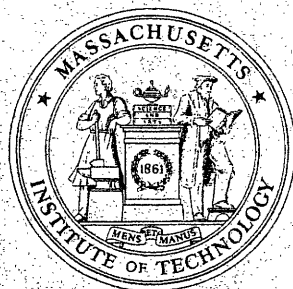


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**MASSACHUSETTS INSTITUTE  
OF TECHNOLOGY**

Extended Vacation Systems and the Universality  
of the M/G/1/K Blocking Formula

by

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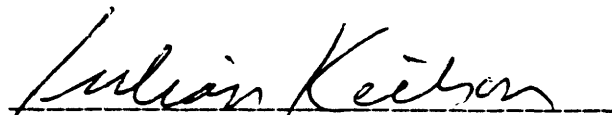
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"This document is a GTE Laboratories Technical Memorandum. It describes interim research results and preliminary conclusions deduced from them. The ideas and views put forth by the author(s) have been reviewed and accepted by the appropriate Department Manager (s)."

  
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## ABSTRACT

A simple blocking formula  $B(K) = (1 - \rho)E_K [1 - \rho E_K]^{-1}$  relates the probability of blocking for the finite capacity M/G/1/K to  $E_K$ , the steady state occupancy tail probability of the same system with infinite capacity. The validity of this formula is demonstrated for M/G/1 vacation systems augmented by an idle state, an umbrella for a host of priority systems and vacation systems related to M/G/1. A class of occupancy level dependent vacation systems introduced are shown to require a variant of this blocking formula.

## EXECUTIVE SUMMARY

For a large class of telecommunication systems it is necessary to know the probability that a call arriving will be blocked when there is insufficient buffer capacity to accommodate it.

Infinite capacity systems are always easier to analyze than their finite capacity counterparts. This underlies the importance of an expression relating  $B(K)$ , the probability of blocking  $B(K)$  for a  $K$  capacity system to the distribution of the number in the system  $N$  for the corresponding infinite capacity system and the system load  $\rho$ .

This paper provides the following exact formula relating  $B(K)$  to the distribution of  $N$  for a large class of systems:

$$B(K) = \frac{(1-\rho) \text{Prob } [N \geq K]}{1 - \rho \text{Prob } [N \geq K]}$$

The class include single server systems with Poisson arrivals with or without different priority classes, clocked schedules (such as that found in the Administrative Processor of the GTD-5), and cyclic service queueing systems.

The paper also provides an efficient recursive algorithm for computing the distribution function  $\text{Prob } [N \geq K]$ .

## Section 1. Introduction and Summary

For a large family of vacation systems with Poisson arrivals (e.g. M/G/1/V), a simple relation has been provided [6] between the blocking probability  $B(K)$  for a finite system capacity  $K$  and the set  $\{P[N_\infty = n]\}$  of ergodic occupancy level probabilities for the same system with infinite capacity. The blocking probability  $B(K)$  is given by the simple formula

$$(1.1) \quad B(K) = \frac{(1 - \rho)E_K}{1 - \rho E_K} .$$

Here  $E_K = P[N_\infty \geq K]$  and  $\rho$  is the system utilization. One also has the scaling relation

$$(1.2) \quad P [N_K = n] = \theta_K P [N_\infty = n], \quad n = 0,1,2,\dots, K-1$$

where  $\theta_K = (1 - \rho E_K)^{-1}$  and  $N_K$  is the ergodic system occupancy for system capacity  $K$ .

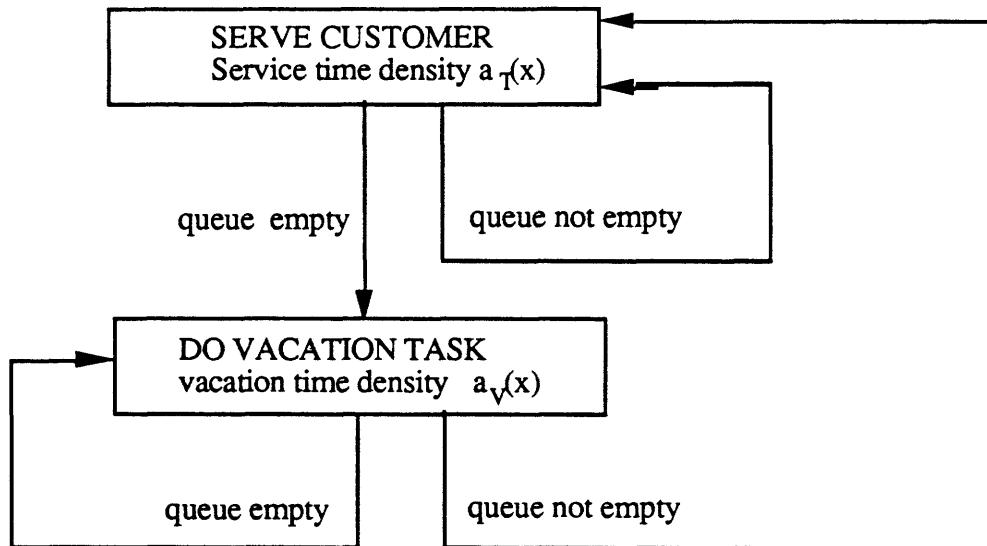
When the distribution of  $N_\infty$  is known, (1.1) and (1.2) then provide the blocking probability for any capacity level  $K$  as well as the distribution of the ergodic system occupancy  $N_K$ . Note that the desired  $B(K)$  is given by  $B(K) = P[N_K = K]$ .

In Section 2 we show that (1.1) and (1.2) are also valid for a richer class of systems, the *extended vacation systems*, which include many of the classical priority and vacation settings. Equations (1.1) and (1.2) require the evaluation of the ergodic system occupancy distribution of the infinite buffer capacity system. In section 3 an efficient recursive algorithm is provided. In Section 4, a second class of occupancy-level dependent systems introduced by Harris and Marchall [3] is examined. For such systems, equation (1.2) still holds and a variant of the blocking formula is found for which (1.1) is an upper bound. In Section 5 a recursive algorithm is provided for the ergodic system occupancy for an important subset of this class.

