Temporary and Permanent Buyout Prices in Online Auctions

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***Abstract*— Increasingly used in online auctions, buyout prices allow bidders to instantly purchase an item listed. We distinguish between two types of buyout options: a *temporary* option that disappears if a bid above the reserve price is made and a *permanent* option that stays throughout the auction or until it is exercised. In order to develop a methodology for finding temporary and permanent buyout prices that maximize the seller's discounted revenue, and to examine the relative benefit of using each type of option in various environments, we formulate a model featuring time-sensitive bidders with uniform valuations and Poisson arrivals (but endogenous bidding times). We characterize equilibrium bidder strategies in both cases and then solve the problem of maximizing seller's utility by simulation. Our numerical experiments suggest that a seller can increase his revenue significantly by introducing a buyout option. Additionally, while a temporary buyout option promotes early bidding, a permanent option gives an incentive to the bidders to bid late, thus leading to concentrated bids near the end of the auction.**

***Index Terms*— buyout option, online auction, Nash equilibrium**

I. Introduction

IN 2003, the total items listed on *ebay* were around 973 million leading to gross merchandise sales of $24 billion. Although hugely popular, online auctions are typically beset by the problem of large waiting times. An online auction on *ebay*, for example, can last anywhere from one to ten days thus potentially driving away time-sensitive bidders to posted price mechanisms or electronic catalogues. In order to make online auctions more attractive to such bidders, auctioneers have introduced buyout options which offer any potential bidder the opportunity to get the item immediately at the posted price. Equipped with this option an online auction becomes a hybrid between an electronic catalogue and a traditional auction.

Such buyout options are widely used in online auctions. In fact, in the fourth quarter of 2003, fixed income trading (primarily from the "Buy It Now" option) contributed $2.0 billion or 28% of *ebay*'s gross annual merchandise sale during the quarter (url: *http://investor.ebay.com/*). Other examples of buyout options are *Yahoo's* "buy now", *Amazon's* "Take-It" and *ubid's* "uBuy it!".

Depending on how long the option is available, a buyout option is classified as temporary or permanent. A temporary buyout option disappears as soon as a bid above the reserve price is made while a permanent option is available through out the auction. While *ebay's* "Buy It Now" is a temporary buyout option, *Yahoo's* "buy-now", *ubid's* "uBuy it!" and *Amazon's* "Take-It" belong to the latter category.

The presence of two different types of buyout prices motivates the following research questions that we seek to answer: Why does *ebay* use a temporary buyout option while *Yahoo* and *Amazon* prefer the permanent one? From the seller's perspective, does one option always outperform the other? If not, when should the seller prefer one over the another? What is the seller's relative benefit of introducing a buyout option be it temporary or permanent? How does the bidder behavior change when such options are introduced in an online auction? What is an equilibrium bidder strategy in such a case?

The remainder of the paper is organized as follows. A literature survey on the topic is presented in §2. In §3, we describe the model used in this paper along with a discussion on its validity. An equilibrium bidder strategy for a temporary buyout option is derived in §4.1. The seller's problem is formulated in §4.2 and is solved for a particular case of impatient bidders. The permanent buyout case is analyzed in §5. An equilibrium bidder strategy is derived in §5.1. The seller's problem is discussed in §5.2. In §6 we run numerical simulations to determine the relative profit of using a buyout option and address the research questions posed earlier. Concluding remarks are offered in §7. All proofs not included in the main text can be found in the appendix.

II. Literature Survey

While auctions have been extensively studied since the seminal work in auction theory by Vickrey [1], the research work on buyout prices is limited and very recent. Indeed the comprehensive survey on auction literature by Klemperer [2] makes no mention of buyout prices. Furthermore, while Lucking-Reiley [3] does observe the use of buyout prices, he points out that he is "not aware of any theoretical literature which examines the effect of such a buyout price in an auction."

We first review papers that, for tractability purposes, study models with two bidder and/or two valuations framework. Budish and Takeyama [4] show, in such a model, that augmenting an English auction with a buy-price can improve the seller's profit by reducing the risk for some risk averse bidders. Reynolds and Wooders [5] show that when bidders are risk neutral, an auction with a buyout option (temporary or permanent) with a high enough buyout price is equivalent in revenue to the standard English ascending auction. However, when bidders are risk averse, a seller can raise more revenue than the ascending bid auction by introducing a buyout option with an appropriately chosen buyout price. In their model, a permanent buyout price raises more revenue than a temporary one but the bidder utility is same in both cases.

Recently several studies of more general models, with an arbitrary number of bidders, have been conducted. Focusing on a temporary buyout option Mathews [6] and Mathews [7] show that a risk averse or a time impatient seller facing risk neutral buyers will choose a buyout price low enough so that the buyout option is exercised with positive probability. The result also holds if the buyers are risk averse. Mathews [8] compares the welfare of bidders in an auction with a buyout option to a traditional auction with no buyout option. Compared to a traditional auction it is shown that, depending on the distribution of the valuation, either all bidders are weakly better off or bidders with a "relatively" high valuation have a lesser utility.

General models of permanent buyout prices include Kirkegaard and Overgaard [9], who offer justification for use of permanent buyout prices when similar products are offered in a sequence of auctions. They show that, when bidders desire multiple objects, it is profitable for an early seller to introduce a permanent buyout option if similar products are offered later on by other sellers. For the case of a single seller running multiple auctions, they show that the seller's total

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revenue increases if the buyers expect the seller to use the buyout option in the future auctions.

Studying an auction with N bidders having continuously distributed private valuations, Hidvégi et al [10] conclude that, when seller or bidders are risk averse, a permanent buyout price can increase the expected total utility. It is shown that, at equilibrium, bidders with valuation much higher than the buyout price will unconditionally bid the buyout price. All other bidders follow a threshold strategy, i.e. they exercise the buyout price if the current high bid reaches a particular threshold. However, they do not consider time-sensitive bidders and, in addition, do not differentiate bidders based on their arrival time.

Finally a paper very closely related to ours is Caldentey and Vulcano [11]. As in our paper, they assume a dynamic market environment where bidders with independent private valuations arrive as a Poisson process. They consider a seller managing a multi-unit auction with a permanent buyout option. In their model, as in ours, the equilibrium participation strategy of the bidders is a threshold strategy. There are important differences between their work and ours, however,: The model in [11] assumes that when bidders arrive at a multi-unit auction they are informed only about the initial number of units and not the number of units remaining. In a single-unit auction, this would correspond to bidders not knowing whether the item listed is available or not. In addition, bidders are assumed to act immediately (bid or buyout) when they arrive to the auction site. In contrast, we assume a more realistic information structure - an arriving bidder not only knows whether the auction is running or not but also the second highest bid at that instant. Also, in our model, bidding times are endogenous. Finally, we analyze, in parallel, temporary and permanent buyout prices.

III. Model

In this section, we first describe our model and then discuss its realism relative to real world online auctions. The model description is divided into two sections - market environment and auction mechanism. We first describe the market environment.

A. Market Environment

We model the bidder arrival process as a Poisson process with a constant rate λ which is exogenously defined. Bidders have independent private valuations which are distributed, with cdf F, over the support $[\underline{v}, \bar{v}]$ (define $m = \bar{v} - \underline{v}$). The arrival rate λ, distribution function F and the reserve price $\underline{v}$ are assumed to be common knowledge.

We assume rational risk-neutral bidders who seek to maximize their expected utility from participating in the auction. A bidder, arriving at time t, with valuation v who gets the item at time τ at price x, is assumed to have the following utility:

$$U(v, t, \tau) = e^{-\beta(\tau - t)}(v - x) \tag{1}$$

while a losing bidder is assumed to have zero utility from the auction. As evident from the above expression, bidders have a discount factor of β (assume $\beta > 0$). Similarly the seller, earning revenue R at time t has utility:

$$U_S(R, t) = e^{-\alpha t} R \tag{2}$$

where α (assume $\alpha > 0$) is the seller's discounting factor. We next describe the auction mechanism.

B. Auction Mechanism

We model an online auction as a second-price single item English auction where the bidder with the highest bid wins the auction but pays the second highest bid price. The auction is assumed to have a reserve price $\underline{v}$, i.e. if the highest bid does not exceed the reserve price, the item is not sold. We can thus restrict our attention to bidders that have a valuation greater than $\underline{v}$. Additionally, in case there is only one bidder with a valuation above the reserve price, he wins the auction and pays the reserve price for the item. The auction runs for T units of time and has a hard deadline, meaning that the auction ends at a prefixed deadline (like *ebay*) rather than have a floating deadline (like an auction on *amazon.com* where the deadline is automatically extended if a late bid arrives).

When a bidder arrives to the auction site, he is assumed to receive information I_t which is the second highest proxy bid at time t. Assume that $I_t = 0$ indicates that no bid has been placed up to time t. If there is one bid up to time t then $I_t = \underline{v}$, the reserve price. A bidder decides whether to bid in the auction (either immediately or at any later time in the auction) or exercise the buyout option (if available) based on his valuation, arrival time and the information he receives when he arrives to the auction site. Alternatively the bidder may decide to wait before making a decision.

C. Discussion

An online auction is a complex, dynamic and interactive process; thus while our model effectively captures some of the key features of such an interaction, there are some others it does not model so well.

Actual online auctions are a first-price mechanism where the highest bidder wins and pays his bid. However, auction sites typically feature "proxy bidding" systems which work as follows, [12],: bidders enter the maximum amount they are willing to pay for the item. The system then bids on behalf of the bidder, using as much of his bid as is necessary, to maintain his position as the highest bidder. If the bidder has the highest maximum, he wins and pays an amount equal to the second highest maximum; otherwise he is outbid and loses the auction. As observed by Lucking-Reiley [3], an online auction with a proxy bidding system can be effectively modeled by a second-price English auction.

In addition, though our model assumes a constant exogenously defined bidder arrival process, bidders may wait and so the arrival process of bids is endogenous. In practice, it has been observed that bids tend to surge near the end of the auction (see Roth and Ockenfels [13] for an empirical study on this trend). As will become clear from our analysis, our model does, capture this phenomenon. Also, the upper bound on the valuation $\bar{v}$ can be justified as the posted price at which the same item is available elsewhere.

