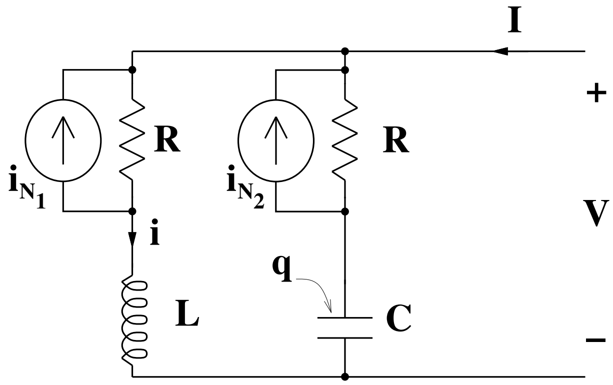

Figure 3: Noisy bridge circuit, Norton form

3 Linear Noisy Case

To analyze the noise behavior of the linear bridge circuit, a Norton equivalent current noise source is associated with each resistor, as in Figure 3. The noise sources i_{N_1} and i_{N_2} are independent, white, Gaussian random processes with spectral density $2kT/R$. The differential equations for the circuit become

$$\frac{dq}{dt} = \frac{V}{R} - \frac{1}{RC}\,q - i_{N_2}, \tag{7}$$

$$L\frac{di}{dt} = V - R\,(i + i_{N_1})\,. \tag{8}$$

The port current is

$$I = i + \frac{dq}{dt} = i + \frac{V}{R} - \frac{1}{RC}\,q - i_{N_2}. \tag{9}$$

The goal of this section is to show that the short-circuit ($V = 0$) power spectral density for I is $2kT/R$, under the matching condition $L/C = R^2$. In anticipation of the nonlinear case, where frequency-domain techniques fail, time-domain techniques are used to find the autocorrelation function for I; the Fourier transform then yields the power spectral density. Since there are no explicit time dependencies, the random process I is stationary, and the autocorrelation only depends on the difference between the two time points.

$$\begin{aligned}
R_{II}(\tau) &= E\left\{I(t)I(t+\tau)\right\} \\
&= E\left\{\left[i(t) - \frac{1}{RC}\,q(t) - i_{N_2}(t)\right]\left[i(t+\tau) - \frac{1}{RC}\,q(t+\tau) - i_{N_2}(t+\tau)\right]\right\} \\
&= E\Big\{i(t)\,i(t+\tau) - \frac{1}{RC}\,i(t)\,q(t+\tau) - i(t)\,i_{N_2}(t+\tau) \\
&\quad -\frac{1}{RC}\,q(t)\,i(t+\tau) + \frac{1}{R^2C^2}\,q(t)\,q(t+\tau) + \frac{1}{RC}\,q(t)\,i_{N_2}(t+\tau) \\
&\quad -i_{N_2}(t)\,i(t+\tau) + \frac{1}{RC}\,i_{N_2}(t)\,q(t+\tau) + i_{N_2}(t)\,i_{N_2}(t+\tau)\Big\}
\end{aligned}$$

For $V = 0$, the two branches of the circuit do not interact. Since the two noise sources are independent, $i(\cdot)$ is not correlated with either $q(\cdot)$ or $i_{N_2}(\cdot)$. For $\tau > 0$, $q(t)$ is uncorrelated with $i_{N_2}(t+\tau)$ (the capacitor charge now is not affected by the current noise in the future). Correspondingly, for $\tau < 0$, $q(t+\tau)$ is uncorrelated with $i_{N_2}(t)$. Since each of the quantities $i(\cdot)$, $q(\cdot)$, and $i_{N_2}(\cdot)$ are individually zero-mean, five of the nine terms in the equation immediately vanish. For $\tau > 0$,

$$\begin{aligned} R_{II}(\tau) &= E\left\{ i(t)\, i(t+\tau) + \frac{1}{R^2C^2}\, q(t)\, q(t+\tau) + i_{N_2}(t)\, i_{N_2}(t+\tau) \right. \\ &\qquad \left. + \frac{1}{RC}\, i_{N_2}(t)\, q(t+\tau) \right\}. \end{aligned} \tag{10}$$

The time arguments of the last term will be switched for $\tau < 0$.

The terms in this expression will be computed individually. First, of course,

$$E\left\{ i_{N_2}(t)\, i_{N_2}(t+\tau) \right\} = \frac{2kT}{R}\, \delta(\tau). \tag{11}$$

The two branches are disjoint for $V = 0$, and the differential equations for the random variables i and q are separable:

$$L\frac{di}{dt} = -R\, i - R\, i_{N_1},$$

which has the solution

$$i(t) = -\frac{R}{L} \int_{-\infty}^{t} e^{-(t-t')R/L}\, i_{N_1}(t')\, dt', \tag{12}$$

and

$$\frac{dq}{dt} = -\frac{1}{RC}\, q - i_{N_2},$$

which has the solution

$$q(t) = -\int_{-\infty}^{t} e^{-(t-t')/RC}\, i_{N_2}(t')\, dt'. \tag{13}$$

These two time-domain, sample-path solutions are the key to computing the remaining expectations. Assuming zero initial conditions ($i(-\infty) = 0$ and $q(-\infty) = 0$) so that the random processes are stationary,

$$\begin{aligned} E\left\{ i(t)\, i(t+\tau) \right\} &= E\left\{ \left[\frac{R}{L} \int_{-\infty}^{t} e^{-(t-t')R/L}\, i_{N_1}(t')\, dt' \right] \left[\frac{R}{L} \int_{-\infty}^{t+\tau} e^{-(t+\tau-t'')R/L}\, i_{N_1}(t'')\, dt'' \right] \right\} \\ &= \frac{R^2}{L^2} \int_{-\infty}^{t} e^{-(t-t')R/L} \int_{-\infty}^{t+\tau} e^{-(t+\tau-t'')R/L}\, E\left\{ i_{N_1}(t')\, i_{N_1}(t'') \right\} dt'\, dt'' \\ &= \frac{R^2}{L^2} \int_{-\infty}^{t} e^{-(t-t')R/L} \int_{-\infty}^{t+\tau} e^{-(t+\tau-t'')R/L}\, \frac{2kT}{R}\, \delta(t'-t'')\, dt'\, dt'' \\ &= \frac{R^2}{L^2} \frac{2kT}{R} \int_{-\infty}^{t} e^{-(t-t'+t+\tau-t')R/L}\, dt' \end{aligned}$$

$$
\begin{aligned}
&= \frac{2kTR}{L^2}\, e^{-(2t+\tau)R/L} \int_{-\infty}^{t} e^{2t'\,R/L}\, dt' \\
&= \frac{2kTR}{L^2}\, e^{-(2t+\tau)R/L} \frac{L}{2R} \left[e^{2t\,R/L} - 0\right] \\
&= \frac{kT}{L}\, e^{-\tau\,R/L}. \qquad (14)
\end{aligned}
$$

In fact, the above derivation assumed $\tau > 0$; the correct expression for any τ is

$$
E\left\{i(t)i(t+\tau)\right\} = \frac{kT}{L}\, e^{-|\tau|\,R/L}. \qquad (15)
$$

Similarly,

$$
\begin{aligned}
E\left\{q(t)\, q(t+\tau)\right\} &= E\left\{\left[-\int_{-\infty}^{t} e^{-(t-t')/RC}\, i_{N_2}(t')\, dt'\right]\left[-\int_{-\infty}^{t+\tau} e^{-(t+\tau-t'')/RC}\, i_{N_2}(t'')\, dt''\right]\right\} \\
&= e^{-(2t+\tau)/RC} \int_{-\infty}^{t}\int_{-\infty}^{t+\tau} e^{(t'+t'')/RC} E\left\{i_{N_2}(t')\, i_{N_2}(t'')\right\}\, dt'\, dt'' \\
&= \frac{2kT}{R}\frac{RC}{2} e^{-|\tau|/RC} \\
&= kTC\, e^{-|\tau|/RC}. \qquad (16)
\end{aligned}
$$

The last term to be computed is

$$
\begin{aligned}
E\left\{i_{N_2}(t)\, q(t+\tau)\right\} &= E\left\{i_{N_2}(t)\left[-\int_{-\infty}^{t+\tau} e^{-(t+\tau-t')/RC}\, i_{N_2}(t')\, dt'\right]\right\} \\
&= -\int_{-\infty}^{t+\tau} e^{-(t+\tau-t')/RC} E\left\{i_{N_2}(t)\, i_{N_2}(t')\right\} dt' \\
&= -e^{-(t+\tau)R/L} \int_{-\infty}^{t+\tau} e^{t'/RC}\, \frac{2kT}{R} \delta(t-t')\, dt' \\
&= -e^{-(t+\tau)/RC}\quad e^{t/RC}\quad \frac{2kT}{R} \\
&= -\frac{2kT}{R}\, e^{-\tau/RC}, \quad \tau > 0. \qquad (17)
\end{aligned}
$$

Unlike the previous terms, this last expectation is zero for $\tau < 0$. For $\tau < 0$, the time arguments on the left-hand side switch places, resulting in

$$
E\left\{i_{N_2}(t+\tau)\, q(t)\right\} = -\frac{2kT}{R}\, e^{\tau/RC}, \quad \tau < 0. \qquad (18)
$$

Assembling Eqs. (11), (15), (16), and (17) and substituting into Eq. (10),

$$\begin{aligned} R_{II}(\tau) &= E\{i(t)\, i(t+\tau)\} + \frac{1}{R^2C^2}\, E\{q(t)\, q(t+\tau)\} + E\{i_{N_2}(t)\, i_{N_2}(t+\tau)\} \\ &\quad + \frac{1}{RC}\, E\{i_{N_2}(t)\, q(t+\tau)\} \\ &= \frac{kT}{L}\, e^{-|\tau|R/L} + \frac{1}{R^2C^2}\, kTC\, e^{-|\tau|/RC} + \frac{2kT}{R}\delta(\tau) - \frac{1}{RC}\,\frac{2kT}{R}\, e^{-|\tau|/RC}. \end{aligned} \tag{19}$$

Eq. (18) ensures that this expression is valid for all τ.

Under the matching condition $L = R^2C$ among the element values, the leading coefficients are related:

$$\frac{kT}{L} = \frac{kT}{R^2C}, \tag{20}$$

and the time constants in the exponential are equal:

$$\frac{1}{RC} = \frac{R}{L}. \tag{21}$$

Therefore, Eq. (19) simplifies to

$$R_{II}(\tau) = \frac{2kT}{R}\, \delta(\tau). \tag{22}$$

The Fourier transform of a delta-function is simple to compute, immediately yielding the desired result,

$$S_{II}(\omega) = \mathcal{F}\{R_{II}(\tau)\} = \frac{2kT}{R}, \tag{23}$$

for all frequencies ω. The noise behavior of the circuit is equivalent to that of a single linear resistor, just as was the case for the deterministic behavior.

This same result may also be obtained without short-circuiting the terminals. If the terminals are driven by a deterministic voltage waveform $V(t)$, the response of the matched bridge circuit will be indistinguishable from the response of a single linear resistor. The autocorrelation of the output current will reduce to

$$R_{II}(\tau) = E\{I(t)\, I(t+\tau)\} = \frac{V(t)\, V(t+\tau)}{R^2} + \frac{2kT}{R}\, \delta(\tau), \tag{24}$$

the deterministic autocorrelation of the applied voltage driving current through a linear resistor, plus the autocorrelation of the noise of a single linear resistor. This fact will be proved in Appendix I.