## Section 2. Extended vacation systems and priority systems.

M/G/1 vacation systems have been discussed in the literature (c.f.[1]). In the M/G/1 vacation system with exhaustive service, as illustrated in Figure 1, the server upon becoming empty

completes i.i.d. vacation tasks iteratively until a queue of one or more regular customers has formed. Regular service is then resumed.



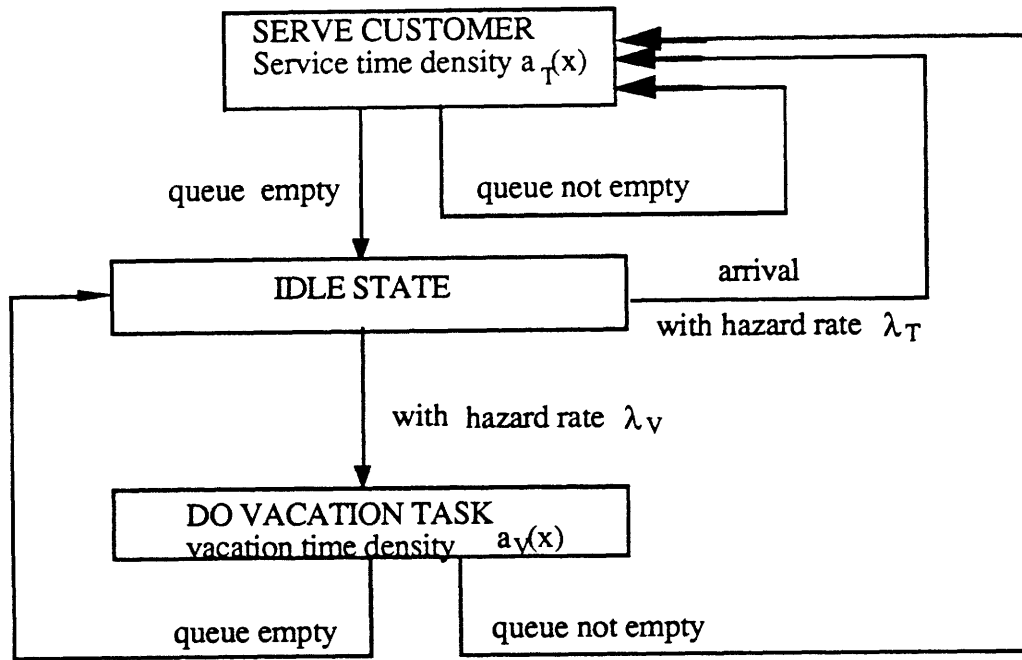
**FIGURE 1: M/G/1 VACATION SYSTEM WITH EXHAUSTIVE SERVICE**

A useful generalization of this M/G/1 vacation system can be obtained by augmenting it with an idle state. The *extended* M/G/1 vacation system resulting is governed by the following rules:

- a) Customers arrive in a Poisson stream at rate  $\lambda_T$ .
- b) At the end of each regular service, another such service is started if any customer is present. Otherwise, the idle state I is entered.
- c) In the idle state I the server is subject to two competing hazard rates  $\lambda_T$  and  $\lambda_V$ . Hazard rate  $\lambda_T$  is associated with arrivals that initiate regular service. Hazard rate  $\lambda_V$  initiates a vacation task.
- d) Vacation task times are i.i.d. After each vacation task, the server again becomes idle if no customers have arrived. If customers have come, regular service is initiated at the end of the vacation task.
- e) The number in the system is  $N(t)$ .

This is illustrated in Figure 2.





**FIGURE 2: THE EXTENDED M/G/1/V VACATION MODEL**

The following four systems all fall within the framework of the extended vacation system.

**A) M/G/1.** This system is governed by arrival rate  $\lambda$  and service time  $T$ , with  $\alpha_T(s) = E[\exp(-sT)]$ .  $N(t)$  is the number in the system.

**B) M/G/1/V with exhaustive service.** This system is governed by arrival rate  $\lambda$ , service time  $T$ , with  $\alpha_T(s) = E[\exp(-sT)]$ , and vacation time  $\bar{V}$  with  $\alpha_V(s) = E[\exp(-s\bar{V})]$ . The vacation time  $\bar{V}$  for this vacation system should not be confused with the vacation time  $V$  for the extended vacation system.  $N(t)$  is the number in the system.

**C) M/G/1 priority system with preempt-resume discipline:** This system is governed by arrival rates  $\lambda_1$  and  $\lambda_2$  and service times  $T_1$  and  $T_2$  with  $\alpha_i(s) = E[\exp(-sT_i)]$ ,  $i = 1, 2$ . Here index 1 denotes the priority class and index 2 the ordinary class. Let  $B_1$  be the priority busy period with  $\sigma_{B_1}(s) = E[\exp(-sB_1)]$ . From Takacs' equation, this satisfies  $\sigma_{B_1}(s) = \alpha_1(s + \lambda_1 - \lambda_1 \sigma_{B_1}(s))$ .  $N_2(t)$  is the number ordinary customers in the system.

**D) M/G/1/V priority system with preempt-resume discipline:** Let the index 1 denote the priority class and let the index 2 the ordinary class. This system is governed by arrival rates  $\lambda_1$  and  $\lambda_2$ , service times  $T_1$  and  $T_2$  with  $\alpha_i(s) = E[\exp(-sT_i)]$ ,  $i = 1, 2$  and a vacation time  $\bar{V}$  with

$\alpha_{\bar{V}}(s) = E[\exp(-s\bar{V})]$ . Index 1 denotes the priority class and index 2 the ordinary class. Let  $B_1$  be the priority busy period with  $\sigma_{B_1}(s) = E[\exp(-sB_1)]$  and  $\sigma_{B_1}(s) = \alpha_1(s + \lambda_1 - \lambda_1 \sigma_{B_1}(s))$  as before.  $N_2(t)$  is the number ordinary customers in the system.

The equivalent parameters and transforms of the extended vacation system for the four systems are displayed in Table 1.

