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Coordinating Inventory Control and Pricing Strategies with Random Demand and Fixed Ordering Cost: The Finite Horizon Case

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We analyze a finite horizon, single product, periodic review model in which pricing and production/inventory decisions are made simultaneously. Demands in different periods are random variables that are independent of each other and their distributions depend on the product price. Pricing and ordering decisions are made at the beginning of each period and all shortages are backlogged. Ordering cost includes both a fixed cost and a variable cost proportional to the amount ordered. The objective is to find an inventory policy and a pricing strategy maximizing expected profit over the finite horizon. We show that when the demand model is additive, the profit-to-go functions are k-concave and hence an (s, S, p) policy is optimal. In such a policy, the period inventory is managed based on the classical (s, S) policy and price is determined based on the inventory position at the beginning of each period. For more general demand functions, i.e., multiplicative plus additive functions, we demonstrate that the profit-to-go function is not necessarily k-concave and an (s, S, p) policy is not necessarily optimal. We introduce a new concept, the symmetric k-concave functions, and apply it to provide a characterization of the optimal policy.

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1. Introduction

Traditional inventory models focus on effective replenishment strategies and typically assume that a commodity's price is exogenously determined. In recent years, however, a number of industries have used innovative pricing strategies to manage their inventory effectively. For example, techniques such as *revenue management* have been applied in the airlines, hotels, and rental car agencies—integrating price, inventory control, and quality of service; see Kimes (1989). In the retail industry, to name another example, dynamically pricing commodities can provide significant improvements in profitability, as shown by Gallego and van Ryzin (1994).

These developments call for models that integrate inventory control and pricing strategies. Such models are clearly important, not only in the retail industry, where price-dependent demand plays an important role, but also in manufacturing environments in which production/distribution decisions can be complemented with pricing strategies to improve the firm's bottom line.

To date, the literature has confined itself mainly to models with variable ordering costs but no fixed costs. Extending some of these models to include a fixed cost component is the focus of this paper. Specifically, we consider a finite horizon, single product, periodic review model with stochastic demand. Demands in different periods are independent of each other and their distributions depend on the product price. Pricing and ordering decisions are made at the beginning of each period, and all shortages are backlogged. The ordering cost includes both a fixed cost and a variable cost proportional to the amount ordered. Inventory holding and shortage costs are convex functions of the inventory level carried over from one period to the next. The objective is to find an inventory policy and pricing strategy maximizing expected profit over the finite horizon.

Our model is similar to the model analyzed by Federgruen and Heching (1999) except that in their model the authors assume that ordering cost is *proportional* to the amount ordered and thus does not include a fixed cost component. They show that in this case a *base-stock list price policy* is optimal. That is, in each period the optimal policy is characterized by an order-up-to level, referred to as the base-stock level, and a price which depends on the initial inventory level at the beginning of the period. If the initial inventory level is below the base-stock level an order is placed to raise the inventory level to the base-stock level. Otherwise, no order is placed and a discount price is offered. This discount price is a nonincreasing function of the initial inventory level.

Of course, many papers address the coordination of replenishment strategies and pricing policies, starting with the work of Whitin (1955), who analyzed the celebrated newsvendor problem with price-dependent demand. For a review, the reader is referred to Eliashberg and Steinberg (1991), Petruzzi and Dada (1999), Federgruen and Heching (1999), or Chan et al. (2001).

The paper by Thomas (1974) considers a model similar to ours, namely, a periodic review, finite horizon model with a fixed ordering cost and stochastic, price-dependent demand. The paper postulates a simple policy, referred to by Thomas as (s, S, p), which can be described as follows. The inventory strategy is an (s, S) policy: If the inventory level at the beginning of period t is below the reorder point, s_t, an order is placed to raise the inventory level to the order-up-to level, S_t. Otherwise, no order is placed. Price depends on the initial inventory level at the beginning of the period. Thomas provides a counterexample which shows that, when price is restricted to a discrete set, this policy may fail to be optimal. Thomas goes on to say: "If all prices in an interval are under consideration, it is conjectured that an (s, S, p) policy is optimal under fairly general conditions" (Thomas 1974, p. 517).

Polatoglu and Sahin (2000) also consider a model similar to ours. However, unlike in our model, they assume unsatisfied demand is lost. They show, under relatively general assumptions, that there could be more than one order-up-to level, and for each order-up-to level there could be more than one reorder interval. Polatoglu and Sahin identify sufficient conditions for the optimality of an (s, S, p) policy but it is not clear from their paper whether or not *there exists any demand function that satisfies these conditions*.

This paper is organized as follows. In §2, we review the main assumptions of our model. In §3, we employ the concept of k-convexity and characterize the optimal inventory and pricing policies for *additive demand functions*. We show that in this case the policy proposed by Thomas is indeed optimal. In §4, we analyze general demand functions which may be *nonadditive*. We demonstrate that in this case the profit-to-go function is not necessarily k-concave and an (s, S, p) policy is not necessarily optimal. We introduce the concept of symmetric k-convex functions and apply it to provide a characterization of the optimal policy for the general demand case. Finally, in §5, we discuss extensions and provide concluding remarks.

2. The Model

Consider a firm that has to make production and pricing decisions over a finite time horizon with T periods. Demands in different periods are independent of each other. For each period t, $t = 1, 2, \ldots, T$, let

$w_t =$ demand in period t,

$p_t =$ selling price in period t,

$\underline{p}_t$, $\bar{p}_t$ are lower and upper bounds on p_t, respectively.

Throughout this paper, we concentrate on demand functions of the following forms:

Assumption 1. *For $t = 1, 2, \ldots, T$, the demand function satisfies*

$$w_t = D_t(p_t, \epsilon_t) := \alpha_t D_t(p_t) + \beta_t, \tag{1}$$

where $\epsilon_t = (\alpha_t, \beta_t)$ and α_t, β_t are two random variables with $E\{\alpha_t\} = 1$ and $E\{\beta_t\} = 0$. The random perturbations, ϵ_t, are independent across time.

The assumptions $E\{\alpha_t\} = 1$ and $E\{\beta_t\} = 0$ can clearly be made without loss of generality. The special cases where $\alpha_t = 1$ and $\beta_t = 0$ are referred to as the *additive* and *multiplicative* cases, respectively. Finally, observe that special cases of the function $D_t(p)$ include $D_t(p) = b_t - a_t p$ ($a_t > 0$, $b_t > 0$) in the additive case and $D_t(p) = a_t p^{-b_t}$ ($a_t > 0$, $b_t > 1$) in the multiplicative case; both are common in the economics literature (see Petruzzi and Dada 1999).

We assume the following.

Assumption 2. *For all t, $t = 1, 2, \ldots, T$, the inverse function of D_t, denoted by D_t^{-1}, is continuous and strictly decreasing. Furthermore, the expected revenue*

$$R_t(d) := dD_t^{-1}(d)$$

is a concave function of expected demand d.

