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ABSTRACT

Given a tree $G = (V, E)$ and a weight function defined on subsets of its nodes, we consider two associated problems. The first, called the "rooted subtree problem", is to find a maximum weight subtree, with a specified root, from a given set of subtrees.

The second problem, called "the subtree packing problem", is to find a maximum weight packing of node disjoint subtrees chosen from a given set of subtrees, where the value of each subtree may depend on its root.

We show that the complexity status of both problems is related, and that the subtree packing problem is polynomial if and only if each rooted subtree problem is polynomial. In addition we show that the convex hulls of the feasible solutions to both problems are related: the convex hull of solutions to the packing problem is given by "pasting together" the convex hulls of the rooted subtree problems.

We examine in detail the case where the set of feasible subtrees rooted at node i consists of all subtrees with at most k nodes. For this case we derive valid inequalities, and specify the convex hull when $k \leq 4$.

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1 Introduction

Given a tree $G = (V, E)$ with $n = |V|$ nodes, we consider two closely related problems. The first, called the "constrained i-rooted subtree problem", is to find a maximum weight i-rooted subtree (i.e a tree containing node i): max $\{c^i(z^i) : z^i \in X^i\}$ defined over a set of incidence vectors X^i of feasible i-rooted subtrees (z^i is the incidence vector of the nodes of the chosen subtree). We shall also be interested in the case arising in applications with auxiliary variables w^i when the value function is given by $c^i(z^i) = d^i z^i +$ max$\{e^i w^i : (z^i, w^i) \in W^i\}$.

The second problem, called "the (constrained) subtree packing problem", involves a function c^i and a feasible set X^i for each root node $i \in V$. The problem is to find a maximum weight packing of node disjoint subtrees: max $\{\sum_{i \in V} c^i(z^i) : (z^1, \dots, z^n) \in Z\}$ defined over the set $Z = \{(z^1, \dots, z^n) : \sum_{i \in V} z^i \leq 1, z^i \in X^i$ for $i \in V\}$. To correctly model the fact that we do not need to choose any subtree rooted at node i, we assume that $0 \in X^i$ and that $c^i(0) = 0$. In treating the special case when each X^i consists of all i-rooted subtrees, Barany et al(1986) describe conv(Z). In a telecommuncations study of capacity expansion for a local area network, Balakrishnan et al.(1991) formulate a version of this problem as a constrained subtree packing problem and obtain practical results by tightening the formulation and using Lagrangian relaxation. Aghezzaf and Wolsey(1990) have further examined one particular aspect of their model, the question of how to correctly model a piecewise linear concave objective function.

In this paper we clarify and generalize these earlier studies by showing that the polyhedral characterizations of the rooted subtree and subtree packing problems are closely related. In particular we show in Section 2 that conv$(Z) = \{(z^1, \dots, z^n) : \sum_{i \in V} z^i \leq 1, z^i \in$ conv (X^i) for $i \in V\}$. More generally we show that if $c^i(z^i)$ is derived as above by optimizing over W^i, the linear program:

$$max\{\sum_{i \in V}(d^i z^i + e^i w^i) : \sum_{i \in V} z^i \leq 1, (z^i, w^i) \in conv(W^i) \text{ for } i \in V\}$$

solves the tree packing problem.

In Sections 3 and 4 we study a particular set of i-rooted subtrees, namely those in which X^i represents the set of subtrees containing at most k nodes, or more generally of subtrees constrained by a knapsack constraint over their nodes. This model is also motivated by the study of Balakrishnan at al. Johnson and Niemy(1983) and Lukes(1974) have described efficient algorithms for this problem, and Boyd(1990) has derived some valid inequalities for a slightly more general model. We study the polyhedral structure in the case of a cardinality constraint. In Section 3 we derive three families of valid inequalities, and in Section 4 we derive a complete characterization of conv(X^i) when $k \leq 4$. We also examine the close relationship between the polyhedra of cardinality and knapsack constrained subtrees.

2 Relationship between the two Problems.

To examine the complexity of the rooted subtree problem and the subtree packing problem, we first consider the question of how to describe conv(Z) when $Z = \{(z^1, \ldots, z^n) : \sum_{i \in V} z^i \leq 1, z^i \in X^i \text{ for } i \in V\}$. A description can be obtained based on a result linking subtrees of trees and chordal graphs. In polyhedral terms, this result takes the following form:

Theorem 1 : *(Golumbic(1980)) Given a family of subtrees of a tree, if A is the corresponding node-subtree incidence matrix, then the polyhedron $\{x : Ax \leq 1, x \geq 0\}$ is integral.*

Theorem 2 : *conv$(Z) = \{(z^1, \ldots, z^n) : \sum_{i \in V} z^i \leq 1, z^i \in$ conv (X^i) for $i \in V.\}$*

Proof. If A^i is the $0-1$ matrix whose columns are the points in X^i, the polyhedron $\{(x^1, \ldots, x^n) : \sum_{i \in V} A^i x^i \leq 1, x^i \geq 0 \text{ for } i \in V\}$ is integral by Theorem 1, and since each subtree rooted at i contains node i, we can write this set as: $\{(x^1, \ldots, x^n) : \sum_{i \in V} A^i x^i \leq 1, 1x^i \leq 1, x^i \geq 0 \text{ for } i \in V\}$. Now let $z^i = A^i x^i$. The set $Q = \{(z^1, \ldots, z^n) : \sum_{i \in V} A^i x^i \leq 1, z^i = A^i x^i, 1x^i \leq 1, x^i \geq 0 \text{ for } i \in V\}$ is integral since z^i is integral when x^i is integral. But conv$(X^i) = \{z^i : z^i = A^i x^i, 1x^i \leq 1, x^i \geq 0\}$

by definition. Thus $Q = \{(z^1, \ldots, z^n) : \sum_{i \in V} z^i \leq 1, z^i \in \text{conv}\ (X^i)$ for $i \in V\}$ is integral. ∎

This theorem shows that we can use any convex hull representation of the sets X^i to describe the convex hull of the tree packing polyhedron Z; we are not restricted to using the subtree-tree incidence matrix representation. Some special cases obtained earlier in the literature can now be derived very simply. The following result for the rooted subtree problem when all subtrees are feasible is easy to verify.

