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MINIMIZING THE NUMBER OF VEHICLES TO MEET A FIXED PERIODIC SCHEDULE: AN APPLICATION OF PERIODIC POSETS

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1. INTRODUCTION

In this paper we consider and solve in polynomial time the problem of minimizing the number of vehicles to meet a fixed periodic schedule. For example, consider an airline that wishes to assign airplanes to a set of fixed daily-repeating flights (e.g., San Francisco at 10 p.m. to Boston at 6 a.m.) so as to minimize the number of airplanes, and deadheading between airports is allowed. This example may easily be extended to both bus scheduling and train scheduling.

The finite horizon version of the above vehicle scheduling problem was solved by Dantzig and Fulkerson [DF]. The periodic version in which deadheading is forbidden was solved by Bartlett [Ba] and by Bartlett and Charnes [BC]. (If deadheading is forbidden, the only question is how to start the schedule. Once in operation, a FIFO scheduling procedure is optimal). Finally, Orlin [O2] has solved the more general problem of minimizing the average linear cost per day of flying a schedule subject to a fixed number of airplanes. This last paper involved a solution technique substantially different from the technique presented here, although both involve solutions induced from finite minimum-cost network flows.

The periodic vehicle scheduling problem may be expressed in terms of task scheduling as follows: What is the minimum number of individuals to meet a fixed periodically repeating set of tasks? (For airplane scheduling, the tasks are flights and the individuals are airplanes.) This problem is shown in Section 4 to be a special case of the "minimum chain-cover problem for periodic partially ordered sets." This latter problem is formulated in Section 2 and a large subclass of this problem is solved as a minimum cost network flow problem in Section 3. The entire problem is solved in the appendix. These results generalize the work of Ford and Fulkerson [FF] who showed that the finite version of the above task scheduling problem may be solved as a special case of the minimum chain-cover problem for finite partially ordered sets.

Periodic Partially Ordered Sets

A partially ordered set (poset) is a set with a transitive anti-symmetric order relation $\succ$. A "periodic poset" is a special type of infinite poset in which the relations occur periodically. A "chain" is a set of elements every two of which are related, and a "chain-cover" is a decomposition of the elements of the partially ordered set into chains. Periodic posets are very structured, and in Section 3 we exploit this structure to obtain a polynomial time algorithm that determines a minimum cardinality chain-cover for a subclass of periodic posets. The technique is similar to that used by Orlin [O1] to obtain maximum-throughput dynamic network flows, in that the solution is obtained by reinterpreting a minimum cost flow in a related finite network.

Periodic Interval Graphs and Circular Arc Graphs

An intersection graph is a graph whose vertices are associated with subsets of a set, and two vertices are adjacent if the corresponding sets have a non-empty intersection. A periodic interval graph is an intersection graph in which the associated subsets are intervals that are periodically spaced over the real line. A circular arc graph is an intersection graph in which the associated subsets are arcs on a circle. The problems of coloring circular arc graphs and (finite) interval graphs have both been studied extensively, e.g., [GJMP], [Go], [LB], [OBB] and [Tu].

In Section 4, we show that the coloring problem for periodic interval graphs is a special case of the task scheduling problem, and thus it is solvable as a network flow problem. We also observe that the circular arc coloring problem, which was proved NP-complete by Garey et. al. [GJMP], is a special case of the task scheduling problem under the added restriction that each instance of the same task is carried out by the same person, proving that this latter problem is NP-hard. These results contrast with a recent result by this author [O3] showing that the coloring problem for "periodic graphs" is polynomial-space complete.

2. PERIODIC POSETS AND DILWORTH'S THEOREM

A partially ordered set (poset) is a set with a transitive, anti-symmetric relation $\succ$. A chain of a poset P is a (possibly infinite) subset of elements of P that are pairwise related, and an anti-chain is a subset of elements of P that are pairwise unrelated. A chain-cover of P is a partition (or decomposition) of the elements of P into chains. It is obvious that the number of chains in any chain-cover is a least the number of elements in any anti-chain. This inequality leads to the min-max result proved by Dilworth [Di] in 1950.