Let x_t be the inventory level at the beginning of period t, just before placing an order. Similarly, y_t is the inventory level at the beginning of period t after placing an order. Define

$$\delta(u) := \begin{cases} 1 & \text{if } u > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The ordering cost function includes both a fixed cost and a variable cost and is calculated for every t, $t = 1, 2, \ldots, T$, as

$$k\delta(y_t - x_t) + c_t(y_t - x_t).$$

Note that while the variable cost function is time dependent, the fixed cost function, k, is time independent. In fact, as we observe at the end of the paper, all our results can be extended to situations in which the fixed cost is a *nonincreasing function* of time.

Unsatisfied demand is backlogged. Let x be the inventory level carried over from period t to period $t+1$. Because we allow backlogging, x may be positive or negative. A cost $h_t(x)$ is incurred at the end of period t which represents inventory holding cost when $x > 0$ and penalty cost if $x < 0$. Denote by

$$G_t(y, p) = E\{h_t(y - D_t(p, \epsilon_t))\}.$$

For technical reasons, we need the following assumptions regarding properties of function $G_t(y, p)$ and the finiteness of the moments of the demand functions. These assumptions are similar to those in Federgruen and Heching (1999).

Assumption 3. *For each t, $t = 1, 2, \ldots, T$, $h_t(x)$ is a convex function of the inventory level x at the end of period t. Furthermore,* $\lim_{y\to\infty} G_t(y, p) = \lim_{y\to-\infty}[c_t y +$

$G_t(y, p)] = \lim_{y\to\infty}[(c_t - c_{t+1})y + G_t(y, p)] = \infty$ *for all* $p \in [\underline{p}_t, \bar{p}_t]$.

Assumption 4. $0 \leqslant G_t(y, p) = O(|y|^\rho)$ *for some integer* ρ.

Assumption 5. $E\{D_t(p, \epsilon_t)\}^\rho < \infty$ *for all* $p \in [\underline{p}_t, \bar{p}_t]$.

The objective is to decide on ordering and pricing policies so as to maximize total expected profit over the entire planning horizon. Note that Assumption 1 implies that there is a one-to-one correspondence between the selling price $p_t \in [\underline{p}_t, \bar{p}_t]$ and the expected demand $E\{w_t\} = D_t(p_t) \in [\underline{d}_t, \bar{d}_t]$, where

$$\underline{d}_t = D_t(\bar{p}_t) \quad \text{and} \quad \bar{d}_t = D_t(\underline{p}_t).$$

Thus, our problem can be formulated as one of selecting, in each period t, an inventory level y and an expected demand level d, such that the total expected profit over the entire planning horizon is maximized.

Denote by $v_t(x)$ the profit-to-go function at the beginning of time period t with inventory level x. A natural dynamic program for the above maximization problem is as follows. Let $v_{T+1}(x) = 0$ for all x and, hence, for each $t = 1, 2, \ldots, T$, we have

$$v_t(x) = c_t x + \max_{y\geqslant x} -k\delta(y-x) + g_t(y, d_t(y)), \tag{2}$$

where

$$g_t(y, d) = R_t(d) - c_t y + E\{-h_t(y - \alpha_t d - \beta_t) + v_{t+1}(y - \alpha_t d - \beta_t)\} \tag{3}$$

and $d_t(y)$ is the expected demand associated with the best selling price for a given inventory level y, i.e.,

$$d_t(y) \in \operatorname*{arg\,max}_{\bar{d}\geqslant d\geqslant \underline{d}} g_t(y, d).$$

We now relate our problem to the celebrated stochastic inventory control problem discussed by Scarf (1960). In that problem demand is assumed to be exogenously determined, while in our problem demand depends on price. Other assumptions regarding the framework of the model are similar to those made by Scarf. For the classical stochastic inventory problem Scarf showed that an (s, S) policy is optimal. In this policy, the optimal decision in period t is characterized by two parameters, the reorder point, s_t, and the order-up-to level, S_t. An order of size $S_t - x_t$ is made at the beginning of period t if the initial inventory level at the beginning of the period, x_t, is smaller than s_t. Otherwise, no order is placed.

To prove that an (s, S) policy is optimal Scarf uses the concept of k-convexity.

Definition 2.1. A real-valued function f is called k-convex for $k \geqslant 0$, if for any $z \geqslant 0$, $b > 0$, and any y we have

$$k + f(z+y) \geqslant f(y) + \frac{z}{b}(f(y) - f(y-b)). \tag{4}$$

A function f is called k-concave if $-f$ is k-convex.

For the purpose of the analysis of our model, we find it useful to apply another, yet equivalent, definition of k-convexity; see Porteus (1971).

Definition 2.2. A real-valued function f is called k-convex for $k \geqslant 0$, if for any $x_0 \leqslant x_1$ and $\lambda \in [0, 1]$,

$$f((1-\lambda)x_0 + \lambda x_1) \leqslant (1-\lambda)f(x_0) + \lambda f(x_1) + \lambda k. \tag{5}$$

One significant difference between k-convexity and traditional convexity is that (5) is not symmetric with respect to x_0 and x_1.

It turns out that this asymmetry is the main barrier when trying to identify the optimal policy to our problem for nonadditive demand functions. Indeed, in §4 we provide counterexamples to show that the profit-to-go function is *not* necessarily k-concave and an (s, S, p) policy is not necessarily optimal. This motivates the development of a new concept, the *symmetric* k-concave function, which allows us to characterize the optimal policy in the general demand case.

However, under the additive demand model analyzed in §3, this concept is not needed. Indeed, we prove that for additive demand functions, the profit-to-go function is k-concave and, hence, the optimal policy for problem (2) is an (s, S, p) policy.

We summarize properties of k-convex functions as follows; see Bertsekas (1995) for details.

Lemma 1. (a) *A real-valued convex function is also 0-convex and, hence, k-convex for all $k \geqslant 0$. A k_1-convex function is also a k_2-convex function for $k_1 \leqslant k_2$.*

(b) *If $g_1(y)$ and $g_2(y)$ are k_1-convex and k_2-convex, respectively, then for $\alpha, \beta \geqslant 0$, $\alpha g_1(y) + \beta g_2(y)$ is $(\alpha k_1 + \beta k_2)$-convex.*

(c) *If $g(y)$ is k-convex and w is a random variable, then $E\{g(y-w)\}$ is also k-convex, provided $E\{|g(y-w)|\} < \infty$ for all y.*

(d) *If g is a continuous k-convex function and $g(y) \to \infty$ as $|y| \to \infty$, then there exists scalars s and S with $s \leqslant S$ such that*

(i) *$g(S) \leqslant g(y)$ for all scalars y,*
(ii) *$g(S) + k = g(s) < g(y)$ for all $y < s$,*
(iii) *$g(y)$ is a decreasing function on $(-\infty, s)$, and*
(iv) *$g(y) \leqslant g(z) + k$ for all y, z with $s \leqslant y \leqslant z$.*

3. Additive Demand Functions

In this section, we focus on additive demand functions, i.e., demand functions of the form

$$w_t = D_t(p_t) + \beta_t,$$

where β_t is a random variable.

In the following, we show, by induction, that $g_t(y, d_t(y))$ is a k-concave function of y and $v_t(x)$ is a k-concave function of x. Therefore, the optimality of an (s, S, p) policy follows directly from Lemma 1.

To prove that v_t is k-concave we need the following lemma.

