Product Selection Policies for Perishable Inventory Systems

by

J. Keilson and A. Seidmann

OR 226-90 August 1990

Abstract

A comparison is made of the performance characteristics of perishable inventory systems with LIFO and FIFO selection policies. For each system one has Poisson arrivals to inventory and Poisson demand epochs with known rates and deterministic shelf life. The comparison requires development of a new analytical framework for the LIFO systems and extension of earlier results on the FIFO systems via Green's function methods. The performance characteristics derived for both policies are spoilage rate, loss sales rate, mean time between stockouts, inventory level on hand and the distribution of age of items delivered. In particular, the dependance of these characteristics on the replenishment rate, demand rate and shelf life are evaluated both theoretically and numerically. Several important management implications are explored and discussed.

1. Introduction

Perishable or outdating items are known to require special management attention due to their high obsolescence or perishability costs. Typical examples of items with limited shelf-life include fresh produce, drugs, chemicals and films (Nahmias (1989)). Our study of product selection policies for perishable items has been motivated by related problems arising in the management of special consumable materials and assembly components in the aerospace industry. These highly specialized items include, for example, structural adhesives, dry film lubricants, fuel cell sealers, batteries or gas energized launchers and ejectors. Compliance with standard age control requirements can limit the shelf life of these items to anywhere between nine months to two or three years. Long term supply contracts from multiple sources are established to assure the desired availability and quality levels. Interleaved deliveries from multiple sources, variability in shipment times, sporadic rejection by the receiving quality inspection tests - altogether give rise to random supply times. The typical demand pattern for many of these items is also random. In such systems, where the replenishment rates and demand rates are random, the determination of the distribution of the stock level is difficult because such evaluation must include the stock level at every age layer.

Several studies are devoted to the management of perishable inventory systems (Peterson and Silver (1979)). One can differentiate between two types of models used in the analysis of such systems. *Generative* (or prescriptive) models aim at developing useful inventory control policies. *Evaluative* (or descriptive) models, aim at characterizing the stochastic system behavior. Several authors (Veinott (1960), Bulinskaya (1964), Pierskala and Roach (1972), Fries (1975), Nahmias (1978), Schmidt and Nahmias (1985) and Pegeles (1986), among others) present prescriptive models leading to the development of optimal and heuristic ordering policies for items having a given utility during their constant lifetime and zero utility after that. These items must be tracked from the time they arrive until they are issued on demand or perish. These studies address both continuous and periodic review policies for deterministic and stochastic demand rates. Ghare and Schrader (1963), Emmons (1968), Nahmias (1977), Freidman and Hoch (1978), Jaiswal and Dave (1980), Dave and Pandya (1985), and Perry and Posner(1990) present dynamic lot-sizing, periodic review, and input control models for items having age dependent or exponential decay rates. Nose et al (1981, 1984) and Nose (1987) treat the cases of different selling prices under zero or single time unit stochastic lead times. Most of the studies cited above assume FIFO issuing policy. Other issuing policies for perishable items are considered by

Derman and Klein (1958), Cohen and Pekelman (1978) and by Martin (1986). Allocation and distribution models for perishable products include Prastacos (1978), and Federgruen et al (1986). A comprehensive review of generative models for perishable inventories and relevant applications can be found in Nahmias (1982a).

The first evaluative model of perishable systems is presented by Pegeles and Jelmert (1970). Their objective is to figure out the effects of the issuing policy on the average inventory level and on the average age of the issued items. Brodheim et al (1975) develop a model of a system with scheduled deliveries of a fixed amount. A constant inventory level is considered by Chazan and Gal (1977). Graves (1982) develops the steady state stock distribution of a constant replenishment inventory system with compound Poisson or unit demand requests. Nahmias (1982b) uses the results of this study to determine the replenishment rate which minimizes the total operating costs per time unit. Kaspi and Perry (1983,1984) compute the generating function of the limiting distribution of the number of items in certain FIFO systems having unit input process and poisson demand. It turns out that each one of the existing models concentrate on a single product selection policy and uses a unique set of operational assumptions. Therefore, one cannot use the results of these studies to investigate the impact of switching from one product selection policy to another.

The following central managerial problem motivates this paper: How should a firm designate either a LIFO or a FIFO product selection policy for issuing perishable items in a way that considers such basic trade-offs as the inventory holding costs, shortage costs, and the mean age at delivery (or the "freshness" of the items) when operating with stochastic supply and demand rates? These policies are typically associated with conflicting objectives: FIFO is commonly used when the suppliers control the issuing sequence, while customers prefer picking the freshest items (LIFO) when the expiration dates are known and when it does not involve an inordinate additional cost. Using LIFO one can expect higher customer's utility as compared with FIFO. This can be translated to higher sales revenues. On the other hand, LIFO results in higher perishing rates and lower service levels. Establishing the most suitable selection policy one must examine the impact of changing the product selection policy on a broad set of performance measures. To our knowledge, the analytical framework presented here constitutes the first such comparative study of LIFO and FIFO using *identical* operating assumptions.

The analytical framework developed here has two goals. The first is to characterize the stock dynamics needed to evaluate various performance measures for the two candidate policies: FIFO and LIFO. These analytical results are mandatory for choosing the desired replenishment rate and could be used by managers to explicitly consider purchase, service level, tax, and inventory holding costs trade-offs. The second goal is to compare the two policies using several parametrized data sets. The evaluation contrasts the relative impact of changes in the demand or supply rates on the service level, mean on hand inventories, time between stockouts and the age of items delivered.

Section 2 states the assumptions characterizing the two models. In Section 3, the LIFO model is studied and in Section 4, the FIFO model. Section 5 compares the performance of the two policies numerically. Conclusions are presented in Section 6.

2. The Inventory System

The perishable inventory system of interest has the following characteristics: The arrival of fresh items follows a Poisson process with mean λ arrivals per unit time. The demand process is a Poisson process independent of the arrival process. The mean demand rate is μ requests per unit time. Each demand request is for one unit at a time. Demand requests arriving when the inventory system is empty are lost. The stored items have a constant shelf life of **D** time units. An item which is not used to meet a demand request within **D** time units perishes (exits the system).

3. The LIFO Policy

Considered first is the following inventory system characterized by Last In First Out (LIFO) selection discipline. Items arrive to inventory one at a time in a Poisson stream of rate λ. One has a sequence of demand epochs with iid customer separation times T_{Cj} and associated demand rate $\mu = \frac{1}{E[T_{Cj}]}$. At each demand epoch, if the inventory is not empty, the freshest item is chosen. Each item has a deterministic shelf life **D**.

To analyze this system caution is needed. Most of the processes describing the system are not Markov. The number of items in storage, for example is not Markov. To make it Markov, the process must be supplemented by the ages of all items in inventory and analysis is then too difficult.

3.1 The loss process $X(t)$

The following loss process $\mathbf{X(t)}$ is Markovian and provides a stepping stone to the analysis of the FIFO system. At $\mathbf{t = 0}$, a fresh item with infinite shelf life arrives. Let

X(t) be the anticipated time to selection *of that item* if no other items were to come, given that at time **t** that item has not been selected. The process terminates when the item is selected.

Theorem 3.1 : The loss process **X(t)** is Markov.

Proof : Consider the motion of **X(t)** on the state space N = {x: 0 < x < ∞}. This motion may be described as follows: At **t=0, X(0)** is distributed as the forward recurrence time T_F of the time T_C between demand epoches, and **X(t)** decreases at unit rate between new arrivals. There is a constant hazard rate λ for new arrivals of fresh items. Each new arrival delays the selection of the original item by a new independent random amount T_{Cj}, i.e. **X(t)** increases by T_{Cj}. The process **X(t)** is therefore Markovian. §§

The motion of **X(t)** on the state space N is shown in Figure 1 for the case of finite shelf life **D** .

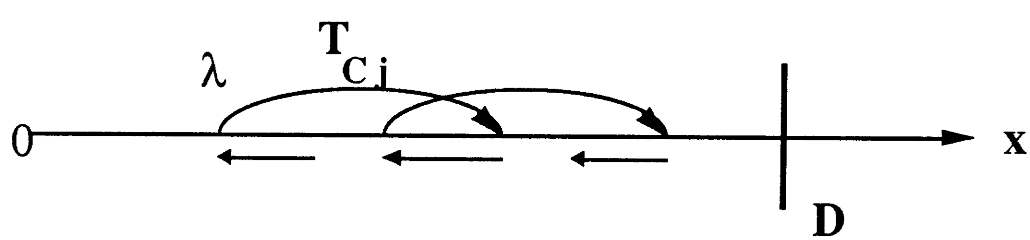


Figure 1. Motion on the state-space N.

Clearly, when the shelf life is infinite, the distribution of the time a new item spends in inventory is equal in distribution to the duration of a special busy period **BP*** for a single server queueing system **M/G/1** with arrival rate λ, service time distributed as T_C, and initial backlog that of the forward recurrence time of T_C.

When the shelf life is finite with maximal life **D**, the time in system T_{SD} is given instead by

$$T_{SD} = \min[BP^*, D] , \tag{3.1}$$

and the distribution of T_{SD} is evaluated below. When $D \to \infty$ one needs $\lambda < \mu$ to assure system stability. For $D < \infty$, the idle state is positive recurrent and the system is always stable due to the outdating of excess stock.

3.2 Fundamental performance relationships

It will be assumed in all that follows that T_C is exponentially distributed with mean μ^{-1}. The forward recurrence time then has the distribution of T_C and BP^*, and the familiar busy period **BP** then coincide in distribution.

The spoilage rate of arriving items is

$$\text{Spoilage Rate} = \lambda \, P[BP > D] . \tag{3.2}$$

The probability of spoilage for an individual item is

$$\Pr [\text{ Spoilage }] = P[BP > D]. \tag{3.3}$$

We assume that demand epochs taking place when the system is empty are ignored and the customers are lost. In order to get the rate of lost sales we denote by e_∞ the probability of having an empty stock.

Getting e_∞ we first note that the long term sales rate from the on-hand inventory is the demand rate minus the lost sales rate. This long term sales rate must equal the arrival rate minus the spoilage rate.

Consequently, we get the following ***inventory balance equation*** for our perishable inventory systems :

$$\lambda - \mu = \textbf{spoilage rate} - \textbf{lost sales rate} = \textbf{spoilage rate} - \mu\, e_\infty \,. \quad (3.4)$$

Thus one has

$$\textbf{Spoilage Rate} = \lambda - \mu(1-e_\infty) = \lambda\, P[BP > D]\,. \qquad (3.5)$$

Similarly

$$\textbf{Loss Sales Rate} = \mu\, e_\infty = \mu - \lambda + \lambda\, P[BP > D], \qquad (3.6)$$

and

$$e_\infty = \frac{\mu - \lambda + \lambda\, P[BP > D]}{\mu}\,. \qquad (3.7)$$

Property 3.1 : When $\mu \longrightarrow 0$, it will be seen subsequently that $e_\infty \longrightarrow e^{-\lambda D}$ since this is true *for any selection policy* when there is no demand.

The inventory system **service level**, or the probability that stock is on hand, is

$$\textbf{Service Level} = 1 - e_\infty .$$

$$\textbf{E[number of items in system]} = \lambda E[T_{SD}]\,. \qquad (3.8)$$

Lemma 3.2: The expected time between stockouts $E[T_G]$ is given by

$$E[T_G] = \frac{1}{\lambda}\left(\frac{1}{e_\infty} - 1\right). \qquad (3.9)$$

Proof: We use the following ergodic argument: Periods of stockout alternate with periods of stock availability. The stockout periods are exponentially distributed with mean duration equal to $\frac{1}{\lambda}$. It follows that the alternating time intervals should satisfy :

$$e_\infty = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + E[T_G]}.$$

This leads the desired result above. §§.

Property 3.2 : The expected time between stockouts satisfies $E[T_G]$ --> **D** as $\mu \dashrightarrow \lambda$.

Property 3.3 : The duration of the stockout period is of course exponentially distributed with mean $\frac{1}{\lambda}$ because there are no backorders.

3.3 Evaluation of the busy period distribution

In order to get the distribution of the time in system T_{SD} we need to evaluate the busy period distribution of the analogous single server queueing system **M/M/1** described above. It is known (Takacs [1962]) that for M/M/1 the busy period **pdf** is given by

$$S_{BP}(x) = 2\mu \; e^{-[(\lambda+\mu)\,x]} \frac{I_1(2x\sqrt{(\lambda\mu)})}{2\sqrt{(\lambda\mu)}\,x} \qquad (3.10)$$

where $I_1(.)$ is the modified Bessel function of the first kind and order 1.