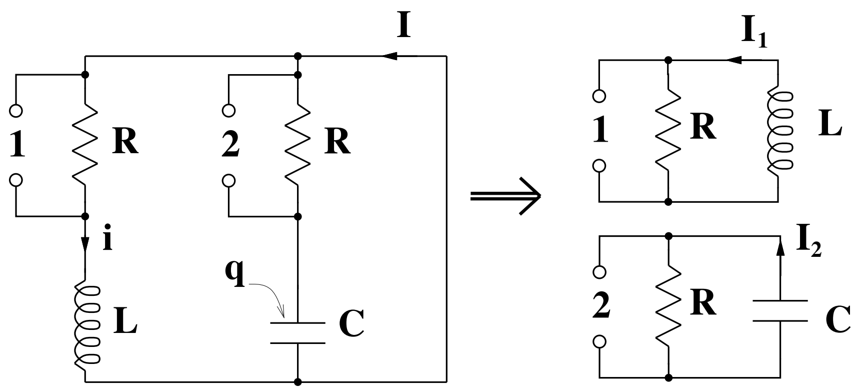

Figure 4: Linear bridge circuit in the frequency domain

4 Frequency Domain

The result of Section 3 can also be obtained using standard frequency-domain techniques for the power spectral density. The Norton-form noise sources drive an RC or an RL filter and the responses are combined to form the output noise current.

First, the matching condition will be re-derived using the complex impedance in the frequency domain. Let Z_1 be the impedance of the RL branch of Figure 1:

$$Z_1 = Z_R + Z_L = R + j\omega L, \tag{25}$$

where $j = \sqrt{-1}$, and let Z_2 be the impedance of the RC branch:

$$Z_2 = Z_R + Z_C = R + 1/(j\omega C). \tag{26}$$

The port impedance $Z = V/I$ is given by the parallel combination of these two impedances

$$\begin{aligned} Z &= \frac{Z_1\, Z_2}{Z_1 + Z_2} = \frac{(R + j\omega L)\,(R + 1/(j\omega C))}{R + j\omega L + R + 1/(j\omega C)} \\ &= \frac{R^2 + R/(j\omega C) + j\omega L\, R + L/C}{2R + j\omega L + 1/(j\omega C)} \\ &= R\,\frac{R + L/(RC) + j\omega L + 1/(j\omega C)}{2R + j\omega L + 1/(j\omega C)}. \end{aligned} \tag{27}$$

Clearly, the fraction is unity if $L/(RC) = R$, which is the matching condition.

Now, associate with each resistor a Norton-form Gaussian white-noise current source with power spectral density $2kT/R$. The sources will be connected to input ports 1 and 2 in Figure 4. Because of the short-circuit condition, the bridge can be broken into two parts, as shown, for the purposes of computing the current due to each source. The output current is a combination of these two currents, and since the two noise sources are independent, their respective contributions to the output current power spectral density add.

For the noise source connected to port 1, its current is divided between the resistor and inductor, and its contribution to the output current is only the current passing through the inductor. If a filter $H(\omega)$ is applied to an input $X(\omega)$, the output will have power spectral density $Y(\omega) = |H(\omega)|^2 X(\omega)$. The filter in this case has the shape $Z_R/(Z_R + Z_L)$ as dictated by current division. Thus, the output current power spectral density for I_1 is

$$S_{I_1 I_1}(\omega) = \frac{2kT}{R}\left|\frac{Z_R}{Z_R + Z_L}\right|^2 = \frac{2kT}{R}\left|\frac{R}{R + j\omega L}\right|^2 = \frac{2kT}{R}\frac{R^2}{R^2 + \omega^2 L^2} = \frac{2kTR}{R^2 + \omega^2 L^2}. \tag{28}$$

For the noise source connected to port 2, similarly,

$$S_{I_2 I_2}(\omega) = \frac{2kT}{R}\left|\frac{Z_R}{Z_R + Z_C}\right|^2 = \frac{2kT}{R}\left|\frac{R}{R + 1/(j\omega C)}\right|^2 = \frac{2kTR}{R^2 + 1/(\omega^2 C^2)}. \tag{29}$$

Since the two noise sources are independent, the output voltage power spectral density is simply the sum of the two half-circuit power spectral densities:

$$\begin{aligned}
S_{II}(\omega) &= \frac{2kTR}{R^2 + \omega^2 L^2} + \frac{2kTR}{R^2 + 1/(\omega^2 C^2)} \\
&= 2kTR\left(\frac{R^2 + 1/(\omega^2 C^2) + R^2 + \omega^2 L^2}{[R^2 + \omega^2 L^2][R^2 + 1/(\omega^2 C^2)]}\right) \\
&= 2kTR\left(\frac{2R^2 + 1/(\omega^2 C^2) + \omega^2 L^2}{R^4 + R^2/(\omega^2 C^2) + R^2\,\omega^2 L^2 + L^2/C^2}\right) \\
&= \frac{2kTR}{R^2}\left(\frac{2R^2 + 1/(\omega^2 C^2) + \omega^2 L^2}{R^2 + L^2/(R^2 C^2) + 1/(\omega^2 C^2) + \omega^2 L^2}\right) \\
&= \frac{2kT}{R},
\end{aligned} \tag{30}$$

where the last equality follows because the fraction inside the large parentheses on the previous line is unity under the matching condition $L^2/(R^2 C^2) = R^2$.

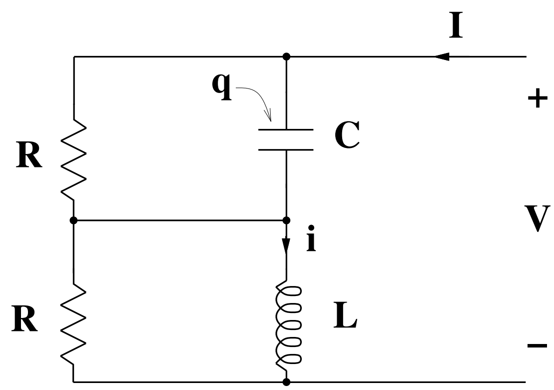

Figure 5: Dual bridge circuit

5 Dual Circuit

The circuit of Figure 5 is the dual of the circuit in Fig. 1: the parallel combination of two branches, each containing two elements in series, has become a series combination of two loops, each containing two elements in parallel. Resistors are their own duals, and the inductor and capacitor are each other's dual. Lastly, for analyzing the noise, the open-circuit voltage noise takes the place of the short-circuit current noise. The construction of a dual circuit is best explained in Ref. [1], chapter 10, section 4. One can also construct the dual circuit for the nonlinear bridge presented later in this paper; for the duals of nonlinear elements, the reader is referred to Ref. [7].

The differential equation for the capacitor charge is

$$\frac{dq}{dt} = I - \frac{q}{RC}, \tag{31}$$

and for the inductor flux

$$\frac{d\phi}{dt} = \left(I - \frac{\phi}{L}\right) R. \tag{32}$$

(The state variable for the inductor has switched to flux in this section in preparation for the nonlinear sections to follow, in which flux is a more natural state variable.) These follow directly from the constitutive relations of the elements and Kirchoff's Laws. Consider the quantity $\phi/L - q/RC$. Using the two previous equations, the time evolution of this quantity is

$$\begin{aligned} \frac{d}{dt}\left(\frac{\phi}{L} - \frac{q}{RC}\right) &= \left[\frac{1}{L}\left(I - \frac{\phi}{L}\right)R\right] - \frac{1}{RC}\left[I - \frac{q}{RC}\right] \\ &= I\left[\frac{R}{L} - \frac{1}{RC}\right] - \frac{R}{L^2}\,\phi + \frac{1}{(RC)^2}\,q. \end{aligned} \tag{33}$$

Using the same matching condition as before, namely, $\frac{R}{L} = \frac{1}{RC}$, Eq. (33) reduces to

$$\frac{d}{dt}\left(\frac{\phi}{L} - \frac{q}{RC}\right) = -\frac{R}{L}\left(\frac{\phi}{L} - \frac{q}{RC}\right). \tag{34}$$

Corresponding to the line $i = q/RC$ in the original circuit, trajectories for the dual circuit are exponentially attracted to the line $\phi/L = q/RC$ in the (ϕ, q) plane, and of course for any initial conditions on the line (including the origin), the trajectory stays on the line for all time. Further, by Kirchoff's Voltage Law,

$$\begin{aligned} V &= \frac{q}{C} + \frac{d\phi}{dt} = \frac{q}{C} + \left(I - \frac{\phi}{L}\right) R = R\left(\frac{q}{RC} - \frac{\phi}{L}\right) + I\,R \\ &= IR, \end{aligned} \tag{35}$$

so that the entire dual circuit also appears as simply a single resistor of value R.

The dual of the short-circuit output current is the open-circuit voltage. The open-circuit voltage power spectral density could be computed by either of the previous two methods, namely, via the expectation over sample paths or by frequency-domain filtering. Instead, a third technique will be used in preparation for the nonlinear cases of later chapters. In the nonlinear case, the differential equations cannot be solved for the sample paths, and frequency-domain techniques are not applicable.

The noise sources v_1 and v_2 will be Thévenin form voltage sources with power spectral density $2kTR$, one placed in series with each resistor. The open-circuit differential equations become

$$\frac{dq}{dt} = -\frac{q}{RC} + \frac{1}{R}\,v_{N_1} \tag{36}$$

and

$$\frac{d\phi}{dt} = -\frac{R\,\phi}{L} + v_{N_2}. \tag{37}$$

These are stochastic differential equations. Rather than solving the differential equations explicitly for sample paths and then taking expectations, the expected paths can be solved for directly. Since expectation and differentiation are linear operations, their order may be interchanged, resulting in differential equations for the means,

$$E\left\{\frac{dq}{dt}\right\} = E\left\{-\frac{q}{RC}\right\} + E\left\{\frac{1}{R}\,v_{N_1}\right\} \Rightarrow \frac{d}{dt}E\left\{q(t)\right\} = -\frac{1}{RC}E\left\{q(t)\right\} + 0$$

and

$$E\left\{\frac{d\phi}{dt}\right\} = E\left\{-\frac{R\phi}{L}\right\} + E\left\{v_{N_2}\right\} \Rightarrow \frac{d}{dt}E\left\{\phi(t)\right\} = -\frac{R}{L}E\left\{\phi(t)\right\} + 0.$$

These simple linear differential equations can be solved explicitly:

$$E\left\{q(\tau)\right\} = E\left\{q(t)\right\} e^{-(\tau - t)/(RC)}, \quad \tau \geq t, \tag{38}$$

and

$$E\left\{\phi(t+\tau)\right\} = E\left\{\phi(t)\right\} e^{-(\tau - t)\,R/L}, \quad \tau \geq t. \tag{39}$$

The variance will also be used in the calculations. The quickest way to obtain the variance is to appeal to the equipartition theorem, which says to each degree of freedom corresponds an energy of $\frac{1}{2}kT$. The capacitor charge is a degree of freedom, so the capacitor will have an expected stored energy at equilibrium

$$E\left\{\frac{q^2}{2C}\right\} = \frac{1}{2}kT \Rightarrow E\left\{q^2(t)\right\} = CkT, \tag{40}$$

and the expected energy stored in the inductor is

$$E\left\{\frac{\phi^2}{2L}\right\} = \frac{1}{2}kT \Rightarrow E\left\{\phi^2(t)\right\} = LkT. \tag{41}$$

Because the noise processes v_{N_1} and v_{N_2} are stationary, these variances have fixed values for all t.