Proposition 3 : *(Groeflin et al.(1982)) If X is the set of incidence vectors of all subtrees rooted at node r, $conv(X) = \{z \in R^n_+ : z_r \leq 1, z_j \leq z_{p(j,r)}$ for $j \in V \setminus \{r\}\}$, where $p(j,r)$ is the predecessor of j on the path from j to the root r.*

As a corollary of this result and Theorem 2, we have:

Proposition 4 : *(Barany et al. (1986)) When, for all $i \in V$, X^i is the set of incidence vectors of all i-rooted subtrees , $conv(Z) = \{(z^1, \ldots, z^n) \in R^{n^2}_+, \sum_{i \in V} z^i \leq \mathbf{1}, z^i_i \leq 1, z^i_j \leq z^i_{p(j,i)}$ for $i, j \in V.\}$*

More generally, suppose that the objective function $c^i(z^i)$ is nonlinear, and the value of an i-rooted subtree is given by:

$$c^i(z^i) = d^i z^i + max_{w^i}\{e^i w^i : (z^i, w^i) \in W^i\} \tag{1}$$

with $proj_{z^i}(W^i) = X^i$ a set of subtrees rooted at i. If T^i is a rooted subtree of G with incidence vector z^i , we shall write $c^i(T^i) = c^i(z^i)$. The formulation (1) permits us to consider problem situations in which our underlying model contains variables w^i in addition to the node variables z^i. In the telecommunications model considered by Balakrishnan et al., these variables represent investment in added arc (cable) capacity.

Theorem 5 : *If $W = \{(z^1, w^1, \ldots, z^n, w^n) : \sum_{i \in V} z^i \leq 1, (z^i, w^i) \in W^i$ for $i \in V\}$, then $conv(W) = \{(z^1, w^1, \ldots, z^n, w^n) : \sum_{i \in V} z^i \leq 1, (z^i, w^i) \in conv(W^i)$ for $i \in V\}$.*

Proof. Consider an arbitrary partial ordering $\preceq$ of the underlying tree graph $G = (V, E)$, defined by choosing a root node r and setting $u \preceq v$ if and only if u lies on the path from v to the root. So $r \preceq v$ for all $v \in V$. Given the ordering, let $R(v) = \{u : v \preceq u\}$ denote the subtree induced by v and its successors, $S(v)$ denote the set of immediate successors of v, $S(T)$ be the immediate successors of subtree T and $r(T)$ denote the root of subtree T with respect to the ordering.

We let $H(v)$ denote the optimal value of the packing problem entirely restricted to subtrees in $R(v)$: $H(v) = max\{\sum_{i\in V}(d^i z^i + e^i w^i) : \sum_{i\in V} z^i \leq 1, (z^i, w^i) \in W^i$ for $i \in R(v), z^i_j = 0$ if $i \notin R(v)$ or $j \notin R(v)\}$. Thus the optimal value of the complete packing problem is $H(r)$.

Now we consider the linear program:

$$\zeta_{LP} = max\{\sum_{i\in V}(d^i z^i + e^i w^i) : \sum_{i\in V} z^i \leq 1, (z^i, w^i) \in conv(W^i) \text{ for } i \in V\}. \quad (LP)$$

Dualizing the packing constraints with multipliers π, we obtain the upper bound

$$\nu(\pi) = \sum_{i\in V} \nu^i(\pi) + \sum_{j\in V} \pi_j$$

where $\nu^i(\pi) = max\{(d^i - \pi)z^i + e^i w^i : (z^i, w^i) \in conv(W^i)\}$. Now let T^i be the optimal i-rooted subtree for the i^{th} subproblem, so $\nu^i(\pi) = c^i(T^i) - \sum_{j\in T^i} \pi_j$, and let r^i be the root of T^i with respect to the partial ordering.

Since H is the optimal value function

$$H(r^i) \geq c^i(T^i) + \sum_{j\in S(T^i)} H(j).$$

Taking $\pi_j = H(j) - \sum_{k\in S(j)} H(k)$, we see that

$$\nu^i(\pi) = c^i(T^i) - \sum_{j\in T^i} \pi_j = c^i(T^i) - (H(r^i) - \sum_{j\in S(T^i)} H(j)) \leq 0.$$

Note that $\sum_{j\in V} \pi_j = H(r)$, as the term $H(v)$ appears twice in the sum with opposite signs for all nodes v other than the root r. Combining these observations, we see that $\zeta_{LP} \leq \nu(\pi) = \sum_{i\in V} \nu^i(\pi) + \sum_{j\in V} \pi_j \leq 0 + H(r)$. Thus the linear program has an upper bound of $H(r)$, which is the optimal value of the subtree packing problem. Since the linear program is a relaxation of this problem, its value is also an upper bound on $H(r)$, and consequently the linear program solves the subtree packing problem. The claim follows. ∎

It follows from Theorems 2 and 5, respectively, that the separation problems for conv(Z) and conv(W) are polynomial if and only if the separation problems for conv(X^i) and conv(W^i) are polynomial for each $i \in V$.

Theorem 6 : *The linear optimization problems over Z and W, respectively, are polynomial if and only if the linear optimization problems over each X^i and W^i are polynomial.*

Algorithmically this means that a cutting plane algorithm to solve the tree packing problem consists just of a cutting plane algorithm for each rooted subtree problem.

Theorem 5 and its proof suggest a natural column generation algorithm for solving the packing problem, which by Theorem 5 is equivalent to solving the linear program (LP). The subproblems in this algorithmic approach are again:

$$\nu^i(\pi) = max\{(d^i - \pi)z^i + e^i w^i) : (z^i, w^i) \in W^i\}$$

and the restricted master program consists of a $0-1$ packing problem: max $\{cx : Ax \leq 1, x \geq 0\}$ whose columns are the incidence vectors of subtrees. By Theorem 1 this linear program has an integer optimal solution with each $x_j = 0$ or 1. Therefore its solution is a set of disjoint subtrees. We can solve it using a dynamic programming recursion

$$H(v) = max\{max_{T^v}\{c(T^v) + \sum_{j \in S(T^v)} H(v)\}, \sum_{j \in S(v)} H(j)\}$$

with T^v restricted to subtrees that have root v in the ordering $(V, \preceq)$ and that correspond to a column of A. After using this dynamic program to solve for $H(r)$, the algorithm sends the dual variables $\pi_j = H(j) - \sum_{k \in S(j)} H(k)$ to the subproblems, which in turn return subtrees T^i for each $i \in V$ if $\nu^i(\pi) > 0$. To resolve the Master problem, we then update the $H(v)$ values using the same recursion.