The assumption on the information structure is what differentiates our model from most of the other research work done in modeling buyout prices. In an online auction, an arriving bidder is informed about the status of the auction and the second highest bid price, which closely matches our assumption on the information received by a bidder when he arrives to the auction site. Furthermore, while we assume bidders to be risk neutral, they do have a non-zero time-discounting factor. Since waiting, in an online auction, leads to increased risk for a bidder, the time-discounting factor of a bidder can be used as a proxy for his risk aversion. In figure 1, a snapshot of the main webpage of an online auction is shown along with its connection with our model.

A limitation of our model is the assumption of a monopolistic seller, thus not accounting for the effect of other auctions offering the same item simultaneously. Usually, as there are many such

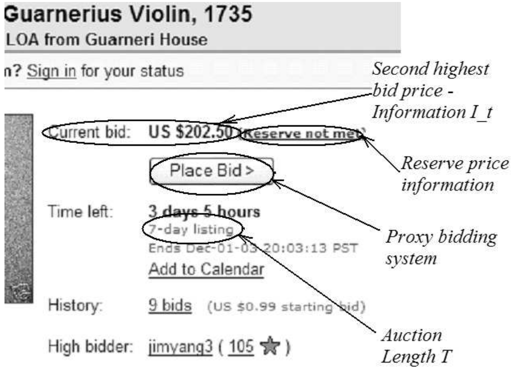


Fig. 1. Snapshot of an *ebay* online auction webpage

simultaneous auctions (even on the same website), the arrival rate of bidders is not exogenous but endogenous, as defined by factors such as presence of a reserve price, the current bid price, advertising etc. Additionally, the presence of other auctions for the same item may prevent bidders from committing to an auction early and thus promote late bidding. Another factor not captured by our model is the effect of network congestion, whereby bids near the end of the auction may not pass through, leading to an inefficient auction. This leads to a decrease in utility for the bidders and the seller. Also, while we assume that the reserve price $\underline{v}$ is common knowledge this is usually not true in practice. For example on *ebay*, although the bidders know whether the auction has a reserve price or not, the exact reserve price is not known. Furthermore, we assume that the reserve price is fixed while, in practice, it is usually negotiable in the sense that if the highest bid is below the reserve price, the seller can negotiate with the highest bidder.

In summary, while we attempt to model an online auction as closely as possible, there are several features of an actual auction that our model does not capture. Thus our results should be interpreted with caution, in light of the assumptions made.

IV. Temporary Buyout Option

We now look at the temporary buyout option case with a buy-price $P \in [\underline{v}, \bar{v}]$. As previously discussed, a temporary option disappears once a bid is made.

A. Bidder Participation Strategy

Suppose that any bidder, with valuation v, who arrives at time t and receives information I_t uses a bidding strategy belonging to the following family of threshold strategies $\mathcal{TS}$:

$$\mathcal{TS}(v,t,I_t): \begin{cases} \text{Buyout Immediately} & \text{if buyout option available} \\ & \text{and } v > v_{th}^1(t, I_t); \\ \text{Bid } v \text{ immediately} & \text{if buyout option available} \\ & \text{and } v \leq v_{th}^1(t, I_t); \\ \text{Bid } v & \text{otherwise,} \end{cases}$$

where $v_{th}^1(t,I_t) : [0,T] \times \{0\} \cup [\underline{v}, \bar{v}] \to [\underline{v}, \bar{v}]$ is a threshold valuation function. Recall that I_t is the second highest bid at time t. While $v_{th}^1(t, I_t)$ is a function of the buy-price P, the dependence will not be shown explicitly as P remains constant throughout the analysis.

Assuming other bidders follow a threshold strategy, we will prove that the best response is indeed a threshold strategy and then characterize the threshold function by imposing the Nash equilibrium requirements.

Consider a bidder, say A, with valuation v and arriving at time t. Assume that all bidders except him follow the strategy $\mathcal{TS}$. We now derive the best response strategy for the bidder A.

Suppose that A is not the first bidder. Then, the first bidder would have acted immediately (strategy $\mathcal{TS}$) and so bidder A will not see the buyout option. Hence his weakly dominant strategy is, as shown in [1], to bid his true valuation.

Note that the buyout option is available at time t only if no bidders arrive in the interval $(0,t)$. In that case, no bids are placed in the interval $(0,t)$ and $I_t = 0$. Thus the threshold function is defined only when $I_t = 0$ and so, in the remaining analysis, we drop the explicit dependence of the threshold valuation on I_t. Also if $I_t > 0$ then, by definition, a bid has been placed in the auction. Hence, the buyout option is not available and, as argued above, the weakly dominant strategy is to bid the true valuation which is independent of I_t. Thus the strategy $\mathcal{TS}$ is independent of I_t and we drop that dependence.

Suppose now that A is indeed the first bidder and so he sees the buyout option. He has three options:

1) Bid in the auction immediately
2) Buyout immediately
3) Wait before making a decision

If the bidder A chooses to bid immediately, the buyout option disappears. Since all other bidders follow the strategy $\mathcal{TS}$, they will bid their true valuation. So, if there are $N-1$ more arrivals (where N is Poisson with parameter $\lambda(T-t)$), it can be shown that, $\mathbf{E}[U_{Bid}(v,t|N)]$, the expected utility of the first bidder from bidding is:

$$\mathbf{E}[U_{Bid}(v,t|N)] = e^{-\beta(T-t)} \frac{(v-\underline{v})}{N} \left(\frac{v-\underline{v}}{m} \right)^{N-1}$$

Taking the expectation over N, we get the expected utility of bidding as:

$$\mathbf{E}[U_{Bid}(v,t)] = e^{-\beta(T-t)} \frac{m \cdot e^{-\lambda(T-t)}}{\lambda(T-t)} \left(e^{\frac{\lambda(T-t)(v-\underline{v})}{m}} - 1 \right) \quad (3)$$

The utility from exercising the buyout option immediately is:

$$U_{Buy}(v) = v - P \quad (4)$$

Thus if the bidder decides to act immediately (i.e. chooses between option (1) or (2)) his expected utility from the auction, $\mathbf{E}[U(v,t)]$, is $\max\left\{\mathbf{E}[U_{Bid}(v,t)], U_{Buy}(v)\right\}$.

We now rule out the third option by proving the following lemma.

Lemma 1: When other bidders follow a threshold strategy $\mathcal{TS}$, the first bidder cannot increase his utility by waiting before making a decision.

The intuition behind the result is that, assuming bidders follow strategy $\mathcal{TS}$, the utility from bidding depends only on the initial arrival time of the bidder and not the time of the bid. The utility from buying out, however, decreases if the first bidder waits and thus his utility from the auction (which is maximum of the two utilities) if he waits is at most equal to the utility from acting immediately .

Thus bidder A must choose between either bidding immediately or buying out immediately. Define excess utility as the difference between U_{Bid} and U_{Buy}. We have

$$e(v,t,P) = v - P - e^{-\beta(T-t)} \frac{m \cdot e^{-\lambda(T-t)}}{\lambda(T-t)} \left(e^{\frac{\lambda(T-t)(v-\underline{v})}{m}} - 1 \right) \quad (5)$$

A utility maximizing bidder will choose to buyout if and only if the utility from buying out is more than the utility from bidding, i.e. $e(v,t,P) \geq 0$. To characterize the optimal policy, firstly notice that $e(v,t,P)$ is strictly increasing in v for $v \in [\underline{v}, \bar{v}]$ for $t \in (0,T)$ since

$$\frac{\partial e(v,t,P)}{\partial v} = 1 - e^{\left(-\beta - \frac{\lambda(\bar{v}-v)}{m}\right)\left(T-t\right)} > 0$$

for all $v \in [\underline{v}, \bar{v}]$ and $t \in (0,T)$

Since $e(v,t,P)$ is increasing in v and $e(\underline{v},t,P) \leq 0$, for all $t \in (0,T)$ there either exists a unique $v^*(t) \in [\underline{v},\bar{v}]$ such that $e(v^*(t),t,P) = 0$ or $e(\bar{v},t,P) < 0$. Define the threshold valuation $v'_{th}(t)$ as:

$$v'_{th}(t) = \begin{cases} v^*(t) & \text{if } \exists\ v^*(t) \in [\underline{v},\bar{v}] \text{ such that } e(v^*(t),t,P) = 0 \\ \bar{v} & \text{otherwise} \end{cases}$$

Thus the best response strategy for bidder A is a threshold strategy of the following form:

$$\mathcal{B}(\mathcal{TS})(v,t) : \begin{cases} \text{Buyout immediately} & \text{if buyout option available} \\ & \text{and } v > v'_{th}(t) \\ \text{Bid } v \text{ immediately} & \text{if buyout option available} \\ & \text{and } v \leq v'_{th}(t) \\ \text{Bid } v & \text{otherwise} \end{cases}$$

Now for $\mathcal{TS}$ to be an equilibrium strategy, the best response to $\mathcal{TS}(v,t)$ must be the strategy itself. Note that $\mathcal{B}(\mathcal{TS})$ and $\mathcal{TS}$ have the same form and for the two to be equal, we require:

$$v'_{th}(t) = v^1_{th}(t)$$

This leads us to the following equilibrium strategy.

Theorem 1: In an English auction with a temporary buyout option, the threshold strategy $\mathcal{TS}(v,t)$ is a Nash equilibrium, where the threshold valuation function $v^1_{th}(t)$ is defined as:

$$v^1_{th}(t) = \begin{cases} v^*(t) & \text{if } \exists\ v^*(t) \in [\underline{v},\bar{v}] \text{ such that } e(v^*(t),t,P) = 0 \\ \bar{v} & \text{otherwise} \end{cases}$$

The solution to the equation $e(v^*(t),t,P) = 0$ is given by:

$$v^*(t) = P - \frac{m}{\lambda(T-t)}\Big(LambertW\Big(-e^{-e^{-(\lambda+\beta)(T-t)}} \times e^{\frac{(P-\underline{v})\lambda(T-t)}{m} - (\lambda+\beta)(T-t)}\Big) + e^{-(\lambda+\beta)(T-t)}\Big)$$

where *LambertW* is Lambert's W function [1] .

We will now formulate the seller's optimization problem and solve it for a particular case of impatient bidders.

B. Seller's Optimization Problem

The seller seeks to maximize his utility from the auction by optimally pricing the buyout option. The parameters λ, α, β and $\bar{v}$ are assumed to be known while $\underline{v}$ and T are assumed to be exogenously defined. Notice that we do not optimize over T since, in practice, the seller has a very limited choice in selecting the auction length.