The open-circuit output voltage V can be expressed as

$$V = \frac{q}{C} + \frac{d\phi}{dt} = \frac{q}{C} - \frac{R\,\phi}{L} + v_{N_2}, \tag{42}$$

so the autocorrelation, for $\tau \geq t$, is

$$\begin{aligned}
R_{VV}(t-\tau) &= E\left\{V(t)V(\tau)\right\} \\
&= E\left\{\left[\frac{q(t)}{C} - \frac{R\phi(t)}{L} + v_{N_2}(t)\right]\left[\frac{q(\tau)}{C} - \frac{R\phi(\tau)}{L} + v_{N_2}(\tau)\right]\right\} \\
&= \frac{1}{C^2}E\Big\{q(t)\,q(\tau)\Big\} - \frac{R}{LC}E\Big\{q(t)\,\phi(\tau)\Big\} + \frac{1}{C}E\Big\{q(t)\,v_{N_2}(\tau)\Big\} \\
&\quad -\frac{R}{LC}E\Big\{\phi(t)\,q(\tau)\Big\} + \frac{R^2}{L^2}E\Big\{\phi(t)\,\phi(\tau)\Big\} - \frac{R}{L}E\Big\{\phi(t)\,v_{N_2}(\tau)\Big\} \\
&\quad +\frac{1}{C}E\Big\{v_{N_2}(t)\,q(\tau)\Big\} - \frac{R}{L}E\Big\{v_{N_2}(t)\,\phi(\tau)\Big\} + E\Big\{v_{N_2}(t)\,v_{N_2}(\tau)\Big\} \\
&= \frac{1}{C^2}E\Big\{q(t)\,q(\tau)\Big\} + \frac{R^2}{L^2}E\Big\{\phi(t)\,\phi(\tau)\Big\} - \frac{R}{L}E\Big\{\phi(t)\,v_{N_2}(\tau)\Big\} \\
&\quad -\frac{R}{L}E\Big\{v_{N_2}(t)\,\phi(\tau)\Big\} + E\Big\{v_{N_2}(t)\,v_{N_2}(\tau)\Big\}.
\end{aligned} \tag{43}$$

The last equality follows because, since v_{N_1} and v_{N_2} are independent, $q(\cdot)$ is uncorrelated with either $\phi(\cdot)$ or $v_{N_2}(\cdot)$. Further, for $\tau \geq t$, because of the independent increments property of the noise process, $\phi(t)$ is uncorrelated with $v_{N_2}(\tau)$, so

$$E\left\{\phi(t)\,v_{N_2}(\tau)\right\} = 0, \quad \tau \geq t. \tag{44}$$

For the Gaussian white-noise process $v_{N_2}(t)$,

$$E\Big\{v_{N_2}(t)\,v_{N_2}(\tau)\Big\} = 2kTR\,\delta(t-\tau). \tag{45}$$

Using the means and variances found above, the other expectations are easily computed for $\tau > t$.

$$\begin{aligned} E\Big\{q(t)\,q(\tau)\Big\} &= E\Big\{q(t)\,E\Big\{q(\tau)\;\Big|\;q(t)\Big\}\Big\} \\ &= E\Big\{q(t)\;\big[q(t)e^{-(\tau-t)/(RC)}\big]\Big\} \\ &= E\Big\{q(t)\,q(t)\Big\}\;e^{-(\tau-t)/(RC)} \\ &= CkT\;e^{-(\tau-t)/(RC)}, \quad \tau \geq t \end{aligned} \tag{46}$$

$$\begin{aligned} E\Big\{\phi(t)\,\phi(\tau)\Big\} &= E\Big\{\phi(t)\,E\Big\{\phi(\tau)\;\Big|\;\phi(t)\Big\}\Big\} \\ &= E\Big\{\phi(t)\;\big[\phi(t)e^{-(\tau-t)\,R/L}\big]\Big\} \\ &= E\Big\{\phi(t)\,\phi(t)\Big\}\;e^{-(\tau-t)\,R/L} \\ &= LkT\;e^{-(\tau-t)\,R/L}, \quad \tau \geq 0 \end{aligned} \tag{47}$$

The last remaining term is a little more tricky to calculate. Starting with the differential equation (37) and multiplying through by $\phi(\tau)$ yields

$$\frac{d\phi(t)}{dt}\,\phi(\tau) = -\frac{R\,\phi(t)}{L}\,\phi(\tau) + v_{N_2}(t)\,\phi(\tau). \tag{48}$$

Take expectations on both sides of the equation,

$$E\left\{\frac{d\phi(t)}{dt}\,\phi(\tau)\right\} = -E\left\{\frac{R\,\phi(t)}{L}\,\phi(\tau)\right\} + E\left\{v_{N_2}(t)\,\phi(\tau)\right\}, \tag{49}$$

and then interchange the order of differentiation and expectation

$$\frac{d}{dt}E\left\{\phi(t)\,\phi(\tau)\right\} = -\frac{R}{L}E\left\{\phi(t)\,\phi(\tau)\right\} + E\left\{v_{N_2}(t)\,\phi(\tau)\right\}. \tag{50}$$

The term $E\left\{\phi(t)\,\phi(\tau)\right\}$ was just calculated in Eq. (47). Using it and its derivative,

$$\frac{R}{L}LkT\;e^{-(\tau-t)R/L} = -\frac{R}{L}\left(LkT\;e^{-(\tau-t)\,R/L}\right) + E\left\{v_{N_2}(t)\,\phi(\tau)\right\}, \quad \tau \geq t,$$

or equivalently,

$$E\Big\{v_{N_2}(t)\,\phi(\tau)\Big\} = 2RkT\;e^{-(\tau-t)\,R/L}, \quad \tau \geq t. \tag{51}$$

Now that the expectations have been calculated, Eqs. (45), (46), (47), and (51) can be substituted into Eq. (43),

$$\begin{aligned} R_{VV}(t-\tau) &= \frac{1}{C^2}\left(CkT\;e^{-(\tau-t)/(RC)}\right) + \frac{R^2}{L^2}\left(LkT\;e^{-(\tau-t)\,R/L}\right) - \frac{R}{L}\left(2RkT\;e^{-(\tau-t)\,R/L}\right) \\ &\quad +2kTR\,\delta(t-\tau), \quad \tau \geq t. \end{aligned} \tag{52}$$

Under the matching condition $\frac{R}{L} = \frac{1}{RC}$, the time constants are equal, and further, the coefficients are related by

$$\frac{kT}{C} = \frac{kTR^2}{L}.$$

Thus, the autocorrelation simplifies to

$$R_{VV}(t-\tau) = 2kTR\,\delta(t-\tau), \quad \tau \geq t. \tag{53}$$

For $\tau < t$, the calculations must be redone. In this linear case, it is simple to re-express the differential equations for the means, under the condition $\tau < t$. For example, Eq. (38) becomes

$$E\{q(t)\} = E\{q(\tau)\}\, e^{-(t-\tau)/(RC)}, \quad \tau < t. \tag{54}$$

Proceeding along these lines will show that the expressions (46) and (47) should have $-|\tau - t|$ rather than $-(\tau - t)$ in the exponents. Such is not the case for (51), which vanishes for $\tau < t$ by the independent increments property. However, the corresponding term is found in $E\{\phi(t)\, v_{N_2}(\tau)\}$, which no longer vanishes as asserted in (44), but instead yields

$$E\Big\{\phi(t)\, v_{N_2}(\tau)\Big\} = 2RkT\; e^{-(t-\tau)\,R/L}, \quad \tau < t. \tag{55}$$

In the nonlinear case, a more complicated reversibility argument will be necessary to calculate this last expectation.

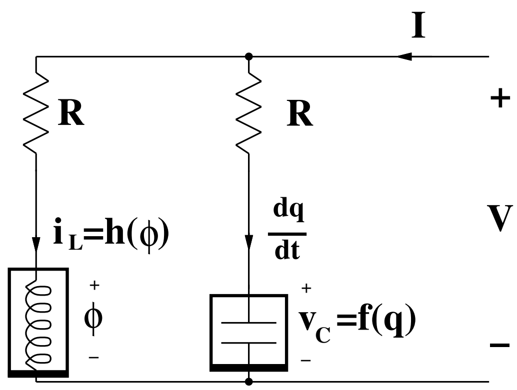

Figure 6: Nonlinear noise-free bridge circuit

6 Nonlinear, Noise-Free Case

In this section, the capacitor and inductor are allowed to be nonlinear, as shown in Figure 6. The functions $f(\cdot)$ and $h(\cdot)$ are assumed to satisfy the following standard assumptions [8, 9, 10]:

- $f(0) = 0, \; h(0) = 0$
- h and f are continuously differentiable functions, and for all values of the arguments and some fixed $\epsilon > 0$,

$$\frac{df}{dq} \geq \epsilon > 0 \qquad \text{and} \qquad \frac{dh}{d\phi} \geq \epsilon > 0.$$

The function $h(\cdot)$ need not be odd. This system is a special case of the situation considered in Ref. [10], where the inductors were required to have odd characteristics in order that the flux random processes be reversible. The system matrix is diagonal and hence symmetric because the states are decoupled for any driving voltage V (that does not depend on I).

The circuit differential equations are

$$\frac{dq}{dt} = \frac{V - f(q)}{R}, \tag{56}$$

$$\frac{d\phi}{dt} = V - R\,h(\phi), \tag{57}$$

and the port current is

$$I = h(\phi) + \frac{dq}{dt} = h(\phi) + \frac{V}{R} - \frac{f(q)}{R}. \tag{58}$$

The goal again is to show that $I = V/R$, which requires

$$h(\phi) = \frac{f(q)}{R}. \tag{59}$$

This is not a sufficient condition, because as yet, there is no relation between q and ϕ.

The two parallel branches can be used to write two distinct expressions for the port voltage V:

$$V = R\,h(\phi) + \frac{d\phi}{dt} \quad (60)$$

$$= R\,\frac{dq}{dt} + f(q). \quad (61)$$

If $f(q) = R\,h(\phi)$, then these last two expressions combine to yield

$$\frac{dq}{dt} = \frac{1}{R}\frac{d\phi}{dt}. \quad (62)$$

With zero initial conditions for q and ϕ, it follows that

$$q(t) = \frac{\phi(t)}{R}, \quad (63)$$

and therefore the full matching condition is

$$h(\phi) = \frac{f\left(\frac{\phi}{R}\right)}{R} \quad \text{or} \quad f(q) = R\,h(Rq). \quad (64)$$

This condition is really a constraint on the functions f and h, and not on the dummy variables q or ϕ. It may be more helpful to express it as

$$f(x) = R\,h(R\,x).$$

Alternately, if one considers $f'(q) = 1/C(q)$ as the reciprocal of the incremental capacitance and $h'(\phi) = 1/L(\phi)$ as the reciprocal of the incremental inductance, then

$$L(\phi) = R^2\,C(q)|_{q=\phi/R}, \quad (65)$$

a local version of the linear matching condition $L = R^2 C$. Under the full matching condition with zero initial conditions, the deterministic terminal behavior of the matched nonlinear bridge is $I = V/R$: the second-order nonlinear circuit reduces to a simple linear resistor.