Ext. Vac. System parameters:	$\lambda_T$	$\alpha_T(s)$	$\lambda_V$	$\alpha_V(s)$	$N(t)$
M/G/1	$\lambda$	$\alpha_T(s)$	0	arbitrary	$N(t)$
M/G/1/V, exhaustive service	$\lambda$	$\alpha_T(s)$	$\infty$	$\alpha_{\bar{V}}(s)$	$N(t)$
M/G/1 with PR priorities	$\lambda_2$	$\alpha_2(s + \lambda_1 - \lambda_1 \sigma_{B_1}(s))$	$\lambda_1$	$\alpha_{B_1}(s)$	$N_2(t)$
M/G/1/V with PR priorities	$\lambda_2$	$\alpha_2(s + \lambda_1 - \lambda_1 \sigma_{B_1}(s))$	$\infty$	$\alpha_{\bar{V}}(s + \lambda_1 - \lambda_1 \sigma_{B_1}(s))$	$N_1(t)$

TABLE 1

### Basic relationships

When the ergodic probability of the number in the system for infinite capacity is known, the ergodic probability of the number in the system for finite system capacity  $K$  may be obtained in a few steps. Basic tools for our results are the following two lemmas.

#### Lemma A (cf. [6])

Suppose that:

- $\underline{X}(t)$  is an ergodic multivariate Markov process in continuous time having a partitioned state space  $\mathcal{N} = G \cup B$ ;
- $A \subset G \subset \mathcal{N}$ . The state  $\underline{l} \in G$  is the only regenerative state for entry into  $G$ . These entries have renewal rate  $i(\underline{l})$ ;
- $e(A)$  is the ergodic probability of  $A$  for  $\underline{X}(t)$  and  $T(A)$  is the mean time spent in  $A$  between regenerations at  $\underline{l}$ .

Then

$$(2.1) \quad e(A) = i(\underline{l}) T(A).$$

**Proof:** Let  $M(T)$  be the number of entries into the set  $G$  during  $[0, T]$  and let  $T_{A_j}$  be the time spent in  $A$  between the  $j^{\text{th}}$  and  $(j+1)^{\text{th}}$  regeneration. One has

$$e(A) = \lim_{T \rightarrow \infty} \left\{ \frac{1}{T} \sum_{j=1}^{M(T)} T_{A_j} \right\} = \lim_{T \rightarrow \infty} \left\{ \frac{M(T)}{T} \right\} \lim_{T \rightarrow \infty} \left\{ \frac{1}{M(T)} \sum_{j=1}^{M(T)} T_{A_j} \right\} = i(\mathbf{l}) T(A). \diamond$$

**Lemma B.** Suppose that:

- $\underline{X}_K(t)$  and  $\underline{X}_\infty(t)$  are two multivariate Markov processes having partitioned state spaces  $\mathcal{N}_K = G_K \cup B_K$  and  $\mathcal{N}_\infty = G_\infty \cup B_\infty$  respectively;
- the state  $\mathbf{l}$  is the only regenerative entry state to the sets  $G_K$  and  $G_\infty$ . The ergodic renewal rates at  $\mathbf{l}$  are  $i_K(\mathbf{l})$  and  $i_\infty(\mathbf{l})$  for  $\underline{X}_K(t)$  and  $\underline{X}_\infty(t)$ , respectively;
- for  $A \subset G_K \cap G_\infty$ ,  $T_K(A)$  and  $T_\infty(A)$  are the mean times in the  $A$  between regenerations for  $\underline{X}_K(t)$  and  $\underline{X}_\infty(t)$  respectively;
- $T_K(A) = T_\infty(A)$  and  $\theta_K = i_K(\mathbf{l})/i_\infty(\mathbf{l})$ .

Then the ergodic probabilities of  $A$  for the processes  $\underline{X}_K(t)$  and  $\underline{X}_\infty(t)$ ,  $e_K(A)$  and  $e_\infty(A)$  respectively, are related by

$$(2.2) \quad e_K(A) = \theta_K e_\infty(A).$$

**Proof:** From Lemma A,  $e_K(A) = i_K(\mathbf{l}) T_K(A) = i_K(\mathbf{l}) T_\infty(A) = \theta_K i_\infty(\mathbf{l}) T_\infty(A) = \theta_K e_\infty(A). \diamond$

### Analysis of the extended vacation system

The extended vacation system with system capacity  $K$  can be viewed as a multivariate Markov process  $\underline{X}_K(t) = (N_K(t), X_K(t), J_K(t))$  with state space  $I \cup \{(n, x, j) : 0 \leq n \leq K; 0 < x; j = S, V, I\}$ . Here  $N_K(t)$  is the number of customers in the system,  $J_K(t) = S$  if a service is in progress,  $J_K(t) = V$  if a vacation is in progress, and  $J_K(t) = I$  if the server is idle.  $X_K(t)$  is the time since the last service or vacation began when  $J_K(t) = S$  or  $V$ . If  $J_K(t) = I$  then  $X_K(t) = 0$ .

To discuss the extended vacation system the following notation will be employed:

$$e_n^{(K)} = P[N_K(\infty) = n] \quad e_V^{(K)} = P[J_K(\infty) = V] + P[J_K(\infty) = I] \quad e_K(A) = P[\underline{X}_K(\infty) \in A]$$

**Proposition 1.** For the extended vacation system with capacity  $K$  and the associated parent vacation system with infinite capacity

$$(2.3) \quad e_n^{(K)} = \theta_K e_n^{(\infty)}, \quad n = 0, 1, 2, \dots, K-1,$$

and

$$(2.4) \quad e_K(I) = \theta_K e_{\infty}(I)$$

where  $\theta_K = i_K(\mathbf{l})/i_{\infty}(\mathbf{l})$  for  $\mathbf{l} = (K-1, 0, S)$ .