This expression can be further simplified to obtain :

$$S_{BP}(y) = 2\,\mu\, e^{-[(\zeta+1)y]} \frac{I_1(y)}{y} \tag{3.11}$$

where

$$y = 2\sqrt{(\lambda\mu)}\; x \tag{3.12}$$

and

$$\zeta = \frac{[\sqrt{\lambda} - \sqrt{\mu}]^2}{2\sqrt{(\lambda\mu)}} \;. \tag{3.13}$$

This expression for $S_{BP}(x)$ *is valid for both* $\lambda < \mu$ *and* $\lambda > \mu$. .

We note that (3.11) will have

$$\Pr\,[BP < \infty] = \begin{cases} 1, & \text{if } \lambda < \mu \\ \frac{\mu}{\lambda}, & \text{if } \lambda > \mu\,. \end{cases} \tag{3.14}$$

If a high service level is wanted, one needs to have $\lambda > \mu$. In that case one gets $\zeta << 1$. For example, if $\lambda/\mu = 1.1$ than ζ is about **.005.**

3.4 The mean age at delivery

For items which are delivered to demanding customers, the mean age at delivery is :

$$E[\text{age at delivery}] = \frac{\int_0^D x\, S_{BP}(x)\, dx}{\int_0^D S_{BP}(x)\, dx}$$

$$= \frac{1}{2\sqrt{(\lambda\mu)}} \; \frac{\int_0^Z y\, e^{-(\zeta+1)\,y} \frac{I_1(y)}{y} dy}{\int_0^Z e^{-(\zeta+1)\,y} \frac{I_1(y)}{y}\, dy} \tag{3.15}$$

where **Z** is given by

$$\mathbf{Z} = 2\sqrt{(\lambda\mu)}\mathbf{D}. \tag{3.16}$$

The quantity **Z** is equal to twice the geometric mean of the number of arrivals and demands in a shelf life. It is approximately equal to twice the mean number of arrivals in a shelf life. In practice we expect to have **Z** >> **1**.

3.5 The probability of perishing

Similarly, the probability that an individual item will perish is:

$$P[BP>D] = \begin{cases} \sqrt{\frac{\mu}{\lambda}} \int_{Z}^{\infty} \exp[-(\zeta+1)\, y] \frac{I_1(y)}{y}\, dy\,, & \text{if } \frac{\mu}{\lambda} > 1, \\ \sqrt{\frac{\mu}{\lambda}} \int_{Z}^{\infty} \exp[-(\zeta+1)\, y] \frac{I_1(y)}{y}\, dy + (1 - \frac{\mu}{\lambda}), & \text{if } \frac{\mu}{\lambda} < 1. \end{cases} \tag{3.17}$$

3.6 The mean inventory level on hand

A Little's Law type argument provides a general expression for the mean inventory level :

$$E(N(\infty)) = \lambda \{ P[BP>D]D + (1 - P[BP>D])E[\text{age at delivery}]\} .$$

4. The FIFO Inventory System

We next treat the perishable inventory system with single poisson arrivals to inventory of rate λ and deterministic shelf life **D**. Again one has a sequence of demand epochs with i.i.d. separation times T_{Cj} and associated demand rate $\mu = \frac{1}{E[T_{Cj}]}$. The intervals between arrivals will be designated by T_{Aj} and these too are i.i.d. with expectation λ^{-1}. At each demand epoch, if the inventory is not empty, the oldest item is chosen. The system is said to be FIFO in that items which have not perished are delivered on a first-in first-out basis. The system is easiest to discuss when the demand epochs come in a Poisson stream and only this case will be treated. The earlier results of Kaspi and Perry (1983) are extended here via the Green's function method for the comparison of many performance characteristics with those developed above for the LIFO policy.

4.1 The age of the oldest item process X(t)

The process describing the number of items in the system is not Markovian and is not tractable. Instead one works with the age $X(t)$ of the oldest item using the convention that $X(t) = 0$ when the system is empty.

Theorem 4.1 : The process $X(t)$ is Markov for FIFO discipline.

Proof : The process $X(t)$ is Markov, for the five following reasons: a) when an item of age **D** perishes, the age of the oldest item becomes $\max[0, D - T_{Aj}]$ where T_{Aj} is the arrival lag between that item and the next arrival; b) when an item of age $x < D$ is selected, the age of the oldest item becomes $\max[0, x - T_{Aj}]$; c) when $X(t) = 0$, an arrival gives rise to age $0+$; d) between demand epochs, $X(t)$ increases at rate 1; e) there is a constant hazard rate λ for an arrival epoch and hazard rate μ for a demand epoch. The process **X(t)** is therefore Markovian. §§

When **D** is finite, the process **X(t)** is ergodic even when $\lambda > \mu$. Its state space N is the union $N = \{S \cup E\}$ of the set $S = \{(\mathbf{S,x}) : \mathbf{0} \le \mathbf{x} \le \mathbf{D}\}$ where stock is available and of the point state E for stockout. The motion on the state space N is as follows. There is a constant hazard rate λ for new arrivals. If the stock is not empty there is a drift to the right due to aging at unit velocity. At demand epochs, the oldest item is removed and the age of the oldest item jumps to the left by **x** with pdf $\lambda\mathbf{exp(-\lambda x)}$. If the virtual value of **X(t)** after a jump is negative, **X(t)** enters the idle state E. When **X(t)** reaches **D**, the item is removed and **X(t)** jumps to the left by **x** with pdf $\lambda\mathbf{exp(-\lambda x)}$, etc'. In the idle state E there is hazard rate λ for transition to **(S,0+)**. The motion of **X(t)**, on the state space N is shown in Figure 2.

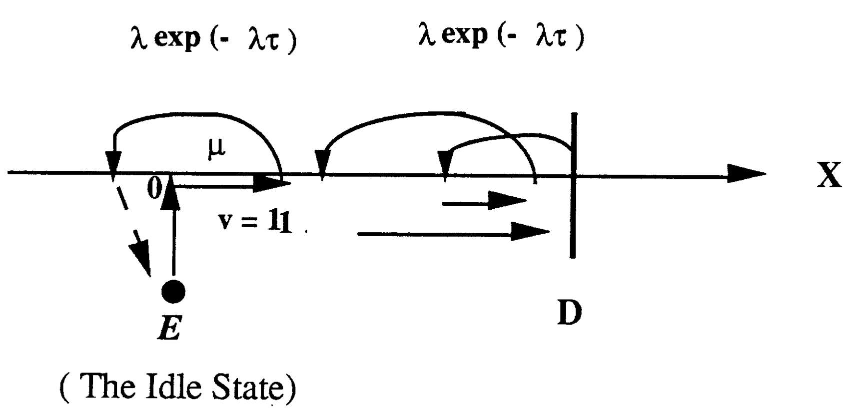


Figure 2. Motion of X(t) on its state space for FIFO.

4.2 Evaluation of the distribution on the state space

The FIFO system can be analyzed by employing the method of green's functions. This method provides a natural tool for the treatment of spatially and temporally homogeneous processes modified by boundaries . A simplified account of the method has been given by Graves and Keilson [1981] in the treatment of a system with a dynamic structure similar to that examined here.

If one looks at the motion of $\mathbf{X(t)}$ on the set $\mathcal{S} = \{(\mathbf{S,x}) : \mathbf{0} \le \mathbf{x} \le \mathbf{D}\}$, it is seen that this motion, apart from the influence of the two boundaries at $\mathbf{x = 0}$ and $\mathbf{x = D}$, consists of exponentially distributed jumps to the left with hazard rate μ and uniform drift to the right at rate unity. Let the spatially homogeneous process in the absence of boundaries starting at $t = 0$ be designated by $\mathbf{X_H(t)}$. This process has a generalized dynamic Green density $g_H(x,t) = \frac{d}{dx} P[X_H(t) \le x \,|X_H(0)=0]$ with Laplace transform

$$\gamma_H(w,t) = E[\exp(-wX_H(t) \,|X_H(0)=0] = \sum_{k=0}^{\infty} e^{-\mu t} \frac{(\mu t)^k}{k!} \left(\frac{\lambda}{\lambda-w}\right)^k e^{-wt}$$

$$= \exp[-\mu t(1-\frac{\lambda}{\lambda-w}) -wt] \;. \tag{4.1}$$

Correspondingly one has the ergodic Green density transform

$$\gamma_{H\infty}(w) = \int_0^{\infty} \gamma_H(w,t)\; dt = \frac{1}{\mu\,[1 - \frac{\lambda}{\lambda-w}] + w} \;. \tag{4.2}$$

The compensation measure needed to recreate the influence of the two boundaries is obtained as follows.When an item reaches age $\mathbf{D}$, it is discarded and is replaced by $[\mathbf{D} - \lambda^{-1}\mathbf{E}]$, where $\mathbf{E}$ is the exponential variate with mean value unity. This is equivalent to an injection of negative mass concentrated at $\mathbf{D}$ to annihilate the item and to injection of positive mass having the distribution of $\mathbf{D} - \lambda^{-1}\mathbf{E}$. The transform of the corresponding compensation measure at the boundary $\mathbf{D}$ is then given by

$$K\left[-1+\frac{\lambda}{\lambda-w}\right]e^{-wD} = K\frac{w}{\lambda-w}\; e^{-wD}, \qquad (4.3)$$

where **K** is a positive constant to be determined. To understand the form of the compensation, one observes it to be equivalent to a delta function at **x=0** with transform **-1** and an exponentially distributed mass with transform $\frac{\lambda}{\lambda-w}$ and a shift of $+D$ corresponding to e^{-wD}.

There is a similar compensation mass localized at the boundary at **0**. There one must have negative mass to annihilate the overshoot mass on $(-\infty,0)$ and positive mass concentrated at **0** representing the entry into the upper part of the state space S after passing through the idle state. The renewal rate for departures from stockout is λe_∞. The transform of the compensation measure at **0** is then of the form $\lambda e_\infty[1-\frac{\lambda}{\lambda-w}] = -\lambda e_\infty \frac{w}{\lambda-w}$ and the transform of the total compensation measure is

$$\chi_\infty(w) = -\lambda e_\infty \frac{w}{\lambda-w} + K\frac{w}{\lambda-w}\; e^{-wD}\;, \qquad (4.4)$$

with **K** positive and yet to be determined. The negative exponentially distributed compensation from the **D** boundary which overshoots **0** has the same structure as the compensation at **0** and blends with it since the arrival rate exceeds the perishing rate.

4.3 The probability of having an empty stock

The compensation method states that the ergodic distribution on S is the convolution of the compensation and the ergodic Green's function. With the convention that $X(t) = 0$ when the system is empty, the ergodic distribution of $X(t)$ which will be designated by X_∞ has the transform.

$$\phi_\infty(w) = e_\infty + \chi_\infty(w)\; \gamma_{H\infty}(w) = e_\infty + \frac{-\lambda e_\infty \frac{w}{\lambda-w} + K\frac{w}{\lambda-w}\; e^{-wD}}{\mu[1-\frac{\lambda}{\lambda-w}] + w}$$

$$= e_\infty + \frac{-\lambda e_\infty + K e^{-wD}}{-\mu + \lambda - w}, \tag{4.5}$$

where $e_\infty = P[X(\infty) = 0]$. Note that X_∞ has all support on $[0,D]$ so that $\phi_\infty(w)$ is entire. The zero in the denominator must be counteracted by a zero in the numerator . One also has $\phi_\infty(0) = 1$. Algebra then gives

$$\phi_\infty(w) = e_\infty + \lambda e_\infty \frac{1 - e^{(\lambda-\mu)D} e^{-wD}}{w - \lambda + \mu}, \tag{4.6}$$

with

$$e_\infty = \frac{\lambda-\mu}{\lambda e^{(\lambda-\mu)D} - \mu}. \tag{4.7}$$

It should be noted that $0 < e_\infty < 1$ whether or not $\lambda > \mu$; When $\lambda = \mu$ then $e_\infty = \dfrac{1}{1 + \lambda D}$.

Property 4.1 : When μ=0, one has $e_\infty = e^{-\lambda D}$.

This result is true for any selection policy, when there is no demand (i.e.μ=0), as one can see from $M/G/\infty$ when each item arriving is given its own shelf (server) and awaits outdating.

