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by

A. Schulz, R.H. Mohring
F. Stork, and M. Uetz

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A Note on Scheduling Problems with Irregular Starting Time Costs

Rolf H. Möhring* Andreas S. Schulz** Frederik Stork* ‡ Marc Uetz* §

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Abstract

In [9], Maniezzo and Mingozzi study a project scheduling problem with irregular starting time costs. Starting from the assumption that its computational complexity status is open, they develop a branch-and-bound procedure, and identify special cases that are solvable in polynomial time. In this note, we review three previously established, related results which show that the general problem is solvable in polynomial time.

1 Introduction

Maniezzo and Mingozzi [9] consider the problem of finding a minimum-cost schedule for a set $V = \{1, \ldots, n\}$ of precedence-constrained jobs which have arbitrary, starting time dependent costs. A schedule must respect the given precedence constraints, and each job $j \in V$ incurs a cost of w_{jt} if it is started at time t. Here, $t \in \{0, 1, \ldots, T\}$, and T denotes the planning horizon. This problem owes its significance to a good part to its appearance as a subproblem in the computation of lower bounds for different resource-constrained project scheduling problems, see, e.g., [3, 5, 10]. Maniezzo and Mingozzi suggest that the computational complexity status of this problem is open. On this account, they discuss special cases which can be solved in polynomial time, namely costs w_{jt} which are monotonic in t, absence of precedence constraints, and precedence constraints which form an out-tree. In addition, they develop a lower bound as well as a branch-and-bound procedure for the general case. Their lower bound is obtained by extracting an out-tree from the given precedence constraints, and by penalizing the violation of the neglected constraints in a Lagrangian fashion. For the special case of an out-tree, they propose a dynamic programming algorithm of running time $O(nT)$.

*Technische Universität Berlin, Fachbereich Mathematik, Sekr. MA 6-1, Straße des 17. Juni 136, D-10623 Berlin, Germany. Email: {moehring, stork, uetz}@math.tu-berlin.de

**M.I.T., Sloan School of Management and Operations Research Center, E53-361, Cambridge, MA 02139, USA. Email: schulz@mit.edu

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In this note, we give a brief historical synopsis of three previously established, inter-related results which show that the general problem considered in [9] is solvable in polynomial time. In fact, the integrality of the linear programming relaxations of two popular integer programming formulations implies that even a more general problem can be solved in polynomial time, namely instances with temporal constraints in the form of arbitrary time lags instead of precedence constraints. We also report on a reduction to a minimum cut problem in an appropriately defined digraph, which results in an algorithm with running time $O(nmT^2 \log T)$. Here, m is the number of temporal constraints.

Let us briefly introduce some notation needed for the subsequent account of results. A temporal constraint between two jobs i and j is an inequality of the form $S_j \geq S_i + d_{ij}$, where S_j and S_i denote the starting times of jobs j and i, respectively, and $-\infty \leq d_{ij} < \infty$ imposes a time lag between them. In contrast, ordinary precedence constraints arise as the special case $S_j \geq S_i + p_i$, where $p_i \geq 0$ denotes the processing time of job i. We assume throughout the text that the given temporal constraints are consistent, i.e., the digraph $G = (V,A)$ with $A = \{(i,j) | d_{ij} > -\infty\}$ and arc lengths d_{ij} does not contain a dicycle of positive length. Given the temporal constraints and the time horizon T, it is easy to compute earliest and latest starting times for each job $j \in V$. For convenience of notation, however, we simply assume (without stating explicitly) that variables with time indices outside these boundaries are fixed at values which ensure that no job is started at an infeasible time.

2 Solution Techniques

Integer programming formulation I. The following integer program represents one way of formulating the project scheduling problem with irregular starting time costs. We use binary variables z_{jt}, $j \in V$, $t = 0, \ldots, T$, with the intended meaning that $z_{jt} = 1$ if job j is started in or before time period t and $z_{jt} = 0$, otherwise. (Note that period t starts at time t and ends at time $t+1$.) The problem then reads as follows.

$$
\begin{array}{llrlll}
\text{minimize} & & \displaystyle\sum_j \sum_t w'_{jt} z_{jt} & & & (1) \\
\text{subject to} & & z_{jT} & = 1, & j \in V, & (2) \\
& & z_{jt} - z_{j,t+1} & \leq 0, & j \in V,\ t = 0, \ldots, T, & (3) \\
& & z_{j,t+d_{ij}} - z_{it} & \leq 0, & (i,j) \in A,\ t = 0, \ldots, T, & (4) \\
& & z_{jt} & \geq 0, & j \in V,\ t = 0, \ldots, T, & (5) \\
& & z_{jt} & \text{integer}, & j \in V,\ t = 0, \ldots, T. & (6)
\end{array}
$$

Here, $w'_{jt} := w_{jt} - w_{j,t+1}$ for all $j \in V$ and $t \in \{0, \ldots, T-1\}$, and $w'_{jT} := w_{jT}$. It follows from the work of Gröflin, Liebling, and Prodon [7] on pipeline scheduling with tree-like precedence constraints that the constraint matrix of (2) – (4) is totally unimodular. This implies that the linear programming relaxation of the above integer program is integral and hence the scheduling problem is solvable in polynomial time. More precisely, the pipeline scheduling problem considered in [7] can be interpreted as a project scheduling problem with irregular starting time costs and zero time lags ($d_{ij} = 0$), which form an out-tree. The constraint matrix of the corresponding linear program as considered in [7] is the arc-node incidence matrix of a digraph.

