A NOTE ON THE PRIMAL-DUAL AND OUT-OF-KILTER ALGORITHMS FOR NETWORK OPTIMIZATION PROBLEMS

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In the literature on network optimization problems, there is often an equivalence or near equivalence assumed between the primal-dual and out-of-kilter network algorithms (see Dantzig [2; Chapter 20], Glover et al [6], Jewell [10], Simonnard [14]). Although both algorithms make iterative use of the efficient maximal flow labeling algorithm, the purpose of this note is to expose a fundamental asymmetry between them. Moreover, the primal-dual algorithm has the capability, which the out-of-kilter algorithm appears not to possess, of being directly approximated by subgradient optimization methods. Thus, it is easy and natural to construct a hybrid network optimization algorithm consisting of the primal-dual algorithm and the essentially heuristic subgradient optimization methods. These latter methods can be used, however, to provide an advanced dual solution for starting the out-of-kilter algorithm as implemented by Barr, Glover and Klingman [1], and Glover, Karney and Klingman [6].

Subgradient optimization methods were first used successfully by Held and Karp in [8] to approximately solve the traveling salesman problem, and has subsequently been used successfully on assignment problems (Held, Wolfe and Crowder [9], set partitioning and covering problems (Marsten, Northup and Shapiro [12]), and scheduling problems (Fisher [3]).

Let $G = [\underline{\bar{N}},\alpha]$ denote a directed network with node set $\underline{\bar{N}} = \{1,\ldots,m\}$ and arc set α consisting of arcs (i,j) for some $(i,j)\varepsilon\underline{\bar{N}}$. For convenience, we will work with a special class of network optimization problems defined on the network G called <u>circulation</u> problems. This problem is

$$z^* = \min \sum_{(i,j)\varepsilon\alpha} c_{ij}x_{ij}$$

$$\text{s.t.} \sum_{(i,j)\varepsilon\alpha} x_{ij} - \sum_{(k,i)\varepsilon\alpha} x_{ki} = 0, \quad i=1,\ldots,m, \tag{1}$$

$$\ell_{ij} \le x_{ij} \le q_{ij} \text{ for all } (i,j)\varepsilon\alpha$$

where $\ell_{ij} \ge 0$ for all $(i,j)\varepsilon\alpha$ and all the c_{ij}, ℓ_{ij}, q_{ij} coefficients are assumed to be integer. A wide variety of network optimization problems can be converted to this form.

Listed below are the primal-dual complementary slackness conditions for the circulation problem (1). Let $u\varepsilon R^m$ be any vector of dual variables. The dual solution is optimal in the dual to problem (1) and the following primal solution x_{ij}, $(i,j)\varepsilon\alpha$ is also optimal in (1) if and only if

$$\sum_{(i,j)\varepsilon\alpha} x_{ij} - \sum_{(k,i)\varepsilon\alpha} x_{ki} = 0, \quad i=1,\ldots,m, \tag{2a}$$

$$c_{ij} - u_i + u_j < 0 \Rightarrow x_{ij} = q_{ij} \tag{2b}$$

$$c_{ij} - u_i + u_j = 0 \Rightarrow \ell_{ij} \le x_{ij} \le q_{ij} \tag{2c}$$

$$c_{ij} - u_i + u_j > 0 \Rightarrow x_{ij} = \ell_{ij} . \tag{2d}$$

The primal-dual algorithm maintains (2b), (2c), (2d) at each iteration while trying to attain the feasibility condition (2a). The out-of-kilter algorithm takes the opposite approach and maintains only (2a) and seeks to attain the others.

The details of the out-of-kilter algorithm are well known (see Fulkerson [6], Simonnard [15]), and we omit further details except to mention that the maximal flow labelling algorithm is used iteratively to monotonically reduce the infeasibilities or lack of complementary slackness in the arcs (i,j) violating conditions (2.b), (2.c), (2.d). We give more detail about the primal-dual algorithm for the circulation problem (1) in order to show how it can be integrated with subgradient optimization methods.