The inverse bilateral Laplace transform of $\phi_\infty(w)$ is, using the indicator function $I_{[0,D]}(x) = 1,\ x \in [0,D]$, zero elsewhere,

$$f_\infty(x) \begin{cases} = e_\infty\, \delta(x) + \lambda e_\infty\, e^{(\lambda-\mu)x}\, I_{[0,D]}(x); & \lambda < \mu, \\ = e_\infty\, \delta(x) + \lambda e_\infty\, e^{(\lambda-\mu)(x-D)}\, I_{[0,D]}(x); & \lambda > \mu, \\ = e_\infty\, \delta(x) + \lambda e_\infty\, I_{[0,D]}(x); & \lambda = \mu. \end{cases} \tag{4.8}$$

One can see from these results that the pdf of the oldest item *decreases monotonically* from $\lambda\, \mathbf{e}_\infty$ on **[0,D]** when $\lambda < \mu$ and *increases monotonically* towards $\lambda\, \mathbf{e}_\infty$ on **[0,D]** when $\lambda > \mu$. It is equal to $\frac{\mathbf{1+\lambda}}{\mathbf{1+\lambda D}}$ when $\lambda = \mu$. Figure 3 illustrates these three cases for $\{\lambda = 0.95,\ \mu = 1.0\}$, $\{\lambda = 1.05,\ \mu = 1.0\}$, and $\{\lambda = 1.0,\ \mu = 1.0\}$ with **D**= 8.

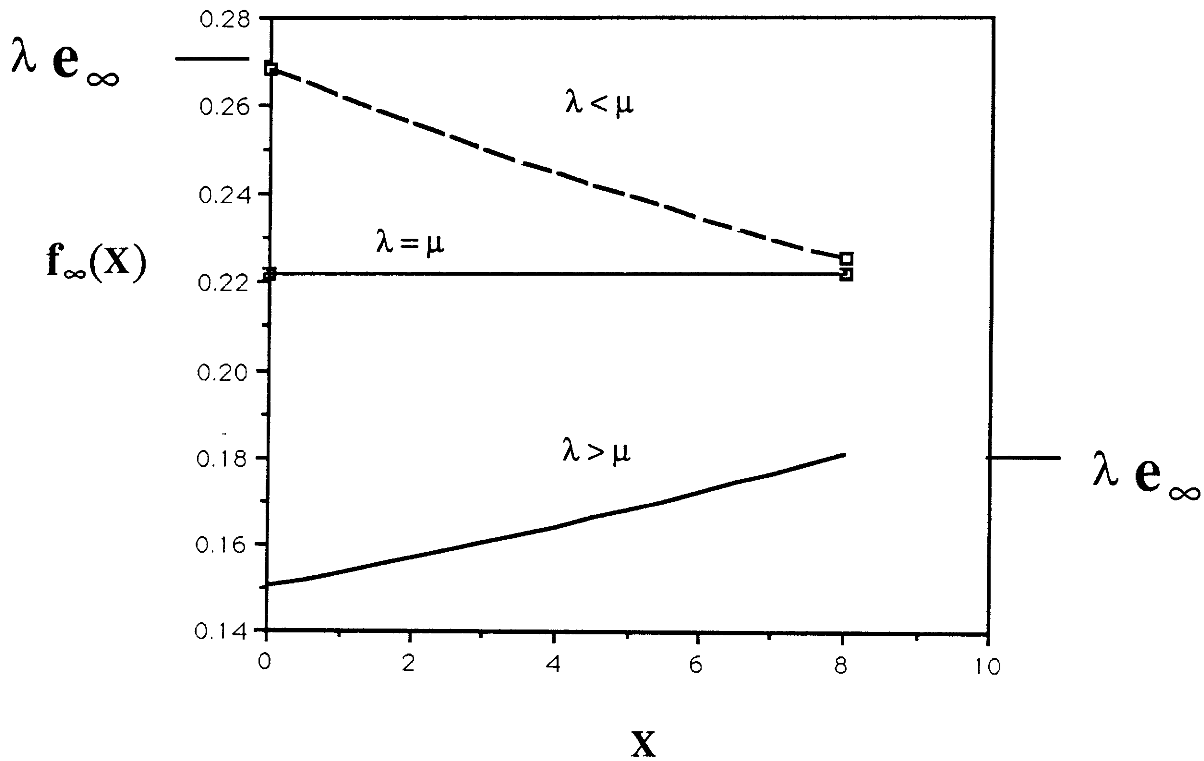


Figure 3. The pdf of the oldest item for FIFO.

The probability that the system is not empty is clearly

$$\textbf{Service Level} = \mathbf{1\text{-}e_\infty} = \mathbf{1}\text{-}\frac{\lambda\text{-}\mu}{\lambda e^{(\lambda-\mu)\mathbf{D}}\text{-}\mu} = \frac{\lambda e^{(\lambda-\mu)\mathbf{D}}\text{-}\lambda}{\lambda e^{(\lambda-\mu)\mathbf{D}}\text{-}\mu}\ . \tag{4.9}$$

Since the spoilage rate is λ - μ**P[not empty]**, simple algebra gives

$$\textbf{Spoilage Rate} = \lambda \frac{\lambda-\mu}{\lambda-\mu e^{-(\lambda-\mu)D}} \tag{4.10}$$

Note that the ratio of spoilage rate to arrival rate is a positive number smaller than one as it should be. This ratio decreases with **D** and decreases with μ.

4.4. The distribution of the age of the items delivered

The age of the item delivered in the steady state is the conditional distribution of X_∞ for X_∞ positive. One then has from $\phi_\infty(w)$, with $\theta = \lambda-\mu$,

$$\phi_{X_\infty \mid X_\infty > 0}(w) = \frac{\dfrac{1 - e^{(\theta-w)D}}{w-\theta}}{\dfrac{1 - e^{\theta D}}{-\theta}} \; . \tag{4.11}$$

The expected value of the age of items delivered is then

$$E[X_\infty|X_\infty>0] = -\phi'_{X_\infty \mid X_\infty > 0}(0)$$

$$= D \frac{d}{dv} \log\left(\frac{e^v-1}{v}\right)\Big|_{v=\theta D} = D\,[1-v^{-1} + (e^v-1)^{-1}]. \tag{4.12}$$

One also sees that $\log\left(\frac{e^v-1}{v}\right) = v + \log\left(\frac{1-e^{-v}}{v}\right)$. The function $\left(\frac{1-e^{-v}}{v}\right)$ is the bilateral Laplace transform of the uniform distribution on **[0,1]**. Since mixtures of log-convex functions are log-convex [Artin(1931)], one sees that $\log\left(\frac{1-e^{-v}}{v}\right)$ is *strictly convex* in **v** on $(-\infty,\infty)$, Hence

$$\frac{\partial}{\partial v} E[X_\infty|X_\infty>0] = D \frac{\partial^2}{\partial v^2} \log\left(\frac{1-e^{-v}}{v}\right)\Big|_{v=\theta D} > 0. \tag{4.13}$$

It follows from (4.13) that $E[X_\infty|X_\infty>0]$ increases with $\theta D = (\lambda-\mu)D$ on $(-\infty,\infty)$.

4.4. Structural properties

One can verify the following structural properties of the FIFO policy :

(a) when θ**D --> -**∞, the mean age at delivery **--> 0;**

(b) when θ**D --> +**∞, the mean age at delivery **--> D;**

(c) when λ **-->** μ, the mean age at delivery **-->** $\frac{1}{2}$ **D ;**

(d) when λ **-->** ∞ and μ **<**∞ the mean age at delivery **--> D;**

and (e) when μ **-->** ∞ and λ **<**∞.the mean age at delivery **--> 0.**

5. Numerical Results

Several numerical examples are presented in this section. These examples illustrate some salient properties of the two selection policies analyzed above. The inventory system considered has a shelf life **D** of 8 time units and a mean demand rate μ of one item per time unit. The age of items delivered $\mathbf{E[X_\infty | X_\infty > 0]}$ is computed using (3.15) and (4.12) and is depicted in Figure 4. The mean age at delivery under FIFO is greater than under the LIFO selection policy. Under FIFO the mean age at delivery *decreases* with the increase of the demand rate λ. This is due to the fact that the mean time in storage declines with the increase in λ. On the other hand, in the LIFO case the mean age at the delivery is not a monotone function of the demand rate. Increasing the initially small demand rate results in supplies of items from older age layers: Further increases in the demand rate lead to minimal storage levels and very short time intervals between replenishments and deliveries. At that point, any increase in the demand rate results in smaller shelf storage times.

Figure 5 illustrates the mean time between stockouts $\mathbf{E[T_G]}$, and Figure 6 illustrates the service probability $(\mathbf{1\text{-}e_\infty})$. The FIFO policy results in relatively less spoilage and, therefore, higher service probability and longer times between stockouts. Note that in the LIFO case a service level of 95% or higher requires a supply rate which is at least twice the demand rate.

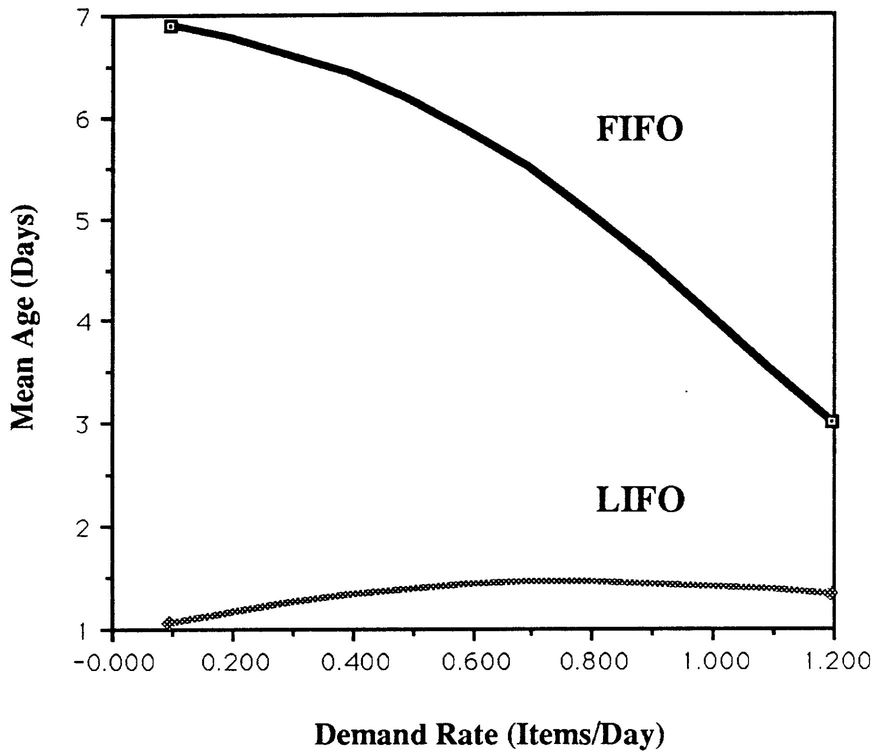


Figure 4. The mean age of items delivered.

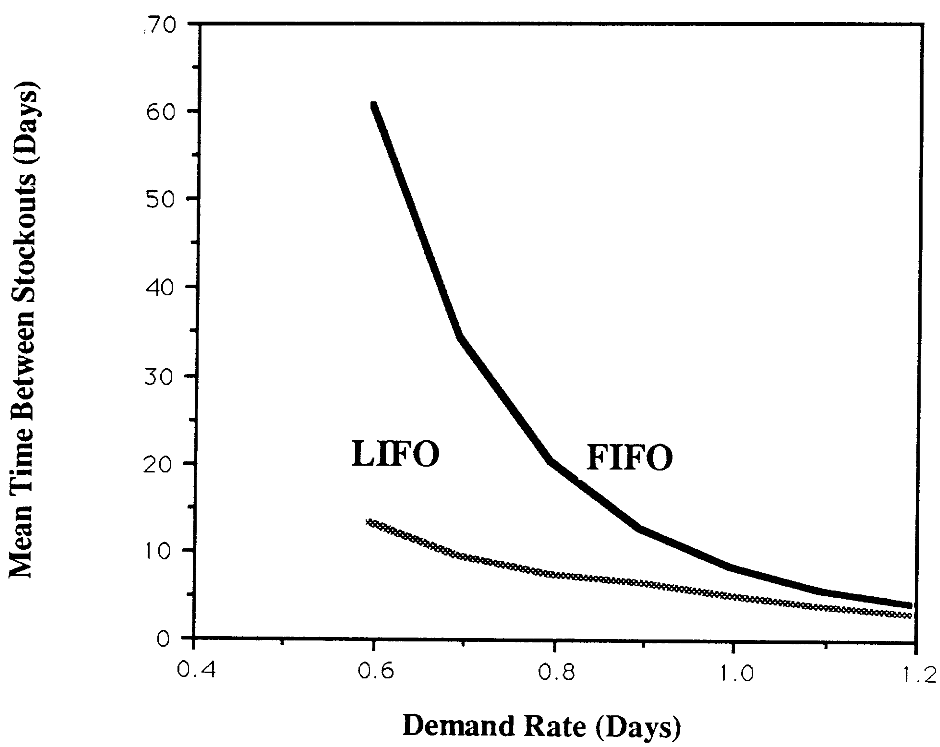


Figure 5. The mean time between stockouts.