Lemma 2. *Suppose that $g_t(y,d)$ is jointly continuous in (y,d). Then, there exists a $d_t(y)$ which maximizes $g_t(y,d)$ for any given y, such that $y-d_t(y)$ is a nondecreasing function of y.*

Proof. Define

$$\begin{aligned}\tilde{g}_t(y,d) &= g_t(y,y-d)\\ &= R_t(y-d)-c_ty+E\{-h_t(d-\beta_t)+v_{t+1}(d-\beta_t)\}.\end{aligned}$$

Then, Assumption 2 implies that function $\tilde{g}_t(y,d)$ has increasing differences in y and d. The lemma thus follows from Topkis (1998, Theorem 2.4.3 and Lemma 2.8.1). □

The lemma thus implies that the higher the inventory level at the beginning of time period t, y_t, the higher the expected inventory level at the end of period t, $y_t-d_t(y_t)$. We are now ready to prove our main results for the additive demand model.

Theorem 3.1. (a) *For $t=T,T-1,\ldots,1$, $g_t(y,d)=O(|y|^\rho)$ and $v_t(x)=O(|x|^\rho)$.*

(b) *For $t=T,T-1,\ldots,1$, $g_t(y,d)$ is continuous in (y,d) and $\lim_{|y|\to\infty} g_t(y,d)=-\infty$ for any $d\in[\underline{d}_t,\bar{d}_t]$. Hence, for any fixed y, $g_t(y,d)$ has a finite maximizer, denoted by $d_t(y)$.*

(c) *For any $t=T,T-1,\ldots,1$, $g_t(y,d_t(y))$ and $v_t(x)$ are k-concave.*

(d) *For $t=T,T-1,\ldots,1$, there exist s_t and S_t with $s_t\leqslant S_t$ such that it is optimal to order S_t-x_t and set the expected demand level $d_t=d_t(S_t)$ when $x_t<s_t$, and not to order anything and set $d_t=d_t(x_t)$ otherwise.*

Proof. By induction. The proof of part (a) follows from the one-to-one correspondence between the expected demand and the selling price, Assumptions 3, 4, and 5, and is similar to that of Theorem 1 in Federgruen and Heching (1999). So we omit its proof.

Assume that parts (a), (b), (c), and (d) hold for $t+1$. The continuity of $g_t(y,d)$ on (y,d) is easy to check. From part (d),

$$v_{t+1}(x)=\begin{cases}-k+g_{t+1}(S_{t+1},d_{t+1}(S_{t+1}))+c_{t+1}x & \text{if } x\leqslant s_{t+1},\\ g_{t+1}(x,d_{t+1}(x))+c_{t+1}x & \text{if } x\geqslant s_{t+1}.\end{cases}$$

This equation implies that $E\{v_{t+1}(y-d_t-\beta_t)-c_{t+1}(y-d_t-\beta_t)\}\leqslant v_{t+1}(S_{t+1})-c_{t+1}S_{t+1}$ and because $G_t(y,D_t^{-1}(d))$ is jointly convex in (y,d), we have that $\lim_{|y|\to\infty} g_t(y,g)=-\infty$ for any $d\in[\underline{d}_t,\bar{d}_t]$ uniformly by Assumption 3. Hence, for any fixed y, $g_t(y,d)$ has a finite maximizer $d_t(y)$. Thus, part (b) holds for period t.

We now focus on part (c). We show that $g_t(y,d_t(y))$ and $v_t(x)$ are k-concave based on the assumption that $v_{t+1}(x)$ is k-concave.

For any $y<y'$ and $\lambda\in[0,1]$, we have by Lemma 2 and the assumption that v_{t+1} is k-concave that

$$\begin{aligned}v_{t+1}&((1-\lambda)(y-d_t(y)-\beta_t)+\lambda(y'-d_t(y')-\beta_t))\\ &\geqslant(1-\lambda)v_{t+1}(y-d_t(y)-\beta_t)\\ &\quad+\lambda v_{t+1}(y'-d_t(y')-\beta_t)-\lambda k.\end{aligned}\tag{6}$$

In addition, the concavity of $R_t(d)$ implies that

$$\begin{aligned}R_t&((1-\lambda)d_t(y)+\lambda d_t(y'))\\ &\geqslant(1-\lambda)R_t(d_t(y))+\lambda R_t(d_t(y')).\end{aligned}$$

Because $h_t(x)$ is convex we also have

$$\begin{aligned}-h_t&((1-\lambda)(y-d_t(y)-\beta_t)+\lambda(y'-d_t(y')-\beta_t))\\ &\geqslant-(1-\lambda)h_t(y-d_t(y)-\beta_t)-\lambda h_t(y'-d_t(y')-\beta_t).\end{aligned}$$

Adding the last three inequalities and taking expectation we get

$$\begin{aligned}g_t&((1-\lambda)y+\lambda y',(1-\lambda)d_t(y)+\lambda d_t(y'))\\ &\geqslant(1-\lambda)g_t(y,d_t(y))+\lambda g_t(y',d_t(y'))-\lambda k.\end{aligned}$$

Because $d_t((1-\lambda)y+\lambda y')$ maximizes $g((1-\lambda)y+\lambda y',d)$ for $d\in[\underline{d},\bar{d}]$, we have

$$\begin{aligned}g_t&((1-\lambda)y+\lambda y',d_t((1-\lambda)y+\lambda y'))\\ &\geqslant g_t((1-\lambda)y+\lambda y',(1-\lambda)d_t(y)+\lambda d_t(y')),\end{aligned}$$

and hence,

$$\begin{aligned}g_t&((1-\lambda)y+\lambda y',d_t((1-\lambda)y+\lambda y'))\\ &\geqslant(1-\lambda)g_t(y,d_t(y))+\lambda g_t(y',d_t(y'))-\lambda k,\end{aligned}\tag{7}$$

that is, $g_t(y,d_t(y))$ is a k-concave function of y.

Thus, using Lemma 1 part (d) we have from the k-concavity of $g_t(y,d_t(y))$ that there exists $s_t<S_t$, such that S_t maximizes $g_t(y,d_t(y))$ and s_t is the smallest value of y for which $g_t(S_t,d_t(S_t))=g_t(y,d_t(y))+k$, and

$$v_t(x)=\begin{cases}-k+g_t(S_t,d_t(S_t))+c_tx & \text{if } x\leqslant s_t,\\ g_t(x,d_t(x))+c_tx & \text{if } x\geqslant s_t.\end{cases}$$

The k-concavity of v_t can be checked directly from the k-concavity of $g_t(y,d_t(y))$; see Bertsekas (1995) for a proof.

Part (d) follows directly from part (c) and Lemma 1. □

An interesting question is whether the optimal selling price is a nonincreasing function of the initial inventory level, as is the case for a similar model with no fixed cost; see Federgruen and Heching (1999). Unfortunately, this property does not hold for our model.

Proposition 1. *The optimal price, $p_t(y)$, is not necessarily a nonincreasing function of the initial inventory level y.*

The proof is provided in Appendix A. Similar behavior holds for the lost sales case; see Polatoglu and Sahin (2000).