**Proof:** Consider the extended vacation system with capacity  $K$  whose state space is partitioned  $\mathcal{N}_K = G_K \cup B_K$  and the corresponding infinite capacity system whose state space is partitioned  $\mathcal{N}_{\infty} = G_{\infty} \cup B_{\infty}$ . For both systems the only state at which entry can occur into the common set  $G = G_K = G_{\infty} = \{(n, x, j) : 0 \leq n \leq K-1\}$  is  $(K-1, 0, S)$ . Since the dynamics of the two systems are the same on  $G$  for both the processes  $\underline{X}_K(t)$  and  $\underline{X}_{\infty}(t)$  one has  $T_K(A) = T_{\infty}(A)$  for all  $A \subset G$ . The proposition then follows from Lemma B by setting  $A$  to  $\{(m, x, j) : m = n\}$  or  $A$  to  $I$ . ♦

**Proposition 2.**

$$(2.5) \quad e_V^{(K)} = \theta_K e_V^{(\infty)}$$

where

$$\theta_K = i_K(\mathbf{l}) / i_{\infty}(\mathbf{l}) \text{ for } \mathbf{l} = (K-1, 0, S).$$

**Proof:** Let  $G = G_K = G_{\infty} = I \cup \{(n, x, j) : j = V\}$ . Using the observation that  $T_K(I) = T_{\infty}(I)$ , Lemma A, and (2.4) we find that

$$i_K(I) = i_K(I)T_K(I)/T_{\infty}(I) = e_K(I) / T_{\infty}(I) = \theta_K e_{\infty}(I) / T_{\infty}(I) = \theta_K i_{\infty}(I).$$

Observe that the mean time in the set  $G$  between regenerations is the same for the processes  $\underline{X}_K(t)$  and  $\underline{X}_{\infty}(t)$ , i.e.,

$$T_K(G) = T_{\infty}(G).$$

But, if  $i_K(I) = \theta_K i_{\infty}(I)$  and  $T_K(G) = T_{\infty}(G)$  then, from Lemma B,

$$e_V^{(K)} = e_K(G) = \theta_K e_{\infty}(G) = \theta_K e_V^{(\infty)}. \diamond$$

**Proposition 3**

$$(2.6) \quad \rho (1 - e_K^{(K)}) = 1 - e_V^{(K)},$$

$$(2.7) \quad e_V^{(\infty)} = 1 - \rho,$$

and

$$(2.8) \quad \sum_{n=0}^{\infty} e_n^{(K)} = 1,$$

where  $\rho = -\lambda_T \alpha_T'(0)$ .

**Proof:** Equation (2.6) is true because both sides are equal to the fraction of time the server is providing service to customers. Equation (2.7) follows from equation (2.6) and the observation that  $\lim_{K \rightarrow \infty} e_K^{(K)} = 0$ . Equation (2.8) states that there is unit mass in the state space. ♦

We can now derive the blocking formula.

**Theorem 1:** The blocking probability  $B(K)$  is

$$(2.9) \quad B(K) = e_K^{(K)} = \frac{(1 - \rho)E_K}{1 - \rho E_K}$$

where  $E_K = P[N_{\infty} \geq K] = \sum_{n=K}^{\infty} e_n^{(\infty)}$  and  $\rho = -\lambda_T \alpha_T'(0)$  is the system utilization.

**Proof:** From (2.3) and (2.8),

$$(2.10) \quad \rho \theta_K (1 - E_K) = \rho \theta_K \sum_{n=0}^{K-1} e_n^{(\infty)} = \rho \sum_{n=0}^{K-1} e_K^{(K)} = \rho (1 - e_K^{(K)})$$

Moreover, from (2.5), (2.6), and (2.7)

$$(2.11) \quad \rho (1 - e_K^{(K)}) = 1 - \theta_K e_V^{(\infty)} = 1 - \theta_K (1 - \rho).$$

Hence,  $\rho \theta_K (1 - E_K) = 1 - \theta_K (1 - \rho)$  and one has

$$(2.12) \quad \theta_K = (1 - \rho E_K)^{-1}.$$

Finally from (2.11) and (2.12) equation (2.9) follows. ♦

### Section 3: A recursive algorithm for the occupancy tail probabilities $E_K$ for the extended vacation system

Equation (2.9) expresses the blocking probability  $B(K)$  for the extended vacation system with capacity  $K$  in terms of the ergodic occupancy tail probabilities  $E_K$  for the parent system with infinite capacity. In this section, a recursive equation is derived to compute  $E_K$ . The recursion is applicable to all of the systems in Table 1 because they belong to the extended vacation family.

The process  $N(t)$  in the extended vacation system satisfies the hypotheses required for the Fuhrmann- Cooper decomposition of [2] or [7]. From [7], the probability generating function of the number in the system is given by

$$(3.1) \quad \pi_S(u) = \sum_{n=0}^{\infty} e_n^{(\infty)} u^n = \frac{(1-\rho)\alpha_T(\lambda_T - \lambda_T u)\pi_B(u)}{1 - \alpha_T^*(\lambda_T - \lambda_T u)}$$

where

$$e_n^{(\infty)} = \text{Prob} [N_{\infty} = n],$$

$$\alpha_T(s) = E[\exp(-sT)],$$

$$\alpha_T^*(s) = - (1 - \alpha_T(s)) / (s\alpha_T'(0)),$$

and  $\pi_B(u)$  is the conditional p.g.f.

$$\pi_B(u) = E[u^{N_S(t)} | J = V \text{ or } I].$$

The p.g.f.  $\pi_B(u)$  is computed as follows using the arguments of [7]: Whenever the system enters the state  $J = V$  the number in the system is 0. Hence, at an arbitrary point in time when  $J=V$  the number in the system is equal in distribution to the number of Poisson arrivals during the backward (or forward) recurrence time of the vacation duration and the corresponding p.g.f. is  $\alpha_V^*(\lambda_T - \lambda_T u)$ . Whenever  $J = I$ , the p.g.f. of the number in the system is simply 1. One then has

$$\pi_B(u) = \text{Prob} [J=I | J=\{I \text{ or } V\}] + \text{Prob} [J=V | J=\{I \text{ or } V\}] \alpha_V^*(\lambda_T - \lambda_T u).$$

Hence

$$(3.2) \quad \pi_B(u) = \frac{e_{\infty}(I)}{e_{\infty}(I) + e_{\infty}(V)} + \frac{e_{\infty}(V)}{e_{\infty}(I) + e_{\infty}(V)} \alpha_V^*(\lambda_T - \lambda_T u)$$

where  $\alpha_V^*(s)$  is the forward recurrence time of  $V$ . The ergodic renewal rate into the set  $\{(n, x, j) : j = V\}$  is  $\lambda_V e_{\infty}(I)$  and the mean time in the vacation state is  $E[V]$ . Hence, from (2.1),  $e_{\infty}(V) = \lambda_V e_{\infty}(I) E[V]$ . Equation (3.2) then simplifies to

$$(3.3) \quad \pi_B(u) = \frac{1 + \lambda_V E[V] \alpha_V^*(\lambda_T - \lambda_T u)}{1 + \lambda_V E[V]}$$