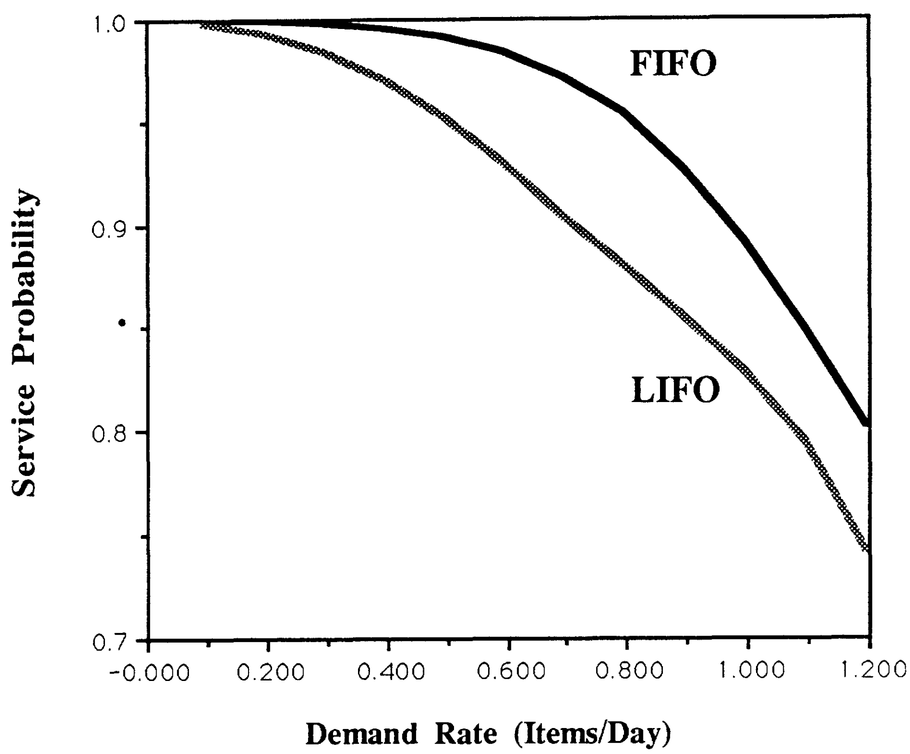


Figure 6. The service probability (level).

The impact of the selection policy on the optimal replenishment rate μ^* and on the net operating income is evaluated next. Evaluating these issues let **h** be the inventory holding cost per item per time unit, **p** be the sales price, **s** be the salvage value of an outdated item, **l** be the cost of lost sales, and **c** be the purchasing price in $ per item. The net profit rate per time unit **N** is given by

$$\mathbf{N} = \mu\ (\mathbf{1}-\mathbf{e}_{\infty})\ \mathbf{p} + (\lambda - \mu\ (\mathbf{1}-\mathbf{e}_{\infty}))\ \mathbf{s} - \lambda\ \mathbf{E}(\mathbf{N}(\infty))\ \mathbf{h} - \lambda\ \mathbf{p} - \mu\ \mathbf{e}_{\infty}\ \mathbf{l}\,. \tag{5.1}$$

Figure 7 presents the values of **N** as a function of λ for **h** = 1, **p** = 80, s = 8, **l** = 100, and **c** = 10 $/item. It shows that FIFO results in a greater maximal net profit rate since it is associated with higher service probability and the unit revenues are age independent. The optimal supply rate for FIFO is a bit smaller than the one for LIFO, and the net profit rate declines sharply when the supply rate is reduced below the optimal levels.

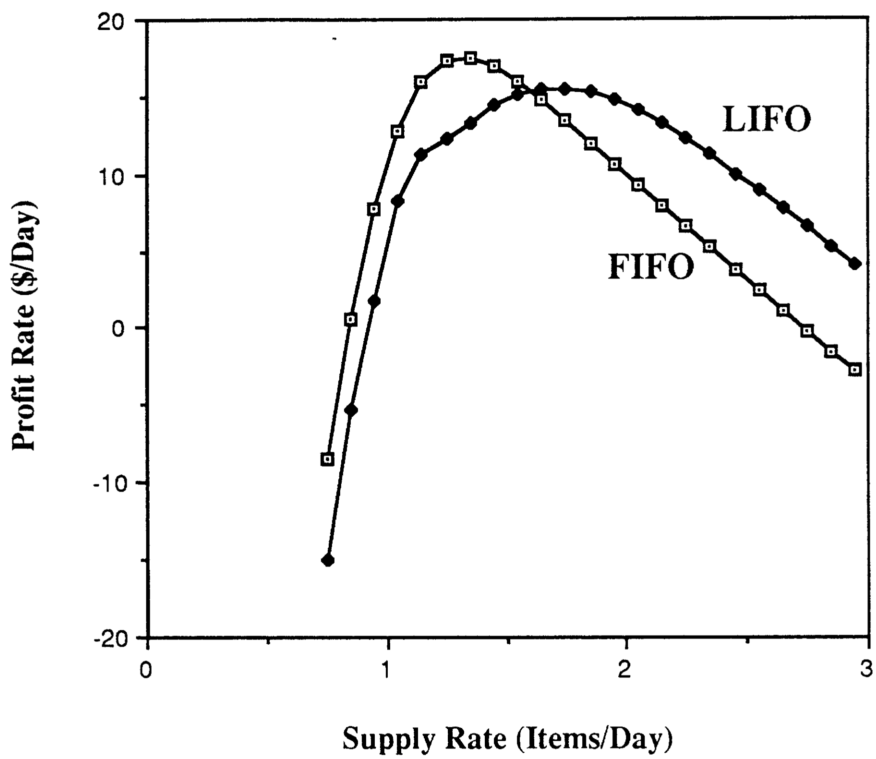


Figure 7. The profit rate with age-independendt sales price.

The LIFO policy is more appropriate when product reliability, or utility, is closely related to age. In those situations where customers can observe the expiration dates one can think of a pricing scheme that results in declining sales prices as a function of the products' age. A typical case in point is the well known discount on "day old bread." Investigating a similar option, consider a case where the sales price **p** declines linearly over the age of the products delivered within the interval **[0-D]**.

Figure 8 shows the values of **N** when the sales price declines from $120 to $80. In this case the LIFO policy dominates since there is an income premium for delivering fresher products as compared with the FIFO policy. Again, the optimal supply rate for FIFO is smaller than the one for LIFO.

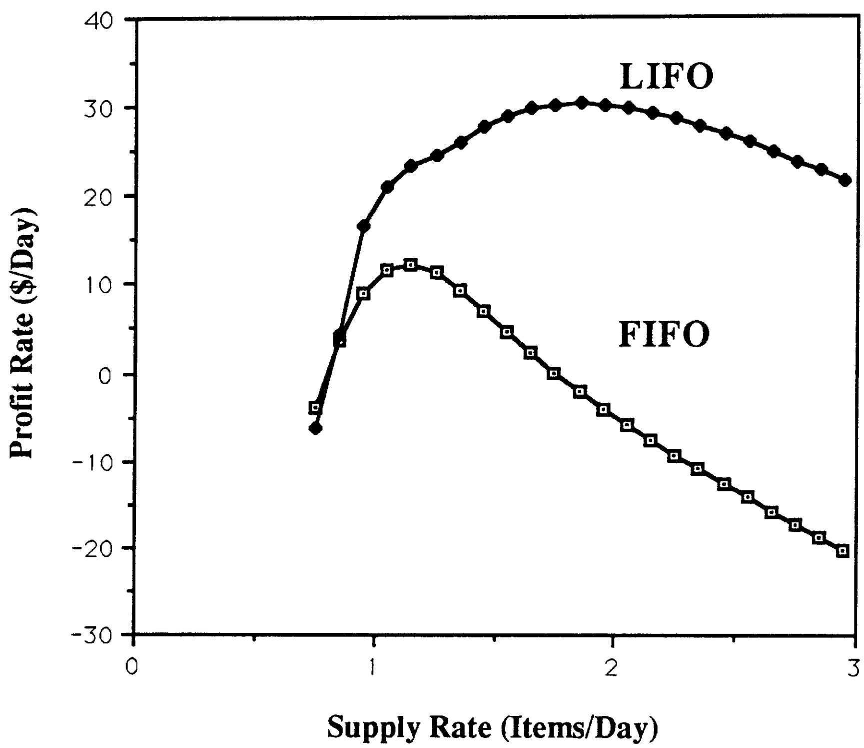
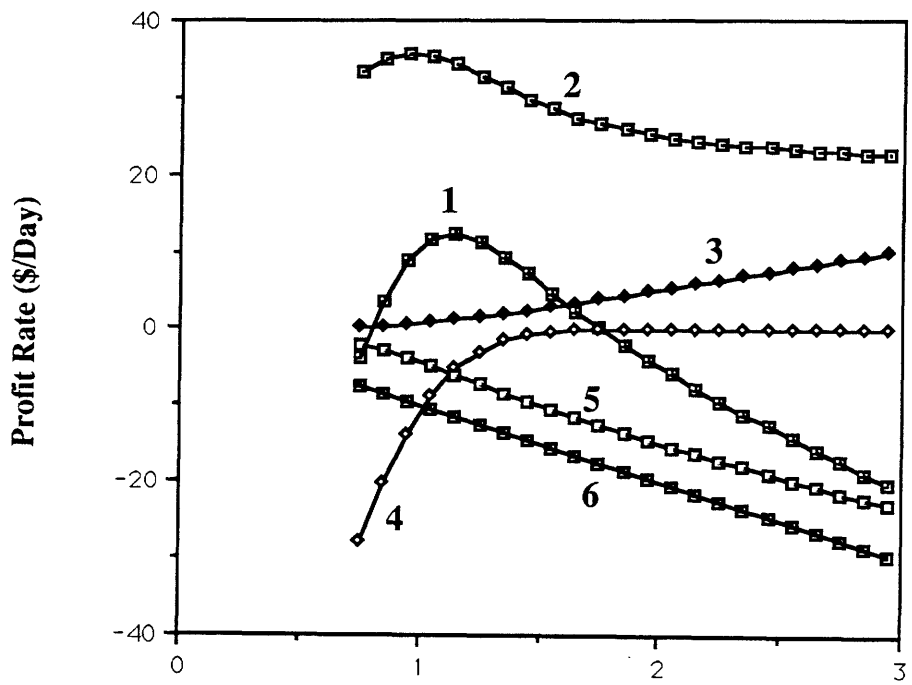


Figure 8. The profit rate with age-dependendt sales price.

Figures 9 and 10 present a breakdown of the various cost components for the FIFO and LIFO policies, respectively. These Figures show that under LIFO the sales revenues increase with the supply rate and the cost of lost sales approaches zero asymptotically. The sales revenues under FIFO tend to be nonmonotone: in our example they tend to increase for $\lambda < \mu$, (to counter the cost of lost sale) and start declining rapidly for $\lambda < \mu$ due to the significant increase in the age of items delivered (and the reduced income for older items.)



Supply Rate (Items/Day)

Figure 9. The profit rate components with FIFO and age-dependendt sales price (1- net profit, 2- sales revenue, 3- salvage income, 4- cost of lost sales, 5- inventory holding cost, 6- purchasing cost.)

