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working paper

MASSACHUSETTS INSTITUTE
OF TECHNOLOGY

# DUALITY AND SENSITIVITY ANALYSIS FOR FRACTIONAL PROGRAMS

by

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WP # OR 035-74 May, 1974

Supported in part by the U.S. Army Research Office (Durham) under Contract No. DAHC04-73-C-0032

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## A B S T R A C T

Duality theory is considered for optimization problems, called fractional, whose objective function is a ratio of two real valued functions $n(x)$ and $d(x)$. Results are derived by applying Lagrangian duality to the problem of maximizing $[n(x) - vd(x)]$, where $v$ is the optimal value of the fractional problem. This leads to a global saddlepoint theory with a fractional Lagrangian function and specializes to a Wolfe type dual under differentiability assumptions. Sensitivity procedures are presented for the linear fractional problem together with a primal-dual algorithm for parametric right-hand-side analysis. A slight variant of Dinkelbach's algorithm for the general fractional problem is also given.

## Introduction

The problem of maximizing (or minimizing) the ratio of two real valued functions $n(\cdot)$ and $d(\cdot)$ over a subset $F$ of $R^n$ will be called a <u>fractional program</u>. When $n(x)$ and $d(x)$ are affine functions and the feasible region $F$ is a polyhedral set this reduces to the much analyzed linear fractional program. Our purpose here is to develop saddlepoint duality results for the fractional problem and to provide sensitivity procedures for the linear fractional case. We also present a primal-dual procedure for parametric right-hand-side analysis for the linear fractional problem and introduce an algorithm for the general fractional problem that is a variant of a procedure developed previously by Dinkelbach [12].

Much previous research on the fractional program has dealt with the linear fractional case. Two distinct approaches have been taken. The well known algorithm of Charnes and Cooper [7] transforms the problem into a linear program by modifying the feasible region. It applies only when the feasible region is bounded. The other approach solves a sequence of linear programs (or at least one pivot step of each linear program) over the original feasible region F by updating the linear programs objective function. Algorithms in this category are all related to ideas originally proposed by Isbell and Marlow [18] and have been proposed by Abadie and Williams [1], Bitran and Novaes [3], Dorn [13], Gilmore and Gomory [17] and Martos [23]. These algorithms can be viewed as a specialization of the Frank-Wolfe approach for nonlinear objective functions [15] or Martos' adjacent vertex programming methods [24].

The second class of algorithms can also be viewed as specializations of Dinkelbach's [12] algorithm for the general fractional problem, though they exploit the underlying linearity of the linear fractional model. In the next section we review this algorithm and mention a modification that extends its scope of applicability. For an approach to the fractional problem that generalizes the Charnes and Cooper algorithm and applies when $n(x)$ and $d(x)$ are homogeneous of degree one to within a constant see Bradley and Frey [6].

The saddlepoint duality theory that we develop in section 2 introduces a new Lagrangian function with multipliers dependent upon the primal variables. It leads to a dual problem that is again a fractional program. To our knowledge, these results provide one of the few instances where saddlepoint duality applies to a large class of nonconvex problems, geometric programming duality [14] being a notable illustration.

When the fractional program has an optimizing feasible solution and consists of differentiable functions, our Lagrangian result provides a Wolfe type dual [22] which has been obtained elsewhere [20] for the linear fractional problem. For another result on linear fractional duality see [9].

In section 3, sensitivity procedures are developed for variations in the problem data of a linear fractional program. The results are analogous to the usual sensitivity procedures of linear programming, but now include variations in both the numerator and denominator of the objective function as well as right-hand-sides of the constraints. The next section continues this study by introducing a primal-dual algorithm for parametric right-hand-side analysis. This algorithm quite naturally motivates a branch and bound procedure for the integer programming version of the linear fractional problem. Proofs and more details for this material appear in [4].

The fractional model arises naturally for maximizing return per unit time in dynamic situations or return per unit trip in transportation settings [16]. It also results when minimizing the ratio of return to risk in financial applications [6].

For a number of reasons, applications of the linear fractional model have been far less numerous than those of linear programming. The essential linearity of many models is certainly a contributing factor. In addition, linear fractional applications are easily disguised as linear programs when the feasible region is bounded. In this case, the fractional model can be reduced to a linear program through convenient artifacts. Finally, the fact that no direct sensitivity analysis has been given for the fractional model may have some bearing on potential applications. In any event, the model has been applied to study changes in the cost coefficients of a transportation problem [8], to the cutting stock problem [17], to Markov decision processes resulting from maintenance and repair problems [11], [19] and to fire programming games [18]. It has also been applied to a marine transportation problem [16], arises in Chebyshev maximization problems [5] and arises in primal-dual approaches to decomposition procedures [2], [21].

## 1. Preliminaries

In this paper we consider the problem

$$v = \sup\{f(x) = \frac{n(x)}{d(x)} : x \in F\} \tag{P}$$

where $F = \{x \in X \subseteq R^n : g(x) \geq 0\}$ and $n(\cdot)$, $d(\cdot)$ and the component functions $g_i(\cdot)$, $i = 1,\ldots,m$ are real valued functions defined on $R^n$. We assume that $d(x) > 0$ for all $x \in X$ so that the problem is well posed (if $d(\cdot) < 0$ on $X$ write $f(x)$ as $\frac{-n(x)}{-d(x)}$) .

The key to our analysis is the simple observation that $v > k$ if and only if $n(x) - kd(x) > 0$ for some point $x \in F$ . Consequently, given any $k = f(x)$ with $x \in F$ or merely $k = \lim_{j \to \infty} f(x^j)$ with $x^j \in F$, we may determine whether or not $k = v$ by solving the auxiliary optimization problem

$$v(k) = \sup\{r(x,k) = n(x) - kd(x) : x \in F\}. \tag{A}$$

There are two possible outcomes to (A)

(1) $v(k) \leq 0$ so that $v = k$, or

(2) $v(k) > 0$ . In solving the auxiliary problem we will identify a point $y \in F$ with $f(y) > k$, i.e., $n(y) - kd(y) > 0$ .

When F is a convex set, $n(x)$ is concave on F, and either (i) $d(x)$ is convex on F and $k \geq 0$, or (ii) $d(x)$ is concave on F and $k \leq 0$, then $r(x,k)$ will be concave and (A) can be solved by concave programming techniques. In particular if $n(\cdot)$, $d(\cdot)$ and $g(\cdot)$ are also differentiable and the auxiliary problem satisfies a constraint qualification, it can be solved via the corresponding Kuhn-Tucker conditions.

On the other hand, if $r(x,k)$ is a quasi-convex function and F is convex and bounded, then the auxiliary problem will solve at an extreme point of F. In particular, taking $k = v$, the original problem (P) will solve at an extreme point of F as well.

These conditions are related, but not equivalent, to quasi-convex and quasi-concave properties for $f(\cdot)$.

We might note that (P) may have local optimum when (A) does not and conversely as illustrated by the following examples.