By conditioning on S_1, the arrival time of the first bidder, the expected discounted utility from the auction can be written as:

$$\mathbf{E}[U_S(P)] = \int_0^T e^{-\alpha T}\mathbf{E}[\text{Rev}|S_1 = t \cap \text{Bid}]Pr(\text{Bid}|S_1 = t)f_{S_1}(t)dt + \int_0^T e^{-\alpha t}\mathbf{E}[\text{Rev}|S_1 = t \cap \text{Buyout}]Pr(\text{Buyout}|S_1 = t)f_{S_1}(t)dt \quad (6)$$

where $\mathbf{E}[\text{Rev}|S_1 = t \cap \text{Bid}]$ is the expected revenue given that the first bidder arrives at time t and bids in the auction. The revenue earned if the buyout option is exercised, $\mathbf{E}[\text{Rev}|S_1 = t \cap \text{Buyout}]$, is the buy-price P. Assuming that bidders follow the equilibrium strategy $\mathcal{TS}(v,t)$, we have:

$$Pr(\text{Buyout}|S_1 = t) = 1 - F\big(v^1_{th}(t)\big) \quad (7)$$

where F is the uniform cumulative distribution function.

[1]Lambert's W function, also called the Omega function, is the inverse function of $f(w) = w.e^w$

Substituting (7) in the expression for seller's discounted expected utility, we get:

$$\mathbf{E}[U_S(P)] = \int_0^T e^{-\alpha t}P\Big(1 - F\big(v^1_{th}(t)\big)\Big)f_{S_1}(t)dt + \int_0^T e^{-\alpha T}\mathbf{E}[\text{Rev}|S_1 = t \cap \text{No Buyout}]F\big(v^1_{th}(t)\big)f_{S_1}(t)dt \quad (8)$$

The seller's problem of maximizing his utility from the auction given that the bidders follow the equilibrium strategy $\mathcal{TS}$ derived earlier, can be written mathematically as:

$$\begin{aligned} \max_P \quad & \mathbf{E}[U_S(P)] \\ \text{subject to} \quad & e(v^1_{th}(t),t,P) = 0 \end{aligned}$$

Here the equality constraint incorporates the fact that the bidders follow the equilibrium strategy.

Unfortunately, we cannot solve the optimization problem analytically for the general case. We now look at the case of impatient bidders where we can derive an approximate optimal buy-price.

Impatient bidders ($\beta \to \infty$)

If the participating bidders are very time-sensitive, i.e. $\beta \to \infty$, the excess utility function becomes

$$e(v,t,P) = v - P$$

Thus the optimal strategy for the first bidder is to buyout if his valuation is above P and bid otherwise.

To calculate the optimal buy-price we need to approximate $\mathbf{E}[\text{Rev}|S_1 = t \cap \text{No Buyout}]$, the expected revenue given the first bidder arrives at time t and does not exercise the buyout option by $\mathbf{E}[\text{Rev}|S_1 = t]$, the expected revenue given that the first bidder arrives at time t (thus ignoring the information that his valuation v is less than the threshold valuation $v^1_{th}(t)$).

The approximate seller's utility, $\mathbf{E}[\widetilde{U}_S(P)]$, can then be determined, using equation (8), as:

$$\mathbf{E}[\widetilde{U}_S(P)] = \frac{P(\bar{v} - P)}{m}\frac{\lambda}{\lambda + \alpha}(1 - e^{-(\lambda+\alpha)T}) + \alpha^T\frac{(P - \underline{v})}{m}R(T)$$

where $R(T)$ is the expected revenue from an English auction (without a buy-price) with the same parameters $(\lambda, \underline{v}, \bar{v}, T)$ which is given as following

$$R(T) = \bar{v}(1 - e^{-\lambda T}) - \frac{2m}{\lambda T}\Big(1 - e^{-\lambda T} - \lambda T e^{-\lambda T}\Big)$$

Proposition 1: The optimal buy-price in an auction with a temporary buyout option, and with bidder time-sensitivity $\beta \to \infty$, lies in the interval $[\underline{v},\bar{v}]$.

Proof: The threshold valuation in this case is $v^1_{th}(t) = P$. Thus if the buyout price $P \geq \bar{v}$ no bidder will exercise it. Hence setting a buyout price greater than $\bar{v}$ is equivalent to setting it equal to $\bar{v}$. If $P = \underline{v}$ then the first bidder will exercise the option with probability 1 (ignoring the zero probability event that the bidder has a valuation equal to $\underline{v}$). Thus setting the buyout price below $\underline{v}$ will only decrease the revenue from the auction without changing the probability of buyout. ■

It can be shown that $\mathbf{E}[\widetilde{U}_S(P)]$ is concave in P and the unconstrained maximum occurs at:

$$P^* = \frac{\bar{v}}{2} + \frac{\alpha^T R(T)}{\frac{2\lambda}{\lambda+\alpha}\big(1 - e^{-(\lambda+\alpha)T}\big)}$$

Thus $\widetilde{P}_{opt}$, the approximate optimal buy-price is:

$$\widetilde{P}_{opt} = \begin{cases} \underline{v} & P^* < \underline{v} \\ P^* & \underline{v} \leq P^* \leq \bar{v} \\ \bar{v} & P^* > \bar{v} \end{cases} \quad (9)$$

In the numerical results section, we examine the performance of this buyout price in an auction with bidders who have a finite time sensitivity.

V. PERMANENT BUYOUT OPTION

We now consider a permanent buyout option with buyout price $P \in [\underline{v}, \bar{v}]$. In this case the option is available through out the auction or until it is exercised. We derive the bidder participation assuming a generic valuation distribution $F(\cdot)$ over the interval $[\underline{v}, \bar{v}]$.

A. Bidder Participation Strategy

Let $v^2_{th}(t, I_t) : [0, T] \times \{0\} \cup [\underline{v}, \bar{v}] \rightarrow [\underline{v}, \bar{v}]$ be a continuous function in t and I_t. We assume that the threshold is increasing in t but decreasing in I_t. The intuition behind why the threshold should be decreasing in I_t is that an arriving bidder will be willing to pay higher for the buyout option, if he sees a higher second highest bid. Thus if I_t is higher, he will have a lower threshold valuation.

Recall, that we assume that $I_t = 0$ indicates that no bid has been made up to time t. If there is one bid till time t then $I_t = \underline{v}$, the reserve price. Thus $t \in (0, T]$ while $I_t \in [\underline{v}, \bar{v}] \cup \{0\}$. Again notice that we do not show the explicit dependence of $v^2_{th}(t, I_t)$ on buy-price P.

Consider a family of threshold strategies $\mathcal{PS}$ of the following form:

$$\mathcal{PS}(v, \tau, I_\tau) : \begin{cases} \text{Bid true valuation at } T & \text{if } v \leq v^2_{th}(\tau, I_\tau) \\ \text{Buyout immediately} & \text{if } v > v^2_{th}(\tau, I_\tau) \end{cases}$$

where τ is the arrival time of the bidder. By "bidding at time T", we mean that the bidder will bid right at the end of the auction - too late for other bidders to respond to his bid.

We will show that the best response strategy to $\mathcal{PS}$ is a threshold strategy of the same form as $\mathcal{PS}$. We then characterize a threshold valuation such that the best response to a profile where all bidders play the corresponding threshold strategy is the strategy itself.

Consider a bidder, call him A, with valuation v arriving at time t and receiving information I_t. We first show, in Lemma 2, that if the bidder A decides to bid he must bid at time T, and next rule out the option of waiting in Lemma 3. Subsequently we show, in Lemma 4 and Lemma 5, that the best response is indeed a threshold strategy and then combine the above results to derive the best response strategy in Theorem 2. The threshold valuation, such that $\mathcal{PS}$ is an equilibrium strategy, is determined by equating the best response strategy to the strategy $\mathcal{PS}$. The proof is then completed by showing, in Lemma 7 that the threshold function, as obtained above, indeed satisfies the assumptions made initially.

We will first derive bidder A's utility if he bids at time T. Then since all bidders bid at time T, we have

$$I_\tau = 0 \qquad \forall \tau \in [0, T)$$

Suppose that there are k bidders in $(0, t)$ and l bidders in $(t, T]$ and they arrive at time $0 < t^1_1 < t^1_2 < .. < t^1_k < t$ and $t < t^2_1 < t^2_2 < .. < t^2_l \leq T$ respectively. Denote this as event $\mathcal{E}$. Now, A wins the auction if no bidder exercises the buyout option and if every bidder has a valuation less than $A's$ valuation, i.e. every bidder has a valuation less than $\min(v^2_{th}(\tau, I_\tau), v)$ where τ is the arrival time of the bidder. Since the auction is still running at time t, all bidders arriving before t have their valuation less than the threshold valuation. Then probability that A wins the auction is:

$$Pr(A \text{ wins}|\mathcal{E}) = \frac{\prod_{i=1}^k F\big(\min(v^2_{th}(t^1_i, 0), v)\big)}{\prod_{i=1}^k F\big(v^2_{th}(t^1_i, 0)\big)} \times \prod_{i=1}^l F\big(\min(v^2_{th}(t^2_i, 0), v)\big)$$

The first term of the product is the probability that all bidders that have arrived have a valuation less than $\min(v^2_{th}(\tau, I_\tau), v)$ given their valuation is less than the threshold. The second term is the probability that all future bidders have a valuation less than $\min(v^2_{th}(\tau, I_\tau), v)$. Here, and in the remainder of the paper, we assume that if $k = 0$ then $\prod_{i=1}^k(\cdot) = 1$. Now, given the bidder A wins the auction, the distribution of the highest bid among the other bidders is:

$$F_{\max|\mathcal{E}, A \text{ wins}}(x) = \frac{\prod_{i=1}^k F\big(\min(v^2_{th}(t^1_i, 0), x)\big)}{\prod_{j=1}^k F\big(\min(v^2_{th}(t^1_j, 0), v)\big)} \times \frac{\prod_{i=1}^l F\big(\min(v^2_{th}(t^2_i, 0), x)\big)}{\prod_{j=1}^l F\big(\min(v^2_{th}(t^2_j, 0), v)\big)} \quad \forall x \in [0, v]$$

Thus the expected highest bid, among the other bidders, is:

$$\begin{aligned}\mathbf{E}[\text{Max|A wins \& } \mathcal{E}] &= \int_0^v \Big(1 - F_{\max|\mathcal{E}, A \text{ wins}}(x)\Big)dx \\ &= v - \int_{\underline{v}}^v F_{\max|\mathcal{E}, A \text{ wins}}(x)dx\end{aligned}$$

where the second equality follows since $F_{\max|\mathcal{E}, A \text{ wins}}(x) = 0$ for all $x < \underline{v}$. If A loses the auction, his utility from the auction is zero. Thus the expected discounted utility from bidding at time T for bidder A is:

$$\mathbf{E}[U_{Bid}(v, t, I_t = 0)|\mathcal{E}] = e^{-\beta(T-t)}\Big(\int_{\underline{v}}^v F_{\max|\mathcal{E}}(x)dx)\Big) Pr(A \text{ wins})$$

which on simplification yields:

$$\mathbf{E}[U_{Bid}(v, t, I_t = 0)|\mathcal{E}] = e^{-\beta(T-t)}\Big(\int_{\underline{v}}^v \frac{\prod_{i=1}^k F\big(\min(v^2_{th}(t^1_i, 0), x)\big)}{\prod_{i=1}^k F\big(v^2_{th}(t^1_i, 0)\big)} \times \prod_{j=1}^l F\big(\min(v^2_{th}(t^2_j, 0), x)\big)dx\Big) \quad (10)$$

The unconditional expected utility is obtained by taking expectation of the expression in (10) over $K, T^1_1, T^1_2, .., T^1_K, L, T^2_1, T^2_2, .., T^2_L$ which are the random variables corresponding to the realizations $k, t^1_1, t^1_2, .., t^1_k, l, t^2_1, t^2_2, .., t^2_l$ respectively.