3 Valid Inequalities for the Cardinality Constrained Problem

Given a rooted directed tree $G = (V, A)$ with root $1 \in V$, node weights $c \in R^{|V|}$, and a nonnegative integer k, the cardinality constrained subproblem can be formulated as the following integer program (I_k):

$$max \sum_{j \in V} c_j x_j \tag{2}$$

$$x_{p(j)} \geq x_j \text{ for } j \in V \tag{3}$$

$$\sum_{j \in V} x_j \leq k \tag{4}$$

$$x_j \in \{0, 1\} \text{ for } j \in V. \tag{5}$$

In this formulation $p(j)$ denotes the predecessor of j, and by convention $x_0 = x_{p(1)} = 1$. We refer to any subtree F of G as *rooted* if F contains the root. For any $u, v \in V$, we let $Path(u, v)$ denote the path in G connecting nodes u and v.

Observe that there is a simple polynomial dynamic programming algorithm to solve problem (I_k), which is a slight variant of the recursion we described in the last section. See also Johnson and Niemi(1983) and Lukes(1974). Let (X_k^{ST}) denote the set of feasible solutions of (I_k). In this section we describe several families of valid inequalities for (X_k^{ST}). First we assume that no node is at a distance k or more from the root, and thus conv(X_k^{ST}) is full-dimensional. The following observations concerning the inequalities defining (X_k^{ST}) are easy to verify.

The inequality (4) is dominated by $\sum_{j \in V} x_j \leq k x_1$.

The inequality $x_j \geq 0$ is dominated unless j is a leaf, and the inequality $x_j \leq 1$ is dominated unless $j = 1$.

Tree Cover Inequalities

Definition 1 : A set $C \subseteq V$ is a *tree cover* if the subgraph induced by C is a subtree rooted at 1, and $|C| = k + 1$.

Proposition 7 : *The tree cover inequality*

$$\sum_{j \in C} (x_{p(j)} - x_j) \geq 1 \tag{6}$$

is valid for X_k^{ST}.

Proof. If $\sum_{j \in C}(x_{p(j)} - x_j) = 0$, then the fact that $x_{p(1)} = 1$ implies that $x_j = 1$ for all $j \in C$. As $|C| = k + 1$, this point is infeasible. Therefore all feasible points satisfy the inequality. ∎

A useful alternative way to write the inequality is:

$$\sum_{j \in C}(1 - |S_C(j)|)x_j \leq 0 \tag{7}$$

where $S_C(j)$ is the set of successors of node j in C.

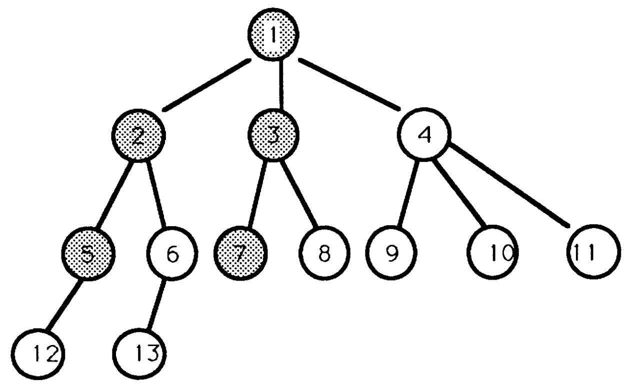
The next question we consider is when the cover inequality (6) defines a facet of conv(X_k^{ST}). We need some notation. For $j \in V \setminus C$, $d_C(j)$ is the distance of j from C as measured by the number of arcs in the path connecting node j to C.

Proposition 8 *: The tree cover inequality (6) defines a facet of conv(X_k^{ST}) if and only if, for all $v \in V \setminus C$, some subtree of C hanging off the path from node v to the root contains at least $d_C(v) + 1$ nodes.*

Proof. First suppose that there is no such subtree for some $v \in V \setminus C$. If the inequality defines a facet, there exists a tight point with $x_v = 1$. Whenever $\sum_{j \in C}(x_{p(j)} - x_j) = 1$, exactly one node $t \in C$ satisfies the condition $x_{p(t)} = 1$ and $x_t = 0$. Since $x_v = 1, x_j = 1$ for every node on $Path(1, v)$. This path includes at least $d_C(v)$ nodes not in C. Node t necessarily lies in some subtree of C hanging off $Path(1, v)$. Thus at most $d_C(j)$ nodes of C have $x_j = 0$. Such a point has cardinality at least $|C| + d_C(j) - d_C(j) \geq |C|$ and is thus infeasible. Thus all tight points satisfy $x_v = 0$, and the inequality does not define a facet.

Conversely suppose that for each $v \in V \setminus C$, there exists a subtree with $d_C(v) + 1$ or more nodes. It is easy to verify that the inequality defines a facet for the tree restricted to C. Sequential lifting (see Nemhauser and Wolsey(1988)) of each node $v \in V \setminus C$ then gives a lifted coefficient of 0, and proves the claim. ∎

Example 1. Consider the graph shown in Figure 1 with node 1 as the root and suppose $k = 4$.



Tree cover for k=4 .

Figure 1.

Taking $\{1,2,3,5,7\}$ as the tree cover, Proposition 7 shows that

$$x_1 \geq x_5 + x_7$$

is a valid inequality. From Proposition 8 the inequality does not define a facet, because, taking $j = 13$, no subtree of C off the path $\{1,2,5,13\}$ contains three or more nodes. However, the inequality defines a facet of $X_4^{ST} \cap \{x_j = 0 \text{ for } j \notin C\}$.

Now sequential lifting of the variables $x_{12}, x_6, x_{13}, x_8, x_4$ in that order shows that

$$x_1 \geq x_5 + x_7 + 0x_{12} + 0x_6 + 1x_{13} + 0x_8 + 0x_4$$

defines a facet of $X_4^{ST} \cap \{x_j = 0 \text{ for } j = 9, 10, 11\}$.

Next simultaneous lifting of x_9, x_{10}, x_{11} shows that

$$x_1 \geq x_5 + x_7 + x_{13} + \frac{1}{2}x_9 + \frac{1}{2}x_{10} + \frac{1}{2}x_{11}$$

defines a facet of X_4^{ST}.

Leaf Inequalities

The second class of inequalities we call *leaf inequalities.* Given a rooted subtree F with node set $V(F)$, we let $L(F)$ denote the leaves of F, and for $S \subseteq V$ we let $C(S)$, the *closure* of S, be the set of nodes on any path from $j \in S$ to the root.

Proposition 9 *: If, for some q, $|C(S)| = k+1$ for all $S \subseteq L(F)$ with $|S| = q$, then the inequality*

$$\sum_{u \in L(F)} x_u \le (q-1)x_1 \tag{8}$$

is valid for X_k^{ST}.