Dilworth's Theorem . Let P be a finite or countably infinite partially ordered set. Then the minimum cardinality of a chain-cover is the maximum cardinality of an anti-chain. □

Let N be an index set, let Z by the set of integers, and let $P = \{i^r : i \in N, r \in Z\}$ be a partially ordered set. We say that P is periodically closed if it satisfies relation (1) below.

$$i^p \succ j^r \text{ if and only if } i^{p+1} \succ j^{r+1} \quad . \tag{1}$$

If $P = \{i^r : i \in N, r \in Z\}$ and S is any set of relations on P (not necessarily a partial order), then the set of relations induced by S and (1) is called the periodic closure of S. For example, the periodic closure of the singleton set $\{i^r \succ j^p\}$ is the set of relations $\{i^k \succ j^{k+p-r} : k \in Z\}$. The transitive closure of set S is the set of relations $u > v$ such that there is a finite sequence $u = u_1, \ldots, u_k = v$ of elements of P such that $u_i > u_{i+1}$ is a relation of S for $i = 1, \ldots, k-1$. The periodic-transitive closure of S is the transitive closure of the periodic closure of S .

Remark 1 . The periodic-transitive closure is periodically closed. □

We say that P is a periodic partially ordered set if P is a partially ordered set with elements $\{i^r : i \in N, r \in Z\}$, and the set of partial order relations is the periodic-transitive closure of some finite set.

Example 1 . (A periodic poset). Let $P = \{1^r, 2^r : r \in Z\}$ such that (1) $1^r \succ 1^p$ for $r > p$, and (2) $2^r \succ 2^p$ for $r < p$, and (3) $1^r \succ 2^p$ for $r, p \in Z$. Then this partially ordered set is the periodic-transitive closure of the set $\{1^1 \succ 1^0, 2^0 \succ 2^1, 1^0 \succ 2^0\}$.

Example 2 . (A partially ordered set that is periodically closed but is not a periodic poset). Let $P = \{1^r, 2^r : r \in Z\}$ such that $i^r \succ 2^p$ for all $r, p \in Z$, and all other elements are unrelated. It is clear that P is not the periodic-transitive closure of a finite set of relations.

If the periodic poset P is the periodic transitive closure of set S of relations, we say that S generates P and that S is a generating set of P .

The main theoretical result of this paper is a polynomial time algorithm for finding a minimum cardinality chain-cover and a maximum cardinality anti-chain in a periodic poset. Here, polynomial time means that the number of elementary operations to determine the chain-cover and anti-chain is polynomially bounded in the length of the input for the generating set, which we assume is given as input.

An Alternative Description of Periodic Posets

Let P be a partially ordered set whose elements are the set of integers. We say that P is periodic with period n if relation (1') holds.

$$i > j \text{ if and only if } i + n > j + n . \tag{1'}$$

There is a 1:1 relation between the sets $P' = Z$ satisfying (1') and partially ordered sets $P = \{i^r : i = 1, \ldots, n, r \in Z\}$ satisfying (1), which is as follows: we associate the element $i^r \in P$ with element $i + rn \in P'$.

If we consider partially ordered sets with $P' = Z$ as above, the notation is somewhat simpler. However, we will continue using the previous notation because it helps to make the proofs and applications more transparent.

3. FINDING MINIMUM CARDINALITY CHAIN-COVERS AND MAXIMUM CARDINALITY ANTI-CHAINS IN FORWARD-DOMINATING PERIODIC POSETS.

Forward and Backward Dominating Indices

In a periodic poset, index i is said to be forward dominating (resp., backward dominating) if there is an integer $p > 0$ (resp., $p < 0$) such that $i^0 \succ i^p$.

Remark 2. If some index i of a periodic poset P is neither forward nor backward dominating, then the set $\{i^r : r \in Z\}$ is an anti-chain of P. □

Remark 3. No index of a periodic poset is both forward and backward dominating.