4. General Demand Functions

In this section, we focus on general demand functions $w_t = \alpha_t D_t(p_t) + \beta_t$. Our objective is two-fold. First, we demonstrate that under demand functions (1), $v_t(x)$ may not be k-concave and an (s, S, p) policy may fail to be optimal for problem (2). Second, we show that in this case the optimal policy satisfies a more general structure.

To characterize the optimal policy for the demand functions (1), one might consider using the same approach applied in §3. Unfortunately, in this case, the function $y - \alpha_t d_t(y)$ is not necessarily a nondecreasing function of y for all possible α_t, as is the case for additive demand functions. Hence, the approach employed in §3 does not work in this case.

Specifically, the next lemma, whose proof is given in Appendix B, illustrates that the profit-to-go function is in general not k-concave.

Lemma 3. *There exists an instance of problem* (2) *with a multiplicative demand function and time independent parameters such that the functions $g_{T-1}(y, d_{T-1}(y))$ and $v_{T-1}(x)$ are not k-concave.*

Of course, it is entirely possible that even if the functions $g_t(y, d_t(y))$ and $v_t(x)$ are not k-concave for some period t, the optimal policy is still an (s, S, p) policy. The next lemma, whose proof is given in Appendix C, shows that this is not true in general.

Lemma 4. *There exists an instance of problem* (2) *with multiplicative demand functions where an (s, S, p) policy is not optimal.*

This lemma, of course, is consistent with the observation made by Polatoglu and Sahin (2000) for the lost sales model.

To overcome these difficulties, we propose a weaker version of k-convexity, referred to as symmetric k-convexity:

Definition 4.1. A real-valued function f is called sym-k-convex for $k \geqslant 0$, if for any x_0, x_1 and $\lambda \in [0, 1]$,

$$f((1-\lambda)x_0 + \lambda x_1) \leqslant (1-\lambda)f(x_0) + \lambda f(x_1) + \max\{\lambda, 1-\lambda\}k. \tag{8}$$

A function f is called sym-k-concave if $-f$ is sym-k-convex.

Observe that k-convexity is a special case of sym-k-convexity. The following lemma describes properties of sym-k-convex functions—properties that are parallel to those stated in Lemma 1.

Lemma 5. (a) *A real-valued convex function is also sym-0-convex and, hence, sym-k-convex for all $k \geqslant 0$. A sym-k_1-convex function is also a sym-k_2-convex function for $k_1 \leqslant k_2$.*

(b) *If $g_1(y)$ and $g_2(y)$ are sym-k_1-convex and sym-k_2-convex, respectively, then for $\alpha, \beta \geqslant 0$, $\alpha g_1(y) + \beta g_2(y)$ is sym-$(\alpha k_1 + \beta k_2)$-convex.*

(c) *If $g(y)$ is sym-k-convex and w is a random variable, then $E\{g(y-w)\}$ is also sym-k-convex, provided $E\{|g(y-w)|\} < \infty$ for all y.*

(d) *If g is a continuous sym-k-convex function and $g(y) \to \infty$ as $|y| \to \infty$, then there exists scalars s and S with $s \leqslant S$ such that*

(i) *$g(S) \leqslant g(y)$ for all y,*

(ii) *s is the smallest value x such that $g(x) = g(S) + k$; therefore, $g(y) > g(s)$ for all $y < s$, and*

(iii) *$g(y) \leqslant g(z) + k$ for all y, z with $(s+S)/2 \leqslant y \leqslant z$.*

Proof. Parts (a), (b), and (c) follow directly from the definition of symmetric k-convexity. Hence, we focus on part (d). Because g is continuous and $g(y) \to \infty$ as $|y| \to \infty$, there exist x and S with $x \leqslant S$ such that $g(S) \leqslant g(y)$ for all y and $g(x) = g(S) + k$. Let $s = \min\{x\colon g(x) = g(S) + k\}$.

To prove part (d)(iii) we consider two cases. First, for any $S \leqslant y \leqslant z$, there exists $\lambda \in [0, 1]$ such that $y = (1-\lambda)S + \lambda z$, and we have from the definition of sym-k-convex that

$$g(y) \leqslant (1-\lambda)g(S) + \lambda g(z) + \max\{\lambda, 1-\lambda\}k \leqslant g(z) + k,$$

where the second inequality follows from the fact that S minimizes $g(x)$ implying that $g(S) \leqslant g(z)$ and because $\max\{\lambda, 1-\lambda\} \leqslant 1$.

In the second case, consider y such that $S \geqslant y \geqslant (s+S)/2$. In this case, there exists $1 \geqslant \lambda \geqslant 1/2$ such that $y = (1-\lambda)s + \lambda S$ and from the definition of sym-k-convex, we have that

$$g(y) \leqslant (1-\lambda)g(s) + \lambda g(S) + \lambda k = g(S) + k \leqslant g(z) + k$$

because $g(s) = g(S) + k$. Hence, (i)–(iii) hold. □

We are ready to show the main result for the general demand model. In the following, we show, by induction, that $g_t(y, d_t(y))$ is a sym-k-concave function of y and $v_t(x)$ is a sym-k-concave function of x. Hence, a characterization of the optimal pricing and ordering policies follows from Lemma 5.

Theorem 4.1. (a) *For $t = T, T-1, \ldots, 1$, $g_t(y, d) = O(|y|^\rho)$ and $v_t(x) = O(|x|^\rho)$.*

(b) *For $t = T, T-1, \ldots, 1$, $g_t(y, d)$ is continuous in (y, p) and $\lim_{|y|\to\infty} g_t(y, d) = -\infty$ for any $d \in [\underline{d}_t, \bar{d}_t]$. Hence, for any fixed y, $g_t(y, d)$ has a finite maximizer, denoted by $d_t(y)$.*

(c) *For any $t = T, T-1, \ldots, 1$, $g_t(y, d_t(y))$ and $v_t(x)$ are sym-k-concave.*

(d) *For $t = T, T-1, \ldots, 1$, there exist s_t and S_t with $s_t \leqslant S_t$ and a set $A_t \subset [s_t, (s_t + S_t)/2]$ such that it is optimal to order $S_t - x_t$ and set the expected demand level $d_t = d_t(S_t)$ when $x_t < s_t$ or $x_t \in A_t$, and not to order anything and set $d_t = d_t(x_t)$ otherwise.*

Proof. The proof of parts (a) and (b) is similar to the proof of the same parts in Theorem 3.1. We now focus on part (c).

By induction. $v_{T+1}(x) = 0$ is sym-k-concave. Assuming that v_{t+1} is sym-k-concave, we need to show that both $g_t(y, d_t(y))$ and $v_t(x)$ are sym-k-concave. To show that $g_t(y, d_t(y))$ is sym-k-concave, we obtain inequality (6) with the last term reduced to $-\max(\lambda, 1-\lambda)k$ (which does not depend on Lemma 2). Proceeding as in the proof of Theorem 3.1, we obtain inequality (7), with the same correction for the last term, i.e., $g_t(y, d_t(y))$ is a sym-k-concave function.