We now show that the bidder A will neither bid immediately nor wait before making a decision.

Lemma 2: When faced with bidders playing the strategy $\mathcal{PS}$, if a bidder decides to bid he must do so only at time T.

The intuition behind the lemma is that while bidding earlier does not increase the utility of a bidder it reveals information about his valuation to other bidders, who can use it this information their advantage. We next address the issue of waiting before making a decision.

Lemma 3: When facing bidders who follow strategy $\mathcal{PS}$, a bidder is weakly better off making a decision immediately i.e. as soon as he first arrives to the auction site.

Proof: Here we give an intuitive argument. A formal proof can be constructed on the lines of the proof of Lemma 1. Suppose the bidder A decides to wait until time $\tau > t$ before making a decision. Then, since no one else bids in the auction, the information available at τ will be the same as that at time t, i.e. $I_t = I_\tau$. Thus the bidder will not gain information by waiting and so his expected utility from bidding will depend only on t the time he first arrived at the auction site. Hence waiting does not change the utility from bidding.

Additionally since the buyout price remains constant throughout the auction, waiting decreases the bidder's utility from buying out because of his time-discounting factor. Thus while the utility from waiting remains constant, the utility from exercising the buying option decreases and so a bidder cannot increase his utility by waiting. ∎

Thus we have shown, Lemma 2 and Lemma 3, that the bidder A must either bid at time T or exercise the buyout option immediately. We now show that the optimal strategy is in fact a threshold strategy. For proving that we first show the following lemma.

Lemma 4: When bidders follow strategy $\mathcal{PS}$, the expected utility from bidding, $\mathbf{E}[U_{Bid}(v,t,I_t=0)]$, is a non-decreasing function of bidder valuation with a slope less than 1 for $t \in (0,T)$.

It implies that, as v increases, the utility from buying out increases more rapidly than utility from bidding.

Recall that the utility from exercising the buyout option is:

$$U_{Buy}(v) = v - P$$

Note that $\frac{\partial}{\partial v}\Big(U_{Buy}(v)\Big) = 1$. Combining Lemma 2 with the fact that for valuation $\underline{v}$, bidding is more attractive, we can prove the following.

Lemma 5: For bidder A, facing bidders following the strategy $\mathcal{PS}$, there exists a unique valuation $v^*(t)$ such that both the options: bid or buyout are equally attractive.

Notice that $v^*(t)$ may be greater than the upper support $\bar{v}$, in which case, since the bidder has a valuation less than $v^*(t)$, he will bid in the auction.

Combining the above results, we get the following best response strategy for bidder A.

Theorem 2: The best response to strategy $\mathcal{PS}$, $\mathcal{B}(\mathcal{PS})$ is:

$$\mathcal{B}(\mathcal{PS})(v,t,I_t=0): \begin{cases} \text{Bid true valuation at } T & \text{if } v \le v^*(t) \\ \text{Buyout immediately} & \text{if } v > v^*(t) \end{cases}$$

where $v^*(t)$ is such that

$$U_{Buy}(v^*(t)) = \mathbf{E}[U_{Bid}(v^*(t),t,0)]$$

Proof: We have shown in Lemma 3 that a bidder is weakly better off making a decision immediately. Now by Lemma 4 and 5,

$$U_{Buy}(v) \le \mathbf{E}[U_{Bid}(v,t,I_t=0)] \quad \text{if } v \le v^*(t)$$

Hence a utility maximizing bidder with valuation $v \le v^*(t)$ will choose to bid. Since this is a second-price auction, a weakly dominant strategy is to bid one's true valuation. We have already argued in Lemma 2 that if a bidder decides to bid in the auction, he must do so only at time T.

For $v > v^*(t)$:

$$\mathbf{E}[U_{Bid}(v,t,I_t=0)] < U_{Buy}(v)$$

and thus it is profitable to buyout. ■

For the strategy $\mathcal{PS}$ to be an equilibrium strategy, the best response to the strategy must be the same strategy, i.e. $\mathcal{B}(\mathcal{PS})(v,t,I_t=0) = \mathcal{PS}(v,t,I_t=0)$. Both the strategies have the same form and thus for the two to be same, we must have that

$$v^*(t) = v_{th}^2(t, I_t = 0)$$

Combining this with the fact that $U_{Buy}(v^*(t)) = \mathbf{E}[U_{Bid}(v^*(t),t,0)]$, we get the following equation for the threshold function

$$v_{th}^2(t,0) - P = \mathbf{E}\Big[e^{-\beta(T-t)}\Big(\int_{\underline{v}}^{v_{th}^2(t,0)} \frac{\prod_{i=1}^k F\big(\min(v_{th}^2(t_i^1,0),x)\big)}{\prod_{i=1}^k F\big(v_{th}^2(t_i^1,0)\big)} \times \prod_{j=1}^{l} F\big(\min(v_{th}^2(t_j^2,0),x)\big)dx\Big)\Big] \tag{11}$$

where the expectation is over $K, T_1^1, T_2^1, .., T_K^1, L, T_1^2, T_2^2, .., T_L^2$.

Thus if the threshold function satisfies equation (11) then the corresponding strategy $\mathcal{PS}$ is indeed an equilibrium strategy. We must now show that the threshold function is indeed non-decreasing in t. For that we first need the following result.

Lemma 6: Assuming bidders follow the strategy $\mathcal{PS}$, the utility from bidding is higher for a bidder arriving later in the auction, i.e. $\mathbf{E}[U_{Bid}(v,t',0)] > \mathbf{E}[U_{Bid}(v,t,0)]$ for $t' > t$.

The reason why Lemma 6 holds, is that a bidder arriving later in the auction has a smaller waiting cost. Additionally, an arriving bidder receives information that all bidders arriving before him have a valuation less than the threshold. This leads to a higher non-discounted utility from bidding for a bidder arriving later in the auction as compared to one arriving earlier. Combining the two effects gives the desired result.

We now use Lemma 6 to show the following result.

Lemma 7: The threshold function, satisfying equation (11) is non-decreasing in t.

We have thus shown that the strategy $\mathcal{PS}$ is an equilibrium strategy where the threshold function satisfies:

$$v_{th}^2(t) - P = \mathbf{E}\Big[e^{-\beta(T-t)}\Big(\int_{\underline{v}}^{v_{th}^2(t)} \frac{\prod_{i=1}^k F\big(\min(v_{th}^2(t_i^1),x)\big)}{\prod_{i=1}^k F\big(v_{th}^2(t_i^1)\big)} F(x)^l dx\Big)\Big] \tag{12}$$

Since no bidder bids early in the auction the information at any instant t, I_t, is zero and so we drop that explicit dependence for the sake of brevity. Also, since $v_{th}^2(t)$ is non-decreasing function in t, we have

$$\min(v_{th}^2(t_j^2), x) = x, \quad \text{if } x \in [\underline{v}, v_{th}^2(t)], \forall j = 1, ..., l$$

since $t_j^2 > t, \forall j = 1, .., l$.

Notice that we can equivalently set the threshold to be $\min(v_{th}^2(t), \bar{v})$ since no bidder has valuation greater than $\bar{v}$.

The expression in equation (12) involves taking expectation over arrival times in a non-homogeneous Poisson process which can be potentially tricky. Hence we now state and prove the following proposition that gives a differential equation for the threshold function.

Proposition 2: Assuming that the bidder valuations are uniformly distributed, the threshold function $v_{th}^2(t)$, defined in (12), satisfies the differential equation:

$$\frac{dv_{th}^2(t)}{dt} = \frac{\Big(\beta + \lambda\Big(1 - F\big(v_{th}^2(t)\big)\Big)\Big)\Big(v_{th}^2(t) - P\Big)}{1 - e^{-\Big(\beta+\lambda\Big(1-F\big(v_{th}^2(t)\big)\Big)\Big)\Big(T-t\Big)}} \tag{13}$$

along with the initial value

$$v_{th}^2(0) = P - \frac{m}{\lambda T}\Big(\mathit{LambertW}\Big(-e^{-(\beta+\lambda)T} \times e^{-\frac{-(P-\underline{v})\lambda T + me^{-(\beta+\lambda)T}}{m}}\Big) + e^{-(\beta+\lambda)T}\Big) \tag{14}$$

B. Seller's Optimization Problem

As before, the seller maximizes his utility from the auction by optimizing over the buyout price P. The nonlinear optimization problem so obtained cannot be solved analytically and so we perform the optimization using a simulation-based line search method. The threshold valuation is determined by numerically solving the differential equation, derived in Proposition (2), and is used to characterize bidder behavior. The results of this optimization along with the results from the temporary buyout case are presented in the next section.

VI. Numerical Results

We calculate, by simulation, the optimal temporary and permanent buyout price and the corresponding benefit of using such an option. Furthermore we examine the dependence of the optimal buyout price and the corresponding additional profit, obtained by introducing a buyout option, on the average number of bidders in the auction and the time sensitivity of the seller and buyers.