Proof. Let p and r be positive integers. If $i^0 \succ i^{-p}$ then $i^{rp} \succ i^0$. If $i^0 \succ i^r$ then $i^0 \succ i^{rp}$. It is therefore impossible that both $i^0 \succ i^{-p}$ and $i^0 \succ i^r$. □

If index i of a partially ordered set is forward (resp., backward) dominating then we also say that element i^p is forward (resp., backward) dominating for each $p \in Z$. If every element of the periodic poset P is forward (resp., backward) dominating then we say that P is forward (resp., backward) dominating.

An Overview

Below we give a polynomial time algorithm for finding a minimum cardinality chain cover in either a forward or backward dominating periodic poset P. To do this, we associate a finite network with each generating set for P, and we show that certain periodic chains may be induced from cycles in the finite network. Finally we show that a minimum cardinality chain-cover of P may be induced from an optimal covering of the nodes of the associated finite network by directed cycles. At the same time we determine a maximum cardinality anti-chain in P. These results are applied in the next section to solve the periodic task scheduling problems.

Suppose $P = P^* \cup P'$, where P^* (resp., P') is the subset of forward (resp., backward) dominating elements of periodic poset P. The union of the minimum chain-covers for P^* and P' does not, in general, form a minimum chain-cover for P. In the appendix we show how to transform this union of chain-covers into a minimum cardinality chain-cover by pairing a number of chains.

Generating Networks

Let P be a periodic poset with element set $\{i^r : i \in N, r \in Z\}$, and let S be a finite set of relations that generates P. We associate with S a generating network $G^S = (N, A^S)$, where N is the set of nodes and A is an arc set that is constructed as follows: for each relation $i^r \succ j^p$ in S, there is an associated arc a in A directed from i to j and with an associated <u>length</u> $c_a = p - r$. Of course, G^S may contain multiple arcs.

<u>Example 3</u>. Let $P = \{1^r, 2^r : r \in Z\}$ be a periodic poset such that (1) $1^r \succ 2^p$ if $p - r > 5$, (2) $1^r \succ 1^p$ if $p - r = 3k$ for some positive integer k and (3) all other elements are unrelated. Then P is generated by the

set of relations $S = \{1^0 \succ 1^3,\ 1^0 \succ 2^5,\ 1^0 \succ 2^6,\ 1^0 \succ 2^7\}$. The generating network G^S is portrayed in Figure 3.1. The numbers on the arcs are the arc lengths.

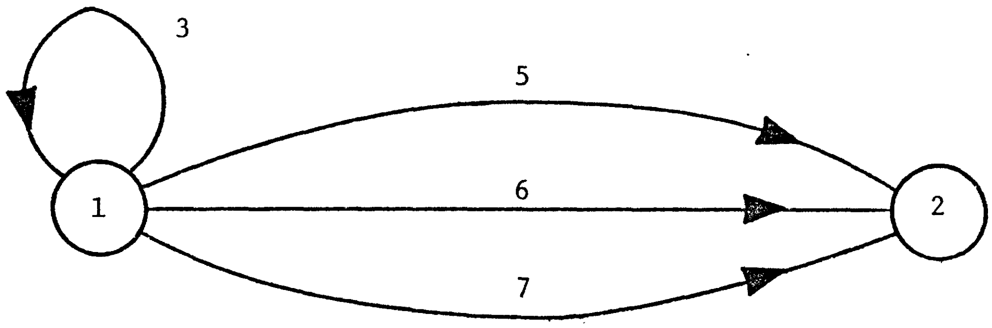

Figure 3.1. A Generating Network

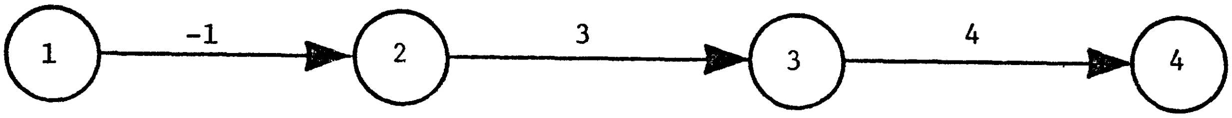

Figure 3.2. A Directed Path of Length 6.