Hence, its dual can be solved as a minimum-cost flow problem, which is also the case for (1) – (5). In fact, Gröflin, Liebling, and Prodon presented an algorithm that solves the pipeline scheduling problem with tree-like precedence constraints in $O(nT)$ time. Their algorithm also applies to project scheduling problems with irregular starting time costs and out-tree precedence constraints.

Integer programming formulation II. Another integer programming formulation of the same problem is based on binary variables x_{jt}, $j \in V$, $t = 0, \dots, T$, where $x_{jt} = 1$ if job j is started in period t and $x_{jt} = 0$, otherwise.

$$
\begin{array}{llll}
\text{minimize} & \sum_j \sum_t w_{jt} x_{jt} & & (7) \\
\text{subject to} & \sum_t x_{jt} = 1, & j \in V, & (8) \\
& \sum_{s=t}^{T} x_{is} + \sum_{s=0}^{t+d_{ij}-1} x_{js} \leq 1, & (i,j) \in A,\ t = 0, \dots, T, & (9) \\
& x_{jt} \geq 0, & j \in V,\ t = 0, \dots, T, & (10) \\
& x_{jt} \text{ integer}, & j \in V,\ t = 0, \dots, T. & (11)
\end{array}
$$

Chaudhuri, Walker and Mitchell [2] showed that the linear programming relaxation of this integer program is integral as well. For this, they made use of the following graph-theoretic interpretation of the problem: Identify with every job-time pair (j,t) a node in an undirected graph. There are two different types of edges. First, all pairs of nodes which belong to the same job are connected, for any job. Second, for each temporal constraint $S_j \geq S_i + d_{ij}$ and each time t, there are edges between (i,t) and all pairs (j,s) with $s < t + d_{ij}$. In the resulting graph, any stable set (a set of pairwise non-adjacent nodes) of cardinality n corresponds to a feasible solution of the original scheduling problem: Job j is started at time t if node (j,t) belongs to the stable set. Consequently, if we assign the cost coefficients w_{jt} as weights to the nodes (j,t), a minimum-weight stable set of cardinality n yields an optimum schedule. Because the so-defined graph can easily be transitively oriented and therefore is a comparability graph, its corresponding fractional stable set polytope is integral (see, e.g., [8, Chapter 9]). Since the inequalities (8) – (10) define a face of the fractional stable set polytope, it follows that LP relaxation (7) – (10) is integral as well. This fact can also be proved from the integrality of LP relaxation (1) – (5) by a linear transformation between z- and x-variables which preserves integrality, as was observed by de Souza and Wolsey [4] as well as Cavalcante et al. [1] in the context of labor-constrained scheduling.

Maniezzo and Mingozzi have also considered an integer programming formulation in x-variables. Instead of using (9), they have modeled temporal constraints as follows:

$$\sum_t t\,(x_{jt} - x_{it}) \geq d_{ij}. \tag{12}$$

However, LP relaxation (7), (8), (10), and (12) is weaker than (7) – (10); in particular, it is not integral.

Reduction to a minimum cut problem. A reduction of the scheduling problem under consideration to a minimum-cut problem has been presented in Möhring et al. [10]. The underlying digraph is closely related to the constraints (3) and (4) of the z-formulation. Every variable z_{jt} corresponds to a node (j,t) in the digraph, and the constraint matrix of (3) and (4) is interpreted as an arc-node incidence matrix. Thus, inequalities (3) define directed chains $((j,0),(j,1)),((j,1),(j,2)),\ldots,((j,T-1),(j,T))$ for any j, and inequalities (4) define arcs between chains which correspond to temporal constraints. The cost w_{jt} is interpreted as the arc capacity of $((j,t),(j,t+1))$, for all j and t. The capacity of all remaining arcs is set to infinity. Then, after the introduction of a dummy source a and sink b, a solution of the original scheduling problem can be computed as a minimum a-b-cut in that digraph. Using a push-relabel maximum-flow algorithm [6], this results in an algorithm for solving the project scheduling problem with irregular starting time costs and arbitrary time lags of running time $O(nmT^2 \log T)$.

Remark. Note that the polynomiality results discussed in this note refer to instances which require an encoding length of $\Omega(nT)$. This is clearly the case for problems with general costs w_{jt}. However, this does not imply polynomial-time algorithms for problems which allow a more succinct encoding.

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