Proof. The condition shows that if the inequality is violated, the point is infeasible in X_k^{ST}. ∎

Proposition 10 *: The inequality (8) defines a facet of $X_k^{ST} \cap \{x : x_j = 0 \text{ for } j \in V \setminus V(F)\}$ if and only if*

$$\cap_{\{S \subseteq L(F):|S|=q-1\}} C(S) = \{1\}.$$

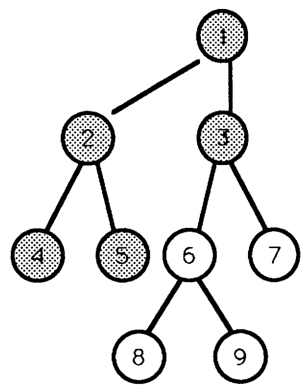
Proof. If $v \in \cap_{\{S \subseteq L(F):|S|=q-1\}} C(S) - \{1\}$, then all tight points satisfy $x_1 = x_v$, and the inequality is not facet-defining. Conversely suppose that$\cap_{\{S \subseteq L(F):|S|=q-1\}} C(S) = \{1\}$, and that all tight points satisfy $\sum_{j \in V(F)} \pi_j x_j \le \pi_0$ at equality. As $x = 0$ is tight, $\pi_0 = 0$. Consider $j \in V(F) \setminus (L(F) \cup \{1\})$. From the condition, there exists $S_j \subset L(F)$ with $|S_j| = q-1$ and $j \notin C(S_j)$. Take $i \in L(F) \setminus S_j$ with j on the path from 1 to i. As $|C(S_j \cup \{i\})| = k+1$, the points $C(S_j) \cup Path(1,j)$ and $C(S_j) \cup Path(1,p(j))$ are feasible and tight. Thus $\pi_j = 0$ for $j \in V(F) \setminus (L(F) \cup \{1\})$.

Now consider $i,j \in L(F)$. Choose $S^j \subset L(F) - \{j\}$ with $i \in S^j$ and $|S^j| = q-1$. As $C(S^j) \cup Path(1,j) \setminus \{j\} \subset C(S^j \cup \{j\})$ and $C(S_j) \cup Path(1,j) \setminus \{i\} \subset C(S^j \cup \{j\})$, both are tight feasible solutions and thus $\pi_j = \pi_i = \pi^*$ for all $i,j \in L(F)$. Finally considering any point $S \subseteq L(F)$ with $|S| = q-1$, we have that $\pi_1 + \pi^*(q-1) = 0$, and the claim follows. ∎

Example 2. Consider the graph shown in Figure 2 with k=4 and q=3. If F is induced by nodes $\{1,2,\ldots,5\}$, we see from Proposition 9 that the leaf inequality:

$$x_3 + x_4 + x_5 \le 2x_1$$

is valid. From Proposition 10 it does not define a facet for the tree induced by $V(F)$ as $C(45) \cap C(34) \cap C(35) = \{1,2\}$, and all tight points satisfy



L(F)={3,4,5} are the leaves of F.

Figure 2.

$x_1 = x_2$. Lifting in variables $6, 7, 8, 9$ in that order gives a facet-defining inequality for X_4^{ST}:

$$x_3 + x_4 + x_5 + x_6 \leq 2x_1$$

while the order $7, 6, 8, 9$ gives the facet-defining inequality

$$x_3 + x_4 + x_5 + x_7 + x_8 + x_9 \leq 2x_1.$$

Depth Inequalities

The third class of inequalities we call *depth inequalities.* Let F be a rooted subtree of G with $p \leq k$ nodes. Let $Q(F) = \{u \in V \setminus V(F) : |Path(1,u) \cup V(F)| = k+1\}$, and T^v denote the subtree rooted at v.

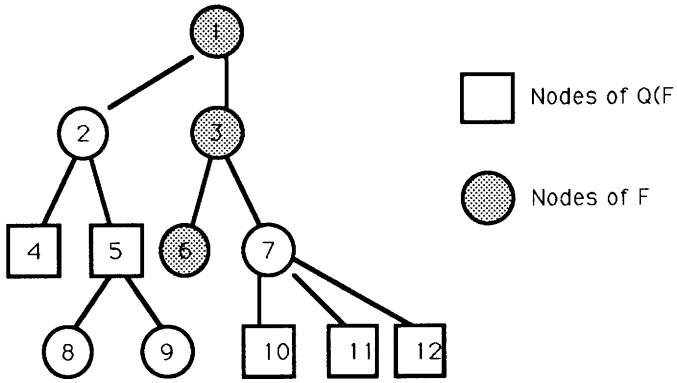
Proposition 11 *: The inequality*

$$\sum_{u \in V(F)\setminus\{1\}} x_u + \sum_{v \in Q(F)} \sum_{u \in V(T^v)} x_u \leq (p-1)x_1 \tag{9}$$

is valid for X_k^{ST}.

Proof. Let y be any incidence vector that violates the depth inequality (9), and for any set S of nodes, let $y(S) = \sum_{u \in S} y_u$. Let $V^Q = \{j \in \cup_{v \in Q(F)} V(T^v) : y_j = 1\}$ be the set of variables set to one in the second term of (9). Since y violates (9) and $y_1 = 1$,

$$y(V(F)) + y(V^Q) > p = |V(F)|$$



Depth inequality with k=4.

Figure 3.

and so

$$y(V(F)) + y(V^Q \setminus \{v\}) \geq |V(F)| \text{ for all } v \in Q(F).$$

Since $y(V(F)) \leq |V(F)|$, some $v \in Q(F)$ satisfies $y_v = 1$ and by definition of $Q(F)$

$$|\mathrm{path(1, v)} \setminus \mathrm{V(F)}| + |\mathrm{V(F)}| = \mathrm{k} + 1.$$

Note that the last two expressions, $y_v = 1$, and the fact that $\text{path}(1, v) \setminus V(F), V(F)$ and $V^Q \setminus \{v\}$ are disjoint implies that

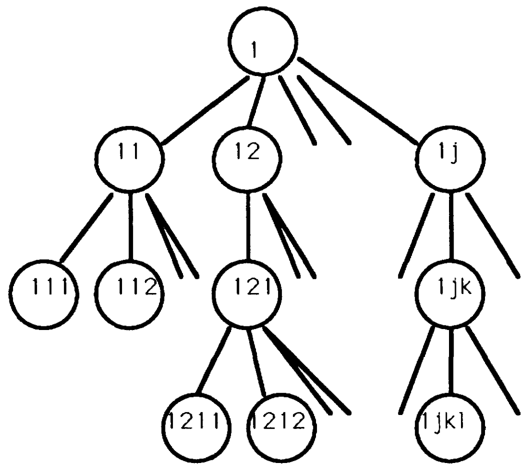
$$\begin{aligned} y(V) &\geq y(\mathrm{path(1,v)} \setminus \mathrm{V(F)}) + [\mathrm{y(V(F))} + \mathrm{y(V^Q} \setminus \{\mathrm{v}\}] \\ &\geq |\text{path}(1, v) \setminus V(F)| + |V(F)| = k + 1. \end{aligned}$$

and so the point y is infeasible. Therefore, the inequality (9) cuts off no feasible point, and so it is valid. ∎