A <u>directed path</u> in network G is an alternating sequence $i_1, a_1, \ldots, a_{k-1}, i_k$ of nodes and arcs such that arc a_j is directed from node i_j to node i_{j+1} for $j = 1, \ldots, k-1$. The <u>length</u> of a directed path is the sum of the lengths of the arcs of the path. A <u>directed cycle</u> is a directed path in which the initial node is the same as the terminal node, <u>and</u> no other node appears twice on the path. Figure 3.2 represents a directed path of length 6. Figure 3.3 is a directed cycle of length 3.

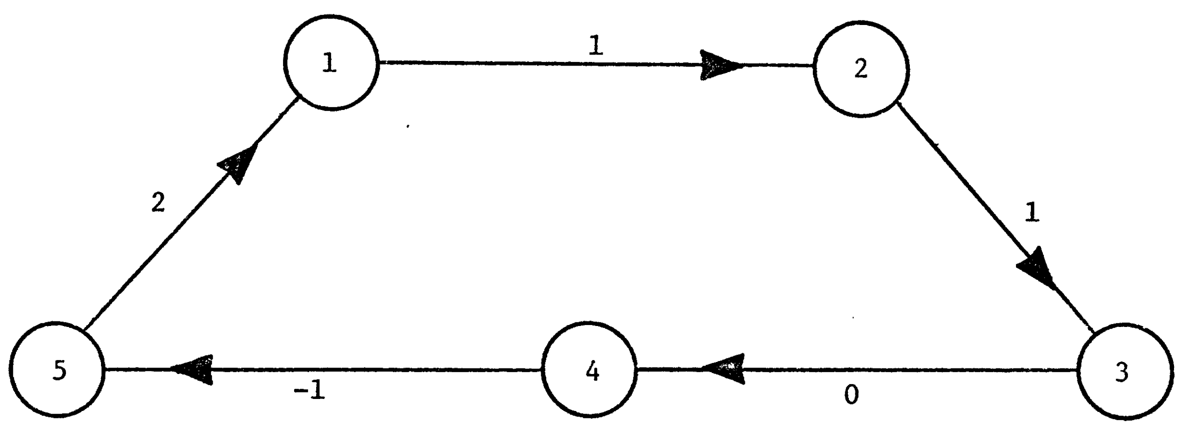

Figure 3.3. A Directed Cycle of Length 3.

<u>Lemma 1</u>. Let P be a periodic poset with generating set S. Then $i^r \succ j^p$ in P if and only if there is a path in the generating network G^S from node i to node j with length $p - r$.

<u>Proof</u>. Let PC(S) be the periodic closure of set S. It is clear that $i^r \succ j^p \in PC(S)$ if and only if there is an arc $a = (i, j) \in A^S$ with length $d_a = p - r$. By definition of the periodic-transitive closure, $i^r \succ j^p$ in P if and only if there is a finite sequence of elements $i^r = i_1^{r_1}, \ldots, i_k^{r_k} = j^p$

in P such that the relation $i_\ell^{r_\ell} > i_{\ell+1}^{r_{\ell+1}}$ is in PC(S) for $\ell = 1, \ldots, k-1$. The above is true if and only if there are corresponding arcs a_ℓ from i_ℓ to $i_{\ell+1}$ with length $r_{\ell+1} - r_\ell$, and this is true if and only if the path $i_1, a_1, \ldots, a_{k-1}, i_k$ is a path in G^S from i to j with length $p - r$. □

Corollary 1 . Let C be a simple cycle with length r in the generating network G^S. Let P' be the periodic poset generated by C. Then elements of P' partition into $|r|$ chains such that no element in one chain is related to an element in another chain.