What happens under non-zero initial conditions for q and ϕ? Observe from Eqs. (56) and (57) that, under the matching condition (64) but irrespective of $V(\cdot)$,

$$\begin{aligned}\frac{d(\phi - Rq)}{dt} &= -R\,h(\phi) + f(q) \\ &= -R\,h(\phi) + R\,h(Rq) \\ &= -R\,h(\xi)\,(\phi - Rq)\,, \quad (66)\end{aligned}$$

where ξ lies between ϕ and Rq, by application of the Mean Value Theorem. It follows that

$$\begin{aligned}\frac{d}{dt}\left[\phi - Rq\right]^2 &= -2R\,h(\xi)\,(\phi - Rq)^2 \\ &\leq -2R\,\epsilon\,\left(\phi - Rq\right)^2, \quad (67)\end{aligned}$$

using the assumed lower bound on the derivative of h. Therefore, the quantity $(\phi - Rq)$ decays to zero at least exponentially quickly. After a sufficiently long time, the matched nonlinear bridge circuit is again indistinguishable from a linear resistor.

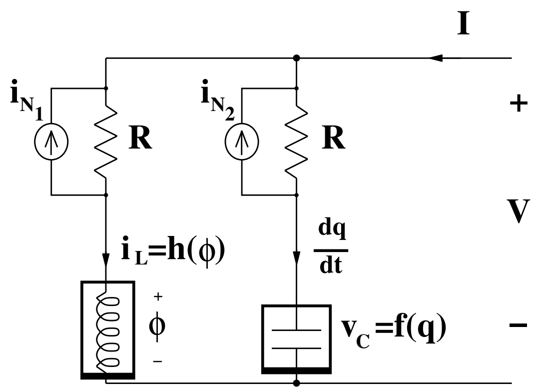

Figure 7: Nonlinear bridge circuit with noise

7 Nonlinear Noisy Case

For this section, a Norton-form Nyquist-Johnson noise model is associated with each resistor in the circuit, as in Figure 7. We would like to show that the terminal current noise of the matched bridge is the same as that for a single linear resistor. To first order, this result is clear. Recall that the incremental capacitance and inductance satisfy $L(\phi) = R^2\, C(q)|_{q=\phi/R}$. A linearization about the noise-free equilibrium operating point (q, ϕ) for a d.c. applied voltage of a nonlinear matched circuit will yield a matched linear circuit. By superposition, the noise current for the linearized circuit is unaffected by the applied voltage. The point of this section is to show that this equivalence holds *exactly*, even for high temperatures or strong nonlinearities for which the noise could drive the circuit out of the valid region of linearization.

The circuit is described by stochastic differential equations (SDE's):

$$\frac{dq}{dt} = \frac{V - f(q)}{R} - i_{N_2} \tag{68}$$

$$\frac{d\phi}{dt} = V - R\,h(\phi) - R\,i_{N_1}, \tag{69}$$

where i_{N_1} and i_{N_2} are independent Gaussian white noise processes with power spectral density $2kT/R$. The port current is

$$\begin{aligned} I &= h(\phi) + \frac{dq}{dt} \\ &= h(\phi) + \frac{V}{R} - \frac{f(q)}{R} - i_{N_2}. \end{aligned} \tag{70}$$

One might be tempted to use the matching condition (59) and immediately conclude $I = V/R - i_{N_2}$. However, the full matching condition (64) does *not* hold, because the derivation was for a different excitation: q and ϕ no longer satisfy $q = \phi/R$, because they are now driven by independent noise sources. So, the power spectrum of I must be calculated in a more methodical way.

Before proceeding to study the noise power spectrum, we show that the nonlinear inductor and capacitor cannot "rectify" the noise. Rectification would cause incorrect "average" behavior, or first-order statistics of the circuit, such that it would be pointless to study the second-order statistic of the power spectral density. We will show that

$$E\{I\} = V/R \tag{71}$$

for any (deterministic) V; then specifically for $V = 0$, I will be zero-mean.

Taking expectations on both sides of Eq. (70),

$$E\{I\} = E\{h(\phi)\} + \frac{V}{R} - \frac{E\{f(q)\}}{R} - 0. \tag{72}$$

The expectations of $f(q)$ and $h(\phi)$ may be computed by integration against the densities for q and ϕ. For a random variable driven by Gaussian white noise, the Fokker-Planck equation (FPE) [11, 12] converts the stochastic differential equation into a partial differential equation in the density of the random variable. For the circuit of this section, the densities obey

$$\frac{dq}{dt} = \frac{V - f(q)}{R} - i_{N_2} \quad \Rightarrow \quad \frac{\partial \rho_q}{\partial t} = -\frac{\partial}{\partial q}\left[\frac{V - f(q)}{R}\rho_q\right] + \frac{kT}{R}\frac{\partial^2 \rho_q}{\partial q^2}, \tag{73}$$

$$\frac{d\phi}{dt} = V - R\,h(\phi) - R\,i_{N_1} \quad \Rightarrow \quad \frac{\partial \rho_\phi}{\partial t} = -\frac{\partial}{\partial \phi}\Big[\Big(V - R\,h(\phi)\Big)\rho_\phi\Big] + kTR\frac{\partial^2 \rho_\phi}{\partial \phi^2}. \tag{74}$$

Using the matching condition $f(q) = R\,h(Rq)$, these two equations become identical up to a scaling. Suppose a density $\rho_\phi(\phi, t)$ satisfies Eq. (74). Then use the change-of-variables formula for probability densities to define

$$\rho_q(t, q) = \rho_\phi\left(t, \phi(q)\right)\frac{\partial \phi(q)}{dq} = \rho_\phi(t, Rq)\,R \tag{75}$$

and substitute this definition into (73) along with the matching condition (59):

$$\frac{\partial}{\partial t}\left[R\rho_\phi(t, Rq)\right] = -\frac{\partial}{\partial q}\left[\frac{V - Rh(Rq)}{R}R\rho_\phi(t, Rq)\right] + \frac{kT}{R}\frac{\partial^2}{\partial q^2}\Big[R\rho_\phi(t, Rq)\Big]. \tag{76}$$

One may divide by R to remove that factor from the left-hand side, then group a factor of R into each $\partial/\partial q$:

$$\frac{\partial}{\partial t}\rho_\phi(t, Rq) = -\frac{\partial}{\partial (qR)}\left[\frac{V - Rh(Rq)}{R}\rho_\phi(t, Rq)\right] + \frac{kT}{R}R^2\frac{\partial^2}{\partial (qR)^2}\rho_\phi(t, Rq). \tag{77}$$

The variables q and ϕ in the Fokker-Planck equations are dummy variables; the change of variables according to $\phi = qR$ does not assert any relationship between the waveforms $q(t)$ and $\phi(t)$ in the actual circuit. We have just transformed Eq. (73) into Eq. (74). The densities corresponding to zero initial conditions (delta functions) also satisfy Eq. (75) at $t = 0$:

$$\delta(q) = R\,\delta(\phi)|_{\phi = Rq} \tag{78}$$

The densities

$$\rho_q(q) = A_q \exp\left[\frac{1}{kT}\int_0^q (V - f(\tilde{q}))\, d\tilde{q}\right] \quad (79)$$

$$\rho_\phi(\phi) = A_\phi \exp\left[\frac{1}{kT}\int_0^\phi \left(\frac{V}{R} - h(\tilde{\phi})\right) d\tilde{\phi}\right], \quad (80)$$

where A_q and A_ϕ are normalization constants, are the steady-state solutions to Eqs. (73) and (74), and also satisfy Eq. (75) at $t = 0$. (See Appendix II for a derivation of these densities and a discussion on the meaning of steady-state.) Thus, the solutions of Eqs. (73) and (74) satisfy Eq. (75) for all time. Therefore,

$$\begin{aligned} E\{f(q)\} &= \int_{-\infty}^{+\infty} f(x)\rho_q(x)\, dx = \int_{-\infty}^{+\infty} R\, h(Rx)\, \rho_\phi(R\, x)\, R\, dx \\ &= R\int_{-\infty}^{+\infty} h(y)\, \rho_\phi(y)\, dy \\ &= RE\{h(\phi)\}, \end{aligned} \quad (81)$$

where the dummy variable $y = Rx$ (with $dy = R\, dx$) was used to go between the first and second lines. Therefore, the relationship $E\{I\} = V/R$ has been established.

The short-circuit ($V = 0$) power spectral density is the Fourier transform of the autocorrelation of I, denoted R_{II}. For $V = 0$ and $\tau > t$,

$$\begin{aligned} R_{II}(t-\tau) &= E\left\{\left[h(\phi(t)) - \frac{f(q(t))}{R} - i_{N_2}(t)\right]\left[h(\phi(\tau)) - \frac{f(q(\tau))}{R} - i_{N_2}(\tau)\right]\right\} \\ &= E\Big\{h(\phi(t))\, h(\phi(\tau))\Big\} + E\Big\{\frac{f(q(t))}{R}\, \frac{f(q(\tau))}{R}\Big\} + E\Big\{i_{N_2}(t)\, \frac{f(q(\tau))}{R}\Big\} \\ &\quad + E\Big\{i_{N_2}(t)\, i_{N_2}(\tau)\Big\}. \end{aligned} \quad (82)$$

The perhaps surprising simplification under which five of the nine terms generated by the product disappear occurs for the same two reasons as in the linear case. Firstly, for $V = 0$, the two circuit branches are disjoint, so that $h(\phi(\cdot))$ is not correlated with $f(q(\cdot))$ or $i_{N_2}(\cdot)$, regardless of the time arguments. Secondly, $f(q(t))$ depends only on past values of $i_{N_2}(t)$ and hence is uncorrelated with $i_{N_2}(\tau)$ for $\tau > t$ by causality and the independent-increments property of $i_{N_2}(\cdot)$.

Using the relation between the densities for q and ϕ again,

$$E\{f(q(t))\, f(q(\tau))\} = R^2\, E\{h(\phi(t))\, h(\phi(\tau))\}, \quad (83)$$

even though the expectations have not been calculated explicitly (see Appendix III, however, for an explicit verification that this identity holds). Therefore, if it can be shown that

$$\frac{2}{R^2}\, E\{f(q(t))\, f(q(\tau))\} = \frac{-1}{R}\, E\{i_{N_2}(t)\, f(q(\tau))\}, \quad (84)$$

this would leave $E\{i_{N_2}(t)\, i_{N_2}(\tau)\}$, which is exactly the desired autocorrelation.