Example 3. Consider the tree of Figure 3 and take $k = 4$. By Proposition 11, if $V(F) = \{1, 3, 6\}$ and $Q(F) = \{4, 5, 10, 11, 12\}$ we obtain the subtree inequality (9):

$$x_3 + x_6 + x_4 + x_5 + x_8 + x_9 + x_{10} + x_{11} + x_{12} \leq 2x_1.$$

In the next section we derive all facet-defining inequalities for small values of k. This analysis permits us to see the relative importance of the



Node numbering for small master polytopes.

Figure 4.

three families of valid inequalities. It turns out that for $k = 2$ and $k = 3$, all such inequalities are lifted tree cover inequalities, and for $k = 4$ only one inequality is not a lifted tree cover inequality. The latter inequality can be described either as a lifted leaf inequality or as a lifted depth inequality.

4 Small Master Polytopes

Here we derive a complete characterization of $\text{conv}(X_k^{ST})$ for $k = 2, 3$ and 4. For notational simplicity we number the nodes as shown in Figure 4.

Theorem 12 *: conv(X_2^{ST})is described by the inequalities:*

$$\begin{aligned} x_1 &\leq 1 \\ x_{1j} &\geq 0 \text{ for all } j \\ x_1 &\geq \sum_j x_{1j} \end{aligned}$$

Proof. The feasible integer points give X_2^{ST}, and the system is of the form $Ax \leq b, 0 \leq x \leq 1$ with A totally unimodular. ∎

For the cases $k = 3$ and 4, we consider the optimization problem: $z = max\{cx : x \in X_k^{ST}\}$ and let $M(c)$ denote the set of optimal solutions.

Theorem 13 : *$conv(X_3^{ST})$ is described by the inequalities:*

$$\begin{aligned}
x_1 &\leq 1 \\
x_{1j} &\geq \sum_k x_{1jk} \text{ for all } j \\
x_{1jk} &\geq 0 \text{ for all } j, k \\
x_1 &\geq x_{1j} + \sum_{j' \neq j} \sum_k x_{1j'k} \text{ for all } j \\
2x_1 &\geq \sum_j x_{1j} + \sum_j \sum_k x_{1jk}
\end{aligned}$$

Proof. We suppose $c \neq 0$. If $z > 0, M(c) \subseteq \{x : x_1 = 1\}$. If $z = 0$ (so $c_1 \leq 0$), we consider three cases. Namely we find the deepest node u in G such that $c_u < 0$, but all nodes v below u have $c_v \geq 0$.
Case 1. $u = 1jk$. As $c_{1jk} < 0, M(c) \subseteq \{x : x_{1jk} = 0\}$.
Case 2. $u = 1j$. As $c_{1j} < 0$ and $c_{1jk} \geq 0$ for all $k, M(c) \subseteq \{x : x_{1j} = \sum_k x_{1jk}\}$.
Case 3. $u = 1$. Here there are two subcases.

Case 3a: There exists a j such that $c_1 + c_{1j} = 0$. If $c_{1j'} > 0$ for any $j' \neq j, c_1 + c_{1j} + c_{1j'} > 0$ and so $z > 0$, a contradiction. But $c_{1j'} \geq 0$ by hypothesis, so $c_{1j'} = 0$ for all $j' \neq j$. Thus $c_1 < 0$ and $c_1 + c_{1j'} < 0$ for $j' \neq j$, and thus $M(c) \subseteq \{x : x_1 = x_{1j} + \sum_{j' \neq j} \sum_k x_{1j'k}\}$.

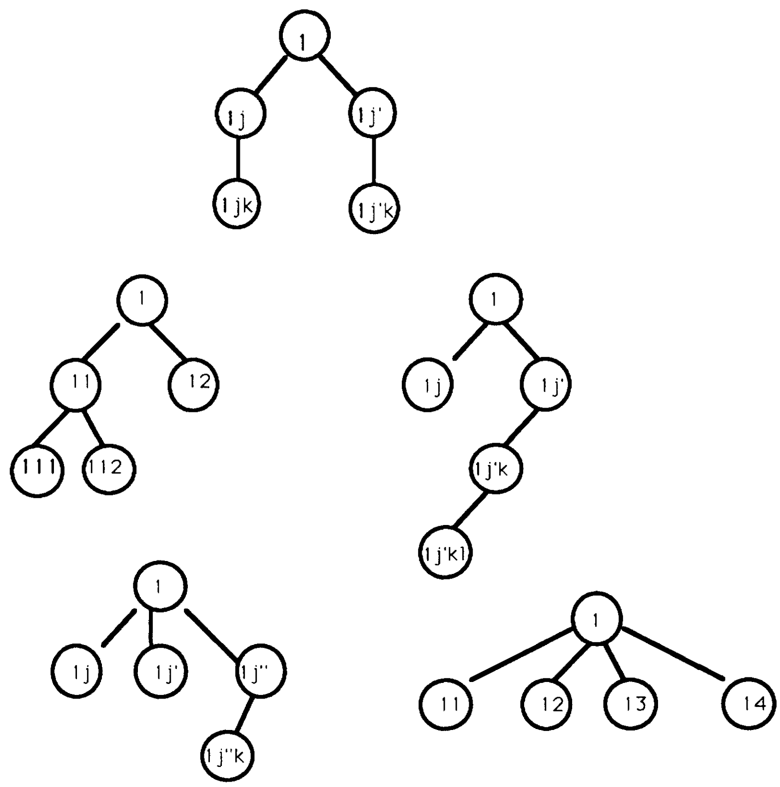
Case 3b: $c_1 + c_{1j} < 0$ for all j. Thus all optimal solutions have cardinality 0 or 3 and $M(c) \subseteq \{x : 2x_1 = \sum_j x_{1j} + \sum_j \sum_k x_{1jk}\}$.