Proof . Let the cycle be a path from i to i with length r. Then C induces a chain in P from i^p to i^{p+r} for all integers r. Let S_j denote the chain obtained by concatening the chains from i^p to i^{p+r} for $\{p : p = rk + j,\ k \in Z\}$. Then the chains $S_1, \ldots, S_{|r|}$ cover all elements ℓ^p for node ℓ of C and $p \in Z$, and no elements in distinct chains are related by Lemma 1. □

Example 4 . Let P be the periodic poset whose generating network is portrayed in Figure 3.3. Then the elements of P decompose into the three chains $S_j = \{1^{j+3k}, 2^{j+1+3k}, 3^{j+2+3k}, 4^{j+2+3k}, 5^{j+1+3k} : k \in Z\}$ for $j = 1, 2, 3$.

Let $G = (N, A)$ be a generating network for a periodic poset. A cycle-cover for G is a union of cycles (not necessarily disjoint) that contain all of the nodes of G. The length of a cycle-cover is the sum of the lengths of the cycles.

Theorem 1 . Let P be a forward dominating periodic poset, and let $G = (N, A)$ be a generating network for P. Then a minimum length cycle-cover for G induces a minimum cardinality chain-cover of P. Furthermore, a minimum length cycle-cover may be determined as an optimal solution to linear program (2) below.

$$\text{Minimize} \quad z = \sum_{a \varepsilon A} c_a x_a \tag{2.1}$$

$$\text{Subject to} \quad \sum_{a \varepsilon T_i} x_a - q_i = 0 \quad \text{for } i \varepsilon N , \tag{2.2}$$

$$q_i - \sum_{a \varepsilon H_i} x_a = 0 \quad \text{for } i \varepsilon N , \tag{2.3}$$

$$q_i \geq 1 \quad \text{for } i \varepsilon N , \tag{2.4}$$

$$x_a \geq 0 \quad \text{for } a \varepsilon A , \tag{2.5}$$

where T_i (resp., H_i) is the set of arcs of G whose tail (resp. head) is node i.

Proof . We first show that there is an optimal solution x to (2) that induces a chain-cover C of P whose cardinality is the objective value of x. If there is no feasible solution to (2), then there is a node i that cannot be covered by any cycle and thus by Remark 2, the set $\{i^p : p \varepsilon Z\}$ is an anti-chain. Henceforth, we assume that there is a feasible solution to (2).

Each feasible solution to (2) is a circultion with the property that the flow into (and out of) each node i is $q_i \geq 1$. It is well known (see, for example, Ford and Fulkerson [FF]) that a circulation may be decomposed into flows around cycles. The flow around any cycle has a positive cost because each cycle has positive length, and thus the objective value of (2.1) is bounded from below. Therefore, there is an optimal solution x^* to (2) that is a basic feasible flow.

The basic flow x^* is integer-valued because the constraint matrix is totally unimodular. Therefore, x^* decomposes into unit flows around directed cycles, and thus x^* induces a cycle-cover of G whose length is equal to the objective value for x^*, which we denote henceforth as r. This cycle-cover in turn induces a chain-cover C of P with cardinality r by Corollary 1.

In the remainder of the proof we produce an anti-chain S whose cardinality is the cardinality of C. Then by Dilworth's Theorem, the chain-cover has minimum cardinality while the anti-chain has maximum cardinality.

Consider the linear program (3) below, which is the dual to linear program (2).

$$\text{Maximize} \quad w = \sum_{i \in N} y_i \tag{3.1}$$

$$\text{Subject to} \quad -u_i + v_j \le c_a \quad \text{for} \quad a = (i, j) \in A , \tag{3.2}$$

$$-u_i + v_i - y_i \ge 0 \quad \text{for} \quad i \in N , \tag{3.3}$$

$$y_i \ge 0 \quad \text{for} \quad i \in N . \tag{3.4}$$

Because (2) has an optimal solution x^*, linear program (3) also has an optimal solution (u, v, y) and by linear programming duality the objective value of (u, v, y) is r. Furthermore, the constraint matrix is totally unimodular, and thus we may assume that (u, v, y) is a integer-valued basic solution. Let

$$S = \bigcup_i \{i^{u_i}, \ldots, i^{v_i - 1}\} .$$

Then $|S| = y_1 + \cdots + y_n = r$. We now claim that S is an anti-chain, and this proves the theorem.