Example 1: (P) has a nonoptimal local maximum but (A) does not

(P) $$v = \sup\{\frac{x - 4}{(1/x)} = x^2 - 4x\}$$

$$\text{s.t.} \quad 1 \leq x \leq 4 \qquad [v = 0]$$

(A) $$\sup\{r(x,v) = x - 4\}$$

$$\text{s.t.} \quad 1 \leq x \leq 4$$

Example 2: (P) does not have a nonoptimal local maximum but (A) has

(P) $$v = \sup\{f(x) = \frac{x^2}{x} = x\}$$

$$\text{s.t.} \quad 1 \leq x \leq 4 \qquad [v = 4]$$

(A) $$\sup\{r(x,v) = x^2 - 4x\}$$

$$\text{s.t.} \quad 1 \leq x \leq 4$$

Dinkelbach [12] has previously used the auxiliary problem to develop an algorithm for the fractional program when F is compact. From our observations above we know that if $x^j \in F$ and $k_j = f(x^j)$ satisfy

$$v(k_j) = \sup\{r(x,k_j) = n(x) - k_j d(x) : x \in F\} > 0$$

then $f(x^{j+1}) > f(x^j)$ for every solution $x^{j+1}$ to this problem. In fact, if $d(\cdot)$ is continuous then there exist $\bar{x} \in F$ and $\bar{\bar{x}} \in F$ with $d(\bar{x}) = \min\{d(x) : x \in F\}$ and $d(\bar{\bar{x}}) = \max\{d(x) : x \in F\}$. Consequently,

$$v(k_j) = \sup\{r(x,k_j) : x \in F\} \le \delta \qquad (\delta \ge 0)$$

implies that

$$\frac{n(x)}{d(x)} \le k_j + \frac{\delta}{d(x)} \le k_j + \frac{\delta}{d(\bar{x})} \qquad \text{for all } x \in F$$

and thus that

$$v \le k_j + \frac{\delta}{d(\bar{x})} \quad .$$

Similarly

$$r(x^{j+1},k_j) > \delta$$

implies that

$$f(x^{j+1}) = \frac{n(x^{j+1})}{d(x^{j+1})} > k_j + \frac{\delta}{d(x^{j+1})} \ge f(x^j) + \frac{\delta}{d(\bar{\bar{x}})} \quad .$$

From these inequalities we see that if $v < +\infty$ then starting with any $x^0 \in F$ and $j = 0$ the following algorithm solves the fractional problem

(a) Let $k_j = f(x^j)$ and solve the auxiliary problem

$$v(k_j) = \max\{n(x) - k_j d(x) : x \in F\}$$

and let $x^{j+1}$ be any solution.

(b) (i) If $r(x^{j+1},k_j) \leq \delta$ then terminate.

(ii) If $r(x^{j+1},k_j) > \delta$ then return to (1) with j incremented by one.

Since $f(x^{j+1}) > f(x^j) + \frac{\delta}{d(\bar{\bar{x}})}$ for each occurance of step b(ii), the algorithm must terminate in step b(i) after a finite number of applications of (a). Also at step b(i), $v \leq f(x^j) + \varepsilon$ where $\varepsilon = \frac{\delta}{d(\bar{x})}$ and thus after a finite number of solutions to the auxiliary problem we can obtain an $\varepsilon$-optimal solution.

By slightly modifying this algorithm, we can insure the same conclusion, but without the compactness assumption on F as long as $v < +\infty$. In step (a) we take $k_j = f(x^j) + \varepsilon$ and we let $\delta = 0$. Then we note that step b(ii) will imply that

$$f(x^{j+1}) = \frac{n(x^{j+1})}{d(x^{j+1})} \geq f(x^j) + \varepsilon$$

and that step b(i) implies that

$$\frac{n(x)}{d(x)} \leq f(x^j) + \varepsilon \qquad \text{for all } x \varepsilon F$$

and thus

$$v \leq f(x^j) + \varepsilon \; .$$

To be implemented we assume that a solution can be found to each auxiliary problem encountered in step (a).

## 2. Duality

Ordinary Lagrangian saddlepoint duality theory is inadequate for dealing with fractional programs even in the special case of linear fractional programming. In this section, we introduce a new Lagrangian function with dual variables depending upon the fractional variables x and show that this new Lagrangian will lead to a saddlepoint duality theory. We begin with an illustration.

Example 3: $v = \sup_{x\varepsilon X}\{\frac{1}{x+1} : x - 1 \geq 0\} = \frac{1}{2}$ where $X = \{x \varepsilon R : x \geq 0\}$

The usual Lagrangian dual is $\inf_{u\geq 0} \sup_{x\varepsilon X}\{\frac{1}{x+1} + u(x-1)\} = 1$ with the infimum being attained at $u = 0$. Replacing the dual variables u with $\frac{u}{x+1}$, however, gives a dual value $\inf_{u\geq 0} \sup_{x\varepsilon X}\{\frac{1}{x+1} + \frac{u}{x+1}(x-1)\} = \frac{1}{2}$ by selecting $u = \frac{1}{2}$. This new dual value now agrees with the primal value $v = \frac{1}{2}$.

With this example as motivation we define the "fractional Lagragian" $L(x,u)$ by

$$L(x,u) = f(x) + \frac{u}{d(x)}\, g(x) = \frac{n(x) + ug(x)}{d(x)}$$

and the fractional (saddlepoint) dual problem as

$$w = \inf_{u\geq 0} \sup_{x\varepsilon X}[L(x,u)] \, . \qquad \text{(D)}$$

We call the original problem

$$v = \sup_{x\varepsilon X}\{f(x) : g(x) \geq 0\} \text{ where } f(x) = \frac{n(x)}{d(x)} \qquad \text{(P)}$$

the primal problem.

Observe that the supremum problem in the dual which we write as

$$D(u) = \sup_{x \in X}[L(x,u)]$$

is itself a fractional program for each fixed value of u. Also note that we are not making any differentiability or even continuity assumptions.

Our results will be obtained by utilizing the auxiliary problem with $k = v$. That is, we consider the auxiliary problem

$$r = \sup_{x \in X}\{[n(x) - vd(x)] : g(x) \geq 0\} \tag{A1}$$

Observe that in this case $r = 0$.

First we note by standard arguments that:

Lemma 1: $v = \sup_{x \in X} \inf_{u \geq 0}[L(x,u)]$. Furthermore, for any $x \in F$, $\inf_{u \geq 0}[L(x,u)] = f(x)$.

Proof: If $x \in X$ and $x \notin F$, then some component $g_i(x)$ of $g(x)$ is negative. Letting $u_i \to +\infty$, the infimum will be $-\infty$. Consequently, points $x \in X$ with $x \notin F$ are inoperative in the sup inf and

$$\sup_{x \in X} \inf_{u \geq 0}[L(x,u)] = \sup_{x \in F} \inf_{u \geq 0}[L(x,u)] \tag{i}$$

If $x \in F$, $\frac{ug(x)}{d(x)} \geq 0$ for any $u \geq 0$ since $g(x) \geq 0$ and $d(x) > 0$. Thus the infimum takes $u_i = 0$ if $g_i(x) > 0$ so that

$$\frac{ug(x)}{d(x)} = 0 \quad \text{and} \quad \inf_{u \geq 0}[L(x,u)] = f(x) \tag{ii}$$

Combining (i) and (ii) we have:

$$\sup_{x \in X} \inf_{u \geq 0}[L(x,u)] = \sup_{x \in F} f(x) = v . \qquad \square$$

Note that this result holds when $F = \emptyset$ with the convention $v = -\infty$ in this case.

Lemma 2 (weak duality): Assume $\bar{x}\varepsilon F$ and $u\varepsilon R^m$, $u \geq 0$. Then $f(\bar{x}) \leq D(u)$. Consequently, $v \leq w$ .

Proof:
$$D(u) = \sup_{x\varepsilon X}[f(x) + \frac{ug(x)}{d(x)}] \geq f(\bar{x}) + \frac{ug(\bar{x})}{d(\bar{x})} \geq f(\bar{x})$$

taking the infimum of the left hand side over all $u\varepsilon R^m$ and the supremum of the right hand side over all $x\varepsilon F$ gives $v \leq w$ . $\square$

Next we show that in most instances the fractional problem (P) inherits strong duality from the auxiliary problem (A1). That is, in general when ordinary Lagrangian duality holds for the auxiliary problem (A1) strong duality holds for the fractional Lagrangian dual of (P).

Remark: The following theorem requires that there is a dual variable $\bar{u}$ solving the Lagrangian dual of (A1).