We have shown that for any objective function $c \neq 0$ one of the proposed constraints is tight (i.e. satisfied at equality by all the optimal solutions). Thus the set of inequalities contains all the facet-defining inequalities. ∎

Theorem 14 : *$conv(X_4^{ST})$ is described by the inequalities:*

$$
\begin{aligned}
x_1 &\leq 1 \\
x_{1jkl} &\geq 0 \text{ for all } j,k,l \\
x_{1jk} &\geq \sum_l x_{1jkl} \text{ for all } j,k \\
x_{1j} &\geq x_{1jk} + \sum_{k'\neq k} \sum_l x_{1jk'l} \text{ for all } j,k \\
2x_{1j} &\geq \sum_k x_{1jk} + \sum_k \sum_l x_{1jkl} \text{ for all } j \\
2x_1 &\geq 2\sum_{j\in J_1} x_{1jk} + 2\sum_{j\in J_1} \sum_{k'\neq k} \sum_l x_{1jk'l} \\
&\quad + \sum_{j\notin J_1} \sum_k (x_{1jk} + \sum_l x_{1jkl}) \\
&\quad \text{for all } J_1 \neq \phi, \quad \text{all } k \\
x_1 + \sum_{j\in J_1} x_{1j} &\geq \sum_{j\in J_1} \sum_k x_{1jk} + x_{1j'} + \sum_{j\neq j'} \sum_k \sum_l x_{1jkl} \\
&\quad \text{for all } J_1 \neq \phi, j' \notin J_1 \\
x_1 &\geq x_{1j} + \sum_{j'\neq j} \sum_k \sum_l x_{1j'kl} \text{ for all } j \\
2x_1 &\geq x_{1j'} + x_{1j''} + \sum_k \sum_l (x_{1j'kl} + x_{1j''kl}) \\
&\quad d + \sum_{j\neq j',j''} \sum_k (x_{1jk} + \sum_l x_{1jkl}) \text{for all } j', j'', j' \neq j'' \\
3x_1 &\geq \sum_j x_{1j} + \sum_j \sum_k x_{1jk} + \sum_j \sum_k \sum_l x_{1jkl} \\
2x_1 &\geq x_{1j'} + x_{1j'k'} + \sum_{j\neq j'} \sum_k x_{1jk} + \sum_{j\neq j'} \sum_k \sum_l x_{1jkl} \\
&\quad + \sum_{k\neq k'} \sum_l x_{1j'kl} \quad \text{for all } j', k'
\end{aligned}
$$

Proof. All but the first and last inequalities in the above list are lifted tree cover inequalities. The corresponding tree covers for the last five such inequalities are shown in order in Figure 5. The last inequality can be viewed either as a lifted leaf or a lifted depth inequality. We leave it to the reader to check validity.



Five tree covers for the Master Polytope with k=4

Figure 5.

Case 1. If $z > 0$, then $M(c) \subseteq \{x : x_1 = 1\}$.

Otherwise $z = 0$. Now as before we choose u as low as possible in the tree such that $c_u < 0$ and $c_v \geq 0$ for all its descendants. By renumbering, we can assume that u is any node at its depth in the tree.

Case 2. $u = (1111)$. $M(c) \subseteq \{x : x_{1111} = 0\}$.

Case 3. $u = (111)$. $c_{111l} \geq 0$. As $c_{111} < 0, M(c) \subseteq \{x : x_{111} = \sum_l x_{111l}\}$.

$u = (11)$. We consider first the case when $c_1 = 0$, and then consider four cases with $c_1 < 0$.

Case 4. $c_{11} < 0, c_1 = 0$. By optimality (i.e., since $z = 0$), $c_{1j} \leq 0$ for all $j \neq 1$. Now consider an optimal solution with $x_{11} = 1$. By optimality $x_{11k} = 1$ for at least one k. Also if $x_{1j} = 1$ in such a solution for $j \neq 1$, necessarily $c_{1j} = 0$ and $\sum_{j \neq 1} \sum_k x_{1jk} = 0$. Thus every optimal solution with $x_{11} = 1$ is optimal in the subtree rooted at (11) for $k = 3$. If $x_{11} = 0$ in any optimal solution, then all of its successors have value zero as well. Therefore from case 3 of Proposition 13, $M(c) \subseteq \{x : x_{11} = x_{11k} + \sum_{k' \neq k} \sum_l x_{11k'l}\}$ for some k, or $M(c) \subseteq \{x : 2x_{11} = \sum_k x_{11k} + \sum_k \sum_l x_{11kl}\}$.

Case 5. $c_{11} < 0, c_1 < 0, c_1 + c_{11} + c_{111} = 0$. By optimality $c_{1j} \leq 0$ for every $j \neq 1$, so $c_1 + \sum_{j \in S} c_{1j} < 0$ for any set S. Let $J_1 = \{j : c_1 + c_{1j} + c_{1j1} = 0\} \neq \phi$, and $J_2 = \{j : c_1 + c_{1j} + c_{1jk} < 0 \text{ for all } k\}$. Again by optimality $c_{1jk} \leq 0$ for all $j \in J_1$ and $k \neq 1$, so $c_1 + c_{1j} + c_{1jk} < 0$ for all $j \in J_1, k \neq 1$. Thus $M(c) \subseteq \{x : 2x_1 = 2\sum_{j \in J_1}(x_{1j1} + \sum_{k \neq 1} \sum_l x_{1jkl}) + \sum_{j \in J_2} \sum_k (x_{1jk} + \sum_l x_{1jkl})\}$.

Since $c_{11} < 0$, note that if $x_{11} = 1$, then by optimality $x_{11j} = 1$ for some j. Case 5 includes all situations with $x_{11} = 1$ and the solution is of cardinality 3. We next consider situations in which $c_{11} < 0, c_1 < 0$ and the problem has at least *two* solutions of the form $(1, 11, 1j, 11k)$ and $(1, 11, 1j, 11k')$.

Case 6. $c_{11} < 0, c_1 < 0, c_1 + c_{11} + c_{11k} < 0$ for all k, $c_1 + c_{11} + c_{111} + c_{12} = 0$ and $c_1 + c_{11} + c_{112} + c_{12} = 0$. Let $J_1 = \{j : c_{1j} < 0\} \neq \phi$. We claim that $M(c) \subseteq \{x : x_1 + \sum_{j \in J_1} x_{1j} = x_{12} + \sum_{j \in J_1} \sum_k x_{1jk} + \sum_{j \neq 2} \sum_k \sum_l x_{1jkl}\}$.