The primal-dual algorithm starts with any dual solution $u \varepsilon R^m$. The arc set α is partitioned into three sets relative to u

$$\alpha^-(u) = \{(i,j) \; \varepsilon \alpha | c_{ij} - u_i + u_j < 0\},$$

$$\alpha^o(u) = \{(i,j) \; \varepsilon \alpha | c_{ij} - u_i + u_j = 0\},$$

$$\alpha^+(u) = \{(i,j) \; \varepsilon \alpha | c_{ij} - u_i + u_j > 0\}.$$

The variables x_{ij} for (i,j) $\varepsilon \alpha^-(u)$ are set at their upper bounds q_{ij} and the variables x_{ij} for (i,j) $\varepsilon \alpha^+(u)$ are set at their lower bounds ℓ_{ij}. The variables x_{ij}, (i,j) $\varepsilon \alpha^o(u)$ are free to vary between upper and lower bounds and the primal-dual algorithm tries to select them so that the flow equations (2a) are satisfied in which case an optimal solution to (1) has been found.

To this end, define

$$b_i(u) = \sum_{(k,i)\varepsilon\alpha^-(u)} q_{ki} + \sum_{(k,i)\varepsilon\alpha^+(u)} \ell_{ki} - \sum_{(i,j)\varepsilon\alpha^-(u)} q_{ij} -$$

$$\sum_{(i,j)\varepsilon\alpha^+(u)} \ell_{ij} \qquad i=1,\ldots,m.$$

The selection of the $x_{ij}, (i,j)\ \varepsilon\alpha^o(u)$ is according to the phase one network optimization problem

$$\min \sum_{i=1}^{m} y_i^+ + y_i^-$$

$$\text{s.t.} \quad \sum_{(i,j)\varepsilon\alpha^o(u)} x_{ij} - \sum_{(k,i)\varepsilon\alpha^o(u)} x_{ki} + y_i^+ - y_i^- = b_i(u) \qquad (3)$$

$$i=1,\ldots,m,$$

$$\ell_{ij} \le x_{ij} \le q_{ij},\ (i,j)\ \varepsilon\alpha^o(u)$$

$$y_i^+ \geq 0,\ y_i^- \geq 0,\quad i=1,\ldots,m.$$

Problems of this type can be solved by the maximal flow labeling algorithm; e.g., see Johnson [10].

As a result of the upper bound substitutions for $(i,j)\ \varepsilon\alpha^-(u) \cup \alpha^+(u)$ and the maximal flow calculation, we have a primal solution x_{ij} to the circulation problem which satisfies along with u all of the conditions (2) except probably (2a). If this solution satisfies the circulation equations (2a), or equivalently, if the minimal objective function value in (3) is zero, then it is optimal.

When the minimal objective function value in (3) is greater than zero, then by necessity we have an optimal solution $v \neq 0$ to the dual of (3) which satisfies the following conditions

$$x_{ij} = q_{ij} \text{ for } (i,j) \ \varepsilon\alpha^o(u) \text{ and } v_j - v_i < 0 \tag{4a}$$

$$x_{ij} = \ell_{ij} \text{ for } (i,j) \ \varepsilon\alpha^o(u) \text{ and } v_j - v_i > 0 \tag{4b}$$

$$\ell_{ij} \leq x_{ij} \leq q_{ij} \text{ for } (i,j) \ \varepsilon\alpha^o(u) \text{ and } v_j - v_i = 0 \tag{4c}$$

The m-vector v serves as a direction of change of the dual solution u. Specifically, u is replaced by $u + \theta^* v$ where θ^* is the largest positive value of θ consistent with

$$c_{ij} - (u_i + \theta v_i) + (u_j + \theta v_j) \begin{cases} \geq 0 \text{ if } (i,j) \ \varepsilon\alpha^+(u) \\ \leq 0 \text{ if } (i,j) \ \varepsilon\alpha^-(u) \end{cases}$$

If $\theta^* = +\infty$, the circulation problem has no feasible solution.