Theorem 1: Suppose that $F \neq \emptyset$, $v < +\infty$ and that

$$\sup_{x\varepsilon F}[n(x) - vd(x)] = \min_{u\geq 0}\sup_{x\varepsilon X}[n(x) - vd(x) + ug(x)]$$

then

$$v = \min_{u\geq 0}\sup_{x\varepsilon X}[f(x) + u\,\frac{g(x)}{d(x)}] = w\ .$$

Proof: From our previous observations about the auxiliary problem,

$$\sup_{x\varepsilon F}[n(x) - vd(x)] = 0$$

Thus by hypothesis there is a $\bar{u} \geq 0$ satisfying

$$\sup_{x\varepsilon X}[n(x) - vd(x) + \bar{u}g(x)] = 0 \quad \text{so that}$$

$$n(x) - vd(x) + \bar{u}g(x) \leq 0 \quad \text{for all } x\varepsilon X$$

since $d(x) > 0$ for $x \varepsilon X$, dividing by $d(x)$ preserves the inequality

$$\frac{n(x)}{d(x)} + \bar{u}\,\frac{g(x)}{d(x)} \le v \qquad \text{for all } x \varepsilon X .$$

Thus

$$w = \inf_{u \ge 0} \sup_{x \varepsilon X} [L(x,u)] \le \sup_{x \varepsilon X} [L(x,u)] \le v .$$

But, weak duality states that $w \ge v$ and consequently $w = v$ .

The existence of a dual variable $\bar{u}$ solving the Lagrangian dual to the auxiliary problem is guaranteed, for example, if the auxiliary problem is concave and satisfies the Slater condition [22], or is a linear program, or more generally a concave quadratic programming program (remember that in Theorem 1 we require $v < +\infty$ so that $r = 0$).

Formally these results are recorded as:

<u>Corollary 1.1</u>: Suppose that $v < +\infty$, that $n(x) - vd(x)$ and $g(x)$ are concave functions, that $X$ is a convex set and that there is $x \varepsilon X$ with $g_i(x) > 0$ , $i = 1,\ldots,m$ (the Slater condition). Then:

$$v = \min_{u \ge 0} \sup_{x \varepsilon X} \left[f(x) + \frac{ug(x)}{d(x)}\right] = w$$

<u>Corollary 1.2</u>: Assume that the fractional objective is the ratio of two affine functions, $n(x) = c_0 + cx$ and $d(x) = d_0 + dx$, that $g(x) = b - Ax$ and that $X = \{x \varepsilon R^n \colon x \ge 0 \text{ and } d_0 + dx > 0\}$ with $F \ne \emptyset$ . Then:

$$v = \min_{u \ge 0} \sup_{x \varepsilon X} \left\{\frac{c_0 + (c - uA)x + ub}{d_0 + dx}\right\}$$

We can also obtain a complementary slackness condition for fractional duality in the same manner used to develop complementary slackness for ordinary Lagrangian duality.

Theorem 2: Suppose that $\bar{x}$ solves the primal problem (P), that $\bar{u}$ solves the dual problem (D) and that $w = v$. Then $v = \sup_{x \in X}[L(x,\bar{u})] = L(\bar{x},\bar{u})$. Moreover, $\bar{u}g(\bar{x}) = 0$.

Proof: $$L(\bar{x},\bar{u}) = f(\bar{x}) + \frac{\bar{u}g(\bar{x})}{d(\bar{x})} \leq \sup_{x \in X}[L(x,\bar{u})] = w = v = f(\bar{x}).$$

Since $\bar{u} \geq 0$, $g(\bar{x}) \geq 0$ and $d(\bar{x}) \geq 0$ this implies that $\bar{u}g(\bar{x}) = 0$ and consequently that $L(\bar{x},\bar{u}) = f(\bar{x}) = v$. ▨

This result immediately provides a Wolfe type duality theorem for the fractional program.

Corollary 2.1: Suppose that $\bar{x}$ solves the primal problem (P), that $\bar{u}$ solves the dual problem (D) and that $w = v$. Then if $n(x)$, $d(x)$ and $g(x)$ are differentiable at $\bar{x}$ and $X$ is open (e.g., if $X = \{x \in R^n: d(x) > 0\}$ and $d(\cdot)$ is continuous) then

$$\nabla f(\bar{x}) + \frac{\bar{u}}{d(\bar{x})} \nabla g(\bar{x}) = 0$$

$$\bar{u}g(\bar{x}) = 0 .$$

(Here $\nabla f(\cdot)$ and $\nabla g(\cdot)$ denote the gradients of f and g.)

Proof: By the previous theorem

$$L(\bar{x},\bar{u}) = \sup_{x \in X}[L(\bar{x},\bar{u})] .$$

Since X is open this implies that

$$\nabla_x L(x,\bar{u})\Big|_{x=\bar{x}} = \nabla f(\bar{x}) + \bar{u}\left[\frac{d(\bar{x})\nabla g(\bar{x}) - g(\bar{x})\nabla d(\bar{x})}{d(\bar{x})^2}\right] = 0 .$$

Using $\bar{u}g(\bar{x}) = 0$ from theorem 1, this reduces to the desired result.

Another duality result can be given when Lagrangian duality holds for the auxiliary problem, but without optimizing dual variables, as long as $d(x)$ is bounded away from zero on $X$.

<u>Theorem 3</u>: Suppose that $F \neq \emptyset$, that $v < +\infty$ and that

$$\sup_{x\in F}[n(x) - vd(x)] = \inf_{u\geq 0}\sup_{x\in X}[n(x) - vd(x) + ug(x)]$$

If there is a $\delta > 0$ such that $d(x) \geq \delta$ for all $x\in X$, then

$$v = \inf_{u\geq 0}\sup_{x\in X}\left[\frac{n(x)}{d(x)} + u\,\frac{g(x)}{d(x)}\right] = w\ .$$

<u>Proof</u>: As in Theorem 1, $\sup_{x\in F}[n(x) - vd(x)] = 0$ and thus by hypothesis given $\varepsilon > 0$ there is a $u^{\varepsilon} \geq 0$ satisfying

$$\sup_{x\in X}[n(x) - vd(x) + u^{\varepsilon}g(x)] \leq \varepsilon$$

since $d(x) \geq \delta > 0$ for all $x\in X$ this implies that

$$\frac{n(x)}{d(x)} + u^{\varepsilon}\,\frac{g(x)}{d(x)} \leq v + \frac{\varepsilon}{d(x)} \leq v + \frac{\varepsilon}{\delta} \qquad \text{for all } x\in X$$

But $\varepsilon > 0$ is arbitrary and consequently

$$w = \inf_{u\geq 0}\sup_{x\in X}\left[\frac{n(x)}{d(x)} + u\,\frac{g(x)}{d(x)}\right] \leq v$$

Coupled with weak duality $w \geq v$, this provides the desired result.

<u>Remark</u>: If $v = +\infty$, then weak duality states that $w = +\infty$ so that each of the previous results apply in this case as well.

We conclude this section with three additional examples. Example 4 shows that some requirement such as the hypothesis in Theorem 2 is needed to assure that fractional duality $v = w$ holds. On the other hand, example 5 shows that fractional duality may hold when $d(x)$ is not bounded from below over $X$. The final example illustrates a situation when ordinary Lagrangian duality applies and fractional Lagrangian duality does not. This example does not violate any of the above results since $[n(x) - vd(x)]$ is not concave.

| Example | $X$ | $n(x)$ | $d(x)$ | $g(x)$ | $v$ | $r = \sup_{x\varepsilon X}[n(x) - vd(x)]$ | $w = \inf_{u\geq 0} \sup_{x\varepsilon X}\left[\frac{n(x) + ug(x)}{d(x)}\right]$ |
|---|---|---|---|---|---|---|---|
| 4 | $\{(x_1,x_2)\varepsilon R^2: x_1 \geq 0, x_2 \leq \sqrt{x_1}\}$ | $x_2$ | $\sqrt{x_1}$ if $x_1 > 0$<br>$1$ if $x_1 = 0$ | $-x_1$ | 0 | 0 | 1 |
| 5 | $\{(x_1,x_2)\varepsilon R^2: x_1 \geq 0, x_2 \leq x_1^{3/4}\}$ | $x_2$ | $\sqrt{x_1}$ if $x_1 > 0$<br>$1$ if $x_1 = 0$ | $-x_1$ | 0 | 0 | 0 |
| 6 | $\{x\varepsilon R: x \geq 1\}$ | $-x$ | $1 + x^2$ | $2 - x$ | $-\frac{2}{5}$ | 0 | 0 |

Observe that ordinary Lagrangian duality does hold for the auxiliary problem of example 4 and that there is no optimizing dual variables to the Lagrangian dual to the auxiliary problem of example 5. Also, for the last example the usual Lagrangian gives $\inf_{u\geq 0} \sup_{x\varepsilon X}[f(x) + ug(x)] = -\frac{2}{5}$ by taking $u = .12$ .