First we observe that $\sum_{J_1} x_{1j} \leq 1$, as otherwise the solution can be improved by reducing some x_{1j} to zero. We next show that if $j \notin J_1 \cup \{2\}$, then $c_{1j} < c_{12}$. Suppose the contrary, so $c_{13} = c_{12}$ with $3 \notin J_1$ ($c_{13} > c_{12}$ is not possible, since otherwise $z > 0$). Since by hypothesis $\{1, 11, 112, 12\}$ is an optimal solution (with cost $z = 0$), the cost of the solution $\{1, 11, 111, 112\}$ must be no more than zero and so $c_{111} \leq c_{12}$. Since the value of the solution $\{1, 12, 13\}$ must be no more than zero as well, comparing this solution with the solution $\{1, 11, 111, 12\}$ implies that $c_{13} = c_{12} \leq c_{11} + c_{111}$. Together these inequalities give $c_{111} \leq c_{12} \leq c_{11} + c_{111}$, so $c_{11} \geq 0$, a contradiction.

a) Suppose $\sum_{J_1} x_{1j} = 1$ with $x_{1j'} = 1$.

i) If $x_{12} = 0, \{1, 1j', 1j'k\}$ cannot be optimal as $c_{12} > 0$. Also $\{1, 1j', 1j'k, 1j''\}$ cannot be optimal with $j'' \notin J_1 \cup \{2\}$ as $c_{1j''} < c_{12}$. Thus $\sum_{J_1} \sum_k (x_{1jk} + \sum_l x_{1jkl}) = 2$.

ii) If $x_{12} = 1$, necessarily $\sum_k x_{1j'k} = 1$ as $c_{1j'} < 0$, and therefore the solution is of the form $\{1, 1j', 12, 1j'k'\}$.

b) Suppose $\sum_{J_1} x_{1j} = 0$.

i) If $x_{12} = 1$, the equality in the definition of $M(c)$ holds.

ii) If $x_{12} = 0$, we need to show that $\sum_{j \notin J_1 \cup \{2\}} \sum_k \sum_l x_{1jkl} = 1$. The points $\{1, 1j, 1j'\}$ and $\{1, 1j, 1jk\}$ with $j, j' \notin J_1 \cup \{2\}$ cannot be optimal as $c_{12} > 0$. The points $\{1, 1j, 1j', 1j'k\}$ and $\{1, 1j, 1j', 1j''\}$ with $j, j', j'' \notin J_1 \cup \{2\}$ cannot be optimal as $c_{1j} < c_{12}$. Finally, consider points of the form $\{1, 1j, 1jk, 1jk'\}$ with $j \notin J_1 \cup \{2\}$. If this point is optimal, $c_{1jk} \geq c_{12}$ and $c_{1jk'} \geq c_{12}$. Also as $\{1, 1j, 1jk, 12\}$ defines a feasible subtree, $c_1 + c_{1j} + c_{1jk} + c_{12} \leq c_1 + c_{11} + c_{111} + c_{12}$, so $c_{1j} + c_{1jk} \leq c_{11} + c_{111}$. As $c_{1j} \geq 0$ and $c_{11} < 0$, this gives $c_{111} > c_{1jk} \geq c_{12}$. But now $c_1 + c_{11} + c_{111} + c_{112} > 0$, a contradiction as $\{1, 11, 111, 12\}$ is feasible and $z = 0$.

For the remaining cases with $c_1 < 0$ and $c_{11} < 0$, there is no optimal solution with $x_{11} = 1$ of cardinality 3, and there is at most one optimal solution with $x_{11} = 1$ of the form $\{1, 11, 11k, 1j\}$.

Case 7. $c_{11} < 0, c_1 < 0, c_1 + c_{11} + c_{11k} < 0$ for all k, $c_1 + c_{11} + c_{11k} + c_{1j} < 0$ for all j, k. There is no optimal solution with $x_{11} = 1$ of the form $\{1, 11, 111, 1j\}$. As $c_{11} < 0$, and there is no optimal solution with $x_{11} = 1$ of cardinality 3, $M(c) \subseteq \{x : 2x_{11} = \sum_k (x_{11k} + \sum_l x_{11kl})\}$.

Case 8. $c_{11} < 0, c_1 < 0, c_1+c_{11}+c_{11k} < 0$ for all k, $c_1+c_{11}+c_{111}+c_{12} = 0$ and $c_1 + c_{11} + c_{11j} + c_{12} < 0$ for all $j \neq 1$. There is exactly one optimal solution with $x_{11} = 1$ of the form $\{1, 11, 111, 12\}$. Then $c_{1j} \leq c_{12}$ for $j \neq 1, 2$ and $c_{11k} < c_{111}$ for $k \neq 1$.

If $x_{11} = 1$, then as $c_{11} < 0, \sum_k x_{11k} \geq 1$. If $x_{111} = 0$, the points $\{1, 11, 11k\}, \{1, 11, 11k, 1j\}$ and $\{1, 11, 11k, 11k'\}$ for $j \neq 2, k, k' \neq 1$ cannot be optimal as $c_{1j} \leq c_{12}$ and $c_{11k} < c_{111}$. Thus if $x_{11k} = 1$ for some $k \neq 1$ then $x_{11kl} = 1$ for some l, and therefore $M(c) \subseteq \{x : x_{11} = x_{111} + \sum_{k\neq 1} \sum_l x_{11kl}\}$.

Finally we have $u = \{1\}$, so $c_1 < 0$ and c_{1j}, c_{1jk} and $c_{1jkl} \geq 0$.

Case 9. $c_1 + c_{11} = 0$. Then $c_{1j} = 0$ for all $j \neq 1$ and and $c_{1jk} = 0$ all j, k. Thus $M(c) \subseteq \{x : x_1 = x_{11} + \sum_{j\neq 1} \sum_k \sum_l x_{1jkl}\}$.