It remains to prove that the function $v_t(x)$ is sym-k-concave. Denote by $v_t^*(x) := v_t(x) - c_t x$. From Lemma 5 we have

$$v_t^*(x) = \begin{cases} -k + g_t(S_t, d_t(S_t)) & \text{if } x \in I_t, \\ g_t(x, d_t(x)) & \text{if } x \notin I_t, \end{cases}$$

where S_t is the maximizer of $g_t(y, d_t(y))$ and $I_t = \{y \leqslant S_t\colon g_t(y, d_t(y)) \leqslant g_t(S_t, d_t(S_t)) - k\}$. Furthermore, $v_t^*(x) \geqslant g_t(x, d_t(x))$ for any x and $v_t^*(x) \geqslant -k + g_t(S_t, d_t(S_t))$ for any $x \leqslant S_t$.

Let s_t be defined as the smallest value of y for which $g_t(S_t, d_t(S_t)) = g_t(y, d_t(y)) + k$. Note that from Lemma 5, $(-\infty, s_t] \subset I_t$ and $[(s_t + S_t)/2, \infty) \subset I_t^c$, the complement of I_t.

We now prove that $v_t(x)$ is sym-k-concave. It suffices to prove that $v_t^*(x) := v_t(x) - c_t x$ is sym-k-concave, because $c_t x$ is linear in x. For any $x_0 \leqslant x_1$ and $\lambda \in [0, 1]$, denote by $x_\lambda = (1-\lambda)x_0 + \lambda x_1$.

We consider four different cases:

Case 1. If $x_0, x_1 \notin I_t$, then $v_t^*(x_\mu) = g_t(x_\mu, d_t(x_\mu))$ for $\mu = 0, 1$ and $v_t^*(x) \geqslant g_t(x, d_t(x))$ for any x implying that

$$\begin{aligned} v_t^*(x_\lambda) &\geqslant g_t(x_\lambda, d_t(x_\lambda)) \\ &\geqslant (1-\lambda) g_t(x_0, d_t(x_0)) + \lambda g_t(x_1, d_t(x_1)) \\ &\quad - \max\{\lambda, 1-\lambda\}k \\ &= (1-\lambda) v_t^*(x_0) + \lambda v_t^*(x_1) - \max\{\lambda, 1-\lambda\}k, \end{aligned}$$

where the second inequality holds because $g_t(y, d_t(y))$ is sym-k-concave.

Case 2. If $x_1 \in I_t$, then $x_\lambda \leqslant S_t$ because $x_0 \leqslant x_1 \leqslant S_t$ and therefore

$$\begin{aligned} v_t^*(x_\lambda) &\geqslant -k + g_t(S_t, d_t(S_t)) \\ &= (1-\lambda) g_t(S_t, d_t(S_t)) \\ &\quad + \lambda(-k + g_t(S_t, d_t(S_t))) - (1-\lambda)k \\ &\geqslant (1-\lambda) v_t^*(x_0) + \lambda v_t^*(x_1) - \max\{\lambda, 1-\lambda\}k, \end{aligned}$$

where the second inequality holds because $x_1 \in I_t$ and S_t is the maximizer of $g_t(y, d_t(y))$.

Case 3. If $x_1 \notin I_t$, $x_0 \in I_t$, and $x_\lambda \leqslant S_t$, we have

$$\begin{aligned} v_t^*(x_\lambda) &\geqslant -k + g_t(S_t, d_t(S_t)) \\ &= (1-\lambda)(-k + g_t(S_t, d_t(S_t))) + \lambda g_t(S_t, d_t(S_t)) - \lambda k \\ &\geqslant (1-\lambda) v_t^*(x_0) + \lambda v_t^*(x_1) - \max\{\lambda, 1-\lambda\}k, \end{aligned}$$

where the second inequality holds because $x_0 \in I_t$ and S_t is the maximizer of $g_t(y, d_t(y))$.

Case 4. If $x_1 \notin I_t$, $x_0 \in I_t$, and $x_\lambda \geqslant S_t$, there exists $0 \leqslant \mu \leqslant \lambda$ such that $x_\lambda = (1-\mu)S_t + \mu x_1$ and

$$\begin{aligned} v_t^*(x_\lambda) &= g_t(x_\lambda, d_t(x_\lambda)) \\ &\geqslant (1-\mu) g_t(S_t, d_t(S_t)) + \mu g_t(x_1, d_t(x_1)) \\ &\quad - \max\{\mu, 1-\mu\}k \\ &\geqslant (1-\lambda) g_t(S_t, d_t(S_t)) + \lambda g_t(x_1, d_t(x_1)) \\ &\quad + (\lambda-\mu)(g_t(S_t, d_t(S_t)) - g_t(x_1, d_t(x_1))) - k \\ &\geqslant (1-\lambda)(-k + g_t(S_t, d_t(S_t))) \\ &\quad + \lambda g_t(x_1, d_t(x_1)) - \lambda k \\ &\geqslant (1-\lambda) v_t^*(x_0) + \lambda v_t^*(x_1) - \max\{\lambda, 1-\lambda\}k, \end{aligned}$$

where the first inequality follows from the definition of sym-k-concavity of $g_t(y, d_t(y))$, the third inequality from the fact that $\mu \leqslant \lambda$, and S_t maximizes $g_t(y, d_t(y))$.

Therefore, $v_t(x)$ is sym-k-concave.

Part (d) follows from Lemma 5 and part (c) by defining

$$A_t = I_t \cap [s_t, (s_t + S_t)/2]. \quad \square$$

Theorem 4.1 thus implies that the optimal policy for problem (2) is an (s, S, A, p) policy. Such a policy is characterized by two parameters s_t and S_t and a set $A_t \subset [s_t, (s_t + S_t)/2]$ possibly empty. When the inventory level x_t at the beginning of period t is less than s_t or x_t is in the set A_t, an order of size $S_t - x_t$ is made. Otherwise, no order is placed. Thus, if an order is placed, it is always to raise the inventory level to S_t. Note that in the lost sales case, there might exist multiple order-up-to levels, as demonstrated by Polatoglu and Sahin (2000). Thus, the optimal policy for the backorder model, (s, S, A, p) proven in this paper, is different than the structure of the policy for the lost sales model.

5. Extensions and Concluding Remarks

In this section, we report on some important extensions of the model and results.

- **Nonincreasing Fixed Cost:** The analysis so far assumes a time-independent fixed cost function. In fact, Lemma 1 part (a) and Lemma 5 part (a) imply that our results can be carried over to nonincreasing fixed cost functions.
- **Infinite Time Horizon:** The analysis of the infinite time horizon is significantly more complex but the main results remain the same. This analysis is presented in a companion paper; see Chen and Simchi-Levi (2004).
- **Markovian Demand Model:** The results obtained in this paper can be extended to Markovian demand models where the demand distribution at every time period is determined by an exogenous Markov chain. Specifically, our results hold under assumptions similar to those employed by Sethi and Cheng (1997) on state-dependent holding costs as well as fixed and variable ordering costs.