Note: Other duality results, obtained using Charnes and Cooper approach, have been developed independently in a recent paper "Fractional Programming: Transformations, Duality and Algorithm Aspects", by Siegfried Schaible, Technical Report 73-9, Stanford University, Operations Research House, November 1973.

## 3. Sensitivity Analysis for the Linear Fractional Program

### Linear Fractional Algorithm

The linear fractional model assumes that $f(x)$ is the ratio of two affine functions $n(x) = c_0 + cx$ and $d(x) = d_0 + dx$, that $g(x) = Ax - b$ and that $X = \{x \in R^n: x \geq 0\}$. (We assume that $d_0 + dx > 0$ for $x \in F$ and may include the redundant condition $d_0 + dx > 0$ in $X$ if we like, i.e., replace $X$ with $X \cap \{x \in R^n: d_0 + dx > 0\}$, to conform with the basic assumptions that we have previously imposed upon $X$.) In this case, the auxiliary problem becomes

$$v(k) = \sup_{x \in F}\{r(x,k) = (c_0 - kd_0) + (c - kd)x\} \tag{A2}$$

where $$F = \{x \in R^n: x \geq 0 \text{ and } Ax = b\} .$$

Since this is a linear program much of the usual linear programming sensitivity analysis can be extended to this more general setting. In this section we outline this analysis. For notational convenience, we assume in this section that the linear constraints are specified in equality form.

Let us first review the linear fractional algorithm [1]. It is initiated with any feasible point $x \in F$, determined possibly by phase I of the simplex method. Setting $k = f(x)$, the auxiliary problem (A2) is solved. There are two possible outcomes:

(a) It has an optimal solution $x'$. From our preliminary observations concerning the auxiliary problem, if $v(k) \leq 0$ then $v = k$. Otherwise $r(x',k) > 0$ and $f(x') > k$, (A2) is re-solved with $k = f(x')$ and the appropriate case (a) or (b) is applied again.

(b) It is unbounded, so by linear programming theory $F$ has an extreme ray $r$ with $(c - kd)r > 0$ . For any $x \in F$,

$$f(x + \lambda r) = \frac{c_0 + cx + \lambda cr}{d_0 + dx + \lambda dr} .$$

If $dr = 0$, then $cr > 0$ and $f(x + \lambda r) \to +\infty$ as $\lambda \to +\infty$ so that $v = +\infty$. If $dr \neq 0$, then $f(x + \lambda r) \to \frac{cr}{dr} > k$. (A2) is re-solved with $k = \frac{cr}{dr}$ and the appropriate case (a) or (b) is applied again.

After solving (A2) a finite number of times, v will be determined by either case (a) or (b). This convergence is a direct consequence of the fact that the extreme points and extreme rays of F are finite in number.

For a relaxation of the condition $d_0 + dx > 0$ for all $x \epsilon F$ and more details about this algorithm see [1], [3], [4], [18], [23]. Also, (A2) need not be solved to completion, but merely until a point x is generated with $r(x,k) > 0$. If $k = f(x')$ and (A2) is initiated with the basis corresponding to x', then assuming nondegeneracy, x will be generated after a single interation of the simplex method. For details see [23].

From the algorithm, we see that the linear fractional problem solves either at an extreme point or extreme ray of the feasible region. The sensitivity analysis will indicate when the solution remains optimal. First let us set some notation. Suppose that $A = [B, A^N]$ where B is a feasible basis and that x, c and d are partitioned conformally as $(x^B, x^N)$, $(c^B, c^N)$ and $(d^B, d^N)$. Then $x^B = \bar{b} - \bar{N}x^N$ where $\bar{b} = B^{-1}b$, $\bar{N} = B^{-1}A^N$ and f(x) can be written in terms of $x^N$ as

$$f(x) = \frac{c_0 + c^B(\bar{b} - \bar{N}x^N) + c^N x^N}{d_0 + d^B(\bar{b} - \bar{N}x^N) + d^N x^N} = \frac{\bar{c}_0^N + \bar{c}^N x^N}{\bar{d}_0^N + \bar{d}^N x^N}$$

where

$$\bar{c}_0^N = c_0 + c^B\bar{b} \qquad \bar{d}_0^N = d_0 + d^B\bar{b} \qquad \bar{c}^N = c^N - c^B\bar{N} \qquad \bar{d}^N = d^N - d^B\bar{N}.$$

We also let

$$d(\bar{x}) = \inf_{x \epsilon F} d(x) \qquad d(\bar{\bar{x}}) = \sup_{x \epsilon F} d(x) \qquad -\delta_1 = \inf_{x \epsilon F} \frac{d(x)}{x_j} \qquad -\delta_2 = \sup_{x \epsilon F} \frac{d(x)}{x_j}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

The simplex multipliers corresponding to the basis B for (A2) are $\pi = (c^B - kd^B)B^{-1} = \pi^n - k\pi^d$ where $\pi^n = c^B B^{-1}$ and $\pi^d = d^B B^{-1}$. Note that this data for $\bar{c}_0^N$, $\bar{c}^N$ and $\pi^n$ is precisely that carried by the simplex method with objective function $c_0 + cx$ and similarly $\bar{d}_0^N$, $\bar{d}^N$, and $\pi^d$ is the data carried with objective function $d_0 + dx$.

Sensitivity Analysis

The nature of the linear fractional algorithm suggests extensions of the usual linear programming sensitivity analysis to this more general setting. Suppose that the fractional problem solves at an extreme point $x^o = (x^{oB}, x^{oN})$ of the feasible region and that B is the corresponding basis of A. This extreme point remains optimal as long as the coefficients of the nonbasic variables in the final canonical form of the auxiliary linear programming problem remain nonpositive. The coefficients are given by

$$\bar{t}^N = \bar{c}^N - f(x^o)\bar{d}^N \quad \text{where} \quad f(x^o) = \frac{c_o + c^B B^{-1} b}{d_o + d^B B^{-1} b}$$

Thus it is required that $\bar{t}^N + \Delta\bar{t}^N \leq 0$ with

$$\Delta\bar{t}^N = \Delta\bar{c}^N - f(x^o)\Delta\bar{d}^N - [\Delta f(x^o)]\bar{d}^N - \Delta f(x^o)\Delta\bar{d}^N \tag{1}$$

Table 1 summarizes the results of altering the initial data $c_o$, $d_o$, $c^B$, $d^B$, $c^N$, $d^N$ and b by translations parametrized by the variable $\delta$. Observe that except for case 6, the interval of variation of $\delta$, for which the basis remains optimal, are specified by linear inequalities in $\delta$. These results are derived by substitution in (1) and the usual arguments of linear programming sensitivity analysis.