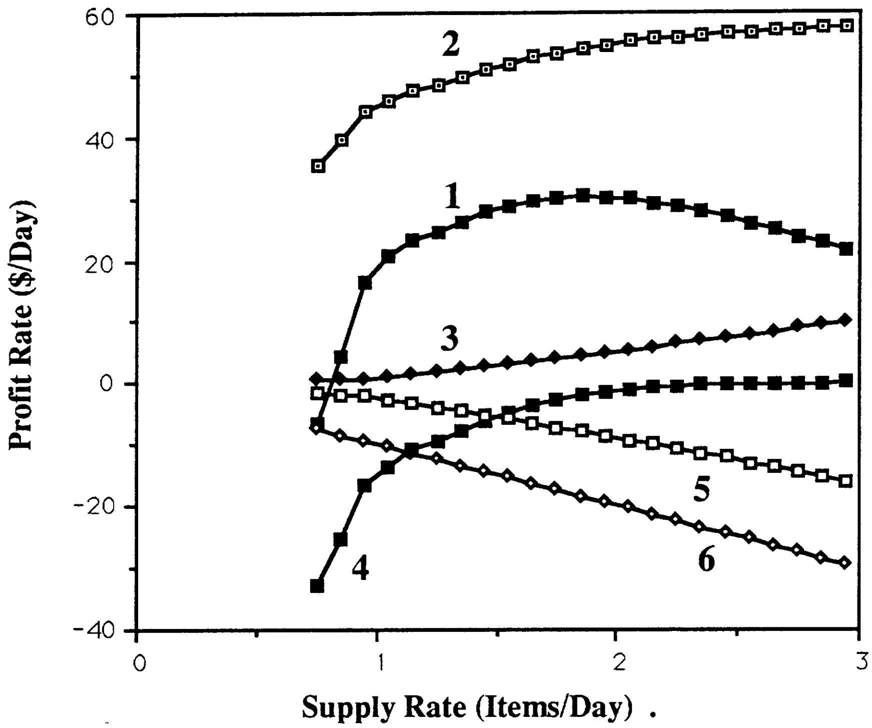


Figure 10. The profit rate components with LIFO and age-dependendt sales price (1- net profit, 2- sales revenue, 3- salvage income, 4- cost of lost sales, 5- inventory holding cost, 6- purchasing cost.)

6. Conclusions

This paper has presented an analysis of perishable inventory systems under either FIFO or LIFO product selection policies. In these systems one has to follow the age of each unit in stock to keep track of the available inventory level. This results in a multidimensional state space that complicates the analysis and the actual monitoring of such systems. The results presented here illustrate the potentially great impact of the product selection policies on the following performance measures (**a**) spoilage rate, (**b**) mean age at delivery, (**c**) expected time between stockouts, (**d**) service level, and (**e**) mean on hand inventory level. In particular, it is shown that LIFO results in a much lower age of items delivered and on hand inventories, while FIFO assures a higher service level and longer times between stockouts. The mean age at delivery for FIFO goes down with the demand rate, and is not monotone with the demand rate for LIFO.

Optimization of the supply rate of both policies reveals that, whenever the sales price is age-independent, FIFO can bring higher profits with lower supply rates. When the utility, and the sales price, of new items is higher than for older ones it may be that LIFO policy is more economical. When customers can observe the expiration dates and select the newest items they may impose the LIFO policy on the vendors . Perhaps the most intriguing result is the observation that under both policies the supply rate must be significantly greater than the demand rate to obtain a reasonable service level and to account for losses due to perishability. This observation is particularly important to logistic and distribution systems managers seeking to figure out the basic tradeoffs between spoilage costs, service level, transportation and shipping time delays and the useful shelf life for perishable products.

References

1. Artin E., (1931) *The Gamma Function*, Holt Reinhart &Winston,NY.

2. Brodheim, E., Derman, C., and Prastacos, G.,(1975), " On the Evaluation of a Class of Inventory Policies for Perishable products Such as Blood," *Management Science* , Vol. 21, No. 11, pp. 1320-1325.

3. Bulinskaya, E., (1964) "Some Results Concerning Inventory Policies," *Theory of Probability and its Applications* , Vol. 6, No. 6, pp.389-403.

4. Chazan,D., and Gal, S.,(1977), " A Markovian Model for a Perishable Product Inventory,"*Management Science* , Vol. 23, No. 5, pp. 512-521.

5. Cohen,M.,A., and Pekelman, D.(1978) "LIFO Inventory Systems," *Management Science* , Vol. 24, No. 11, pp. 1150-1162.

6. Dave, U., and Pandya, B.(1985) "Inventory Returns and Special Sales in a Lot-Size System with Constant Rate of Deterioration," *European Journal of Operational Research* , *Management Science* ,Vol. 19, No. 3, pp. 305-312

7. Derman, C. and Klein, M.,(1958) "Inventory Depletion Management,"*Management Science* , Vol. 4, No. 5, pp. 450-456.

8. Emmons, H.(1968) "A Replenishment Model for Radioactive Nuclide Generators," *Management Science* ,Vol. 14, No. 5, pp. 263-274.

9. Federgruen,A., Prastacos,G.,and Zipkin, P.H (1986) "An Allocation and Distribution Model for Perishable Products,".*Operations Research* ,Vol. 34, No. 1, pp.75-82.

10. Freidmann, Y. and Hoch, Y.(1978) "A Dynamic Lot Size Model with Inventory Deterioration,"INFOR, Vol. 16, No. 3, pp.183-188.

11. Fries, B. (1975) "Optimal Ordering Policy for a Perishable Commodity with ,Fixed Lifetime," *Operations Research* ,Vol. 23, No. 1, pp.46-61.

12. Ghare,P. and Schrader,G. (1963) "Model for an Exponentially Decaying Inventory,",*Journal of Industrial Engineering* , *Management Science* , Vol. 23, No. 5, pp. 512-521

13. Graves,S.C., (1982), " The Application of Queueing Theory to Continuous Perishable Inventory Systems," *Management Science* , Vol. 28, No., 4, pp. 400-406.

14. Graves,S.C., and Keilson, J.,(1981), "The Compensation Method Applied to a One-Product Production Inventory Model," *Math. of Operations Research* , Vol. 6, No. 2, pp.246-262.

15. Jaiswal, M.C., and Dave, U. (1980) "An Order Level Inventory Model for Deteriorating Items Subject to Discrete Time Variable," Working Paper, Dept. of Statistics, Gujarat University, Ahmedabad, India.

16. Kaspi, H., and Perry, D.(1983) “Inventory Systems of Perishable Commodities,” *Advances in Applied Probability* , Vol. 15, pp.674-685.

17. Kaspi, H., and Perry, D.(1984) “Inventory Systems for Perishable Commodities with Renewal Input and Poisson Output” *Advances in Applied Probability* , Vol. 16, pp.402-421.

18. Keilson, J.,*Green's Function Methods in Probability Theory* , Charles Griffin & Co', London, England, 1965.

19. Jan, B.B., Jaiswal, M.C.,and Shah, Y.K.,(1978) "(s,q_p) System Inventory Model for Deteriorating Items," *International Journal of Production Research* , Vol. 16, No. 1. pp. 1-9.

20. Perry,D., and Posner, M.J.M.(1990) “Control of Inputs and Demand Rates in Inventory Systems of Perishable Inventories,” *Naval Research Logistics* , Vol. 37, No., 2,pp.85-97.

21. Pierskala,W. and Roach, C.,(1972) "Optimal Issuing Policy for Perishable Inventory,"*Management Science* ,Vol. 18, No. 11, pp. 603-614.

22. Martin G.E. (1986) "An Optimal Decision Model for Disposal of Perishable Inventories,"International Journal of Production Research , Vol. 24, No. 1, pp.73-80.

23.Nahmias S.(1977) "On Ordering Perishable Inventory When Both the Demand and Lifetime are Random," *Management Science* ,Vol. 24, No. 2, pp. 82-90

24. Nahmias S.(1978) "The Fixed Charge Perishable Inventory Problem,"*Operations Research* ,Vol. 26, No. 7, pp.464-481

25. Nahmias, S.(1982a) "Perishable Inventory Theory: A Review,"*Operations Research*,Vol. 30, No. 4, pp.680-708.

26. Nahmias, S.(1982b) "Queueing Models for Controlling Perishable Inventories," in *Economics of Management of Inventories* , A. Chikan (ed),Proceedings of the First International Conference on Inventories. Elsevier, Amsterdam.

27. Nahmias, S.(1989) *Production and Operations Analysis,* Irwin, Boston, MA.

28. Nose, T. (1987) "Studies of Optimal Ordering Policies for Perishable Commodities,"Proceedings of the IX International Conference on Production Research,Cincinnati, Ohio,pp.1365-1371.

29. Nose, T., Kuriyama , S. and Fukui, H.(1981) "Perishable Inventory Management Subject to Price Discounting," *Japan Management Association* ,Vol. 32, No. 3, pp. 208-212.

30. Nose, T.Ishii, H.,and Nishida,T.(1984) "Perishable Inventory Management with Stochastic Leadtimes and Different Selling Prices," *European Journal of Operational Research* , *Management Science* ,Vol. 18, No. 11, pp. 603-614.

31 Pegeles, C.,C., (1986) "A Probabilistic Batch Size Model for Perishable Products," *International Journal of Operations and Production Management* (IJOPM), Vol. 6, No. 2, pp.36-46.

32. Pegeles.C. C.,and Jelmert, A.E. (1970) "An Evaluation of Blood-Inventory Policies: A Markov Chain Application," *Operations Research* , Vol. 18, No. 6, pp. 1087-1098.

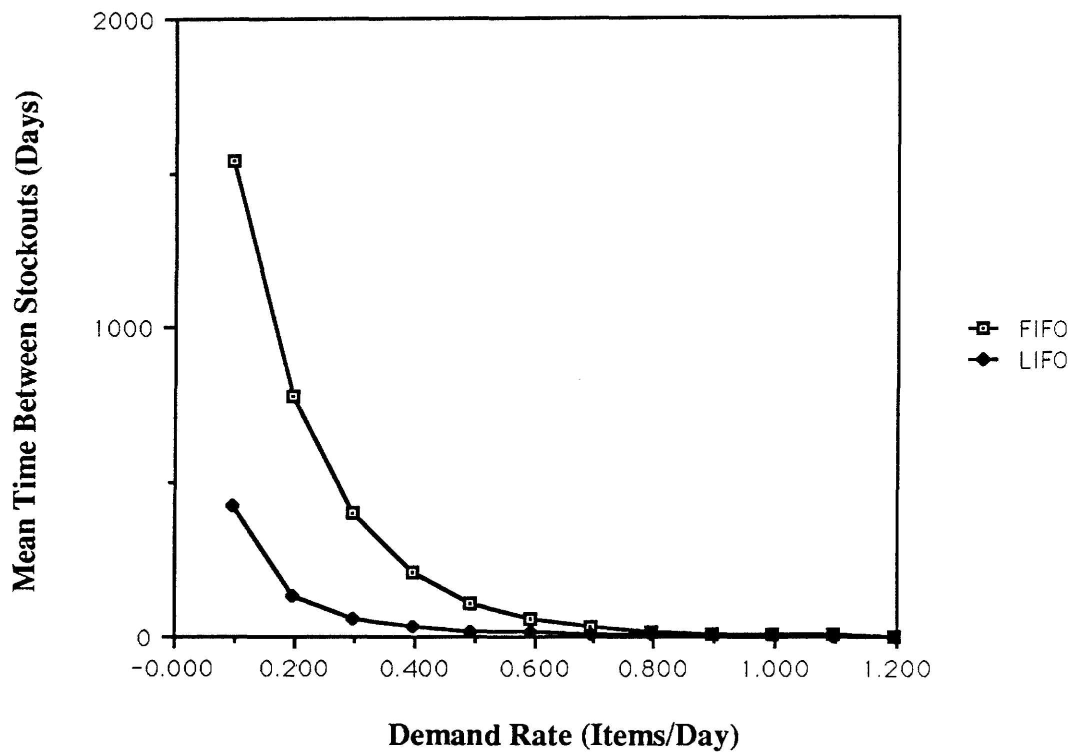
33. Peterson.R., and Silver, E., A.(1979) Decision Systems for Inventory Management and Production Planning,Wiley,NY.

34. Prastacos, G.P.(1978) "Optimal Myopic Allocation of a Product with Fixed Lifetime,"*J.of Operations Research* , Vol. 29, No. 3.pp.185-193.