In the $V = 0$ situation, the stochastic differential equation (SDE) for q becomes

$$\frac{dq(t)}{dt} = -\frac{f(q(t))}{R} - i_{N_2}(t). \tag{85}$$

If we multiply through by $f(q(\tau))$ and take expectations,

$$E\left\{\frac{dq(t)}{dt}\, f(q(\tau))\right\} = -\frac{1}{R}E\Big\{f(q(t))\, f(q(\tau))\Big\} - E\Big\{i_{N_2}(t)\, f(q(\tau))\Big\}. \tag{86}$$

The dummy time indices t and τ may be interchanged, corresponding to writing the SDE in τ and multiplying through by $f(q(t))$, to get

$$E\left\{\frac{dq(\tau)}{d\tau}\, f(q(t))\right\} = -\frac{1}{R}E\Big\{f(q(\tau))\, f(q(t))\Big\} - E\Big\{i_{N_2}(\tau)\, f(q(t))\Big\}. \tag{87}$$

By causality and independent-increments, for $\tau > t$,

$$E\left\{i_{N_2}(\tau)\, f(q(t))\right\} = 0.$$

Since differentiation and expectation are linear operations, their order can be interchanged.

$$E\left\{\frac{dq(t)}{dt}\, f(q(\tau))\right\} = \frac{d}{dt}E\left\{q(t)\, f(q(\tau))\right\} \quad \text{and} \quad E\left\{\frac{dq(\tau)}{d\tau}\, f(q(t))\right\} = \frac{d}{d\tau}E\left\{q(\tau)\, f(q(t))\right\}$$

Further, since the stochastic process i_{N_2} is stationary, the process $q(t)$ is also, and thus the expectation is only a function of the difference $(t - \tau)$. Let $F(\cdot)$ be defined by

$$F(t-\tau) \stackrel{\Delta}{=} E\left\{q(t)\, f(q(\tau))\right\} \quad \Rightarrow \quad E\left\{q(\tau)\, f(q(t))\right\} = F(\tau - t).$$

As a consequence of the assumptions on $f(\cdot)$ and Eq. (85), $q(t)$ is a reversible process [10]. This means that for all t_1 and t_2,

$$\Pr\left[\alpha \le q(t_1) \le \alpha + d\alpha, \beta \le q(t_2) \le \beta + d\beta\right] = \Pr\left[\beta \le q(t_1) \le \beta + d\beta, \alpha \le q(t_2) \le \alpha + d\alpha\right]$$

As a consequence of this reversibility, F is an even function:

$$\begin{aligned} F(t-\tau) &= E\left\{q(t)\, f(q(\tau))\right\} = \iint a f(b)\, p(q(t)=a, q(\tau)=b)\, da\, db \\ &= \iint a f(b)\, p(q(\tau)=a, q(t)=b)\, da\, db = E\left\{q(\tau)\, f(q(t))\right\} \\ &= F(\tau - t), \end{aligned} \tag{88}$$

where $p(\cdot\,,\,\cdot)$ represents the joint probability density of its two arguments, and equality between the first and second lines follows from reversibility. Since $F(\cdot)$ is an even function, $F'(\cdot)$ must be odd. Hence, $F'(t-\tau) = -F'(\tau - t)$, and

$$\begin{aligned} \frac{d}{dt}E\left\{q(t)\, f(q(\tau))\right\} &= \frac{d}{dt}F(t-\tau) = F'(t-\tau) \\ &= -F'(\tau - t) = -\frac{d}{d\tau}F(\tau - t) \\ &= -\frac{d}{d\tau}E\left\{q(\tau)\, f(q(t))\right\}. \end{aligned} \tag{89}$$

Thus, by adding together (86) and (87), the left-hand sides cancel, so that

$$0 = -\frac{1}{R}E\left\{f(q(t))\,f(q(\tau))\right\} - \frac{1}{R}E\left\{f(q(\tau))\,f(q(t))\right\} - E\left\{i_{N_2}(t)\,f(q(\tau))\right\},$$

or, equivalently,

$$\frac{2}{R}\,E\left\{f(q(t))f(q(\tau))\right\} = -E\left\{i_{N_2}(t)\,f(q(\tau))\right\}, \tag{90}$$

which differs from (84) by only a common factor of R. Therefore,

$$R_{II}(t-\tau) = E\{I(t)\,I(\tau)\} = E\left\{i_{N_2}(t)\,i_{N_2}(\tau)\right\} = \frac{2kT}{R}\,\delta(t-\tau).$$

The nonlinear noisy matched bridge circuit has a short-circuit noise current spectral density precisely the same as a single linear resistor. Appendix IV will show that the noise process is Gaussian white noise.

8 Nonequilibrium Situations

The noise results so far have all been equilibrium results. For nonequilibrium situations, the noise component of the total current is

$$n(t) = I(t) - \frac{V(t)}{R}.$$

As mentioned in the introduction, there are some situations in which the noise current of the matched bridge is not statistically equivalent to the noise of a single linear resistor. However, the result does hold in some specific nonequilibrium situations. For example, if the two resistors of the matched linear bridge are maintained at the same constant temperature by a large thermal reservoir, the noise component of the output current will be Gaussian white noise with power spectral density $2kT/R$, for any driving voltage $V(t)$. This is a result of superposition, in which the deterministic driven response is decoupled from the noise response. Since the proper behavior has been proven for the responses individually, the total response will be correct. The explicit calculations are found in Appendix I.

In the nonlinear case, we cannot use superposition to establish our result. Nevertheless, it is interesting that the result will still hold if the driving voltage is constant. Unfortunately, the result does not hold in a time-varying circuit, and this failure casts doubts on the hopes of establishing the general nonlinear nonequilibrium result for time-varying voltage.

8.1 Non-zero drive voltage

Consider the circuit of Figure 4, again described by Eqs. (68) to (70) with the matching condition (59) in force, but now with an arbitrary driving voltage applied to its terminals. Denote by R_{nn} the autocorrelation of $n(t) = I(t) - V(t)/R$. Then

$$R_{nn}(t-\tau) = \frac{2kT}{R}\,\delta(t-\tau), \tag{91}$$

provided that either (a) the circuit is in fact linear, or (b) the voltage $V(t)$ is constant.

For an arbitrary driving voltage $V(t)$, the autocorrelation of $n(t)$ is

$$\begin{aligned}
R_{nn}(t,\tau) &= E\Big\{\Big[h(\phi(t)) - \frac{f(q(t))}{R} - i_{N_2}(t)\Big]\Big[h(\phi(\tau)) - \frac{f(q(\tau))}{R} - i_{N_2}(\tau)\Big]\Big\} \\
&= E\Big\{h(\phi(t))\,h(\phi(\tau))\Big\} - E\Big\{h(\phi(t))\,\frac{f(q(\tau))}{R}\Big\} - E\Big\{h(\phi(t))\,i_{N_2}(\tau)\Big\} \\
&\quad -E\Big\{\frac{f(q(t))}{R}\,h(\phi(\tau))\Big\} + E\Big\{\frac{f(q(t))}{R}\,\frac{f(q(\tau))}{R}\Big\} + E\Big\{\frac{f(q(t))}{R}\,i_{N_2}(\tau)\Big\} \\
&\quad -E\Big\{i_{N_2}(t)\,h(\phi(\tau))\Big\} + E\Big\{i_{N_2}(t)\,\frac{f(q(\tau))}{R}\Big\} + E\Big\{i_{N_2}(t)\,i_{N_2}(\tau)\Big\}.
\end{aligned}$$

As in the short-circuit case, $h(\phi(\cdot))$ is not correlated with $i_{N_2}(\cdot)$, so that

$$E\Big\{h(\phi(t))\,i_{N_2}(\tau)\Big\} = E\Big\{h(\phi(t))\Big\}\,E\Big\{i_{N_2}(\tau)\Big\} = 0.$$

The process $h(\phi(\cdot))$ is also uncorrelated with $f(q(\cdot))$, but these processes are no longer zero-mean, so we can only separate the expectation of the product into the product of the expectations,

$$E\Big\{h(\phi(t))\ \frac{f(q(\tau))}{R}\Big\} = E\Big\{h(\phi(t))\Big\}\ E\Big\{\frac{f(q(\tau))}{R}\Big\}.$$

Because of the similarity of the Fokker-Planck equations (73) and (74),

$$E\Big\{f(q(t))\Big\} = R\,E\Big\{h(\phi(t))\Big\},$$

for all times t (or τ), and also

$$E\Big\{h(\phi(t))\ h(\phi(\tau))\Big\} = E\Big\{\frac{f(q(t))}{R}\ \frac{f(q(\tau))}{R}\Big\}.$$

The autocorrelation can thus be simplified to

$$\begin{aligned} R_{nn}(t,\tau) \;=\;& 2E\Big\{\frac{f(q(t))}{R}\ \frac{f(q(\tau))}{R}\Big\} - 2E\Big\{\frac{f(q(t))}{R}\Big\}\ E\Big\{\frac{f(q(\tau))}{R}\Big\} \\ & + E\Big\{\frac{f(q(t))}{R}\ i_{N_2}(\tau)\Big\} + E\Big\{i_{N_2}(t)\ \frac{f(q(\tau))}{R}\Big\} + E\Big\{i_{N_2}(t)\ i_{N_2}(\tau)\Big\}. \end{aligned} \tag{92}$$

Using the trick of Section 7, multiplying both sides of the differential equation for $q(t)$ by $f(q(\tau))$, we obtain

$$\frac{d}{dt}E\Big\{f(q(\tau))\ q(t)\Big\} = \frac{V(t)}{R}\ E\Big\{f(q(\tau))\Big\} - E\Big\{f(q(t))\ f(q(\tau))\Big\} - E\Big\{i_{N_2}(t)\ f(q(\tau))\Big\}. \tag{93}$$

Define

$$F(t,\tau) = E\Big\{f(q(\tau))\ q(t)\Big\},$$

so that the autocorrelation may be expressed

$$\begin{aligned} R_{nn}(t,\tau) \;=\;& \left[\frac{V(t)}{R} - E\left\{\frac{f(q(t))}{R}\right\}\right] E\left\{\frac{f(q(\tau))}{R}\right\} + \left[\frac{V(\tau)}{R} - E\left\{\frac{f(q(\tau))}{R}\right\}\right] E\left\{\frac{f(q(t))}{R}\right\} \\ & - \left[\frac{dF(t,\tau)}{dt} + \frac{dF(t,\tau)}{d\tau}\right] + E\Big\{i_{N_2}(t)\ i_{N_2}(\tau)\Big\}. \end{aligned} \tag{94}$$

For arbitrary time-varying $V(t)$, no further simplification is apparent. However, if V is constant, *i.e.*, the system is at steady-state, then

$$E\left\{\frac{dq}{dt}\right\} = 0,$$

and by taking expectations of both sides of the differential equation (68),

$$E\left\{\frac{dq}{dt}\right\} = 0 = E\left\{\frac{V(t) - f(q(t))}{R}\right\} + E\left\{i_{N_2}(t)\right\},$$

so that

$$V(t) = E\Big\{ f(q(t)) \Big\}.$$

Further, $q(t)$ is again a stationary random process, so that $F(t,\tau) = F(t-\tau)$ and

$$\left[\frac{dF(t,\tau)}{dt} + \frac{dF(t,\tau)}{d\tau} \right] = 0,$$

and therefore

$$R_{nn}(t,\tau) = E\Big\{ i_{N_2}(t)\, i_{N_2}(\tau) \Big\} = \frac{2kT}{R}\,\delta(t-\tau).$$

8.2 Linear time-varying elements

Up to this point, every case that we have been able to solve has yielded the correct result, and we cannot conclude anything from our failure to simplify the equations for the nonlinear case with time-varying drive voltage. The following case will find our first nontrivial exception.