Suppose $\theta^* < +\infty$ and consider the above analysis at the new dual solution $u + \theta^* v$. By construction, the solution x_{ij}, $(i,j) \ \varepsilon\alpha$, given by (4) plus $x_{ij} = q_{ij}$ for $(i,j) \ \varepsilon\alpha^-(u)$, $x_{ij} = \ell_{ij}$ for $(i,j)\varepsilon\alpha^+(u)$, satisfies the conditions (2), except (2a), at $u + \theta^* v$. To see why this is so, consider x_{ij} defined in (4a). Since $c_{ij} - u_i + u_j = 0$, we have

$c_{ij} - (u_i + \theta v_i) + (u_j + \theta v_j) = \theta(v_j - v_i) < 0$ for all $\theta > 0$ implying $(i,j) \varepsilon \alpha^-(u + \theta^* v)$ and $x_{ij} = q_{ij}$ is the correct setting at $u + \theta^* v$. The same argument shows $(i,j) \varepsilon \alpha^+(u + \theta^* v)$ for x_{ij} defined in (4b) and $(i,j) \varepsilon \alpha^\circ(u + \theta^* v)$ for x_{ij} defined in (4c). Similarly, $\theta^* > 0$ is chosen small enough that, except for one arc, $(i,j) \varepsilon \alpha^-(u)$ implies $(i,j) \varepsilon \alpha^-(u + \theta^* v)$ and $(i,j) \varepsilon \alpha^+(u)$ implies $(i,j) \varepsilon \alpha^+(u + \theta^* v)$. The exceptional arc, say $(s,t) \varepsilon \bar{\alpha}(u)$ is chosen such that $c_{st} - (u_s + \theta^* v_s) + (u_t + \theta^* v_t) = 0$. Thus, $x_{st} = q_{st}$ is a correct setting at $u + \theta^* v$ since the variable prices out zero.

The analysis at $u + \theta^* v$ proceeds by exploiting the variable x_{st} since now it is free to vary in the range $\ell_{st} \le x_{st} \le q_{st}$. In terms of the phase one arc network optimization problem (3), the variable x_{st} is added, it prices out negatively, and the minimization of the artificial variables continues. It can be shown that the maximal flow labeling algorithm can continue from its previous termination by the addition of the arc (s,t). The entire process described above is repeated. The primal-dual algorithm converges because there is a monotone decrease (barring degeneracy) in the phase one objective function value.

Subgradient optimization can also be initiated at any dual solution $u \varepsilon R^m$. A complementary primal solution x_{ij}, $(i,j) \varepsilon \alpha$, is selected according to (2.b), (2.c), (2.d) without recourse to problem (3). This solution is used to calculate the direction of change v of the dual solution u by

$$\begin{aligned} v_i = & \sum_{(k,i)\varepsilon\alpha^-(u)} q_{ki} - \sum_{(i,j)\varepsilon\alpha^-(u)} q_{ij} \\ & + \sum_{(k,i)\varepsilon\alpha^+(u)} \ell_{ki} \quad \sum_{(i,j)\varepsilon\alpha^+(u)} q_{ij} \\ & + \sum_{(k,i)\varepsilon\alpha^\circ(u)} x_{ki} \quad \sum_{(i,j)\varepsilon\alpha^\circ(u)} x_{ij} \end{aligned} \tag{5}$$

$i=1,\ldots,m.$

If $v = 0$, then the given solution x_{ij}, $(i,j)\varepsilon\alpha$, satisfies (2a) and is optimal a fortiori In the usual case when $v \neq 0$, a step is taken to the new dual solution $u + tv$ where the positive scalar t must satisfy certain conditions.

Specifically, let

$$u^{k+1} = u^k + t_k v^k, \qquad k=1,2,\ldots,$$

be the sequence of dual solutions and let x_{ij}^k, $(i,j)\varepsilon\alpha$, be the corresponding sequence of primal solution determined by u^k and the conditions (2b), (2c), (2d), if the positive scalars t_k are chosen such that $t_k \to 0$ and $\sum_{k=1}^{\infty} t_k = +\infty$, then it can be shown ((Poljak [13]) that $\sum_{(i,j)} (c_{ij} - u_i^k + u_j^k) x_{ij}^k$ converges to the minimal objective function value z^* of the circulation problem (1). Finite convergence of $\sum_{(i,j)\varepsilon\alpha} (c_{ij} - u_i^k + u_j^k) x_{ij}^k$ to any value $\bar{z} < z^*$ can be achieved if for all k

$$t_k = \lambda_k \frac{(\bar{z} - \sum_{(i,j)\varepsilon\alpha} (c_{ij} - u_j^k + u_j^k) x_{ij}^k)}{||v^k||^2}$$

where $||v^k||$ denotes Euclidean norm and $\varepsilon < \lambda_k \leq 2$ for $\varepsilon > 0$ (Poljak [14]).