If a fractional problem solves at an extreme ray r the analysis changes somewhat. If $v = \bar{\bar{f}} = +\infty$ then $cr > 0$, $dr = 0$ and the solution remains

$+\infty$ as long as these conditions are maintained. When $v = \bar{f} < +\infty$ the final auxiliary problem is

$$\bar{v} = v(\bar{f}) = \sup_{x \in F}\{(c_o - \bar{f}d_o) + (c - \bar{f}d)x\} \qquad \text{(A2)}$$

Let $x^o$ be the optimal extreme point solution to this problem and let B be the corresponding basis. B will be optimal whenever it is feasible and

$$\bar{t}^N = \bar{c}^N - \bar{f}\bar{d}^N = (c^N - \bar{f}d^N) - \pi A^N \leq 0 \qquad (2)$$

The optimal value to (A2) determined by this solution will be nonpositive if

$$\bar{v} = v(\bar{f}) = (c_o - \bar{f}d_o) + (c - \bar{f}d)x^o = (c_o - \bar{f}d_o) + \pi b \leq 0 \qquad (3)$$

Consequently, the ray r remains optimal whenever (2) and (3) are satisfied after the data change. The reader should note that the optimal basis B to (A2) can change, yet maintaining $\bar{v} \leq 0$ and r optimal. Considering only (2) and (3) provides conservative bounds for $\bar{f}$ to remain optimal. These can be read directly from the final LP tableau to (A2). When the optimal basis B changes, the sensitivity analysis can be continued by pivoting to determine the new optimal basis.

Table 2 summarized the results when the problem "solves" at an extreme ray with value $\bar{f} \leq +\infty$ . The changes that involve a variation in $d(x) = d_o + dx$ include also the condition $d(x) > 0$ in an appropriate form, i.e., for $-d(\bar{\bar{x}}) < \delta < -d(\bar{x})$ or $\delta_2 < \delta < \delta_1$ the problem becomes unbounded due to the creation of feasible points $x^1$, $x^2$ such that $d(x^1) > 0$ and $d(x^2) < 0$ . In the last case of table 2 the condition $d_o + dx > 0$ can be controlled by solving simultaneously the problem

$$s(\delta) = \min_{x \in F(\delta)} d(x)$$

where $F(\delta)$ is the set $\{x \epsilon R^n: x \geq 0 \quad Ax \leq b + \delta e_p\}$ and $e_p$ is the unit p vector.

When $\bar{f} = +\infty$ cases 11) and 13) of Table 2 present special characteristics.

<u>Case 11 and 13</u>: When j is such that $r_j = 0$ in either case, $\hat{d}r = dr + \delta r_j = 0$ and the problem remains unbounded. However, if $r_j \neq 0$, $\hat{d}r \neq 0$ for any $\delta \neq 0$ so that the extreme ray does not continue to give $\bar{f} = +\infty$. In this instance, it can be shown that the problem remains unbounded for $\delta_2 < \delta \leq \delta_1 = 0$ and a new optimal solution can be found for $\delta > 0$ by solving the following problem

$$\max[c - \frac{cr}{\delta r_j}(d + \delta e_j)x: x \epsilon F]. \tag{4}$$

We omit the details and refer the reader to [3].

| DATA CHANGE | $\Delta \bar{c}_k^N$ | $\Delta \bar{d}_k^N$ | $\Delta f(x^o)$ | Necessary and sufficients conditions for the optimal basis to remain optimal. $x^o$ is the optimal extreme point. |
|---|---|---|---|---|
| 1) $\Delta c_o = \delta$ | 0 | 0 | $\frac{\delta}{d(x^o)}$ | $\bar{t}^N - \frac{\delta}{d(x^o)} \bar{d}^N \leq 0$ |
| 2) $\Delta d_o = \delta$ | 0 | 0 | $\frac{-f(x^o)\delta}{d(x^o) + \delta}$ | $\bar{t}^N + \frac{\delta f(x^o)\bar{d}^N}{d(x^o) + \delta} \leq 0$ <br> $\delta > - d(\bar{x})$ |
| 3) $\Delta c_j = \delta$ ($x_j^o$ nonbasic) | $\delta$ for $k = j$ <br> 0 for $k \neq j$ | 0 | 0 | $\bar{t}_j^N + \delta \leq 0$ |
| 4) $\Delta d_j = \delta$ ($x_j^o$ nonbasic) | 0 | $\delta$ for $k = j$ <br> 0 for $k \neq j$ | 0 | $\bar{t}_j^N - \delta f(x^o) \leq 0$ <br> $\delta > \delta_1$ |
| 5) $\Delta c_j = \delta$ ($x_j^o$ ith basic variable) | $-\delta B_i^{-1} . A^N$ | 0 | $\frac{\delta x_j^o}{d(x^o)}$ | $\bar{t}^N - \delta B_i^{-1} A^N - \frac{\delta x_j^o}{d(x^o)} \bar{d}^N \leq 0$ |
| 6) $\Delta d_j = \delta$ ($x_j^o$ ith basic variable) | 0 | $-\delta B_i^{-1} . A^N$ | $- \frac{f(x^o)\delta x_j^o}{d(x^o) + \delta x_j^o}$ | $\bar{t}^N + f(x^o)\delta B_i^{-1} . A^N + \frac{f(x^o)\delta x_j^o}{d(x^o) + \delta x_j^o}[\bar{d}^N - \delta B_i^{-1} . A^N] \leq 0$ <br> $\delta > \delta_1$ |
| 7) $\Delta b_i = \delta$ | 0 | 0 | $\frac{\pi_i \delta}{d(x^o) + \pi_i^d \delta}$ | $\bar{t}^N - \frac{\pi_i \delta}{d(x^o) + \pi_i^d} \bar{d}^N \leq 0$ <br> $x^{oB} + \delta B_{\cdot i}^{-1} \geq 0$ and $d_o + dx > 0$ for all $x \varepsilon F(\delta)$ |