An online auction is simulated as follows: Time-sensitive bidders with uniformly distributed random valuations in the interval $[50, 500]$ arrive according to a Poisson process of rate λ and play the equilibrium strategy described in §4 and §5 when faced with temporary and permanent buyout option case respectively. The seller's utility from the auction is calculated by discounting the second highest valuation (or the buyout price, in case the option is exercised) by an appropriate discounting factor depending on the time of sale of the item. The seller's utility is then maximized, and the corresponding optimal buyout price determined, by running a simulation-based line search over the buyout price P.

The auction is assumed to run for a length of $T = 16$ units. The auction is simulated for different values of λ, α and β. The simulations were run long enough to ensure that the 95% confidence interval width is within 1% of the plotted and tabulated values.

A. Performance of $\widetilde{P}_{opt}$

In an auction with a temporary buyout price and with bidders having a finite time-sensitivity β, we propose using the approximate optimal price, $\widetilde{P}_{opt}$, as derived in equation (9) and assess its sub-optimality. Define $\mathbf{E}^S[U_S(\widetilde{P}_{opt})]$ and $\mathbf{E}^S[U_S(P_{opt})]$ to be the simulated expected utility of the seller if he sets the buyout price to be $\widetilde{P}_{opt}$ and P_{opt} respectively and $\mathbf{E}^S[U_S]$ to be the simulated expected utility without a buyout price.

In Table 1, we compare $\Delta U_S[(P_{opt})]\Big(= \frac{\mathbf{E}^S[U_S(P_{opt})]-\mathbf{E}^S[U_S]}{\mathbf{E}^S[U_S]} \times 100\Big)$ and $\Delta U_S[(\widetilde{P}_{opt})]\Big(= \frac{\mathbf{E}^S[U_S(\widetilde{P}_{opt})]-\mathbf{E}^S[U_S]}{\mathbf{E}^S[U_S]} \times 100\Big)$, the percent increase in seller's utility (over an auction without a buyout price), achieved by using $\widetilde{P}_{opt}$ and P_{opt}, for different arrival rates λ and buyer time sensitivity β (with $\alpha = 0.03$).

λT	β	$\Delta U_S[(P_{opt})]$	$\Delta U_S[(\widetilde{P}_{opt})]$
2	0.01	10.81%	2.05%
	0.03	16.03%	10.5%
	0.05	19.66%	16.31%
4	0.01	11.21 %	9.19 %
	0.03	13.12%	11.77%
	0.05	14.63%	13.79%
8	0.01	10.18%	9.88%
	0.03	10.38%	10.17%
	0.05	10.42%	10.26%

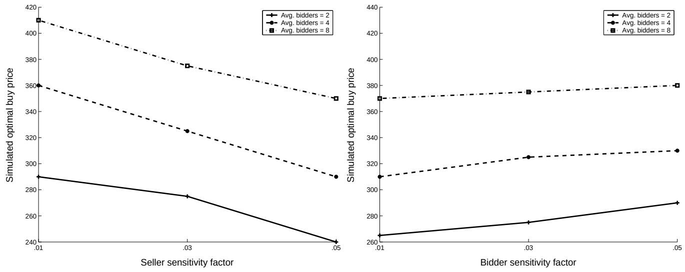
TABLE I

PERCENT UTILITY INCREASE ACHIEVED BY THE OPTIMAL AND APPROXIMATE BUYOUT PRICE

From the table, it is evident that $\widetilde{P}_{opt}$ performs well if the average number of bidders in the auction (λT) is high. However for low values of λT the increase in seller's utility achieved by using the approximate buyout price is significantly lower than the maximum achievable.

B. Variation of optimal buyout price with auction parameters

The simulated optimal temporary buyout price is plotted as a function of seller sensitivity α in figure 2(a) (with $\beta = 0.03$). The optimal buyout price decreases with an increasing seller time sensitivity (i.e. increasing α) as a more time-sensitive seller will prefer selling the product at a lower price early in the auction rather than waiting for the auction to end. In figure 2(b), the simulated optimal temporary buyout price is plotted as a function of bidder sensitivity β (with $\alpha = 0.03$). The buyout price increases with β, since a more time-sensitive bidder will be willing to pay a higher



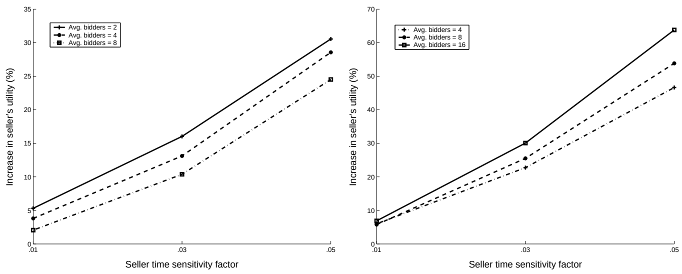
(a) (b)

Fig. 2. Variation of simulated optimal buyout price with seller and bidder time sensitivity

price for getting the product earlier. Figure 2 (both (a) and (b)) also show the obvious fact that the seller will set a higher buyout price if there are more bidders. The simulated optimal permanent buyout price follows similar patterns as the temporary.

C. Variation of seller's utility with auction parameters

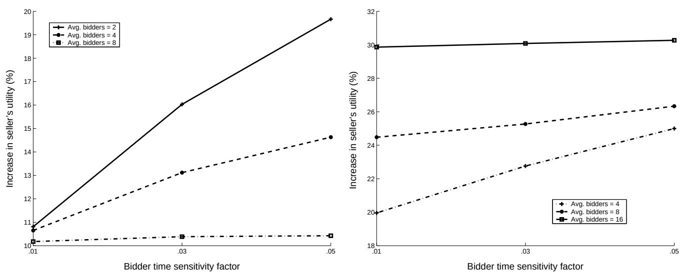
We next look at the additional benefit of introducing a buyout option. The percent increase in seller's simulated utility over an auction without a buyout price, for both temporary and permanent case, is plotted as a function of seller's time-sensitivity α in figure 3 (with $\beta = 0.03$). As seller's time-sensitivity increases (i.e. α increases), the increase in utility, for both cases, is higher since a seller with a higher time-sensitivity finds it more advantageous to sell the product earlier.



(a) Temporary (b) Permanent

Fig. 3. Variation of seller's utility increase with seller time sensitivity

Figure 4 plots percent increase in seller's simulated utility, for both cases, as a function of β, bidder time sensitivity (with $\alpha = 0.03$). The increase in utility is higher for more time-sensitive bidders since such bidders will be willing to pay more for getting the product earlier.



(a) Temporary (b) Permanent

Fig. 4. Variation of seller's utility increase with bidder time sensitivity

As evident from figure 3(a) and figure 4(a) the percent increase in seller's utility, for the temporary case, decreases as the average number of bidders increase. However this effect is reversed for the permanent buyout option case. This is due to the fact that while the temporary option is available only to the first bidder, all arriving bidders see the option in the permanent case.

From the simulation results it can be concluded that, in case the bidders/seller is time-sensitive, the seller can substantially (by as much as 60%) increase his utility from the auction by introducing a buyout option. The increase in utility obtained by introducing a temporary buyout option is significant (as high as 20%) when there are lesser number of bidders in the auction, but decreases as the number of bidders in the auction increases. The permanent buyout option, on the other hand, not only outperforms the temporary option in every scenario, but also leads to a higher increase in utility as the number of bidders increase.

However, as suggested by the equilibrium strategy, when a permanent buyout option is used, all bids are concentrated near the end of the auction. This is generally undesirable for various reasons not captured by our model. Firstly, because of network congestion, some bids may not pass through and the actual revenue of the seller may be lesser than the theoretically predicted values. Secondly, since bidders don't need to commit themselves to this auction, they may choose other simultaneously running auctions, for a similar item, thus leading to a loss in revenue. Thus, in cases when the revenue with a permanent option is not significantly better than the temporary case, it may be better to use a temporary option.

VII. Concluding Remarks

We analyze the strategy of rational time-sensitive bidders in an auction with a buyout option and show that, when bidders seek to maximize their expected discounted utility, a threshold strategy is an equilibrium for both the temporary and permanent buyout option case. However, while bidders may bid early in an auction with a temporary buyout option, they will necessarily bid just at the end in the permanent case. Thus there is a surge of bids near the end of an auction with a permanent buyout option.

Assuming that bidders follow the equilibrium strategies, as derived before, the seller's problem of maximizing utility is solved using simulation. Additionally, for the case of impatient bidders, in an auction with a temporary buyout price, the approximate optimal buyout price is determined analytically.

The simulation results show that, when any of the auction agents is time-sensitive, the seller can increase his utility from the auction significantly by introducing a buyout option. Introducing a temporary buyout option in an online auction can increases the seller's utility by as much as 20% in some cases. More importantly such an option gives the first bidder an incentive to bid early by giving him a huge advantage over the other bidders - he can either exercise the buyout option or make it unavailable for other bidders by bidding first. Some auction sites, *Amazon* for example, offer the first bidder a 10% discount for achieving the same effect.

In case the number of bidders in the auction is high, the permanent buyout option can increase the seller's utility by as much as 60%, thus making it very attractive in such cases. However, by penalizing early bidding, such an option promotes late bidding which may lead to undesirable effects like network congestion or bidders choosing a different auction. These effects, not captured by our model, can decrease the seller's utility significantly.

It must be mentioned that all our conclusions must be interpreted keeping in mind in light of the limitations of the model. While the model incorporates many features of an actual online auction, there are several that are not captured accurately. Consequently, it is important to validate our model predictions with empirical data and we are currently working on it.

Our current research focuses on analyzing the permanent buyout option in multi-unit auctions. We are also looking at a buyout option whose price varies dynamically as the auction progresses. Both these auction features are not widespread in practice and we hope that our research in this area will aid auctioneers who are considering to implement such auctions.

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Appendix

Proof of Lemma 1

Proof: Suppose the bidder waits till time $\tau(\tau > t)$ before deciding. Since the bidder is time-sensitive, he will discount his utility at time τ by the factor $e^{-\beta(\tau-t)}$.