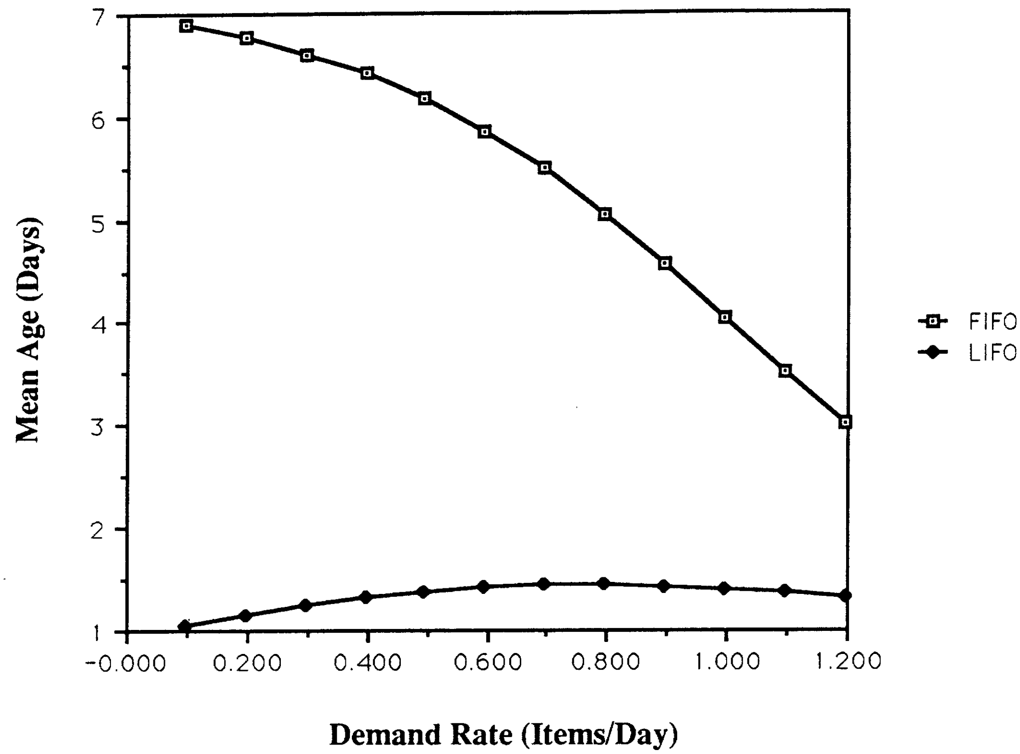
35.Schmidt, C.P., and Nahmias, S. (1985) "(S-1,S) Policies for Perishable Inventories," *Management Science* ,Vol. 31, No. 6, pp. 719-728.

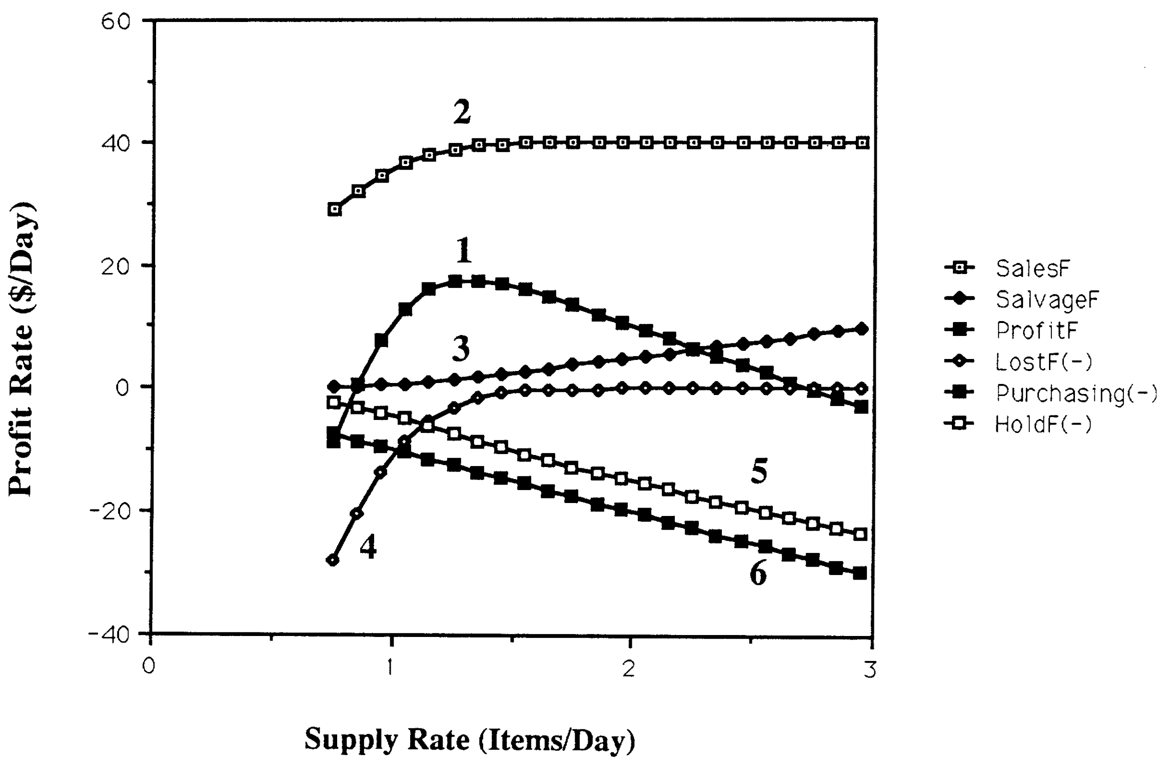
36. Takacs, L.(1962) *Introduction to the Theory of Queues* , Oxford University Press, London.

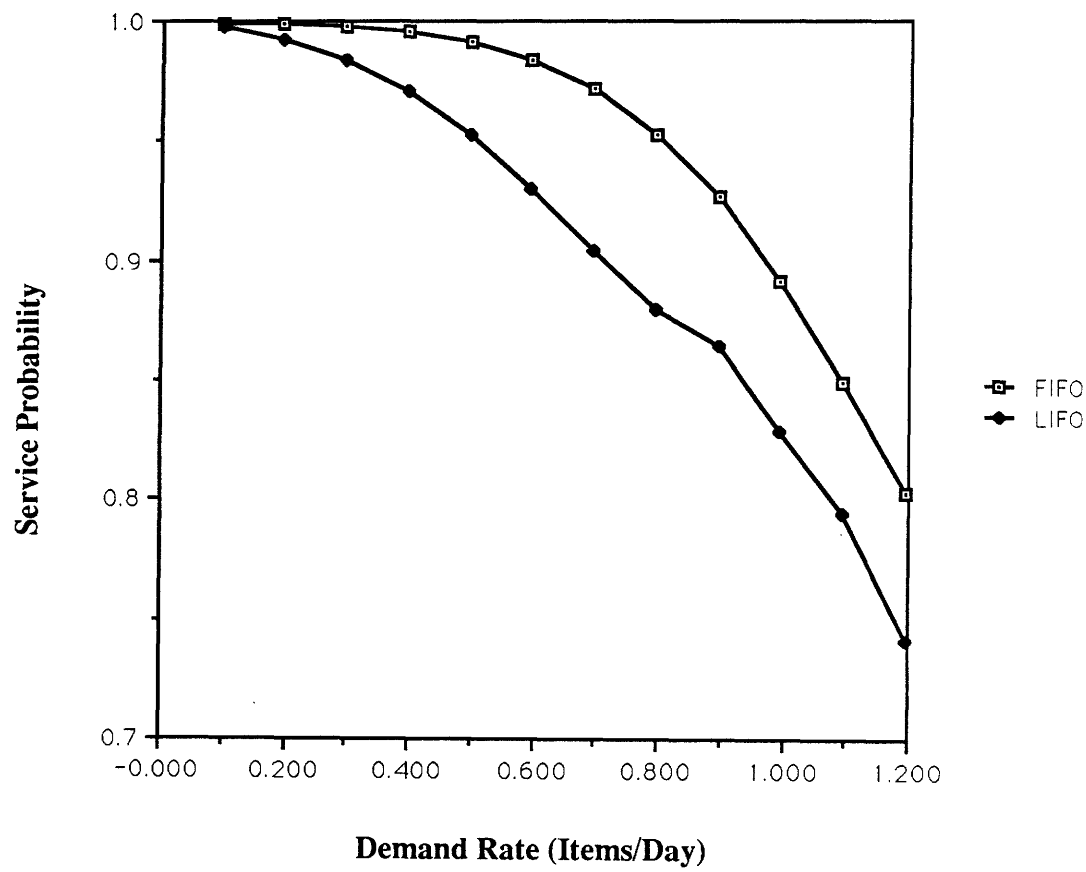
37. Veinott, A.F.(1960) "Optimal Ordering, Issuing and Disposal of Inventory with Known demand," unpublished Ph.D. dissertation, Columbia University.



Mean Time Btwn Stockouts706.pic

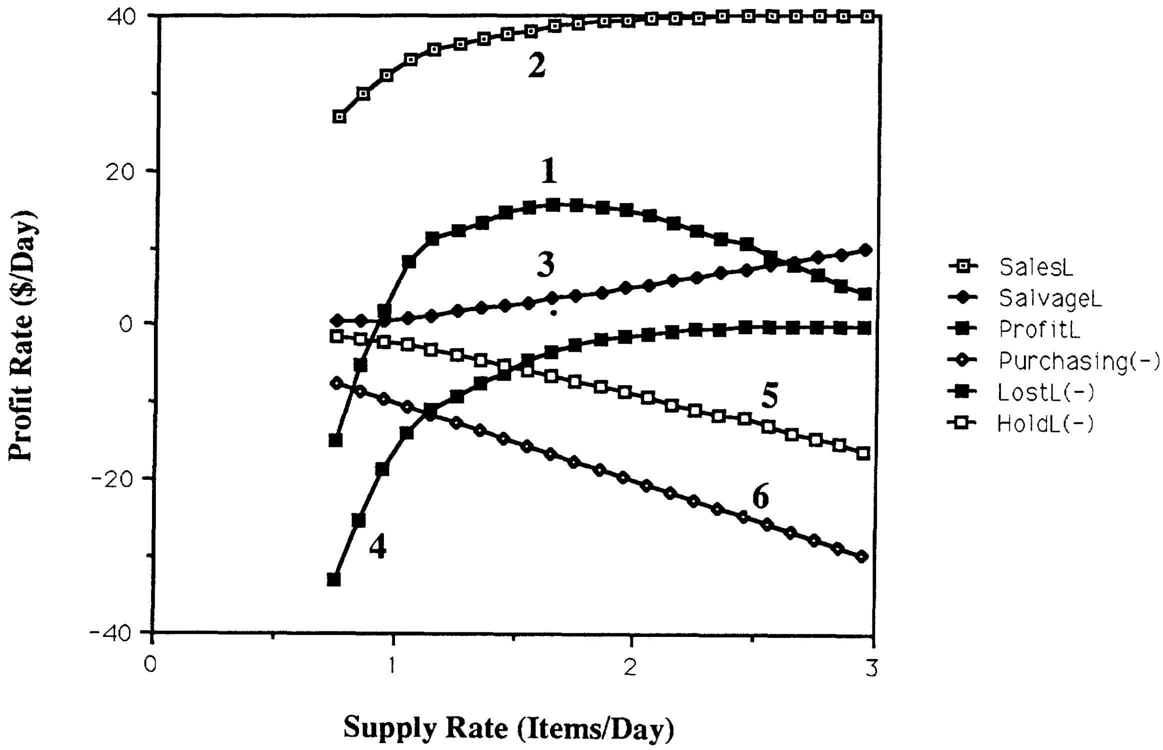

Mean Age (Days)
7
6
5
4
3
2
1
-0.000
0.200
0.400
0.600
0.800
1.000
1.200
Demand Rate (Items/Day)
FIFO
LIFO


Profit Rate ($/Day)
60
40
20
0
-20
-40
0
1
2
3
Supply Rate (Items/Day)
1
2
3
4
5
6
SalesF
SalvageF
ProfitF
LostF(-)
Purchasing(-)
HoldF(-)