• **Markdown Model:** In this case we assume that price in period t, p_t, is constrained by $p_t \leqslant p_{t-1}$ for $t = 2, 3, \ldots, T$. In this case, the dynamic program (2) must be modified and it can be written as

$$v_t(x,d') = c_t x + \max_{y\geqslant x,\, \max\{\underline{d}_t,\, d'\}\leqslant d\leqslant \bar{d}_t} -k\delta(y-x) + R_t(d) - c_t y + E\{-h_t(y-\alpha_t d-\beta_t) + v_{t+1}(y-\alpha_t d-\beta_t, d)\}.$$

It turns out that Theorem 4.1 holds for the modified function $v_t(x, d')$ and, hence, the policy introduced in §4 is optimal under the markdown setting. This is true because the sym-k-convexity property can be easily extended to multivariable functions. Of course, the optimal s and S vary with d'.

• **Applications of Symmetric K-Convexity:** Recently, we showed that the concept of symmetric K-convexity provides a natural tool to analyze another classical problem, the stochastic cash balance problem with fixed costs; see Chen (2003) and Chen and Simchi-Levi (2003).

We also point out that our results imply that in the special case with zero fixed ordering cost and general demand functions, i.e., the model analyzed by Federgruen and Heching (1999), a base-stock list price policy is optimal. Indeed, the optimality of a base-stock policy follows from the concavity of the function $g_t(y, d)$ while the optimality of a list price policy follows from a similar argument to the one in Lemma 2 because $g_t(y, d)$ has increasing differences in y and d. Thus, our analysis extends the results of Federgruen and Heching (1999) to the general demand case. In fact, a key assumption in Federgruen and Heching implied by their Lemma 1 is that the demand function, $D_t(p, \epsilon_t)$, is a linear function of the price.

It is appropriate to point out three important limitations of the model analyzed in this paper. The first is the lack of capacity constraints. Indeed, it is well known that even with a single price a modified (s, S) policy fails to be optimal under capacity constraints. The second is the assumption that the backorder cost is independent of the selling price. A more realistic backorder cost function would be a nondecreasing function of the selling price. Of course, while this criticism may be valid, our model, as well as the one by Federgruen and Heching, can serve as a valid approximation.

Finally, the assumption that a customer pays the prevailing price when an order is placed rather than when the product is received may not be appropriate in some cases. Indeed, because our model allows for backlogging, it is possible that some customers receive products a few periods after they placed their orders, perhaps at a time in which price is lower than the price quoted when they ordered the products.

Of course, an important challenge is the analysis of continuous review inventory pricing models. A step in that direction is the recent paper by Feng and Chen (2002) who considered a continuous review model in which the interarrival time is assumed to be exponential and a function of the selling price. Prices are restricted to a discrete set and demand is assumed to be of unit size. For this model, the authors characterize the structure of the optimal policy. They show that in this case, inventory is managed based on an (s, S) policy and price is a function of the inventory level when a decision is made.

Appendix A

Proof of Proposition 1. The following example shows that for additive demand functions, the optimal price $p_t(y)$ is not necessarily nonincreasing.

Example. Consider the last two time periods of problem (2). Let

$$k=1,\quad c_T=0,\quad h_T(x)=|x|,\quad d_T=4-p,$$
$$\bar{p}_T=\underline{p}_T=1,\quad c_{T-1}=0,$$
$$h_{T-1}(x)=\max\{0,-x\}+\tfrac{1}{2}\max\{0,x\},\quad d_{T-1}=1-p,$$
$$\bar{p}_{T-1}=1,\quad \underline{p}_{T-1}=0.$$

Then,

$$v_T(x)=\begin{cases}3-|x-3| & \text{for } x\geqslant 2,\\ 2 & \text{otherwise,}\end{cases}$$

and

$$f_{T-1}(y,p)=p(1-p)+\begin{cases}2+(y-1+p) & \text{for } y-1+p\leqslant 0,\\ 2-\frac{1}{2}(y-1+p) & \text{for } y-1+p\in[0,2],\\ \frac{1}{2}(y-1+p) & \text{for } y-1+p\in[2,3],\\ 6-\frac{3}{2}(y-1+p) & \text{otherwise,}\end{cases}$$

where $f_{T-1}(y,p)=g_{T-1}(y,4-p)$.

Figure 1 depicts the functions $v_T(y)$, $f_{T-1}(y, p_{T-1}(y))$, and $v_T(y) - h_{T-1}(y)$ and while Figure 2 presents the optimal selling price $p_{T-1}(y)$. In Figure 2, the dash-dotted line is $p_{T-1}(y)$ before making the decision to order up to S_{T-1} and solid line represents the optimal price after making the ordering decision.

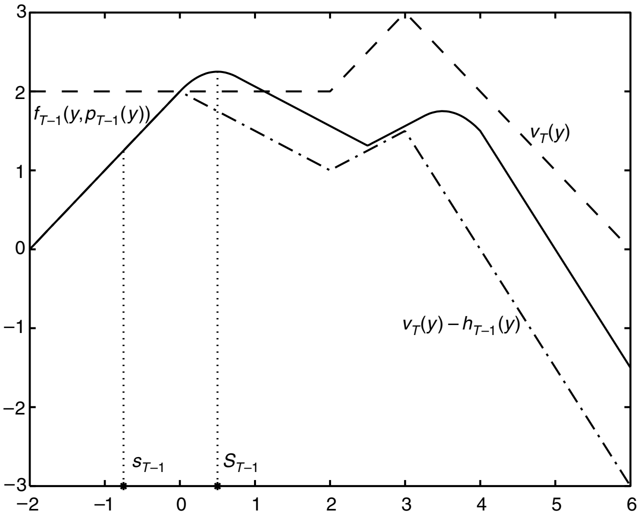
For instance, if

$$y=1,\quad y-1+p=p\in[0,1],$$
$$f_{T-1}(1,p)=\tfrac{1}{2}p-p^2+2,\quad\text{and}\quad p_{T-1}(1)=\tfrac{1}{4},$$

while when

$$y=3,\quad y-1+p=2+p\in[2,3],$$
$$f_{T-1}(3,p)=\tfrac{3}{2}p-p^2+1,\quad\text{and}\quad p_{T-1}(3)=\tfrac{3}{4}.$$

Figure 1. $v_T(y)$, $f_{T-1}(y, p_{T-1}(y))$, and $v_T(y) - h_{T-1}(y)$.



Appendix B

Proof of Lemma 3. Consider an instance with stationary input data for the last two periods of problem (2): for $t = T, T-1$,

$$c_t = 0, \quad \beta_t = 0, \quad \underline{d}_t = 0, \quad \bar{d}_t = b,$$
$$D_t(p) = b - ap, \quad R_t(d) = d(b-d)/a,$$
$$h_t(x) = h_+ \max\{0, x\} + h_- \max\{0, -x\}$$

and

$$\alpha_t = \begin{cases} \underline{\alpha} & \text{with probability } q, \\ \bar{\alpha} & \text{with probability } 1-q, \end{cases}$$

where $h_+ \gg h_- > 0$ are fixed, $\bar{\alpha} > 1 > \underline{\alpha} > 0$, and $q\underline{\alpha} + (1-q)\bar{\alpha} = 1$. We will choose $b \gg h_+$ and $a \gg b^2$.