The subgradient optimization approach to solving the circulation problem (1) requires less work at each dual solution u than the primal-dual because it does not require a reoptimization of the maximal flow problem (3) to obtain a new primal solution and direction of change v, and it does not require the calculation of θ^* to obtain the new dual solution $u + \theta^* v$. It does require some sorting of the arcs at each dual solution to determine the sets $\alpha^-(u)$, $\alpha^o(u)$ and $\alpha^+(u)$, but the primal-dual requires some sorting to compute θ^*. The out-of-kilter algorithm requires less sorting of the arcs because it can work with any out-of-kilter arc at each iteration, rather than requiring the most out-of-kilter arc although the latter arc is the most desirable to work on.

The implementation of subgradient optimization requires artistry which we will not go into here but refer the reader to [5, 8, 9, 12]. Note that at each dual solution u, the algorithm has the option of taking a subgradient optimization step or a primal-dual step, depending on recent performance of the two methods on the problem.

References

1. R.S. Barr, F. Glover, D. Klingman, "An Improved Version of the Out-of-Kilter Method and a Comparative Study of Computer Codes," Mathematical Programming, 7, No. 1, 1974, pp 60-86.

2. G.B. Dantzig, Linear Programming and Extensions, Princeton U. Press, 1963.

3. M.L. Fisher, "A Dual Algorithm for the One-Machine Scheduling Problem," Graduate School of Business Report, University of Chicago, 1974.

4. M.L. Fisher and J.F. Shapiro, "Constructive Duality in Integer Programming," SIAM J. on Applied Mathematics, 27. No. 1, 1974, pp 31-52.

5. M.L. Fisher, W.D. Northup and J.F. Shapiro, "Using Duality to Solve Discrete Optimization Problems: Theory and Computational Experience," Working Paper OR 030-74, Operations Research Center, M.I.T., January 1974 (to appear in Mathematical Programming).

6. D.R. Fulkerson, "An Out-of-Kilter Method for Solving Minimal Cost Flow Problems," Journal for the Society for Industrial and Applied Mathematics, 9, No. 1, 1961, pp. 18-27.

7. F. Glover, D. Karney and D. Klingman, "Implementation and Computational Comparisons of Primal, Dual and Primal-Dual Computer Codes for Minimum Cost Network Flow Problems," Networks, 4, No. 3, 1974, pp 191-212.

8. M. Held and R.M. Karp, "The Traveling Salesman Problem and Minimum Spanning Trees: Part II," Mathematical Programming, 1, No. 1, 1971, pp. 6-25.

9. M. Held, P. Wolfe, and H.P. Crowder, "Validation of Subgradient Optimization," Mathematical Programming, 6, No. 1, 1974, pp. 62-88.

10. W.S. Jewell, "Optimal Flow through Networks with Gains," Operations Research, 10, 1962, pp. 476-499.

11. E.L. Johnson, "Programming in Networks and Graphs, Working Paper ORC 65-1, Operations Research Center, University of California, Berkeley, 1965.

12. R.E. Marsten, W.D. Northup and J.F. Shapiro, "Subgradient Optimization for Set Partitioning and Covering Problems," (in preparation).

13. B.T. Poljak, "A General Method for Solving Extremum Problems," Soviet Mathematics Doklady, 8, 1967, pp 5-3-597.

14. B.T. Poljak, "Minimizational of Unsmooth Functionals," U.S.S.R. Computational Mathematics and Mathematical Physics, 9, 1969, pp. 4-29.

15. M. Simonnard, Linear Programming, Prentice-Hall, 1966.