Suppose the energy storage elements in Figure 4 were linear, but time varying. It is sufficient to consider the short-circuit (undriven) behavior. The circuit differential equations are

$$\frac{d\phi}{dt} = -\frac{R\,\phi(t)}{L(t)} - R\, i_{N_1}(t) \tag{95}$$

$$\frac{dq}{dt} = -\frac{q(t)}{R\,C(t)} - i_{N_2}(t), \tag{96}$$

and the port current is

$$I(t) = \frac{\phi(t)}{L(t)} - \frac{q(t)}{R\,C(t)} - i_{N_2}(t). \tag{97}$$

The corresponding matching condition is of course

$$L(t) = R^2\, C(t). \tag{98}$$

The differential equation for $q(t)$ can be solved explicitly in terms of sample paths of the noise process $i_{N_2}(t)$:

$$q(t) = \exp\left[-\int_0^t \frac{ds}{RC(s)} \right] \left(q(0) - \int_0^t i_{N_2(\sigma)}\, \exp\left[-\int_0^\sigma \frac{ds}{RC(s)} \right] d\sigma \right). \tag{99}$$

The autocorrelation function for the port current for $\tau > t$ is

$$\begin{aligned} R_{II}(t,\tau) &= E\left\{ \left[\frac{\phi(t)}{L(t)} - \frac{q(t)}{R\,C(t)} - i_{N_2}(t) \right] \left[\frac{\phi(\tau)}{L(\tau)} - \frac{q(\tau)}{R\,C(\tau)} - i_{N_2}(\tau) \right] \right\} \\ &= \frac{1}{R^2 C(t) C(\tau)} E\Big\{ q(t)\, q(\tau) \Big\} + \frac{1}{L(t)L(\tau)} E\Big\{ \phi(t)\, \phi(\tau) \Big\} + \frac{1}{RC(\tau)} E\Big\{ q(\tau)\, i_{N_2}(t) \Big\} \\ &\quad + E\Big\{ i_{N_2}(t)\, i_{N_2}(\tau) \Big\}, \end{aligned}$$

where the other terms vanish because the variables are uncorrelated (and zero-mean) or by causality, as argued previously. Again by appeal to the Fokker-Planck equations and the matching condition (98), it can be shown that

$$\frac{1}{R^2\, C(t)\, C(\tau)}\; E\Big\{q(t)\; q(\tau)\Big\} = \frac{1}{L(t)\, L(\tau)}\; E\Big\{\phi(t)\; \phi(\tau)\Big\}.$$

Thus, in order that the short-circuit current noise have the proper autocorrelation, it must be shown that

$$\frac{2}{R^2\, C(t)\, C(\tau)}\; E\Big\{q(t)\; q(\tau)\Big\} + \frac{1}{RC(\tau)}\; E\Big\{q(\tau)\, i_{N_2}(t)\Big\} = 0. \tag{100}$$

Two quick calculations from Eq. (99) yield

$$\begin{aligned} E\left\{q(t)\; q(\tau)\right\} \;&=\; \exp\left[-\int_0^t \frac{ds}{RC(s)}\right] \exp\left[-\int_0^\tau \frac{1}{RC(s)} ds\right] \\ &\quad \times \left(E\left\{q^2(0)\right\} + \frac{2kT}{R}\int_0^t \exp\left[2\int_0^\sigma \frac{ds}{RC(s)}\right] d\sigma\right) \end{aligned} \tag{101}$$

and

$$E\left\{q(\tau)\; i_{N_2}(t)\right\} = -\frac{2kT}{R} \exp\left[-\int_0^\tau \frac{ds}{RC(s)}\right] \exp\left[\int_0^t \frac{ds}{RC(s)}\right]. \tag{102}$$

Substituting these into Eq. (100), the test is

$$\begin{aligned} 0 \;&\stackrel{?}{=}\; \frac{2}{R^2 C(t) C(\tau)} \exp\left[-\int_0^t \frac{ds}{RC(s)}\right] \exp\left[-\int_0^\tau \frac{1}{RC(s)} ds\right] \\ &\qquad \times \left(E\left\{q^2(0)\right\} + \frac{2kT}{R}\int_0^t \exp\left[2\int_0^\sigma \frac{ds}{RC(s)}\right] d\sigma\right) \\ &\quad -\frac{1}{RC(\tau)} \frac{2kT}{R} \exp\left[-\int_0^\tau \frac{ds}{RC(s)}\right] \exp\left[\int_0^t \frac{ds}{RC(s)}\right] \\ &= \exp\left[-\int_0^t \frac{1}{RC(s)} ds\right] \left(E\left\{q^2(0)\right\} + \frac{2kT}{R}\int_0^t \exp\left[2\int_0^\sigma \frac{ds}{RC(s)}\right] d\sigma\right) \\ &\quad -\frac{RC(t)}{2}\frac{2kT}{R} \exp\left[\int_0^t \frac{ds}{RC(s)}\right] \\ &= E\left\{q^2(0)\right\} + \frac{2kT}{R}\int_0^t \exp\left[2\int_0^\sigma \frac{ds}{RC(s)}\right] d\sigma - \frac{RC(t)}{2}\frac{2kT}{R} \exp\left[2\int_0^t \frac{ds}{RC(s)}\right] \end{aligned} \tag{103}$$

Differentiating by t will yield a necessary condition for the equation to be true:

$$\begin{aligned} 0 \;&\stackrel{?}{=}\; \frac{2kT}{R} \exp\left[2\int_0^t \frac{ds}{RC(s)}\right] d\sigma - \frac{dC(t)}{dt}\frac{R}{2}\frac{2kT}{R} \exp\left[2\int_0^t \frac{ds}{RC(s)}\right] \\ &\quad -\frac{RC(t)}{2}\frac{2kT}{R} \exp\left[2\int_0^t \frac{ds}{RC(s)}\right] \left(\frac{2}{RC(t)}\right) \\ &= -\frac{dC(t)}{dt}\; kT\; \exp\left[2\int_0^t \frac{ds}{RC(s)}\right]. \end{aligned}$$

Thus, C must be a constant. In this case, the integrals in Eq. (103) can be computed, and if the system starts at equilibrium, *i.e.*, $E\{q^2(0)\} = kTC$, then this condition is sufficient as well as necessary. Of course, if C is a constant, then the bridge is simply the linear, time-invariant case that was already considered in Section 3. A time-varying bridge circuit can be distinguished from a single linear resistor, although it is hard to imagine the black box of the layman's test corresponding specifically to this case.

The importance of this case comes from the following analysis: For a driving voltage $V(t)$ significantly larger than the noise, one could solve the deterministic system and then compute an approximation for the noise behavior by linearization about this time-varying solution. This approximation would behave like the time-varying linear system described above. Since the second-order statistics for that system are incorrect, we believe that the second-order statistics for the nonlinear system driven by a time-varying voltage will not match the statistics of a single linear resistor driven by that same voltage.

9 Conclusions

This paper has considered the application of the fluctuation-dissipation theorem (or, in circuit theory terms, a result relating impedances to noise spectra) to many variations of a particular bridge circuit.

The linear results were previously known, but we are unaware of a reference that includes all of the special cases (the circuit and its dual) and different approaches to the solution (sample-path, frequency-domain, and Fokker-Planck equation as a special case of the nonlinear results).

Further, we have indicated an extension of this fluctuation-dissipation theorem to a nonlinear situation. The spectral calculations for this case have been nontrivial, calling on a reversibility idea and martingale theory.

The positive results hold for a specific time-invariant bridge circuit, linear or nonlinear, in thermal equilibrium or at d.c. steady-state. The linear results have also been shown to hold for the dual circuit and for an arbitrary time-varying drive voltage.

The negative results in Section 7 show that our original fluctuation-dissipation conjecture is not correct as stated and must be limited to exclude time-varying networks and nonlinear networks with time-varying inputs. Is the modified form below correct? This remains an open question in the field, and some of the ideas in [10] may be of assistance.

> *Modified Fluctuation-Dissipation Conjecture for Circuits*: No two zero-state deterministically equivalent *time-invariant* networks can be distinguished by the terminal noise currents *at any d.c. voltage input* when the networks are in statistical steady-state.

The assumptions here remain those in the paragraph preceding the initial formulation (see the Introduction), including LTI Nyquist-Johnson resistors and nonlinear inductors and capacitors. Additional assumptions may be required to guarantee reversibility of the charge or flux random processes.

The further extensions to include nonlinear resistor noise models or multiterminal circuits remain completely unexplored, so far as we know. (For nonlinear resistors, progress is hampered by lack of a universally-accepted noise model for nonlinear devices.)

Appendix I: Linear Noisy Case with Non-zero Drive

The result of Section 3 may also be obtained without short-circuiting the terminals. If the terminals of the linear bridge circuit are driven by a deterministic voltage waveform $V(t)$, the autocorrelation of the output current must reduce to the autocorrelation of the current driven by $V(t)$ applied across a noisy resistor, namely,

$$R_{II}(\tau) = E\left\{I(t)\, I(t+\tau)\right\} = \frac{V(t)\, V(t+\tau)}{R^2} + \frac{2kT}{R}\,\delta(\tau). \tag{104}$$

This result can be claimed immediately as a result of superposition, since in this linear case, the deterministic response and the noise behavior are decoupled. Some calculations verify that claim.

The solutions to the differential equations (7) and (8) with drive terms are

$$q(t) = \int_{-\infty}^{t} e^{-(t-t')/RC} \left[\frac{V(t')}{R} - i_{N_2}(t')\right] dt', \tag{105}$$

$$i(t) = \int_{-\infty}^{t} e^{-(t-t')R/L} \left[\frac{V(t')}{L} - \frac{R}{L} i_{N_1}(t')\right] dt'. \tag{106}$$

The autocorrelation is

$$\begin{aligned}
R_{II}(\tau) &= E\Bigg\{\left[i(t) + \frac{V(t)}{R} - \frac{1}{RC}\, q(t) - i_{N_2}(t)\right] \\
&\qquad \times \left[i(t+\tau) + \frac{V(t+\tau)}{R} - \frac{1}{RC}\, q(t+\tau) - i_{N_2}(t+\tau)\right]\Bigg\} \\
&= E\left\{i(t)\, i(t+\tau)\right\} - \frac{1}{RC}\, E\left\{i(t)\, q(t+\tau)\right\} - \frac{1}{RC}\, E\left\{q(t)\, i(t+\tau)\right\} \\
&\quad + \frac{1}{R^2C^2}\, E\left\{q(t)\, q(t+\tau)\right\} + \frac{1}{RC}\, E\left\{q(t)\, i_{N_2}(t+\tau)\right\} + \frac{1}{RC}\, E\left\{i_{N_2}(t)\, q(t+\tau)\right\} \\
&\quad + E\left\{i(t)\right\} \frac{V(t+\tau)}{R} + \frac{V(t)}{R}\, E\left\{i(t+\tau)\right\} \\
&\quad - \frac{1}{RC}\, E\left\{q(t)\right\} \frac{V(t+\tau)}{R} - \frac{V(t)}{R^2C}\, E\left\{q(t+\tau)\right\} \\
&\quad - E\left\{i(t)\, i_{N_2}(t+\tau)\right\} - E\left\{i_{N_2}(t)\, i(t+\tau)\right\} \\
&\quad - \frac{V(t)}{R}\, E\left\{i_{N_2}(t+\tau)\right\} - E\left\{i_{N_2}(t)\right\} \frac{V(t+\tau)}{R} \\
&\quad + \frac{V(t)\, V(t+\tau)}{R^2} + E\left\{i_{N_2}(t)\, i_{N_2}(t+\tau)\right\}.
\end{aligned}$$

Working backwards, it will be shown that all terms but the two in the last line vanish or cancel. Of course, $E\{i_{N_2}(t)\} = 0$, removing two terms. A quick calculation using Eq. (106) shows that $i(\cdot)$ is uncorrelated with $i_{N_2}(\cdot)$, even for $V \neq 0$.