Let $\mathcal{E}$ be the event that no bidder arrives in the interval (t,τ). In this case the buyout option is available to bidder A. The bidder A is effectively the first bidder (arriving at time τ) except that he has waited for $(\tau - t)$ time . Let $\mathbf{E}[U_{Bid}(v,t)|\mathcal{E}]$ be the expected utility from bidding given event $\mathcal{E}$ happens. The expression for $\mathbf{E}[U_{Bid}(v,t)]$ is

$$\mathbf{E}[U_{Bid}(v,t)] = e^{-\beta(T-t)} \frac{m \cdot e^{-\lambda(T-t)}}{\lambda(T-t)} \Big(e^{\frac{\lambda(T-t)(v-\underline{v})}{m}} - 1 \Big) \tag{A.15}$$

Then $\mathbf{E}[U_{Bid}(v,t)|\mathcal{E}]$ can be obtained by replacing t by τ in (A.15). Similarly the utility from buying out is $v - P$. The complementary event $\bar{\mathcal{E}}$ is the event that one or more arrivals occurred in the interval (t,τ). In this case the buyout option is no longer available. Let $\mathbf{E}[U_{Bid}(v,t)|\bar{\mathcal{E}}]$ be the expected utility from bidding if event $\bar{\mathcal{E}}$ happens. Then the expected utility of the bidder A, if he waits till time $\tau(\tau > t)$, is

$$\begin{aligned}\mathbf{E}[U_\tau(v,t)] = e^{-\beta(\tau-t)} \Big(& \max\Big\{ v - P, \mathbf{E}[U_{Bid}(v,\tau)] \Big\} \cdot Pr(\mathcal{E}) \\ & + \mathbf{E}[U_{Bid}(v,t)|\bar{\mathcal{E}}] \cdot Pr(\bar{\mathcal{E}}) \Big) \end{aligned} \tag{A.16}$$

The discounting factor $e^{-\beta(\tau-t)}$ incorporates the waiting cost. As argued earlier, $\mathbf{E}[U_{Bid}(v,t)|\mathcal{E}] = \mathbf{E}[U_{Bid}(v,\tau)]$. Notice that the event $\bar{\mathcal{E}}$ also includes the event that another bidder buys out. In that case the auction is closed and the utility from bidding is zero.

Using the law of conditional expectation, we also have:

$$\mathbf{E}[U_{Bid}(v,t)] = e^{-\beta(\tau-t)}\Big(\mathbf{E}[U_{Bid}(v,t)|\mathcal{E}]\cdot Pr(\mathcal{E}) + \mathbf{E}[U_{Bid}(v,t)|\bar{\mathcal{E}}]\cdot Pr(\bar{\mathcal{E}})\Big) \quad \text{(A.17)}$$

Notice that the time-discounting factor incorporates the accumulated waiting cost.

Using (A.15), it can be verified that

$$e^{-\beta(\tau-t)}\mathbf{E}[U_{Bid}(v,\tau)] \geq \mathbf{E}[U_{Bid}(v,t)] \quad \text{(A.18)}$$

This coupled with equation (A.17) implies that

$$e^{-\beta(\tau-t)}\mathbf{E}[U_{Bid}(v,t)|\bar{\mathcal{E}}] \leq \mathbf{E}[U_{Bid}(v,t)] \quad \text{(A.19)}$$

(notice that $Pr(\mathcal{E}) \in (0,1)$).

Using the above result, we now show that the expected utility of bidder A at τ is almost equal to the expected utility obtained from acting immediately, i.e. $\mathbf{E}[U_\tau(v,t)] \leq \mathbf{E}[U(v,t)]$. Consider the following two cases:

- **Case 1:** $v - P \leq \mathbf{E}[U_{Bid}(v,\tau)]$
  In this case (A.16) becomes

$$\begin{aligned}
\mathbf{E}[U_\tau(v,t)] &= e^{-\beta(\tau-t)}\Big(\max\Big\{v-P, \mathbf{E}[U_{Bid}(v,\tau)]\Big\}\cdot Pr(\mathcal{E}) + \mathbf{E}[U_{Bid}(v,t)|\bar{\mathcal{E}}]\cdot Pr(\bar{\mathcal{E}})\Big)\\
&= e^{-\beta(\tau-t)}\Big(\mathbf{E}[U_{Bid}(v,\tau)]\cdot Pr(\mathcal{E}) + \mathbf{E}[U_{Bid}(v,t)|\bar{\mathcal{E}}]\cdot Pr(\bar{\mathcal{E}})\Big)\\
&= e^{-\beta(\tau-t)}\Big(\mathbf{E}[U_{Bid}(v,t)|\mathcal{E}]\cdot Pr(\mathcal{E}) + \mathbf{E}[U_{Bid}(v,t)|\bar{\mathcal{E}}]\cdot Pr(\bar{\mathcal{E}})\Big)\\
&= \mathbf{E}[U_{Bid}(v,t)]\\
&\leq \max\big(\mathbf{E}[U_{Bid}(v,t)], U_{Buy}(v)\big) = \mathbf{E}[U(v,t)]
\end{aligned}$$

  The third equality follows from the fact that $\mathbf{E}[U_{Bid}(v,t)|\mathcal{E}] = \mathbf{E}[U_{Bid}(v,\tau)]$ while the fourth equality follows from (A.17).
- **Case 2:** $v - P > \mathbf{E}[U_{Bid}(v,\tau)]$
  In this case (A.16) becomes

$$\mathbf{E}[U_\tau(v,t)] = e^{-\beta(\tau-t)}\Big((v-P)\cdot Pr(\mathcal{E}) + \mathbf{E}[U_{Bid}(v,t)|\bar{\mathcal{E}}]\cdot Pr(\bar{\mathcal{E}})\Big)$$

  Now notice that

$$\begin{aligned}
e^{-\beta(\tau-t)}(v-P) &> e^{-\beta(\tau-t)}\mathbf{E}[U_{Bid}(v,\tau)]\\
&\geq \mathbf{E}[U_{Bid}(v,t)]\\
&\geq e^{-\beta(\tau-t)}\mathbf{E}[U_{Bid}(v,t)|\bar{\mathcal{E}}]
\end{aligned} \quad \text{(A.20)}$$

  The second and third inequality follow from (A.18) and (A.19) respectively.
  Using (A.20) we get

$$\begin{aligned}
\mathbf{E}[U_\tau(v,t)] &= e^{-\beta(\tau-t)}\Big((v-P)\cdot Pr(\mathcal{E}) + \mathbf{E}[U_{Bid}(v,t)|\bar{\mathcal{E}}]\cdot Pr(\bar{\mathcal{E}})\Big)\\
&< e^{-\beta(\tau-t)}\Big((v-P)\cdot Pr(\mathcal{E}) + (v-P)\cdot Pr(\bar{\mathcal{E}})\Big)\\
&= e^{-\beta(\tau-t)}(v-P)\\
&< (v-P) \leq \max\big(\mathbf{E}[U_{Bid}(v,t)], U_{Buy}(v)\big)\\
&= \mathbf{E}[U(v,t)]
\end{aligned}$$

Thus in both cases $\mathbf{E}[U_\tau(v,t)] \leq \mathbf{E}[U(v,t)]$ and so the bidder A should not wait. ■

Proof of Lemma 2

Proof: Suppose the bidder A bids immediately. This reveals information about his valuation to bidders arriving after him. Since he bids in the auction at time t and all other bidders bid at time T, we have $I_\tau = \underline{v}$ for $\tau \in (t,T)$.

In this case, the probability that bidder A wins the auction is:

$$Pr_{(t)}(A \text{ wins}|\mathcal{E}) = \frac{\prod_{i=1}^k F\big(\min(v_{th}^2(t_i^1,0),v)\big)}{\prod_{i=1}^k F\big(v_{th}^2(t_i^1,0)\big)} \times \prod_{i=1}^{l} F\big(\min(v_{th}^2(t_i^2, I_{t_i^2}),v)\big)$$

And the discounted expected utility is:

$$\mathbf{E}[U_{Bid(t)}(v,t,0)|\mathcal{E}] = e^{-\beta(T-t)}\Big(\int_{\underline{v}}^{v} \frac{\prod_{i=1}^k F\big(\min(v_{th}^2(t_i^1,0),x)\big)}{\prod_{i=1}^k F\big(v_{th}^2(t_i^1,0)\big)} \times \prod_{j=1}^{l} F\big(\min(v_{th}^2(t_j^2,\underline{v}),x)\big)dx\Big) \quad \text{(A.21)}$$

By assumption, the threshold function is a decreasing function of I_t. Thus, we have:

$$F\big(\min(v_{th}^2(t_j^2,\underline{v}),x)\big) \leq F\big(\min(v_{th}^2(t_j^2,0),x)\big) \qquad \forall j = 1,2,..,l$$

Thus comparing equation (A.21) with equation (10) we get:

$$\mathbf{E}[U_{Bid(t)}(v,t,I_t=0)|\mathcal{E}] \leq \mathbf{E}[U_{Bid}(v,t,I_t=0)|\mathcal{E}]$$

This is true for the event $\mathcal{E}$ and in fact for any realization of the random bidder arrival process. Thus, taking the expectation over $K, T_1^1, T_2^1, .., T_K^1, L, T_1^2, T_2^2, .., T_L^2$, we have

$$\mathbf{E}[U_{Bid(t)}(v,t,I_t=0)] \leq \mathbf{E}[U_{Bid}(v,t,I_t=0)]$$

Hence the expected utility if the bidder bids immediately is less than or equal to his utility if he bids at time T. ■

Proof of Lemma 4

Proof: Differentiating the conditional utility from bidding, $\mathbf{E}[U_{Bid}(v,t,I_t=0)|\mathcal{E}]$, with respect to bidder valuation v, we get:

$$\frac{\partial}{\partial v}\mathbf{E}[U_{Bid}(v,t,I_t=0)|\mathcal{E}] = e^{-\beta(T-t)}\Big(\frac{\prod_{i=1}^k F\big(\min(v_{th}^2(t_i^1,0),v)\big)}{\prod_{i=1}^k F\big(v_{th}^2(t_i^1,0)\big)} \times \prod_{j=1}^{l} F\big(\min(v_{th}^2(t_j^2,0),v)\big)\Big) \quad \text{(A.22)}$$

Thus $0 \leq \frac{\partial}{\partial v}\mathbf{E}[U_{Bid}(v,t,I_t=0)|\mathcal{E}] < 1$ for all $v \geq \underline{v}$ and $t \in (0,T)$. Since the derivative exists and is finite for all $t_i^1, i = 1,..,k, t_j^2, j = 1,..,l, k, l$, we have

$$\frac{\partial}{\partial v}\mathbf{E}[U_{Bid}(v,t,I_t=0)] = \mathbf{E}\Big[\frac{\partial}{\partial v}\mathbf{E}[U_{Bid}(v,t,I_t=0)|\mathcal{E}]\Big]$$

where the outer expectation in the right term is over $T_i^1, i = 1,..,K$ and $T_j^2, j = 1,..,L$. Using equation (A.22), we have $0 \leq \frac{\partial}{\partial v}\mathbf{E}[U_{Bid}(v,t,I_t=0)] < 1$ for all $v \geq \underline{v}$ and $t \in (0,T)$. ■

Proof of Lemma 5

Proof: For valuation $v = \underline{v}$, we have:

$$U_{Buy}(\underline{v}) = \underline{v} - P \leq 0 = \mathbf{E}[U_{Bid}(\underline{v},t,0)]$$

In Lemma 4, we have shown that, with increasing v, $\mathbf{E}[U_{Bid}(v,t,0)]$ increases less rapidly than U_{Buy}, i.e. $0 \leq \frac{\partial}{\partial v}\mathbf{E}[U_{Bid}(v,t,I_t=0)] < 1 = \frac{\partial}{\partial v}\Big(U_{Buy}(v)\Big)$. Thus there exists a unique $v^*(t) \geq \underline{v}$ such that:

$$U_{Buy}(v^*(t)) = \mathbf{E}[U_{Bid}(v^*(t),t,0)] \quad \text{(A.23)}$$

■

Proof of Lemma 6

Proof: Let us compare the utility from bidding (at T) for a bidder if his first arrival time (the first time he visits the auction site) is t' $(> t)$ instead of t.