TABLE 1

TABLE 2

| DATA CHANGE | $\Delta c_o$ | $\Delta d_o$ | $\Delta c_\ell$ | $\Delta d_\ell$ | $\Delta \bar{f}$ | $\Delta \pi$ | Conditions for the optimal ray $r$ to remain optimal. The optimal value is $\bar{f} \cdot B^{-1} b = x^{oB}$, $x^o$ is the optimal extreme point solving (A1) with $k = \bar{f}$. | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | $\bar{f} = +\infty$ | $\bar{f} < +\infty$<br>$\bar{v} = (c_o - \bar{f} d_o) + \pi b$ , $\bar{t}^N = c^N - \bar{f} d^N - \pi A^N$ |
| 8<br>$.c_o = \delta$ | $\delta$ | 0 | 0 | 0 | 0 | 0 | The problem is unbounded for any $\delta$ | $\bar{v} + \delta \leq 0$ |
| 9<br>$.d_o = \delta$ | 0 | $\delta$ | 0 | 0 | 0 | 0 | $\delta > -d(\bar{x})$ | $\bar{v} - \bar{f}\delta \leq 0$<br>$\delta > -d(\bar{x})$ |
| 10<br>$.g = \delta$<br>nonbasic | 0 | 0 | $\delta \quad \ell = j$<br>$0 \quad \ell \neq j$ | 0 | $\frac{\delta r_j}{dr}$ | $-\frac{\delta r_j}{dr} \pi^d$ | $\hat{c}r = cr + \delta r_j > 0$ | $\bar{t}_i^N + \delta \delta_{ij} - \frac{\delta r_j}{dr} \bar{d}_i^N \leq 0$ for all $i$<br>$\bar{v} - \frac{\delta r_j}{dr}(d_o + \pi^d b) \leq 0$ |
| 11<br>$.d_j = \delta$<br>nonbasic | 0 | 0 | 0 | $\delta \quad \ell = j$<br>$0 \quad \ell \neq j$ | $-\frac{cr\delta r_j}{dr(dr+\delta r_j)}$ | $+\frac{cr\delta r_j}{dr(dr+\delta r_j)} \pi^d$ | $\hat{d}r = dr + \delta r_j = \delta r_j = 0$<br>(The problem remains unbounded if $r_j = 0$) | $\bar{t}_i^N + \frac{cr\delta r_j}{dr(dr+\delta r_j)} (\bar{d}_i^N + \delta \delta_{ij}) - \delta_{ij} \bar{f} \delta \leq 0$ for all $i$<br>$\bar{v} + \frac{cr\delta r_j}{dr(dr+\delta r_j)} (d_o + \pi^d b) \leq 0$ $\quad \delta > \delta_1$ |
| 12<br>$c_j = \delta$<br>$p^{th}$ basic variable | 0 | 0 | $\delta \quad \ell = j$<br>$0 \quad \ell \neq j$ | 0 | $\frac{\delta r_j}{dr}$ | $\delta B_{p.}^{-1} - \frac{\delta r_j}{dr} \pi^d$ | $\hat{c}r = cr = \delta r_j > 0$ | $\bar{t}_i^N - \frac{\delta r_j}{dr} \bar{d}_i^N - \delta B_{p.}^{-1} A_i^N \leq 0$ for all $i$<br>$\bar{v} - \frac{\delta r_j}{dr}(d_o + \pi^d b) + \delta B_{p,}^{-1} b \leq 0$ |
| 13<br>$.d_j = \delta$<br>$p^{th}$ basic variable | 0 | 0 | 0 | $\delta \quad \ell = j$<br>$\ell \neq j$ | $-\frac{cr\delta r_j}{dr(dr+\delta r_j)}$ | $-\bar{f}\delta B_{p.}^{-1}$<br>$+\frac{cr\delta r_j}{dr(dr+\delta r_j)}(\pi^d + \delta B_{p.}^{-1})$ | $\delta r_j = 0$ | $\bar{t}_i^N + \frac{cr\delta r_j}{dr(dr+\delta r_j)} [\bar{d}_i^N - \delta B_{p.}^{-1} A_i^N] + \bar{f}\delta B_{p.}^{-1} A_i^N \leq 0$<br>$\bar{v} + \frac{cr\delta r_j}{dr(dr+\delta r_j)} [d_o + (\pi^d + \delta B_{p,}^{-1}) b] - \bar{f}\delta B_{p,}^{-1} b \leq 0$<br>$\delta > \delta_1$ |
| 14<br>$.b_p = \delta$ | 0 | 0 | 0 | 0 | 0 | 0 | The problem is unbounded for any $\delta$ such that the feasible region is not empty. | $\bar{v} + \pi_i \delta \leq 0$<br>$x^{oB} + \delta B_{.i}^{-1} \geq 0$<br>$do + dx > 0$ |

## 4. A Primal-Dual Parametric Algorithm

Once a linear fractional problem has been solved, parametric right-hand-side variations can be investigated by a primal-dual algorithm. The linear fractional algorithm terminates by solving the linear program ($v < +\infty$)

$$\max[(c_0 - kd_0) + (c - kd)x : x \geq 0 \text{ and } Ax \leq b] \tag{A2}$$

with $k = v$, the optimal solution to the problem.

Suppose that $g$ is a fixed vector and that the right-hand-side of the equality constraints is parameterized by a scalar $\Theta$ as $b + \Theta g$. For each fixed value of $\Theta$, the optimal solution can be found by first applying the dual simplex algorithm to (A2) with $b$ replaced by $b + \Theta g$, obtaining a feasible solution $x^0$. This solution can then be used as a starting point for an application of the linear fractional algorithm. This approach ignores the dependency of $k$ on $x$ when solving for $x^0$ by the dual simplex algorithm.

In constrast, a primal-dual algorithm can be developed extending ideas of the self-parametric algorithm of linear programming [10] by using both primal and dual simplex pivots to maintain an optimal basis as the parameter $\Theta$ is varied. Suppose that $\bar{x}$ is an optimal extreme point to the fractional problem at $\Theta = 0$ and that $B$ is the corresponding basis. This basis remains optimal as long as it is both

$$\text{primal feasible, i.e., } \bar{b} + \Theta\bar{g} \geq 0 \qquad (\bar{b} + B^{-1}b, \bar{g} = B^{-1}g) \tag{5}$$

and dual feasible, i.e., $\bar{t}^N = \bar{c}^N - f(x(\Theta))\bar{d}^N \leq 0$

where

$$f(x(\Theta)) = \frac{c_0 + c^B\bar{b} + \Theta c^B\bar{g}}{d_0 + d^B\bar{b} + \Theta d^B\bar{g}}$$

For the previous analysis to be valid, we require, in addition, that $d_0 + dx$ remains positive on the feasible region. This may require solving $\min[d_0 + dx]$ s.t. $Ax \leq b + \Theta g$, $x \geq 0$. Below we always assume that this condition is met.

Using the notation and techniques of the previous section, $\bar{t}^N$ as a function of $\Theta$ is easily seen to be

$$\bar{t}^N - \frac{\Theta \pi g}{d(\bar{x}) + \Theta \pi^d g} \bar{d}^N \tag{6}$$

and the dual feasible condition $\bar{t}^N \leq 0$ becomes

$$\Theta[\pi^d g \bar{t}^N - \pi g \bar{d}^N] \leq -\bar{t}^N d(\bar{x}) \tag{7}$$

As long as conditions (5) and (7) hold, the basic feasible solution given by (5) is optimal for the parameterized problem. To study the solution behavior for $\Theta \geq 0$, $\Theta$ is increased until at $\Theta = \Theta_0$ equality occurs in either (5) or (7) and any further increase in $\Theta$ causes one of these inequalities to be violated. Thus, either some basic variable reaches zero or some nonbasic objective coefficient reaches zero. We make the nondegeneracy assumption that exactly one primal or dual condition in (5) or (7) is constraining at $\Theta = \Theta_0$ and distinguish two cases:

<u>Case 1</u>: The $i^{th}$ basic variable reaches zero, i.e., $\bar{b}_i + \Theta_0 \bar{g}_i = 0$ and $\bar{g}_i < 0$. The new value for $\bar{t}^N$ is computed by (6) and then a usual dual simplex pivot is made in row i (if every constraint coefficient in row i is nonnegative, linear programming duality theory [10] shows that the problem is infeasible for $\Theta > \Theta_0$ and the procedure terminates). Since the $i^{th}$ basic variable is zero at $\Theta_0$, this pivot simply re-expresses the extreme point given by (5) at $\Theta = \Theta_0$ in terms of a new basis. Consequently, $k = f(x(\Theta_0))$ does not vary during the pivot.

<u>Case 2(a)</u>: The $j^{th}$ reduced cost $\bar{t}_j^N$ reaches zero. A primal simplex pivot is made in column j (the unbounded situation is considered below). Since $\bar{t}_j^N = 0$ and $\bar{t}_i^N < 0$ for $i \neq j$ at $\Theta = \Theta_0$, after this pivot the resulting basis will be optimal.

By our nondegeneracy assumption, the new basis $B_0$ determined after the pivot in either case will be optimal for $\Theta$ in some interval $[\Theta_0, \Theta_1]$, $\Theta_1 > \Theta_0$. Expressing $b + \Theta g$ for $\Theta \geq \Theta_0$ as $(b + \Theta_0 g) + (\Theta - \Theta_0)g$, the parametric analysis given above can be repeated with $(b + \Theta_0 g)$ replacing b. The procedure will continue in this fashion by increasing $\Theta$ and successively reapplying cases 1 and 2. Since both conditions (5) and (7) are linear in $\Theta$, a given basis will be optimal (i.e., both primal and dual feasible) for some interval of $\Theta$. Evoking nondegeneracy, this implies that no basis will repeat as $\Theta$ increases and establishes finite convergence of the method.

To complete the analysis, let us see how to proceed in case 2 if:

<u>Case 2(b)</u>: Every constraint coefficient in column j is nonpositive, so that no primal pivot can be made. Then an extreme ray r is identified which for $\Theta = \Theta_0$ satisfies

$$f(x(\Theta_0) + \lambda r) \to \frac{cr}{dr} = f(x(\Theta_0)) \quad \text{as } \lambda \text{ approaches } +\infty .$$

Thus at $\Theta = \Theta_0$ the extreme ray r becomes optimal and $k = \frac{cr}{dr}$. Since k does not depend upon $\Theta$, (A2) is now solved by the dual simplex algorithm as a linear program with a parametric right-hand-side. A lemma below will show that the ray r remains optimal as $\Theta$ is increased as long as the problem remains feasible.