Suppose there is a chain containing two elements i^r, j^p of S. By Lemma 1, there is a path in G from node i to node j of length $p - r$, and suppose it is the path $i = i_1, a_1, \ldots, a_{k-1}, i_k = j$. Then the length of the path may also be written as follows:

$$p - r = \sum_{\ell=1}^{k-1} c_{a_\ell} \geq \sum_{\ell=1}^{k-1} (v_{i_{\ell+1}} - u_{i_\ell})$$

$$= v_j - u_i + \sum_{\ell=2}^{k-1} (v_{i_\ell} - u_{i_\ell}) \geq v_j - u_i .$$

If $i^r \in S$ then $r \geq u_i$ and hence $p \geq v_j$, contradicting that $j^p \in S$, and proving that S is an anti-chain. □

In order to find maximum cardinality anti-chains and minimum cardinality chain-covers in backward dominating periodic posets, it is easy to transform the poset into a forward dominating poset as follows.

<u>Remark 4</u> . Let P be a backward dominating periodic poset with generating set S. Let P' be the forward dominating periodic poset derived by P by reversing the direction of all relations. Then P' is generated by the set

S' obtained from S by reversing the direction of all relations. Furthermore, each chain (resp., anti-chain of P) is a chain (resp., anti-chain) of P'. □

At this point the following conjecture may seem plausible: "a minimum cardinality chain-cover for periodic poset P may be obtained by finding minimum cardinality chain-covers for the forward dominating elements and the backward dominating elements, and then taking the union." This decomposition approach fails because, as in example 1, sometimes a chain of forward dominating elements may be paired with a chain of backward dominating elements to yield a single chain. A decomposition argument plus a pairing approach is given in the appendix to determine a minimum chain-cover for periodic posets with both forward and backward dominating elements.

4. THE MINIMAL NUMBER OF INDIVIDUALS TO MEET A FIXED PERIODIC SCHEDULE OF TASKS.

Ford and Fulkerson [FF] showed how to find the minimum number of individuals needed to meet a fixed schedule of tasks by reducing that problem to a special case of finding a minimum cardinality chain-cover in a poset. Here we extend their results to the case in which there are a finite number of tasks that must be performed periodically over an infinite horizon.

Let $T_1, \ldots, T_n$ be a set of tasks that must be carried out periodically, and let p denote the period length. Associated with task T_i are non-negative real numbers a_i and b_i such that T_i must be processed by an individual during the time interval $(a_i + k_p, b_i + kp)$ for $k = 0, 1, 2, \ldots$. We refer to the k^{th} iteration of task T_i as the k^{th} instance of T_i. (We allow that $a_i + p < b_i$, which corresponds to the case that two instances of the same task are processed in overlapping intervals.) The individuals (processors) are identical, and thus any individual can carry out any task. Finally, there is a set-up time r_{ij} between the successive processings of instances of task T_i and task T_j. We assume that $r = (r_{ij})$ satisfies the triangle inequalities: $r_{ik} \leq r_{ij} + r_{jk}$ for all i, j, k.

We transform the task scheduling problem into the chain-covering problem as follows. Let j^r denote the r^{th} instance of task T_j, which is carried out in interval $(a_j + pr, b_j + pr)$. We then induce a periodic partial order as follows: $i^q \succ j^s$ if the q^{th} instance of task T_i may be the immediate predecessor of the s^{th} instance of task T_j ; i.e., $b_i + pq + r_{ij} \leq a_j + ps$. So long as r_{ii} is finite for each i, it is clear that the above periodic poset is forward dominating. Furthermore, each chain of P is a number of instances of tasks that can be carried out by the same individual. Thus the minimum number of individuals needed to carry out all of the tasks is the minimum cardinality of a chain-cover of the induced periodic poset. Furthermore, we have the following interpretation of Dilworth's Theorem:

> The minimum number of individuals needed to carry out a fixed number of periodically repeating tasks is equal to the maximum number of instances of tasks such that no two of them may be carried out by the same individual.