From (3.1),

$$(3.4) \quad \pi_S(u) = (1 - \rho) \alpha_T(\lambda_T - \lambda_T u) \pi_B(u) + \rho \alpha_T^*(\lambda_T - \lambda_T u) \pi_S(u)$$

Inverting (3.4) and noting that multiplication corresponds to convolution,

$$(3.5) \quad e_n^{(\infty)} = (1 - \rho) \sum_{m=0}^n [a_T(m) p_B(n-m)] + \rho \sum_{m=0}^n [a_{T^*}(n-m) e_m^{(\infty)}]$$

where  $e_n^{(\infty)}$ ,  $p_B(n)$ ,  $a_T(m)$ , and  $a_{T^*}(m)$  are the ergodic probabilities associated with the pgf's  $\pi_Q(u)$ ,  $\pi_B(u)$ ,  $\alpha_T(\lambda_T - \lambda_T u)$ , and  $\alpha_T^*(\lambda_T - \lambda_T u)$ , respectively.

From (3.5),

$$(3.6) \quad e_0^{(\infty)} = \frac{(1 - \rho) a_T(0) p_B(0)}{1 - \rho a_{T^*}(0)} = (1 - \rho) p_B(0)$$

and, for  $n=1,2,\dots$ , we have the following recursive relationship

$$(3.7) \quad e_n^{(\infty)} = \frac{(1 - \rho) \sum_{m=0}^n [a_T(m) p_B(n-m)] + \rho \sum_{m=0}^{n-1} [a_{T^*}(n-m) e_m^{(\infty)}]}{1 - \rho a_{T^*}(0)}$$

Note that  $E_n = \text{Prob} [N_\infty \geq n]$  satisfies the equation

$$(3.8) \quad e_n^{(\infty)} = E_n - E_{n+1}$$

so, from (3.7) and (3.8),

$$E_n - E_{n+1} = \frac{(1 - \rho) \sum_{m=0}^n [a_T(m) p_B(n-m)] + \rho \sum_{m=0}^{n-1} [a_{T^*}(n-m) (E_m - E_{m+1})]}{1 - \rho a_{T^*}(0)}$$

Hence, we have the recursion in  $E_m$ , for  $n = 1,2,\dots$

$$(3.9) \quad E_{n+1} = E_n - \frac{(1-\rho) \sum_{m=0}^n [a_T(m) p_B(n-m)] + \rho \sum_{m=0}^{n-1} [a_{T^*}(n-m)(E_m - E_{m+1})]}{1 - \rho a_{T^*}(0)},$$

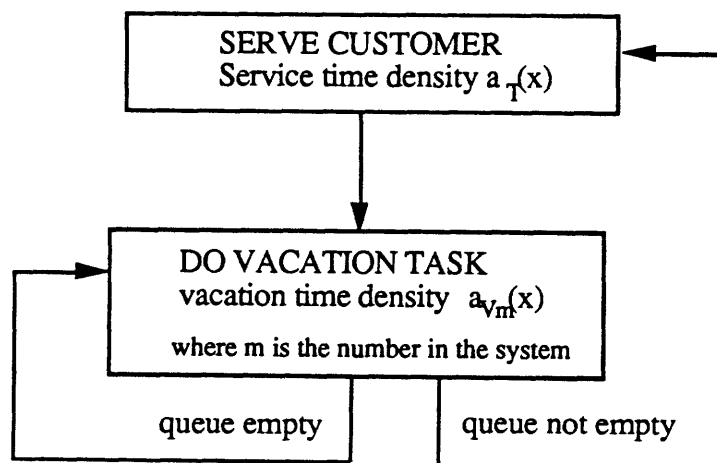
$$(3.10) \quad E_0 = \text{Prob} [N_\infty \geq 0] = 1,$$

and, from (3.6),

$$(3.11) \quad E_1 = 1 - e_0^{(\infty)} = 1 - (1 - \rho) p_B(0).$$

#### Section 4. The state dependent vacation system (SDVS)

In [3], a state dependent vacation systems (SDVS) was introduced by Harris and Marchall which provides a common framework for: a) M/G/1; b) M/G/1/V with single service, exhaustive service, or Bernoulli service; c) M/G/1/V with exhaustive service and preempt-resume priorities; d) M/G/1/V with a Bernoulli schedule and Preempt-Resume priorities. This SDVS system is governed by an arrival rate  $\lambda$ , a service time T with  $\alpha_T(s) = E[\exp(-sT)]$  and a set of vacation times  $\bar{V}_m$  with  $\alpha_{\bar{V}_m}(s) = E[\exp(-s\bar{V}_m)]$ . A vacation  $\bar{V}_m$  is initiated after a service completion if the system has m customers and a vacation  $\bar{V}_0$  is initiated after either a service or vacation completion if the system is empty. The system is illustrated in Figure 3. In [3] the ergodic system occupancy is examined for the infinite capacity case. It will be shown next that the blocking probability formula, equation (1.1), can be generalized to the SDVS system with the original formula (1.1) serving as an upper bound.