Suppose that there are k bidders in $(0,t)$ and l bidders in $(t,T]$ and they arrive at time $0 < t_1^1 < t_2^1 < .. < t_k^1 < t$ and $t < t_1^2 < t_2^2 < .. < t_l^2 < T$ respectively. Suppose also that j $(0 \le j \le l)$ bidders arrive in the interval (t,t'). Then the bidder arriving at t' has information about $k+j$ bidders and his utility from bidding is

$$\begin{aligned}&\mathbf{E}[U_{Bid}(v,t',0)|k,l,j,t_1^1,..,t_1^2,..] = \\ &\quad e^{-\beta(T-t')}\Big(\int_{\underline{v}}^{v}\frac{\prod_{i=1}^{k}F\big(\min(v_{th}^2(t_i^1,0),x)\big)}{\prod_{i=1}^{k}F\big(v_{th}^2(t_i^1,0)\big)} \\ &\times\frac{\prod_{i=1}^{j}F\big(\min(v_{th}^2(t_i^2,0),x)\big)}{\prod_{i=1}^{j}F\big(v_{th}^2(t_i^2,0)\big)}\prod_{i=j+1}^{l}F\big(\min(v_{th}^2(t_i^2,0),x)\big)dx\Big)\end{aligned}$$

which can be written as

$$\begin{aligned}&\mathbf{E}[U_{Bid}(v,t',0)|k,l,j,t_1^1,..,t_1^2,..] = \\ &\quad e^{-\beta(T-t')}\Big(\int_{\underline{v}}^{v}\frac{\prod_{i=1}^{k}F\big(\min(v_{th}^2(t_i^1,0),x)\big)}{\prod_{i=1}^{k}F\big(v_{th}^2(t_i^1,0)\big)} \\ &\qquad\qquad\times\frac{\prod_{i=1}^{l}F\big(\min(v_{th}^2(t_i^2,0),x)\big)}{\prod_{i=1}^{j}F\big(v_{th}^2(t_i^2,0)\big)}\Big)\end{aligned}$$

The corresponding expected utility for a bidder arriving at t is:

$$\begin{aligned}&\mathbf{E}[U_{Bid}(v,t,I_t=0)|k,l,t_1^1,..,t_1^2,..] = \\ &\quad e^{-\beta(T-t)}\Big(\int_{\underline{v}}^{v}\frac{\prod_{i=1}^{k}F\big(\min(v_{th}^2(t_i^1,0),x)\big)}{\prod_{i=1}^{k}F\big(v_{th}^2(t_i^1,0)\big)} \\ &\qquad\qquad\times\prod_{i=1}^{l}F\big(\min(v_{th}^2(t_i^2,0),x)\big)dx\Big)\end{aligned}$$

Since $F(\cdot) \le 1$ and $e^{-\beta(T-t')} > e^{-\beta(T-t)}$, we have:

$$\mathbf{E}[U_{Bid}(v,t',0)|k,l,j,t_1^1,..,t_1^2,..] > \mathbf{E}[U_{Bid}(v,t,0)|k,l,j,t_1^1,..,t_1^2,..]$$

This is true for any sequence of arrivals and hence is true if we take the expectation over the bidder arrival times. Thus, we have

$$\mathbf{E}[U_{Bid}(v,t',0)] > \mathbf{E}[U_{Bid}(v,t,0)]$$

Hence the utility from bidding is higher for bidders arriving later in the auction than for bidders arriving earlier. ■

Proof of Lemma 7

Proof: By definition of the threshold function, we have

$$v_{th}^2(t,0) - P = \mathbf{E}[U_{Bid}(v_{th}^2(t,0),t,0)] \tag{A.24}$$

Assume for contradiction that $v_{th}^2(t,0) < v_{th}^2(t-dt,0)$. Thus there exists a $dv > 0$, such that

$$v_{th}^2(t,0) = v_{th}^2(t-dt,0) - dv \tag{A.25}$$

Using Lemma 6, we have

$$\mathbf{E}[U_{Bid}(v_{th}^2(t,0),t,0)] \ge \mathbf{E}[U_{Bid}(v_{th}^2(t,0),t-dt,0)] \tag{A.26}$$

Substituting equation (A.25) in equation (A.24) and using (A.26), we get

$$\begin{aligned}v_{th}^2(t-dt,0) - dv - P &= \mathbf{E}[U_{Bid}(v_{th}^2(t-dt,0)-dv,t,0)] \\ &\ge \mathbf{E}[U_{Bid}(v_{th}^2(t-dt,0)-dv,t-dt,0)]\end{aligned} \tag{A.27}$$

By the definition of $v_{th}^2(t-dt,0)$, we have

$$v_{th}^2(t-dt,0) - P = \mathbf{E}[U_{Bid}(v_{th}^2(t-dt,0),t-dt,0)] \tag{A.28}$$

Using equation (A.28) in (A.27), we get

$$\mathbf{E}[U_{Bid}(v_{th}^2(t-dt,0),t-dt,0)] - \mathbf{E}[U_{Bid}(v_{th}^2(t-dt,0)-dv,t-dt,0)] \ge dv$$

On dividing both sides by dv and taking the limit as dv goes to zero, we get that $\frac{\partial}{\partial v}\mathbf{E}[U_{Bid}(v,t-dt,0)] \ge 1$ which is a contradiction to Lemma 4. Hence the threshold valuation $v_{th}^2(t,0)$ is a non-decreasing function of t. ■

Proof of Proposition 2

Proof: Consider the threshold at time t and $t+\Delta t$. We have

$$v_{th}^2(t) - P = \mathbf{E}\Big[U_{Bid}(v_{th}^2(t),t)\Big]$$

$$v_{th}^2(t+\Delta t) - P = \mathbf{E}\Big[U_{Bid}(v_{th}^2(t+\Delta t),t+\Delta t)\Big]$$

To get a differential equation, we first calculate the expected utility from bidding at time t and $t+\Delta t$. We then subtract the two and divide by Δt. Taking the limit as Δt approaches zero, we get:

$$\begin{aligned}\frac{dv_{th}^2(t)}{dt} &= \lim_{\Delta t\to 0}\frac{v_{th}^2(t+\Delta t) - v_{th}^2(t)}{\Delta t} \\ &= \lim_{\Delta t\to 0}\frac{\mathbf{E}\Big[U_{Bid}(v_{th}^2(t+\Delta t),t+\Delta t)\Big] - \mathbf{E}\Big[U_{Bid}(v_{th}^2(t),t)\Big]}{\Delta t}\end{aligned} \tag{A.29}$$

First consider a bidder, call him A, arriving at time t and having a valuation equal to the threshold valuation $v_{th}^2(t)$. He has information about all the bidders arriving before him, i.e. in the interval (0,t). In particular every bidder i whose arrival time $t_i \in (0,t)$ has valuation $v_i \le v_{th}^2(t_i)$. Thus for bidder A the arrival process of other bidders is:

1) Non-homogeneous Poisson process in (0,t) with arrival rate $\lambda(\tau) = \lambda F\big(v_{th}^2(\tau)\big)$
2) Homogeneous Poisson process in (t,T] with arrival rate λ.

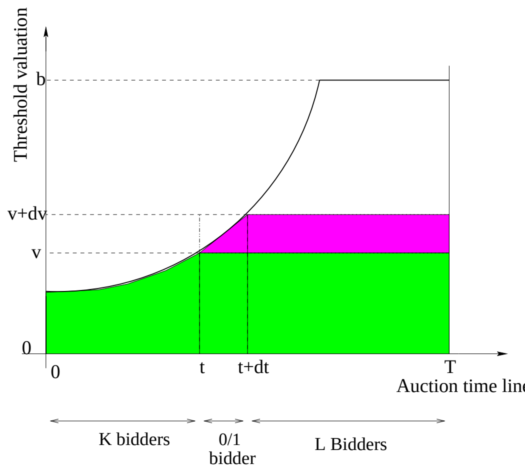

Fig. 5. Threshold valuation

To calculate $\mathbf{E}[U_{bid}(v_{th}^2(t),t)]$, we will condition on the number of arrivals in the interval $(t,t+\Delta t)$. For notational convenience let $t' = t+\Delta t$. Suppose that k bidders arrived in (0,t) at time $t_1^1,t_2^1,..,t_k^1$ respectively and L bidders arrive in $[t',T]$ where L is a Poisson random variable with parameter $\lambda(T-t')$.

For the sake of brevity, let

$$\prod = \frac{\prod_{i=1}^{k}F\big(\min(v_{th}^2(t_i^1),x)\big)}{\prod_{i=1}^{k}F\big(v_{th}^2(t_i^1)\big)}$$

Now suppose that there was one arrival in (t,t'). The probability of this event is $\lambda\Delta t$, since the arrival process is Poisson. The utility from bidding in this case is:

$$\mathbf{E}[U_{bid}\big(v_{th}^2(t), t|\text{Arrival in } (t,t')\big)] = \sum_{l=0}^{\infty}\mathbf{E}\Big[e^{-\beta(T-t)}\Big(\int_{\underline{v}}^{v_{th}^2(t)}\prod F(x)\times F(x)^l dx\Big)\Big]\times Pr(L=l)$$

Notice that here we first calculate the expected utility assuming $L=l$ and then sum over all possible values l. The expectation $\mathbf{E}$ is over $K, T_1, T_2, ..., T_k$.