For period T: Given $h_+ \gg h_- > 0$, choose $b \gg h_+$ and $a \gg b^2$. In this case, it is optimal to choose a feasible d such that $y - \underline{\alpha} d$ is as close to 0 as possible. Therefore,

$$d_T(y) = \begin{cases} 0 & \text{for } y \leqslant 0, \\ y/\underline{\alpha} & \text{for } 0 \leqslant y \leqslant \underline{\alpha} b, \\ b & \text{for } y \geqslant \underline{\alpha} b, \end{cases}$$

and

$$g_T(y, d_T(y)) = \begin{cases} h\_y & \text{for } y \leqslant 0, \\ y/\underline{\alpha}(b - y/\underline{\alpha})/a + h\_(y - y/\underline{\alpha}) & \\ \qquad \text{for } 0 \leqslant y \leqslant \underline{\alpha} b, \\ -qh_+(y - \underline{\alpha} b) + (1-q)h\_(y - \bar{\alpha} b) & \\ \qquad \text{for } \underline{\alpha} b \leqslant y \leqslant \bar{\alpha} b, \\ -h_+(y-b) & \text{for } y \geqslant \bar{\alpha} b. \end{cases}$$

Under these assumptions, we have that $S_T = 0$, $g_T(0,0) = 0$, and

$$v_T(x) = \begin{cases} -k & \text{for } y \leqslant -k/h\_, \\ g_T(y, d_T(y)) & \text{for } y \geqslant -k/h\_. \end{cases}$$

For period T − 1: We have that

$$-h_{T-1}(x) + v_T(x)$$
$$= \begin{cases} -k + h\_x & \text{for } x \leqslant -k/h\_, \\ 2h\_x & \text{for } -k/h\_ \leqslant x \leqslant 0, \\ x/\underline{\alpha}(b - x/\underline{\alpha})/a + h\_(x - x/\underline{\alpha}) - h_+ x & \\ \qquad \text{for } 0 \leqslant x \leqslant \underline{\alpha} b, \\ -qh_+(x - \underline{\alpha} b) + (1-q)h\_(x - \bar{\alpha} b) - h_+ x & \\ \qquad \text{for } \underline{\alpha} b \leqslant x \leqslant \bar{\alpha} b, \\ -2h_+ x + h_+ b & \text{for } x \geqslant \bar{\alpha} b. \end{cases}$$

Because $b \gg h_+ \gg h_- > 0$ and $a \gg b^2$, it is optimal to choose a feasible d such that $y - \underline{\alpha} d$ is as close to 0 as possible. Therefore,

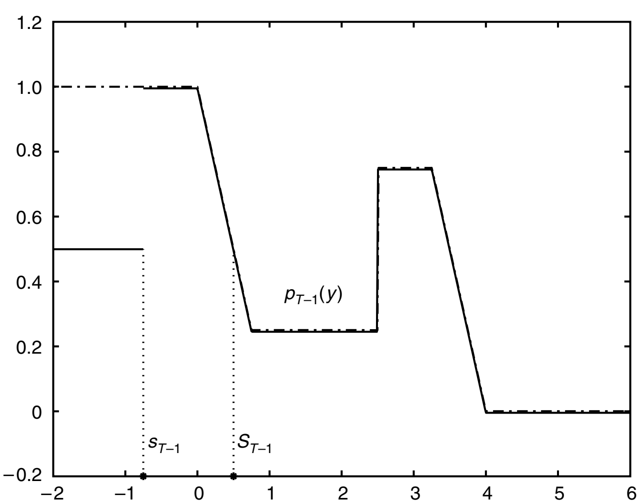
$$d_{T-1}(y) = d_T(y)$$

and

$$g_{T-1}(y, d_{T-1}(y))$$
$$= \begin{cases} -k + h\_y & \text{for } y \leqslant -k/h\_, \\ 2h\_y & \text{for } -k/h\_ \leqslant y \leqslant 0, \\ y/\underline{\alpha}(b - y/\underline{\alpha})/a + 2h\_(y - y/\underline{\alpha}) & \\ \qquad \text{for } 0 \leqslant y \leqslant \underline{\alpha} k/(h\_(\bar{\alpha} - \underline{\alpha})), \\ y/\underline{\alpha}(b - y/\underline{\alpha})/a + h\_(y - y/\underline{\alpha}) - (1-q)k & \\ \qquad \text{for } \underline{\alpha} k/(h\_(\bar{\alpha} - \underline{\alpha})) \leqslant y \leqslant \underline{\alpha} b, \\ \cdots & \text{for } y \geqslant \underline{\alpha} b. \end{cases}$$

Observe that $g_{T-1}(y, d_{T-1}(y))$ is decreasing for $y \geqslant 0$ and directionally differentiable for any y. For $y \geqslant \underline{\alpha} b$, there

Figure 2. $p_{T-1}(y)$.



exists some constant $\tau > 0$ such that $g'_{T-1}(y, d_{T-1}(y)) \leqslant -\tau h_+$ and hence it is much smaller than that for $y < \underline{\alpha} b$, because $h_+ \gg h_- > 0$.

Denote by

$$\underline{y} = \underline{\alpha} k/(h_-(\bar{\alpha} - \underline{\alpha})) = \lambda \underline{\alpha} b$$

for some $\lambda \in [0, 1]$. For $\underline{y} \leqslant y \leqslant \underline{\alpha} b$, we have that

$$g_{T-1}(y, d_{T-1}(y)) = y/\underline{\alpha}(b - y/\underline{\alpha})/a + h_-(y - y/\underline{\alpha}) - (1-q)k.$$

It remains to show that $g_{T-1}(y, d_{T-1}(y))$ is not k-concave. Observe that for $y = 0$, $d_{T-1}(y) = 0$ and $g_{T-1}(0, 0) = 0$. If $g_{T-1}(y, d_{T-1}(y))$ is k-concave, then for $x_0 = 0$, $x_1 = \underline{\alpha} b$, we have from the definition of k-concavity that

$$\underline{y}/\underline{\alpha}(b - \underline{y}/\underline{\alpha})/a + h_-(\underline{y} - \underline{y}/\underline{\alpha}) - (1-q)k \geqslant \lambda(h_- b(\underline{\alpha} - 1) - (1-q)k) - \lambda k,$$

which implies that

$$\underline{y}/\underline{\alpha}(b - \underline{y}/\underline{\alpha})/a - (1-q)k \geqslant -\lambda(2-q)k.$$

However, if we increase a, b and keep b^2/a very small, the above inequality does not hold because $\lambda \to 0+$, which is a contradiction. Hence, $g_{T-1}(y, d_{T-1}(y))$ is not k-concave. Furthermore, under the above assumptions, one can see that $S_{T-1} = 0$ and $g_{T-1}(0, 0) = 0$. Therefore, $v_{T-1}(x)$ is not k-concave, because $v_{T-1}(x) = g_{T-1}(x, d_{T-1}(x))$ for $x \geqslant 0$. □

Appendix C

Proof of Lemma 4. We extend the example of Appendix B by investigating time period $T-2$. Note that

$$v_{T-1}(x) = \begin{cases} -k & \text{for } y \leqslant -k/(2h_-), \\ g_{T-1}(y, d_{T-1}(y)) & \text{for } y \geqslant -k/(2h_-). \end{cases}$$

For period $T-2$: Let

$$c_{T-2} = \beta_{T-2} = 0, \quad \underline{d}_{T-2} = 0, \quad \bar{d}_{T-2} = b',$$
$$D_{T-2}(p) = b' - a'p, \quad R_{T-2}(d) = d(b'-d)/a',$$
$$h_{T-2}(x) = \rho \max\{0, -x\} + \epsilon \max\{0, x\}$$

and

$$\alpha_{T-2} = \begin{cases} \underline{\alpha} & \text{with probability } q, \\ \bar{\alpha} & \text{with probability } 1-q, \end{cases}$$

where we choose $1 < \bar{\alpha} < 2$ and recall that $q\underline{\alpha} + (1-q)\bar{\alpha} = 1$.