The single term expectations may be computed from the pathwise solution (105) for q:

$$\begin{aligned} E\{q(t)\} &= E\left\{\int_{-\infty}^{t} e^{-(t-t')/RC} \left[\frac{V(t')}{R} - i_{N_2}(t')\right] dt'\right\} \\ &= \int_{-\infty}^{t} e^{-(t-t')/RC} \left[\frac{V(t')}{R} - E\{i_{N_2}(t')\}\right] dt' \\ &= \int_{-\infty}^{t} e^{-(t-t')/RC} \frac{V(t')}{R}\, dt', \end{aligned} \tag{107}$$

and (106) for i:

$$\begin{aligned} E\{i(t)\} &= E\left\{\int_{-\infty}^{t} e^{-(t-t')R/L} \left[\frac{V(t')}{L} - \frac{R}{L} i_{N_1}(t')\right] dt'\right\} \\ &= \int_{-\infty}^{t} e^{-(t-t')R/L} \left[\frac{V(t')}{L} - \frac{R}{L} E\{i_{N_1}(t')\}\right] dt' \\ &= \int_{-\infty}^{t} e^{-(t-t')R/L} \frac{V(t')}{L}\, dt'. \end{aligned} \tag{108}$$

Therefore, under the matching condition $R/L = 1/RC$,

$$\begin{aligned} E\{i(t)\} \frac{V(t+\tau)}{R} &= \left[\int_{-\infty}^{t} e^{-(t-t')R/L} \frac{V(t')}{L}\, dt'\right] \frac{V(t+\tau)}{R} \\ &= \frac{1}{RC}\left[\int_{-\infty}^{t} e^{-(t-t')/RC} \frac{V(t')}{R}\, dt'\right] \frac{V(t+\tau)}{R} \\ &= \frac{1}{RC} E\{q(t)\} \frac{V(t+\tau)}{R}, \end{aligned} \tag{109}$$

and the same holds if the time arguments are switched. (Note that $E\{i(t)\} \neq E\{i(t+\tau)\}$ in general, but the cancellation does not require this.) This leaves a more manageable set of terms in the autocorrelation.

$$\begin{aligned} R_{II}(\tau) &= E\{i(t)\, i(t+\tau)\} - \frac{1}{RC} E\{i(t)\, q(t+\tau)\} - \frac{1}{RC} E\{q(t)\, i(t+\tau)\} \\ &\quad + \frac{1}{R^2C^2} E\{q(t)\, q(t+\tau)\} + \frac{1}{RC} E\{q(t)\, i_{N_2}(t+\tau)\} + \frac{1}{RC} E\{i_{N_2}(t)\, q(t+\tau)\} \\ &\quad + \frac{V(t)\, V(t+\tau)}{R^2} + E\{i_{N_2}(t)\, i_{N_2}(t+\tau)\}. \end{aligned} \tag{110}$$

Then, by using the pathwise solutions (105) and (106), the remaining expectations can be calculated just as they were in the previous section.

$$
\begin{aligned}
E\{i(t)\, i(t+\tau)\} &= E\Bigg\{\left[\int_{-\infty}^{t} e^{-(t-t')R/L}\, \left(\frac{V(t')}{L} - \frac{R}{L} i_{N_1}(t')\right)\, dt'\right] \\
&\quad \times \left[\int_{-\infty}^{t+\tau} e^{-(t+\tau-t'')R/L}\, \left(\frac{V(t'')}{L} - \frac{R}{L} i_{N_1}(t'')\right)\, dt''\right]\Bigg\} \\
&= \int_{-\infty}^{t}\int_{-\infty}^{t+\tau} e^{-(2t+\tau-t'-t'')R/L}\, \left[\frac{V(t')\, V(t'')}{L^2} + \frac{R^2}{L^2}\frac{2kT}{R}\, \delta(t'-t'')\, dt'\, dt''\right] \\
&= \int_{-\infty}^{t}\int_{-\infty}^{t+\tau} e^{-(2t+\tau-t'-t'')R/L} \frac{V(t')\, V(t'')}{L^2}\, dt'\, dt'' + \frac{kT}{L}\, e^{-|\tau|R/L} \qquad (111)
\end{aligned}
$$

$$
\begin{aligned}
E\{q(t)\, q(t+\tau)\} &= E\Bigg\{\left[\int_{-\infty}^{t} e^{-(t-t')/RC}\, \left(\frac{V(t')}{R} - i_{N_2}(t')\right)\, dt'\right] \\
&\quad \times \left[\int_{-\infty}^{t+\tau} e^{-(t+\tau-t'')/RC}\, \left(\frac{V(t'')}{R} - i_{N_2}(t'')\right)\, dt''\right]\Bigg\} \\
&= \int_{-\infty}^{t}\int_{-\infty}^{t+\tau} e^{-(2t+\tau-t'-t'')/RC}\, \left[\frac{V(t')\, V(t'')}{R^2} + \frac{2kT}{R}\, \delta(t'-t'')\, dt'\, dt''\right] \\
&= \int_{-\infty}^{t}\int_{-\infty}^{t+\tau} e^{-(2t+\tau-t'-t'')/RC}\, \frac{V(t')\, V(t'')}{R^2}\, dt'\, dt'' + kTC\, e^{-|\tau|/RC} \qquad (112)
\end{aligned}
$$

Under the matching condition, $\frac{1}{L^2} = \frac{1}{R^2C^2}\frac{1}{R^2}$, $\frac{1}{L} = \frac{1}{R^2C^2}\, C$, and $R/L = 1/RC$, so that

$$
E\{i(t)\, i(t+\tau)\} = \frac{1}{R^2C^2} E\{q(t)\, q(t+\tau)\}\,. \qquad (113)
$$

The expectation

$$
E\{i_{N_2}(t)\, q(t+\tau)\} = -\frac{2kT}{R}\, e^{-\tau/RC}, \quad \tau > 0, \qquad (114)
$$

found in Eq. (17) is still valid because $V(\cdot)$ is uncorrelated with $i_{N_2}(\cdot)$. However, the two remaining terms of Eq. (110) were not calculated in the previous case because $q(\cdot)$ and $i(\cdot)$ were uncorrelated. This is no longer true with a non-zero applied voltage; instead,

$$
\begin{aligned}
E\{i(t)\, q(t+\tau)\} &= E\Bigg\{\left[\int_{-\infty}^{t} e^{-(t-t')R/L}\, \left(\frac{V(t')}{L} - \frac{R}{L} i_{N_1}(t')\right)\, dt'\right] \\
&\quad \times \left[\int_{-\infty}^{t+\tau} e^{-(t+\tau-t'')/RC}\, \left(\frac{V(t'')}{R} - i_{N_2}(t'')\right)\, dt''\right]\Bigg\} \\
&= \int_{-\infty}^{t}\int_{-\infty}^{t+\tau} e^{-(t-t')R/L}\, e^{-(t+\tau-t'')/RC}\, \frac{V(t')\, V(t'')}{R^2}, \qquad (115)
\end{aligned}
$$

because the deterministic voltage driving each process is the same (that is, completely correlated), even though the random current processes driving them are independent. Since the time constants are equal under the matching conditions,

$$
E\{i(t)\, q(t+\tau)\} = E\{q(t)\, i(t+\tau)\}\,. \qquad (116)
$$

The term

$$\int_{-\infty}^{t}\int_{-\infty}^{t+\tau} e^{-(t-t')R/L}\, e^{-(t+\tau-t'')/RC}\, V(t')\, V(t'')$$

appears in four terms of Eq. 110: twice positively ($E\{i(t)\, i(t+\tau)\}$ and $E\{q(t)\, q(t+\tau)\}$) and twice negatively ($E\{i(t)\, q(t+\tau)\}$ and $E\{q(t)\, i(t+\tau)\}$). The matching condition shows that the coefficients are equal in magnitude, so that these terms drop out of the final expression for $R_{II}(\tau)$.

Similarly, just as in the undriven case, the term

$$\frac{kT}{L}\, e^{-|\tau|R/L} = \frac{1}{R^2C^2}\, kTC\, e^{-|\tau|/RC}$$

appears in four terms: positively in $E\{i(t)\, i(t+\tau)\}$ and $E\{q(t)\, q(t+\tau)\}$, and negatively but doubled in $E\{i_{N_2}(t)\, q(t+\tau)\}$ for $\tau > 0$ or $E\{i_{N_2}(t+\tau)\, q(t)\}$ for $\tau < 0$. Therefore, these terms again cancel, proving the claim that

$$R_{II}(\tau) = \frac{V(t)\, V(t+\tau)}{R^2} + \frac{2kT}{R}\, \delta(\tau). \tag{117}$$

Appendix II: Steady-State Densities

In this appendix, we are searching for the steady-state densities solving the Fokker-Planck equations (73) and (74). But first, a word about what steady-state means, and what the solutions to our equations look like.

In the absence of noise, the circuit equations for the nonlinear bridge circuit are

$$\frac{dq}{dt} = \frac{V - f(q)}{R} \tag{118}$$

$$\frac{d\phi}{dt} = V - R\,h(\phi). \tag{119}$$

A *d.c. equilibrium* or *equilibrium operating point* for a fixed (d.c.) voltage V is any point (q, ϕ) for which $\frac{dq}{dt} = \frac{d\phi}{dt} = 0$. Solutions to the circuit differential equations are trajectories in the (q, ϕ) plane.

In the presence of noise, the circuit equations for the nonlinear bridge circuit are stochastic differential equations:

$$\frac{dq}{dt} = \frac{V - f(q)}{R} - i_{N_2} \tag{120}$$

$$\frac{d\phi}{dt} = V - R\,h(\phi) - R\,i_{N_1}. \tag{121}$$

The solutions to these equations can be expressed in two different ways. *Sample-path* solutions are trajectories in the (q, ϕ) plane that show the behavior of the circuit to specific realizations of the noise processes i_{N_1} and i_{N_2}. Sometimes, as in Section 3, the trajectories may be computed explicitly. For most nonlinear stochastic differential equations, however, explicit solutions cannot be calculated.

In such cases, one turns instead to Fokker-Planck equations [11, 12]. In fact, the FPE is also frequently used in the linear case, when one is interested in the statistical behavior of the circuit rather than individual sample paths. The FPE's corresponding to the stochastic differential equations above are, respectively,

$$\frac{\partial \rho_q}{\partial t} = -\frac{\partial}{\partial q}\left[\frac{V - f(q)}{R}\rho_q\right] + \frac{kT}{R}\frac{\partial^2 \rho_q}{\partial q^2}, \tag{122}$$

$$\frac{\partial \rho_\phi}{\partial t} = -\frac{\partial}{\partial \phi}\left[\Big(V - R\,h(\phi)\Big)\rho_\phi\right] + kTR\frac{\partial^2 \rho_\phi}{\partial \phi^2}. \tag{123}$$

The solution to an FPE is a probability density ρ which lives in the infinite-dimensional space of real-valued nonnegative smooth functions (a density is also normalized such that its integral over all space is 1).

A *steady-state* density satisfies $\frac{d\rho}{dt} = 0$. *Thermal equilibrium* (not to be confused with a d.c. equilibrium point) for this circuit is the steady state with $V = 0$.