**FIGURE 3: THE M/G/1 STATE DEPENDENT VACATION SYSTEM (SDVS)**

The above systems fit into the SDVS model in keeping with the following chart:



SDVS parameters:	$\lambda$	$\alpha_T(s)$	$\alpha_{V_m}(s)$	$N(t)$
M/G/1	$\lambda$	$\alpha_T(s)$	1	$N(t)$
M/G/1/V with exh. Service	$\lambda$	$\alpha_T(s)$	1, $m \neq 0$ ; $\alpha v(s)$ , $m=0$	$N(t)$
M/G/1/V Bernoulli schedule	$\lambda$	$\alpha_T(s)$	$p+(1-p)\alpha v(s)$	$N(t)$
M/G/1/V with single service	$\lambda$	$\alpha_T(s)$	$\alpha v(s)$	$N(t)$
M/G/1/V with PR priorities	$\lambda_2$	$\alpha_2(s + \lambda_1 - \lambda_1 \sigma_{B_1}(s))$	$\alpha v(s + \lambda_1 - \lambda_1 \sigma_{B_1}(s))$	$N_2(t)$
M/G/1/V Bern. sch. with PR	$\lambda_2$	$\alpha_2(s + \lambda_1 - \lambda_1 \sigma_{B_1}(s))$	$p+(1-p)\alpha v(s + \lambda_1 - \lambda_1 \sigma_{B_1}(s))$	$N_2(t)$

TABLE 2

Note that the extended vacation model, considered in section 2, does *not* fall into the framework of the SDVS because the extended vacation model has an idle state whereas the SDVS system does not. However, as indicated in Section 5, the homogeneous SDVS system (where  $V_m = V$  for all  $m$ ) is closely related to the extended vacation model.

The analysis of SDVS again requires a partitioning of the state state  $\mathcal{N} = G \cup B$ . However, now Lemmas A and B must be generalized to accommodate *multiple* regeneration points into  $G$ . With this generalization Propositions 1 and 2 can be modified to address the SDVS.

**Lemma A'.**

Suppose that:

- $\mathbf{X}(t)$  is an ergodic multivariate Markov process in continuous time having a partitioned state space  $\mathcal{N} = G \cup B$ .
- $A \subset G \subset \mathcal{N}$ . The states  $\mathbf{r}_j \in G$  are regenerative states for entry into  $G$  from its exterior  $B$ . These entries have renewal rate  $i(\mathbf{r}_j)$ .
- $e(A)$  is the ergodic probability of  $\mathbf{X}(t)$  of  $A$  and  $T_j(A)$  is the mean time spent in  $A$  between a regeneration at  $\mathbf{r}_j$  and the next regeneration .

Then

$$(4.1) \quad e(A) = \sum_j i(\mathbf{r}_j) T_j(A).$$

**Proof:** Let  $M_j(T)$  be the number of entries into the set  $G$  via  $\mathbf{r}_j$  during  $[0, T]$  and let  $T_{ji}(A, G)$  be the  $j^{\text{th}}$  time spent in  $A$  between the  $i^{\text{th}}$  regeneration starting at  $\mathbf{r}_j$  and the next regeneration. Then

$$\begin{aligned}
e(A) &= \lim_{T \rightarrow \infty} \left\{ \frac{1}{T} \sum_j \sum_{i=1}^{M_j(T)} T_{ji}(A, G) \right\} \\
&= \sum_j \left[ \lim_{T \rightarrow \infty} \left\{ \frac{M_j(T)}{T} \right\} \lim_{T \rightarrow \infty} \left\{ \frac{1}{M_j(T)} \sum_{i=1}^{M_j(T)} T_{ji}(A, G) \right\} \right] \\
&= \sum_j i(\mathbf{r}_j) T_j(A, G) \quad \blacklozenge
\end{aligned}$$

**Lemma B'.** Suppose that:

a)  $\underline{X}_K(t)$  and  $\underline{X}_\infty(t)$  are two multivariate Markov processes having partitioned state spaces  $\mathcal{N}_K = G_K \cup B_K$  and  $\mathcal{N}_\infty = G_\infty \cup B_\infty$  respectively;

b) the sets of regenerative entry states for  $G_K$  and  $G_\infty$  are identical, i.e. every entry state of  $G_K$  is an entry state of  $G_\infty$  and vice versa. For this common set  $\{\mathbf{r}_j\}$ , the renewal rates at ergodicity are  $i_K(\mathbf{r}_j)$  and  $i_\infty(\mathbf{r}_j)$  for  $\underline{X}_K(t)$  and  $\underline{X}_\infty(t)$ , respectively;

c) for  $A \subset G_K \cap G_\infty$ , let  $T_{Kj}(A)$  and  $T_{\infty j}(A)$  be the mean times spent in  $A$  between a regeneration at  $\mathbf{r}_j$  and the next regeneration for  $\underline{X}_K(t)$  and  $\underline{X}_\infty(t)$  respectively;

d) For all  $j$ ,  $T_{Kj}(A) = T_{\infty j}(A)$  and  $i_K(\mathbf{r}_j) = \theta_K i_\infty(\mathbf{r}_j)$

Then the ergodic probabilities of  $A$  for the processes  $\underline{X}_K(t)$  and  $\underline{X}_\infty(t)$ ,  $e_K(A)$  and  $e_\infty(A)$  respectively are related by

$$(4.2) \quad e_K(A) = \theta_K e_\infty(A)$$

**Proof:** From Lemma A',

$$e_K(A) = \sum_i i_K(\mathbf{r}_i) T_{Ki}(A) = \sum_i \theta_K i_\infty(\mathbf{r}_i) T_{Ki}(A) = \theta_K \sum_i i_\infty(\mathbf{r}_i) T_{\infty i}(A) = \theta_K e_\infty(A). \quad \blacklozenge$$

The state space for the SDVS system is the set  $\{(n, x, j): 0 \leq n \leq K; 0 < x; j = S \text{ or } V(m)\}$ .

The state  $(n, x, V(m))$  corresponds to  $n$  customers in the system, the server on vacation with the vacation mode initiated when  $m$  customers were present, and elapsed vacation task time  $x$  since that initiation. The state  $(n, x, S)$  corresponds to  $n$  customers in the system, a service in progress with an elapsed service time of  $x$ . Again one considers two processes,  $\underline{X}_K(t)$  and  $\underline{X}_\infty(t)$  corresponding to the capacity  $K$  and infinite capacity respectively.

With a minimal changes, Proposition 1 is shown to be applicable to the SDVS system.