Having completed the analysis if the problem solves at an extreme point at $\Theta = 0$, let us now suppose that the extreme ray r is optimal at $\Theta = 0$. The dual simplex procedure of case 2(a) is applied until at $\Theta = \Theta_0$, the objective value reaches zero and further increase of $\Theta$ causes it to become positive. In this case the optimal basis at $\Theta = \Theta_0$ corresponds to an extreme point satisfying $f(x(\Theta)) > \frac{cr}{dr}$ for $\Theta = \Theta_0^+$ and we revert to cases 1 and 2 above with this extreme point to continue the analysis for $\Theta > \Theta_0$.

A final point to note here is that the optimal value of the linear fractional problem as a function of $\Theta$ is quasi-concave. Formally,

<u>Lemma</u>: Let $v(\Theta) = \sup\{\frac{c_0 + cx}{d_0 + dx} : x \geq 0$ and $Ax \leq b + \Theta g\}$ and suppose that $v(0) < +\infty$ (or equivalently that $v(\Theta) < +\infty$ for all $\Theta$). Then $v(\Theta)$ is quasi-concave on $\{\Theta : x \geq 0$ and $Ax \leq b + \Theta g$ is feasible$\}$. That is, for any given scalar k, $\{\Theta : v(\Theta) \geq k\}$ is a convex set.

<u>Proof</u>: Let $F(\Theta) = \{x \varepsilon R^n : x \geq 0$ and $Ax \leq b + \Theta g\}$ and suppose that

$$x^1 \varepsilon F(\Theta^1), \quad x^2 \varepsilon F(\Theta^2)$$

with
$$\frac{c_0 + cx^1}{d_0 + dx^1} \geq \bar{k} \quad \text{and} \quad \frac{c_0 + cx^2}{d_0 + dx^2} \geq \bar{k} .$$

If
$$0 \leq \alpha \leq 1 \quad \text{and} \quad \Theta = \alpha\Theta^1 + (1 - \alpha)\Theta^2 \quad \text{then}$$

$$\frac{c_0 + c[\alpha x^1 + (1 - \alpha)x^2]}{d_0 + d[\alpha x^1 + (1 - \alpha)x^2]} \geq \bar{k} \quad \text{and} \quad \alpha x^1 + (1 - \alpha)x^2 \; \varepsilon \; F(\Theta) .$$

But this implies the desired conclusion, for if $v(\Theta^1) \geq k$ and $v(\Theta^2) \geq k$ then for any $\varepsilon > 0$, $x^1$ and $x^2$ can be found for $\bar{k} = k - \varepsilon$. Thus $v(\Theta) \geq k - \varepsilon$ for any $\varepsilon > 0$, i.e., $v(\Theta) \geq k$, and $\{\Theta : v(\Theta) \geq k\}$ is a convex set. $\square$

Remarks: (1) This lemma is valid for multiparameter variations. That is, the same proof applies if $\Theta$ is a vector, and g a given matrix so that $\Theta g = \Theta_1 g^1 + \Theta_2 g^2 + \dots + \Theta_k g^k$ for vectors $g^1, \dots, g^k$. In fact, the result is also valid in infinite dimensional spaces.

(2) The lemma also shows that the extreme ray maximizing $\frac{cr}{dr}$ is optimal for all $\Theta$ in some intervals $(-\infty, \Theta_0]$, $[\Theta_1, -\infty)$ (possibly $\Theta_0 = -\infty$ and or $\Theta_1 = +\infty$) as long as the problem remains feasible. This is a direct result of $v(\Theta) \geq \frac{cr}{dr}$ and quasi-concavity of $v(\Theta)$.

Corollary: Suppose $g \geq 0$. Then $v(\Theta)$ is nondecreasing in its argument $\Theta$ and if an extreme ray r is optimal for some $\Theta$ there is a $\Theta_0 \geq -\infty$ such that r is optimal for $\Theta \varepsilon (-\infty, \Theta_0)$.

Proof: $F(\Theta) \subseteq F(\Theta_1)$ for $\Theta_1 > \Theta$ so $v(\Theta_1) \geq v(\Theta)$. The last conclusion is a consequence of the theorem.

Of course if $g \leq 0$ a similar result can be stated. These results include, as a special case, variations in only one component of the right-hand-side.

Example: The following example illustrates many of the above results. For simplicity, we present only the problem geometry, omitting the detailed pivoting calculations.