To show that an (s, S, p) policy is not necessarily optimal for this period, we let

$$a = n^7, \quad b = n^3, \quad h_+ = n^2, \quad h_- = 1, \quad \rho = n^3,$$
$$a' = n, \quad \epsilon = n^{-1}, \quad \delta = n^{-1}$$

for some scalar $n > 1$. Also, let

$$b' = (\gamma + \epsilon')a' + 2\sqrt{(1 + q(1-q) - \delta)ka'}, \tag{9}$$

where $\gamma = q(\bar{\alpha} - \underline{\alpha})h_-(1/\underline{\alpha} - 1)$ and $\epsilon' = q(\bar{\alpha} - \underline{\alpha})\epsilon$. One can see that $b' = O(n)$.

To simplify notation, we omit the term $y/\underline{\alpha}(b - y/\underline{\alpha})/a$ in $g_{T-1}(y, d_{T-1}(y))$ for $0 \leqslant y \leqslant \underline{\alpha} b$. This is possible, because $b^2 \ll a$ implying that $y/\underline{\alpha}(b - y/\underline{\alpha})/a \to 0_+$ as $n \to \infty$ and thus does not impact the argument below.

Assuming $n \to \infty$, it is optimal to choose a feasible d such that $y - \bar{\alpha} d \geqslant 0$ for $y \geqslant 0$ because $\rho \gg a', b'$ as n tends to infinity. Therefore, we have that $d \leqslant y/\bar{\alpha}$. Because $1 < \bar{\alpha} < 2$ we have that $q(\bar{\alpha} - \underline{\alpha}) = \bar{\alpha} - 1 < 1$ and, hence, $\gamma < h_-(1/\underline{\alpha} - 1)$. Thus, one can prove that

$$d_{T-2}(y) = \begin{cases} y/\bar{\alpha} & \text{for } 0 \leqslant y \leqslant \bar{\alpha} b', \\ b' & \text{for } y \geqslant \bar{\alpha} b' \end{cases}$$

by some simple calculation. This is true because for $y \geqslant 0$, $\partial g_{T-2}(y, d)/\partial d = 0$ implies that $d \geqslant b'$. We only prove the case with $\underline{\alpha} b \geqslant y - \underline{\alpha} d \geqslant y - \bar{\alpha} d \geqslant \underline{\alpha} k/(h_-(\bar{\alpha} - \underline{\alpha}))$. (The other cases can be proven by following a similar argument.) In this case, we have

$$g_{T-2}(y, d) = d(b'-d)/a' - \epsilon(y-d) + h_-(1 - 1/\underline{\alpha})(y-d) - (1-q)k.$$

Thus, $\partial g_{T-2}(y, d)/\partial d = 0$ implies that $d = (b' + a'(h_-(1/\underline{\alpha} - 1) + \epsilon))/2 \geqslant b'$ for sufficiently large n, because $\gamma < h_-(1/\underline{\alpha} - 1)$. Hence, $d_{T-2}(y) = \min(y/\bar{\alpha}, b')$.

Denote by

$$y^* = (\bar{\alpha}(b' - (\gamma + \epsilon')a'))/2 = 2\sqrt{(1 + q(1-q) - \delta)ka'}$$

and

$$\hat{y} = (\bar{\alpha}(b' - (2\gamma + \epsilon')a'))/2.$$

It is clear that for n sufficiently large,

$$(1 - \underline{\alpha}/\bar{\alpha})\,\underline{y} \leqslant y^* \leqslant \bar{\alpha} b' \tag{10}$$

and

$$\hat{y} < 0. \tag{11}$$

In the following, we prove that y^* is the global optimal solution of $g_{T-2}(y, d_{T-2}(y))$ by distinguishing between four cases depending on the value of y.

(a) If $y \geqslant \underline{\alpha} b'$, it is easy to check that $g_{T-2}(\bar{\alpha} b', b') \geqslant g_{T-2}(y, b')$ because $-h_{T-2}(x) + v_{T-1}(x)$ is nonincreasing for $x \geqslant 0$.

(b) If

$$\underline{y}/(1-\underline{\alpha}/\bar{\alpha}) \leqslant y \leqslant \bar{\alpha} b', \tag{12}$$

then

$$g_{T-2}(y, d_{T-2}(y)) = y/\bar{\alpha}(b' - y/\bar{\alpha})/a' - (\gamma + \epsilon')y/\bar{\alpha} - q(1-q)k.$$

One can see from the first-order optimality condition that y^* maximizes $g_{T-2}(y, d_{T-2}(y))$ for y satisfying (12) and

$$g_{T-2}(y^*, d_{T-2}(y^*)) = \frac{(b' - (\gamma+\epsilon')a')^2}{4a'} - q(1-q)k = (1-\delta)k. \tag{13}$$

(c) For

$$0 \leqslant y \leqslant \underline{y}/(1-\underline{\alpha}/\bar{\alpha}), \tag{14}$$

we have that

$$g_{T-2}(y, d_{T-2}(y)) = y/\bar{\alpha}(b' - y/\bar{\alpha})/a' - (2\gamma + \epsilon')y/\bar{\alpha}.$$

The first-order optimality condition implies that $y = 0$ maximizes $g_{T-2}(y, d_{T-2}(y))$ for y satisfying (14) because $g'_{T-2}(\hat{y}, d_{T-2}(\hat{y})) = 0$ and $\hat{y} < 0$.

(d) If $y \leqslant 0$, then it is clear that $g_{T-2}(y, d_{T-2}(y)) \leqslant 0$ because $\rho \gg a', b'$ as $n \to \infty$.

Thus, (a)–(d) imply that y^* is the global maximizer of $g_{T-2}(y, d_{T-2}(y))$.

Finally, we have that for sufficiently large n,

$$g_{T-2}(\underline{y}, d_{T-2}(\underline{y})) = -\underline{y}^2/\bar{\alpha}^2 a' - \gamma \underline{y}/\bar{\alpha} < -\delta k.$$

Thus, from (13), it is optimal to order up to y^* when the inventory level is $\underline{y}$ and not to order when the inventory level is $y = 0$ because $g_{T-2}(0, d_{T-2}(0)) = g_{T-2}(0, 0) = 0$. This implies that

$$s_{T-2} < 0 < \underline{y} < S_{T-2} = y^*$$

and, hence, any (s, S) inventory policy is not optimal in this case.

Therefore, the example shows that (s, S, p) policies are not necessarily optimal. $\square$

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