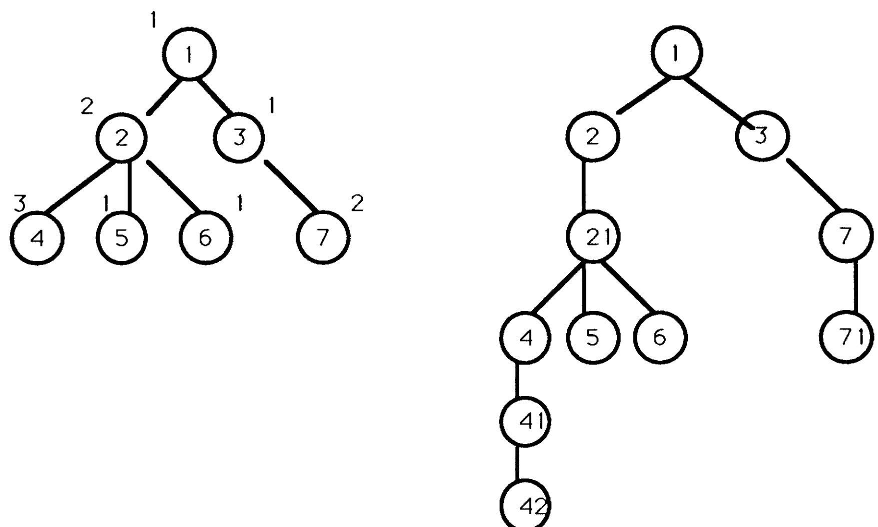
Case 10. $c_1+c_{11}+c_{12} = 0$ with $c_{11}, c_{12} > 0$. Then $c_{1j} = 0$ for $j \neq 1, 2$ and $c_{11k} = c_{12k} = 0$ for all k. Points such as $(1, 11, 11k), (1, 11, 13), (1, 11, 13, 14)$ and $(1, 13, 13k)$ are not optimal as either 12 can be added, or 13 can be replaced by 12. Thus $M(c) \subseteq \{x : 2x_1 = x_{11} + x_{12} + \sum_k \sum_l (x_{11kl} + x_{12kl}) + \sum_{j\neq 1,2} \sum_k x_{1jk} + \sum_{j\neq 1,2} \sum_k \sum_l x_{1jkl}\}$.

Case 11. $c_1 + c_{11} + c_{111} = 0$ with c_{11} and $c_{111} > 0$. Then $c_{1j} = 0$ for $j \neq 1, c_{11k} = 0$ for $k \neq 1$ and $c_{111l} = 0$ for all l. Now if a point is optimal with $x_{11} = 1$ and $x_{111} = 0$, necessarily $\sum_{j\neq 1} \sum_k x_{1jk} + \sum_{k\neq 1} \sum_l x_{11kl} = 1$. Now since $c_{11} > 0$, points $(1, 1j, 1j'), (1, 1j, 1j', 1j")$ and $(1, 1j, 1jk)$ are not optimal and thus $M(c) \subseteq \{x : 2x_1 = x_{11} + x_{111} + \sum_{j\neq 1} \sum_k x_{1jk} + \sum_{k\neq 1} \sum_l x_{11kl} + \sum_{j\neq 1} \sum_k \sum_l x_{1jkl}\}$.

Case 12. $c_1 + c_{11} + c_{111} = 0$ with $c_{11} = 0$. The argument here is as in case 5, with $M(c) \subseteq \{x : 2x_1 = 2\sum_{j\in J_1} (x_{1j1} + \sum_{k\neq 1} \sum_l x_{1jkl}) + \sum_{j\in J_2} \sum_k (x_{1jk} + \sum_l x_{1jkl})\}$.

Case 13. The previous cases include all situations when the solutions with $x_{11} = 1$ have cardinality 1 to 3 (i.e., in Case 9, $x_1 = x_{11} = 1$ can be a solution). All that remains is the case where all solutions are of cardinality 0 or 4. $M(c) \subseteq \{x : 3x_1 = \sum_j x_{1j} + \sum_j \sum_k x_{1jk} + \sum_j \sum_k \sum_l x_{1jkl}\}$.

∎



(a) Tree with weights a_j (b) Tree with unit weights

Figure 6.

Knapsack Constrained Trees

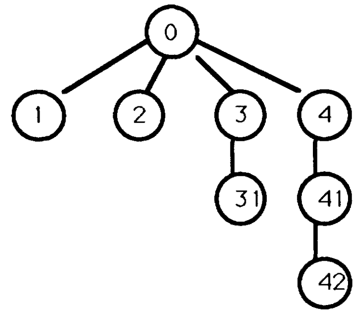
We now briefly consider what happens when the cardinality constraint is replaced by a knapsack constraint: $\sum_{j \in V} a_j x_j \leq k$ with $a_j \in Z^1_+ \setminus \{0\}, a_j \leq k$. It is easiest to demonstrate this extension by example.

Example 4. The weights a_j are shown adjacent to the nodes $j \in V$ in Figure 6a. The problem reduces to the cardinality constrained case if we transform to the graph shown in Figure 6b with the additional constraints $x_2 = x_{21}, x_4 = x_{41} = x_{42}$ and $x_7 = x_{71}$.

The standard knapsack problem can also be viewed as a very special case of the cardinality constrained subtree problem.

Example 5. The knapsack problem

$$x_1 + x_2 + 2x_3 + 3x_4 \leq 3, x_j \in \{0, 1\}, j = 1, \ldots, 4$$



Knapsack problems as cardinality problems.

Figure 7.

can be modelled using the tree shown in Figure 7, the cardinality constraint

$$x_0 + x_1 + x_2 + x_3 + x_{31} + x_4 + x_{41} + x_{42} \leq 4, x \in \{0, 1\}$$

and the additional constraints $x_0 = 1, x_3 = x_{31}, x_4 = x_{41} = x_{42}$.

As the inequalities $x_{p(j)} \geq x_j$ are faces of X_k^{ST}, the master polytope for X_k^{ST} gives the convex hull of any knapsack constraint with right hand side $k-1$. This close tie between the knapsack and cardinality constrained problems suggests that the polyhedral structure of conv(X_k^{ST}) is unlikely to have a simple form, an observation that the inequalities of the previous section somewhat confirm.

5 Conclusions

The results in Section 2 provide a rare example of a situation where given integral polyhedra $\{P^i\}$, and linking constraints $\sum_i A^i x^i \leq b$, the intersection $\{x : \sum_i A^i x^i \leq b, x^i \in P^i \text{ for all } i\}$ is integral. It is natural to look for other examples of this phenomena. In addition the discussion at the end of the section suggests that the results may have computational as well as theoretical value. A next step would be to test empirically the decomposition approach we have suggested. Furthermore, as the subtree packing problem has close ties with "location problems on trees" for which there is an abundant literature (e.g., see Mirchandani and Francis(1990), this suggests examining the implications of Theorems 2 and 5 in this area.

It is perhaps surprising that the polyhedral structure of the cardinality constrained subtree problem examined in Sections 3 and 4 appears complicated. However the close ties to knapsack polyhedra provide some justification for the belief that the intersection of an integral polyhedron and a cardinality constraint will typically be nontrivial. Aghezzaf(1992) studies other examples of such intersections. In particular he considers a variant of the subtree packing problem in which the number p of subtrees is limited. Here again the polyhedral structure appears to be very complicated, although the problem remains polynomially solvable, and has an extended formulation based on dynamic programming. Ward et al.(1987) provide a complete description when $p = 2$.

Finally, the question of finding good separation heuristics for the lifted tree cover inequalities and the two other families of inequalities is still under investigation.

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