If there was no arrival in (t,t') (probability = $1-\lambda\Delta t$), the expected utility is

$$\mathbf{E}[U_{bid}(v_{th}^2(t), t|\text{ No Arrival in } (t, t+\Delta t))] = \sum_{l=0}^{\infty}\mathbf{E}\Big[e^{-\beta(T-t)}\Big(\int_{\underline{v}}^{v_{th}^2(t)}\prod F(x)^l dx\Big)\Big]\times Pr(L=l)$$

The probability of more than one arrival in an interval of length Δt is $o(\Delta t)$, where $o(\Delta t)$ indicates any function $f(\delta)$ such that $\lim_{\delta\to 0}\frac{f(\delta)}{\delta}=0$, and thus when we divide by Δt and take the limit $\Delta t\to 0$ (as in equation A.29) the $o(\Delta t)$ terms will disappear. Hence ignoring the terms corresponding to more than one arrival, the unconditional expected utility is:

$$\mathbf{E}[U_{bid}(v_{th}^2(t), t)] = \mathbf{E}\Big[U_{bid}\big(v_{th}^2(t), t|\text{Arrival in } (t,t')\big)\Big]\times(\lambda\Delta t) + \mathbf{E}\Big[U_{bid}\big(v_{th}^2(t), t|\text{No Arrival in}(t,t')\big)\Big]\times(1-\lambda\Delta t)$$

Substituting for the terms, we get

$$\begin{aligned}\mathbf{E}[U_{bid}(v_{th}^2(t), t)] =& \Big(\sum_{l=0}^{\infty}\mathbf{E}\Big[e^{-\beta(T-t)}\Big(\int_{\underline{v}}^{v_{th}^2(t)}\prod F(x)^{l+1}dx\Big)\Big]Pr(L=l)\Big)(\lambda\Delta t)\\ &+\Big(\sum_{l=0}^{\infty}\mathbf{E}\Big[e^{-\beta(T-t)}\Big(\int_{\underline{v}}^{v_{th}^2(t)}\prod F(x)^l dx\Big)\Big]Pr(L=l)\Big)(1-\lambda\Delta t)\end{aligned}\tag{A.30}$$

We now use the same technique to get $\mathbf{E}[U_{bid}(v_{th}^2(t'), t')]$. Consider a bidder, call him B, arriving at a time t' having a valuation $v_{th}^2(t')$. He has information about bidders arriving before him, i.e. in the interval $(0,t')$. In particular every bidder i whose arrival time $t_i\in(0,t')$ has valuation $v_i\le v_{th}^2(t_i)$. Thus for bidder B the arrival process of other bidders is:

1) Non-homogeneous Poisson process in $(0,t')$ with arrival rate $\lambda(\tau)=\lambda F\big(v_{th}^2(\tau)\big)$
2) Homogeneous Poisson process in $(t',T]$ with arrival rate λ.

Now to calculate $\mathbf{E}[U_{bid}(v_{th}^2(t'), t')]$, we will condition on the number of arrivals in the interval (t,t'). Suppose that k bidders arrived in (0,t] at time $t_1, t_2, .., t_k$ and L bidders arrive in $(t',T]$. Recall that L is a Poisson random variable with parameter $\lambda(T-t')$.

First suppose that there was an arrival at $\tau\in(t,t')$. The probability of this event is $\lambda F\big(v_{th}^2(\tau)\big)\Delta t$, since the arrival process is a non-homogeneous Poisson process. Then the utility from bidding is:

$$\mathbf{E}[U_{bid}(v_{th}^2(t'), t')|\text{Arrival in } (t,t')] = \sum_{l=0}^{\infty}\mathbf{E}\Big[e^{-\beta(T-t')}\Big(\int_{\underline{v}}^{v_{th}^2(t')}\prod\frac{F\big(\min(v_{th}^2(\tau),x)\big)}{F\big(v_{th}^2(\tau)\big)}\times F(x)^l dx\Big)\Big]\times Pr(L=l)$$

Now if there was no arrival in (t,t') $\Big($probability = $\big(1-\lambda F(v_{th}^2(\tau))\Delta t\big)\Big)$, the expected utility is:

$$\mathbf{E}[U_{bid}(v_{th}^2(t'), t')|\text{No Arrival in } (t,t')] = \sum_{l=0}^{\infty}\mathbf{E}\Big[e^{-\beta(T-t')}\Big(\int_{\underline{v}}^{v_{th}^2(t')}\prod F(x)^l dx\Big)\Big]Pr(L=l)$$

Again ignoring the possibility of more than one arrival, the unconditional expected utility is:

$$\begin{aligned}\mathbf{E}[U_{bid}(v_{th}^2(t'), t')] =& \mathbf{E}\Big[U_{bid}(v_{th}^2(t'), t')|\text{Arrival in } (t,t')\Big]\Big(\lambda F\big(v_{th}^2(\tau)\big)\Delta t\Big)\\ &+\mathbf{E}\Big[U_{bid}(v_{th}^2(t'), t')|\text{No Arrival in } (t,t')\Big]\Big(1-\lambda F\big(v_{th}^2(\tau)\big)\Delta t\Big)\end{aligned}$$

Substituting for the terms we get:

$$\begin{aligned}\mathbf{E}[U_{bid}(v_{th}^2(t'), t')] =& \sum_{l=0}^{\infty}\Bigg(\mathbf{E}\Big[e^{-\beta(T-t')}\Big(\int_{\underline{v}}^{v_{th}^2(t')}\prod\frac{F\big(\min(v_{th}^2(\tau),x)\big)}{F\big(v_{th}^2(\tau)\big)}F(x)^l dx\Big)\Big]\\ &\times Pr(L=l)\Big(\lambda F\big(v_{th}^2(\tau)\big)\Delta t\Big)\Bigg)\\ &+\sum_{l=0}^{\infty}\Bigg(\mathbf{E}\Big[e^{-\beta(T-t')}\Big(\int_{\underline{v}}^{v_{th}^2(t')}\prod F(x)^l dx\Big)\Big]\\ &\times Pr(L=l)\Big(1-\lambda F\big(v_{th}^2(\tau)\big)\Delta t\Big)\Bigg)\end{aligned}\tag{A.31}$$

To proceed further we need to make the following approximation: Replace $F\big(\min(v_{th}^2(\tau),x)\big)$ by $F(x)$ in the first term of the equation (A.31). The replacement effectively assumes that one bidder has valuation in the interval $[\underline{v}, v_{th}^2(t')]$ instead of $[\underline{v}, v_{th}^2(\tau)]$. When calculating the maximum valuation among the bidders, this assumption can lead to an error e which is bounded as follows:

$$0\le e\le v_{th}^2(t')-v_{th}^2(\tau)\le v_{th}^2(t')-v_{th}^2(t)$$

Assuming that $\frac{dv_{th}^2(t)}{dt}$ is finite, i.e. there exists a C such that $\Big|\frac{dv_{th}^2(t)}{dt}\Big|\le C$, the error e can be bounded above by $C(\Delta t)$. Thus if we let T_1 to be the first term in equation (A.31) and T_1^a be the corresponding approximate expression, we have:

$$\begin{aligned}0\le T_1^a-T_1\le& \sum_{l=0}^{\infty}\Bigg(\mathbf{E}\Big[e^{-\beta(T-t')}\Big(v_{th}^2(t')-v_{th}^2(\tau)\Big)\Big]\\ &\times Pr(L=l)*\lambda F\big(v_{th}^2(\tau)\big)\Delta t\Bigg)\\ \le& e^{-\beta(T-t')}C\lambda F\big(v_{th}^2(\tau)\big)(\Delta t)^2\end{aligned}$$

Dividing by Δt and taking the limit $\Delta t\to 0$, it is seen that the error goes to zero.

Now getting back to our original derivation. The expression in (A.31), after a little simplification, becomes:

$$\begin{aligned}\mathbf{E}[U_{bid}(v_{th}^2(t'), t')] =& \sum_{l=0}^{\infty}\mathbf{E}\Big[e^{-\beta(T-t')}\Big(\int_{\underline{v}}^{v_{th}^2(t')}\prod F(x)^{l+1}dx\Big)\Big]Pr(L=l)(\lambda\Delta t)\\ &+\sum_{l=0}^{\infty}\Bigg(\mathbf{E}\Big[e^{-\beta(T-t')}\Big(\int_{\underline{v}}^{v_{th}^2(t')}\prod F(x)^l dx\Big)\Big]\\ &\times Pr(L=l)\Big(1-\lambda F\big(v_{th}^2(\tau)\big)\Delta t\Big)\Bigg)\end{aligned}\tag{A.32}$$

Subtracting equation (A.30) from equation (A.32), dividing by Δt and taking the limit $\Delta t \to 0$, we get, after some simplification:

$$\frac{dv_{th}^2(t)}{dt} = \frac{\Big(\beta + \lambda\Big(1 - F\big(v_{th}^2(t)\big)\Big)\Big)\Big(v_{th}^2(t) - P\Big)}{1 - e^{-\Big(\beta+\lambda\Big(1-F\big(v_{th}^2(t)\big)\Big)\Big)\Big(T-t\Big)}}$$

As before, we can set the threshold to be $\min(v_{th}^2(t), \bar{v})$. Since we assume the bidder valuations to be uniformly distributed in $[\underline{v}, \bar{v}]$, $\Big(1 - F\big(v_{th}^2(t)\big)\Big) = \frac{\bar{v} - v_{th}^2(t)}{m}$ for $v_{th}^2(t) \in [\underline{v}, \bar{v}]$.

Substituting $t = 0$ in equation (12), we get the initial value for the above differential equation:

$$v_{th}^2(0) = P - \frac{m}{\lambda T}\Big(LambertW\Big(- e^{-(\beta+\lambda)T} \times e^{-\frac{-(P-\underline{v})\lambda T + me^{-(\beta+\lambda)T}}{m}}\Big) + e^{-(\beta+\lambda)T}\Big)$$

■