An Application to Airplane Scheduling

Consider an airline that must schedule a minimum number of airplanes to meet a fixed daily-repeating set of flights, where deadheading is permitted. This problem is easily seen to be a special case of the above task scheduling problem. The tasks are flights, and the period length is one day. The set-up time r_{ij} is the amount of set-up time required between the arrival of flight i and the departure of flight j, assuming that the same airplane flies both flights. We allow that the arrival site s for flight i is different from the departure site t for flight j, in which case the set up time r_{ij} would include the deadhead time from airport s to airport t.

We observe that the flight schedule that we obtain from Section 3 is daily-repeating. However, it is not the case that each individual airplane flies a daily repeating schedule. Recall that the chain-cover of a periodic poset is induced by cycles in a generating network. A cycle of length 5 will be interpreted in the airplane scheduling problem as a periodic route that repeats every five days. To fly this route daily, we need five different airplanes.

If we add the restriction that the schedule for each airplane is daily-repeating, then the resulting problem is NP-hard, as demonstrated below in the subsection on "circular arc graphs."

In terms of airplane scheduling, Dilworth's Theorem states that the minimum number of airplanes needed to meet a fixed daily-repeating schedule is the maximum number of instances of flights, no two of which may be flown by the same airplane.

The above problem and solution technique generalizes that of Dantzig and Fulkerson [DF], who solved a finite horizon version of the airplane scheduling problem. The approach here also generalizes the work of Bartlett [Ba] and Charnes [BC] who solved the airplane scheduling problem in the case that deadheading is not permitted. This latter case is significantly easier because there are no relevant decisions to make once the schedule is initialized, as airplanes may fly only the required routes.

Coloring Periodic Interval Graphs

Another "application" of the task scheduling problem is to a coloring problem in graph theory. An intersection graph G is the graph derived by a set of subsets S as follows: we associate each vertex of G with a subset in S, and two vertices of G are adjacent if the corresponding subsets have a non-empty intersection. An interval graph is the intersection graph of a set of intervals on the real line. An interval graph is periodic if the corresponding set of intervals is an infinite set spaced periodically over the real line, i.e., the set of intervals may be written as follows: $\{(a_i + kp, b_i + kp) : i \in N, k \in Z\}$. In other words, it is the set of intervals in which tasks may be carried out for the periodic task scheduling problem.

Interval graphs were introduced into the literature by Lekkerkerker and Boland [LB] in 1962, and have been studied quite extensively. For a recent book that surveys the literature on interval graphs see Golumbic [Go].

It is easy to see that the minimum number of colors needed to color the vertices of a periodic interval graph is exactly the number of individuals needed to carry out the tasks of the corresponding task scheduling problem, assuming $r_{ij} = 0$ for all i,j.

Coloring Circular Arc Graphs

Consider the problem of coloring periodic interval graphs with the added restriction that the set of vertices corresponding to the intervals $\{(a_i + kp, b_i + kp) : k \in Z\}$ be colored the same color for any fixed i. We may reinterpret the coloring as follows. Consider a circle whose points are real numbers in the interval $(0,p)$ extending clockwise around the circle. Let (a,b) denote an arc of the circle extending clockwise from point a to point b, and let $S = \{(a_i, b_i) : i \in N\}$. Then the intersection graph for S is a circular arc graph, and any k-coloring of the graph may be extended to a k-coloring of the periodic interval graph such that $(a_i + kp, b_i + kp)$ is given the same color for each p.

In terms of the scheduling problem, such a coloring corresponds to an assignment of tasks to individuals so that each instance of task i is assigned to the same individual, or each instance of a flight is flown by the same plane. Recently Garey et. al. [GJMP] proved that the problem of coloring circular arc graphs is NP-hard. Since the circular arc coloring problem is a special case of the airplane scheduling problem with the restriction that schedules for each airplane repeat daily, this latter problem is also NP-hard.