**Proposition 1'.** For the SDVS system with capacity  $K$  and the associated parent system with infinite capacity

$$(4.3) \quad e_n^{(K)} = \theta_K e_n^{(\infty)}, \quad n = 0, 1, \dots, K-1 ;$$

$$(4.4) \quad e_K(N(\underline{\mathbf{r}}_m)) = \theta_K e_\infty(N(\underline{\mathbf{r}}_m)) \quad m = 0, 1, \dots, K-1$$

where:

a) for  $m = 0, 1, \dots, K-1$ , for some small  $\varepsilon > 0$ ,  $N(\underline{\mathbf{r}}_m) = \{(n, x, j): n = m, j = V(m), 0 \leq x \leq \varepsilon\}$

and

b)  $\theta_K = i_K(\underline{\mathbf{r}})/i_\infty(\underline{\mathbf{r}})$  for  $\underline{\mathbf{r}} = (K-1, 0, V(K-1))$

**Proof:** Consider the SDVS system with capacity  $K$  whose state space is partitioned  $\mathcal{N}_K = G_K \cup B_K$  and the corresponding infinite capacity system whose state space is partitioned  $\mathcal{N}_\infty = G_\infty \cup B_\infty$ . For both systems the only state at which entry can occur into the common set  $G = G_K = G_\infty = \{(n, x, j): 0 \leq n \leq K-1\}$  is  $(K-1, 0, V(K-1))$ . Since the dynamics of the two systems are the same on  $G$  for both the processes  $\underline{X}_K(t)$  and  $\underline{X}_\infty(t)$ , one has  $T_K(A) = T_\infty(A)$  for all  $A \subset G$ . The proposition then follows from Lemma B by setting the set  $A$  to  $\{(m, x, j) : m = n\}$  or  $N(\underline{\mathbf{r}}_m)$ . ♦

**Remark:** Note that the sets  $N(\underline{\mathbf{r}}_m)$ ,  $m = 0, 1, \dots, K-1$  in Proposition 1' correspond to neighborhoods of the regenerative entry states into the set  $\{(n, x, j): j = V(i), i \leq K-1\}$ . These points will be of special interest for Proposition 2'.

Proposition 2 must also be modified to accommodate the SDVS system.

**Proposition 2':**

For the SDVS system

$$(4.5) \quad e_V^{(K)} = \theta_K [ e_V^{(\infty)} - e_{VK} ]$$

where

$$a) \quad e_{VK} = \sum_{m=K}^{\infty} P[J(\infty) = V(m)] \quad \text{for the infinite buffer system}$$

$$b) \quad e_V^{(\infty)} = \sum_{m=0}^{\infty} P[J(\infty) = V(m)] \quad \text{for the infinite buffer system}$$

$$c) \quad e_V^{(K)} = \sum_{m=0}^{K-1} P[J(\infty) = V(m)] \quad \text{for the } K \text{ buffer system}$$

$$d) \quad \theta_K = i_K(\mathbf{r})/i_\infty(\mathbf{r}) \quad \text{for } \mathbf{r} = (K-1, 0, V(K-1))$$

**Proof:** Let  $G = G_K = G_\infty = \{(n, x, j) : j = V(i), i \leq K-1\}$ . Let  $\mathbf{r}_j = (j, 0, V(j))$  for  $j = 0, 1, \dots$ . Note that in the SDVS system, the neighborhood of the regeneration states,  $N(\mathbf{r}_j)$ , defined formally in Proposition 1', is not reachable from the regenerative state  $\mathbf{r}_j$  if  $j \neq i$ . Hence, using the definition of  $T_{Ki}(N(\mathbf{r}_j))$  and  $T_{\infty i}(N(\mathbf{r}_j))$  in Lemma B',

$$(4.6) \quad T_{Ki}(N(\mathbf{r}_j)) = T_{\infty i}(N(\mathbf{r}_j)) = 0, \quad j \neq i.$$

Note also that the mean time in the neighborhood of a regeneration state between successive entries into  $G$  is the same for the  $K$  capacity system and its infinite capacity parent, i.e.,

$$(4.7) \quad T_{Kj}(N(\mathbf{r}_j)) = T_{\infty j}(N(\mathbf{r}_j)) \quad j = 0, 1, \dots, K-1.$$

In preparation for the use of Lemma B' we must prove that

$$(4.8) \quad i_K(\mathbf{r}_j) = \theta_K i_\infty(\mathbf{r}_j), \quad j = 0, 1, \dots, K-1.$$

This is done as follows: For  $j = 0, 1, \dots, K-1$ ,

$$\begin{aligned} i_K(\mathbf{r}_j) &= i_K(\mathbf{r}_j) T_{Kj}(N(\mathbf{r}_j)) / T_{\infty j}(N(\mathbf{r}_j)), && \text{from (4.7)} \\ &= \sum_{i=0}^{K-1} i_K(\mathbf{r}_i) T_{Ki}(N(\mathbf{r}_j)) / T_{\infty j}(N(\mathbf{r}_j)) && \text{from (4.6)} \\ &= e_K(N(\mathbf{r}_j)) / T_{\infty j}(N(\mathbf{r}_j)) && \text{from (4.1)} \\ &= \theta_K e_\infty(N(\mathbf{r}_j)) / T_{\infty j}(N(\mathbf{r}_j)) && \text{from (4.4)} \\ &= \theta_K \sum_{i=0}^{K-1} i_\infty(\mathbf{r}_i) T_{\infty i}(N(\mathbf{r}_j)) / T_{\infty j}(N(\mathbf{r}_j)) && \text{from (4.1)} \\ &= \theta_K i_\infty(\mathbf{r}_j) T_{\infty j}(N(\mathbf{r}_j)) / T_{\infty j}(N(\mathbf{r}_j)) && \text{from (4.6)} \\ &= \theta_K i_\infty(\mathbf{r}_j) && \text{from (4.7)}. \end{aligned}$$

Observe that the mean time in the set G between regenerations is the same for the processes  $X_K(t)$  and  $X_\infty(t)$ , i.e.,

$$(4.9) \quad T_{Kj}(G) = T_{\infty j}(G), \quad j = 0, 1, \dots, K-1.$$

Hence, from (4.8) and (4.9) and Lemma B',

$$e_V^{(K)} = e_K(G) = \theta_K e_\infty(G) = \theta_K (e_V^{(\infty)} - e_{VK}). \quad \blacklozenge$$

**Remark:** For the exhaustive service schedule (as well as the non-vacation system)  $e_{VK} = 0$  so that the more familiar formula,  $e_V^{(K)} = \theta_K e_V^{(\infty)}$  is true.