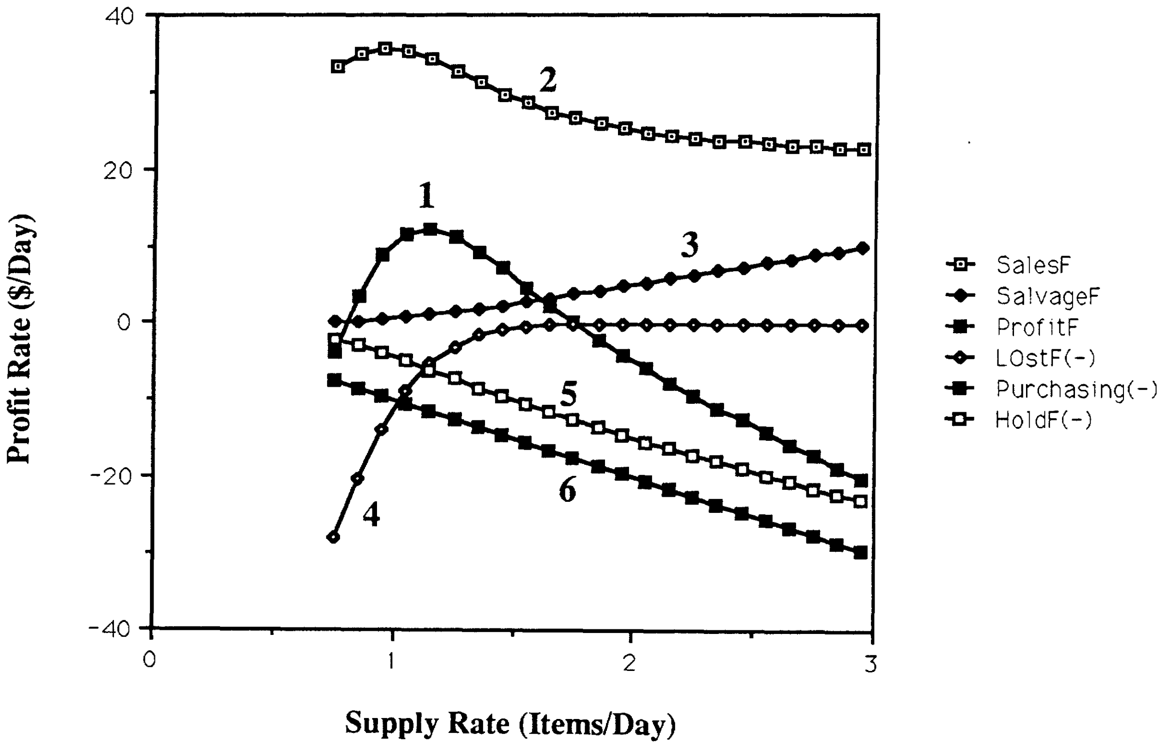


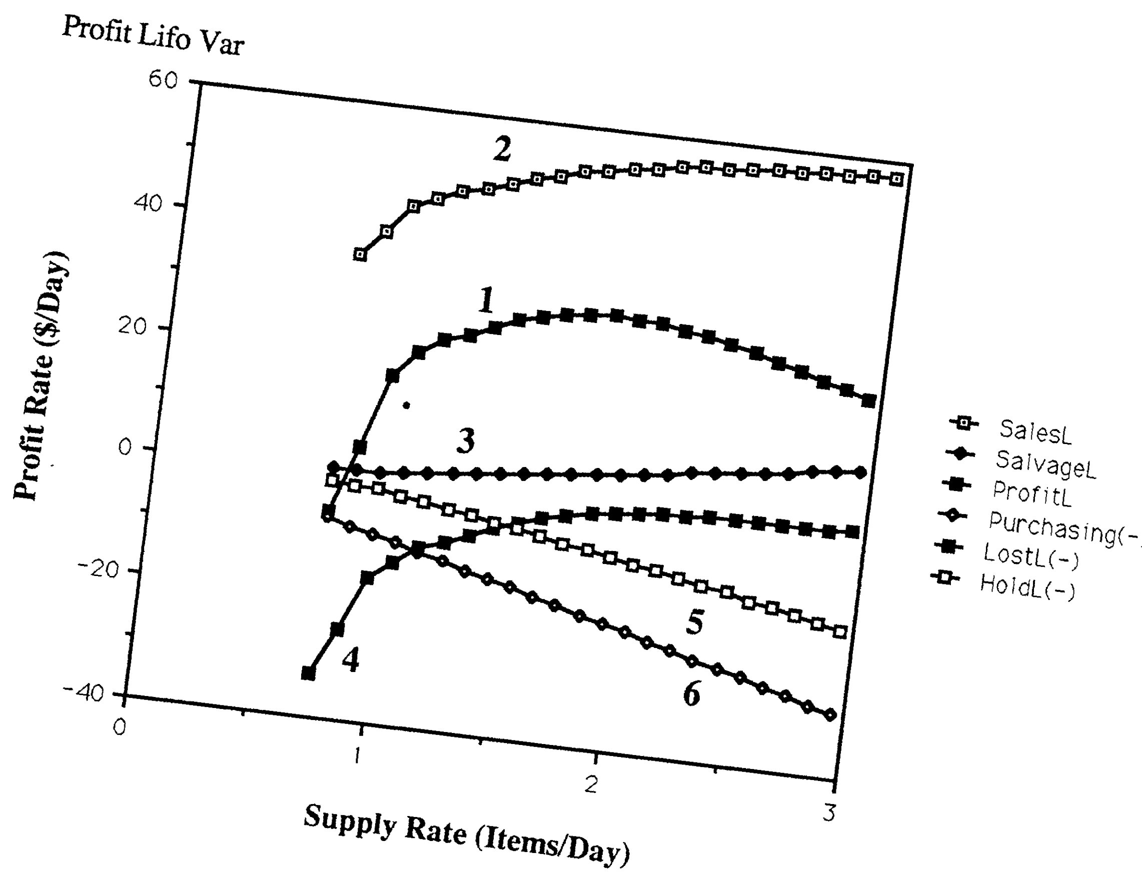
1.0
0.9
0.8
0.7
-0.000
0.200
0.400
0.600
0.800
1.000
1.200
Service Probability
Demand Rate (Items/Day)
FIFO
LIFO

profit lifo (fixed)

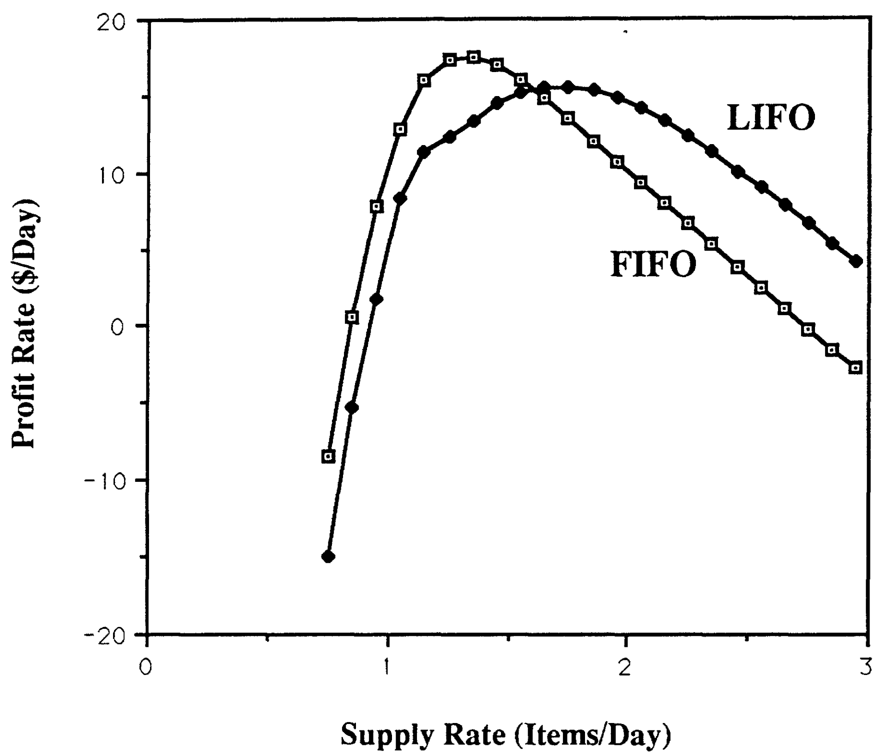


Profit Fifo Var

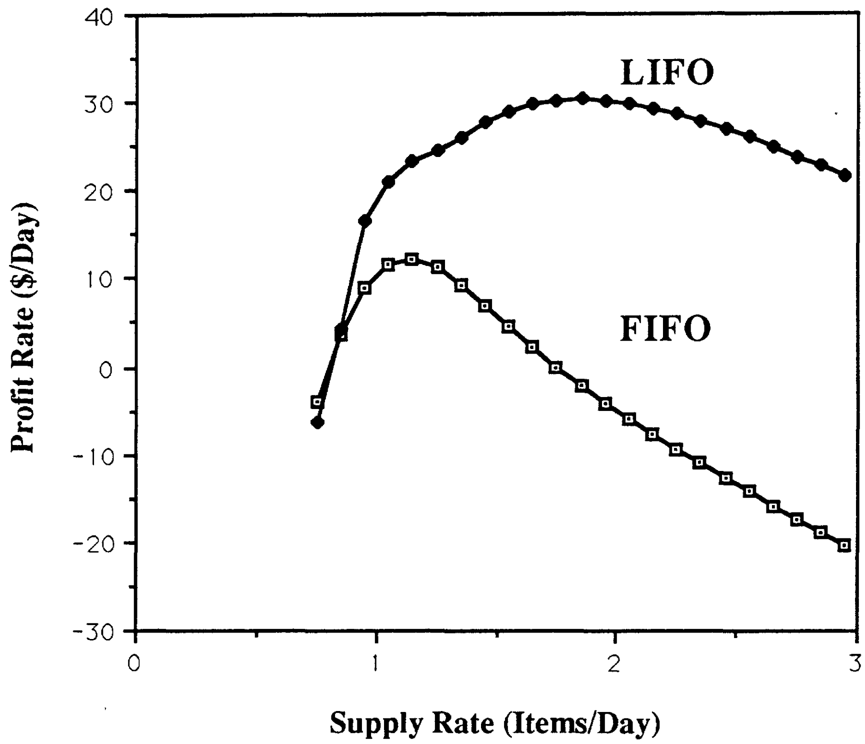


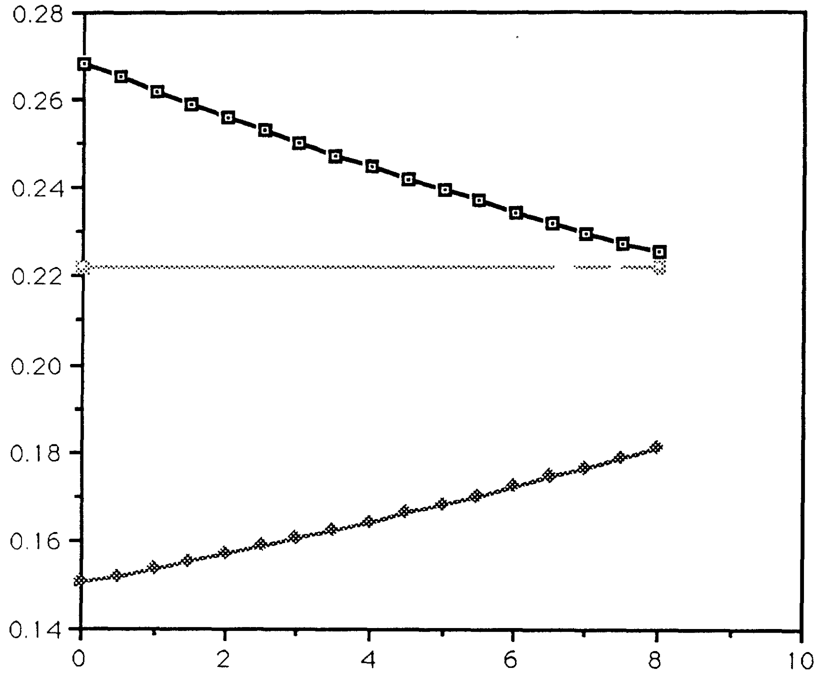

Profit Lifo Var
Profit Rate ($/Day)
60
40
20
0
-20
-40
0
1
2
3
Supply Rate (Items/Day)
1
2
3
4
5
6
SalesL
SalvageL
ProfitL
Purchasing(-)
LostL(-)
HoldL(-)