We are now prepared to search for the steady-state solutions to the FPE's above for a constant V. Starting with the first equation, for the probability density of the capacitor charge,

$$0 = -\frac{\partial}{\partial q}\left[\frac{V-f(q)}{R}\rho_q\right] + \frac{kT}{R}\frac{\partial^2\rho_q}{\partial q^2} \tag{124}$$

$$= -\frac{\partial}{\partial q}\left(\left[\frac{V-f(q)}{R}\rho_q\right] - \frac{kT}{R}\frac{\partial\rho_q}{\partial q}\right). \tag{125}$$

The density ρ must decay to zero at $q = \pm\infty$ so that ρ is integrable; its derivative $\partial\rho/\partial q$ hence necessarily also decays to zero. Therefore, at steady-state, not only must the inside of the large parentheses be a constant with respect to q, that constant must be zero. (For multidimensional systems, one must require reversibility or detailed balance, which are deep physical concepts that essentially say that one cannot tell if time is running forwards or backwards. One could tell the direction of time if probability were exiting at $q = \infty$ but entering at $q = -\infty$ to maintain a total probability of unity.) Thus, we are looking only for solutions to

$$0 = \left(\left[\frac{V-f(q)}{R}\rho_q\right] - \frac{kT}{R}\frac{\partial\rho_q}{\partial q}\right),$$

or, equivalently,

$$\frac{\partial\rho_q}{\partial q} = \frac{V-f(q)}{kT}\ \rho_q. \tag{126}$$

The solution is

$$\rho_q(q) = A_q \exp\left[\frac{1}{kT}\int_0^q (V - f(\tilde{q}))\, d\tilde{q}\right] \tag{127}$$

where A_q is a normalization constant.

Similarly, solving for the probability density of the inductor flux from

$$0 = -\frac{\partial}{\partial\phi}\left[\Big(V - R\,h(\phi)\Big)\rho_\phi\right] + kTR\frac{\partial^2\rho_\phi}{\partial\phi^2}. \tag{128}$$

reduces to

$$\frac{\partial\rho_\phi}{\partial\phi} = \frac{V - R\,h(\phi)}{RkT}\ \rho_\phi. \tag{129}$$

The steady-state solution is then

$$\rho_\phi(\phi) = A_\phi \exp\left[\frac{1}{kT}\int_0^\phi \left(\frac{V}{R} - h(\tilde{\phi})\right) d\tilde{\phi}\right], \tag{130}$$

where A_ϕ is a normalization constants.

Appendix III: Explicit Covariance Calculation

The identity to be proven in this section is

$$E\left\{f(q(t))\ f(q(\tau))\right\} = R^2\ E\left\{h(\phi(t))\ h(\phi(\tau))\right\}. \tag{131}$$

In earlier drafts of this paper, the identity was simply asserted to follow from the Fokker-Planck equations for the densities of q and ϕ, Eqs. (73) and (74), respectively. Further, the FPE is really a conditional density evolution equation, which assumes some initial density that then evolves in time. So, it should not trouble us that we shall be splitting the left-hand side into

$$E\left\{f(q(t))\ f(q(\tau))\right\} = E\Big\{f(q(\tau))\ E\left\{f(q(t))|q(\tau)\right\}\Big\}$$

for $t > \tau$. We shall use the notation

$$\rho_{q(t)}(x)$$

for the probably density of the random variable $q(t)$ evaluated at x, and

$$\rho_{q(\tau)|q(t)}(y|x)$$

for the probably density of the random variable $q(t)$ evaluated at x, conditional on $q(\tau) = x$. Then,

$$\begin{aligned}
E\left\{f(q(t))\ f(q(\tau))\right\} &= E\Big\{f(q(\tau))\ E\left\{f(q(t))|q(\tau)\right\}\Big\} \\
(1) \quad &= \int_{-\infty}^{+\infty} f(x)\rho_{q(\tau)}(x)\left[\int_{-\infty}^{+\infty} f(y)\rho_{q(t)|q(\tau)}(y|x)dy\right]dx \\
(2) \quad &= \int_{-\infty}^{+\infty} R\,h(Rx)\rho_{q(\tau)}(x)\left[\int_{-\infty}^{+\infty} R\,h(Ry)\rho_{q(t)|q(\tau)}(y|x)dy\right]dx \\
(3) \quad &= R^2\int_{-\infty}^{+\infty} h(Rx)R\ \rho_{\phi(\tau)}(Rx)\left[\int_{-\infty}^{+\infty} h(Ry)\rho_{q(t)|q(\tau)}(y|x)dy\right]dx \\
(4) \quad &= R^2\int_{-\infty}^{+\infty} h(x')\ \rho_{\phi(\tau)}(x')\left[\int_{-\infty}^{+\infty} h(Ry)\rho_{q(t)|q(\tau)}\left(y\,\middle|\,\frac{x'}{R}\right)dy\right]dx' \\
(5) \quad &= R^2\int_{-\infty}^{+\infty} h(x')\ \rho_{\phi(\tau)}(x')\left[\int_{-\infty}^{+\infty} h(y')\rho_{q(t)|q(\tau)}\left(\frac{y'}{R}\,\middle|\,\frac{x'}{R}\right)dy'\right]dx'
\end{aligned}$$

where:

(1) is writing the expectations in terms of densities

(2) uses the matching condition (64) $f(x) = R\,h(Rx)$

(3) uses the change-of-variables (75) for single-time densities, $\rho_q(x) = R\,\rho_\phi(Rx)$

(4) rescales according to $x' = Rx$

(5) rescales according to $y' = Ry$

We now write out the explicit integral for the right-hand side of our identity.

$$\begin{aligned} E\{h(\phi(t))\, h(\phi(\tau))\} &= E\{h(\phi(\tau))\, E\{h(\phi(t))|\phi(\tau)\}\} \\ &= \int_{-\infty}^{+\infty} h(x')\ \rho_{\phi(\tau)}(x') \left[\int_{-\infty}^{+\infty} h(y')\rho_{\phi(t)|\phi(\tau)}\left(y'|x'\right) dy'\right] dx' \end{aligned}$$

So, all that remains to showing our identity is to show that

$$\rho_{q(t)|q(\tau)}\left(\frac{y'}{R}\middle|\frac{x'}{R}\right) = \rho_{\phi(t)|\phi(\tau)}\left(y'|x'\right) \tag{132}$$

If we start with the FPE (73) expressed in the dummy variable y,

$$\frac{\partial\rho_q(s,y)}{\partial s} = -\frac{\partial}{\partial y}\left[\frac{V-f(y)}{R}\rho_q(s,y)\right] + \frac{kT}{R}\frac{\partial^2\rho_q(s,y)}{\partial y^2} \tag{133}$$

we can compute the conditional density as follows:

$$\begin{aligned} \rho_{q(t)|q(\tau)}\left(\frac{y'}{R}\middle|\frac{x'}{R}\right) &= \int_t^\tau \frac{\partial\rho_q(s,y'/R)}{\partial s}\,ds + \rho_{q(\tau)|q(\tau)}\left(\frac{y'}{R}\middle|\frac{x'}{R}\right) \\ &= \int_t^\tau -\frac{\partial}{\partial(y'/R)}\left[\frac{V-f(y'/R)}{R}\rho_q(s,y'/R)\right] + \frac{kT}{R}\frac{\partial^2\rho_q(s,y'/R)}{\partial(y'/R)^2}\,ds \\ &\quad +\delta\left(\frac{y'}{R}-\frac{x'}{R}\right) \\ &= \int_t^\tau -\frac{\partial}{\partial y'}\left[\Big(V-f(y'/R)\Big)\rho_q(s,y'/R)\right] + kTR\frac{\partial^2\rho_q}{\partial y'^2}(s,y'/R)\,ds \\ &\quad +R\,\delta\left(y'-x'\right) \\ &= \int_t^\tau -\frac{\partial}{\partial y'}\left[\Big(V-Rh(y')\Big)\rho_q(s,y'/R)\right] + kTR\frac{\partial^2\rho_q}{\partial y'^2}(s,y'/R)\,ds \\ &\quad +R\,\delta\left(y'-x'\right) \end{aligned} \tag{134}$$

Similarly, from Eq. (74),

$$\begin{aligned} \rho_{\phi(t)|\phi(\tau)}\left(y'|x'\right) &= \int_t^\tau \frac{\partial\rho_\phi(s,y')}{\partial s}\,ds + \rho_{\phi(\tau)|\phi(\tau)}\left(y'|x'\right) \\ &= \int_t^\tau -\frac{\partial}{\partial y'}\left[\Big(V-Rh(y')\Big)\rho_\phi(s,y')\right] + kTR\frac{\partial^2\rho_\phi}{\partial y'^2}(s,y')\,ds \\ &\quad +\delta\left(y'-x'\right) \end{aligned} \tag{135}$$

The initial conditions are scaled,

$$\rho_{q(\tau)|q(\tau)}\left(\frac{y'}{R}\middle|\frac{x'}{R}\right) = \delta\left(\frac{y'}{R}-\frac{x'}{R}\right) = R\,\delta\left(y'-x'\right) = R\rho_{\phi(\tau)|\phi(\tau)}\left(y'|x'\right)$$

and the unconditioned density appears linearly in the integrals, so that the scaling will be preserved as we evolve away from the initial conditions.

Appendix IV: Martingale Theory

A technical question remains: whether the short-circuit current noise process is in fact Gaussian white noise, or whether it just has the same power spectral density. Let $x(t)$ be defined by

$$x(t) = \int_a^t I(\tau)\, d\tau.$$

Then $x(t)$ is a martingale on any finite interval $a \le t \le b$: denoting by $\mathcal{F}_s$ the history of the process up to time s,

$$E\left\{x(t)\,|\,\mathcal{F}_s\right\} = x(s) + E\left\{\int_s^t I(\tau)\, d\tau\right\} = x(s) + \int_s^t E\left\{I(\tau)\right\} d\tau = x(s),$$

because $I(\tau)$ is zero-mean (for $V = 0$). The sample paths of $x(t)$ are continuous, since the right-hand side of (70) consists of all continuous functions [12]. The variance of $x(t)$ is finite:

$$\begin{aligned} E\left\{x^2(t)\right\} &= E\left\{\left[\int_a^t I(\tau)\, d\tau\right]\left[\int_a^t I(\tau')\, d\tau'\right]\right\} = \int_a^t \int_a^t E\left\{I(\tau)I(\tau')\right\} d\tau\, d\tau' \\ &= \int_a^t \int_a^t \frac{2kT}{R}\,\delta(\tau-\tau')d\tau\, d\tau' = \frac{2kT}{R}\,(t-a) < \infty. \end{aligned}$$

And further, for $t > s$,

$$\begin{aligned} E\left\{(x(t)-x(s))^2 \,\middle|\, \mathcal{F}_s\right\} &= E\left\{x^2(t) - 2x(t)x(s) + x^2(s)\,|\,\mathcal{F}_s\right\} \\ &= E\left\{x^2(t)\,|\,\mathcal{F}_s\right\} - 2E\left\{x(t)\,|\,\mathcal{F}_s\right\} + x^2(s) \\ &= x^2(s) + E\left\{\left[\int_s^t I(\tau)\, d\tau\right]\left[\int_s^t I(\tau')\, d\tau'\right]\right\} - 2x(s)x(s) + x^2(s) \\ &= \int_s^t \int_s^t E\left\{I(\tau)I(\tau')\right\} d\tau\, d\tau' = \int_s^t \int_s^t \frac{2kT}{R}\,\delta(\tau-\tau')d\tau\, d\tau' \\ &= \frac{2kT}{R}\,(t-s). \end{aligned}$$

The process $x(t)$ satisfies all of the hypotheses of Theorem 11.9 of Ref. [13], from which we conclude that it is a Brownian motion. Therefore the short-circuit noise current current process is Gaussian white noise.

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