The exact statement of Proposition 3 and its proof is directly applicable to the SDVS system. We can now derive the blocking formula.

**Theorem 2:** The blocking probability  $B(K)$  is

$$(4.10) \quad B(K) = e_K^{(K)} = \frac{(1 - \rho)E_K - e_{VK}}{1 - \rho E_K - e_{VK}} \leq \frac{(1 - \rho)E_K}{1 - \rho E_K}.$$

where

$$e_{VK} = \sum_{m=K}^{\infty} P[J_\infty(\infty) = V(m)],$$

$$E_K = P[N_\infty \geq K] = \sum_{n=K}^{\infty} e_n^{(\infty)}$$

and

$$\rho = -\lambda_T \alpha_T'(0).$$

**Proof:** From (2.8) and (4.3),

$$(4.11) \quad \rho \theta_K (1 - E_K) = \rho \theta_K \sum_{n=0}^{K-1} e_n^{(\infty)} = \sum_{n=0}^{K-1} e_n^{(K)} = \rho (1 - e_K^{(K)})$$

From (2.6), (2.7) and (4.5),

$$(4.12) \quad \rho (1 - e_K^{(K)}) = 1 - e_V^{(K)} = 1 - \theta_K (e_V^{(\infty)} - e_{VK}) = 1 - \theta_K (1 - \rho) + \theta_K e_{VK}.$$

Hence,  $\rho \theta_K (1 - E_K) = 1 - \theta_K (1 - \rho) + \theta_K e_{VK}$ , so

$$(4.13) \quad \theta_K = [1 - \rho E_K - e_{VK}]^{-1}.$$

Finally, from (4.11) and (4.13), the equality of (4.10) follows. But

$$\frac{(1 - \rho)E_K - e_{VK}}{1 - \rho E_K - e_{VK}} = 1 - \frac{1 - E_K}{1 - \rho E_K - e_{VK}}$$

which monotonically increases as  $e_{VK}$  decreases. Hence the inequality of (4.10) follows. ♦

### Extensions

The proof of Theorem 2 (and the associated propositions) could be easily generalized to be applicable to class of systems larger than the SDVS System. If, for example an idle state was augmented to the system, the service times also had state dependent durations(c.f. [6]), the arrival process was batch Poisson and/or the interarrival times has an Erlang distribution, then equations (4.3) and (4.10) would still be valid relationships between the infinity buffer system and the associated finite buffer system.

### Section 5: A recursive equations for the infinite capacity system for the homogeneous SDVS system

Section 4 described the blocking probability in terms of the number in the system for the infinite buffer parent system. In the general case the infinite buffer parent system in analyzed in [3]. In this section, a recursive equation is derived to compute this quantity for the special case of the SDVS system where  $V_m = V$  for all  $m$ . This section is applicable each of the models in Table 2 because they are simply subsets of these model.

In the case of  $V_m = V$  for all  $m$ , the SDVS vacation system is identical to the M/G/1 vacation system with single service. From [4], the time in the *queue* has Laplace Transform

$$(5.1) \quad \alpha_Q(s) = \frac{(1 - \rho_{EFF})}{1 - \rho_{EFF}\alpha_{TEFF}^*(s)}$$

where

$\alpha_{TEFF}(s)$  = the Laplace Transform of  $T + V$ ,

$\alpha_{TEFF}^*(s) = -(1 - \alpha_{TEFF}(s)) / (s\alpha'_{TEFF}(0))$ ,

and

$\rho_{EFF} = -\lambda\alpha'_{TEFF}(0)$ .

From [5], the number in the *system* has a p.g.f.,

$$(5.2) \quad \pi_S(u) = \alpha_Q(\lambda - \lambda u)\alpha_T(\lambda - \lambda u) = \frac{(1 - \rho_{EFF})\alpha_T(\lambda - \lambda u)}{1 - \rho_{EFF}\alpha_{TEFF^*}(\lambda - \lambda u)}$$

Hence,

$$(5.3) \quad \pi_S(u) = (1 - \rho_{EFF})\alpha_T(\lambda - \lambda u) + \rho_{EFF}\alpha_{TEFF^*}(\lambda - \lambda u)\pi_S(u).$$

Inverting (5.3) yields,

$$(5.4) \quad e_n^{(\infty)} = (1 - \rho_{EFF}) a_T(n) + \rho_{EFF} \sum_{m=0}^n [a_{TEFF^*}(n-m) e_m^{(\infty)}]$$

where  $e_n^{(\infty)}$ ,  $a_T(n)$ ,  $a_{TEFF}(m)$ , and  $a_{TEFF^*}(m)$  are the ergodic probabilities associated with the pgf's  $\pi_S(u)$ ,  $\alpha_T(\lambda - \lambda u)$ ,  $\alpha_{TEFF}(\lambda - \lambda u)$ , and  $\alpha_{TEFF^*}(\lambda - \lambda u)$ .

From (5.4),

$$(5.5) \quad e_0^{(\infty)} = \frac{(1 - \rho_{EFF})a_T(0)}{1 - \rho_{EFF} a_{TEFF^*}(0)} = 1 - \rho_{EFF}$$

and for  $n=1,2,\dots$  we have the following recursive relationship

$$(5.6) \quad e_n^{(\infty)} = (1 - \rho_{EFF}) a_T(n) + \frac{\rho_{EFF} \sum_{m=0}^{n-1} [a_{TEFF^*}(n-m) e_m^{(\infty)}]}{1 - \rho_{EFF} a_{TEFF^*}(0)}.$$

Using the argument of the derivation of (3.9),  $E_n = \text{Prob}[N_\infty = n]$  satisfies the recursive equation, for  $n = 1,2,\dots$ ,

$$(5.7) \quad E_{n+1} = E_n - (1 - \rho_{EFF}) a_T(n) + \frac{\rho_{EFF} \sum_{m=0}^{n-1} [a_{TEFF^*}(n-m) (E_m - E_{m+1})]}{1 - \rho_{EFF} a_{TEFF^*}(0)},$$

$$(5.8) \quad E_0 = \text{Prob}[N_\infty \geq 0] = 1,$$

and, from (5.5),

$$(5.9) \quad E_1 = 1 - e_0^{(\infty)} = \rho_{EFF}.$$

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