There have been partial results on coloring circular arc graphs. For example, Tucker [Tu] analyses several heuristics for coloring and reduces the problem to a multi-commodity flow. Recently Orlin, Bonuccelli and Bovet [OBB] gave a polynomial time algorithm for the special case of circular arc coloring in which no arc is contained within another.

APPENDIX

Minimum Cardinality Chain-Covers in Periodic Posets

In Section 3, we showed how to obtain in polynomial time a minimum cardinality chain-cover and a maximum cardinality anti-chain in a forward (resp., backward) dominating periodic poset. The proof depended critically on the fact that all cycles of a generating network have positive (resp., negative) length. In this section, we show how to extend the results to periodic posets that are neither forward nor backward dominating.

The gist of the procedure is as follows. Partition the elements of P into sets P' and P^* of forward and backward dominating elements. Then find minimum cardinality chain-covers S' and S^* for P' and P^* using the procedure in Section 3. A forward dominating chain C and a backward dominating chain D are compatible if $C \cup D$ is a chain. A minimum cardinality chain-cover may be obtained by taking $S' \cup S^*$ and then pairing as many compatible pairs of chains as is possible. This procedure may be carried out by solving a matching problem on an associated bipartile graph that has $|S'| + |S^*|$ vertices, as detailed below.

Henceforth, we consider only those periodic posets in which each element is either forward or backward dominating. By Remark 2, the minimum cardinality of a chain-cover in all other periodic posets is ∞ .

We say that a pair C, D of chains is compatible if $C \cup D$ is a chain. To pair two compatible chains C and D is to create the single chain $C \cup D$.

Theorem 2 . Let $P = P' \cup P^*$ be a periodic poset with forward (resp., backward) subset of elements P' (resp., P^*). Let S' and S^* be minimum cardinality chain covers for P' and P^*. Then a minimum cardinality chain-cover for P may be obtained from $S' \cup S^*$ by pairing a maximum number of compatible chains.

<u>Proof</u>. As a preliminary we prove the following lemma.

<u>Lemma 2</u>. If C' and C^* are chains of S' and S^* respectively, then either C' and C^* are compatible or else no element of C' is related to an element of C^*.

<u>Proof</u>. Suppose $i^r \in C'$, $j^p \in C^*$ and $i^r \succ j^p$. (The proof in the case that $j^p \succ i^r$ is symmetric to the one below.) Chains C' and C^* were induced by cycles in the corresponding generating networks as in Corollary 1. Let ℓ and $-k$ be the lengths of these cycles. It follows that for each integer t we have $i^{r+\ell kt} \in C'$ and $j^{p+\ell kt} \in C^*$. Because P is periodic, we also have $i^{r+\ell kt} \succ j^{p+\ell kt}$ for each integer t. Then for every $u \in C'$ and $v \in C^*$, we can choose t sufficiently large so that $u \succ i^{r+\ell kt} \succ j^{p+\ell kt} \succ v$.
□

To complete the proof of Theorem 2, we let $U = \{u_1, . . ., u_r\}$ (resp., $V = \{v_1, . . ., v_p\}$) be a maximum cardinality anti-chain of P' (resp., P^*). Of course, $r = |S'|$, $p = |S^*|$. Furthermore, for every $s \in U \cup V$ there is a unique associated chain C in $S' \cup S^*$ such that $s \in C$ because $|S'| = |U|$ and $|S^*| = |V|$.

Let B be a bipartite graph with vertex set $U \cup V$, and where u_i is adjacent to v_j if the corresponding elements of P are related. Let I a maximum cardinality independent set in B, and let M be a maximum cardinality matching. By our construction, I is an anti-chain of P. In fact, we claim that I is a maximum cardinality anti-chain. To see this, let S denote the chain-cover of P derived from $S' \cup S^*$ by pairing compatible chains C' and C^* whenever the associated vertices of B are matched. (This pairing is legal by Lemma 2.) The cardinality of this chain-cover is $|S'| + |S^*| - |M|$, which is equal to $|I|$ by a direct extension of the König-Egervary duality theorem for bipartite matchings. Therefore, the chain-cover S has minimum cardinality, and the anti-chain I has maximum cardinality. □

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