Profit Fixed

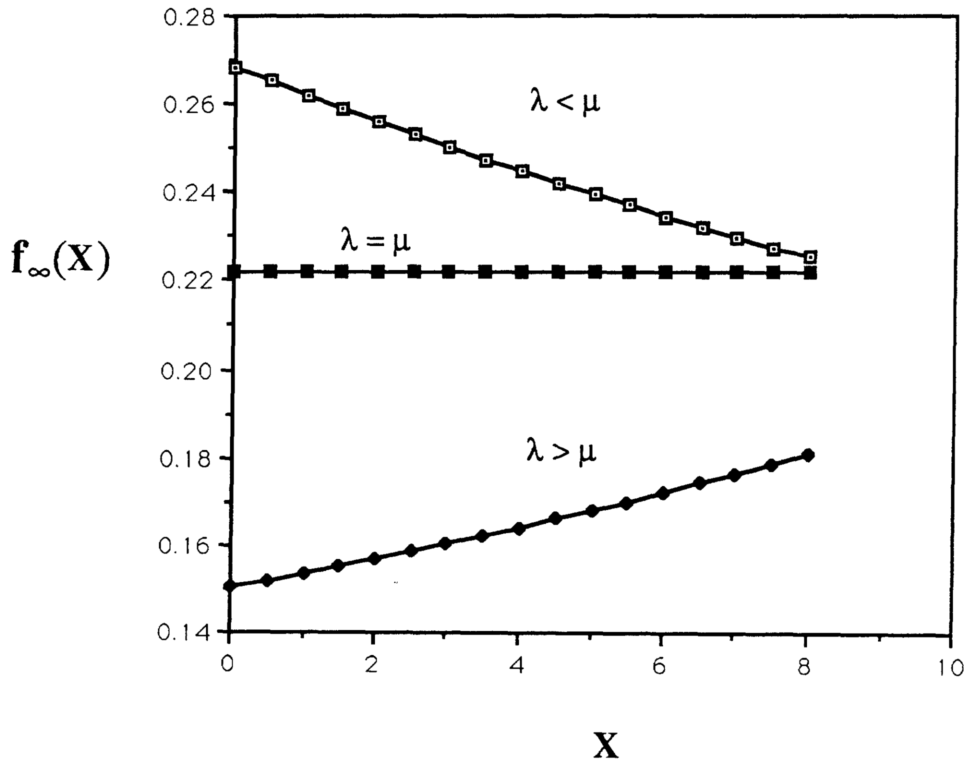


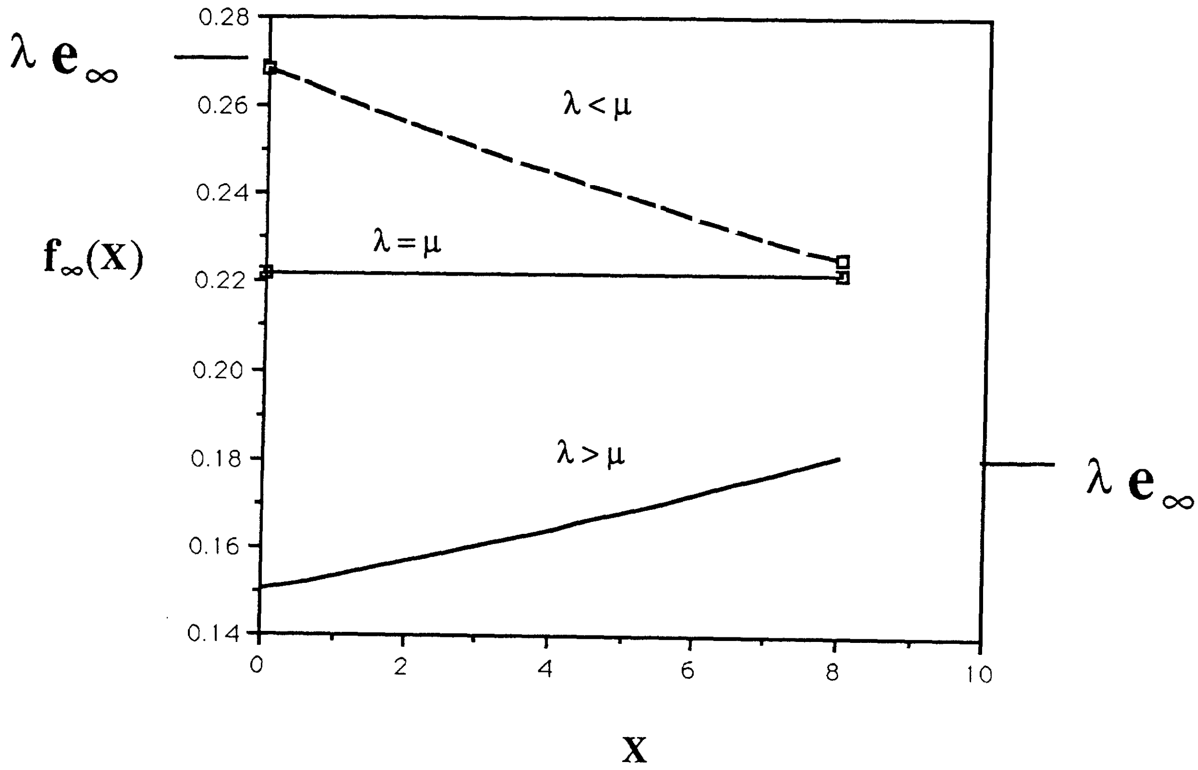
Profit Variable



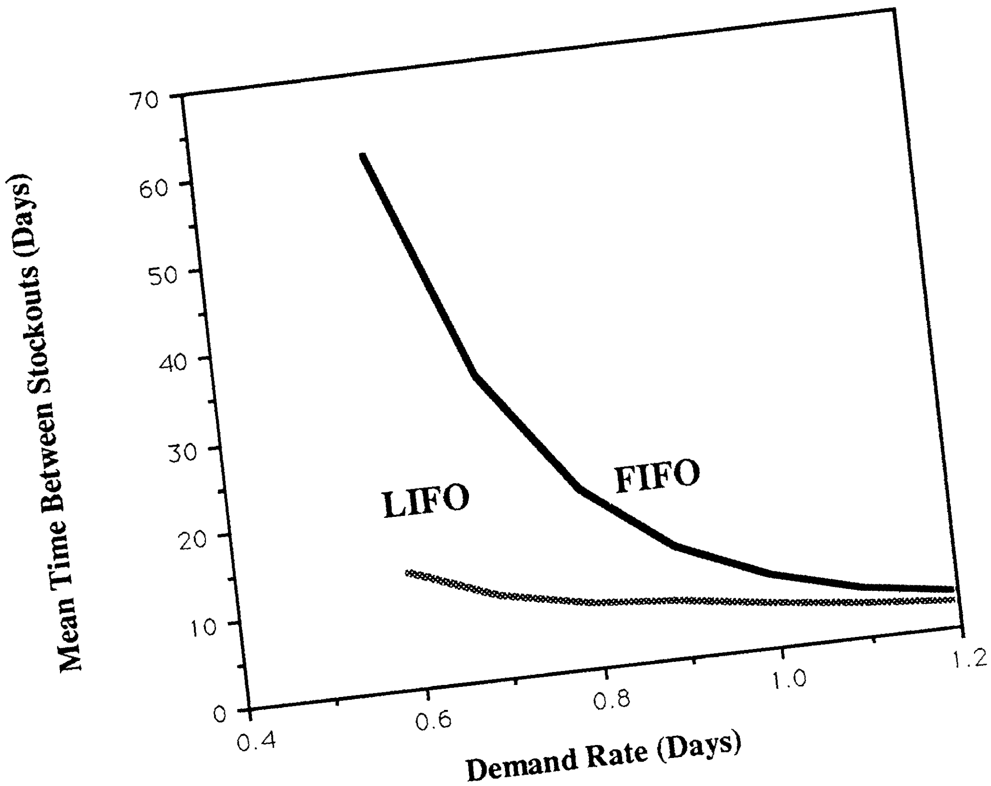

0.28
0.26
0.24
0.22
0.20
0.18
0.16
0.14
0
2
4
6
8
10

d=8, λ=1.05, 0.95, μ=1

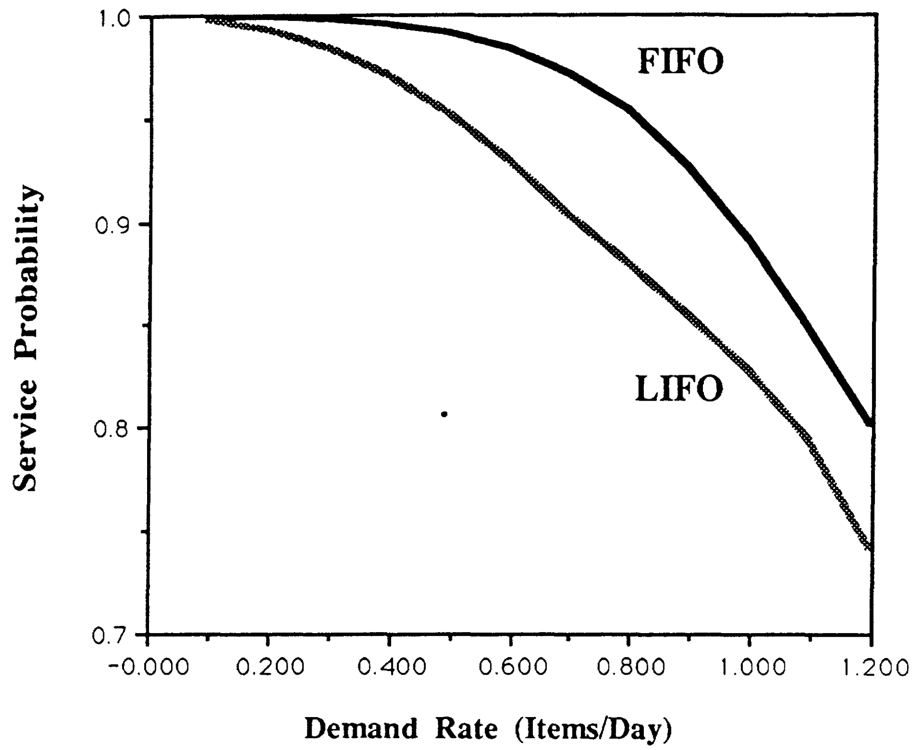




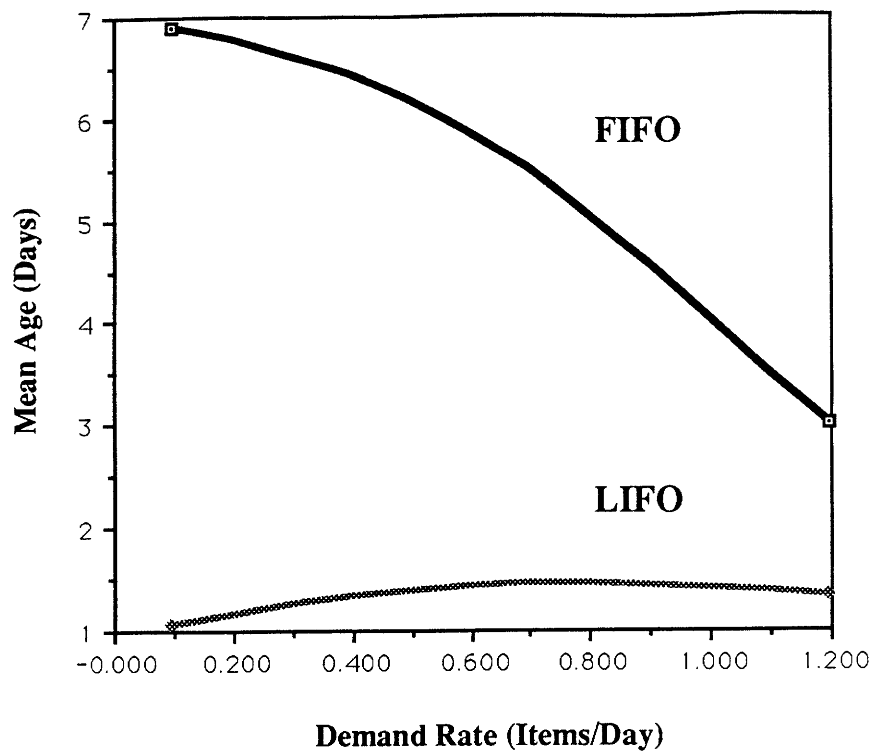
λ e∞
λ e∞
f∞(X)
λ < μ
λ = μ
λ > μ
0.28
0.26
0.24
0.22
0.20
0.18
0.16
0.14
0
2
4
6
8
10
X



Mean Time Between Stockouts (Days)
70
60
50
40
30
20
10
0
0.4
0.6
0.8
1.0
1.2
LIFO
FIFO
Demand Rate (Days)



FIFO
LIFO
Service Probability
1.0
0.9
0.8
0.7
-0.000
0.200
0.400
0.600
0.800
1.000
1.200
Demand Rate (Items/Day)


7
6
5
4
3
2
1
Mean Age (Days)
FIFO
LIFO
-0.000
0.200
0.400
0.600
0.800
1.000
1.200
Demand Rate (Items/Day)