$$\max\{f(x) = \frac{n(x)}{d(x)} = \frac{-x_1 + 5}{x_2}\}$$

subject to

$$-x_1 + x_2 \geq 0 - 2\Theta$$

$$x_1 + x_2 \geq 4$$

$$x_1 \geq 1 + 2\Theta$$

$$x_1 \leq 6$$

$$x_1 \geq 0, \; x_2 \geq 0$$

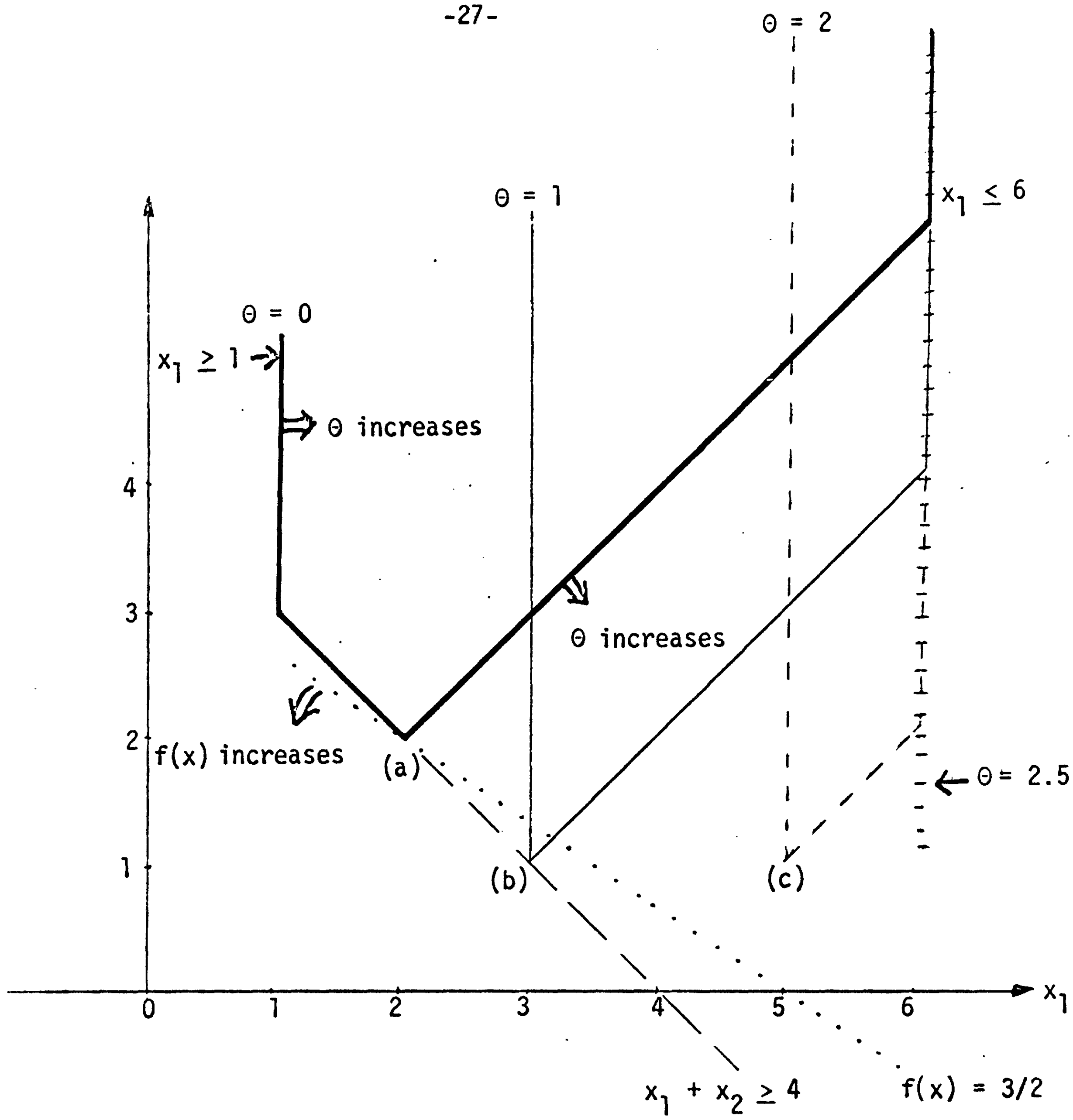
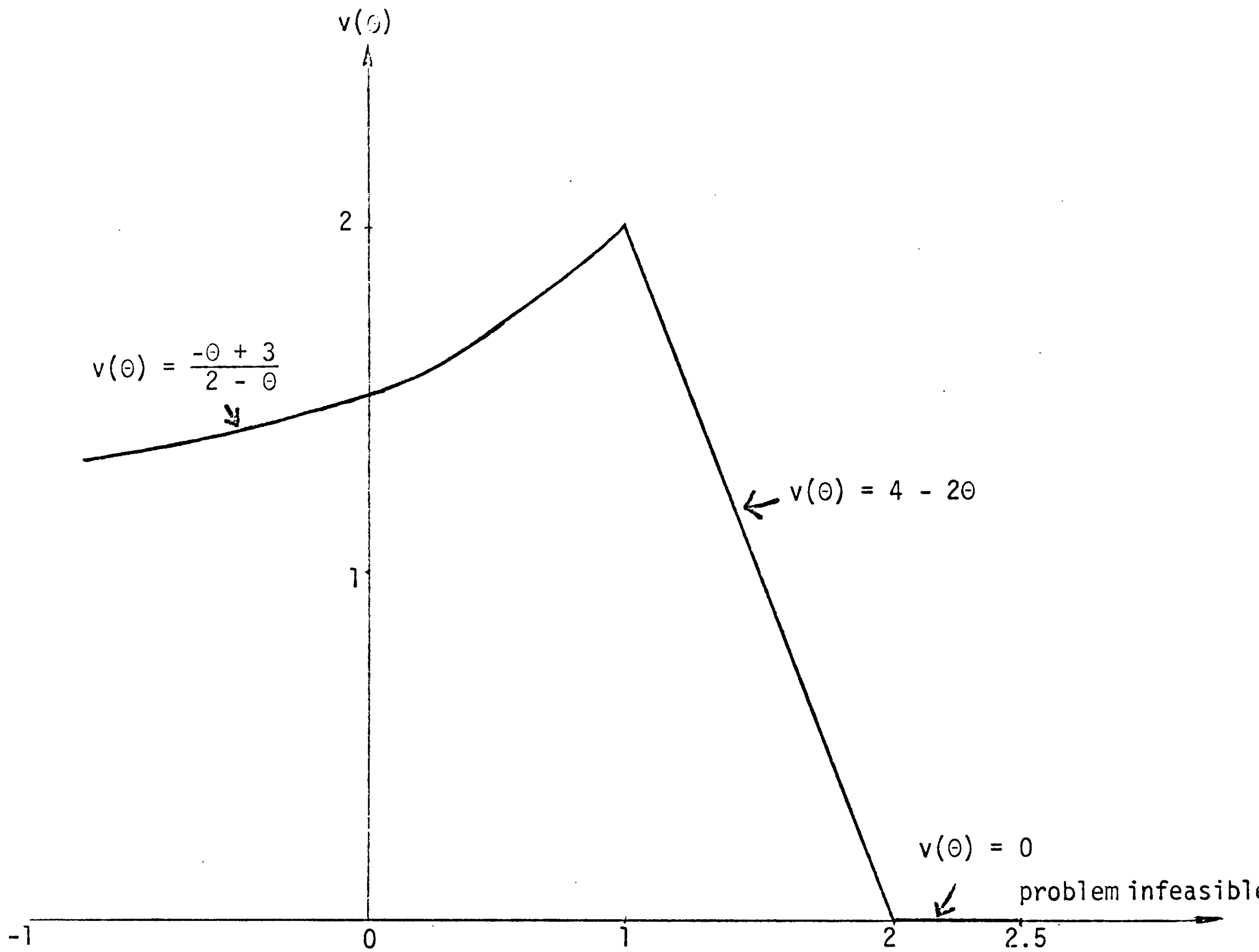

Θ = 2
Θ = 1
$x_1 \leq 6$
Θ = 0
$x_1 \geq 1$
Θ increases
Θ increases
f(x) increases
(a)
(b)
(c)
Θ = 2.5
0
1
2
3
4
5
6
$x_1$
1
2
3
4
$x_1 + x_2 \geq 4$
f(x) = 3/2

Figure 1



v(Θ)
2
1
$v(\Theta) = \frac{-\Theta + 3}{2 - \Theta}$
$v(\Theta) = 4 - 2\Theta$
$v(\Theta) = 0$
problem infeasible
-1
0
1
2
2.5


Figure 2

The feasible region is indicated for $\Theta = 0,1,2$ and 2.5 by respectively the bold faced region, solid region, dashed region and hatched region on figure 1. As shown in [4], the region where $f(x) = k$ is given by a hyperplane [in this case a line in $(x_1,x_2)$ space] and as k varies this hyperplane rotates about the point where $n(x) = d(x) = 0$ .

For $\Theta = 0$, the solution is at point (a) and the basis is determined by equality of the first two constraints. As $\Theta$ increases (or for $\Theta \leq 0$) this basis remains optimal until at point (b) increasing $\Theta$ above 1 causes it to be infeasible. A dual simplex pivot is made to replace the third constraint's surplus variable in the basis with the surplus variable of the second constraint.

The new basis remains optimal until at point (c) with $\Theta = 2$ the extreme ray $r = (0,1)$ becomes optimal. This ray is optimal for $2 \leq \Theta \leq 2.5$. For $\Theta > 2.5$, the problem is infeasible.

By using the tight constraints for $\Theta \leq 2$ to solve for $x_1$ and $x_2$ in terms of $\Theta$ and noting that $\frac{cr}{dr} = 0$ for $r = (0,1)$, the optimal objective value $v(\Theta)$ is plotted in figure 2. It is quasi-concave.

One immediate application of this primal-dual algorithm is for branch and bound when integrality conditions are imposed upon the variables of a linear fractional model. If, for example, $x_j$ is restricted to be an integer and solving problem P without the integrality conditions gives $x_j$ basic at value 3.5, two new problems $P_1$ and $P_2$ are constructed by adding, respectively, the constraints $x_j \leq 3$ and $x_j \geq 4$ . Taking $P_1$ for example, we can suppose that the problem was initially formulated with the $(m + 1)^{\underline{st}}$ constraint $x_j + s_{m+1} = 3 + \Theta$ . Eliminating the basic variable from this constraint gives an updated equation with right-hand-side equal to $(-.5 + \Theta)$. For $\Theta \geq .5$ the optimal basis to P together with $s_j$ forms an optimal basis to the parameterized problem and the parametric algorithm can be applied to decrease

$\Theta$ to 0 and solve the modified problem. A similar procedure is applied to $P_2$ and other modified problems generated by the branching rules. In every other way, the branch and bound procedure is the same as that for (mixed) integer linear programs and all the usual fathoming tests can be applied.

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